Chapter 1

Tensors

1.1 Tensor Index Notation

Tensors are a generalization of scalars, vectors, and matrices. The order of a tensor represents how many 'axis' the tensor has. For example, a scalar would be a 0^{th} order tensor meanwhile a vector would be a 1^{st} order tensor and a matrix would be 2^{nd} order tensor. Tensors of higher orders are permitted though a visual representation of them is meaningless. One can alternatively imagine tensors as multi-dimensional arrays, much like the case in a programming language.

The tensor index notation comprises of 2 main indices: A free index and a dummy index. A free index corresponds to the positioning of a certain value in a tensor. For example, the i^{th} component of a vector $\bar{v}$ is usually represented as v_i. That is an example of a free index usage. A dummy index is an index that is used for summation. Dummy indices occur in pairs and a pair of dummy indices imply summation. For example in the case of a dot product, A_jB_j represents scalar multiplication between the j^{th} components of vectors $\bar{A}$ and $\bar{B}$, added all together for the enirety of the length of vector $\bar{A}$ and vector $\bar{B}$.

Since what specific name one gives to a an index is arbitrary, this leads to index renaming rules. Dummy indices may be renamed within a single term. For example $A_jB_j = A_iB_i$. Free indices however, must be renamed across all algebraically summed terms. For example, $A_iB_pC_p + D_iE_qF_q = A_jB_pC_p + D_jE_qF_q$

1.2 Kronecker-Delta & Permutation Tensor

The kronecker-delta is a function that maps 2 integers to a 1 or 0. A mathematical description of the kronecker-delta function δ_{ij} is shown below,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

The permutation tensor ϵ_{ijk} a 3^{rd} order tensor and is anti-symmetric in any 2 of the indices. The indices can accept a range of integers from 1 until 3. Therefore, ϵ_{123}, ϵ_{213} are both valid but ϵ_{352} is not. The permutation tensor has a cyclic property described below,

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

Switching any 2 index of the permutaton tensor makes it negative. This property is described below,

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

The exact value of the permutation tensor,

$$\epsilon_{123} = 1 \quad , \quad \epsilon_{213} = -1$$

The other cases of i, j, and k are all obtainable by applying the properties above.

1.3 Common Vector Operations

Let $\bar{A}$ and $\bar{B}$ be vector fields, and ϕ, ψ be scalar fields. Let A_i and B_i represent the i^{th} component of the vector $\bar{A}$ and $\bar{B}$ respectively.

1.3.1 Scalar Multiplication

Since scalar multiplication simplt multiples all components of a vector by some scalar,

$$\left[\phi \bar{A}\right]_i = \phi A_i$$

wherein the LHS represents the vector notation and the RHS represents the index notation equivalent. Note that the $[]_i$ is used to denote the i^{th} index of the vector notation.

1.3.2 Dot Product

Dot products can be represented very elegantly in tensor index notation,

$$\bar{A} \cdot \bar{B} = A_j B_j$$

The repeated index j here makes j a dummy index which is used for counting. A repeated index such as j, implies summation. Therefore,

$$A_j B_j = A_1 B_1 + A_2 B_2 + A_3 B_3$$

1.3.3 Cross Product

The cross product of 2 vectors is defined with the permutation tensor,

$$\left[\bar{A} \times \bar{B}\right]_i = \epsilon_{ijk} A_j B_k$$

1.4 Tensor Index Identities

Let $\bar{A}$ and $\bar{B}$ be vector fields, and ϕ, ψ be scalar fields. Let $\bar{\bar{\mu}}$ and $\bar{\bar{\gamma}}$ represent second order tensors,

1.4.1 Symmetric-Antisymmetric Tensor

Let $\bar{\bar{\mu}}$ be a symmetric tensor and $\bar{\bar{\gamma}}$ be an anti-symmetric tensor. By the properties of the symmetric and anti-symmetric tensors,

$$\mu_{ij} = \mu_{ji} \quad , \quad \gamma_{ij} = -\gamma_{ji}$$

Consider the following,

$$\mu_{ij}\gamma_{ij} = -\mu_{ji}\gamma_{ji}$$

Here, the dummy indices have been switched, and this is true due to the symemtric and anti-symmetric definitions of μ and γ. The dummy indices are renamed, $j \to p$, $i \to q$,

$$\mu_{ij}\gamma_{ij} = -\mu_{pq}\gamma_{pq} \tag{1.1}$$

Next, start with $m\bar{\bar{u}}$ and $\bar{\bar{\gamma}}$ again, but this time rename them based on a different set of variable change. $i \to p$ and $j \to q$. This seems illegal, but it is not.

Remember, the naming are arbitrary and we have not violated any of the rules. Therefore,

$$\mu_{ij}\gamma_{ij} = \mu_{pq}\gamma_{pq} \tag{1.2}$$

Susbtituting $\mu_{ij}\gamma_{ij}$ out from euqation 1.1 and equation 1.2,

$$\mu_{pq}\gamma_{pq} = -\mu_{pq}\gamma_{pq}$$

Therefore,

$$0 = \mu_{pq}\gamma_{pq}$$

Hence, the element-wise multiplication of a symmetric and anti-symmetric tensor added together for the entire tensor would yield zero.

1.4.2 Double Permutation Tensor

Arguably one of the most important identities for tensor indices,

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

1.4.3 Kronecker-Delta Renaming

The kronecker-delta function can be used to rename the indices of a tensor,

$$\delta_{ij}A_i = A_j$$

This is because when $i \neq j$, the kronecker-delta function is zero, which means that $\delta_{ij}A_i$ is only non-zero when $i = j$, which renames the dummy variable of i in A_i into j.

1.4.4 Curl of Scalar Gradient

The curl of a scalar gradient is zero,

$$0 = \nabla \times (\nabla\phi)$$

Let ,

$$LHS = 0 \quad , \quad RHS = \nabla \times (\nabla\phi)$$

Converting LHS and RHS into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[\frac{\partial \phi}{\partial x_k} \right] = \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_k} (\phi)$$

Since partial derivative operators are commutative, $\frac{\partial^2}{\partial x_j \partial x_k} (\phi)$ is a symmetry tensor. If i is held constant, the permutation tensor ϵ_{ijk} is anti-symmetric. The element-wise multiplication of a symmetric tensor and anti-symmetric tensor added up together yields zero. Therefore,

$$RHS_i = 0$$

Since $LHS_i = RHS_i$, the claim is proven to be true.

1.4.5 Divergence of Vector Curl

The divergence of the curl of a vector field is zero,

$$0 = \nabla \cdot (\nabla \times \bar{A})$$

Let,

$$LHS = 0 \quad , \quad RHS = \nabla \cdot (\nabla \times \bar{A})$$

Converting LHS and RHS into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \frac{\partial}{\partial x_j} \left[\epsilon_{jkl} \frac{\partial}{\partial x_k} (A_l) \right] = \epsilon_{jkl} \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_k} (A_l) \right] = \epsilon_{jkl} \frac{\partial^2}{\partial x_j \partial x_k} (A_l)$$

Since ϵ_{jkl} is an anti-symmetric tensor and $\frac{\partial^2}{\partial x_j \partial x_k} (A_l)$ is a symmetric tensor, then $RHS_i = 0$. Since $LHS_i = RHS_i$, then the claim is proven to be true.

1.4.6 Curl of 2 Vector Cross Products

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Let,

$$LHS = \nabla \times (\bar{A} \times \bar{B}) \quad , \quad RHS = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Converting RHS into index notation,

$$RHS_i = B_j \frac{\partial}{\partial x_j}(A_i) + A_i \frac{\partial}{\partial x_j}(B_j) - A_j \frac{\partial}{\partial x_j}(B_i) - B_i \frac{\partial}{\partial x_j}(A_j)$$

Converting LHS into index notation,

$$LHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} [\epsilon_{klm} A_l B_m] = \epsilon_{ijk}\epsilon_{klm} \frac{\partial}{\partial x_j} [A_l B_m]$$

Using the cylcic permutation property of the permutation tensor $\epsilon_{ijk} = \epsilon_{kij}$.
Therefore,

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm}$$

Using the double permutation tensor identity,

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Substituting into LHS_i,

$$LHS_i = \epsilon_{ijk}\epsilon_{klm} \frac{\partial}{\partial x_j} [A_l B_m] = \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right] \frac{\partial}{\partial x_j} [A_l B_m] = \delta_{il}\delta_{jm} \frac{\partial}{\partial x_j} [A_l B_m] - \delta_{im}\delta_{jl} \frac{\partial}{\partial x_j} [A_l B_m]$$

$$LHS_i = \delta_{jm} \frac{\partial}{\partial x_j} [A_i B_m] - \delta_{jl} \frac{\partial}{\partial x_j} [A_l B_i] = \frac{\partial}{\partial x_j} \left[A_i B_j\right] - \frac{\partial}{\partial x_j} \left[A_j B_i\right]$$

Expanding using product rule,

$$LHS_i = A_i \frac{\partial}{\partial x_j} \left[B_j\right] + B_j \frac{\partial}{\partial x_j} [A_i] - \left\{ A_j \frac{\partial}{\partial x_j} [B_i] + B_i \frac{\partial}{\partial x_j} \left[A_j\right] \right\}$$

$$LHS_i = A_i \frac{\partial}{\partial x_j} \left[B_j\right] + B_j \frac{\partial}{\partial x_j} [A_i] - A_j \frac{\partial}{\partial x_j} [B_i] - B_i \frac{\partial}{\partial x_j} \left[A_j\right]$$

Since $LHS_i = RHS_i$, the vector identity is proven to be true.

1.4.7 Double Curl of Vector

$$\nabla \times \left(\nabla \times \bar{A}\right) = \nabla \left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Let

$$LHS = \nabla \times \left(\nabla \times \bar{A}\right) \quad , \quad RHS = \nabla \left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Converting LHS into index notation,

$$LHS_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{kmn}\frac{\partial}{\partial x_m}A_n$$

Since the permutation tensor ϵ_{kmn} is a constant in x_j and x_m,

$$LHS_i = \epsilon_{ijk}\epsilon_{kmn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}A_n$$

Using the permutation tensor cyclic identity,

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn}$$

Using the double permutation tensor identity,

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

Substituting,

$$LHS_i = [\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}]\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n)$$

$$LHS_i = \delta_{im}\delta_{jn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n) - \delta_{in}\delta_{jm}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n)$$

Using the renaming identity of the kronecker-delta function,

$$LHS_i = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Since partial derivatives are commutative with one another, $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(A_j) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j)$. Susbtituting,

$$LHS_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Reiterating RHS,

$$RHS = \nabla\left(\nabla \cdot \bar{A}\right) - \nabla^2\bar{A}$$

Converting RHS into index notation,

$$RHS_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Since $LHS_i = RHS_i$, then the identity is proven to be true.

1.4.8 Curl of Vector Scalar

$$\nabla \times (\phi \bar{A}) = \phi \nabla \times \bar{A} + (\nabla \phi) \times \bar{A}$$

Let

$$LHS = \nabla \times (\phi \bar{A}) \quad , \quad RHS = \phi \nabla \times \bar{A} + (\nabla \phi) \times \bar{A}$$

Converting LHS into index notation,

$$LHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\phi A_k)$$

Using product rule,

$$LHS_i = \epsilon_{ijk} \left[\phi \frac{\partial}{\partial x_j} (A_k) + A_k \frac{\partial}{\partial x_j} (\phi) \right]$$

$$LHS_i = \epsilon_{ijk} \phi \frac{\partial}{\partial x_j} (A_k) + \epsilon_{ijk} A_k \frac{\partial}{\partial x_j} (\phi)$$

Converting RHS into index notation,

$$RHS_i = \phi \epsilon_{ijk} \frac{\partial}{\partial x_j} (A_k) + \epsilon_{ijk} \left[\frac{\partial}{\partial x_j} (\phi) \right] A_k$$

$$RHS_i = \phi \epsilon_{ijk} \frac{\partial}{\partial x_j} (A_k) + \epsilon_{ijk} A_k \left[\frac{\partial}{\partial x_j} (\phi) \right]$$

Since $LHS_i = RHS_i$, the identity is proven to be true.

1.4.9 Triple Curl of Vector

$$\nabla \times [\nabla \times (\nabla \times \bar{A})] = -\nabla^2 (\nabla \times \bar{A})$$

Let,

$$LHS = \nabla \times [\nabla \times (\nabla \times \bar{A})] \quad , \quad RHS = -\nabla^2 (\nabla \times \bar{A})$$

In index notation,

$$(\nabla \times \bar{A})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (A_k)$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \epsilon_{lmi} \frac{\partial}{\partial x_m} \left[\epsilon_{ijk} \frac{\partial}{\partial x_j} (A_k) \right]$$

Since ϵ_{ijk} is simply a constant in x_m or x_j,

$$[\nabla \times (\nabla \times \bar{A})]_l = \epsilon_{lmi} \epsilon_{ijk} \frac{\partial}{\partial x_m} \left[\frac{\partial}{\partial x_j} (A_k) \right]$$

Using the cyclic property of the permutation tensor,

$$\epsilon_{lmi} \epsilon_{ijk} = \epsilon_{ilm} \epsilon_{ijk}$$

Using the double permutation tensor identity,

$$\epsilon_{lmi} \epsilon_{ijk} = \epsilon_{ilm} \epsilon_{ijk} = \delta_{lj} \delta_{mk} - \delta_{lk} \delta_{jm}$$

Substituting for the double permutation tensor identity,

$$[\nabla \times (\nabla \times \bar{A})]_l = [\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{jm}] \frac{\partial}{\partial x_m} \left[\frac{\partial}{\partial x_j} (A_k) \right]$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \delta_{lj} \delta_{mk} \frac{\partial}{\partial x_m} \left[\frac{\partial}{\partial x_j} (A_k) \right] - \delta_{lk} \delta_{jm} \frac{\partial}{\partial x_m} \left[\frac{\partial}{\partial x_j} (A_k) \right]$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \frac{\partial}{\partial x_k} \left[\frac{\partial}{\partial x_l} (A_k) \right] - \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_j} (A_l) \right]$$

$$\left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_p = \epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_k} \left[\frac{\partial}{\partial x_l} (A_k) \right] - \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

$$\left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_p = \epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_k} \left[\frac{\partial}{\partial x_l} (A_k) \right] \right\} - \epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

Since partial derivative operations are commutative with one another,

$$\epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_k} \left[\frac{\partial}{\partial x_l} (A_k) \right] \right\} = \epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_l} \left[\frac{\partial}{\partial x_k} (A_k) \right] \right\}$$

The permutation tensor is anti-symmetric in any 2 of its indices, and $\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_l}\left[\frac{\partial}{\partial x_k}(A_k)\right]\right\}$ is symmetric in q and l due to the commutativity of the partial differential operator. Since this would mean a symmetric tensor multiplied by an anti-symmetric element-wise and added together,

$$0 = \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right]\right\}$$

Therefore,

$$\left\{\nabla \times [\nabla \times (\nabla \times \bar{A})]\right\}_p = -\epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

renaming the free index $p \to i$,

$$\left\{\nabla \times [\nabla \times (\nabla \times \bar{A})]\right\}_i = -\epsilon_{iql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

Since $LHS_i = \left\{\nabla \times [\nabla \times (\nabla \times \bar{A})]\right\}_i$,

$$LHS_i = -\epsilon_{iql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

Reiterating definition of RHS,

$$RHS = -\nabla^2(\nabla \times \bar{A})$$

Converting RHS into index notation,

$$RHS_i = -\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}\left[\epsilon_{iql}\frac{\partial}{\partial x_q}(A_l)\right]\right\}$$

Since the permutation tensor is a constant in x_j and x_q and that partial derivative operations are commutative with one another,

$$RHS_i = -\epsilon_{iql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

Since $LHS_i = RHS_i$, the identity is proven to be true.