Chapter 1

Long-Term Behaviour of Circuits

1.1 Final Value Theorem

The final value theorem for an arbitrary function $f(t)$ is written as,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

wherein $F(s)$ represents the laplace transform of $f(t)$. The proof of this particular form of the final value theorem is shown below,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{s\to 0}\left\{\int_0^\infty e^{-st}f'(t)\,dt\right\} = \int_0^\infty f'(t)\,dt$$

By fundamental theorem of calculus, and change of variable,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{r\to\infty}\left\{f(r) - f(0)\right\} = \lim_{t\to\infty}\left\{f(t) - f(0)\right\}$$

Taking the approach of the Laplace Transform of derivatives,

$$\mathcal{L}\left[f'(t)\right] = s\mathcal{L}f(t) - f(0) = sF(s) - f(0)$$

Therefore, taking the limits as before,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{s\to 0}\left\{sF(s)\right\}\} - f(0)$$

By equating the $\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\}$ to each other,

$$\lim_{t\to\infty}\left\{f(t)\right\} - f(0) = \lim_{s\to 0}\left\{sF(s)\right\}\} - f(0)$$

This completes the proof,

$$\lim_{t\to\infty}\left\{f(t)\right\} = \lim_{s\to 0}\left\{sF(s)\right\}\}$$

1.2 Impulse Response

The transfer function $G(s)$ is defined as

$$G(s) = \frac{Y(s)}{U(s)}$$

wherein the $Y(s)$ is the output and $U(s)$ is the input, both in the laplace domain. In the laplace domain, the delta-dirac impulse function with arbitrary magnitude μ_0,

$$\mathcal{L}[\mu_0\,\delta(t-c)] = \mu_0 e^{-cs}$$

If the system described by the transfer function $G(s)$ has an input of the dirac delta function of arbitrary magnitude,

$$G(s) = \mu_0 e^{-cs}\,Y(s)$$

Manipulating for $Y(s)$ to be the subject of the equation,

$$Y(s) = \left(\frac{e^{cs}}{\mu_0}\right) G(s)$$

To find the long term behaviour of the output of the system $\lim_{t\to\infty}\big[y(t)\big]$, we should consult the final value theorem. The final value theorem for an arbitrary function $f(t)$,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

By performing the substitution $f(t) = y(t)$ and $F(s) = Y(s)$, wherein $y(t)$ represents the output function in the time domain and $Y(s)$ represents the output function in the laplace domain accordingly,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}[sY(s)]$$

If the input function is the dirac-delta function as shown previously, then $Y(s) = \left(\frac{e^{cs}}{\mu_0}\right) G(s)$. Therefore,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}\left[s\left(\frac{e^{cs}}{\mu_0}\right) G(s)\right]$$

A famous identity for limits of two arbitrary functions $q(t)$ and $p(t)$,

$$\lim_{t\to t_0}\big[p(t)\times q(t)\big] = \lim_{t\to t_0}\big[p(t)\big] \times \lim_{t\to t_0}\big[q(t)\big]$$

By the identity above,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}\big[sG(s)\big] \times \lim_{s\to 0}\left[\frac{e^{cs}}{\mu_0}\right] = \frac{1}{\mu_0}\lim_{s\to 0}\big[sG(s)\big]$$

1.3 Step Response

Let $G(s)$ represent the transfer function of some system in the laplace domain, $Y(s)$ represent the output of some system in the laplace domain, and $U(s)$ represent the input in the laplace domain.

$$Y(s) = G(s)U(s)$$

Consider the heaviside function $u(t-c)$ wherein c is some arbitrary time the heaviside function "turns on". The laplace transform of the arbitray heaviside function,

$$\mathcal{L}[u(t-c)] = \frac{e^{-}cs}{s}$$

If the input $u(t) = \mu_0$ wherein μ_0 is a constant for all t, then

$$\mu_0 G(0) = \lim_{t\to\infty}[y(t)]$$

1.4 Pure Sinusoid Response

The convolution of two arbitrary functions $f(t)$ and $g(t)$ are defined as

$$f * g(t) = g * f(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

Suppose a system has a transfer function $G(s)$ and the output and input in the laplace domain respectively is $Y(s)$ and $U(s)$.

$$G(s) = \frac{Y(s)}{U(s)}$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\left[G(s)U(s)\right] = \mathcal{L}^{-1}\left[G(s)\right] * \mathcal{L}^{-1}\left[U(s)\right] = g(t) * u(t) = \int_0^t g(\tau)u(t-\tau)\,d\tau$$

wherein $\mathcal{L}^{-1}G(s) = g(t)$, and $\mathcal{L}^{-1}U(s) = u(t)$. Here, the function $u(t)$ does not represent the heaviside unit function. Assuming that the input function is reasonably well-behaved and without loss of generality,

$$u(t) = \sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0 t}\right] \quad , \quad a_k = \frac{1}{\tau}\int_0^\tau e^{ik\omega_0 t} f(t)dt \quad , \quad \omega_0 = 2\pi/\tau$$

Substituting the Fourier representation of the input function,

$$y(t) = \int_0^t g(\tau) \sum_{k=-\infty}^{\infty} \left[a_k e^{-ik\omega_0(t-\tau)}\right] d\tau = \int_0^t g(\tau) \sum_{k=-\infty}^{\infty} \left[a_k e^{ik\omega_0\tau} e^{-ik\omega_0 t}\right] d\tau$$

$$y(t) = \sum_{k=-\infty}^{\infty} \left[\int_0^t g(\tau) e^{ik\omega_0\tau}\, d\tau a_k e^{-ik\omega_0 t}\right]$$

Observing the long term-behaviour of the output, $t = \infty$. Therefore,

$$y(t) = \sum_{k=-\infty}^{\infty} \left[\int_0^\infty g(\tau) e^{ik\omega_0\tau}\, d\tau a_k e^{-ik\omega_0 t}\right]$$

The term $\int_0^\infty g(\tau) e^{ik\omega_0\tau}\, d\tau$ represents a laplace transform with $s = -ik\omega_0$. Therefore,

$$G(-ik\omega_0) = \int_0^\infty g(\tau) e^{ik\omega_0\tau}\, d\tau$$

By substitution,

$$y(t) = \sum_{k=-\infty}^{\infty} \left[G(-ik\omega_0) a_k e^{-ik\omega_0 t}\right]$$

For real sinusoidal inputs $u(t) = k\sin(\omega t)$,

$$\lim_{t\to\infty} [y(t)] = k|G(i\omega)| \sin\{\omega t + arg[G(i\omega)]\}$$