0.1 Problem 1

0.1.1 Part a

The continuity and momentum equations in vector form,

$$\nabla \cdot u = 0 \quad , \quad \frac{Du}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

The continuity equation in index form,

$$0 = \frac{\partial u_j}{\partial x_j}$$

Simplifying the momentum equation in vector form,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

Converting the momentum equation into index form,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(u_i)$$

0.1.2 Part b

Renaming the dummy indices in the index momentum equation $j \to k$,

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(u_i)$$

Taking the derivative of the index momentum equation with respect to x_j,

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}(u_i) + \frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(p) + \nu \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(u_i)$$

Here, the fluid is assumed to be incompressible, hence ρ is a simple known fluid property. Since the partial derivative operator is commutative,

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x_j}(u_i) + \frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}(u_i)$$

Substituting $e_{ij} = \dfrac{\partial u_i}{\partial x_j}$,

$$\frac{\partial}{\partial t}e_{ij} + \frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}e_{ij}$$

Simplifying the convective acceleration term by applying chain rule,

$$\frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = u_k \frac{\partial}{\partial x_j}\left[\frac{\partial u_i}{\partial x_k}\right] + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}\left[u_k\right]$$

Due to the partial derivative operator being commutative,

$$\frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = u_k\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}(u_i) + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}[u_k]$$

Substituting $e_{ij} = \dfrac{\partial u_i}{\partial x_j}$,

$$\frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = u_k\frac{\partial}{\partial x_k}e_{ij} + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}[u_k]$$

Based on the definition of e_{ij},

$$e_{ik} = \frac{\partial u_i}{\partial x_k} \quad , \quad e_{kj} = \frac{\partial u_k}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = u_k\frac{\partial}{\partial x_k}e_{ij} + e_{ik}e_{kj}$$

Susbtituting the convective acceleration into the momentum equation,

$$\frac{\partial}{\partial t}e_{ij} + u_k\frac{\partial}{\partial x_k}e_{ij} + e_{ik}e_{kj} = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}e_{ij}$$

Manipulating the equation further,

$$\frac{\partial}{\partial t}e_{ij} + u_k\frac{\partial}{\partial x_k}e_{ij} = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj} + \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}e_{ij}$$

$$\frac{\partial}{\partial t}(e_{ij}) + u_k\frac{\partial}{\partial x_k}(e_{ij}) - \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj}$$

0.1.3 Part c

Reiterating the momentum equation in index form,

$$\frac{\partial}{\partial t}(e_{ij}) + u_k\frac{\partial}{\partial x_k}(e_{ij}) - \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj}$$

Renaming the indices, $i \to j$, and $j \to i$,

$$\frac{\partial}{\partial t}(e_{ji}) + u_k\frac{\partial}{\partial x_k}(e_{ji}) - \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ji}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_j}(p) - e_{jk}e_{ki}$$

Adding the 2 equations above together and taking into account that $\dfrac{\partial^2}{\partial x_i \partial x_j}(p) = \dfrac{\partial^2}{\partial x_j \partial x_i}(p)$,

$$\frac{\partial}{\partial t}(e_{ij}+e_{ji}) + u_k\frac{\partial}{\partial x_k}(e_{ij}+e_{ji}) - \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}+e_{ji}) = -\frac{2}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj} - e_{jk}e_{ki}$$

$$\frac{\partial}{\partial t}(e_{ij}+e_{ji}) + u_k\frac{\partial}{\partial x_k}(e_{ij}+e_{ji}) - \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}+e_{ji}) = -\frac{2}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - (e_{ik}e_{kj} + e_{jk}e_{ki})$$

Multiplying both sides by half,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(e_{ij}+e_{ji})\right]+u_k\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij}+e_{ji})\right]-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij}+e_{ji})\right]=-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)-\frac{1}{2}(e_{ik}e_{kj}+e_{jk}e_{ki})$$

The symmetric strain rate tensor is defined as, $S_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$. Expressing the symmetric strain rate tensor in terms of e_{ij} and e_{ji}, $S_{ij}=\frac{1}{2}\left(e_{ij}+e_{ji}\right)$ Substituting for the symmetric strain rate tensor into the momentum equation,

$$\frac{\partial}{\partial t}\left[S_{ij}\right]+u_k\frac{\partial}{\partial x_k}\left[S_{ij}\right]-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[S_{ij}\right]=-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)-\frac{1}{2}(e_{ik}e_{kj}+e_{jk}e_{ki})$$

Simplifying further,

$$\left\{\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\right\}S_{ij}=-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)-\frac{1}{2}(e_{ik}e_{kj}+e_{jk}e_{ki})$$

$\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}$ represents the substantive derivative in index notation meanwhile $\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}$ represents the laplacian in index notation.

0.1.4 Part d

Performing the same steps as the previous part but instead of adding 2 equations together, the equations are subtracted off each other,

$$\frac{\partial}{\partial t}(e_{ij}-e_{ji})+u_k\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})=-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)+\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)-e_{ik}e_{kj}+e_{jk}e_{ki}$$

Since the partial differential operators are commutative, $-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)+\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)=0$. Substituting,

$$\frac{\partial}{\partial t}(e_{ij}-e_{ji})+u_k\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})=-e_{ik}e_{kj}+e_{jk}e_{ki}$$

Multiplying both sides by half,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(e_{ij}-e_{ji})\right]+u_k\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij}-e_{ji})\right]-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij}-e_{ji})\right]=-\frac{1}{2}\left[e_{ik}e_{kj}-e_{jk}e_{ki}\right]$$

The rotation rate tensor is defined as $\Omega_{ij}=\frac{1}{2}(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i})$. Substituting for $e_{ij}=\frac{\partial u_i}{\partial x_j}$ and $e_{ji}=\frac{\partial u_j}{\partial x_i}$, $\Omega_{ij}=\frac{1}{2}(e_{ij}-e_{ji})$ Substituting the rotation rate tensor,

$$\frac{\partial}{\partial t}\left[\Omega_{ij}\right]+u_k\frac{\partial}{\partial x_k}\left[\Omega_{ij}\right]-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\Omega_{ij}\right]=-\frac{1}{2}\left[e_{ik}e_{kj}-e_{jk}e_{ki}\right]$$

$$\left\{\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\right\}\Omega_{ij}=-\frac{1}{2}\left[e_{ik}e_{kj}-e_{jk}e_{ki}\right]$$

Just as before, $\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}$ represents the substantive derivative in index notation meanwhile $\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}$ represents the laplacian in index notation. Interestingly, here the expression is independent of the pressure gradient tensor sinc ethe pressure gradient tensor is symmetric.