Chapter 1

Vorticity Equation

Let $\bar{A}$ and $\bar{B}$ be vector fields, and ϕ, ψ be scalar fields,

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Let,

$$LHS = \nabla \times (\bar{A} \times \bar{B}) \quad , \quad RHS = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Converting RHS into index notation,

$$RHS_i = B_j \frac{\partial}{\partial x_j}(A_i) + A_i \frac{\partial}{\partial x_j}(B_j) - A_j \frac{\partial}{\partial x_j}(B_i) - B_i \frac{\partial}{\partial x_j}(A_j)$$

Converting LHS into index notation,

$$LHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[\epsilon_{klm} A_l B_m\right] = \epsilon_{ijk} \epsilon_{klm} \frac{\partial}{\partial x_j} \left[A_l B_m\right]$$

Using the cylcic permutation property of the permutation tensor $\epsilon_{ijk} = \epsilon_{kij}$. Therefore,

$$\epsilon_{ijk} \epsilon_{klm} = \epsilon_{kij} \epsilon_{klm}$$

Using the double permutation tensor identity,

$$\epsilon_{ijk} \epsilon_{klm} = \epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

Substituting into LHS_i,

$$LHS_i = \epsilon_{ijk} \epsilon_{klm} \frac{\partial}{\partial x_j} \left[A_l B_m\right] = \left[\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\right] \frac{\partial}{\partial x_j} \left[A_l B_m\right] = \delta_{il} \delta_{jm} \frac{\partial}{\partial x_j} \left[A_l B_m\right] - \delta_{im} \delta_{jl} \frac{\partial}{\partial x_j} \left[A_l B_m\right]$$

$$LHS_i = \delta_{jm} \frac{\partial}{\partial x_j} \left[A_i B_m\right] - \delta_{jl} \frac{\partial}{\partial x_j} \left[A_l B_i\right] = \frac{\partial}{\partial x_j} \left[A_i B_j\right] - \frac{\partial}{\partial x_j} \left[A_j B_i\right]$$

Expanding using product rule,

$$LHS_i = A_i \frac{\partial}{\partial x_j} \left[B_j\right] + B_j \frac{\partial}{\partial x_j} \left[A_i\right] - \left\{ A_j \frac{\partial}{\partial x_j} \left[B_i\right] + B_i \frac{\partial}{\partial x_j} \left[A_j\right] \right\}$$

$$LHS_i = A_i \frac{\partial}{\partial x_j}\left[B_j\right] + B_j \frac{\partial}{\partial x_j}\left[A_i\right] - A_j \frac{\partial}{\partial x_j}\left[B_i\right] - B_i \frac{\partial}{\partial x_j}\left[A_j\right]$$

Since $LHS_i = RHS_i$, the vector identity is proven to be true.
The curl of a scalar gradient is zero,

$$0 = \nabla \times \nabla \phi$$

Let ,

$$LHS = 0 \quad , \quad RHS = \nabla \times \nabla \phi$$

Converting LHS and RHS into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j}\left[\frac{\partial \phi}{\partial x_k}\right] = \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_k}(\phi)$$

Since partial derivative operators are commutative, $\frac{\partial^2}{\partial x_j \partial x_k}(\phi)$ is a symmetry tensor. If i is held constant, the permutation tensor ϵ_{ijk} is anti-symmetric. The element-wise multiplication of a symmetric tensor and anti-symmetric tensor added up together yields zero. Therefore,

$$RHS_i = 0$$

Since $LHS_i = RHS_i$, the claim is proven to be true.
The divergence of the curl of a vector field is zero,

$$0 = \nabla \cdot (\nabla \times \bar{A})$$

Let,

$$LHS = 0 \quad , \quad RHS = \nabla \cdot (\nabla \times \bar{A})$$

Converting LHS and RHS into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \frac{\partial}{\partial x_j}\left[\epsilon_{jkl} \frac{\partial}{\partial x_k}(A_l)\right] = \epsilon_{jkl} \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_k}(A_l)\right] = \epsilon_{jkl} \frac{\partial^2}{\partial x_j \partial x_k}(A_l)$$

Since ϵ_{jkl} is an anti-symmetric tensor and $\frac{\partial^2}{\partial x_j \partial x_k}(A_l)$ is a symmetric tensor, then $RHS_i = 0$.
Since $LHS_i = RHS_i$, then the claim is proven to be true.
The substantive derivative of fluid velocity $\bar{v_f}$ appears in the non-conservative form of the momentum equation. The substantive derivative of fluid velocity $\bar{v_f}$ in vector form,

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f}$$

By conjecture,

$$\bar{v_f} \cdot \nabla \bar{v_f} = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f}\right)$$

Let,

$$LHS = \bar{v_f} \cdot \nabla \bar{v_f} \quad , \quad RHS = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

wherein $\bar{\omega_f}$ represents the fluid vorticity. Fluid vorticity is defined as the curl of fluid velocity, $\bar{\omega_f} = \nabla \times \bar{v_f}$.

Expressing LHS in index notation,

$$LHS_i = v_j \frac{\partial v_i}{\partial x_j}$$

wherein v_i represents the i^{th} component of the velocity vector $\bar{v_f}$.

Expressing RHS in index notation,

$$RHS_i = \epsilon_{ijk} \omega_j v_k + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

wherein ω_j represents the j^{th} component of fluid vorticity vector $\bar{\omega_f}$. Expressing the definition of vorticity as curl of fluid velocity in index notation,

$$\omega_j = \epsilon_{jlm} \frac{\partial v_m}{\partial x_l}$$

Substituting ω_j into RHS_i,

$$RHS_i = \epsilon_{ijk} \epsilon_{jlm} \frac{\partial v_m}{\partial x_l} v_k + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = \epsilon_{ijk} \epsilon_{jlm} v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

Based on the cyclic permutation properties of the permutation tensors ϵ_{ijk}, $\epsilon_{ijk} = \epsilon_{jki}$. Therefore,

$$\epsilon_{ijk} \epsilon_{jlm} = \epsilon_{jki} \epsilon_{jlm}$$

Based on the double permutation tensor identity,

$$\epsilon_{ijk} \epsilon_{jlm} = \epsilon_{jki} \epsilon_{jlm} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il}$$

Substituting into RHS_i,

$$RHS_i = \left[\delta_{kl} \delta_{im} - \delta_{km} \delta_{il} \right] v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = \delta_{kl} \delta_{im} v_k \frac{\partial v_m}{\partial x_l} - \delta_{km} \delta_{il} v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

$$RHS_i = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_i} \left[v_j v_j \right]$$

$$RHS_i = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + \frac{1}{2} \left[v_j \frac{\partial}{\partial x_i} \left(v_j \right) + v_j \frac{\partial}{\partial x_i} \left(v_j \right) \right] = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + v_j \frac{\partial v_j}{\partial x_i} = v_l \frac{\partial v_i}{\partial x_l}$$

Renaming the dummy index $l \rightarrow j$,

$$RHS_i = v_j \frac{\partial v_i}{\partial x_j}$$

Since $LHS_i = RHS_i$, then the conjecture shown below must be true,

$$\bar{v_f} \cdot \nabla \bar{v_f} = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Susbtituting into the substantive derivative of fluid velocity,

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f} = \frac{\partial \bar{v_f}}{\partial t} + \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Taking the curl of the substantive derivative of fluid velocity,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \nabla \times \left(\frac{\partial \bar{v_f}}{\partial t} \right) + \nabla \times (\bar{\omega_f} \times \bar{v_f}) + \nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$$

The curl operations and the partial derivative operations are commutative. Therefore,

$$\nabla \times \left(\frac{\partial \bar{v_f}}{\partial t} \right) = \frac{\partial}{\partial t} (\nabla \times \bar{v_f})$$

Substituting for the definition of fluid vorticity $\bar{\omega_f} = \nabla \times \bar{v_f}$,

$$\nabla \times \left(\frac{\partial \bar{v_f}}{\partial t} \right) = \frac{\partial}{\partial t} (\bar{\omega_f}) = \frac{\partial \bar{\omega_f}}{\partial t}$$

Substituting into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{\partial \bar{\omega_f}}{\partial t} + \nabla \times (\bar{\omega_f} \times \bar{v_f}) + \nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$$

$\bar{v_f} \cdot \bar{v_f}$ is a scalar. The curl of a scalar gradient is zero. Therefore,

$$0 = \nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$$

Negletcting the $\nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$ term,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{\partial \bar{\omega_f}}{\partial t} + \nabla \times (\bar{\omega_f} \times \bar{v_f})$$

This is a vector identity,

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Let $\bar{A} = \bar{\omega_f}$ and $\bar{B} = \bar{v_f}$,

$$\nabla \times (\bar{\omega_f} \times \bar{v_f}) = \bar{v_f} \cdot \nabla \bar{\omega_f} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f} - \bar{v_f} \nabla \cdot \bar{\omega_f}$$

Based on the definition of fluid vorticity $\bar{\omega_f} = \nabla \times \bar{v_f}$,

$$\bar{v_f}\nabla \cdot \bar{\omega_f} = \bar{v_f}\nabla \cdot (\nabla \times \bar{v_f})$$

Since the divergence of a vector field curl is zero, $\nabla \cdot (\nabla \times \bar{v_f}) = 0$. Therefore,

$$0 = \bar{v_f}\nabla \cdot \bar{\omega_f}$$

Neglecting the $\bar{v_f}\nabla \cdot \bar{\omega_f}$ term,

$$\nabla \times (\bar{\omega_f} \times \bar{v_f}) = \bar{v_f} \cdot \nabla\bar{\omega_f} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

Substituting $\nabla \times (\bar{\omega_f} \times \bar{v_f})$, into the substantive derivative of vorticity,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{\partial\bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla\bar{\omega_f} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

The definition of substantive derivative of vorticity is shown below,

$$\frac{D\bar{\omega_f}}{Dt} = \frac{\partial\bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla\bar{\omega_f}$$

Substituting for the substantive derivative of vorticity into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

The symmetric strain rate tensor is defined as,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

The symmetric strain rate tensor is symmetric.
The rotation rate tensor is defined as,

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

The rotation rate tensor is anti-symmetric.
The summation of the strain rate tensor and the rotation rate tensor,

$$S_{ij}+\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}\right)$$

$$S_{ij} + \Omega_{ij} = \frac{\partial v_i}{\partial x_j}$$

Therefore, this shows that the fluid velocity gradient tensor $\frac{\partial v_i}{\partial x_j}$ can be decomposed into an algebraic sum of a symmetric tensor S_{ij} and an anti-symmetric tensor Ω_{ij}.

The rotation rate tensor is somewhat related to the fluid vorticity vector. Consider i^{th} component of the fluid vorticity vector the i^{th} component of fluid velocity curl,

$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Using the fluid vorticity to contract the permutation tensor on along its third dimension,

$$\epsilon_{lmi}\omega_i = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Due to the permutation cyclic property, $\epsilon_{lmi} = \epsilon_{ilm}$. Therefore,

$$\epsilon_{lmi}\epsilon_{ijk} = \epsilon_{ilm}\epsilon_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$$

Substituting into the permutation tensor contraction,

$$\epsilon_{lmi}\omega_i = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j} = [\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}]\frac{\partial v_k}{\partial x_j} = \delta_{lj}\delta_{mk}\frac{\partial v_k}{\partial x_j} - \delta_{lk}\delta_{mj}\frac{\partial v_k}{\partial x_j} = \delta_{mk}\frac{\partial v_k}{\partial x_l} - \delta_{mj}\frac{\partial v_l}{\partial x_j}$$

$$\epsilon_{lmi}\omega_i = \frac{\partial v_m}{\partial x_l} - \frac{\partial v_l}{\partial x_m}$$

Under an index varible change $l \to i$, $m \to j$, $i \to k$,

$$\epsilon_{ijk}\omega_k = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}$$

This form is already similar to the rotation rate tensor. In essence, contracting the permutation tensor along any dimension would allow the usage of the double permutation tensor identity. The third dimension was chosen in order to obtain the 'alternating' pattern similar to the rotation rate tensor. Minor algebraic manipulatins can then be performed to match $\epsilon_{ijk}\omega_k$ to the rotation rate tensor,

$$-\epsilon_{ijk}\omega_k = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$$

$$-\frac{1}{2}\epsilon_{ijk}\omega_k = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

The RHS matches the rotation rate tensor. Therefore,

$$-\frac{1}{2}\epsilon_{ijk}\omega_k = \Omega_{ij}$$

By conjecture, using the fluid vorticity vector $\bar{\omega_f}$ to contract the rotation rate tensor along the second dimension would yield zero. This claim is expressed in index notation,

$$0 = \omega_j\Omega_{ij}$$

Let

$$LHS_i = 0 \quad , \quad RHS_i = \omega_j\Omega_{ij}$$

Substituting the definition of the rotation rate tensor in terms of the fluid vorticity vector,

$$RHS_i = \omega_j \Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k$$

Since ϵ_{ijk} is an anti-symmetric tensor, and $\omega_j\omega_k$ is a symmetric tensor due to multiplication being a commutative operation,

$$RHS_i = \omega_j \Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k = 0$$

Therefore, the claim is proven to be true.

Reiterating the last form of the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$

Converting the last term in RHS into index notation,

$$\left(\bar{\omega_f} \cdot \nabla \bar{v_f}\right)_i = \omega_j \frac{\partial v_i}{\partial x_j}$$

Expressing the velocity gradient tensor $\frac{\partial v_i}{\partial x_j}$ in terms of its symmetric and anti-symmetric components,

$$\left(\bar{\omega_f} \cdot \nabla \bar{v_f}\right)_i = \omega_j \frac{\partial v_i}{\partial x_j} = \omega_j[S_{ij} + \Omega_{ij}] = \omega_j S_{ij} + \omega_j \Omega_{ij}$$

Based on previous work, the contraction of the rotation tensor on the second index using the fluid vorticity vector yields zero,

$$0 = \omega_j \Omega_{ij}$$

Neglecting the $\omega_j \Omega_{ij}$ term,

$$\left(\bar{\omega_f} \cdot \nabla \bar{v_f}\right)_i = \omega_j S_{ij}$$

Converting into vector notation,

$$\bar{\omega_f} \cdot \nabla \bar{v_f} = \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

wherein $\bar{\bar{S}}_f$ represents the symmetric strain rate tensor for the fluid velocity vector field.

Substituting into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

The differential continuity equation is shown below,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v_f})$$

Converting the differential continuity equation into tensor index notation,

$$0 = \frac{\partial}{\partial t}[\rho] + \frac{\partial}{\partial x_j}(\rho v_j)$$

Using chain rule,

$$0 = \frac{\partial}{\partial t}[\rho] + \rho\frac{\partial}{\partial x_j}(v_j) + v_j\frac{\partial}{\partial x_j}(\rho)$$

Manipulating further,

$$\rho\frac{\partial}{\partial x_j}(v_j) = -\frac{\partial}{\partial t}[\rho] - v_j\frac{\partial}{\partial x_j}(\rho)$$

$$\frac{\partial}{\partial x_j}(v_j) = -\frac{1}{\rho}\left[\frac{\partial}{\partial t}(\rho) + v_j\frac{\partial}{\partial x_j}(\rho)\right]$$

The substantive derivative of fluid density ρ in vector notation,

$$\frac{D}{Dt}(\rho) = \frac{\partial}{\partial t}(\rho) + \bar{v_f}\cdot\nabla\rho$$

Converting the substatntive derivative of fluid density into index notation,

$$\left[\frac{D}{Dt}(\rho)\right]_i = \frac{\partial}{\partial t}(\rho) + v_j\frac{\partial}{\partial x_j}(\rho)$$

Substituting,

$$\frac{\partial}{\partial x_j}(v_j) = -\frac{1}{\rho}\left[\frac{D}{Dt}(\rho)\right]_i$$

Converting into vector index notation,

$$\nabla\cdot\bar{v_f} = -\frac{1}{\rho}\frac{D}{Dt}(\rho)$$

Substituting the divergence of fluid velocity $\bar{v_f}$ into the fluid velocity substantive derivative,

$$\nabla\times\left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho) - \bar{\omega_f}\cdot\bar{\bar{S}}_f$$

By conjecture,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho)$$

Let

$$LHS = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] \quad , \quad RHS = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho)$$

Using quotient rule,

$$LHS = \rho \times \frac{1}{\rho^2}\left\{\rho \frac{D}{Dt}\left[\omega_f\right] - \bar{\omega_f}\frac{D}{Dt}\left[\rho\right]\right\} = \frac{1}{\rho}\left\{\rho \frac{D}{Dt}\left[\omega_f\right] - \bar{\omega_f}\frac{D}{Dt}\left[\rho\right]\right\} = \frac{1}{\rho}\rho\frac{D}{Dt}\left[\omega_f\right] - \frac{1}{\rho}\bar{\omega_f}\frac{D}{Dt}\left[\rho\right]$$

$$LHS = \frac{D}{Dt}\left[\omega_f\right] - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}\left[\rho\right]$$

Since $LHS = RHS$, the claim is proven.
Substituting for this simplification,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] - \bar{\omega_f}\cdot\bar{\bar{S}}_f$$

The non-conservative form of the momentum equation,

$$\rho\frac{D\bar{v_f}}{Dt} = \nabla\cdot\bar{\bar{T}}_f + \bar{g}_b$$

Making the substantive derivative of fluid velocity the subject of the equation,

$$\frac{D\bar{v_f}}{Dt} = \frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f + \frac{1}{\rho}\bar{g}_b$$

Taking the curl of the resulting expression so that it might be substituted into the main equation,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f + \frac{1}{\rho}\bar{g}_b\right] = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

Substituting the complete stress tensor and external acceleration into the main equation,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] - \bar{\omega_f}\cdot\bar{\bar{S}}_f = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] - \bar{\omega_f}\cdot\bar{\bar{S}}_f = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

Hence, the 'basic' vorticity equation,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] = \bar{\omega_f}\cdot\bar{\bar{S}}_f + \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

The complete stress tensor $\bar{\bar{T}}_f$ defined in index notation,

$$T_{ij} = -P_r\delta_{ij} + \mu\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] + \lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}$$

wherein $\left(\bar{\bar{T}}_f\right)_{ij} = T_{ij}$, μ is the dynamic viscosity and λ is the second coefficient of viscosity. The viscous stress tensor $\bar{\bar{\tau}}_f$ has a rank of 2 and its ij component is referred as τ_{ij}. The viscous stress tensor components in index form is defined to be,

$$\tau_{ij} = \mu\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] + \lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}$$

Therefore, the complete stress tensor can be expressed in terms of the viscous stress tensor,

$$T_{ij} = -P_r \delta_{ij} + \tau_{ij}$$

Expressing the complete stress tensor term in index notation,

$$\left\{ \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] \right\}_i = \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (T_{ml}) \right]$$

Renaming the indices $i \to m$ $j \to l$,

$$T_{ml} = -P_r \delta_{ml} + \tau_{ml}$$

Substituting for the complete stress tensor,

$$\left\{ \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] \right\}_i = \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (-P_r \delta_{ml} + \tau_{ml}) \right]$$

$$\left\{ \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] \right\}_i = \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (-P_r \delta_{ml}) \right] + \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (\tau_{ml}) \right]$$

By applying the kronecker-delta contraction,

$$\left\{ \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] \right\}_i = -\epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial P_r}{\partial x_l} \right] + \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (\tau_{ml}) \right]$$

By applying product rule on the pressure-related term,

$$\left\{ \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] \right\}_i = -\epsilon_{ijl} \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\frac{\partial P_r}{\partial x_l} \right] - \epsilon_{ijl} \frac{\partial P_r}{\partial x_l} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \right] + \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (\tau_{ml}) \right]$$

Due to the partial derivative operations being commutative $\frac{\partial}{\partial x_j} \left[\frac{\partial P_r}{\partial x_l} \right]$ is a symmetric tensor of rank 2. Since the permuation tensor ϵ_{ijl} is anti-symmetric,

$$0 = -\epsilon_{ijl} \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\frac{\partial P_r}{\partial x_l} \right]$$

Neglecting the term,

$$\left\{ \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] \right\}_i = -\epsilon_{ijl} \frac{\partial P_r}{\partial x_l} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \right] + \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (\tau_{ml}) \right]$$

Applying chain rule, $\frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \right] = -\frac{1}{\rho^2} \frac{\partial \rho}{\partial x_j}$. Substituting,

$$\left\{ \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] \right\}_i = \epsilon_{ijl} \frac{\partial P_r}{\partial x_l} \frac{1}{\rho^2} \frac{\partial \rho}{\partial x_j} + \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (\tau_{ml}) \right] = \epsilon_{ijl} \frac{1}{\rho^2} \frac{\partial \rho}{\partial x_j} \frac{\partial P_r}{\partial x_l} + \epsilon_{ijl} \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \frac{\partial}{\partial x_m} (\tau_{ml}) \right]$$

Converting into index notation,

$$\nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{T}}_f \right] = \frac{1}{\rho^2} \nabla \rho \times \nabla P_r + \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{\tau}}_f \right]$$

Substituting into the basic vorticity equation,

$$\rho \frac{D}{Dt} \left[\frac{\bar{\omega}_f}{\rho} \right] = \bar{\omega}_f \cdot \bar{\bar{S}}_f + \frac{1}{\rho^2} \nabla \rho \times \nabla P_r + \nabla \times \left[\frac{1}{\rho} \nabla \cdot \bar{\bar{\tau}}_f \right] + \nabla \times \left[\frac{1}{\rho} \bar{g}_b \right]$$