Chapter 1

Gauss Law

1.1 Gravitational Fields

The divergence theorem for any field $\bar{F}$ is denoted as
$\iiint \nabla \cdot F \, dV = \oiint F \cdot \bar{n} \, ds$, wherein $div[F] = \nabla \cdot F$
For a spherical mass m_1, the gravitational field generated by the mass m_1 at any point: $\bar{F_{field}} = -\frac{Gm_1}{r^2}\hat{r}$, wherein $\hat{r}$ represents unit vector with direction from mass to object.

$$\hat{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\hat{r} = \begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{1}{2}} \end{pmatrix}$$

Therefore, given that $\quad x \neq 0, \quad y \neq 0, \quad z \neq 0$

$$\bar{F_f} = -Gm_1 \begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\nabla \cdot \bar{F_f} = -Gm_1 \nabla \cdot \left[\begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \end{pmatrix} \right]$$

$$\text{Let } k = \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ y(x^2+y^2+z^2)^{-\frac{3}{2}} \\ z(x^2+y^2+z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\nabla \cdot k = f_x + g_y + h_z = \nabla \cdot \begin{pmatrix} x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ y(x^2+y^2+z^2)^{-\frac{3}{2}} \\ z(x^2+y^2+z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial f}{\partial x} = -3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} + (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial g}{\partial y} = -3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} + (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial f}{\partial z} = -3z^2(x^2+y^2+z^2)^{-\frac{5}{2}} + (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\nabla \cdot k = -3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-\frac{5}{2}} + 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = -3(x^2+y^2+z^2)^{-\frac{3}{2}} + 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = 0$$

$$\nabla \cdot F_f = -Gm_1 \times 0$$

$$\nabla \cdot F_f = 0$$

Therefore, for any point outside of the sphere, the divergence of the gravitational field is nonexistent. Consider a region R which is the space containing empty space and a mass sphere. Divergence in region R is the addition of divergence of free space and divergence of mass sphere. Since $div(F_f) = 0$ outside of the mass sphere as shown in previous example, divergence in R is only divergence of mass sphere.

$$div(F_R) = div(F_{sphere}) + div(F_{freespace})$$

By previous example, $div(F_{freespace}) = 0$

$$div(F_R) = div(F_{sphere})$$

Consider a point mass of mass M of uniform density with radius R:

$$\bar{f}_f = \frac{GM}{r^2}\hat{r}$$

$$\iiint \nabla \cdot \bar{f}_f \, dV = \oiint \bar{f}_f \cdot \bar{n} \, ds$$

By considering a surface to be enveloping the mass sphere,

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi R^2 \bar{f}_f$$

For all points with radius $r = R$ away from the center of the mass sphere,

$$\bar{f}_f = \frac{GM}{r^2}\hat{r}$$

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi R^2 \frac{GM}{R^2}$$

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi GM$$

By considering the predetermined property of this mass sphere to have uniform density:

$$(\nabla \cdot \bar{f}_f)V = 4\pi GM$$

$$(\nabla \cdot \bar{f}_f) \lim_{V \to 0}[V] = 4\pi G \lim_{M \to 0}[M]$$

$$(\nabla \cdot \bar{f}_f)dV = 4\pi G dM$$

$$\nabla \cdot \bar{f}_f = 4\pi G \frac{dM}{dV}$$

wherein ρ is density,

$$\nabla \cdot \bar{f}_f = 4\pi G\rho$$

By considering the graviational field of any arbitrary object to be the summation of the gravitational field of small components of the object,

$$\bar{F} = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 + \ldots \bar{f}_i$$

$$\bar{F} = \sum_{n=1}^{i} \left[\bar{f}_n\right]$$

$\nabla\cdot$ could be considered as a linear transformation because
$\nabla \cdot (\bar{a} + \bar{b}) = (\nabla \cdot \bar{a}) + (\nabla \cdot \bar{b})$ and also $\nabla \cdot (c\bar{a}) = c(\nabla \cdot \bar{a})$. Therefore,

$$\nabla \cdot \bar{F} = \sum_{n=1}^{i} \left[\nabla \cdot \bar{f}_n\right]$$

$$\nabla \cdot \bar{F} = \sum_{n=1}^{i} \left[4\pi G \rho_n\right]$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = \lim_{\Delta V_n \to\ 0} \left[\sum_{n=1}^{\infty} \left[4\pi G \rho_n \Delta V_n\right]\right]$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = 4\pi G \lim_{\Delta V_n \to\ 0} \left[\sum_{n=1}^{\infty} \left[\rho_n \Delta V_n\right]\right]$$

$$\rho_n \Delta V_n = \left(\frac{dM_n}{dV_n}\right) dV_n = dM_n$$

$$\lim_{\Delta V_n \to\ 0} \left[\sum_{n=1}^{\infty} \left[\rho_n \Delta V_n\right]\right] = \int_0^{m_e} dM = m_e$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = 4\pi G m_e$$

wherein m_e represents the mass enclosed by the surface. By reiterating The Divergence Theorem,

$$\iiint \nabla \cdot \bar{F}\, dV = \oiint \bar{F} \cdot \bar{n}\, ds$$

$$4\pi G m_e = \oiint \bar{F} \cdot \bar{n}\, ds$$

For the special case wherein $\rho = f(r)$, the gravitational field would be perpendicular to an imaginary spherical surface at radius r away from the center of the mass sphere. Therefore,

$$\bar{F} \cdot \bar{n} = |\bar{F}|$$

$$\oiint \bar{F} \cdot \bar{n}\, ds = |\bar{F}| \times 4\pi r^2$$

$$4\pi G m_e = |\bar{F}| \times 4\pi r^2$$

$$|\bar{F}| = \frac{G m_e}{r^2}$$

The $|\bar{F}|$ represents gravitational field produced by a spherical mass with varying radial density, $\rho = f(r)$. m_e represents enclosed mass, which in this case is the entirety of the mass sphere. Since the force of gravity experienced by an object with mass m_2 is $F_{force} = \bar{F}_{field} \times m_2$,

$$F_{force} = \frac{G m_e m_2}{r^2} = \frac{G m_e}{r^2} \int_0^R f(r) dr$$

wherein R is the radius of the mass sphere of varying radial density.