Contents

The famous Galuret's integral is shown below,

$$\int_0^{\pi} \frac{\cos(n\theta_0)}{\cos(\theta_0)-\cos(\theta)}\, d\theta_0 = \frac{\pi\sin(n\theta)}{\sin(\theta)} \quad , \quad n = 0, 1, 2, 3, \ldots \tag{1}$$

0.1 Biot-Savart Law

0.1.1 Variable List

The variable definitions are reiterated below for reference,

- $\bar{v}_i$ represents the induced velocity at a particular point of interest due to a vortex line
- Γ_v represents the constant strength of a vortex. For full generality, the strength of the vortex can change along the vortex line, but for our purposes, this is asssumed to be constant within a single vortex line
- $\bar{l}_l$ represents some vector tanget to the vortex line

- $\bar{r_r}$ represents the relative position vector of the point of interest to some point on the vortex line
- β_v represents the angle between the vector $\bar{r_r}$ and the vortex line
- $\hat{k}$ represents a unit vector into the page
- δ_r represents a scalar which represents the perpendicular distance between the point of interest and the vortex line.
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0.1.2 General Form

The general form of the Biot-Savart law is shown below,

$$d\bar{v_i} = \frac{\Gamma_v}{4\pi} \frac{d\bar{l_l} \times \bar{r_r}}{|\bar{r_r}|^3}$$

wherein $\bar{v_i}$ represents the induced velocity at a particular point of interest due to a vortex line, Γ_v represents the constant strength of a vortex, $\bar{l_l}$ represents some vector tanget to the vortex line, and $\bar{r_r}$ represents the relative position vector of the point of interest to some point on the vortex line. Switching the order of the cross product,

$$d\bar{v_i} = -\frac{\Gamma_v}{4\pi} \frac{\bar{r_r} \times d\bar{l_l}}{|\bar{r_r}|^3}$$

$$d\bar{v_i} = -\frac{\Gamma_v}{4\pi} \frac{\bar{r_r}}{|\bar{r_r}|^3} \times d\bar{l_l} \tag{2}$$

0.1.3 Straight Vortex Line

Figure 1: Schematic of the Vortex Line and the Point of Interest

Reiterating equation 2,

$$d\bar{v_i} = -\frac{\Gamma_v}{4\pi} \frac{\bar{r_r}}{|\bar{r_r}|^3} \times d\bar{l_l}$$

Re-writing the equation above in terms of the vector magnitudes,

$$d\bar{v_i} = -\frac{\Gamma_v}{4\pi} \frac{|\bar{r_r}| \sin(\beta_v)}{|\bar{r_r}|^3} d|\bar{l_l}|\hat{k}$$

wherein β_v represents the angle between the vector $\bar{r_r}$ and the vortex line and $\hat{k}$ represents a unit vector into the page. Simplifying,

$$d\bar{v_i} = -\frac{\Gamma_v}{4\pi} \frac{\sin(\beta_v)}{|\bar{r_r}|^2} d|\bar{l_l}|\hat{k} \tag{3}$$

Examining Figure 1,

$$\delta_r = |\bar{r_r}| \sin(\beta_v) \tag{4}$$

$$\tan(\beta_v) = \frac{\delta_r}{|\bar{l_l}|} \tag{5}$$

wherein δ_r is a scalar which represents the perpendicular distance between the point of interest and the vortex line. Manipulating equation 4 to obtain a transformation between $|\bar{r_r}|$ to β_v,

$$\delta_r = |\bar{r_r}| \sin(\beta_v)$$

$$\frac{\delta_r}{\sin(\beta_v)} = |\bar{r_r}| \tag{6}$$

Manipulating expression 5 to obtain a relation of $d\beta_v$ to $d|\bar{l_l}|$,

$$\tan(\beta_v) = \frac{\delta_r}{|\bar{l_l}|}$$

$$|\bar{l_l}| = \frac{\delta_r}{\tan(\beta_v)}$$

Taking derivative with respect to angle β_v,

$$\frac{d}{d\beta_v}\left[|\bar{l_l}|\right] = \frac{d}{d\beta_v}\left[\frac{\delta_r}{\tan(\beta_v)}\right]$$

$$\frac{d}{d\beta_v}\left[|\bar{l_l}|\right] = \delta_r \frac{d}{d\beta_v}\left[\frac{1}{\tan(\beta_v)}\right]$$

$$\frac{d}{d\beta_v}\left[|\bar{l_l}|\right] = \delta_r \frac{d}{d\beta_v}\left[\cot(\beta_v)\right]$$

$$\frac{d}{d\beta_v}\left[|\bar{l_l}|\right] = \delta_r \times -\frac{1}{\sin^2(\beta_v)}$$

$$\frac{d|\bar{l_l}|}{d\beta_v} = -\delta_r \frac{1}{\sin^2(\beta_v)}$$

$$d|\bar{l_l}| = -\delta_r \frac{1}{\sin^2(\beta_v)}\, d\beta_v \tag{7}$$

Reiterating equation 3

$$d\bar{v_i} = -\frac{\Gamma_v}{4\pi}\frac{\sin(\beta_v)}{|\bar{r_r}|^2} d|\bar{l_l}|\hat{k}$$

$$d\bar{v_i} = -\frac{\Gamma_v}{4\pi}\sin(\beta_v)\frac{1}{|\bar{r_r}|^2} d|\bar{l_l}|\hat{k}$$

Susbtituting equation 6 into the expression above,

$$\frac{\delta_r}{\sin(\beta_v)} = |\bar{r_r}|$$

$$\frac{\sin(\beta_v)}{\delta_r} = \frac{1}{|\bar{r_r}|}$$

$$\frac{\sin^2(\beta_v)}{\delta_r^2} = \frac{1}{|\bar{r_r}|^2}$$

$$d\bar{v}_i = -\frac{\Gamma_v}{4\pi}\sin(\beta_v)\frac{\sin^2(\beta_v)}{\delta_r^2}\,d|\bar{l}_l|\hat{k}$$

Substituting equation 7 for the differentials,

$$d|\bar{l}_l| = -\delta_r\frac{1}{\sin^2(\beta_v)}\,d\beta_v$$

$$d\bar{v}_i = -\frac{\Gamma_v}{4\pi}\sin(\beta_v)\frac{\sin^2(\beta_v)}{\delta_r^2} \times -\delta_r\frac{1}{\sin^2(\beta_v)}\,d\beta_v\hat{k}$$

$$d\bar{v}_i = \frac{\Gamma_v}{4\pi}\sin(\beta_v)\frac{\sin^2(\beta_v)}{\delta_r}\frac{1}{\sin^2(\beta_v)}\,d\beta_v\hat{k}$$

$$d\bar{v}_i = \frac{\Gamma_v}{4\pi}\frac{1}{\delta_r}\sin(\beta_v)\,d\beta_v\hat{k}$$

$$d\bar{v}_i = \frac{\Gamma_v}{4\pi\delta_r}\sin(\beta_v)\,d\beta_v\hat{k}$$

Integrating the expression above,

$$\int d\bar{v}_i = \frac{\Gamma_v}{4\pi\delta_r}\int_{\beta_1}^{\beta_2}\sin(\beta_v)\,d\beta_v\hat{k}$$

$$\bar{v}_i = \frac{\Gamma_v}{4\pi\delta_r} \times -\left[\cos(\beta_v)\right]_{\beta_1}^{\beta_2}\hat{k}$$

$$\bar{v}_i = -\frac{\Gamma_v}{4\pi\delta_r}\left[\cos(\beta_v)\right]_{\beta_1}^{\beta_2}\hat{k}$$

$$\bar{v}_i = -\frac{\Gamma_v}{4\pi\delta_r}\left[\cos(\beta_2) - \cos(\beta_1)\right]\hat{k}$$

$$\bar{v}_i = \frac{\Gamma_v}{4\pi\delta_r}\left[\cos(\beta_1) - \cos(\beta_2)\right]\hat{k} \tag{8}$$

0.1.3.1 Infinite Vortex Line

Reiterating equation 8,

$$\bar{v}_i = \frac{\Gamma_v}{4\pi\delta_r}\left[\cos(\beta_1) - \cos(\beta_2)\right]\hat{k}$$

For the infinite vortex line case, $\beta_1 = 0$ and $\beta_2 = \pi$

$$\bar{v}_i = \frac{\Gamma_v}{4\pi\delta_r}\left[1 + 1\right]\hat{k}$$

$$\bar{v}_i = \frac{2\Gamma_v}{4\pi\delta_r}\hat{k}$$

$$\bar{v}_i = \frac{\Gamma_v}{2\pi\delta_r}\hat{k} \tag{9}$$

0.1.3.2 Semi-Infinite Vortex Line

Reiterating equation 8,

$$\bar{v_i} = \frac{\Gamma_v}{4\pi\delta_r}\left[\cos(\beta_1) - \cos(\beta_2)\right]\hat{k}$$

For the semi-infinite vortex line, $\beta_1 = \pi/2$ and $\beta_2 = \pi$. Therefore,

$$\bar{v_i} = \frac{\Gamma_v}{4\pi\delta_r}\left[0 + 1\right]\hat{k}$$

$$\bar{v_i} = \frac{\Gamma_v}{4\pi\delta_r}\hat{k} \tag{10}$$

0.2 Lifting Line Theory

Figure 2: Schematic of Continuous Vortex Distribution in Lifting Line Theory

0.2.1 Variable List

The variable definitions are reiterated below for reference,

- L_a represents the lift force experienced by a 2-dimensional airfoil
- ρ_∞ represents the free-stream density of the incoming fluid
- v_∞ represents the free-stream velocity of the incoming fluid
- Γ_a represents the circulation for a 2-dimensional airfoil
- L_w represents the total lift experienced by the whole generalized wing
- Γ_w represents the total circulation generated by the wing
- $\bar{w_i}$ is a vector representing the "downwash" velocity
- y_0 represents a particular point of interest at the wing
- D_i represents the total induced drag experienced by the wing
- α_{eff} represents the effective angle of attack that the airfoil sees with the altered flow
- α_{Geo} represents the geometric angle of attack of the airfoil with respect to the free-stream very far away from the airfoil
- $C_{L,2d}$ represents the coefficient of lift for a 2-dimensional airfoil
- α_0 represents the zero lift angle of attack for a 2-dimensional airfoil
- $c_w(y_0)$ represents the chord of the wing at some spanwise location along the wing
- b represents the total span of the wing
- A_w represents the area of the wing

- $$A_R$$
  represents the aspect ratio of a particular wing
- c_0 represents the root chord, which is the maximum chord for a wing with an elliptical planform
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0.2.2 Lift

Based on the Kutta-Joukowski theorem, which can be proven using 2-dimensional conformal mapping,

$$L_a = \rho_\infty v_\infty \Gamma_a$$

wherein L_a represents the lift force experienced by a 2-dimensional airfoil, ρ_∞ represents the free-stream density of the incoming fluid, v_∞ represents the free-stream velocity of the incoming fluid, and Γ_a represents the circulation for a 2-dimensional airfoil. For a generalized wing, wherein there is airfoil geometry variation in the spanwise direction, it seems reasonable to consider the 2-dimensional airfoil properties to hold true for a particular point in the generalized wing. Therefore,

$$L_a = dL_w$$

wherein L_w represents the total lift experienced by the whole generalized wing.

$$\Gamma_a = \Gamma_w dy$$

wherein Γ_w represents the total circulation generated by the wing. Substituting into the Kutta-Joukowski theorem,

$$dL_w = \rho_\infty v_\infty \Gamma_w \, dy \tag{11}$$

Integrating for the entire wing,

$$L_w = \rho_\infty v_\infty \int_{-b/2}^{b/2} \Gamma_w \, dy \tag{12}$$

0.2.3 Induced Drag

Induced drag is specific only to 3-dimensional flows around 3-dimensional wings. Induced drag does not exist for 2-dimensional flows around 2-dimensional wings. Based on the lifting line model, the "legs" of the vortex distribution induces a downward velocity on the wing. THis downward fluid velocity shifts the lift vector slightly and produces a horizontal component of force in the flow direction. This force is the induced drag.

0.2.3.1 Downwash

Downwash is the vertical component of induced velocity due to the "legs" of the vortex distribution stretching to ∞. Reiterating the induced velocity due to application of the Biot-Savart Law for a semi-infinite vortex line (equation 10)

$$\bar{v_i} = \frac{\Gamma_v}{4\pi\delta_r}\hat{k}$$

$$\bar{v_i} = \frac{1}{4\pi\delta_r}\Gamma_v\hat{k}$$

The strenght of the vortex line, Γ_v is an infinitesmially small portion of the total vortex strenght of the wing. Therefore,

$$\Gamma_v = d\Gamma_w$$

Substituting, and then multiplying by 1,

$$\bar{v_i} = \frac{1}{4\pi\delta_r}d\Gamma_w\hat{k}$$

$$\bar{v_i} = \frac{1}{4\pi\delta_r}\frac{d\Gamma_w}{dy}dy\hat{k}$$

The induced velocity when applied to this problem $\bar{v_i}$ takes downwards to be the positive direction. Typically downwash takes "upwards" as positive direction. Therefore,

$$\bar{v_i} = -d\bar{w_i}$$

wherein $\bar{w_i}$ is a vector representing the "downwash" velocity. Substituting,

$$-d\bar{w_i} = \frac{1}{4\pi\delta_r}\frac{d\Gamma_w}{dy}dy\hat{z}$$

$$d\bar{w_i} = -\frac{1}{4\pi\delta_r}\frac{d\Gamma_w}{dy}dy\hat{z}$$

The distance δ_r is taken to be,

$$\delta_r = y_0 - y$$

wherein y_0 represents a particular point of interest at the wing. Substituting,

$$d\bar{w_i} = -\frac{1}{4\pi}\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}dy\hat{z}$$

Integrating for the entire wing,

$$\bar{w_i} = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}dy\hat{z} \tag{13}$$

0.2.3.2 Induced Angle of Attack

The induced angle of attack by convention is a positive quantity when the downwash vector is pointing down. Therefore,

$$\alpha_i = \arctan\left(\frac{|\bar{w}_i|}{v_\infty}\right)$$

By the small angle approximation,

$$\alpha_i \approx \frac{|\bar{w}_i|}{v_\infty}$$

Substituting the expression for downwash velocity vector (equation 13)

$$\bar{w}_i = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy\hat{z}$$

$$|\bar{w}_i| = \frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy$$

$$\alpha_i(y_0) = \frac{1}{v_\infty}\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy$$

$$\alpha_i(y_0) = \frac{1}{4\pi v_\infty}\int_{-b/2}^{b/2} \frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy \tag{14}$$

0.2.3.3 Drag Expression

Examining the 2 dimensional airfoil, with a shifted incoming fluid velocity,

$$dD_i = dL_w \sin(\alpha_i)$$

wherein D_i represents the total induced drag experienced by the wing. Using the small angle approximation,

$$\sin(\alpha_i) \approx \alpha_i$$

Therefore,

$$dD_i = \alpha_i dL_w$$

Using equation 11

$$dL_w = \rho_\infty v_\infty \Gamma_w\,dy$$

$$dD_i = \alpha_i \rho_\infty v_\infty \Gamma_w\,dy$$

Integrating for the whole wing,

$$D_i = \int_{-b/2}^{b/2} \alpha_i \rho_\infty v_\infty \Gamma_w\,dy$$

$$D_i = \rho_\infty v_\infty \int_{-b/2}^{b/2} \alpha_i \Gamma_w\,dy \tag{15}$$

0.2.4 Vortex Distribution

0.2.4.1 Angle of Attack Relations

$$\alpha_{eff}(y_0) = \alpha_{Geo}(y_0) - \alpha_i(y_0)$$

wherein α_{eff} represents the effective angle of attack that the airfoil sees with the altered flow and α_{Geo} represents the geometric angle of attack of the airfoil with respect to the free-stream very far away from the airfoil. These various angles of attack are functions of spanwise distance y_0. Simple algebraic manipulation of the expression above yields,

$$\alpha_{eff} + \alpha_i = \alpha_{Geo} \tag{16}$$

0.2.4.2 Lifting Line Fundamental Equation

Based on results of conformal mapping for 2-dimensional airfoils, the coefficient of lift for a 2-dimensional airfoil,

$$C_{L,2d} = 2\pi(\alpha_{eff} - \alpha_0)$$

wherein $C_{L,2d}$ represents the coefficient of lift for a 2-dimensional airfoil and α_0 represents the zero lift angle of attack for a 2-dimensional airfoil. The total lift experienced by the wing can be thought of as an integral of the infinitesmially small amount of lift generated by a series of 2-dimensional airfoil cross-sections in the spanwise direction. Therefore,

$$\frac{dL_w}{dy} = \frac{1}{2}\rho_\infty v_\infty^2 c_w(y_0) C_{L,2d}$$

wherein $c_w(y_0)$ represents the chord of the wing at some spanwise location along the wing. Reiterating equation 11

$$dL_w = \rho_\infty v_\infty \Gamma_w \, dy$$

$$\frac{dL_w}{dy} = \rho_\infty v_\infty \Gamma_w$$

Substituting into the definition of the 2-dimensional coefficient of lift,

$$\frac{dL_w}{dy} = \frac{1}{2}\rho_\infty v_\infty^2 c_w(y_0) C_{L,2d} = \rho_\infty v_\infty \Gamma_w$$

$$\frac{1}{2} v_\infty^2 c_w(y_0) C_{L,2d} = v_\infty \Gamma_w$$

$$\frac{1}{2} v_\infty c_w(y_0) C_{L,2d} = \Gamma_w$$

$$C_{L,2d} = \frac{2\Gamma_w}{v_\infty c_w}$$

Subsituting into the expression for the 2-dimensional lift coefficient based on angles of attack,

$$C_{L,2d} = \frac{2\Gamma_w}{v_\infty c_w} = 2\pi(\alpha_{eff} - \alpha_0)$$

$$\frac{\Gamma_w}{v_\infty c_w} = \pi(\alpha_{eff} - \alpha_0)$$

$$\frac{\Gamma_w}{\pi v_\infty c_w} = \alpha_{eff} - \alpha_0$$

$$\frac{\Gamma_w}{\pi v_\infty c_w} + \alpha_0 = \alpha_{eff}$$

Substituting the expression above into equation 16,

$$\frac{\Gamma_w}{\pi v_\infty c_w} + \alpha_0 + \alpha_i = \alpha_{Geo}$$

Subsituting the definition for induced angle of attack (equation 14)

$$\alpha_i(y_0) = \frac{1}{4\pi v_\infty} \int_{-b/2}^{b/2} \frac{1}{y_0 - y} \frac{d\Gamma_w}{dy}\, dy$$

$$\frac{\Gamma_w}{\pi v_\infty c_w} + \alpha_0 + \frac{1}{4\pi v_\infty} \int_{-b/2}^{b/2} \frac{1}{y_0 - y} \frac{d\Gamma_w}{dy}\, dy = \alpha_{Geo} \tag{17}$$

The equation shown above is the famous "Fundamental Equation of Prandtl's Lifting Line Theory".

0.2.4.3 Fourier Method to Solving Circulation

Using the coordinate transformation,

$$y = -\frac{b}{2}\cos(\theta) \tag{18}$$

An arbitrary vortex distribution can be expressed as an infinite sum of sine waves as shown below,

$$\Gamma_w = 2bv_\infty \sum_{k=1}^{\infty} \left[A_k \sin(k\theta)\right] \tag{19}$$

wherein b represents the total span of the wing. The constant $2bv_\infty$ was chosen purely for scaling purposes. The Fourier Sine infinite series shown above could be substituted into the "Fundamental Equation of Prandtl's Lifting Line Theory" to obtain an expression of the constants A_j in terms of known quantities of the wing. Solving the resulting equation would yield coefficients A_j which can then be used to solve for lift and drag of the wing.

Determining the derivative of equation 19 with respect to y,

$$\frac{d}{dy}\left[\Gamma_w\right] = \frac{d}{dy}\left\{2bv_\infty \sum_{k=1}^{\infty} \left[A_k \sin(k\theta)\right]\right\}$$

$$\frac{d}{dy}\left[\Gamma_w\right] = 2bv_\infty \sum_{k=1}^{\infty} \left[A_k \frac{d}{dy}\left\{\sin(k\theta)\right\}\right] \tag{20}$$

$$\frac{d}{dy}\left\{\sin(k\theta)\right\} = \frac{d}{dk\theta}\left\{\sin(k\theta)\right\} \times \frac{dk\theta}{dy}$$

$$\frac{d}{dy}\left\{\sin(k\theta)\right\} = \cos(k\theta) \times k\frac{d\theta}{dy}$$

$$\frac{d}{dy}\left\{\sin(k\theta)\right\} = k\cos(k\theta) \times \frac{d\theta}{dy}$$

Reiterating the coordinate transformation used,

$$y = -\frac{b}{2}\cos(\theta)$$

Taking derivative of the coordinate transformation with respect to θ,

$$\frac{dy}{d\theta} = \frac{b}{2}\sin(\theta) \tag{21}$$

$$\frac{d\theta}{dy} = \frac{2}{b}\frac{1}{\sin(\theta)}$$

Substituting,

$$\frac{d}{dy}\left\{\sin(k\theta)\right\} = k\cos(k\theta) \times \frac{2}{b}\frac{1}{\sin(\theta)}$$

$$\frac{d}{dy}\left\{\sin(k\theta)\right\} = \frac{2k}{b}\frac{\cos(k\theta)}{\sin(\theta)}$$

Susbtituting into equation 20

$$\frac{d}{dy}\left[\Gamma_w\right] = 2bv_\infty \sum_{k=1}^{\infty}\left[A_k\frac{2k}{b}\frac{\cos(k\theta)}{\sin(\theta)}\right]$$

$$\frac{d}{dy}\left[\Gamma_w\right] = 2bv_\infty \sum_{k=1}^{\infty}\left[\frac{2A_k k}{b}\frac{\cos(k\theta)}{\sin(\theta)}\right]$$

Reiterating equation 21

$$\frac{dy}{d\theta} = \frac{b}{2}\sin(\theta)$$

Manipulating the expression,

$$dy = \frac{b}{2}\sin(\theta)d\theta$$

Therefore,

$$\frac{d}{dy}\left[\Gamma_w\right] dy = 2bv_\infty \sum_{k=1}^{\infty}\left[\frac{2A_k k}{b}\frac{\cos(k\theta)}{\sin(\theta)}\right] \times \frac{b}{2}\sin(\theta)\, d\theta$$

$$\frac{d}{dy}\left[\Gamma_w\right] dy = 2bv_\infty \sum_{k=1}^{\infty}\left[\frac{2A_k k}{b}\frac{b}{2}\cos(k\theta)\right] d\theta$$

$$\frac{d}{dy}\left[\Gamma_w\right] dy = 2bv_\infty \sum_{k=1}^{\infty}\left[A_k k\cos(k\theta)\right] d\theta$$

Reiterating the coordinate transformation 18,

$$y = -\frac{b}{2}\cos(\theta)$$

Applying the coordinate transformation for some specific point along the wing $y = y_0$,

$$y_0 = -\frac{b}{2}\cos(\theta_0)$$

Therefore,

$$y_0 - y = -\frac{b}{2}\cos(\theta_0) + \frac{b}{2}\cos(\theta)$$

$$y_0 - y = \frac{b}{2}\left[\cos(\theta) - \cos(\theta_0)\right]$$

$$\frac{1}{y_0 - y} = \frac{2}{b}\frac{1}{\cos(\theta) - \cos(\theta_0)}$$

Therefore,

$$\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy = 2bv_\infty \sum_{k=1}^{\infty}\left[\frac{2}{b}\frac{1}{\cos(\theta) - \cos(\theta_0)} \times A_k k\cos(k\theta)\right]\,d\theta$$

$$\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy = 2bv_\infty \sum_{k=1}^{\infty}\left[\frac{2}{b}A_k k\frac{\cos(k\theta)}{\cos(\theta) - \cos(\theta_0)}\right]\,d\theta$$

$$\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy = 4v_\infty \sum_{k=1}^{\infty}\left[A_k k\frac{\cos(k\theta)}{\cos(\theta) - \cos(\theta_0)}\right]\,d\theta$$

When $y = b/2$,

$$\frac{b}{2} = -\frac{b}{2}\cos(\theta)$$

$$-1 = \cos(\theta)$$

$$\theta = \pi$$

When $y = -b/2$,

$$-\frac{b}{2} = -\frac{b}{2}\cos(\theta)$$

$$1 = \cos(\theta)$$

$$\theta = 0$$

Integrating for the whole wing with the bounds determined above,

$$\int_{-b/2}^{b/2}\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy = 4v_\infty \sum_{k=1}^{\infty}\left[A_k k\int_0^{\pi}\frac{\cos(k\theta)}{\cos(\theta) - \cos(\theta_0)}\,d\theta\right]$$

The last term in the expression above is Glauret's integral. Reiterating Glauret's integral (equation 1)

$$\int_0^{\pi}\frac{\cos(n\theta_0)}{\cos(\theta_0) - \cos(\theta)}\,d\theta_0 = \frac{\pi\sin(n\theta)}{\sin(\theta)} \quad , \quad n = 0, 1, 2, 3, \ldots$$

$$\int_{-b/2}^{b/2}\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy = 4v_\infty \sum_{k=1}^{\infty}\left[A_k k\frac{\pi\sin(k\theta_0)}{\sin(\theta_0)}\right]$$

$$\int_{-b/2}^{b/2}\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy = 4v_\infty\pi \sum_{k=1}^{\infty}\left[A_k k\frac{\sin(k\theta_0)}{\sin(\theta_0)}\right] \tag{22}$$

Substituting equation 22 and equation 19 into "Fundamental Lifting Line Theory" (equation 17),

$$\frac{\Gamma_w}{\pi v_\infty c_w} + \alpha_0 + \frac{1}{4\pi v_\infty}\int_{-b/2}^{b/2} \frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\,dy = \alpha_{Geo}$$

$$\frac{\Gamma_w}{\pi v_\infty c_w} + \alpha_0 + \frac{1}{4\pi v_\infty} \times 4v_\infty\pi \sum_{k=1}^{\infty}\left[A_k k\frac{\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\frac{\Gamma_w}{\pi v_\infty c_w} + \alpha_0 + \frac{v_\infty}{v_\infty}\sum_{k=1}^{\infty}\left[A_k k\frac{\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\frac{\Gamma_w}{\pi v_\infty c_w} + \alpha_0 + \sum_{k=1}^{\infty}\left[A_k k\frac{\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\frac{1}{\pi v_\infty c_w}\left\{2bv_\infty\sum_{k=1}^{\infty}\left[A_k\sin(k\theta_0)\right]\right\} + \alpha_0 + \sum_{k=1}^{\infty}\left[A_k k\frac{\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\alpha_0 + \frac{2b}{\pi c_w}\left\{\sum_{k=1}^{\infty}\left[A_k\sin(k\theta_0)\right]\right\} + \sum_{k=1}^{\infty}\left[A_k k\frac{\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\alpha_0 + \sum_{k=1}^{\infty}\left[A_k\frac{2b}{\pi c_w}\sin(k\theta_0)\right] + \sum_{k=1}^{\infty}\left[A_k k\frac{\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\alpha_0 + \sum_{k=1}^{\infty}\left[A_k\frac{2b\sin(k\theta_0)}{\pi c_w}\right] + \sum_{k=1}^{\infty}\left[A_k\frac{k\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\alpha_0 + \sum_{k=1}^{\infty}\left[A_k\frac{2b\sin(k\theta_0)}{\pi c_w} + A_k\frac{k\sin(k\theta_0)}{\sin(\theta_0)}\right] = \alpha_{Geo}$$

$$\alpha_0 + \sum_{k=1}^{\infty}\left\{A_k\left[\frac{2b\sin(k\theta_0)}{\pi c_w} + \frac{k\sin(k\theta_0)}{\sin(\theta_0)}\right]\right\} = \alpha_{Geo}$$

$$\alpha_0 + \sum_{k=1}^{\infty}\left\{A_k\left[\left(\frac{2b\sin(k\theta_0)}{\pi c_w} + \frac{k\sin(k\theta_0)}{\sin(\theta_0)}\right)\right]\right\} = \alpha_{Geo}$$

$$\alpha_0 + \sum_{k=1}^{\infty}\left\{A_k\left[\left(\frac{2b}{\pi c_w} + \frac{k}{\sin(\theta_0)}\right)\sin(k\theta_0)\right]\right\} = \alpha_{Geo} \tag{23}$$

0.2.4.4 Solution Algorithm

0.2.5 Special Case: Elliptical Vortex Distribution

A wing with an elliptical circulation distribution has a circulation distribution as denoted below,

$$\Gamma_w = \Gamma_0\sqrt{1 - \left(\frac{2}{b}y\right)^2}$$

wherein Γ_0 is some constant. Using the coordinate transformation,

$$y = -\frac{b}{2}\cos(\theta)$$

$$\frac{2}{b}y = -\frac{2}{b}\frac{b}{2}\cos(\theta)$$

$$\frac{2}{b}y = -\cos(\theta)$$

$$\left(\frac{2}{b}y\right)^2 = \cos^2(\theta)$$

$$\Gamma_0\sqrt{1-\left(\frac{2}{b}y\right)^2} = \Gamma_0\sqrt{1-\cos^2(\theta)}$$

A famous trigonometric identity,

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$\sin^2(\alpha) = 1 - \cos^2(\alpha)$$

$$\Gamma_0\sqrt{1-\left(\frac{2}{b}y\right)^2} = \Gamma_0\sqrt{\sin^2(\theta)}$$

$$\Gamma_w = \Gamma_0\sqrt{1-\left(\frac{2}{b}y\right)^2} = \Gamma_0\sin(\theta) \tag{24}$$

From a previous algebraic manipulation, When $y = b/2$,

$$\frac{b}{2} = -\frac{b}{2}\cos(\theta)$$

$$-1 = \cos(\theta)$$

$$\theta = \pi$$

When $y = -b/2$,

$$-\frac{b}{2} = -\frac{b}{2}\cos(\theta)$$

$$1 = \cos(\theta)$$

$$\theta = 0$$

Taking derivative of the coordinate transformation,

$$\frac{dy}{d\theta} = \frac{b}{2}\sin(\theta)$$

$$dy = \frac{b}{2}\sin(\theta)\,d\theta \tag{25}$$

Reiterating the Fourier Series used to represent arbitrary vortex distribution (equation 19)

$$\Gamma_w = 2bv_\infty\sum_{k=1}^{\infty}\left[A_k\sin(k\theta)\right]$$

Substituting the expression above into the "Fundamental Equation of Lifting Line Theory" produces the result below (equation 23)

$$\alpha_0 + \sum_{k=1}^{\infty} \left\{ A_k \left[\left(\frac{2b}{\pi c_w} + \frac{k}{\sin(\theta_0)} \right) \sin(k\theta_0) \right] \right\} = \alpha_{Geo}$$

For a wing with an elliptical vortex distribution,

$$\Gamma_w = \Gamma_0 \sin(\theta) = 2bv_\infty \sum_{k=1}^{\infty} \left[A_k \sin(k\theta) \right]$$

From this, $A_2 = A_3 = \cdots = 0$. That is the only way the equality above is true. Therefore,

$$\Gamma_w = \Gamma_0 \sin(\theta) = 2bv_\infty A_1 \sin(\theta)$$

$$\Gamma_0 = 2bv_\infty A_1$$

$$\frac{\Gamma_0}{2bv_\infty} = A_1$$

Truncating equation 23 for just $k = 1$,

$$\alpha_0 + A_1 \left[\left(\frac{2b}{\pi c_w} + \frac{1}{\sin(\theta_0)} \right) \sin(\theta_0) \right] = \alpha_{Geo}$$

$$\alpha_0 + A_1 \left[\left(\frac{2b}{\pi c_w} \sin(\theta_0) + \frac{\sin(\theta_0)}{\sin(\theta_0)} \right) \right] = \alpha_{Geo}$$

$$\alpha_0 + A_1 \left[\frac{2b}{\pi c_w} \sin(\theta_0) + 1 \right] = \alpha_{Geo}$$

$$A_1 \left[\frac{2b}{\pi c_w} \sin(\theta_0) + 1 \right] = \alpha_{Geo} - \alpha_0$$

Substituting for A_1,

$$\frac{\Gamma_0}{2bv_\infty} \left[\frac{2b}{\pi c_w} \sin(\theta_0) + 1 \right] = \alpha_{Geo} - \alpha_0$$

$$\Gamma_0 = \frac{2bv_\infty}{\frac{2b}{\pi c_w} \sin(\theta_0) + 1} (\alpha_{Geo} - \alpha_0) \tag{26}$$

0.2.5.1 Lift

Reiterating the expression for lift experienced by a wing, 12

$$L_w = \rho_\infty v_\infty \int_{-b/2}^{b/2} \Gamma_w \, dy$$

Substituting the circulation distribution (equation 24), the bounds and the differential dy (equation 25),

$$L_w = \rho_\infty v_\infty \int_0^{\pi} \Gamma_0 \sin(\theta) \frac{b}{2} \sin(\theta) \, d\theta$$

$$L_w = \frac{1}{2}\rho_\infty v_\infty b \int_0^\pi \Gamma_0 \sin(\theta)\sin(\theta)\, d\theta$$

$$L_w = \frac{1}{2}\rho_\infty v_\infty \Gamma_0 b \int_0^\pi \sin^2(\theta)\, d\theta$$

Using the double angle trigonometric identity,

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

$$2\sin^2(\theta) = 1 - \cos(2\theta)$$

$$\sin^2(\theta) = \frac{1}{2}\left[1 - \cos(2\theta)\right]$$

$$L_w = \frac{1}{2}\rho_\infty v_\infty \Gamma_0 b \frac{1}{2}\int_0^\pi 1 - \cos(2\theta)\, d\theta$$

$$L_w = \frac{1}{4}\rho_\infty v_\infty \Gamma_0 b \left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^\pi$$

$$L_w = \frac{1}{4}\rho_\infty v_\infty \Gamma_0 b\pi \tag{27}$$

Using the definition for coefficient of lift,

$$C_L = \frac{L_w}{\frac{1}{2}\rho_\infty v_\infty^2 A_w}$$

wherein A_w represents the area of the wing. Manipulating the expression,

$$\frac{1}{2}\rho_\infty v_\infty^2 A_w C_L = L_w$$

Equating the lift force produced by the wing to equation 27,

$$\frac{1}{2}\rho_\infty v_\infty^2 A_w C_L = L_w = \frac{1}{4}\rho_\infty v_\infty \Gamma_0 b\pi$$

$$\frac{1}{2}v_\infty A_w C_L = \frac{1}{4}\Gamma_0 b\pi$$

$$\frac{4}{2}v_\infty A_w C_L = \Gamma_0 b\pi$$

$$2v_\infty A_w C_L = \Gamma_0 b\pi$$

$$\frac{2v_\infty A_w C_L}{b\pi} = \Gamma_0$$

$$\Gamma_0 = \frac{2v_\infty A_w C_L}{b\pi} \tag{28}$$

Reiterating equation 26

$$\Gamma_0 = \frac{2bv_\infty}{\frac{2b}{\pi c_w}\sin(\theta_0) + 1}\left(\alpha_{Geo} - \alpha_0\right)$$

The ellitpical vortex distribution,

$$\Gamma_w = \Gamma_0\sqrt{1 - \left(\frac{2}{b}y\right)^2} = \Gamma_0 \sin(\theta)$$

For a wing with an elliptical planform,

$$c_w = c_0\sqrt{1-\left(\frac{2}{b}y\right)^2} = c_0\sin(\theta)$$

wherein c_0 represents the root chord, which is the maximum chord for a wing with an elliptical planform. Substituting chord c_w into equation 26,

$$\Gamma_0 = \frac{2bv_\infty}{\frac{2b}{\pi c_0\sin(\theta_0)}\sin(\theta_0)+1}\left(\alpha_{Geo}-\alpha_0\right)$$

$$\Gamma_0 = \frac{2bv_\infty}{\frac{2b}{\pi c_0}+1}\left(\alpha_{Geo}-\alpha_0\right)$$

$$\Gamma_0 = \frac{2bv_\infty}{\frac{2b}{\pi c_0}+\frac{\pi c_0}{\pi c_0}}\left(\alpha_{Geo}-\alpha_0\right)$$

$$\Gamma_0 = \frac{2bv_\infty}{\frac{2b+\pi c_0}{\pi c_0}}\left(\alpha_{Geo}-\alpha_0\right)$$

$$\Gamma_0 = \frac{2bv_\infty\pi c_0}{2b+\pi c_0}\left(\alpha_{Geo}-\alpha_0\right)$$

Substituting equation 28

$$\Gamma_0 = \frac{2bv_\infty\pi c_0}{2b+\pi c_0}\left(\alpha_{Geo}-\alpha_0\right) = \frac{2v_\infty A_w C_L}{b\pi}$$

$$\frac{b\pi c_0}{2b+\pi c_0}\left(\alpha_{Geo}-\alpha_0\right) = \frac{A_w C_L}{b\pi}$$

$$\frac{b^2\pi^2 c_0}{2b+\pi c_0}\left(\alpha_{Geo}-\alpha_0\right) = A_w C_L$$

$$\frac{b^2}{A_w}\times\frac{\pi^2 c_0}{2b+\pi c_0}\left(\alpha_{Geo}-\alpha_0\right) = C_L$$

Using the definition of aspect ratio,

$$A_R\frac{\pi^2 c_0}{2b+\pi c_0}\left(\alpha_{Geo}-\alpha_0\right) = C_L$$

$$A_R\frac{\pi^2\frac{c_0}{b}}{2+\pi\frac{c_0}{b}}\left(\alpha_{Geo}-\alpha_0\right) = C_L$$

The area of an ellipse is,

$$A_w = \frac{\pi}{4}bc_0$$

Using the definition for aspect ratio,

$$A_R = \frac{b^2}{A_w} = \frac{4}{\pi}\frac{1}{bc_0}b^2$$

$$A_R = \frac{4}{\pi}\frac{b}{c_0}$$

$$\frac{c_0}{b} = \frac{4}{\pi}\frac{1}{A_R}$$

Substituting,

$$A_R \frac{\pi^2 \frac{4}{\pi}\frac{1}{A_R}}{2 + \pi\frac{4}{\pi}\frac{1}{A_R}} (\alpha_{Geo} - \alpha_0) = C_L$$

$$\frac{4\pi}{2 + 4\frac{1}{A_R}} (\alpha_{Geo} - \alpha_0) = C_L$$

$$\frac{2\pi}{1 + \frac{2}{A_R}} (\alpha_{Geo} - \alpha_0) = C_L$$

$$C_L = \frac{2\pi}{1 + (2/A_R)} (\alpha_{Geo} - \alpha_0) \tag{29}$$

0.2.5.2 Downwash

Reiterating the expression for downwash produced by a wing, 13

$$\bar{w}_i = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\, dy\hat{z}$$

Reiterating the elliptical vortex distribution (equation 24)

$$\Gamma_w = \Gamma_0\sqrt{1 - \left(\frac{2}{b}y\right)^2} = \Gamma_0 \sin(\theta)$$

wherein the coordinate transformation used is shown below (equation 18),

$$y = -\frac{b}{2}\cos(\theta)$$

Taking derivative of the elliptical vortex distribution with respect to y,

$$\frac{d\Gamma_w}{dy} = \frac{d\Gamma_w}{d\theta} \times \frac{d\theta}{dy}$$

$$\frac{d\Gamma_w}{d\theta} = \Gamma_0 \frac{d}{d\theta}\left[\sin(\theta)\right]$$

$$\frac{d\Gamma_w}{d\theta} = \Gamma_0 \cos(\theta)$$

Reiterating the relation of the differentials dy and $d\theta$ (equation 21)

$$\frac{dy}{d\theta} = \frac{b}{2}\sin(\theta)$$

$$\frac{d\theta}{dy} = \frac{2}{b\sin(\theta)}$$

Therefore,

$$\frac{d\Gamma_w}{dy} = \Gamma_0 \cos(\theta) \times \frac{2}{b\sin(\theta)}$$

$$\frac{d\Gamma_w}{dy} = \frac{2\Gamma_0}{b}\frac{\cos(\theta)}{\sin(\theta)}$$

Reiterating the differentials again,

$$\frac{dy}{d\theta} = \frac{b}{2}\sin(\theta)$$

$$dy = \frac{b}{2}\sin(\theta)\, d\theta$$

Therefore,

$$\frac{d\Gamma_w}{dy}\, dy = \frac{2\Gamma_0}{b}\frac{\cos(\theta)}{\sin(\theta)} \times \frac{b}{2}\sin(\theta)\, d\theta$$

$$\frac{d\Gamma_w}{dy}\, dy = \frac{2\Gamma_0}{b}\cos(\theta) \times \frac{b}{2}\, d\theta$$

$$\frac{d\Gamma_w}{dy}\, dy = \Gamma_0 \cos(\theta)\, d\theta$$

Reiterating the coordinate transformation used,

$$y = -\frac{b}{2}\cos(\theta)$$

$$y_0 = -\frac{b}{2}\cos(\theta_0)$$

Therefore,

$$\frac{1}{y_0 - y} = \frac{1}{-\frac{b}{2}\cos(\theta_0) + \frac{b}{2}\cos(\theta)}$$

$$\frac{1}{y_0 - y} = \frac{2}{b}\frac{1}{\cos(\theta) - \cos(\theta_0)}$$

Taking the product with the previous result,

$$\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\, dy = \frac{2\Gamma_0}{b}\frac{1}{\cos(\theta) - \cos(\theta_0)}\cos(\theta)\, d\theta$$

$$\frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\, dy = \frac{2\Gamma_0}{b}\frac{\cos(\theta)}{\cos(\theta) - \cos(\theta_0)}\, d\theta$$

Susbtituting the equation above into the expression for downwash velocity,

$$\bar{w_i} = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y_0 - y}\frac{d\Gamma_w}{dy}\, dy\hat{z}$$

Using the change of bounds that was discussed earlier,

$$\bar{w_i} = -\frac{1}{4\pi}\int_{0}^{\pi} \frac{2\Gamma_0}{b}\frac{\cos(\theta)}{\cos(\theta) - \cos(\theta_0)}\, d\theta\hat{z}$$

$$\bar{w_i} = -\frac{\Gamma_0}{2\pi b}\int_0^{\pi} \frac{\cos(\theta)}{\cos(\theta)-\cos(\theta_0)}\, d\theta\hat{z}$$

Reiterating Glauret's integral (equation 1)

$$\int_0^{\pi} \frac{\cos(n\theta_0)}{\cos(\theta_0)-\cos(\theta)}\, d\theta_0 = \frac{\pi\sin(n\theta)}{\sin(\theta)} \quad , \quad n = 0, 1, 2, 3, \ldots$$

When $n = 1$,

$$\int_0^{\pi} \frac{\cos(\theta_0)}{\cos(\theta_0)-\cos(\theta)}\, d\theta_0 = \frac{\pi\sin(\theta)}{\sin(\theta)}$$

$$\int_0^{\pi} \frac{\cos(\theta_0)}{\cos(\theta_0)-\cos(\theta)}\, d\theta_0 = \pi$$

Making the change of variables, $\theta_0 \to \theta$, and $\theta \to \theta_0$,

$$\int_0^{\pi} \frac{\cos(\theta)}{\cos(\theta)-\cos(\theta_0)}\, d\theta = \pi$$

Substituting into the expression for downwash,

$$\bar{w_i} = -\frac{\Gamma_0}{2\pi b} \times \pi\hat{z}$$

$$\bar{w_i} = -\frac{\Gamma_0}{2b}\hat{z} \tag{30}$$

Susbtituting the maximum vortex strength Γ_0 in equation 28 to the equation above,

$$\Gamma_0 = \frac{2v_{\infty}A_w C_L}{b\pi}$$

$$\bar{w_i} = -\frac{1}{2b} \times \frac{2v_{\infty}A_w C_L}{b\pi}\hat{z}$$

$$\bar{w_i} = -\frac{v_{\infty}C_L A_w}{\pi b^2}\hat{z}$$

$$\bar{w_i} = -\frac{v_{\infty}C_L}{\pi}\frac{A_w}{b^2}\hat{z}$$

The definition of aspect ratio for a wing,

$$A_R = \frac{b^2}{A_w}$$

$$\frac{1}{A_R} = \frac{A_w}{b^2}$$

Therefore,

$$\bar{w_i} = -\frac{v_{\infty}C_L}{\pi}\frac{1}{A_R}\hat{z}$$

$$\bar{w_i} = -\frac{v_{\infty}C_L}{\pi A_R}\hat{z} \tag{31}$$

0.2.5.3 Induced Drag

Reiterating the elliptical vortex distribution (equation 24)

$$\Gamma_w = \Gamma_0 \sqrt{1 - \left(\frac{2}{b}y\right)^2} = \Gamma_0 \sin(\theta)$$

wherein the coordinate transformation used is shown below (equation 18),

$$y = -\frac{b}{2}\cos(\theta)$$

Reiterating the expression for induced angle of attack produced by the trailing vortices,

$$\alpha_i \approx \frac{|\bar{w_i}|}{v_\infty}$$

Reiterating the expression for the downwash experienced by a wing with an elliptical lift distribution (equation 30),

$$\bar{w_i} = -\frac{\Gamma_0}{2b}\hat{z}$$

The magnitude of the downwash,

$$|\bar{w_i}| = \frac{\Gamma_0}{2b}$$

Therefore,

$$\frac{|\bar{w_i}|}{v_\infty} = \frac{\Gamma_0}{2bv_\infty}$$

$$\alpha_i = \frac{\Gamma_0}{2bv_\infty}$$

Reiterating the expression for induced drag experienced by a wing, 15

$$D_i = \rho_\infty v_\infty \int_{-b/2}^{b/2} \alpha_i \Gamma_w \, dy$$

Substituting the induced angle of attack into the drag expression,

$$D_i = \rho_\infty v_\infty \int_{-b/2}^{b/2} \frac{\Gamma_0}{2bv_\infty} \Gamma_w \, dy$$

$$D_i = \frac{\Gamma_0 \rho_\infty v_\infty}{2bv_\infty} \int_{-b/2}^{b/2} \Gamma_w \, dy$$

Substituting the elliptical vortex distribution (equation 24),

$$D_i = \frac{\Gamma_0 \rho_\infty v_\infty}{2bv_\infty} \int_{-b/2}^{b/2} \Gamma_0 \sin(\theta) \, dy$$

Reiterating the relation of the differentials dy and $d\theta$ (equation 21)

$$\frac{dy}{d\theta} = \frac{b}{2}\sin(\theta)$$

$$dy = \frac{b}{2}\sin(\theta)\,d\theta$$

Using the change of coordinates and change of bounds,

$$D_i = \frac{\Gamma_0 \rho_\infty v_\infty}{2bv_\infty}\int_0^\pi \Gamma_0 \sin(\theta)\frac{b}{2}\sin(\theta)\,d\theta$$

$$D_i = \frac{\Gamma_0^2 \rho_\infty v_\infty}{4v_\infty}\int_0^\pi \sin(\theta)\sin(\theta)\,d\theta$$

$$D_i = \frac{\Gamma_0^2 \rho_\infty v_\infty}{4v_\infty}\int_0^\pi \sin^2(\theta)\,d\theta$$

Using the double angle identity,

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

$$2\sin^2(\theta) + \cos(2\theta) = 1$$

$$2\sin^2(\theta) = 1 - \cos(2\theta)$$

$$\sin^2(\theta) = \frac{1}{2}\left[1 - \cos(2\theta)\right]$$

$$D_i = \frac{\Gamma_0^2 \rho_\infty v_\infty}{4v_\infty} \times \frac{1}{2}\int_0^\pi 1 - \cos(2\theta)\,d\theta$$

$$D_i = \frac{\Gamma_0^2 \rho_\infty v_\infty \pi}{8v_\infty}$$

$$D_i = \frac{\Gamma_0^2 \rho_\infty \pi}{8} \tag{32}$$

Susbtituting the maximum vortex strength Γ_0 in equation 28 to the equation above,

$$\Gamma_0 = \frac{2v_\infty A_w C_L}{b\pi}$$

$$D_i = \frac{\rho_\infty \pi}{8} \times \Gamma_0^2$$

$$D_i = \frac{\rho_\infty \pi}{8} \times \left(\frac{2v_\infty A_w C_L}{b\pi}\right)^2$$

$$D_i = \frac{\rho_\infty \pi}{8} \times \frac{4v_\infty^2 A_w^2 C_L^2}{b^2\pi^2}$$

$$D_i = \frac{1}{2}\frac{\rho_\infty v_\infty^2 A_w^2 C_L^2}{b^2\pi}$$

$$D_i = \frac{1}{2}\rho_\infty v_\infty^2 A_w \times \frac{A_w C_L^2}{b^2\pi}$$

$$\frac{D_i}{\frac{1}{2}\rho_\infty v_\infty^2 A_w} = \frac{C_L^2}{\pi} \times \frac{A_w}{b^2}$$

Using the definition for coefficient of drag,

$$C_{D,i} = \frac{C_L^2}{\pi} \times \frac{A_w}{b^2}$$

Using the definition for aspect ratio of a wing,

$$C_{D,i} = \frac{C_L^2}{\pi} \times \frac{A_w}{b^2}$$

$$\frac{1}{A_R} = \frac{A_w}{b^2}$$

$$C_{D,i} = \frac{C_L^2}{\pi} \times \frac{1}{A_R}$$

$$C_{D,i} = \frac{C_L^2}{\pi A_R} \tag{33}$$

0.2.6 Elliptical Distribution Proofs