Quantiled Conditional Moments (QCMs)

QCMs

- Let $\{y_t\}_{t=1}^{T}$ be a time series of interest, and $\mathcal{F}_t = \sigma(y_s; s \leqslant t)$ be the available information set up to time t. Given $\mathcal{F}_{t-1}$, the conditional mean, variance, skewness, and kurtosis of y_t are defined by

$$\mu_t = E(y_t|\mathcal{F}_{t-1}),\ h_t = Var(y_t|\mathcal{F}_{t-1}),\ s_t = E\Big(\Big(\frac{y_t-\mu_t}{\sqrt{h_t}}\Big)^3|\mathcal{F}_{t-1}\Big), \\ \text{and } k_t = E\Big(\Big(\frac{y_t-\mu_t}{\sqrt{h_t}}\Big)^4|\mathcal{F}_{t-1}\Big). \tag{1}$$

- Let $Q_t(\alpha)$ be the conditional quantile of y_t given $\mathcal{F}_{t-1}$ at quantile level $\alpha \in (0,1)$. From the Cornish-Fisher expansion (Cornish and Fisher (1938)), we have

$$Q_t(\alpha) = \mu_t + \sqrt{h_t}\omega_t(\alpha), \tag{2}$$

where

$$\omega_t(\alpha) = x + (x^2-1)\frac{s_t}{6} + (x^3-3x)\frac{k_t-3}{24} \\ + \text{remaining terms on the higher-order CMs,} \tag{3}$$

and $x = \Phi^{-1}(\alpha)$ with $\Phi(\cdot)$ being the distribution function of $N(0,1)$. By ignoring the remaining terms in (3), the results (2)–(3) imply

$$Q_t(\alpha) \approx \mu_t + \sqrt{h_t}\Big[x + (x^2-1)\frac{s_t}{6} + (x^3-3x)\frac{k_t-3}{24}\Big]. \tag{4}$$

QCMs

- Taking some quantile levels α_i, $i = 1, 2, \cdots, n$, the result (4) entails that for each t, we have

$$\begin{pmatrix} Q_t(\alpha_1) \\ Q_t(\alpha_2) \\ \vdots \\ Q_t(\alpha_n) \end{pmatrix} - \begin{pmatrix} 1 & x_1 & x_1^2 - 1 & x_1^3 - 3x_1 \\ 1 & x_2 & x_2^2 - 1 & x_2^3 - 3x_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 - 1 & x_n^3 - 3x_n \end{pmatrix} \begin{pmatrix} \mu_t \\ \sqrt{h_t} \\ \frac{\sqrt{h_t} s_t}{6} \\ \frac{\sqrt{h_t}(k_t - 3)}{24} \end{pmatrix} \triangleq \boldsymbol{Q}_t - \boldsymbol{X}\boldsymbol{M}_t \approx 0, \tag{5}$$

  where $x_i = \Phi^{-1}(\alpha_i)$ for $i = 1, 2, \cdots, n$.
- Denote $\widehat{\boldsymbol{Q}}_t \triangleq (\widehat{Q}_t(\alpha_1), \widehat{Q}_t(\alpha_2), \cdots, \widehat{Q}_t(\alpha_n))'$, where $\widehat{Q}_t(\alpha_i)$ is an estimator of $Q_t(\alpha_i)$ for $i = 1, 2, \cdots, n$. With model (5), we assume that $\widehat{\boldsymbol{Q}}_t$ could be modelled as

$$\widehat{\boldsymbol{Q}}_t = \boldsymbol{X}\boldsymbol{M}_t + \boldsymbol{U}_t, \tag{6}$$

  where $\boldsymbol{U}_t = (u_{1,t}, u_{2,t}, \cdots, u_{n,t})'$ is a vector of independent errors with mean 0 and different variances σ_{it}^2. In view of (6), we could estimate μ_t, h_t, s_t, and k_t by just solving a simple system of linear regressions.
- We call $\widehat{\mu}_t$, $\widehat{h}_t$, $\widehat{s}_t$, and $\widehat{k}_t$ as the quantiled CMs (QCMs) of y_t, since they are implied from the conditional quantiles of y_t.

Validity Checks on QCMs

- We consider a regression model for $\widehat{\mu}_t$:

$$y_t = a_1^\mu + a_2^\mu \widehat{\mu}_t + \epsilon_t^\mu, \tag{7}$$

where a_1^μ and a_2^μ are two coefficients, and ϵ_t^μ is the model error with mean zero. Logically, if $\widehat{\mu}_t$ is a valid estimator of μ_t, we shall have $a_1^\mu = 0$ and $a_2^\mu = 1$ in (7). This motivates us to propose W_μ, a heteroscedasticity-robust Wald test as in White (1980), for detecting the null hypothesis

$$H_0^\mu : a_1^\mu = 0 \text{ and } a_2^\mu = 1.$$

If H_0^μ is rejected by W_μ, we conclude that $\widehat{\mu}_t$ is invalid; otherwise, we conclude that $\widehat{\mu}_t$ is valid.

- Moreover, we use the similar regression-based testing idea as for $\widehat{\mu}_t$ to check the validity of $\widehat{h}_t$, $\widehat{s}_t$, and $\widehat{k}_t$. Specifically, we introduce three regression models

$$\begin{aligned} (y_t - \mu_t)^2 &= a_1^h + a_2^h \widehat{h}_t + \epsilon_t^h, \\ \left(\frac{y_t - \mu_t}{\sqrt{h_t}}\right)^3 &= a_1^s + a_2^s \widehat{s}_t + \epsilon_t^s, \\ \left(\frac{y_t - \mu_t}{\sqrt{h_t}}\right)^4 &= a_1^k + a_2^k \widehat{k}_t + \epsilon_t^k, \end{aligned} \tag{8}$$

Validity Checks on QCMs

and consider three null hypotheses

$$H_0^h : a_1^h = 0 \text{ and } a_2^h = 1,$$
$$H_0^s : a_1^s = 0 \text{ and } a_2^s = 1,$$
$$H_0^k : a_1^k = 0 \text{ and } a_2^k = 1,$$

where ϵ_t^h, ϵ_t^s, and ϵ_t^k are model errors with mean zero. Similar to W_μ, we construct three heteroscedasticity-robust Wald tests W_h, W_s, and W_k to detect H_0^h, H_0^s, and H_0^k, respectively, and the acceptance of these Wald tests implies the validity of corresponding QCM. However, since μ_t and h_t in (8) are unknown, we have to estimate them appropriately before implementing W_h, W_s, and W_k.