Implied conditional moments by Cornish-Fisher expansion and their applications

Ningning Zhang and Ke Zhu

University of Hong Kong

August 19, 2021

Outline

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Important conditional moments

In many applications, it is crucially important to investigate the conditional moments of financial time series over time. Consider $y_t \in \mathbb{R}$, and let $\mathcal{F}_t = \sigma(y_s; s \leqslant t)$ denote the available information at time t. Given $\mathcal{F}_{t-1}$, the conditional mean, variance, skewness and kurtosis of y_t are defined by

$$
\begin{aligned}
\mu_t &= E(y_t|\mathcal{F}_{t-1}), \\
h_t &= Var(y_t|\mathcal{F}_{t-1}), \\
s_t &= E\Big(\Big(\frac{y_t - \mu_t}{\sqrt{h_t}}\Big)^3|\mathcal{F}_{t-1}\Big), \qquad (1.1) \\
k_t &= E\Big(\Big(\frac{y_t - \mu_t}{\sqrt{h_t}}\Big)^4|\mathcal{F}_{t-1}\Big).
\end{aligned}
$$

Conditional mean μ_t and conditional variance h_t

- As for the conditional mean μ_t, the time-varing μ_t may support us to challenge the efficient market hypothesis in Fama (1970, JOF). If μ_t is tested to have dynamic structure, we could use a linear model (e.g., the autoregressive moving-average model) or a nonlinear model (e.g., the threshold autoregressive model) to study μ_t.

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- For the conditional variance h_t, it plays an important role in the option pricing, derivative pricing, portfolio selection and risk management. It has been widely used as a proxy for risk in financial returns. In the literature, many parametric models are proposed to study the dynamic structures of h_t, such as the generalized autoregressive conditional heterosecdasticity (GARCH) model and its variants (see Engle (1982, Econometrica)).

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- The conditional kurtosis is also getting more and more attention since it can be regarded as the variance of variance and served as a judgment for the modeling of returns and conditional variance.
- Yet, only several research works have been involved in the study of conditional skewness and kurtosis (see e.g., Harvey and Siddique (1999, JFQA), Jondeau and Rockinger (2003, JEDC)), which lacks the study compared with a wealth of research for conditional variance.

Implied conditional moments

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- The greatest strength of the implied conditional moments is that they enable us to imply the conditional moments without any parametric assumptions, and the estimation errors in these implied conditional moments are expected to be small once the conditional quantiles are chosen appropriately.

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Implied conditional moments

Given $\mathcal{F}_{t-1}$, $Q_t(\alpha)$ denotes the conditional quantile of y_t at quantile level α. By the Cornish-Fisher (CF) expansion (see Cornish and Fisher (1938, ISI)), we have

$$Q_t(\alpha) = \mu_t + \sqrt{h_t}\omega_t(\alpha), \tag{2.1}$$

where

$$\begin{aligned}\omega_t(\alpha) =& x + (x^2 - 1)\frac{s_t}{6} + (x^3 - 3x)\frac{k_t}{24} \\ & + \text{remaining terms on the higher-order conditional moments,}\end{aligned} \tag{2.2}$$

and $x = \Phi^{-1}(\alpha)$ with $\Phi(\cdot)$ being the unit normal distribution $N(0, 1)$. By ignoring the remaining terms in (2.2), the results in (2.1)-(2.2) imply

$$Q_t(\alpha) \approx \mu_t + \sqrt{h_t}[x + (x^2 - 1)\frac{s_t}{6} + (x^3 - 3x)\frac{k_t}{24}]. \tag{2.3}$$

Taking four different quantile levels α_i, $i = 1, 2, 3, 4$, the result in (2.3) entails that for each t,

Implied conditional moments

$$\begin{pmatrix} Q_t(\alpha_1) \\ Q_t(\alpha_2) \\ Q_t(\alpha_3) \\ Q_t(\alpha_4) \end{pmatrix} - \begin{pmatrix} 1 & x_1 & \frac{x_1^2-1}{6} & \frac{x_1^3-3x_1}{24} \\ 1 & x_2 & \frac{x_2^2-1}{6} & \frac{x_2^3-3x_2}{24} \\ 1 & x_3 & \frac{x_3^2-1}{6} & \frac{x_3^3-3x_3}{24} \\ 1 & x_4 & \frac{x_4^2-1}{6} & \frac{x_4^3-3x_4}{24} \end{pmatrix} \begin{pmatrix} \mu_t \\ \sqrt{h_t} \\ \sqrt{h_t} s_t \\ \sqrt{h_t} k_t \end{pmatrix} \triangleq \boldsymbol{Q}_t - \boldsymbol{X}\boldsymbol{M}_t \approx 0, \tag{2.4}$$

where $x_i = \Phi^{-1}(\alpha_i)$ for $i = 1, 2, 3, 4$. In view of (2.4), it motivates us to estimate $\boldsymbol{M}_t$ by

$$\widehat{\boldsymbol{M}}_t \triangleq \left(\widehat{M}_{1t}, \widehat{M}_{2t}, \widehat{M}_{3t}, \widehat{M}_{4t}\right)' = \boldsymbol{X}^{-1}\boldsymbol{Q}_t, \tag{2.5}$$

provided that $\boldsymbol{X}$ is invertible. Using $\widehat{\boldsymbol{M}}_t$, we then can estimate μ_t, h_t, s_t, and k_t by

$$\widehat{\mu}_t = \widehat{M}_{1t}, \ \widehat{h}_t = \widehat{M}_{2t}^2, \ \widehat{s}_t = \widehat{M}_{3t}/\widehat{M}_{2t}, \text{ and } \widehat{k}_t = \widehat{M}_{4t}/\widehat{M}_{2t} + 3. \tag{2.6}$$

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- Third, the implied conditional moments can apply to the non-stationary and heteroscedastic data as long as the valid $Q_t(\alpha)$ is obtained in this case.
- Fourth, the idea of implied conditional moments can be extended to the multivaraite or high dimensional cases, once the corresponding $Q_t(\alpha)$ for each univariate entry is fairly provided.

Outline

Quantiles estimation methods

Quantiles estimation methods

- **Univariate case**:
  - Conditional Autoregressive Value at Risk (CAViaR) in Engle and Manganelli (2004, JBES)
    1. Adaptive:
    $$Q_t(\alpha) = Q_{t-1}(\alpha) + \beta_1\{[1 + \exp(G[y_{t-1} - Q_{t-1}(\alpha)])]^{-1} - \alpha\}.$$
    2. Symmetric absolute value:
    $$Q_t(\alpha) = \beta_1 + \beta_2 Q_{t-1}(\alpha) + \beta_3|y_{t-1}|.$$
    3. Asymmetric slope:
    $$Q_t(\alpha) = \beta_1 + \beta_2 Q_{t-1}(\alpha) + \beta_3(y_{t-1})^+ + \beta_4(y_{t-1})^-.$$
    4. Indirect GARCH(1,1):
    $$Q_t(\alpha) = (\beta_1 + \beta_2 Q^2_{t-1}(\alpha) + \beta_3 y^2_{t-1})^{1/2}.$$
  - Quantile Autoregression (QAR) in Xiao and Koenker (2009, JASA)
    5. QAR(1):
    $$Q_t(\alpha) = \theta_0(\alpha) + \theta_1(\alpha) y_{t-1}$$

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- **Multivariate case**:
  Multivariate Quantile Autoregression (MQAR) in White et al. (2015, JOE)
  $$Q_{1t}(\alpha) = X_t^{'}\beta_1 + b_{11}Q_{1t-1}(\alpha) + b_{12}Q_{2t-1}(\alpha),$$
  $$Q_{2t}(\alpha) = X_t^{'}\beta_2 + b_{21}Q_{1t-1}(\alpha) + b_{22}Q_{2t-1}(\alpha),$$

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- This gives us the motivation to check the validity of the implied conditional moments based on some regression-based tests. To illustrate the testing idea, we consider a regression model for $\hat{\mu}_t$:

$$y_t = a_1^{\mu} + a_2^{\mu}\hat{\mu}_t + \epsilon_t^{\mu}. \tag{2.7}$$

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- We use the heteroscedasticity-robust Wald test W_μ in the spirit of Mincer and Zarnowitz (1969, NBER), White (1980, Econometrica), Pagan and Schwert (1990, JOE) to detect the null hypothesis

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- If H_0^{μ} is rejected by W_{μ}, then we conclude that $\hat{\mu}_t$ is invalid; otherwise, we conclude that $\hat{\mu}_t$ is valid.
- By applying this test to all choices of $\hat{\mu}_t$ from different quantile estimation methods, we select the best one with the largest p-value.

Validity checks for implied conditional moments

- Following the similar idea, we can further check the validity of $\hat{h}_t$, $\hat{s}_t$, and $\hat{k}_t$ by introducing three regression models

$$
\begin{aligned}
(y_t - \hat{\mu}_t)^2 &= a_1^h + a_2^h \hat{h}_t + \epsilon_t^h, \\
\left(\frac{y_t - \hat{\mu}_t}{\hat{h}_t^{1/2}}\right)^3 &= a_1^s + a_2^s \hat{s}_t + \epsilon_t^s, \\
\left(\frac{y_t - \hat{\mu}_t}{\hat{h}_t^{1/2}}\right)^4 &= a_1^k + a_2^k \hat{k}_t + \epsilon_t^k.
\end{aligned} \tag{2.8}
$$

- Then construct three heteroscedasticity-robust Wald tests W_h, W_s, and W_k, respectively, to detect three null hypotheses

$$
\begin{aligned}
H_0^h &: a_1^h = 0 \text{ and } a_2^h = 1, \\
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- Following the similar idea, we can further check the validity of $\hat{h}_t$, $\hat{s}_t$, and $\hat{k}_t$ by introducing three regression models

$$\begin{aligned}(y_t-\hat{\mu}_t)^2 &= a_1^h + a_2^h\hat{h}_t + \epsilon_t^h,\\ (\frac{y_t-\hat{\mu}_t}{\hat{h}_t^{1/2}})^3 &= a_1^s + a_2^s\hat{s}_t + \epsilon_t^s,\\ (\frac{y_t-\hat{\mu}_t}{\hat{h}_t^{1/2}})^4 &= a_1^k + a_2^k\hat{k}_t + \epsilon_t^k.\end{aligned} \tag{2.8}$$

- Then construct three heteroscedasticity-robust Wald tests W_h, W_s, and W_k, respectively, to detect three null hypotheses

$$\begin{aligned}H_0^h &: a_1^h = 0 \text{ and } a_2^h = 1,\\ H_0^s &: a_1^s = 0 \text{ and } a_2^s = 1,\\ H_0^k &: a_1^k = 0 \text{ and } a_2^k = 1.\end{aligned} \tag{2.9}$$

- Since W_h depends on $\hat{\mu}_t$, and W_s (or W_k) depends on $\hat{\mu}_t$ and $\hat{h}_t$, we will implement W_μ, W_h, and W_s (or W_k) sequentially. That is, we first apply W_μ to pick up the best $\hat{\mu}_t$, and then use this best $\hat{\mu}_t$ to compute W_h whereby the best $\hat{h}_t$ is chosen. In the end, we implement W_s and W_k analogically based on the best $\hat{\mu}_t$ and $\hat{h}_t$, to obtain the best $\hat{s}_t$ and $\hat{k}_t$.

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- The specific procedure for selection of quantile levels:
  1. First, quantile levels $\alpha_1, \alpha_2, \alpha_3$, α_4 respectively take some values from the four intervals $[0.01, 0.05]$, $[0.06, 0.10]$, $[0.90, 0.94]$, $[0.95, 0.99]$ for a collection of quantile levels $\mathcal{A}$.
  2. For each pair of quantile levels in $\mathcal{A}$, the corresponding implied conditional moments could be obtained by our proposed method.
  3. Among various implied conditional moments, carry out the Wald tests W_μ, W_h, W_s, W_k for the validity of the implied conditional moments.
  4. Finally, select the quantile level with highest p-values of the Wald tests of interest.

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Simulated data

The following simulations adopt ARMA-MN-GARCH(1,1) model in Haas et al. (2004, JOFQA) to simulate the real data with sample size $T = 1500$, and all experiments are carried out based on 1000 replications. Consider a time series Y_t is an ARMA process as

$$Y_t = a_0 + a_1 Y_{t-1} + \epsilon_t + b_1 \epsilon_{t-1}, \tag{3.1}$$

where ϵ_t refers to an MN-GARCH process

$$\epsilon_t \sim MN(\lambda_1, \lambda_2, \mu_1, \mu_2, \sigma^2_{1,t}, \sigma^2_{2,t}), \tag{3.2}$$

where $\lambda_i \in (0,1)$, $i = 1,2$, $\lambda_1 + \lambda_2 = 1$, and $\mu_k = -\sum_{i=1}^{k-1} (\lambda_i/\lambda_k)\mu_i$, $k = 1,2$. Further, for $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$, we have

$$\begin{pmatrix} \sigma^2_{1,t} \\ \sigma^2_{2,t} \end{pmatrix} = \begin{pmatrix} \alpha_{10} \\ \alpha_{20} \end{pmatrix} + \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix} \epsilon^2_{t-1} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{22} \end{pmatrix} \begin{pmatrix} \sigma^2_{1,t-1} \\ \sigma^2_{2,t-1} \end{pmatrix}. \tag{3.3}$$

Simulated data

Then, we have its theoretical conditional moments as:

$$\mu_t^0 = \alpha_0 + \alpha_1 Y_{t-1} + b_1 \epsilon_{t-1}$$
$$h_t^0 = \lambda_1(\mu_1^2 + \sigma_{1t}^2) + \lambda_2(\mu_2^2 + \sigma_{2t}^2) - (\lambda_1\mu_1 + \lambda_2\mu_2)^2$$
$$s_t^0 = \frac{\lambda_1(\mu_1^3 + 3\sigma_{1t}^2\mu_1) + \lambda_2(\mu_2^3 + 3\sigma_{2t}^2\mu_2)}{(\lambda_1(\mu_1^2 + \sigma_{1t}^2) + \lambda_2(\mu_2^2 + \sigma_{2t}^2))^{3/2}}; \tag{3.4}$$
$$k_t^0 = \frac{\lambda_1(\mu_1^4 + 6\mu_1^2\sigma_{1t}^2 + 3\sigma_{1t}^4) + \lambda_2(\mu_2^4 + 6\mu_2^2 * \sigma_{2t}^2 + 3\sigma_{2t}^4)}{(\lambda_1(\mu_1^2 + \sigma_{1t}^2) + \lambda_2(\mu_2^2 + \sigma_{2t}^2))^2}$$

and its theoretical conditional quantiles as:

$$Q_t^0(\alpha) = \mu_t^0 + Q_t^\epsilon(\alpha), \tag{3.5}$$

where $Q_t^\epsilon(\alpha)$ satisfies $\lambda_1 F_t(Q_t^\epsilon(\alpha), \mu_1, \sigma_{1,t}) + \lambda_2 F_t(Q_t^\epsilon(\alpha), \mu_2, \sigma_{2,t}) = \alpha$, and $F_t(Q_t^\epsilon(\alpha), \mu_i, \sigma_{i,t})$ represents the normal curriculum distribution function with mean μ_i and standard deviation $\sigma_{i,t}$, evaluated at the values in $Q_t^\epsilon(\alpha)$, $i = 1, 2$.

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Basic results

- First, we generate 1000 sets of simulated data and utilize the theoretical conditional quantiles and CF expansion to estimate implied conditional moments.
- We compare our implied conditional moments with theoretical moments and investigate the accuracy by computing the root mean squared error (RMSE) as

$$\text{RMSE}^{\mu} = \sqrt{\frac{\sum_{t=1}^{T}(\mu_t^0 - \hat{\mu}_t)^2}{T}}, \text{RMSE}^{h} = \sqrt{\frac{\sum_{t=1}^{T}(h_t^0 - \hat{h}_t)^2}{T}},$$

$$\text{RMSE}^{s} = \sqrt{\frac{\sum_{t=1}^{T}(s_t^0 - \hat{s}_t)^2}{T}}, \text{RMSE}^{k} = \sqrt{\frac{\sum_{t=1}^{T}(k_t^0 - \hat{k}_t)^2}{T}}.$$

Basic results

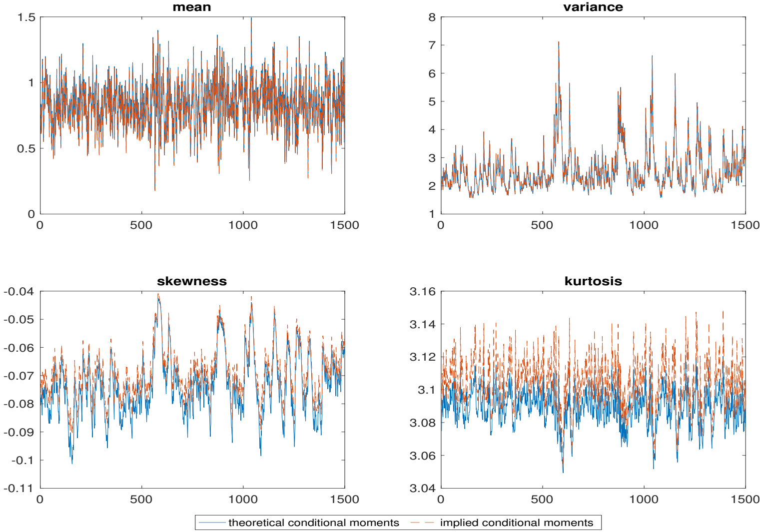

Figure 1: The comparison of theoretical conditional moments and implied conditional moments.

Basic results: RMSE

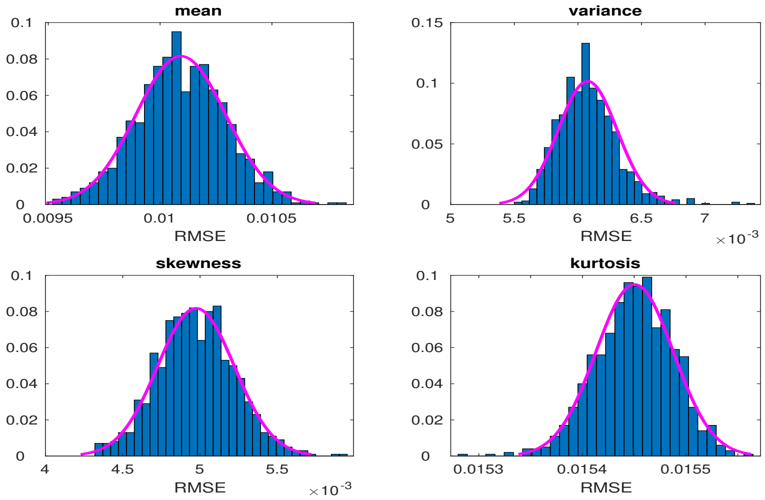

Figure 2: The density of RMSEs between theoretical conditional moments and implied conditional moments.

Basic results: regression-based tests

Table 1: The relative frequencies (%) having valid implied conditional moments at 1%, 5%, and 10% confidence levels.

Confidence levels	Mean	Variance	Skewness	Kurtosis
1%	99.2000	97.2000	99.2000	80.9000
5%	94.6000	91.9000	95.4000	68.3000
10%	89.6000	85.2000	89.3000	61.0000

Remarks: The implied conditional moment is valid if its validity is not rejected by the related Wald test at a given confidence interval.

Outline

Sensitivity on conditional quantiles

In this part, we calculate implied conditional moments based on $\boldsymbol{Q}_t$, where $\boldsymbol{Q}_t = \boldsymbol{Q}_t^0 +$ white noise $\mathrm{N}(0,\ \sigma^2)$, and $\boldsymbol{Q}_t^0$ is the theoretical conditional quantile.

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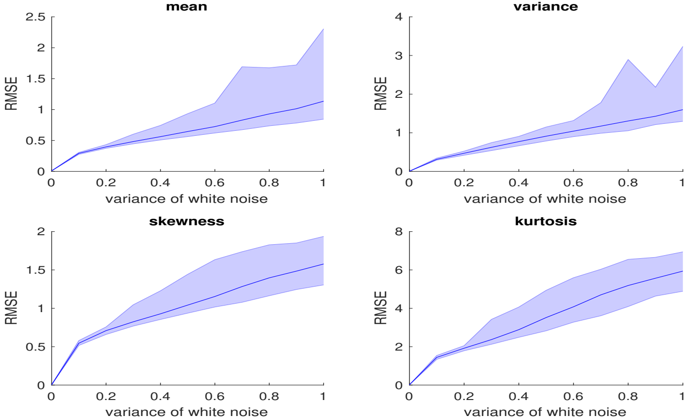

Figure 3: The plots of RMSEs between theoretical conditional moments and implied conditional moments using $\boldsymbol{Q}_t$.

Sensitivity on conditional quantiles

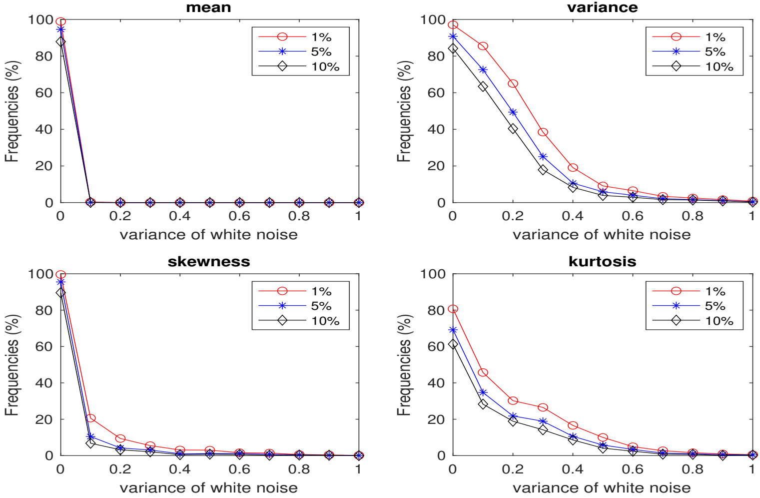

Figure 4: At 1%, 5%, and 10% confidence levels, the relative frequencies (%) having valid implied conditional moments using $\boldsymbol{Q}_t$.

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Sensitivity on quantile levels

- In the previous studies, we fix the quantile level $[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ as $[0.025, 0.1, 0.9, 0.975]$ when implying the conditional moments. In this part, we investigate the sensitivity of our implied conditional moments on various quantile levels.

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- For the sake of simplicity, we set the four quantile levels taking values from the sets $\{0.01, 0.03, 0.05\}$, $\{0.06, 0.08, 0.10\}$, $\{0.90, 0.92, 0.94\}$ and $\{0.95, 0.97, 0.99\}$, respectively. In this manner, there are 81 different quantile levels.

Sensitivity on quantile levels

- In the previous studies, we fix the quantile level $[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ as $[0.025, 0.1, 0.9, 0.975]$ when implying the conditional moments. In this part, we investigate the sensitivity of our implied conditional moments on various quantile levels.
- For the sake of simplicity, we set the four quantile levels taking values from the sets $\{0.01, 0.03, 0.05\}$, $\{0.06, 0.08, 0.10\}$, $\{0.90, 0.92, 0.94\}$ and $\{0.95, 0.97, 0.99\}$, respectively. In this manner, there are 81 different quantile levels.
- We firstly simulate 1000 simulated data sets, and then respectively obtain the implied conditional moments by these 81 sets of quantile levels.

Sensitivity on quantile levels

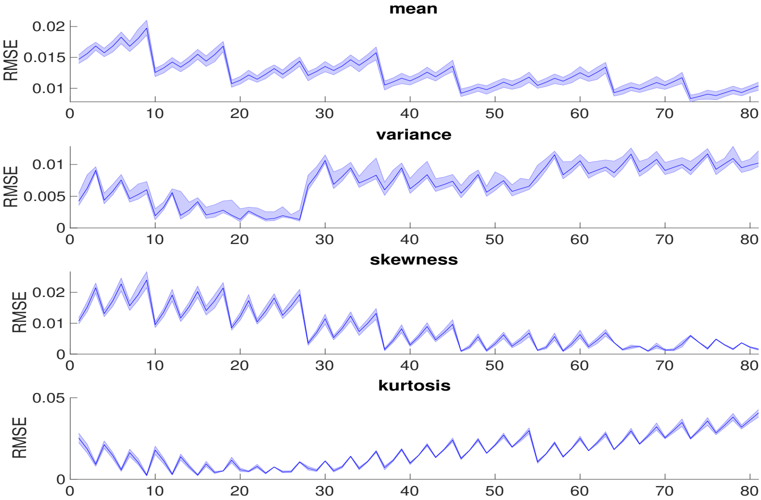

Figure 5: The plots of RMSEs between theoretical conditional moments and implied conditional moments across 81 different quantile levels.

Sensitivity on quantile levels

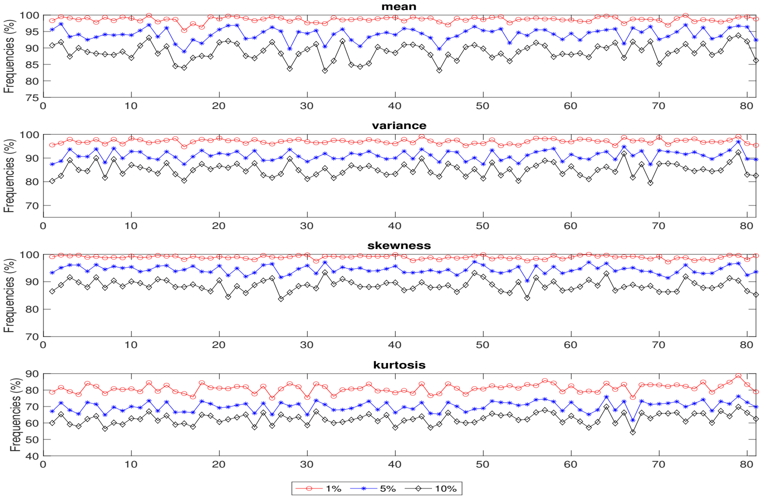

Figure 6: At 1%, 5%, and 10% confidence levels, the plots of relative frequencies (%) having valid conditional moments across 81 different quantile levels.

Outline

Outline

Inference for the unknown news impact functions

- Engle and Ng (1993, JOF) proposed the notion of "news impact function" to exploit how the conditional variance h_t is influenced by the shock ε_{t-1}, where $\varepsilon_t = y_t - \mu_t$. To be specific, suppose h_t could be modeled by

$$h_t = \theta_h h_{t-1} + g_2(\varepsilon_{t-1}), \tag{4.1}$$

in which $g_2(\cdot)$ is the so-called "news impact function", and $\theta_h \in (0, 1)$ is the auto-regressive parameter of h_t.

Inference for the unknown news impact functions

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  in which $g_2(\cdot)$ is the so-called "news impact function", and $\theta_h \in (0, 1)$ is the auto-regressive parameter of h_t.
- (4.1) is the parent model of ARCH-type models, for example, the GARCH model with $g_2(x) = a_0 + a_1 x^2$, the assymmetric GARCH model with $g_2(x) = a_0 + a_1(x + a_2)^2$, and the GJR model with $g_2(x) = a_0 + a_1 x^2 + a_2 x^2 I(x < 0)$, to name just a few.

Inference for the unknown news impact functions

- So many ARCH-type models, which one should we use?

Inference for the unknown news impact functions

- So many ARCH-type models, which one should we use?
- Estimating θ_h and $g_2(\cdot)$ is difficult if h_t and ε_t are unobserved. Making use of $\hat{\mu}_t$ and $\hat{h}_t$, we propose a semiparametric implied conditional variance (Icv) model

$$\hat{h}_t = \theta_h \hat{h}_{t-1} + g_2(\hat{\varepsilon}_{t-1}) + \epsilon_t^{Icv}, \qquad (4.2)$$

  where $\hat{\varepsilon}_t = y_t - \hat{\mu}_t$, and ϵ_t^{Icv} denotes the error resulting from the substitution.

Inference for the unknown news impact functions

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  where $\hat{\varepsilon}_t = y_t - \hat{\mu}_t$, and ϵ_t^{Icv} denotes the error resulting from the substitution.
- The specific procedure for the inference of unknown news impact function g_2:
  - First, the estimation $\hat{g}_2$ and $\hat{\theta}_h$ could be achieved by the classical semiparametric method (see Robinson (1988, Econometrica)) directly.
  - Based on $\hat{g}_2$, we choose a reasonable parametric model to fit it.
  - Check the validity of the parametric hypothesis for g_2 as in Hardle and Mammen (1993, Ann. Stat).

Inference for the unknown news impact functions

- Following the similar idea, we can extend model (4.2) to the conditional skewness and kurtosis for checking their dynamic structures. Similarly, we consider the following models

$$\begin{aligned} s_t &= \theta_s s_{t-1} + g_3(\eta_{t-1}), \\ k_t &= \theta_k k_{t-1} + g_4(\eta_{t-1}), \end{aligned} \tag{4.3}$$

  where $g_3(\cdot)$ and $g_4(\cdot)$ denote the unknown "news impact functions" of conditional skewness and kurtosis, respectively. $\eta_t = (y_t - \mu_t)/\sqrt{h_t}$ serves as a normalized shock.

- Similarly, we propose the following semiparametric implied conditional skewness (Ics) and implied conditional kurtosis (Ick) models, respectively, modeled by

$$\begin{aligned} \hat{s}_t &= \theta_s \hat{s}_{t-1} + g_3(\hat{\eta}_{t-1}) + \epsilon_t^{Ics}, \\ \hat{k}_t &= \theta_k \hat{k}_{t-1} + g_4(\hat{\eta}_{t-1}) + \epsilon_t^{Ick}, \end{aligned} \tag{4.4}$$

  where ϵ_t^{Ics} and ϵ_t^{Ick} denote the model errors resulting from the replacement. Then, conduct similar procedure for the inference of unknown news impact functions g_3 and g_4.

Inference for the unknown news impact functions

Table 2: Quantile levels and p-values of validity checks for implied conditional moments of four stock indexes.

	S&P500	CAC	W5000	N225
quantile levels(α_1)	0.01	0.01	0.03	0.04
quantile levels(α_2)	0.10	0.09	0.06	0.06
quantile levels(α_3)	0.94	0.94	0.91	0.90
quantile levels(α_4)	0.96	0.97	0.96	0.98
p-values ($\hat{\mu}_t$)	0.9902	0.9312	0.9867	0.5250
p-values ($\hat{h}_t$)	0.9932	0.8467	0.7390	0.4147
p-values ($\hat{s}_t$)	0.3516	0.8237	0.9931	0.9103
p-values ($\hat{k}_t$)	0.1790	0.2403	0.1568	0.1740

Table 3: The implied news impact functions by the implied conditional moments.

	$g_2(\cdot)$	$g_3(\cdot)$	$g_4(\cdot)$
S&P500	$\alpha_0 + \alpha_1 \varepsilon_{t-1}^2$	$\beta_0 + \beta_1 \eta_{t-1}^3$	$\gamma_0 + \gamma_1 \eta_{t-1} + \gamma_2 s_{t-1}$ $+\gamma_3 I(\eta_{t-1} < 0)\eta_{t-1}$
CAC	$\alpha_0 + \alpha_1 \varepsilon_{t-1}^2$	$\beta_0 + \beta_1 \lvert\eta_{t-1}\rvert$	$\gamma_0 + \gamma_1 \lvert\eta_{t-1}\rvert + \gamma_2 s_{t-1}$
W5000	$\alpha_0 + \alpha_1 \varepsilon_{t-1}^2$	$\beta_0 + \beta_1 \lvert\eta_{t-1}\rvert$	$\gamma_0 + \gamma_1 \lvert\eta_{t-1}\rvert + \gamma_2 s_{t-1}$
N225	$\alpha_0 + \alpha_1 \varepsilon_{t-1}^2$	$\beta_0 + \beta_1 \lvert\eta_{t-1}\rvert$	$\gamma_0 + \gamma_1 \lvert\eta_{t-1}\rvert + \gamma_s s_{t-1}$

Inference for the unknown news impact functions

Table 4: P-values of testing parametric hypothesis of the three unknown news impact functions with various bandwidth ($c * n^{-0.2}$).

	S&P500		CAC		W5000		N225	
	c	p-values	c	p-values	c	p-values	c	p-values
lcv model	0.2	0.4900	0.3	0.3830	0.2	0.2730	0.3	0.5310
	0.3	0.5030	0.4	0.3430	0.3	0.3690	0.4	0.4250
	0.4	0.2240	0.5	0.2780	0.4	0.4270	0.5	0.3310
	0.5	0.3450	0.6	0.2160	0.5	0.4430	0.6	0.3270
lcs model	0.1	0.5830	0.1	0.2510	0.2	0.9510	0.10	0.7210
	0.2	0.6660	0.2	0.2450	0.3	0.9680	0.15	0.6810
	0.3	0.6410	0.3	0.4150	0.4	0.9460	0.2	0.6920
	0.4	0.5520	0.4	0.4350	0.5	0.8740	0.25	0.5760
lck model	0.4	0.3590	0.2	0.6440	0.3	0.9240	0.25	0.5390
	0.5	0.2600	0.3	0.6100	0.4	0.8930	0.3	0.5200
	0.6	0.2190	0.4	0.4770	0.5	0.8190	0.4	0.5160
	0.7	0.2290	0.5	0.4790	0.6	0.7360	0.5	0.5430

Inference for the unknown news impact functions

Table 5: The parameter estimation for implied news impact functions.

	parameters	S&P500	CAC	W5000	N225
lcv model	α_0	**0.0424**	**0.0468**	**0.0495**	**0.1017**
		(0.0064)	(0.0113)	(0.0066)	(0.0131)
	α_1	**0.1958**	**0.0777**	**0.1672**	**0.0869**
		(0.0029)	(0.0095)	(0.0066)	(0.0082)
	$\alpha_2(\theta_h)$	**0.7683**	**0.8914**	**0.7912**	**0.8504**
		(0.0078)	(0.0102)	(0.0116)	(0.0112)
lcs model	β_0	**-0.0116**	**0.0277**	**0.0291**	-0.0064
		(0.0029)	(0.0037)	(0.0120)	(0.0194)
	β_1	**0.0008**	$-$**0.0461**	$-$**0.0446**	-0.1986
		(0.0002)	(0.0031)	(0.0209)	(0.0066)
	$\beta_2(\theta_s)$	**0.9831**	**0.9889**	**0.7888**	**0.9346**
		(0.0046)	(0.0047)	(0.0161)	(0.0453)
lck model	γ_0	**0.4674**	**0.2769**	**1.0550**	**0.1224**
		(0.0592)	(0.0123)	(0.0547)	(0.0151)
	γ_1	**0.4022**	$-$**0.2127**	**0.3504**	-0.0251
		(0.0092)	(0.0066)	(0.0095)	(0.0177)
	γ_2	**0.8251**	**0.9699**	**0.6473**	**0.9699**
		(0.0136)	(0.0030)	(0.0139)	(0.0013)
	$\gamma_3(\theta_k)$	$-$**0.7457**	–	–	–
		(0.0164)			

Inference for the unknown news impact functions

Based on the implied news impact functions, we propose the following parametric model, called, the implied GARCH-SK model

$$\begin{cases} y_t &= \bar{y}_t + \epsilon_t \\ \epsilon_t &= \eta_t \sqrt{h_t}, \\ h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 h_{t-1} \\ s_t &= \beta_0 + \beta_1 \eta_{t-1}^3 + \beta_2 s_{t-1} \\ k_t &= \gamma_0 + \gamma_1 \eta_{t-1} + \gamma_2 k_{t-1} + \gamma_3 I(\eta_{t-1} < 0)\eta_{t-1} \end{cases} \qquad \text{(for S\&P500)}$$

and
$$\begin{cases} y_t &= \bar{y}_t + \epsilon_t \\ \epsilon_t &= \eta_t \sqrt{h_t}, \\ h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 h_{t-1} \\ s_t &= \beta_0 + \beta_1 |\eta_{t-1}| + \beta_2 s_{t-1} \\ k_t &= \gamma_0 + \gamma_1 |\eta_{t-1}| + \gamma_2 k_{t-1} \end{cases} \quad \text{(for CAC, W5000 and N225).}$$

Inference for the unknown news impact functions

Table 6: The estimation results of implied GARCH-SK model.

	parameters	S&P500	CAC	W5000	N225
Variance equation	α_0	**0.0316** (0.0053)	**0.0270** (0.0075)	**0.0307** (0.0056)	**0.0600** (0.0136)
	α_1	**0.1771** (0.0184)	**0.1010** (0.0122)	**0.1713** (0.0202)	**0.1221** (0.0172)
	α_2	**0.7905** (0.0200)	**0.8761** (0.0160)	**0.7915** (0.0213)	**0.8389** (0.0208)
Skewness equation	β_0	**-0.0359** (0.0193)	-0.0551 (0.0686)	0.0010 (0.0195)	$-$**0.1803** (0.0341)
	β_1	**0.0053** (0.0022)	-0.0042 (0.0342)	$-$**0.0446** (0.0209)	0.0501 (0.0343)
	β_2	**0.8018** (0.0970)	0.2797 (0.6316)	**0.8337** (0.0754)	$6.4028e^{-4}$ (0.0005)
Kurtosis equation	γ_0	**0.6854** (0.2429)	**0.5714** (0.0498)	1.8081 (1.6197)	**0.2400** (0.0481)
	γ_1	**0.2452** (0.0829)	$-$**0.1952** (0.0257)	0.0609 (0.0763)	**0.0935** (0.0204)
	γ_2	**0.7869** (0.0755)	**0.8780** (0.0102)	0.4533 (0.4842)	**0.9065** (0.0157)
	γ_3	-0.1415 (0.1047)	–	–	–
Log-likehood	–	-2989	-3828	-3038	-3863
AIC	–	5998	7675	6095	7744
BIC	–	6056	7727	6147	7796

Inference for the unknown news impact functions

The standard GARCH-SK model in Leon et al. (2005, QREF) is

$$\begin{cases} y_t & = \bar{y}_t + \epsilon_t \\ \epsilon_t & = \eta_t \sqrt{h_t}, \\ h_t & = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 h_{t-1} . \\ s_t & = \beta_0 + \beta_1 \eta_{t-1}^3 + \beta_2 s_{t-1} \\ k_t & = \gamma_0 + \gamma_1 \eta_{t-1}^4 + \gamma_2 k_{t-1} \end{cases}$$

Inference for the unknown news impact functions

Table 7: The estimation results of standard GARCH-SK model.

	parameters	S&P500	CAC	W5000	N225
Variance equation	α_0	**0.0339**	**0.0287**	**0.0310**	**0.0504**
		(0.0055)	(0.0065)	(0.0055)	(0.0140)
	α_1	**0.1765**	**0.1011**	**0.1732**	**0.1191**
		(0.0187)	(0.0131)	(0.0183)	(0.0154)
	α_2	**0.7798**	**0.8794**	**0.7887**	**0.8507**
		(0.0206)	(0.0151)	(0.0199)	(0.0194)
Kurtosis equation	β_0	-0.0393	-0.0047	-0.0507	-0.1182
		(0.0287)	(0.0026)	(0.0372)	(0.2119)
	β_1	**0.0065**	0.0012	**0.0082**	0.0004
		(0.0028)	(0.0009)	(0.0035)	(0.0021)
	β_2	**0.7538**	**0.9740**	**0.7136**	0.0026
		(0.1622)	(0.0147)	(0.1906)	(1.7629)
Skewness equation	γ_0	**2.9990**	**0.1843**	1.7000	**0.1674**
		(1.6008)	(0.0190)	(0.8884)	(0.0099)
	γ_1	**0.0028**	0.0002	**0.0028**	0.0006
		(0.0011)	(0.0005)	(0.0011)	(0.0005)
	γ_2	0.1243	**0.9403**	0.4955	**0.9504**
		(0.4659)	(0.0059)	(0.2626)	(0.0025)
Log-likehood	–	-2997	-3880	-3043	-3866
AIC	–	6012	7777	6103	7751
BIC	–	6064	7830	6156	7803

Inference for the unknown news impact functions

Table 8: The RMSEs between the implied conditional variance $\hat{h}_t$ and estimated conditional variance by different ARCH-type models.

	GARCH	GJR	EGARCH	AGARCH	APGARCH	NGARCH	GARCHSK	implied GARCHSK
S&P500	0.2166	1.2129	1.1585	0.4156	2.3502	2.4223	0.1309	0.1060
CAC	0.6984	1.8399	0.9723	0.8613	1.0579	1.1154	0.5014	0.4797
W5000	0.4139	1.4123	1.0633	0.6178	2.3200	2.3959	0.2834	0.2813
N225	0.6687	1.1898	0.7323	0.7925	0.7789	0.6948	0.5630	0.5249

Remarks: The implied GARCH-SK model has the smallest RMSEs among eight ARCH-type models for all data sets.

Outline

Interaction effects among conditional moments

- For a multivariate time series, the study on the interactive effects among the conditional moments of their entries is important in many applications, and this can be easily achieved by our implied conditional moments. In our work, we consider a bivariate time series $y_t = (y_{1t}, y_{2t})'$, since the method below can be easily extended to other multivariate cases. Firstly, we use the method stated before (White et al. (2015, JOE)) to estimate the multivariate quantiles $Q_{i,t}(\alpha)$ (for $i = 1, 2$). Then, we can obtain $\hat{\mu}_{i,t}$, $\hat{h}_{i,t}$, $\hat{s}_{i,t}$, and $\hat{k}_{i,t}$.
- Using $\hat{\mu}_{i,t}$ and $\hat{h}_{i,t}$, we can firstly study the volatility spillover effect by the regression model

$$\begin{aligned}\begin{pmatrix}\hat{h}_{1,t}\\ \hat{h}_{2,t}\end{pmatrix} &= \begin{pmatrix}a_{10}\\ a_{20}\end{pmatrix} + \sum_{i=1}^{p}\begin{pmatrix}a_{i1,1} & a_{i1,2}\\ a_{i2,1} & a_{i2,2}\end{pmatrix}\begin{pmatrix}\hat{h}_{1,t-i}\\ \hat{h}_{2,t-i}\end{pmatrix}\\ &\quad + \sum_{j=1}^{q}\begin{pmatrix}b_{j1,1} & b_{j1,2}\\ b_{j2,1} & b_{j2,2}\end{pmatrix}\begin{pmatrix}\hat{\varepsilon}^2_{1,t-j}\\ \hat{\varepsilon}^2_{2,t-j}\end{pmatrix} + \begin{pmatrix}\epsilon_{1,t}\\ \epsilon_{1,t}\end{pmatrix},\end{aligned} \tag{4.5}$$

where $\hat{\varepsilon}_{i,t} = y_{i,t} - \hat{\mu}_{i,t}$, $(\epsilon_{1,t}, \epsilon_{2,t})'$ is the model error vector, and all coefficients are positive.

Interaction effects among conditional moments

- The existing methods used some multivariate GARCH models to study the volatility spillover (see Hamao et al. (1990, RFS)). However, the chosen multivariate GARCH models may be mis-specified, and their computation is also unstable especially for large dimension case. By using our regression model, we can simply check the volatility spillover effect by examining whether $a_{i1,2}$, $a_{i2,1}$, $b_{j1,2}$, $b_{j2,1}$, $i = 1, \cdots, p$, $q = 1, \cdots, q$, are significant.
- Similarly, we can also detect the skewness and kurtosis spillover effect:

$$\begin{aligned}\begin{pmatrix}\hat{s}_{1,t}\\ \hat{s}_{2,t}\end{pmatrix} &= \begin{pmatrix}c_{10}\\ c_{20}\end{pmatrix} + \sum_{i=1}^{p}\begin{pmatrix}c_{i1,1} & c_{i1,2}\\ c_{i2,1} & c_{i2,2}\end{pmatrix}\begin{pmatrix}\hat{s}_{1,t-i}\\ \hat{s}_{2,t-i}\end{pmatrix}\\ &\quad + \sum_{j=1}^{q}\begin{pmatrix}d_{j1,1} & d_{j1,2}\\ d_{j2,1} & d_{j2,2}\end{pmatrix}\begin{pmatrix}\hat{\eta}^3_{1,t-j}\\ \hat{\eta}^3_{2,t-j}\end{pmatrix} + \begin{pmatrix}\epsilon_{1,t}\\ \epsilon_{1,t}\end{pmatrix},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\begin{pmatrix}\hat{k}_{1,t}\\ \hat{k}_{2,t}\end{pmatrix} &= \begin{pmatrix}e_{10}\\ e_{10}\end{pmatrix} + \sum_{i=1}^{p}\begin{pmatrix}e_{i1,1} & e_{i1,2}\\ e_{i2,1} & e_{i2,2}\end{pmatrix}\begin{pmatrix}\hat{k}_{1,t-i}\\ \hat{k}_{2,t-i}\end{pmatrix}\\ &\quad + \sum_{j=1}^{q}\begin{pmatrix}f_{j1,1} & f_{j1,2}\\ f_{j2,1} & f_{j2,2}\end{pmatrix}\begin{pmatrix}\hat{\eta}^4_{1,t-j}\\ \hat{\eta}^4_{2,t-j}\end{pmatrix} + \begin{pmatrix}\epsilon_{1,t}\\ \epsilon_{1,t}\end{pmatrix}.\end{aligned} \tag{4.7}$$

Interaction effects among conditional moments

Table 9: Quantile levels and p-values of validity checks for implied conditional moments of four pairs of stock indexes in pre and post crisis.

Pre-crisis	parameters	US	DAX	US	SZ	US	HS	US	ATX
	quantile levels(α_1)	0.01		0.03		0.03		0.02	
	quantile levels(α_2)	0.09		0.09		0.07		0.10	
	quantile levels(α_3)	0.90		0.93		0.90		0.90	
	quantile levels(α_4)	0.96		0.96		0.98		0.99	
	p values (mean)	0.2378	0.6288	0.4167	0.9997	0.2141	0.1677	0.6189	0.4869
	p values (variance)	0.3915	0.7470	0.8052	0.8463	0.9553	0.8469	0.9821	0.7069
	p values (skewness)	0.8546	0.4850	0.7503	0.5058	0.3030	0.7284	0.3300	0.2971
	p values (kurtosis)	0.2689	0.1382	0.0306	0.0120	0.1220	0.1607	0.1996	0.3039
Post-crisis									
	quantile levels(α_1)	0.03		0.01		0.02		0.05	
	quantile levels(α_2)	0.10		0.09		0.06		0.10	
	quantile levels(α_3)	0.90		0.90		0.93		0.90	
	quantile levels(α_4)	0.96		0.95		0.97		0.98	
	p values (mean)	0.1287	0.2195	0.6165	0.9068	0.3724	0.5963	0.7134	0.1018
	p values (variance)	0.5510	0.9139	0.5707	0.5601	0.5829	0.7700	0.3616	0.2615
	p values (skewness)	0.9936	0.5214	0.6561	0.9137	0.7532	0.8784	0.8214	0.7055
	p values (kurtosis)	0.0043	0.0837	0.2391	0.3044	0.4501	0.1737	0.7507	0.1577

Interaction effects among conditional moments

Pre-crisis parameters		US	ATX
variance	a_{10}/a_{20}	**0.0129** (0.0043)	**0.0860** (0.0100)
	$a_{11,1}/a_{12,1}$	**0.9165** (0.0039)	$3.3779e^{-11}$ (0.0047)
	$a_{11,2}/a_{12,2}$	$9.4943e^{-12}$ (0.0039)	**0.7825** (0.0144)
	$b_{11,1}/b_{12,1}$	**0.0669** (0.0025)	**0.0261** (0.0027)
	$b_{11,2}/b_{12,2}$	**0.0050** (0.0017)	**0.0891** (0.0091)
skewness	c_{10}/c_{20}	**0.0102** (0.0033)	**-0.0722** (0.0080)
	$c_{11,1}/c_{12,1}$	**0.9238** (0.0107)	**0.0609** (0.0216)
	$c_{11,2}/c_{12,2}$	**0.0257** (0.0075)	**0.6945** (0.0196)
	$d_{11,1}/d_{12,1}$	-0.0011 (0.0008)	-0.0003 (0.0021)
	$d_{11,2}/d_{12,2}$	-0.0006 (0.0006)	-0.0032 (0.0026)
kurtosis	e_{10}/e_{20}	**0.3904** (0.0440)	**1.0982** (0.1113)
	$e_{11,1}/e_{12,1}$	**0.8776** (0.0105)	$1.1447e^{-17}$ (0.0310)
	$e_{11,2}/e_{12,2}$	$2.4652e^{-12}$ (0.0071)	**0.7105** (0.0198)
	$f_{11,1}/f_{12,1}$	0.0005 (0.0005)	$7.1265e^{-22}$ (0.0008)
	$f_{11,2}/f_{12,2}$	$3.0301e^{-18}$ (0.0004)	**0.0021** (0.0004)

Post-crisis parameters		US	ATX
variance	a_{10}/a_{20}	$1.4232e^{-11}$ (0.0343)	**0.0857** (0.0324)
	$a_{11,1}/a_{12,1}$	**0.8237** (0.0642)	$1.4219e^{-9}$ (0.0586)
	$a_{11,2}/a_{12,2}$	0.0336 (0.0530)	**0.8896** (0.0490)
	$b_{11,1}/b_{12,1}$	**0.0816** (0.0049)	**0.0723** (0.0051)
	$b_{11,2}/b_{12,2}$	**0.0302** (0.0039)	**0.0332** (0.0040)
skewness	c_{10}/c_{20}	-0.0025 (0.0057)	**-0.0360** (0.0034)
	$c_{11,1}/c_{12,1}$	**0.8812** (0.0125)	**-0.0675** (0.0077)
	$c_{11,2}/c_{12,2}$	**0.0723** (0.0134)	**0.9328** (0.0081)
	$d_{11,1}/d_{12,1}$	-0.0019 (0.0015)	0.0002 (0.0008)
	$d_{11,2}/d_{12,2}$	0.0008 (0.0005)	**-0.0015** (0.0007)
kurtosis	e_{10}/e_{20}	0.0209 (0.1214)	**0.1032** (0.0430)
	$e_{11,1}/e_{12,1}$	**0.9429** (0.0133)	**0.0082** (0.0039)
	$e_{11,2}/e_{12,2}$	0.0630 (0.0323)	**0.9493** (0.0118)
	$f_{11,1}/f_{12,1}$	$2.1569e^{-13}$ (0.0016)	$9.5266e^{-11}$ (0.0002)
	$f_{11,2}/f_{12,2}$	$1.8699e^{-15}$ (0.0017)	**0.0019** (0.0008)

Interaction effects among conditional moments

Specially, we focus on the impact of shocks of S&P500 index on the conditional variance of DAX, SZ, HS, and ATX indexes with the increase of order j, $j = 1, \cdots, q$. For this purpose, let $A = \begin{pmatrix} a_{11,1} & a_{11,2} \\ a_{12,1} & a_{12,2} \end{pmatrix}$, $B = \begin{pmatrix} b_{11,1} & b_{11,2} \\ b_{12,1} & b_{12,2} \end{pmatrix}$, and $C = A^{j-1}B$, then the coefficients of interest are $b_{j2,1} = C(2,1)$, $j = 1, \cdots, q$.

Interaction effects among conditional moments

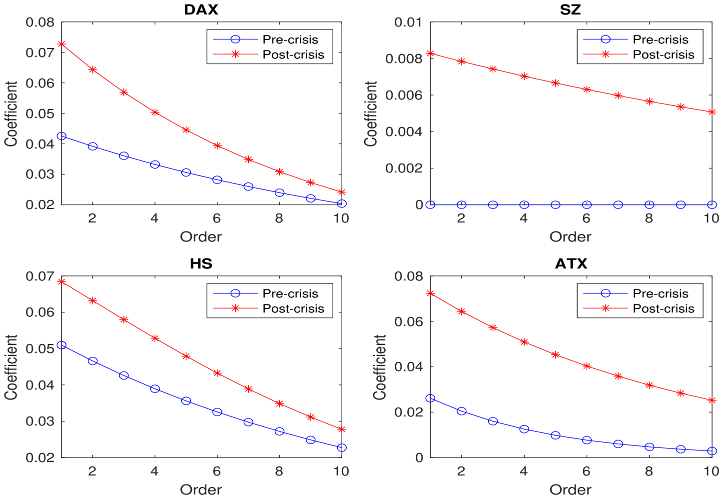

Figure 7: The impacts of shocks of S&P500 index on the conditional variance of DAX, SZ, HS, and ATX indexes with the increase of order, respectively.

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- Simulations show that the implied conditional moments could be good proxies for their unobserved counterparts, and they exhibit robust performances across the choices of quantile estimation and quantile level.
- Finally, the implied conditional moments are applied to two important applications, which unveil news impacts functions and interactive effects among the conditional moments.

Thank you!