Physics Exploratory Notes

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Contents

Chapter 1

Binary Object Systems

1.1 Orbit Trajectory

For the scenario wherein an extraterrestial object of mass m approaches a planet of mass m_1, energy within the system of the planet and the object remains conserved because there is no external force acting on the system. Therefore, the summation of energy,$\sum E$ can be considered a constant in this scenario.

$$\sum E = E_k + E_p$$

$$E_k = \frac{1}{2}m|v|^2$$

By decomposing both radial and tangential components of velocity in polar coordinates:

$$|v|^2 = (\dot{\theta}r)^2 + \dot{r}^2$$

Therefore,

$$E_k = \frac{1}{2}m[(\dot{\theta}r)^2 + \dot{r}^2] = \frac{1}{2}m(\dot{\theta}r)^2 + \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m\dot{\theta}^2r^2 + \frac{1}{2}m\dot{r}^2$$

Since E_p represents gravitational potential energy the object has:

$$E_p = -\frac{Gmm_1}{r}$$

Let $\alpha = -Gmm_1$

$$E_p = \frac{\alpha}{r}$$

By substituting the expressions for kinetic energy and potential energy back into the equation for total energy:

$$\sum E = E_k + E_p = \frac{1}{2}m\dot{\theta}^2r^2 + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r}$$

The expression above is in the form of a differential equation involving two variables, θ and r as a derivative in respect to time. Let L represent angular momentum, and I represent moment inertia:

$$L = I\dot{\theta} \quad , \quad I = mr^2$$

Substituting for the moment of inertia for a single particle,

$$L = mr^2\dot{\theta}$$

making $\dot{\theta}$ subject of the equation,

$$\dot{\theta} = \frac{L}{mr^2}$$

The gravitational forces are always parallel to the radial vector of the object to the planet. Therefore, no net torque is produced on the passing object and angular momentum is conserved for the object. L could be treated as a constant in equation 2. Since L is a constant, equation 2 provides a method to reduce the dual variable problem in equation 1 to a single variable problem in terms of r. By substituting $\dot{\theta}$ from equation 2 to equation 1:

$$\sum E = \frac{1}{2}m\dot{\theta}^2 r^2 + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r}$$

Using a change in notation for convenience, let $\sum E = E$,

$$E = \frac{1}{2}m \left(\frac{L}{mr^2}\right)^2 r^2 + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r} = \frac{1}{2}mr^2 \frac{L^2}{(mr^2)^2} + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r} = \frac{L^2}{2mr^2} + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r}$$

By manipulating the equation to make $\dot{r}$ the subject of the equation:

$$\frac{1}{2}m\dot{r}^2 = E - \frac{L^2}{2mr^2} - \frac{\alpha}{r}$$

$$\dot{r}^2 = \frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{2\alpha}{mr}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{2\alpha}{mr}$$

The expression above involve r as a derivative in respect to time. Manipulating on this expression further would produce r as a function of time. Since the shape of the path traced by the object getting near to the planet is the subject in question, the derivative of r with respect to time must be converted to the derivative of r in respect to θ.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

Fortunately, equation 2 reduces $\dot{\theta}$ or $\frac{d\theta}{dt}$ into an expression purely in terms of known constants and r.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{L}{mr^2}$$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{d\theta}\frac{L}{mr^2}\right)^2 = \left(\frac{dr}{d\theta}\frac{1}{r^2}\right)^2 \frac{L^2}{m^2}$$

By substituting the term $\dot{r}$ from equation 4 into equation 3:

$$\left(\frac{dr}{d\theta}\frac{1}{r^2}\right)^2 \frac{L^2}{m^2} = \frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{2\alpha}{mr}$$

$$\left(\frac{dr}{d\theta}\frac{1}{r^2}\right)^2 = \frac{2E}{m}\frac{m^2}{L^2} - \frac{L^2}{m^2 r^2}\frac{m^2}{L^2} - \frac{2\alpha}{mr}\frac{m^2}{L^2} = \frac{2Em}{L^2} - \frac{1}{r^2} - \frac{2\alpha m}{L^2 r}$$

On the right hand side, there are lots of terms containing $\dfrac{1}{r}$ as well as $\dfrac{1}{r^2}$. For the reason that the left hand side also contain the term $\dfrac{dr}{d\theta}\dfrac{1}{r^2}$, let $u = \dfrac{1}{r}$

$$u = \frac{1}{r}$$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$-\frac{du}{d\theta} = \frac{1}{r^2}\frac{dr}{d\theta}$$

By substituting the term $\dfrac{dr}{d\theta}\dfrac{1}{r^2}$ into the main equation,

$$\left(-\frac{du}{d\theta}\right)^2 = \frac{2Em}{L^2} - \frac{1}{r^2} - \frac{2\alpha m}{L^2 r}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2Em}{L^2} - u^2 - \frac{2\alpha m}{L^2}u = \frac{2Em}{L^2} - \left(u^2 + \frac{2\alpha m}{L^2}u\right)$$

The right hand side takes the form of a quadratic in terms of u. The completing the square method should be performed on the right hand side of the expression to simplify the expression.

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2Em}{L^2} - \left[u^2 + \frac{2\alpha m}{L^2}u + \left(\frac{\alpha m}{L^2}\right)^2 - \left(\frac{\alpha m}{L^2}\right)^2\right] = \frac{2Em}{L^2} + \left(\frac{\alpha m}{L^2}\right)^2 - \left[u^2 + \frac{2\alpha m}{L^2}u + \left(\frac{\alpha m}{L^2}\right)^2\right]$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2EmL^2}{L^4} + \frac{\alpha^2 m^2}{L^4} - \left(u + \frac{\alpha m}{L^2}\right)^2 = \frac{2EmL^2 + \alpha^2 m^2}{L^4} - \left(u + \frac{\alpha m}{L^2}\right)^2$$

$$\text{Let } \beta^2 = \frac{2EmL^2 + \alpha^2 m^2}{L^4}$$

$$\left(\frac{du}{d\theta}\right)^2 = \beta^2 - \left(u + \frac{\alpha m}{L^2}\right)^2$$

It can be noticed that within the right hand side, the completing square form contains the variable u in arithmetic addition with some known constants. In order to "get rid" of these known constants to operate the equation further, let $\gamma = u + \dfrac{\alpha m}{L^2}$

$$\frac{d\gamma}{du} = \frac{d}{du}\left(u + \frac{\alpha m}{L^2}\right)$$

$$\frac{d\gamma}{du} = 1$$

$$d\gamma = du$$

$$\frac{d\gamma}{d\theta} = \frac{du}{d\theta}$$

By substituting γ to the original expression:

$$\left(\frac{d\gamma}{d\theta}\right)^2 = \beta^2 - (\gamma)^2$$

By solving for γ as a function of θ

$$\frac{d\gamma}{d\theta} = \sqrt{\beta^2 - (\gamma)^2}$$

$$\frac{d\theta}{d\gamma} = \frac{1}{\sqrt{\beta^2 - (\gamma)^2}}$$

$$d\theta = \frac{1}{\sqrt{\beta^2 - (\gamma)^2}} d\gamma$$

$$\int d\theta = \int \frac{1}{\sqrt{\beta^2 - (\gamma)^2}} d\gamma$$

By substitution of a inverse trigonometric derivative, let $p = \arcsin\left(\frac{\gamma}{\beta}\right)$

$$\frac{dp}{d\gamma} = \frac{1}{\sqrt{1 - \frac{\gamma^2}{\beta^2}}} \times \frac{1}{\beta} = \frac{1}{\sqrt{\frac{\beta^2 - \gamma^2}{\beta^2}}} \times \frac{1}{\beta}$$

$$\frac{dp}{d\gamma} = \frac{\beta}{\sqrt{\beta^2 - \gamma^2}} \times \frac{1}{\beta}$$

$$d\gamma = \sqrt{\beta^2 - \gamma^2} dp$$

By substuting$d\gamma$ into the original integral:

$$\theta = \int \frac{1}{\sqrt{\beta^2 - \gamma^2}} \sqrt{\beta^2 - \gamma^2} dp$$

$$\theta = p - \theta_0$$

$$\theta = \arcsin\left(\frac{\gamma}{\beta}\right) - \theta_0$$

$$\theta + \theta_0 = \arcsin\left(\frac{\gamma}{\beta}\right)$$

$$\sin\left(\theta + \theta_0\right) = \frac{\gamma}{\beta}$$

$$\beta \sin\left(\theta + \theta_0\right) = \gamma$$

$$\gamma = u + \frac{\alpha m}{L^2}$$

$$\beta \sin\left(\theta + \theta_0\right) = u + \frac{\alpha m}{L^2}$$

$$u = \frac{\beta L^2 sin(\theta + \theta_0) - \alpha m}{L^2}$$

$$u = \frac{1}{r}$$

$$\frac{1}{r} = \frac{\beta L^2 sin(\theta + \theta_0) - \alpha m}{L^2}$$

$$r = \frac{L^2}{\beta L^2 sin(\theta + \theta_0) - \alpha m}$$

$$\beta^2 = \frac{2EmL^2 + \alpha^2 m^2}{L^4}$$

$$\beta = \sqrt{\frac{2EmL^2 + \alpha^2 m^2}{L^4}}$$

$$\beta L^2 = \sqrt{2EmL^2 + \alpha^2 m^2}$$

$$\beta L^2 = \alpha m \sqrt{\frac{2EL^2}{m\alpha^2} + 1}$$

By substituting to the expression for r as a function of θ:

$$r = \frac{L^2}{\alpha m (\sqrt{\frac{2EL^2}{m\alpha^2} + 1}) \sin(\theta + \theta_0) - \alpha m}$$

Let choice of θ_0 be such that $\sin(\theta + \theta_0) = \cos(\theta)$. In the real context, the variable θ_0 provides a method to shift the point of reference rotationally. However, since the aim of the derivation is to investigate the matter of the orbit trajectory of a foreign object, the choice of θ_0 does not affect the shape of the trajectory produced. Since we are free to choose any value of θ_0, the choice of θ_0 such that $\sin(\theta + \theta_0) = \cos(\theta)$ simplifies the problem. Let $\epsilon_0 = \sqrt{\frac{2EL^2}{m\alpha^2} + 1}$,

$$r = -\frac{L^2}{m\alpha(1 - \epsilon_0 \cos(\theta))}$$

Since $\dfrac{L^2}{m\alpha}$ is a constant produced by the expressions of known constants, let $k = -\dfrac{L^2}{m\alpha}$

$$r = \frac{k}{1 - \epsilon_0 \cos(\theta)}$$

For the values of ϵ_0 to be 0, the orbit trajectory of the object on the planet shall be a circle, for the values of ϵ_0 to be $0 < \epsilon_0 < 1$, the orbit trajectory shall be an ellipse. For values of $\epsilon_0 > 1$, the orbit trajectory shall be a hyperbola. By comparing the expression above to the expression for conic sections in polar coordinates, it can be inferred that ϵ_0 takes the value of the eccentricity of the path the orbit takes.

1.2 Time Dependent Quantities

The previous working shows the trajectory of a binary system. The purpose of this working is to establish a relationship between time and radius r. All variables and symbols written below rerpresent the same quantities as the previous working. A reiteration of the previous equation:

$$\dot{r}^2 = \frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{2\alpha}{mr}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2Emr^2 - L^2 - 2\alpha mr}{m^2 r^2}$$

By taking the sqare root and reciprocral of both sides,

$$\frac{dt}{dr} = \frac{mr}{\sqrt{2Emr^2 - 2\alpha mr - L^2}}$$

Using completing the square method and rearranging the equations,

$$\frac{dt}{dr} = \frac{mr}{\sqrt{2Em\left[r^2 - \frac{\alpha}{E}r + \left(\frac{\alpha}{2E}\right)^2 - \left(\frac{\alpha}{2E}\right)^2 - \frac{L^2}{2Em}\right]}}$$

$$\frac{dt}{dr} = \sqrt{\frac{m}{2E}}\left(\frac{r}{\sqrt{r^2 - \frac{\alpha}{E}r + \left(\frac{\alpha}{2E}\right)^2 - \frac{m\alpha^2}{4E^2 m} - \frac{2EL^2}{4E^2 m}}}\right)$$

$$\frac{dt}{dr} = \sqrt{\frac{m}{2E}}\left(\frac{r}{\sqrt{\left(r - \frac{\alpha}{2E}\right)^2 - \frac{2EL^2 + m\alpha^2}{4E^2 m}}}\right)$$

Let

$$k_1 = \sqrt{\frac{m}{2E}} \qquad k_2 = \frac{\alpha}{2E} \qquad k_3 = \frac{2EL^2 + m\alpha^2}{4E^2 m}$$

$$\frac{dt}{dr} = k_1 \frac{r}{\sqrt{(r - k_2)^2 - k_3}}$$

$$\frac{dt}{dr} = k_1 \frac{r - k_2}{\sqrt{(r - k_2)^2 - k_3}} + k_1 \frac{k_2}{\sqrt{(r - k_2)^2 - k_3}}$$

$$\int dt = k_1 \int \frac{r - k_2}{\sqrt{(r - k_2)^2 - k_3}} dr + k_1 \int \frac{k_2}{\sqrt{(r - k_2)^2 - k_3}} dr$$

For the first integrand of the equation above, let $u = r - k_2$

$$\frac{du}{dr} = 1$$

$$\int \frac{r - k_2}{\sqrt{(r - k_2)^2 - k_3}} dr = \int \frac{u}{\sqrt{u^2 - k_3}} du$$

Let $\gamma = \sqrt{u^2 - k_3}$

$$\frac{d\gamma}{du} = \frac{1}{2} \frac{1}{\sqrt{u^2 - k_3}} (2u)$$

$$\int \frac{r - k_2}{\sqrt{(r - k_2)^2 - k_3}} dr = \sqrt{u^2 - k_3} + C$$

$$\int \frac{r - k_2}{\sqrt{(r - k_2)^2 - k_3}} dr = \sqrt{(r - k_2)^2 - k_3} + C$$

For the second integrand of the equation, let $u = r - k_2$

$$\frac{du}{dr} = 1$$

$$\int \frac{k_2}{\sqrt{(r - k_2)^2 - k_3}} = \int \frac{k_2}{\sqrt{u^2 - k_3}} du$$

Using the method of hyperbolic substitution, let $u = \sqrt{k_3} \cosh(\theta)$

$$u^2 = k_3 \cosh^2(\theta) = k_3 \sinh^2(\theta) + k_3$$

$$\int \frac{k_2}{\sqrt{u^2 - k_3}} du = \int \frac{k_2}{\sqrt{k_3 \sinh^2(\theta) + k_3 - k_3}}$$

$$\int \frac{k_2}{\sqrt{u^2 - k_3}} du = \int \frac{k_2}{\sqrt{k_3 \sinh^2(\theta)}} du = \int \frac{k_2}{\sqrt{k_3}} \frac{1}{\sinh(\theta)} du$$

Finding and expression of the differential du in terms of θ,

$$u = \sqrt{k_3} \cosh(\theta)$$

$$\frac{du}{d\theta} = \sqrt{k_3} \sinh(\theta)$$

$$du = \sqrt{k_3} \sinh(\theta) d\theta$$

Substituting the definition of the differential du,

$$\int \frac{k_2}{\sqrt{u^2 - k_3}} du = \frac{k_2}{\sqrt{k_3}} \int \frac{1}{\sinh(\theta)} \sqrt{k_3} \sinh(\theta) d\theta$$

$$\int \frac{k_2}{\sqrt{u^2 - k_3}} du = k_2 \theta + C$$

Expressing θ in terms of u,

$$u = \sqrt{k_3} \cosh(\theta)$$

$$\frac{u}{\sqrt{k_3}} = \cosh(\theta)$$

$$\theta = arcosh\left(\frac{u}{\sqrt{k_3}}\right)$$

Substituing the definition of θ

$$\int \frac{k_2}{\sqrt{u^2 - k_3}} du = k_2 arcosh\left(\frac{r - k_2}{\sqrt{k_3}}\right) + C$$

Therefore, the second integrand is:

$$\int \frac{k_2}{\sqrt{(r - k_2)^2 - k_3}} = k_2 arcosh\left(\frac{r - k_2}{\sqrt{k_3}}\right) + C$$

By placing the two integrands together,

$$\int dt = k_1\sqrt{(r-k_2)^2 - k_3} + k_1 k_2 \, arcosh\left(\frac{r-k_2}{\sqrt{k_3}}\right) + C$$

By substituting the expressions for $k_1, k_2,$ and k_3:

$$t = \sqrt{\frac{m}{2E}}\sqrt{(r-\frac{\alpha}{2E})^2 - \frac{2EL^2 + m\alpha^2}{4E^2 m}} + \sqrt{\frac{m}{2E}}\frac{\alpha}{2E}\, arcosh\left(\frac{r - \frac{\alpha}{2E}}{\sqrt{\frac{2EL^2+m\alpha^2}{4E^2 m}}}\right) + C$$

$$t = \frac{\sqrt{m}}{(2E)^{\frac{3}{2}}}\sqrt{(2Er-\alpha)^2 - \frac{2EL^2 + m\alpha^2}{m}} + \frac{\alpha\sqrt{m}}{(2E)^{\frac{3}{2}}}\, arcosh\left(\frac{\frac{2Er}{2E} - \frac{\alpha}{2E}}{\sqrt{\frac{2EL^2+m\alpha^2}{4E^2 m}}}\right) + C$$

$$t = \frac{\sqrt{m}}{(2E)^{\frac{3}{2}}}\sqrt{(2Er-\alpha)^2 - \frac{2EL^2}{m} - \alpha^2} + \frac{\alpha\sqrt{m}}{(2E)^{\frac{3}{2}}}\, arcosh\left(\frac{2Er-\alpha}{2E} \times \sqrt{\frac{4E^2 m}{2EL^2 + m\alpha^2}}\right) + C$$

$$t = \frac{\sqrt{m}}{(2E)^{\frac{3}{2}}}\sqrt{(2Er-\alpha)^2 - \frac{2EL^2}{m} - \alpha^2} + \frac{\alpha\sqrt{m}}{(2E)^{\frac{3}{2}}}\, arcosh\left[(2Er-\alpha)\sqrt{\frac{m}{2EL^2 + m\alpha^2}}\right] + C$$

Chapter 2

Gauss Law

2.1 Gravitational Fields

The divergence theorem for any field $\bar{F}$ is denoted as $\iiint \nabla \cdot F \, dV = \oiint F \cdot \bar{n} \, ds$, wherein

$$div[F] = \nabla \cdot F$$

For a spherical mass m_1, the gravitational field generated by the mass m_1 at any point:

$\bar{F}_{field} = -\dfrac{Gm_1}{r^2}\hat{r}$, wherein $\hat{r}$ represents unit vector with direction from mass to object.

$$\hat{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\hat{r} = \begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{1}{2}} \end{pmatrix}$$

Therefore, given that $\quad x \neq 0, \quad y \neq 0, \quad z \neq 0$

$$\bar{F}_f = -Gm_1 \begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\nabla \cdot \bar{F}_f = -Gm_1 \nabla \cdot \left[\begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \end{pmatrix} \right]$$

$$\text{Let } k = \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\nabla \cdot k = f_x + g_y + h_z = \nabla \cdot \begin{pmatrix} x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ y(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \right]$$

$$\frac{\partial f}{\partial x} = -3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} + (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left[y(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial g}{\partial y} = -3y^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} + (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[z(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial f}{\partial z} = -3z^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} + (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\nabla \cdot k = -3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-\frac{5}{2}} + 3(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = -3(x^2 + y^2 + z^2)^{-\frac{3}{2}} + 3(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = 0$$

$$\nabla \cdot F_f = -Gm_1 \times 0$$

$$\nabla \cdot F_f = 0$$

Therefore, for any point outside of the sphere, the divergence of the gravitational field is nonexistent. Consider a region R which is the space containing empty space and a mass sphere. Divergence in region R is the addition of divergence of free space and divergence of mass sphere. Since $div(F_f) = 0$ outside of the mass sphere as shown in previous example, divergence in R is only divergence of mass sphere.

$$div(F_R) = div(F_{sphere}) + div(F_{freespace})$$

By previous example, $div(F_{freespace}) = 0$

$$div(F_R) = div(F_{sphere})$$

Consider a point mass of mass M of uniform density with radius R:

$$\bar{f}_f = \frac{GM}{r^2}\hat{r}$$

$$\iiint \nabla \cdot \bar{f}_f \, dV = \oiint \bar{f}_f \cdot \bar{n} \, ds$$

By considering a surface to be enveloping the mass sphere,

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi R^2 \bar{f}_f$$

For all points with radius $r = R$ away from the center of the mass sphere, $\bar{f}_f = \frac{GM}{r^2}\hat{r}$

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi R^2 \frac{GM}{R^2}$$

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi GM$$

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By considering the predetermined property of this mass sphere to have uniform density:

$$(\nabla \cdot \bar{f}_f)V = 4\pi GM$$

$$(\nabla \cdot \bar{f}_f) \lim_{V \to 0}[V] = 4\pi G \lim_{M \to 0}[M]$$

$$(\nabla \cdot \bar{f}_f)dV = 4\pi GdM$$

$$\nabla \cdot \bar{f}_f = 4\pi G\frac{dM}{dV}$$

wherein ρ is density,

$$\nabla \cdot \bar{f}_f = 4\pi G\rho$$

By considering the graviational field of any arbitrary object to be the summation of the gravitational field of small components of the object,

$$\bar{F} = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 + \ldots \bar{f}_i$$

$$\bar{F} = \sum_{n=1}^{i} \left[\bar{f}_n\right]$$

$\nabla\cdot$ could be considered as a linear transformation because $\nabla \cdot (\bar{a} + \bar{b}) = (\nabla \cdot \bar{a}) + (\nabla \cdot \bar{b})$ and also $\nabla \cdot (c\bar{a}) = c(\nabla \cdot \bar{a})$. Therefore,

$$\nabla \cdot \bar{F} = \sum_{n=1}^{i} \left[\nabla \cdot \bar{f}_n\right]$$

$$\nabla \cdot \bar{F} = \sum_{n=1}^{i} \left[4\pi G\rho_n\right]$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = \lim_{\Delta V_n \to 0} \left[\sum_{n=1}^{\infty} \left[4\pi G\rho_n \Delta V_n\right]\right]$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = 4\pi G \lim_{\Delta V_n \to 0} \left[\sum_{n=1}^{\infty} \left[\rho_n \Delta V_n\right]\right]$$

$$\rho_n \Delta V_n = \left(\frac{dM_n}{dV_n}\right) dV_n = dM_n$$

$$\lim_{\Delta V_n \to 0} \left[\sum_{n=1}^{\infty} \left[\rho_n \Delta V_n\right]\right] = \int_0^{m_e} dM = m_e$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = 4\pi Gm_e$$

wherein m_e represents the mass enclosed by the surface. By reiterating The Divergence Theorem,

$$\iiint \nabla \cdot \bar{F}\, dV = \oiint \bar{F} \cdot \bar{n}\, ds$$

$$4\pi Gm_e = \oiint \bar{F} \cdot \bar{n}\, ds$$

For the special case wherein $\rho = f(r)$, the gravitational field would be perpendicular to an imaginary spherical surface at radius r away from the center of the mass sphere. Therefore,

$$\bar{F} \cdot \bar{n} = |\bar{F}|$$

$$\oiint \bar{F} \cdot \bar{n}\, ds = |\bar{F}| \times 4\pi r^2$$

$$4\pi G m_e = |\bar{F}| \times 4\pi r^2$$

$$|\bar{F}| = \frac{G m_e}{r^2}$$

The $|\bar{F}|$ represents gravitational field produced by a spherical mass with varying radial density, $\rho = f(r)$. m_e represents enclosed mass, which in this case is the entirety of the mass sphere. Since the force of gravity experienced by an object with mass m_2 is

$$F_{force} = \bar{F_{field}} \times m_2,$$

$$F_{force} = \frac{G m_e m_2}{r^2} = \frac{G m_e}{r^2} \int_0^R f(r)dr$$

wherein R is the radius of the mass sphere of varying radial density.

Chapter 3

Radial Fields

3.1   Curl and Circulation

Let the general equation denoting force produced by a radial field be: $\bar{F} = kr^n\hat{r}$ wherein k and n are arbitrary constants. Since $\hat{r}$ represents a unit vector in the radial direction,

$$\hat{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$$

$$r^n = \left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}$$

$$\bar{F} = \frac{k\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\bar{F} = k(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\bar{F} = \begin{pmatrix} kx(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} \\ ky(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} \\ kz(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} \end{pmatrix}$$

Stokes theorem states: $\oint_R \bar{F} \cdot \bar{dr} = \oiint_S \nabla \times \bar{F} \cdot \hat{n} dS$ wgereub R represents a closed path in $\mathbb{R}^3$ and S represents the corresponding surface bounded by the closed path similarly in $\mathbb{R}^3$. In conservative vector field, the circulation, $\oint_R \bar{F} \cdot \bar{dr} = 0$, therefore, the curl, $\nabla \times \bar{F} = 0$.

$$\text{Let } \bar{F} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$\nabla \times \bar{F} = \begin{pmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\frac{\partial h}{\partial y} = kyz(n-1)(x^2 + y^2 + z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial g}{\partial z} = kyz(n-1)(x^2 + y^2 + z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} = 0$$

$$\frac{\partial f}{\partial z} = kxz(n-1)(x^2 + y^2 + z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial h}{\partial x} = kxz(n-1)(x^2 + y^2 + z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial g}{\partial x} = kxy(n-1)(x^2 + y^2 + z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial f}{\partial y} = kxy(n-1)(x^2 + y^2 + z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

Therefore, $\nabla \times \bar{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Any radial vector field with the following expression $\bar{F} = kr^n \hat{r}$ is conservative. Since radial vector fields are considered as conservative force fields, a potential function must exist with the following conditions:

$$f = \frac{\partial p}{\partial x} \qquad g = \frac{\partial p}{\partial y} \qquad h = \frac{\partial p}{\partial z}$$

$$p = \int kx(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} dx$$

$$p = \frac{k}{n+1}(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)} + C(y, z) + k_1$$

$C(y, z)$ represents functions strictly in terms of y and z and k_1 represents some arbitrary numerical integration constant.

$$\frac{\partial p}{\partial y} = ky(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} + C'(y, z)$$

The potential function's partial derivative with respect to y represents y component of the force field. Therefore,

$$ky(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} + C'(y, z) = ky(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)}$$

$$C'(y, z) = 0$$

$$\int dC(y, z) = 0 \times \int dy$$

$$C(y, z) = C_2(z) + k_2$$

$$p = \frac{k}{n+1}(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)} + C_2(z) + k_1 + k_2$$

$$\frac{\partial p}{\partial z} = kz(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} + C_2'(z) = kz(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)}$$

$$C_2'(z) = 0$$

$$C_2(z) = k_3$$

$$p = \frac{k}{n+1}(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)} + k_1 + k_2 + k_3$$

By convention, the potential of a force field is defined as 0 infinitely away from the origin. Therefore,

$$0 = p = \frac{k}{n+1} \lim_{(x,y,z)\to\infty} \left[(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)} \right] + k_1 + k_2 + k_3$$

$$0 = k_1 + k_2 + k_3$$

Therefore, the potential function for any given radial force field is given as:

$$p = \frac{k}{n+1}(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)}$$