Knowledge Archives

Ginger Gengar

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Part I

Mathematics

Chapter 1

Linear Differential Equations

1.1 Definition of Differential Operator

Let the differential operator be defined as the following:

$$L = \sum_{i=0}^{n} \left[a_i \frac{d^{n-i}}{dx^{n-i}} \right] = a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1} \frac{d}{dx} + a_n$$

Let the following operator D be defined:

$$D = \frac{d}{dx} \qquad D^k = \frac{d^k}{dx^k}$$

Therefore, the linear differential operator could be defined as

$$L = \sum_{i=0}^{n} \left[a_i D^{n-i} \right] = a_0 \prod_{i=1}^{n} [D - r_i]$$

wherein a_i and r_i are constants albeit complex or real. The notation above is always true because an fundamental theorem of algebra states that n^{th} order polynomial must have n roots. Since linear differential operator could be expressed as a polynomial in terms of the operator D, therefore the notation above would always be true regardless of the choice of a_i and n. The linear differential operator have certain properties associated to them discussed in the propositions,

1.1.1 Proposition 1: Operation on 0

The linear differential operator when operated on a 0 will yield 0:

$$L[0] = \sum_{i=0}^{n} \left[a_i D^{n-i} \right] 0 = a_0 \frac{d^n}{dx^n} 0 + a_1 \frac{d^{n-1}}{dx^{n-1}} 0 + \cdots + a_{n-1} \frac{d}{dx} 0 + a_n 0$$

It is given that $\frac{d}{dx}(0) = 0$, and by reapplying recursively, $\frac{d^k}{dx^k}(0) = 0$. Therefore,

$$L[0] = 0$$

1.1.2 Proposition 2: Operation on Constants

The linear differential operator when operated on a constant will yield some constant provided that $a_n \neq 0$:

$$L[c] = \sum_{i=0}^{n} \left[a_i D^{n-i}\right] c = a_0 \frac{d^n}{dx^n} c + a_1 \frac{d^{n-1}}{dx^{n-1}} c + \cdots + a_{n-1} \frac{d}{dx} c + a_n c$$

$$\text{Considering } \frac{d}{dx} c = \frac{d^k}{dx^k} c = 0,$$

$$L[c] = \sum_{i=0}^{n} \left[a_i D^{n-i}\right] c = a_n c$$

1.1.3 Proposition 3: Operation Commutativity

If there exist two linearly independent differential operators L_1 and L_2, then the solution of the system $0 = L_1 L_2[y]$ must be a linear combination of the solution to the system $0 = L_1[y]$ and $0 = L_2[y]$:

$$L_1 = \prod_{i=0}^{m} [D - \alpha_i] \quad , \quad L_2 = \prod_{j=0}^{n} \left[D - \beta_j\right]$$

Let $y_1(x)$ and $y_2(x)$ be such that:

$$0 = L_1[y_1(x)] = \prod_{i=0}^{m} [D - \alpha_i]\, y_1(x) \quad , \quad 0 = L_2[y_2(x)] = \prod_{j=0}^{n} \left[D - \beta_j\right] y_2(x)$$

Let T_i be the transformation defined as $T_i : f(x) \to g(x) \quad , \quad T_i[f(x)] = (D - r_i) f(x)$, wherein $f(x)$ is some arbitrary continuous function over some interval. Indeed the transformation T_1 is linear:

$$T_i[cu(x)] = (D - r_i) cu(x)$$

$$T_i[cu(x)] = c(D - r_i) u(x)$$

$$cT_i[u(x)] = c(D - r_i) u(x)$$

Therefore, $T_i[cu(x)] = cT_i[u(x)]$ wherein c is some arbitrary constant.

$$T_i[u(x) + v(x)] = (D - r_i)[u(x) + v(x)]$$

$$T_i[u(x) + v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

$$T_i[u(x)] + T_i[v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

Therefore, $T_i[u(x) + v(x)] = T_i[u(x)] + T_i[v(x)]$, and T_i must be a linear transformation. Linear transformations applied compositely form a linear transformation:

$$T_0(u + v) = T_0(u) + T_0(v)$$

$$T_1[T_0(u + v)] = T_1[T_0(u)] + T_1[T_0(v)]$$

$$\prod_{i=0}^{\alpha} [T_i]\, (u + v) = \prod_{i=0}^{\alpha} [T_i]\, (u) + \prod_{i=0}^{\alpha} [T_i]\, (v)$$

Since L_1 and L_2 is only a specific case of the transformation described as T_1 it can be considered that the differential operators of L_1 and L_2 are linear. Therefore, it can be said that $L_1[L_2]$ must be linear.

$$0 = null = L_1[y_1(x)] = \prod_{i=0}^{m} [D - \alpha_i] \, y_1(x) \quad , \quad 0 = null = L_2[y_2(x)] = \prod_{j=0}^{n} \left[D - \beta_j\right] y_2(x)$$

For $y_1(x)$ and $y_2(x)$:

$$0 = null = L_2\{L_1[y_1(x)]\} = \prod_{j=0}^{n} \left[D - \beta_j\right] \prod_{i=0}^{m} [D - \alpha_i] \, y_1(x)$$

$$0 = null = L_1\{L_2[y_2(x)]\} = \prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] y_2(x)$$

$$\prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] = \prod_{j=0}^{n} \left[D - \beta_j\right] \prod_{i=0}^{m} [D - \alpha_i] = L_1[L_2] = L_2[L_1]$$

It follows by definition of linear transformation that, $L_1[k_1 y_1(x)] = k_1 L_1[y_1(x)] = null$ and that $L_2[k_2 y_2(x)] = k_2 L_2[y2(x)] = null$:

$$0 = null = L_2\{L_1[k_1 y_1(x)]\} = \prod_{j=0}^{n} \left[D - \beta_j\right] \prod_{i=0}^{m} [D - \alpha_i] \, k_1 y_1(x)$$

$$0 = null = L_1\{L_2[k_2 y_2(x)]\} = \prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] k_2 y_2(x)$$

$$0 = L_2\{L_1[k_1 y_1(x)]\} + L_2\{L_1[k_2 y_2(x)]\} = L_2\{L_1[k_1 y_1(x) + k_2 y_2(x)]\}$$

$$0 = \prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] [k_1 y_1(x) + k_2 y_2(x)]$$

1.2 Homogenous Differential Equation Cases

Consider the following homogenous differential equation:

$$0 = \sum_{i=0}^{n} \left[a_i \overset{n-i}{y} \right] = a_0 \overset{n}{y} + a_1 \overset{n-1}{y} + \cdots + a_{n-1} \dot{y} + a_n y$$

$$0 = L[y] = \sum_{i=0}^{n} \left[a_i D^{n-i} \right] y = \prod_{i=1}^{n} [D - r_i] \, y$$

1.2.1 Non-Repeated Roots

By principle of superposition verified by proposition 1 and 3, the general solution to the homogenous n^{th} order differential equation,

$$y_c = \sum_{i=1}^{n} [c_i y_i]$$

Wherein c_i represents either complex or real constants, y_i represents solutions to the $0 = [D - r_i]y_i$ system. Considering the partial system,

$$0 = [D - r_i]y_i$$

$$0 = Dy_i - r_i y_i$$

$$\frac{d}{dx}(y_i) = r_i y_i$$

$$\int \frac{1}{y_i} dy_i = \int r_i dx$$

$$\ln y_i = r_i x + C$$

$$y_i = e^{r_i x + C} = c_i e^{r_i x}$$

Therefore for as long as there are no roots with multiplicity greater than 1, the following is true, for some choice of constants,

$$y_c = \sum_{i=1}^{n} [c_i e^{r_i x}]$$

1.2.2 Repeated Roots

Suppose the α^{th} root has a multiplicity of k,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^{n} [D - r_i] (D - r_\alpha)^k y$$

By proposition 3, the general solution to the system must be the linear combination:

$$y_g(x) = c_1 y_1(x) + c_2 y_2(x)$$

wherein $y_1(x)$ is the solution to the system $0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^{n} [D - r_i]\, y$ and $y_2(x)$ is the solution to the system $0 = (D - r_\alpha)^k y$. By conjecture, it is suspected that the $y_2(x) = u(x)e^{r_\alpha x}$ wherein $u(x)$ is some function to be determined.

$$0 = (D - r_\alpha)u(x)e^{r_\alpha x}$$

$$0 = \frac{d}{dx}\left[u(x)e^{r_\alpha x}\right] - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = \overset{1}{u}(x)e^{r_\alpha x} + r_\alpha u(x)e^{r_\alpha x} - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = \overset{1}{u}(x)e^{r_\alpha x}$$

By reapplying the linear differential operator recursively:

$$(D - r_\alpha)^k u(x)e^{r_\alpha x} = \overset{k}{u}(x)e^{r_a x}$$

Therefore, the system would follow:

$$0 = (D - r_\alpha)^k u(x)e^{r_\alpha x} = \overset{k}{u}(x)e^{r_a x}$$

$$0 \neq e^{r_a x} \text{ for all } x$$

$$0 = \overset{k}{u}(x)$$

A function that satisfies the following condition must be a polynomial with at most degree $k-1$. Therefore,

$$u(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right]$$

The general solution $y_{rr}(x)$ to the system $0 = (D - r_\alpha)^k y$:

$$y_{rr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right] e^{r_\alpha x}$$

1.2.3 Complex Roots

Suppose the α^{th} root is a complex root, by the fundamental theorem of algebra, some other root must be its complex conjugate. Let the complex conjugate root of the α^{th} root be ordered next to the α^{th} root in the product notation. Therefore,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{i=\alpha+2}^{n} [D - r_i\ ]\,[D - r_\alpha][D - r_{\alpha+1}]y$$

Let y_{cr} represent the complex root corresponding to the system $0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$. By principle of superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2} [c_i e^{r_i x}] + y_{cr}$$

$$0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$$

By the principles presented earlier,

$$y_{cr}(x) = c_\alpha e^{(a+bi)x} + c_{\alpha+1} e^{(a-bi)x}$$

$$y_{cr}(x) = e^{ax}\left[c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}\right]$$

By De Moivre's theorem,

$$e^{bxi} = \cos(bx) + i\sin(bx) \quad , \quad e^{-bxi} = \cos(bx) - i\sin(bx)$$

Suppose the constants c_α and $c_{\alpha+1}$ are complex numbers,

$$c_\alpha = f_1 + g_1 i \quad , \quad c_{\alpha+1} = f_2 + g_2 i$$

By substituting to the expression for complex solution,

$$y_{cr}(x) = e^{ax}\left[(f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}\right]$$

Let $y_{cr} = e^{ax} y_{co}$,

$$y_{co} = c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}$$

$$y_{co}(x) = (f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}$$
$$y_{co}(x) = (f_1 + g_1 i)[\cos(bx) + i\sin(bx)] + (f_2 + g_2 i)[\cos(bx) - i\sin(bx)]$$

Let

$$A(x) = (f_1 + g_1 i)[\cos(bx) + i\sin(bx)] \quad , \quad B(x) = (f_2 + g_2 i)[\cos(bx) - i\sin(bx)]$$
$$A(x) = f_1\cos(bx) - g_1\sin(bx) + i[f_1\sin(bx) + g_1\cos(bx)]$$
$$B(x) = f_2\cos(bx) + g_2\sin(bx) + i[-f_2\sin(bx) + g_2\cos(bx)]$$
$$y_{co}(x) = A(x) + B(x)$$
$$y_{co}(x) = (f_1 + f_2)\cos(bx) + (g_2 - g_1)\sin(bx) + i[(f_1 - f_2)\sin(bx) + (g_1 + g_2)\cos(bx)]$$

For the complex root $y_{cr}(x)$ to be real, the imaginary component of $y_{cr}(x)$ must be equals to 0. Therefore, the following must hold true,

$$f_1 = f_2 \quad , \quad g_1 = -g_2$$

For as long as the condition above hold true, the two constants c_α and$c_{\alpha+1}$ must be complex conjugates. Considering the case wherein c_α and$c_{\alpha+1}$ as complex conjugates,

$$y_{co}(x) = 2f_1\cos(bx) + 2g_2\sin(bx)$$
$$y_{cr} = 2e^{ax}[f_1\cos(bx) + g_2\sin(bx)]$$

Therefore, the following is true for each complex root and conjugate pair,

$$y_c = \sum_{i=1}^{n-2}[c_i e^{r_i x}] + 2e^{ax}[f_1\cos(bx) + g_2\sin(bx)]$$

1.2.4 Repeated Complex Roots

Suppose the α^{th} root is a complex root with a multiplicity of k. Let its complex conjugate be placed adjacent after said complex root,

$$0 = \prod_{i=1}^{\alpha-1}[D - r_i] \prod_{i=\alpha+2k}^{n}[D - r_i]\,[D - r_\alpha]^k\,[D - \bar{r_\alpha}]^k\,y$$

Let y_{crr} be considered as the solution to the system $0 = [D - r_\alpha]^k\,[D - \bar{r_\alpha}]^k\,y_{crr}$. Based on superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2k}[c_i e^{r_i x}] + y_{crr}$$

Based on the previous work on repeated roots with multiplicity greater than 1,

$$y_{crr}(x) = \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right] C_1 e^{r_\alpha x} + \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right] C_2 e^{\bar{r_\alpha} x}$$

Let $r_\alpha = a + bi$, and $\bar{r_\alpha} = a - bi$,

$$y_{crr}(x) = \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right][C_1 e^{bxi} + C_2 e^{-bxi}]e^{ax}$$

Let $c_1 = f_1 + g_1 i$ and $c_2 = f_2 + g_2 i$. Based on previous work on complex roots,

$$C_1 e^{bxi} + C_2 e^{-bxi} = 2f_1\cos(bx) + 2g_2\sin(bx)$$

Therefore,

$$y_{crr}(x) = \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right][2f_1\cos(bx) + 2g_2\sin(bx)]e^{ax}$$

1.3 General Solutions to Homogenous Differential Equations

Therefore, if an n^{th} order homogeneous differential equation with a real non-repeated roots, b complex root pairs, c real repeated roots with multiplicity γ, and d complex repeated root pairs with multiplicity β

$$\begin{aligned} y_c = &\sum_{l=1}^{a} \left[c_{1,l}e^{r_{1,l}x}\right] \\ &+ \sum_{j=1}^{b}[c_{2,j,1}\cos(b_{1,j}x) + c_{2,j,2}\sin(b_{1,j}x)]e^{a_{1,j}x} \\ &+ \sum_{k=1}^{c}\left[\sum_{m=0}^{\gamma_k-1}\left[c_{3,m,k}x^{\gamma_k-1-m}\right]e^{r_k x}\right] \\ &+ \sum_{i=1}^{d}\left[\sum_{p=0}^{\beta_p-1}\left[c_{4,p,i}x^{\beta_p-1-p}\right][k_{4,i,1}\cos(b_{2,i})x + k_{4,i,2}\sin(b_{2,i})x]e^{r_i x}\right] \end{aligned}$$

The variables a, b, c, d, γ, β, and n are related by the following expression,

$$n = a + 2b + \sum_{i=1}^{c}[\gamma_i] + \sum_{j=1}^{d}[2\beta_j]$$

1.4 Non-Homogenous Differential Equations

Consider the following system:

$$\sum_{i=0}^{n}\left[a_i \overset{n-i}{y}\right] = \sum_{j=0}^{m}\left[c_i f_i(x)\right]$$

wherein $f_i(x)$ represents the i^{th} arbitrary function, and a_i represents the i^{th} arbitrary constant. The following function could be rewritten in terms of the linear differential operator L:

$$L[y] = \sum_{j=0}^{m}\left[c_i f_i(x)\right]$$

Let y_j represent the general solution to the j^{th} system:

$$L[y_j] = c_i f_j(x)$$

By taking the summations of the various solutions to the various sytems:

$$L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j] = c_0 f_0(x) + c_1 f_1(x) + \cdots + c_{j-1}f_{j-1}(x) + c_j f_j(x)$$

Since the differential operator L is linear, as shown in proposition 3:

$$L\left[\sum_{j=0}^{m}(y_j)\right] = L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j]$$

$$L\left[\sum_{j=0}^{m}(y_j)\right] = \sum_{j=0}^{m}\left[c_i f_i(x)\right]$$

Therefore, a solution to the non-homogenous differential equation:

$$y_p(x) = \sum_{j=0}^{m}(y_j)$$

An m^{th} dimensional subspace spanned by m functions must always contain a null element, in this case, a zero function. Let the y_c represent the general solution to the homogenous differential equation $Ly = null = 0$. Then the general solution must follow:

$$y_g(x) = \sum_{j=0}^{m}(y_j) + y_c$$

Chapter 2

Dynamical Systems: Eigenvalues and Eigenvectors

Let A represent a $n \times n$ matrix (a matrix with n rows and n columns), x represent a column vector of n variables and x' represent the derivative of the column vector x. The system below is known as a dynamical system:

$$x' = Ax$$

Consider the dynamical system $x' = kx$ wherein k is some arbitrary constant. Therefore,

$$\frac{dx}{dt} = kx$$

$$dt = \frac{1}{kx}dx$$

$$\int dt = \int \frac{1}{kx}dx$$

$$t = \frac{1}{k}\ln x + C$$

$$\ln x = kt + C$$

$$x = Ce^{kt}$$

Wherein C is a constant determined by the initial conditions.

2.1 Non-Repeated Real Eigenvalues of n × n Case

The previous working gives the conjecture that the general solution set $x(t)$ to the dynamical system $x' = Ax$ is the linear combination of exponential functions analogous to the example shown above. Consider the possibility that one solution to the dynamical system takes the form below:

$$x(t) = \bar{v}_i e^{\lambda_i t}$$

wherein $\bar{v}_i$ represents a vector and λ_i represents a constant. By taking derivative of the solution,

$$x'(t) = \lambda_i \bar{v}_i e^{\lambda_i t}$$

$$Ax(t) = A\bar{v}_i e^{\lambda_i t}$$

By considering that $x(t)$ represents a solution to the dynamical system, $x' = Ax$

$$\lambda_i \bar{v_i} e^{\lambda_i t} = A\bar{v_i} e^{\lambda_i t}$$

Since $e^{\lambda_i t} \neq 0$ for all values of t,

$$A\bar{v_i} = \lambda_i \bar{v_i}$$

This is a familiar equation for eigenvalues and eigenvectors. This shows that each eigenvalue-eigenvector pairs of the matrix A represents a solution set. Therefore, the general solution set is:

$$x(t) = span[\bar{v_1} e^{\alpha_1 t}, \bar{v_2} e^{\alpha_2 t}, \ldots \bar{v_n} e^{\alpha_n t}]$$

$$x(t) = \sum_{i=1}^{n} \left[c_i \bar{v_i} e^{\lambda_i t} \right]$$

wherein c_i are constants determined by the initial value of the problem.

2.2 Non-Repeated Complex Eigenvalues of 2 × 2 Case

Consider the special case wherein the matrix A is a 2×2 matrix and that the eigenvalues are complex, by conjecture,

$$x(t) = c_1 \bar{v_1} e^{\lambda_1 t} + c_2 \bar{v_2} e^{\lambda_2 t} = k_1 Re[\bar{v_1} e^{\lambda_1 t}] + k_2 Im[\bar{v_1} e^{\lambda_1 t}]$$

wherein c_1 and c_2 are complex values meanwhile k_1 and k_2 are real values. There must always be some choice of complex values c_1 and c_2 such that the expression above is true. The proof is shown below,

Let

$$\bar{v_1} = \bar{v_r} + i\bar{v_i} \qquad \lambda_1 = a + bi$$

$$x(t) = (\bar{v_r} + i\bar{v_i}) e^{(a+bi)t}$$

$$x(t) = e^{at}(\bar{v_r} + i\bar{v_i})[\cos(bt) + i\sin(bt)]$$

$$x(t) = e^{at}[\bar{v_r}\cos(bt) + i\bar{v_r}\sin(bt) + i\bar{v_i}\cos(bt) - \bar{v_i}\sin(bt)]$$

$$x(t) = e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)] + ie^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

$$Re[\bar{v_1} e^{\lambda_1 t}] = e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)]$$

$$Im[\bar{v_1} e^{\lambda_1 t}] = e^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

$$LHS = k_1 Re[\bar{v_1} e^{\lambda_1 t}] + k_2 Im[\bar{v_1} e^{\lambda_1 t}]$$

$$LHS = k_1 e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)] + k_2 e^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

$$LHS = e^{at}[k_1\bar{v_r}\cos(bt) - k_1\bar{v_i}\sin(bt) + k_2\bar{v_r}\sin(bt) + k_2\bar{v_i}\cos(bt)]$$

$$LHS = e^{at}\{[k_1\bar{v_r} + k_2\bar{v_i}]\cos(bt) + [k_2\bar{v_r} - k_1\bar{v_i}]\sin(bt)\}$$

$$LHS = e^{at}[k_1\bar{v_r} + k_2\bar{v_i}]\cos(bt) + e^{at}[k_2\bar{v_r} - k_1\bar{v_i}]\sin(bt)$$

It is important to note that eigenvalues and their corresponding eigenvectors occur in conjugate pairs. Therefore, if $\lambda_1 = a + bi$, then $\lambda_2 = \lambda_1^* = a - bi$ and if the eigenvector $\bar{v_1} = \bar{v_r} + i\bar{v_i}$, then $\bar{v_2} = \bar{v_1}^* = \bar{v_r} - i\bar{v_i}$.

Let

$$c_1 = f_1 + g_1 i \qquad c_2 = f_2 + g_2 i$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = (f_1 + g_1 i)(\bar{v}_r + i\bar{v}_i)e^{(a+bi)t} + (f_2 + g_2 i)(\bar{v}_r - i\bar{v}_i)e^{(a-bi)t}$$

For ease of notation,

$$A(t) = (f_1 + g_1 i)(\bar{v}_r + i\bar{v}_i)e^{(a+bi)t} \qquad B(t) = (f_2 + g_2 i)(\bar{v}_r - i\bar{v}_i)e^{(a-bi)t}$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = A(t) + B(t)$$

$$A(t) = e^{at}(f_1 + g_1 i)(\bar{v}_r + i\bar{v}_i)\left[\cos(bt) + i\sin(bt)\right]$$

$$A(t) = e^{at}(f_1\bar{v}_r + if_1\bar{v}_i + ig_1\bar{v}_r - g_1\bar{v}_i)\left[\cos(bt) + i\sin(bt)\right]$$

$$A(t) = e^{at}[f_1\bar{v}_r - g_1\bar{v}_i + i(f_1\bar{v}_i + g_1\bar{v}_r)]\left[\cos(bt) + i\sin(bt)\right]$$

$$A(t) = e^{at}[(f_1\bar{v}_r - g_1\bar{v}_i)\cos(bt) + i(f_1\bar{v}_i + g_1\bar{v}_r)\cos(bt) + i(f_1\bar{v}_r - g_1\bar{v}_i)\sin(bt) - (f_1\bar{v}_i + g_1\bar{v}_r)\sin(bt)]$$

$$B(t) = (f_2 + g_2 i)(\bar{v}_r - i\bar{v}_i)e^{(a-bi)t}$$

$$B(t) = e^{at}(f_2\bar{v}_r - if_2\bar{v}_i + ig_2\bar{v}_r + g_2\bar{v}_i)\left[\cos(-bt) + i\sin(-bt)\right]$$

$$B(t) = e^{at}[(f_2\bar{v}_r + g_2\bar{v}_i) + i(g_2\bar{v}_r - f_2\bar{v}_i)]\left[\cos(bt) - i\sin(bt)\right]$$

$$B(t) = e^{at}[(f_2\bar{v}_r + g_2\bar{v}_i)\cos(bt) + i(g_2\bar{v}_r - f_2\bar{v}_i)\cos(bt) + i(-f_2\bar{v}_r - g_2\bar{v}_i)\sin(bt) + (g_2\bar{v}_r - f_2\bar{v}_i)\sin(bt)]$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = Re[A(t)] + Re[B(t)] + i\{Im[A(t)] + Im[B(t)]\}$$

$$0 = Im[A(t)] + Im[B(t)]$$

$$0 = (f_1\bar{v}_i + g_1\bar{v}_r)\cos(bt) + (f_1\bar{v}_r - g_1\bar{v}_i)\sin(bt) + (g_2\bar{v}_r - f_2\bar{v}_i)\cos(bt) - (f_2\bar{v}_r + g_2\bar{v}_i)\sin(bt)$$

$$0 = (f_1\bar{v}_i + g_1\bar{v}_r + g_2\bar{v}_r - f_2\bar{v}_i)\cos(bt) + (f_1\bar{v}_r - g_1\bar{v}_i - f_2\bar{v}_r - g_2\bar{v}_i)\sin bt$$

$$0 = [(g_1 + g_2)\bar{v}_r + (f_1 - f_2)\bar{v}_i]\cos(bt) + [(f_1 - f_2)\bar{v}_r - (g_1 + g_2)\bar{v}_i]\sin(bt)$$

For as long as the condition below is met, the imaginary component of $A(t) + B(t)$ is negligible.

$$g_1 = -g_2 \qquad f_1 = f_2$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = Re[A(t)] + Re[B(t)]$$

$$\begin{aligned} c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} =& e^{at}(f_1\bar{v}_r - g_1\bar{v}_i)\cos(bt) - (f_1\bar{v}_i + g_1\bar{v}_r)\sin(bt) \\ &+ (f_2\bar{v}_r + g_2\bar{v}_i)\cos(bt) + (g_2\bar{v}_r - f_2\bar{v}_i)\sin(bt) \end{aligned}$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = e^{at}(f_1\bar{v}_r - g_1\bar{v}_i + f_2\bar{v}_r + g_2\bar{v}_i)\cos(bt) + (g_2\bar{v}_r - f_2\bar{v}_i - f_1\bar{v}_i - g_1\bar{v}_r)\sin(bt)$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = e^{at}[(f_1 + f_2)\bar{v}_r + (g_2 - g_1)\bar{v}_i]\cos(bt) + [(g_2 - g_1)\bar{v}_r - (f_1 + f_2)\bar{v}_i]\sin(bt)$$

$$RHS = c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t}$$

$$RHS = e^{at}[(f_1 + f_2)\bar{v}_r + (g_2 - g_1)\bar{v}_i]\cos(bt) + e^{at}[(g_2 - g_1)\bar{v}_r - (f_1 + f_2)\bar{v}_i]\sin(bt)$$

$$LHS = e^{at}[k_1\bar{v}_r + k_2\bar{v}_i]\cos(bt) + e^{at}[k_2\bar{v}_r - k_1\bar{v}_i]\sin(bt)$$

If the conditions below are met, therefore $LHS = RHS$ and the statement $x(t) = c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = k_1 Re[\bar{v}_1 e^{\lambda_1 t}] + k_2 Im[\bar{v}_1 e^{\lambda_1 t}]$ is true.

$$g_1 + g_2 = 0 \qquad f_1 + f_2 - k_1 = 0 \qquad f_1 - f_2 = 0 \qquad g_2 - g_1 - k_2 = 0$$

The corresponding augmented matrix of the following conditions is

$$\begin{array}{ccccccc} f_1 & f_2 & g_1 & g_2 & k_1 & k_2 & C \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The row-reduced echelon form of the corresponding augmented matrix is

$$\begin{array}{ccccccc} f_1 & f_2 & g_1 & g_2 & k_1 & k_2 & C \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

The row-reduced echelon form is unique and is consistent, therefore the system has a consistent solution. This proves that for some special choice of c_1 and c_2, the expression below is correct.

$$x(t) = c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = k_1 Re[\bar{v_1}e^{\lambda_1 t}] + k_2 Im[\bar{v_1}e^{\lambda_1 t}]$$

A restatement of the general real solution set is:

$$x(t) = k_1 e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)] + k_2 e^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

The solution set for all real numbers could be better expressed as a matrix multiplication

$$x(t) = e^{at}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} k_2 \\ k_1 \end{pmatrix}$$

The real and imaginary components of the eigenvector v_1 form a linearly independent set. Therfore, the matrix $\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}$ must be invertible. Through the invertible matrix theorem, the matrix $\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}$ must have a suitable inverse.

$$x(t) = e^{at}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$x(t) = e^{at}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x_0$$

$$\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x(t) = e^{at}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x_0$$

By considering the substitution $y = \begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x(t)$ and $y_0 = \begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x_0$,

$$y = e^{at}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix} y_0$$

wherein e^{at} represents a scaling transformation and $\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}$ represents a rotation. Therefore, for a suitable substitution, the general real solution set of the dynamical system $x' = Ax$ will form a rotation with a scaling component. The rotation is sometimes known as the "hidden rotation". Some possibilities of the solution set may be ellipses, circles, and spirals.

2.3 Non-Repeated Complex Eigenvalues of 3 $\times$ 3 Case

Consider the case wherein $n = 3$

$$x(t) = \sum_{i=1}^{3} \left[c_i \bar{v_i} e^{\lambda_i t} \right]$$

$$x(t) = c_1 \bar{v_1} e^{\lambda_1 t} + c_2 \bar{v_2} e^{\lambda_2 t} + c_3 \bar{v_3} e^{\lambda_3 t}$$

Complex eigenvalues occure in conjugate pairs. When A is a 3×3 matrix, 2 of the eigenvalues will be complex conjugate pairs and the third one will be a real value. Therefore, two of the eigenvectors must be complex vectors with the third eigenvector being a real vector. Therefore, through the similar argument and proof written above,

$$x(t) = k_1 Re \left[\bar{v_1} e^{\lambda_1 t} \right] + k_2 Re \left[\bar{v_1} e^{\lambda_1 t} \right] + k_3 \bar{v_3} e^{\lambda_3 t}$$

$$x(t) = k_1 e^{at} [\bar{v_r} \cos{(bt)} - \bar{v_i} \sin{(bt)}] + k_2 e^{at} [\bar{v_r} \sin{(bt)} + \bar{v_i} \cos{(bt)}] + k_3 \bar{v_3} e^{\lambda_3 t}$$

The following solution set could be factorised as matrix multiplications

$$x(t) = e^{at} \begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix} \begin{pmatrix} \cos{(bt)} & -\sin{(bt)} & 0 \\ \sin{(bt)} & \cos{(bt)} & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} \begin{pmatrix} k_2 \\ k_1 \\ k_3 \end{pmatrix}$$

The vectors $\bar{v_i}, \bar{v_r}, \bar{v_3}$ form a linearly independent set, therefore, the matrix $\begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix}$ is invertible and its inverse must exist.

$$\text{Let } y_0 = \begin{pmatrix} k_2 \\ k_1 \\ k_3 \end{pmatrix}$$

$$\begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix}^{-1} x(t) = e^{at} \begin{pmatrix} \cos{(bt)} & -\sin{(bt)} & 0 \\ \sin{(bt)} & \cos{(bt)} & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} y_0$$

$$\text{Let } y(t) = \begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix}^{-1} x(t)$$

$$y(t) = e^{at} \begin{pmatrix} \cos{(bt)} & -\sin{(bt)} & 0 \\ \sin{(bt)} & \cos{(bt)} & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} y_0$$

y_0 is dependent on the system's initial conditions. This shows that for some suitable substitution, the general solution set forms a helix. The geometrical implication of the solution set is a spiral around the z-axis while it is moving away from the xy plane. The substitution back into the conventional axis x_1, x_2, x_3 could be considered as a transformation that "distorts" the helix.

2.4 Repeated Eigenvalues

Given the matrix A in the system $x' = Ax$ is a matrix with repeated eigenvalues with multiplicity k, a reasonable conjecture is the solution to the system is similar in form to the repeated roots case in the linear differential equation. By conjecture,

$$x(t) = \sum_{i=0}^{k-1} \left[\bar{v_i} t^{k-1-i} e^{\lambda t} \right]$$

$$x'(t) = \sum_{i=0}^{k-1} \left[\bar{v_i} \frac{d}{dt} \left[t^{k-1-i} e^{\lambda t} \right] \right]$$

$$\frac{d}{dt} \left[t^{k-1-i} e^{\lambda t} \right] = (k-1-i) t^{k-2-i} e^{\lambda t} + \lambda t^{k-1-i} e^{\lambda t}$$

$$x'(t) = \sum_{i=0}^{k-1} \left[(k-1-i) t^{k-2-i} \bar{v_i} e^{\lambda t} + \lambda t^{k-1-i} \bar{v_i} e^{\lambda t} \right]$$

Remembering $x'(t) = Ax(t)$,

$$\sum_{i=0}^{k-1} \left[A\bar{v_i} t^{k-1-i} e^{\lambda t} \right] = \sum_{i=0}^{k-1} \left[(k-1-i) t^{k-2-i} \bar{v_i} e^{\lambda t} + \lambda t^{k-1-i} \bar{v_i} e^{\lambda t} \right]$$

Considering that $e^{\lambda t} \neq 0$, therefore,

$$\sum_{i=0}^{k-1} \left[A\bar{v_i} t^{k-1-i} \right] = \sum_{i=0}^{k-1} \left[\lambda t^{k-1-i} \bar{v_i} + (k-1-i) t^{k-2-i} \bar{v_i} \right]$$

For the 0^{th} element,

$$A\bar{v_0} t^{k-1} = \lambda t^{k-1} \bar{v_0}$$

Considering that $t^{k-1} \neq 0$ for as long as $t \neq 0$,

$$A\bar{v_0} = \lambda \bar{v_0}$$

For the α^{th} element,

$$A\bar{v_\alpha} t^{k-1-\alpha} = \lambda t^{k-1-\alpha} \bar{v_\alpha} + \left[k-1-(\alpha-1) \right] t^{k-2-(\alpha-1)} \bar{v_{\alpha-1}}$$

$$A\bar{v_\alpha} t^{k-1-\alpha} = \lambda t^{k-1-\alpha} \bar{v_\alpha} + [k-\alpha] \, t^{k-1-\alpha} \bar{v_{\alpha-1}}$$

For as long as $t \neq 0$, $t^{k-1-\alpha} \neq 0$. Therefore,

$$A\bar{v_\alpha} = \lambda \bar{v_\alpha} + [k-\alpha] \, \bar{v_{\alpha-1}}$$

$$\frac{1}{[k-\alpha]} (A - \lambda I) \bar{v_\alpha} = \bar{v_{\alpha-1}}$$

By applying definition recursively,

$$\frac{1}{\prod_{i=0}^{j-1} [k-i]} (A - \lambda I)^j \bar{v_\alpha} = \bar{v_{\alpha-j}}$$

For when $j = \alpha$,

$$\frac{1}{\prod_{i=0}^{\alpha-1} [k-i]} (A - \lambda I)^\alpha \bar{v_\alpha} = \bar{v_0}$$

2.5 Simple First Order Non-Homogenous System

Suppose, for a non-homogeneous dynamical system, $x' = Ax + k$. The non-homogenous dynamical system could be reduced to a homogenous dynamical system, $y' = Ay$ by an appropriate substitution shown below:

$$y_1 = x_1 + c_1 \qquad y_2 = x_2 + c_2 \qquad \dots \qquad y_n = x_n + c_n$$

wherein $c_1, c_2, c_3 \dots c_n$ are constants

$$y_1' = x_1' \qquad y_2' = x_2' \qquad \dots \qquad y_n' = x_n'$$

Let the columns of matrix A be denoted as $a_1, a_2, a_3, \dots a_n$

$$A = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix}$$

$$Ay = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Ay = \sum_{i=1}^{n} [\bar{a_i} y_i]$$

$$Ay = \sum_{i=1}^{n} \left[\bar{a_i}(x_i + c_i)\right]$$

$$Ay = \sum_{i=1}^{n} [\bar{a_i} x_i + \bar{a_i} c_i]$$

$$Ay = \sum_{i=1}^{n} [\bar{a_i} x_i] + \sum_{i=1}^{n} [\bar{a_i} c_i]$$

$$Ax + k = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$Ax + k = \sum_{i=1}^{n} [\bar{a_i} x_i] + k$$

$$Ay = Ax + k$$

$$\sum_{i=1}^{n} [\bar{a_i} x_i] + \sum_{i=1}^{n} [\bar{a_i} c_i] = \sum_{i=1}^{n} [\bar{a_i} x_i] + k$$

$$\sum_{i=1}^{n} [\bar{a_i} c_i] = k$$

The system above is equivalent to an augmented matrix whose first column until nth column is the columns of the matrix A and its last column is the column vector k. Therefore, the augmented matrix is written below:

$$c_1 \quad c_2 \quad \dots \quad c_n \quad K$$

$$\begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} & k \end{bmatrix}$$

The solution to the augmented matrix will be the values for the constants $c_1, c_2, \dots, c_n$ that would be used in the substitution process in transforming the non-homogenous dynamical system into a homogenous dynamical system. The augmented matrix above would only have a solution for all k in $\mathbb{R}^n$ if the matrix A is invertible. If the matrix A is non-invertible , then k must be in $col[A]$, otherwise, then the augmented system forms an inconsistent system. In otherwords, a substitution with the above methods may not exist for an aribtrary choice of $n \times n$ matrix A and arbitrary column vector k.

2.6 Simple Higher Order System

Suppose the dynamical system follows the expression $\overset{m}{x} = Ax$, a similar technique with eigenvalues and eigenvectors may be employed along with the roots of unity. By conjecture, the partial solution to the dynamical system $\overset{m}{x} = Ax$ follows

$$x_p = \bar{v}_i e^{\alpha_i t}$$

$$\dot{x_p} = \alpha_i \bar{v}_i e^{\alpha_i t}$$
$$\ddot{x_p} = \alpha_i^2 \bar{v}_i e^{\alpha_i t}$$
$$\overset{m}{x_p} = \alpha_i^m \bar{v}_i e^{\alpha_i t}$$
$$Ax_p = \overset{m}{x_p}$$
$$A\bar{v}_i e^{\alpha_i t} = \alpha_i^m \bar{v}_i e^{\alpha_i t}$$
$$A\bar{v}_i = \alpha_i^m \bar{v}_i$$

Since $A\bar{v}_i = \alpha_i^m \bar{v}_i$ wherein λ_i are eigenvalues of A, then $\lambda_i = \alpha_i^m$. Since λ_i may be a complex number, α_i must be the roots of unity to the complex number λ_i. If $\lambda_i = a + bi$

$$\alpha_n = (a^2 + b^2)^{\frac{1}{2m}} cis\left[\frac{1}{m} arctan\left(\frac{b}{a}\right) + \frac{2\pi n}{m}\right]$$

The general solution to the problem must be the linear combination of the partial solutions $\sum_{i=1}^{m} \left[c_i \bar{v}_i e^{\alpha_{in} t}\right]$ wherein c_i are constants determined by the initial conditions and α_{in} represents the n^{th} root of unity of the i^{th} eigenvalue albeit complex or real.

2.7 Simple n^{th} Order Homogenous System

Suppose the differential equation follows the expression:

$$0 = \sum_{i=0}^{m} [A_i \overset{i}{x}] = A_0 x + A_1 \dot{x} + A_2 \ddot{x} + \dots + A_{i-1} \overset{i-1}{x} + A_i \overset{i}{x}$$

The general solution to the system above is a linear combination of the partial solutions, $x(t) = \sum_{j=1}^{n} [c_j \bar{v}_j e^{\lambda_j t}]$ wherein partial solutions are defined as $x_{partial}(t) = c_j \bar{v}_j e^{\lambda_j t}$ and $c_1, c_2 \dots c_n$ are constants determined by the initial value of the problem.

$$x_p(t) = c_j \bar{v}_j e^{\lambda_j t}$$

$$\overset{k}{x_p}(t) = c_j \bar{v}_j \lambda_j^k e^{\lambda_j t}$$

$$0 = \sum_{i=0}^{m} [A_i \bar{v}_j c_j \lambda_j^i e^{\lambda_j t}] = A_0 \bar{v}_j c_j e^{\lambda_j t} + A_1 \bar{v}_j c_j \lambda_j e^{\lambda_j t} + \cdots + A_m \bar{v}_j c_j \lambda_j^m e^{\lambda_j t}$$

For the non-trivial solutions to the homoegenous system of differential equations, $c_j, \bar{v}_j, \lambda_j \neq 0$. The function $e^{\lambda_j t} \neq 0$ for all time. Therefore,

$$0 = \left\{ \sum_{i=0}^{m} [A_i \lambda_j^i] \right\} \bar{v}_j$$

For $\bar{v}_j \neq 0$, the matrix $\sum_{i=0}^{m} [A_i \lambda_j^i]$ must be non-invertible. Therefore, $det \left\{ \sum_{i=0}^{m} [A_i \lambda_j^i] \right\} = 0$

The expressions for λ_j^i could be substituted to the expression $A_i \lambda_j^i \bar{v}_j = 0$ to express vector $\bar{v}_j$ explicitly.

Chapter 3

Fourier Series

3.1 Definition of Inner Product

Let f and h be complex valued vectors,

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad , \quad h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

The inner product is formally defined as:

$$(f, h) = \sum_{k=1}^{n} \left[f_k \bar{h_k} \right]$$

wherein $\bar{h_k}$ represents the complex conjugate of the k^{th} element of the vector h. There are some properties of the inner product:

$$(f, h) = \overline{(h, f)} \quad , \quad (\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h) \quad , \quad (f, f) \geq 0$$

$(f, f) = 0$ if and only if $f_k = 0$ for all k of the vector elements. The magnitude of n^{th} dimensional vectors:

$$\|f\| = \left(\sum_{k=1}^{n} |f_k|^2 \right)^{\frac{1}{2}}$$

3.2 Definition of Lebesgue Space

A function would be in Lebesgue space if

$$\int_0^{\tau} |f(t)|^2 dt < \infty$$

The inner product of a function on the Lebesgue space $L^2(0, \tau)$:

$$(f, g) = \frac{1}{\tau} \int_0^{\tau} f(t)\overline{g(t)} dt$$

The norm of a function in Lebesgue space:

$$\|f\| = \left[\frac{1}{\tau}\int_0^{\tau} |f(t)|^2 dt\right]^{\frac{1}{2}}$$

Therefore, it follows that

$$\|f\|^2 = (f, f) = \frac{1}{\tau}\int_0^{\tau} |f(t)|^2 dt$$

Distance between two functions defined in Lebesgue space:

$$\|f - g\| = \left[\frac{1}{\tau}\int_0^{\tau} |f(t) - g(t)|^2 dt\right]^{\frac{1}{2}}$$

3.3 Exponential Fourier Series

Fourier series in exponential form:

$$f(t) = \sum_{k=-\infty}^{\infty} \left[a_k e^{-ik\omega_0 t}\right]$$

wherein k represent integers and the coefficients a_k could be found by,

$$a_k = \frac{1}{\tau}\int_0^{\tau} e^{ik\omega_0 t} f(t) dt$$

wherein $\omega_0 = 2\pi/\tau$. An orthonormal set in Lebesgue space is defined as a collection of functions that are orthonormal to each other in Lebesgue space and have a magnitude of one. The proof below shows that the complex exponential $e^{ik\omega_0 t}$ forms an orthonormal set. If two functions are orthogonal, then their inner products in Lebesgue space must be zero.

$$(f, g) = \frac{1}{\tau}\int_0^{\tau} f(t)\overline{g(t)} dt$$

Substituting for the complex exponential functions, $f(t) = e^{-ik_1\omega_0 t} \quad , \quad g(t) = e^{-ik_2\omega_0 t}$,

$$(f, g) = \frac{1}{\tau}\int_0^{\tau} e^{-ik_1\omega_0 t} e^{ik_2\omega_0 t} dt = \frac{1}{\tau}\int_0^{\tau} e^{i(k_2-k_1)\omega_0 t} dt$$

For the case wherein $k_2 = k_1$,

$$(f, g) = \frac{1}{\tau}\int_0^{\tau} 1 dt = \frac{1}{\tau}[t]_0^{\tau} = \frac{1}{\tau}(\tau) = 1$$

Using the previous definition of function magnitudes in Lebesgue space, $\|f\|^2 = (f, f) \quad , \quad \|f\| = \sqrt{(f, f)}$. Therefore, $\|f\| = 1$ which shows that the complex exponential function has a magnitude of 1 in Lebesgue space. For the case wherein $k_2 \neq k_1$, the subtraction of the two integers yields another non-zero integer.

$$(f, g) = \frac{1}{\tau}\int_0^{\tau} \cos\left[(k_2 - k_1)\omega_0 t\right] + i \sin\left[(k_2 - k_1)\omega_0 t\right] dt$$

$$(f, g) = \frac{1}{\tau(k_2 - k_1)\omega_0}\left\{\sin\left[(k_2 - k_1)\omega_0 t\right] - i \cos\left[(k_2 - k_1)\omega_0 t\right]\right\}_{t=0}^{t=\tau}$$

$$\text{Substituting for } \omega_0 = \frac{2\pi}{\tau}$$

$$\left\{ \sin\left[\frac{2\pi(k_2-k_1)t}{\tau}\right] \right\}_{t=0}^{t=\tau} = \sin\left[2\pi(k_2-k_1)\right] - \sin\left[0\right] = 0$$

$$\left\{ \cos\left[\frac{2\pi(k_2-k_1)t}{\tau}\right] \right\}_{t=0}^{t=\tau} = \cos\left[2\pi(k_2-k_1)\right] - \cos\left[0\right] = 1 - 1 = 0$$

Therefore, for the case wherein $k_2 \neq k_1$, $(f,g) = 0$. This shows that complex exponentials are form an orthogonal set in Lebesgue space. Since $e^{ik\omega_0 t}$ forms an orthonormal set, $e^{ik\omega_0 t}$ could be used as a basis to represent any function that is in Lebesgue space. The proof below shows the method to find the complex coefficients a_k for an arbitrary function $f(t)$ in Lebesgue space.

$$f(t) = \sum_{k=-\infty}^{\infty} \left[a_k e^{-ik\omega_0 t}\right]$$

For some particular integer k_2,

$$\frac{1}{\tau}\int_0^\tau e^{ik_2\omega_0 t} f(t)dt = \frac{1}{\tau}\int_0^\tau e^{ik_2\omega_0 t} \sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0 t}\right]dt = \sum_{k=-\infty}^{\infty}\left[\frac{a_k}{\tau}\int_0^\tau e^{-ik\omega_0 t} e^{ik_2\omega_0 t}dt\right]$$

The above working is true due to the integral operation being a linear operation. Linear operations are discussed earlier in this document. From the previous findings,

$$(f,g) = \frac{1}{\tau}\int_0^\tau e^{-ik_1\omega_0 t} e^{ik_2\omega_0 t}dt = \frac{1}{\tau}\int_0^\tau e^{i(k_2-k_1)\omega_0 t}dt$$

$(f,g) = 1$ only when f and g are identical to each other. For all other cases, $(f,g) = 0$. Following this, $\frac{1}{\tau}\int_0^\tau e^{-ik_1\omega_0 t} e^{ik_2\omega_0 t}dt = 1$ only when $k = k_2$, otherwise, $\frac{1}{\tau}\int_0^\tau e^{-ik_1\omega_0 t} e^{ik_2\omega_0 t}dt = 0$. Therefore,

$$\frac{1}{\tau}\int_0^\tau e^{ik_2\omega_0 t} f(t)dt = \sum_{k=-\infty}^{\infty}\left[a_k \times \frac{1}{\tau}\int_0^\tau e^{-ik\omega_0 t} e^{ik_2\omega_0 t}dt\right] = a_k$$

3.4 Trigonometric Fourier Series

Any arbitrary function $f(t)$ with a periodicity of L could be expressed as a linear combination of sinusoids of varying frequencie. a_n and b_n, but n are integers $n = 1, 2, 3, 4 \ldots.$ The abritrary function $f(t)$ expressed as a linear combination of trigonometric functions:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)\right]$$

The three equations below is correct and serves as a method to find coefficients a_0, a_n and b_n,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

The proof of each of the three equations is showb. Below is written the list of trigonometric identities relating multiplication of trigonometric functions of differing frequencies that will be important for the proof:

$$2\cos(\theta)\cos(\phi) = \cos(\theta-\phi)+\cos(\theta+\phi)$$
$$2\sin(\theta)\sin(\phi) = \cos(\theta-\phi)-\cos(\theta+\phi)$$
$$2\sin(\theta)\cos(\phi) = \sin(\theta+\phi)+\sin(\theta-\phi)$$

For coefficient a_0,

$$\int_{-L}^{L} f(t)dt = \frac{1}{2}\int_{-L}^{L} a_0 dt + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt\right]$$

$$\frac{1}{2}\int_{-L}^{L} a_0 dt = \frac{1}{2}a_0 \times 2L$$
$$\frac{1}{2}\int_{-L}^{L} a_0 dt = a_0 L$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt = \frac{L}{n\pi}\left[\sin\left(\frac{n\pi}{L}t\right)\right]_{t=-L}^{t=L}$$
$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt = \frac{L}{n\pi}\left[\sin(n\pi)-\sin(-n\pi)\right]$$

Considering that n is an integer, $\sin(n\pi) = \sin(-n\pi) = 0$. Therefore,

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt = 0$$

By similar reasoning, it could be seen that $\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right) = 0$, but the integral is evaluated below anyways,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{L}t\right)\right]_{t=-L}^{t=L}$$
$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = -\frac{L}{n\pi}\left[\cos(n\pi)-\cos(-n\pi)\right]$$

By the even property of the cosine function, $\cos(\theta) = \cos(-\theta)$. Therefore,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = -\frac{L}{n\pi}\left[\cos(n\pi)-\cos(n\pi)\right]$$
$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = 0$$

By reiterating the integration of $f(t)$ from $t = -L$ until $t = L$

$$\int_{-L}^{L} f(t)dt = a_0 L + \sum_{n=1}^{\infty} [a_n \times 0 + b_n \times 0]$$

$$\int_{-L}^{L} f(t)dt = a_0 L$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

For coefficient a_n, let m be some particular integer, either 1, 2, or 3. For some of the terms of the Fourier Series, $m = n$. However, for all other terms, $m \neq n$.

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt = \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt\right]$$
$$+\frac{1}{2}a_0\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt$$

$$2\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right) = \cos\left[\frac{(n-m)\pi}{L}t\right] + \cos\left[\frac{(n+m)\pi}{L}t\right]$$
$$\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right) = \frac{1}{2}\cos\left[\frac{(n-m)\pi}{L}t\right] + \frac{1}{2}\cos\left[\frac{(n+m)\pi}{L}t\right]$$
$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = \frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(n-m)\pi}{L}t\right]dt + \frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(n+m)\pi}{L}t\right]dt$$

Consider the case wherein $m = n$,

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{2n\pi}{L}t\right) + 1dt$$

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}\left[\frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}t\right) + t\right]_{t=-L}^{t=L}$$

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}\left[\frac{L}{2n\pi}\big(\sin(2n\pi) - \sin(-2n\pi)\big) + 2L\right]$$

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}[2L] = L$$

Consider the case wherein $m \neq n$,

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = \frac{L}{2(n-m)\pi}\left\{\sin\left[\frac{(n-m)\pi}{L}t\right]\right\}_{t=-L}^{t=L}$$
$$+\frac{L}{2(n+m)\pi}\left\{\sin\left[\frac{(n+m)\pi}{L}t\right]\right\}_{t=-L}^{t=L}$$

By similar argument mentioned previously that sine of a multiple of π yields 0, then

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = 0$$

For the second term in the summation notation,

$$2\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)=\sin\left[\frac{(n+m)\pi}{L}t\right]+\sin\left[\frac{(n-m)\pi}{L}t\right]$$

$$\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)=\frac{1}{2}\sin\left[\frac{(n+m)\pi}{L}t\right]+\frac{1}{2}\sin\left[\frac{(n-m)\pi}{L}t\right]$$

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt=\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(n+m)\pi}{L}t\right]dt+\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(n-m)\pi}{L}t\right]dt$$

For the case wherein $m\neq n$,

$$\begin{aligned}\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt=&-\frac{L}{2(n+m)\pi}\left\{\cos\left[\frac{(n+m)\pi}{L}t\right]\right\}_{t=-L}^{t=L}\\&-\frac{L}{2(n-m)\pi}\left\{\cos\left[\frac{(n-m)\pi}{L}t\right]\right\}_{t=-L}^{t=L}\end{aligned}$$

Since $\cos(x)$ is an even function, the integral evaliuates to 0. Therefore,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt=0$$

For the case wherein $m=n$, the second sine function is irrelevant because $\sin(0)=0$, due to $n-m=0$. The following is just a degenerate case of the case wherein $m\neq n$.

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt=\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(n+m)\pi}{L}t\right]dt$$

Since the integral above is just a degenerate case of $m\neq n$, then the integral just evaluates to 0. Therefore,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right)dt=0$$

For coefficient b_n,

$$\int_{-L}^{L}f(t)dt=\frac{1}{2}\int_{-L}^{L}a_0dt+\sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt+b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt\right]$$

For the final term in the expression describing the integral of $f(t)\cos\left(\frac{n\pi}{L}t\right)$,

$$\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt=\frac{L}{m\pi}\left[\sin\left(\frac{m\pi}{L}t\right)\right]_{t=-L}^{t=L}$$

Since m is an integer,

$$\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt=0$$

A reiteration of the integral of $f(t)\cos\left(\frac{n\pi}{L}t\right)$,

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt = \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt\right] + \frac{1}{2}a_0\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt$$

By substituing all the known parts from the previous workings,

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt = a_m L$$

wherein $m = n$. Therefore,

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt$$

for $n = 1, 2, 3\ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

The proof for the coefficient b_n could be done in a similar way as for the coefficient a_n. These two coefficients are analogous to each other.

3.5 Discrete Fourier Transform

The usage of the Discrete Fourier Transform Matrix is given below.

$$\begin{bmatrix} p(t_0) \\ p(t_1) \\ p(t_2) \\ \vdots \\ p(t_{v-3}) \\ p(t_{v-2}) \\ p(t_{v-1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{-3} & \lambda_1^{-2} & \lambda_1^{-1} \\ 1 & \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \cdots & \lambda_2^{-3} & \lambda_2^{-2} & \lambda_2^{-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{v-3}^1 & \lambda_{v-3}^2 & \lambda_{v-3}^3 & \cdots & \lambda_{v-3}^{-3} & \lambda_{v-3}^{-2} & \lambda_{v-3}^{-1} \\ 1 & \lambda_{v-2}^1 & \lambda_{v-2}^2 & \lambda_{v-2}^3 & \cdots & \lambda_{v-2}^{-3} & \lambda_{v-2}^{-2} & \lambda_{v-2}^{-1} \\ 1 & \lambda_{v-1}^1 & \lambda_{v-1}^2 & \lambda_{v-1}^3 & \cdots & \lambda_{v-1}^{-3} & \lambda_{v-1}^{-2} & \lambda_{v-1}^{-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{-3} \\ a_{-2} \\ a_{-1} \end{bmatrix}$$

3.6 Higher-Dimensional Fourier Series

The Fourier Series in complex exponential form,

$$f(t) = \sum_{k_i=-\infty}^{\infty}\left[a_{k_i}e^{-ik_i\omega_{k_i}t}\right] \quad , \quad \omega_{k_i} = 2\pi/\tau_{k_i}$$

wherein k_i represent integers. Let the two-dimensional Fourier Series be defined as the total expression of a Fourier Series nested in the coefficients of another Fourier Series,

$$f_2(x_1, x_2) = \sum_{k_1=-\infty}^{\infty}\left[\sum_{k_2=-\infty}^{\infty}\left(a_{k_1,k_2}e^{-ik_2\omega_{k_2}x_2}\right)e^{-ik_1\omega_{k_1}x_1}\right] = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\left(a_{k_1,k_2}e^{-ik_2\omega_{k_2}x_2}e^{-ik_1\omega_{k_1}x_1}\right)$$

$$f_2(x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \left[a_{k_1,\, k_2} e^{-i(k_1 \omega_{k_1} x_1 + k_2 \omega_{k_2} x_2)} \right]$$

wherein the natural frequency ω are defined as,

$$\omega_{k_1} = 2\pi/\tau_{k_1} \quad , \quad \omega_{k_2} = 2\pi/\tau_{k_2}$$

wherein τ_{k_1} represent the outer Fourier interval, and τ_{k_2} represent the inner Fourier interval. Therefore, generalizing to n dimensions,

$$f_m(x_1, x_2, \ldots, x_n) = \prod_{m=1}^{n} \left\{ \sum_{k_m=-\infty}^{\infty} \left[a_{k_1,\, k_2, \ldots, k_n} e^{-i\left[\sum_{r=1}^{n} \left(k_r \omega_{k_r} x_r\right)\right]} \right] \right\}$$

wherein $\prod_{m=1}^{n} \left[\sum_{k_m=-\infty}^{\infty} (obj) \right]$ represents n summations of mathematiical objects nested in each other. The expression above does not represent consecutive multiplications of the product notation. The product notation is just used to represent summation notations placed side by side and implemented consecutively in any order.

Chapter 4

Laplace Transform

4.1 Definition of Laplace Transform

Laplace Transform is defined as the following,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

The Laplace Transform is a linear transform since the integral and product operations are both linear operations as well.

$$\mathcal{L}[\alpha f(t)] = \alpha\mathcal{L}[f(t)] \quad , \quad \mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$$

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)]$$

wherein α, β represent constants and $f(t)$, $g(t)$ represent functions of t.

4.2 Transforms of Derivatives

In General form,

$$\mathcal{L}[\overset{n}{f}(t)] = s^n\mathcal{L}[f(t)] - \sum_{k=0}^{n-1}\left[s^{n-1-k}\overset{k}{f}(0)\right]$$

wherein $\overset{n}{f}(t)$ represents the n^{th} derivative of the function $f(t)$. Proof is shown below,

$$\mathcal{L}[\overset{n}{f}(t)] = \int_0^\infty e^{-st}\overset{n}{f}(t)\,dt$$

$$\int uv'\,dt = uv - \int u'v\,dt$$

$$u = e^{-st} \quad , \quad u' = -se^{-st} \quad , \quad v' = \overset{n}{f}(t) \quad , \quad v = \overset{n-1}{f}(t)$$

$$\mathcal{L}[\overset{n}{f}(t)] = \int_0^\infty e^{-st}\overset{n}{f}(t)dt = \left[e^{-st}\overset{n-1}{f}(t)\right]_0^\infty - \int_0^\infty -se^{-st}\overset{n-1}{f}(t)\,dt = -\overset{n-1}{f}(0) + s\int_0^\infty e^{-st}\overset{n-1}{f}(t)\,dt$$

Generalizing for the second integral term,

$$\int_0^\infty e^{-st}\overset{n-i}{f}(t)\,dt = \left[e^{-st}\overset{n-1-i}{f}(t)\right]_0^\infty + s\int_0^\infty e^{-st}\overset{n-1-i}{f}(t)\,dt = -\overset{n-1-i}{f}(0) + s\int_0^\infty e^{-st}\overset{n-1-i}{f}(t)\,dt$$

By applying substitution recursively,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{k}\left[s^{i}\overset{n-1-i}{f}(0)\right] + \prod_{i=0}^{k}[s]\int_0^\infty e^{-st}\overset{n-1-k}{f}(t)\;dt = -\sum_{i=0}^{k}\left[s^{i}\overset{n-1-i}{f}(0)\right] + s^{k+1}\int_0^\infty e^{-st}\overset{n-1-k}{f}(t)\;dt$$

Substituting the value for $k = n-1$,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{n-1}\left[s^{i}\overset{n-1-i}{f}(0)\right] + s^{n}\int_0^\infty e^{-st}f(t)\,dt$$

A few things should be noted,

$$\int_0^\infty e^{-st}f(t)\,dt = \mathcal{L}[f(t)] \quad , \quad \sum_{i=0}^{n-1}\left[s^{i}\overset{n-1-i}{f}(0)\right] = \sum_{i=0}^{n-1}\left[s^{n-1-i}\overset{i}{f}(0)\right]$$

By substitution of the counting variable i with k,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{n-1}\left[s^{i}\overset{n-1-i}{f}(0)\right] + s^{n}\int_0^\infty e^{-st}f(t)\,dt = s^{n}\mathcal{L}[f(t)] + \sum_{k=0}^{n-1}\left[s^{n-1-k}\overset{k}{f}(0)\right]$$

4.3 Transforms of Integrals

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}[f(t)]$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt$$

$$\int uv'\,dt = uv - \int u'v\,dt$$

$$u = \int_0^t f(\tau)d\tau \quad , \quad u' = f(t) \quad , \quad v' = e^{-st} \quad , \quad v = -\frac{1}{s}e^{-st}$$

$$\int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt = -\left[\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\right]_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}f(t)\,dt$$

It should be noted that since the function $f(t)$ is in exponential order,

$$\left[\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\right]_0^\infty = 0$$

Substituting the uv term with zero,

$$\int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt = \int_0^\infty \frac{1}{s}e^{-st}f(t)\,dt = \frac{1}{s}\int_0^\infty e^{-st}f(t)\,dt = \frac{1}{s}\mathcal{L}[f(t)]$$

4.4 Derivative of Transforms

$$\mathcal{L}[t^n f(t)] = (-1)^n \overset{n}{F}(s) = (-1)^n \frac{d^n}{ds^n}\left\{\mathcal{L}[f(t)]\right\}$$

wherein $F(s)$ represents the laplace transform of the function $f(t)$. By the definition of Laplace Transforms discussed earlier,

$$\mathcal{L}[t^n f(t)] = \int_0^\infty e^{-st} t^n f(t)\, dt \quad , \quad F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

Differentiating the Laplace Transform of $f(t)$ with respect to s iteratively n times,

$$\overset{n}{F}(s) = \frac{d^n}{ds^n}\{\mathcal{L}[f(t)]\} = \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\, dt = (-1)^n \int_0^\infty e^{-st} t^n f(t)\, dt$$

$$(-1)^n \overset{n}{F}(s) = \int_0^\infty e^{-st} t^n f(t)\, dt$$

By substituting the definition for the Laplace Transform of $t^n f(t)$,

$$(-1)^n \overset{n}{F}(s) = \mathcal{L}[t^n f(t)]$$

4.5 Integration of Transforms

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tau)\, d\tau$$

wherein $F(s)$ represents the laplace transform of the function $f(t)$.

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_0^\infty \frac{e^{-st} f(t)}{t}\, dt \quad , \quad F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\int_s^\infty F(\tau)\, d\tau = \int_s^\infty \int_0^\infty e^{-\tau t} f(t)\, dt\, d\tau = \int_0^\infty \int_s^\infty e^{-\tau t} f(t)\, d\tau\, dt = \int_0^\infty \left[-\frac{1}{t} e^{-\tau t} f(t)\right]_{\tau=s}^{\tau=\infty} dt$$

$$\int_s^\infty F(\tau)\, d\tau = -\int_0^\infty \frac{f(t)}{t}\left[e^{-\tau t}\right]_{\tau=s}^{\tau=\infty} dt = -\int_0^\infty \frac{f(t)}{t}\left[\lim_{\tau\to\infty}(e^{-\tau t}) - e^{-st}\right] dt$$

Taking into account that, $\lim_{\tau\to\infty}(e^{-\tau t}) = 0$,

$$\int_s^\infty F(\tau)\, d\tau = -\int_0^\infty \frac{f(t)}{t}\left[-e^{-st}\right] dt = \int_0^\infty \frac{e^{-st} f(t)}{t}\, dt$$

By substituting the definition of the laplace transform of $\frac{f(t)}{t}$,

$$\int_s^\infty F(\tau)\, d\tau = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

4.6 Translation of Transforms

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\mathcal{L}\left[f(t+c)\right]$$

By definition of Laplace Transform,

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_0^\infty e^{-st}u(t-c)f(t)\,dt = \int_c^\infty e^{-st}f(t)\,dt + \int_0^c e^{-st}\times 0\,dt$$

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_c^\infty e^{-st}f(t)\,dt$$

Using the substitution $t = \tau + c$. When $t = \infty \quad , \quad \tau = \infty$ and when $t = c \quad , \quad \tau = 0$. Therefore,

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_{t=c}^{t=\infty} e^{-s(\tau+c)}f(\tau+c)\,dt = \int_{\tau=0}^{\tau=\infty} e^{-s(\tau+c)}f(\tau+c)\,d\tau$$

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_{\tau=0}^{\tau=\infty} e^{-s\tau-sc}f(\tau+c)\,d\tau = \int_{\tau=0}^{\tau=\infty} e^{-cs}e^{-s\tau}f(\tau+c)\,d\tau$$

Since variables s and c are not changing with time, the term e^{-cs} could be treated as some form of constant. Therefore,

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\int_0^\infty e^{-s\tau}f(\tau+c)\,d\tau$$

It should be noted that the change of variables allows,

$$\mathcal{L}\left[f(t+c)\right] = \int_0^\infty e^{-st}f(t+c)\,dt = \int_0^\infty e^{-s\tau}f(\tau+c)\,d\tau$$

By substitution,

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\mathcal{L}\left[f(t+c)\right]$$

4.7 Transforms of Translated Functions

$$\mathcal{L}[e^{ct}f(t)] = F(s-c)$$

Reiterating the definition of laplace transforms,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt = F(s)$$

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{ct}e^{-st}f(t)\,dt = \int_0^\infty e^{-st+ct}f(t)\,dt$$

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c)$$

4.8 Convolution

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

The convolution is a commutative transformation. Therefore,

$$f * g(t) = g * f(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

One useful property of the convolution function,

$$\mathcal{L}\left[f * g(t)\right] = \mathcal{L}\left[f(t)\right] \times \mathcal{L}\left[g(t)\right]$$

wherein

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt \quad , \quad G(s) = \mathcal{L}[g(t)] = \int_0^\infty e^{-st}g(t)\,dt$$

By a substitution of variables $t = u$ it could be re-written,

$$F(s) = \mathcal{L}[f(u)] = \int_0^\infty e^{-su}f(u)\,du \quad , \quad G(s) = \mathcal{L}[g(u)] = \int_0^\infty e^{-su}g(u)\,du$$

Examining the Lapalce Transform of $g(u)$, and making the substitution $u = t - \tau$

$$\mathcal{L}[g(t-\tau)] = \int_{u=0}^{u=\infty} e^{-s(t-\tau)}g(t-\tau)\,dt$$

When $u = \infty \quad , \quad t = \infty$ and when $u = 0 \quad , \quad t = \tau$. Therefore,

$$\mathcal{L}[g(t-\tau)] = \int_{t=\tau}^{t=\infty} e^{-s(t-\tau)}g(t-\tau)\,dt$$

The $e^{\tau s}$ term could be isolated because both variables τ and s in this case are non-changing with t. The next form is identical to the laplace transform at the Translation of Transforms section,

$$\int_{\tau=0}^{\tau=\infty} e^{-s(\tau+c)}f(\tau+c)\,d\tau = e^{-cs}\int_0^\infty e^{-s\tau}f(\tau+c)\,d\tau$$

By substituting τ in the Translation of Transforms section with t, substituting c with $-\tau$, and substituting the arbitrary function g with the arbitrary function f,

$$\int_{t=0}^{t=\infty} e^{-s(t-\tau)}g(t-\tau)\,dt = e^{\tau s}\int_0^\infty e^{-st}g(t-\tau)\,dt$$

Therefore,

$$\mathcal{L}[g(t-\tau)] = G(s) = e^{\tau s}\int_0^\infty e^{-st}g(t-\tau)\,dt$$

Proving the Convolution Property by first examining the product of the two Laplace Transforms,

$$F(s) \times G(s) = G(s)\int_0^\infty e^{-su}f(u)\,du = \int_0^\infty e^{-su}G(s)f(u)\,du$$

The above would be perfectly legal operations because $G(s)$ is a function in terms of s and is unchanging with respect to variable t. Therefore, the function $G(s)$ could be treated as a constant that can be place inside and outside of the integral.

$$F(s) \times G(s) = \int_0^\infty e^{-s\tau} f(\tau) \times e^{\tau s} \int_0^\infty e^{-st} g(t-\tau)\, dt\, d\tau$$

$$F(s) \times G(s) = \int_0^\infty \int_0^\infty e^{-st} f(\tau) g(t-\tau)\, dt\, d\tau$$

By chaging the order of integration,

$$F(s) \times G(s) = \int_0^\infty e^{-st} \int_0^\infty f(\tau) g(t-\tau)\, d\tau\, dt = \mathcal{L}\left[\int_0^\infty f(\tau) g(t-\tau)\, d\tau\right]$$

$$F(s) \times G(s) = \mathcal{L}\left[f * g(t)\right]$$

Chapter 5

Gradient Operators

Given the the arbitrary fucntion f,

5.1 Cartesian Coordinates

5.2 Cylindrical Coordinates

5.3 Spherical Coordinates

Chapter 6

Partial Differential Equations

The conventional gradient operator in cartesian coordinates is typically defined as,

$$\nabla_{xyz} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T \quad , \quad \nabla^n_{xyz} = \begin{pmatrix} \frac{\partial^n}{\partial x^n} & \frac{\partial^n}{\partial y^n} & \frac{\partial^n}{\partial z^n} \end{pmatrix}^T$$

Let the modified gradient operator (${}_m\nabla$) in cartesian coordinates be defined as,

$${}_m\nabla_{xyz} = \begin{pmatrix} \alpha_1\frac{\partial}{\partial x} & \alpha_2\frac{\partial}{\partial y} & \alpha_3\frac{\partial}{\partial z} \end{pmatrix}^T \quad , \quad {}_m\nabla^n_{xyz} = \begin{pmatrix} \beta_1\frac{\partial^n}{\partial x^n} & \beta_2\frac{\partial^n}{\partial y^n} & \beta_3\frac{\partial^n}{\partial z^n} \end{pmatrix}^T$$

This modification allows the gradient operator to be more general. The modified gradient operator for m dimensional cartesian coordinates,

$${}_m\nabla_{xyz} = \begin{pmatrix} \alpha_1\frac{\partial}{\partial x_1} & \alpha_2\frac{\partial}{\partial x_2} & \dots & \alpha_n\frac{\partial}{\partial x_m} \end{pmatrix}^T \quad , \quad {}_m\nabla^n_{xyz} = \begin{pmatrix} \beta_1\frac{\partial^n}{\partial x_1^n} & \beta_2\frac{\partial^n}{\partial x_2^n} & \dots & \beta_3\frac{\partial^n}{\partial x_m^n} \end{pmatrix}^T$$

6.1 Methods in Generalized Cartesian Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{x_1\dots x_q}(u) + {}_{m_2}\nabla_{x_1\dots x_q}(u) = \sum_{i=1}^{q}\left[b_i\frac{\partial^2 u}{\partial x_i^2} + c_i\frac{\partial u}{\partial x_i}\right]$$

$$\text{wherein } {}_{m_1}\nabla^2_{x_1\dots x_q}(u) = {}_{m_1}\nabla_{x_1\dots x_q} \cdot \nabla_{x_1\dots x_q} u$$

6.2 Methods in Cylindrical Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{r\theta z}(u) + {}_{m_2}\nabla_{r\theta z} \cdot (u) =$$

6.3 Methods in Spherical Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{r\theta\phi}(u) + {}_{m_2}\nabla_{r\theta\phi}(u) =$$

Chapter 7

Temperature Distribution of Cartesian Slabs

$$\frac{\partial u}{\partial t} = k\nabla^2_{xy}(u) = k\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

Chapter 8

Temperature Distribution of Polar Slabs

$$\frac{\partial u}{\partial t} = k\nabla_r^2(u) = k\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

Chapter 9

Longitudinal Structural Bar Vibrations

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Chapter 10

Transverse Structural Bar Vibrations

$$\frac{\partial^2 y}{\partial t^2} = -a^4 \frac{\partial^4 y}{\partial x^4}$$

Chapter 11

Natural Frequencies of Beams

$$\frac{\partial^2 y}{\partial t^2} = -a^4 \frac{\partial^4 y}{\partial x^4}$$

Chapter 12

Two-Dimensional Wave Equation in Cartesian Coordinates

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2_{xy}(u) = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

Chapter 13

Two-Dimensional Wave Equation in Polar Coordinates

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2_{r\theta}(u) = c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right]$$

Chapter 14

Spherical Harmonics and Ocean Waves

$$\frac{\partial^2 u}{\partial t^2} = b^2 \nabla^2_{\phi\theta}(u) = b^2 \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} \left[\sin(\phi) \frac{\partial u}{\partial \phi} \right] + \frac{1}{\sin^2(\phi)} \frac{\partial^2 u}{\partial \theta^2} \right\}$$

Chapter 15

Sturm-Liouville Problems

15.1 Definition of Stum-Liouville Problems

A Sturm-Liouville Problem is a problem that satisfies the following equation with the following boundary conditions,

$$0 = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y$$

Alternately,

$$0 = \frac{dy}{dx}\frac{d}{dx}\left[p(x)\right] + p(x)\frac{d^2y}{dx^2} - q(x)y + \lambda r(x)y$$

$$0 = p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} - q(x)y + \lambda r(x)y$$

The initial conditions are shown below,

$$0 = \alpha_1 y(a) - \alpha_2 y'(a) \quad , \quad 0 = \beta_1 y(b) + \beta_2 y'(b)$$

wherein neither α_1 and α_2 both zero nor β_1 and β_2 both zero. The parameter λ is the "eigenvalue" whose possible (constant) values are sought usually via the application of the boundary conditions.

15.2 Eigenvalue Theorem of Sturm-Liouville Problems

Suppose that the functions $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous on the closed interval $[a, b]$ and that $p(x) > 0$ and $r(x) > 0$ at each point of $[a, b]$. Then the eigenvalues of the Sturm–Liouville problem constitute an increasing sequence,

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_\infty$$

of real numbers with

$$\lim_{n\to\infty}[\lambda_n] = \infty$$

To within a constant factor, only a single eigenfunction $y_n(x)$ is associated with each eigenvalue λ_n. Moreover, if $q(x) \geq 0$ on the closed interval $[a, b]$ and the coefficients α_1, α_2, β_1, and β_2 are all non-negative, then the eigenvalues are all non-negative.

15.3 Eigenvalues-Eigenfunctions Series

If the functions $p(x)$, $q(x)$ and $r(x)$ of the Sturm-Liouville problem satisifes the Eigenvalue Theorem, then eigenfunctions $y_i(x)$ and $y_j(x)$ corresponding to eigenvalues λ_i and λ_j wherein $j \neq 1$ are orthogonal with respect to each other relative to the function $r(x)$,

$$0 = \int_a^b y_i(x)y_j(x)r(x)dx$$

For a sturm-liouville problem with infinite eigenvalues, it is possible to represent an arbitrary function $f(x)$ as the infinite sum of the eigenvalues,

$$f(x) = \sum_{m=1}^{\infty} \left[c_m y_m(x)\right]$$

wherein $y_m(x)$ represents the m^{th} eigenfunction of the m^{th} eigenvalue λ_m. Taking the integral in both sides with the product to the eigenfunction $y_n(x)$ relative to the function $r(x)$,

$$\int_a^b f(x)y_n(x)r(x)dx = \int_a^b \sum_{m=1}^{\infty} \left[c_m y_m(x)\right] y_n(x)r(x)dx$$

Using the assumption, $0 = \int_a^b y_i(x)y_j(x)r(x)dx$,

$$\int_a^b f(x)y_n(x)r(x)dx = \int_a^b c_n \left[y_n(x)\right]^2 r(x)dx = c_n \int_a^b \left[y_n(x)\right]^2 r(x)dx$$

Therefore, the constant c_n could be obtained by,

$$c_n = \frac{\displaystyle\int_a^b f(x)y_n(x)r(x)dx}{\displaystyle\int_a^b \left[y_n(x)\right]^2 r(x)dx}$$

This particular theorem could be used to prove under certain reasonable conditions that there exists an infinite series that would allow the boundary conditions to be implemented analytically into the partial differential equations problems.

Chapter 16

Temperature Distribution of a Heated Rod

The function of temperature u at some distance x from the origin of a one-dimensional heated rod of uniform material is governed by the equation below,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

wherein k is some constant related to thermal conductivity. Using the substitution, $u(x,t) = f(x)g(t)$ wherein $f(x)$ is a function purely in x and $g(t)$ is a function purely in terms of t,

$$\frac{\partial[f(x)g(t)]}{\partial t} = k\frac{\partial^2[f(x)g(t)]}{\partial x^2}$$

$$f(x)\frac{\partial[g(t)]}{\partial t} = kg(t)\frac{\partial^2[f(x)]}{\partial x^2}$$

$$f(x)g'(t) = kg(t)f''(x)$$

wherein $g'(t)$ represents the first order derivative of $g(t)$ with respect to time t, and $f''(x)$ represents second order derivative of $f(x)$ with respect to distance x. Further manipulation to yield a left hand side completely in terms of time t and a right hand side completely in terms of displacement x,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = \lambda$$

wherein λ is some constant. The reason why λ is a constant is because x and t are independent variables, therefore a change in one of the values should not affect the other variable. Since the left hand side is and right hand side are in represented completely in independent variables, λ must be a constant if x and t are independent variables. λ is the eigenvalue of the problem whose values are often sought and is inferred from the boundary conditions. This is as far as analysis can go without specifying the boundary conditions.

16.1 Zero-Endpoint Temperatures

Suppose the rod is of length L and that the initial temperature distribution is known. The boundary conditions,

$$u(0,t) = u(L,t) = 0 \quad , \quad u(x,0) = m(x)$$

The first boundary condition is the zero endpoint condition and the second condition is the initial temperature distribution. Reiterating the first boundary condition and substituting $u(x,t)$ as the product of two single variable functions,

$$u(0,t) = u(L,t) = 0 \quad , \quad f(0)g(t) = f(L)g(t) = 0$$

The function $g(t)$ is not trivial, and therefore, $g(t) \neq 0$. Therefore, it follows that,

$$f(0) = f(L) = 0$$

Because the endpoint conditions, it is convenient that the function $f(x)$ be made into a trigonometric function. Consider the eigenvalue to be negative,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = -\lambda$$

Two ordinary differential equation problems can be obtained from this,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = -\lambda \quad , \quad \frac{f''(x)}{f(x)} = -\lambda$$

$$g'(t) = -\lambda k g(t) \quad , \quad f''(x) = -\lambda f(x)$$

The second ordinary differential equation, $f''(x) = -\lambda f(x)$ has the solution,

$$f(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

To satisfy the endpoint condition $f(0) = f(L) = 0$, $c_1 = 0$ and $\sqrt{\lambda}L = n\pi$, wherein n are integers starting from zero. Therefore, $\lambda = n^2\pi^2/L^2$. Substituting the eigenvalues and arbitrary constants c_1,

$$f(x) = c_2 \sin\left(\frac{n\pi}{L}x\right)$$

Substituting the eigenvalue and solving the second ordinary differential equations problem,

$$g'(t) = -\frac{n^2\pi^2 k}{L^2} g(t)$$

$$\int \frac{1}{g(t)} dg(t) = -\frac{n^2\pi^2 k}{L^2} \int dt$$

$$\ln[g(t)] = -\frac{n^2\pi^2 k}{L^2} t + c$$

$$g(t) = Ce^{-\left(\frac{n^2\pi^2 k}{L^2}\right)t}$$

Substituting the two equations together,

$$u(x,t) = C_n e^{-\left(\frac{n^2\pi^2 k}{L^2}\right)t} \sin\left(\frac{n\pi}{L}x\right)$$

Due to the partial differential operator being a linear operator, the superposition principle holds true. Therefore, the general solution to the partial differential equation must be the linear combination of its linearly independent solutions,

$$u_g(x,t) = \sum_{n=1}^{\infty}\left[C_n e^{-\left(\frac{n^2\pi^2 k}{L^2}\right)t} \sin\left(\frac{n\pi}{L}x\right)\right]$$

16.2 Insulated Ends

With the same length of rod L and known initial temperature distribution $m(x)$, the boundary conditions,

$$\frac{\partial}{\partial x}\left[u(0,t)\right] = \frac{\partial}{\partial x}\left[u(L,t)\right] = 0 \quad , \quad u(x,0) = m(x)$$

Substituting the boundary conditions with the definition of u as the product of two single variable functions,

$$f'(0)g(t) = f'(L)g(t) = 0$$

Similarly to the previous case, since $g(t)$ is not the trivial zero function,

$$f'(0) = f'(L) = 0$$

Just in the previous part, it is convenient to choose the eigenvalues to be negative in the two ordinary differential equation problems,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = -\lambda \quad , \quad \frac{f''(x)}{f(x)} = -\lambda$$

This is advantageous because f will take the form of a linear combination of trigonometric functions, at which we can simply choose the cosine series to satisfy the boundary condition above. The general form of $f(x)$,

$$f(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

$$f'(x) = -c_1 \sin\left(\sqrt{\lambda}x\right) + c_2 \cos\left(\sqrt{\lambda}x\right)$$

The only conditions that would satisfy the end-point boundary conditions, $c_2 = 0$, $\sqrt{\lambda}L = n\pi$, $\lambda = n^2\pi^2/L^2$. Substituting the eigenvalues and arbitrary constants would yield,

$$f(x) = c_1 \cos\left(\frac{n\pi}{L}x\right)$$

Solving for $g(t)$ yields,

$$g'(t) = -\lambda k g(t)$$

Familiarly,

$$g(t) = e^{-\lambda k t} = Ce^{\left(-\frac{n^2\pi^2 k}{L^2}t\right)}$$

Similarly to the previous chapter and by principle of superposition,

$$u_g(x,t) = \sum_{n=1}^{\infty}\left[C_n e^{-\left(\frac{n^2\pi^2 k}{L^2}\right)t} \cos\left(\frac{n\pi}{L}x\right)\right]$$

Chapter 17

One-Dimensional Wave Equation

The equation for the one-dimensional wave equation is shown below,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Using the familiar substitution $y(x,t) = u(x) \times r(t)$,

$$\frac{\partial^2}{\partial t^2}\left[u(x)r(t)\right] = a^2 \frac{\partial^2}{\partial x^2}\left[u(x)r(t)\right]$$

$$u(x)r''(t) = a^2 u''(x)r(t)$$

wherein $r''(t)$ and $u''(x)$ represents the second order derivative of $r(t)$ and $u(x)$ respectively. Manipulation of the equation,

$$\frac{1}{a^2}\frac{r''(t)}{r(t)} = \frac{u''(x)}{u(x)}$$

To fit for the somewhat arbitrary conditions,

$$y(0,t) = y(L,t) = 0 \quad , \quad y(x,0) = f(x) \quad , \quad \frac{\partial}{\partial t}\left[y(x,0)\right] = g(x)$$

it is useful to consider the derivative homogenous case (A) and the displacement homogenous case (B) seperately,

$$y_A(0,t) = y_A(L,t) = 0 \quad , \quad y_B(0,t) = y_B(L,t) = 0$$

$$y_A(x,0) = f(x) \quad , \quad y_B(x,0) = 0$$

$$\frac{\partial}{\partial t}[y_A(x,0)] = 0 \quad , \quad \frac{\partial}{\partial t}[y_B(x,0)] = g(x)$$

The somewhat arbitray boundary condition case would be the satisfied by the addition of the derivative homogenous case and the displacement homogenous case. Both function $y_A(x,t)$ and $y_B(x,t)$ satisifes the partial differential equation, therefore, the algebraic addition of them must also satisfy the one dimensional wave equation. When they are added algebraically,

$$y_A(0,t) + y_B(0,t) = y_A(L,t) + y_B(L,t) = 0 + 0$$

$$y_A(x,0) + y_B(x,0) = f(x) + 0 \quad , \quad \frac{\partial}{\partial t}\left[y_A(x,0) + y_B(x,0)\right] = 0 + g(x)$$

Therefore, the algebraic addition of the derivative homogenou and displacement homogenous satisfies the somewhat arbitrary conditions provided earlier.

17.1 Derivative Homogenous Case

The boundary conditions for the derivative homogenous case,

$$y(0,t) = y(L,t) = 0 \quad , \quad y(x,0) = f(x) \quad , \quad \frac{\partial}{\partial t}[y(x,0)] = 0$$

To satisfy the first boundary condition listed above, it would be convenient for the eigenvalues to be considered negative,

$$\frac{1}{a^2}\frac{r''(t)}{r(t)} = \frac{u''(x)}{u(x)} = -\lambda$$

Therefore, the two ordinary differential equations,

$$u''(x) = -\lambda u(x) \quad , \quad r''(t) = -\lambda a^2 r(t)$$

The characteristic equation associated to the displacement differential equation,

$$r^2 = -\lambda \quad , \quad r = \sqrt{\lambda} i$$

Therefore, the displacement function $u(x)$ is in terms of the familiar linear combination of trigonometric functions,

$$u(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

The only constants that will satisfy the condition $y(0,t) = y(L,t) = 0$, $c_1 = 0$, and $\sqrt{\lambda} L = n\pi$. Therefore, the eigenvalues are, $\lambda = \dfrac{n^2\pi^2}{L^2}$. Substituting for eigenvalues and arbitray constant c_1 into $u(x)$,

$$u(x) = c_2 \sin\left(\frac{n\pi}{L} x\right)$$

The characteristic equation associated to the time dependent differential equation,

$$r^2 = -\lambda a^2 \quad , \quad r = a\sqrt{\lambda} i$$

Therefore, the trigonometric solution to the above characteristic equation,

$$r(t) = k_1 \cos\left(a\sqrt{\lambda} t\right) + k_2 \sin\left(a\sqrt{\lambda} t\right)$$

$$r(t) = k_1 \cos\left(\frac{n\pi a}{L} t\right) + k_2 \sin\left(\frac{n\pi a}{L} t\right)$$

17.2 Displacement Homogenous Case

The boundary conditions for the displacement homogenous case,

$$y(0,t) = y(L,t) = 0 \quad , \quad y(x,0) = 0 \quad , \quad \frac{\partial}{\partial t}[y(x,0)] = g(x)$$

Chapter 18

Numerical Methods

18.1 Thomas Method: Tridiagonal Systems

Consider a single row in the tri-diagonal system,

$$\beta_j \phi_{j-1} + D_j \phi_j + \alpha_j \phi_{j+1} = C_j$$

Suppose the perceding row in the tri-diagonal system,

$$D_{j-1}\phi_{j-1} + \alpha_{j-1}\phi_j = C_{j-1}$$

The forward sweep of the thomas algorithm seeks to eliminate the sub-diagonal terms of the tri-diagonal system. For the two rows of the tri-diagonal system shown above, the sub-diagonal term is ϕ_{j-1}. Manpulating the perceding row,

$$D_{j-1}\beta_j \phi_{j-1} + \alpha_{j-1}\beta_j \phi_j = \beta_j C_{j-1}$$

Manipulating the following row,

$$\beta_j D_{j-1}\phi_{j-1} + D_j D_{j-1}\phi_j + \alpha_j D_{j-1}\phi_{j+1} = D_{j-1}C_j$$

Subtracting the following row by the perceding row

$$\beta_j D_{j-1}\phi_{j-1} + D_j D_{j-1}\phi_j + \alpha_j D_{j-1}\phi_{j+1} - D_{j-1}\beta_j \phi_{j-1} - \alpha_{j-1}\beta_j \phi_j = D_{j-1}C_j - \beta_j C_{j-1}$$

$$D_j D_{j-1}\phi_j - \alpha_{j-1}\beta_j \phi_j + \alpha_j D_{j-1}\phi_{j+1} = D_{j-1}C_j - \beta_j C_{j-1}$$

$$[D_j D_{j-1} - \alpha_{j-1}\beta_j]\phi_j + \alpha_j D_{j-1}\phi_{j+1} = D_{j-1}C_j - \beta_j C_{j-1}$$

$$\phi_j + \left[\frac{\alpha_j D_{j-1}}{D_j D_{j-1} - \alpha_{j-1}\beta_j}\right]\phi_{j+1} = \frac{D_{j-1}C_j - \beta_j C_{j-1}}{D_j D_{j-1} - \alpha_{j-1}\beta_j}$$

The results above would Complete the forward sweep of the thomas algorithm for the first row until the sec ond last row. The last row is simply a more speCific case of the expression above wherein $a_j = 0$. Substituting for only the last row,

$$\phi_j = \frac{D_{j-1}C_j - \beta_j C_{j-1}}{D_j D_{j-1} - \alpha_{j-1}\beta_j}$$

The last row in the tri-diagonal system is solved after the forward sweep of the thomas algorithm. After t he forward sweep of the thomas algorithm, the perceding row,

$$\phi_j + \alpha_j \phi_{j+1} = C_j$$

The following row,

$$\phi_{j+1} = C_{j+1}$$

Substituting the following row to the perceding row,

$$\phi_j + \alpha_j C_{j+1} = C_j$$

$$\phi_j = C_j - \alpha_j C_{j+1}$$

This would be true because the main diagonal after the forward sweep of the thomas algorithm would all be just 1. The Thomas algorithm implemented in fortran is shown below,

Chapter 19

Tensors

19.1 Tensor Index Notation

Tensors are a generalization of scalars, vectors, and matrices. The order of a tensor represents how many 'axis' the tensor has. For example, a scalar would be a 0^{th} order tensor meanwhile a vector would be a 1^{st} order tensor and a matrix would be 2^{nd} order tensor. Tensors of higher orders are permitted though a visual representation of them is meaningless. One can alternatively imagine tensors as multi-dimensional arrays, much like the case in a programming language.

The tensor index notation comprises of 2 main indices: A free index and a dummy index. A free index corresponds to the positioning of a certain value in a tensor. For example, the i^{th} component of a vector $\bar{v}$ is usually represented as v_i. That is an example of a free index usage. A dummy index is an index that is used for summation. Dummy indices occur in pairs and a pair of dummy indices imply summation. For example in the case of a dot product, A_jB_j represents scalar multiplication between the j^{th} components of vectors $\bar{A}$ and $\bar{B}$, added all together for the enirety of the length of vector $\bar{A}$ and vector $\bar{B}$.

Since what specific name one gives to a an index is arbitrary, this leads to index renaming rules. Dummy indices may be renamed within a single term. For example $A_jB_j = A_iB_i$. Free indices however, must be renamed across all algebraically summed terms. For example, $A_iB_pC_p + D_iE_qF_q = A_jB_pC_p + D_jE_qF_q$

19.2 Kronecker-Delta & Permutation Tensor

The kronecker-delta is a function that maps 2 integers to a 1 or 0. A mathematical description of the kronecker-delta function δ_{ij} is shown below,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

The permutation tensor ϵ_{ijk} a 3^{rd} order tensor and is anti-symmetric in any 2 of the indices. The indices can accept a range of integers from 1 until 3. Therefore, ϵ_{123}, ϵ_{213} are both valid but ϵ_{352} is not. The permutation tensor has a cyclic property described below,

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

Switching any 2 index of the permutaton tensor makes it negative. This property is described below,

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

The exact value of the permutation tensor,

$$\epsilon_{123} = 1 \quad , \quad \epsilon_{213} = -1$$

The other cases of i, j, and k are all obtainable by applying the properties above.

19.3 Common Vector Operations

Let $\bar{A}$ and $\bar{B}$ be vector fields, and ϕ, ψ be scalar fields. Let A_i and B_i represent the i^{th} component of the vector $\bar{A}$ and $\bar{B}$ respectively.

19.3.1 Scalar Multiplication

Since scalar multiplication simplt multiples all components of a vector by some scalar,

$$\left[\phi\bar{A}\right]_i = \phi A_i$$

wherein the LHS represents the vector notation and the RHS represents the index notation equivalent. Note that the $[]_i$ is used to denote the i^{th} index of the vector notation.

19.3.2 Dot Product

Dot products can be represented very elegantly in tensor index notation,

$$\bar{A} \cdot \bar{B} = A_j B_j$$

The repeated index j here makes j a dummy index which is used for counting. A repeated index such as j, implies summation. Therefore,

$$A_j B_j = A_1 B_1 + A_2 B_2 + A_3 B_3$$

19.3.3 Cross Product

The cross product of 2 vectors is defined with the permutation tensor,

$$\left[\bar{A} \times \bar{B}\right]_i = \epsilon_{ijk} A_j B_k$$

19.4 Tensor Index Identities

Let $\bar{A}$ and $\bar{B}$ be vector fields, and ϕ, ψ be scalar fields. Let $\bar{\bar{\mu}}$ and $\bar{\bar{\gamma}}$ represent second order tensors,

19.4.1 Symmetric-Antisymmetric Tensor

Let $\bar{\bar{\mu}}$ be a symmetric tensor and $\bar{\bar{\gamma}}$ be an anti-symmetric tensor. By the properties of the symmetric and anti-symmetric tensors,

$$\mu_{ij} = \mu_{ji} \quad , \quad \gamma_{ij} = -\gamma_{ji}$$

Consider the following,

$$\mu_{ij}\gamma_{ij} = -\mu_{ji}\gamma_{ji}$$

Here, the dummy indices have been switched, and this is true due to the symemtric and anti-symmetric definitions of μ and γ. The dummy indices are renamed, $j \to p$, $i \to q$,

$$\mu_{ij}\gamma_{ij} = -\mu_{pq}\gamma_{pq} \tag{19.1}$$

Next, start with $m\bar{\bar{u}}$ and $\bar{\bar{\gamma}}$ again, but this time rename them based on a different set of variable change. $i \to p$ and $j \to q$. This seems illegal, but it is not. Remember, the naming are arbitrary and we have not violated any of the rules. Therefore,

$$\mu_{ij}\gamma_{ij} = \mu_{pq}\gamma_{pq} \tag{19.2}$$

Susbtituting $\mu_{ij}\gamma_{ij}$ out from euqation 19.1 and equation 19.2,

$$\mu_{pq}\gamma_{pq} = -\mu_{pq}\gamma_{pq}$$

Therefore,

$$0 = \mu_{pq}\gamma_{pq}$$

Hence, the element-wise multiplication of a symmetric and anti-symmetric tensor added together for the entire tensor would yield zero.

19.4.2 Double Permutation Tensor

Arguably one of the most important identities for tensor indices,

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

19.4.3 Kronecker-Delta Renaming

The kronecker-delta function can be used to rename the indices of a tensor,

$$\delta_{ij}A_i = A_j$$

This is because when $i \neq j$, the kronecker-delta function is zero, which means that $\delta_{ij}A_i$ is only non-zero when $i = j$, which renames the dummy variable of i in A_i into j.

19.4.4 Curl of Scalar Gradient

The curl of a scalar gradient is zero,

$$0 = \nabla \times (\nabla\phi)$$

Let ,

$$LHS = 0 \quad , \quad RHS = \nabla \times (\nabla \phi)$$

Converting LHS and RHS into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[\frac{\partial \phi}{\partial x_k} \right] = \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_k} (\phi)$$

Since partial derivative operators are commutative, $\frac{\partial^2}{\partial x_j \partial x_k}(\phi)$ is a symmetry tensor. If i is held constant, the permutation tensor ϵ_{ijk} is anti-symmetric. The element-wise multiplication of a symmetric tensor and anti-symmetric tensor added up together yields zero. Therefore,

$$RHS_i = 0$$

Since $LHS_i = RHS_i$, the claim is proven to be true.

19.4.5 Divergence of Vector Curl

The divergence of the curl of a vector field is zero,

$$0 = \nabla \cdot (\nabla \times \bar{A})$$

Let,

$$LHS = 0 \quad , \quad RHS = \nabla \cdot (\nabla \times \bar{A})$$

Converting LHS and RHS into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \frac{\partial}{\partial x_j} \left[\epsilon_{jkl} \frac{\partial}{\partial x_k} (A_l) \right] = \epsilon_{jkl} \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_k} (A_l) \right] = \epsilon_{jkl} \frac{\partial^2}{\partial x_j \partial x_k} (A_l)$$

Since ϵ_{jkl} is an anti-symmetric tensor and $\frac{\partial^2}{\partial x_j \partial x_k}(A_l)$ is a symmetric tensor, then $RHS_i = 0$. Since $LHS_i = RHS_i$, then the claim is proven to be true.

19.4.6 Curl of 2 Vector Cross Products

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Let,

$$LHS = \nabla \times (\bar{A} \times \bar{B}) \quad , \quad RHS = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Converting RHS into index notation,

$$RHS_i = B_j \frac{\partial}{\partial x_j} (A_i) + A_i \frac{\partial}{\partial x_j} (B_j) - A_j \frac{\partial}{\partial x_j} (B_i) - B_i \frac{\partial}{\partial x_j} (A_j)$$

Converting LHS into index notation,

$$LHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[\epsilon_{klm} A_l B_m \right] = \epsilon_{ijk} \epsilon_{klm} \frac{\partial}{\partial x_j} \left[A_l B_m \right]$$

Using the cylcic permutation property of the permutation tensor $\epsilon_{ijk} = \epsilon_{kij}$. Therefore,

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm}$$

Using the double permutation tensor identity,

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Substituting into LHS_i,

$$LHS_i = \epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}[A_l B_m] = \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right]\frac{\partial}{\partial x_j}[A_l B_m] = \delta_{il}\delta_{jm}\frac{\partial}{\partial x_j}[A_l B_m] - \delta_{im}\delta_{jl}\frac{\partial}{\partial x_j}[A_l B_m]$$

$$LHS_i = \delta_{jm}\frac{\partial}{\partial x_j}[A_i B_m] - \delta_{jl}\frac{\partial}{\partial x_j}[A_l B_i] = \frac{\partial}{\partial x_j}\left[A_i B_j\right] - \frac{\partial}{\partial x_j}\left[A_j B_i\right]$$

Expanding using product rule,

$$LHS_i = A_i\frac{\partial}{\partial x_j}\left[B_j\right] + B_j\frac{\partial}{\partial x_j}[A_i] - \left\{A_j\frac{\partial}{\partial x_j}[B_i] + B_i\frac{\partial}{\partial x_j}\left[A_j\right]\right\}$$

$$LHS_i = A_i\frac{\partial}{\partial x_j}\left[B_j\right] + B_j\frac{\partial}{\partial x_j}[A_i] - A_j\frac{\partial}{\partial x_j}[B_i] - B_i\frac{\partial}{\partial x_j}\left[A_j\right]$$

Since $LHS_i = RHS_i$, the vector identity is proven to be true.

19.4.7 Double Curl of Vector

$$\nabla \times \left(\nabla \times \bar{A}\right) = \nabla\left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Let

$$LHS = \nabla \times \left(\nabla \times \bar{A}\right) \quad , \quad RHS = \nabla\left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Converting LHS into index notation,

$$LHS_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{kmn}\frac{\partial}{\partial x_m}A_n$$

Since the permutation tensor ϵ_{kmn} is a constant in x_j and x_m,

$$LHS_i = \epsilon_{ijk}\epsilon_{kmn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}A_n$$

Using the permutation tensor cyclic identity,

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn}$$

Using the double permutation tensor identity,

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

Substituting,

$$LHS_i = [\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}]\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n)$$

$$LHS_i = \delta_{im}\delta_{jn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n) - \delta_{in}\delta_{jm}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n)$$

Using the renaming identity of the kronecker-delta function,

$$LHS_i = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Since partial derivatives are commutative with one another, $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(A_j) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j)$.

Susbtituting,

$$LHS_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Reiterating RHS,

$$RHS = \nabla\left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Converting RHS into index notation,

$$RHS_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Since $LHS_i = RHS_i$, then the identity is proven to be true.

19.4.8 Curl of Vector Scalar

$$\nabla \times (\phi\bar{A}) = \phi\nabla \times \bar{A} + (\nabla\phi) \times \bar{A}$$

Let

$$LHS = \nabla \times (\phi\bar{A}) \quad , \quad RHS = \phi\nabla \times \bar{A} + (\nabla\phi) \times \bar{A}$$

Converting LHS into index notation,

$$LHS_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi A_k)$$

Using product rule,

$$LHS_i = \epsilon_{ijk}\left[\phi\frac{\partial}{\partial x_j}(A_k) + A_k\frac{\partial}{\partial x_j}(\phi)\right]$$

$$LHS_i = \epsilon_{ijk}\phi\frac{\partial}{\partial x_j}(A_k) + \epsilon_{ijk}A_k\frac{\partial}{\partial x_j}(\phi)$$

Converting RHS into index notation,

$$RHS_i = \phi\epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k) + \epsilon_{ijk}\left[\frac{\partial}{\partial x_j}(\phi)\right]A_k$$

$$RHS_i = \phi\epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k) + \epsilon_{ijk}A_k\left[\frac{\partial}{\partial x_j}(\phi)\right]$$

Since $LHS_i = RHS_i$, the identity is proven to be true.

19.4.9 Triple Curl of Vector

$$\nabla \times [\nabla \times (\nabla \times \bar{A})] = -\nabla^2(\nabla \times \bar{A})$$

Let,

$$LHS = \nabla \times [\nabla \times (\nabla \times \bar{A})] \quad , \quad RHS = -\nabla^2(\nabla \times \bar{A})$$

In index notation,

$$(\nabla \times \bar{A})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k)$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \epsilon_{lmi}\frac{\partial}{\partial x_m}\left[\epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k)\right]$$

Since ϵ_{ijk} is simply a constant in x_m or x_j,

$$[\nabla \times (\nabla \times \bar{A})]_l = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right]$$

Using the cyclic property of the permutation tensor,

$$\epsilon_{lmi}\epsilon_{ijk} = \epsilon_{ilm}\epsilon_{ijk}$$

Using the double permutation tensor identity,

$$\epsilon_{lmi}\epsilon_{ijk} = \epsilon_{ilm}\epsilon_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm}$$

Substituting for the double permutation tensor identity,

$$[\nabla \times (\nabla \times \bar{A})]_l = [\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm}]\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right]$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \delta_{lj}\delta_{mk}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right] - \delta_{lk}\delta_{jm}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right]$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right] - \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]$$

$$\left\{\nabla \times [\nabla \times (\nabla \times \bar{A})]\right\}_p = \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right] - \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

$$\left\{\nabla \times [\nabla \times (\nabla \times \bar{A})]\right\}_p = \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right]\right\} - \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

Since partial derivative operations are commutative with one another,

$$\epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right]\right\} = \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_l}\left[\frac{\partial}{\partial x_k}(A_k)\right]\right\}$$

The permutation tensor is anti-symmetric in any 2 of its indices, and $\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_l}\left[\frac{\partial}{\partial x_k}(A_k)\right]\right\}$ is symmetric in q and l due to the commutativity of the partial differential operator. Since this would mean a symmetric tensor multiplied by an anti-symmetric element-wise and added together,

$$0 = \epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_k} \left[\frac{\partial}{\partial x_l} (A_k) \right] \right\}$$

Therefore,

$$\left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_p = -\epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

renaming the free index $p \to i$,

$$\left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_i = -\epsilon_{iql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

Since $LHS_i = \left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_i$,

$$LHS_i = -\epsilon_{iql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

Reiterating definition of RHS,

$$RHS = -\nabla^2 (\nabla \times \bar{A})$$

Converting RHS into index notation,

$$RHS_i = -\frac{\partial}{\partial x_j} \left\{ \frac{\partial}{\partial x_j} \left[\epsilon_{iql} \frac{\partial}{\partial x_q} (A_l) \right] \right\}$$

Since the permutation tensor is a constant in x_j and x_q and that partial derivative operations are commutative with one another,

$$RHS_i = -\epsilon_{iql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

Since $LHS_i = RHS_i$, the identity is proven to be true.

19.4.10 Divergence of Vector Scalar

$$\nabla \cdot (\phi \bar{A}) = \phi (\nabla \cdot \bar{A}) + \bar{A} \cdot \nabla \phi$$

Let,

$$LHS = \nabla \cdot (\phi \bar{A}) \quad , \quad RHS = \phi (\nabla \cdot \bar{A}) + \bar{A} \cdot \nabla \phi$$

Converting RHS into index notation,

$$RHS_i = \phi \frac{\partial}{\partial x_j} [A_j] + A_j \frac{\partial}{\partial x_j} [\phi]$$

Converting LHS into index notation,

$$LHS_i = \frac{\partial}{\partial x_j}[\phi A_j]$$

Using product rule,

$$LHS_i = \phi \frac{\partial}{\partial x_j}[A_j] + A_j \frac{\partial}{\partial x_j}[\phi]$$

Since $LHS_i = RHS_i$, the identity is proven to be true.

Part II

Fluid Dynamics

Chapter 20

Basic Definitions

20.1 Dynamic Variables

Name	Symbollic Representation	Units	Description
Lift	L	N	Upward force experienced by the aircraft
Drag	D	N	Backward force experienced by the aircraft

20.2 Geometrical variables

Name	Symbollic Representation	Units	Description
Angle of attack	α	rad	How pitched up or down the wing or horizontal stabilizer is usually, could represent more than just wings or horizontal stabilizers though
Leading Edge	-	-	The front-most edge of the airfoil
Trailing Edge	-	-	The back-most edge of the airfoil
Chord length		m	Length of the chord line, wherein chord line is a line joining the leading edge and trailing edge
Span length		m	The sideways length of the wing. The distance between one wing tip to another wing tip
Mean Camber line			
Chord line			

20.3 Processed Geometry

Name	Symbollic Representation	Units	Description
Aerodynamic Center		-	A specific point in the airfoil wherein the moments acting on the airfoil due to fluid pressures is unchanging with angle of attack
Center of Pressure		-	A specific point in the airfoil wherein the airfoil experiences no resultant moment about this point
Neutral Point			
Aspect Ratio			

20.4 Dimensionless Coefficients

Name	Symbollic Representation	Units	Description
Coefficient of Lift			
Coefficient of Drag			
Coefficient of Moments			

20.5 Definition of Processes

Name	Symbollic Representation	Description
Isothermal	it	Constant temperature
Isobaric	ib	Constant pressure
Isochoric	ic	Constant volume
Adiabatic	ad	No heat exchange with external system
Reversible	rev	No dissipative phenomena, no mass diffusion, no thermal conducitvity, no viscosity
Isentropic	ise	Both Adiabatic and Reversible

Chapter 21

Reynold's Transport Theorem

One variation of Liebniz Rule applicable for volumetric integrals is shown below. for the variable T wherein T may represent a time dependent scalar, vector, or tensor.

$$\frac{d}{dt}\iiint_{R(t)} T\,dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T]dV_o + \iint_{S(t)} T\bar{v}_s\bar{n}dS$$

wherein $R(t)$ represents an arbitray region of space, V_o represents volume, $S(t)$ represents the surface of the region defined by $R(t)$, $\bar{v}_s$ represents the velocity of the moving surface, $\bar{n}$ represents normal vector of the surface. Depending on the variable type T, the operation $T\bar{v}_s\bar{n}$ would depend on a case to case basis.

Using a change of variables,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\,dV_o = \frac{d}{dt}\iiint_{\Gamma} \phi J\,dV_{o,i}$$

wherein $V_s(t)$ represents a control mass region, ϕ may represent some time changing scalar variable, but in general, could represent the elements of an arbitrary tensor as well. dV_o represents infinitesmial volume. Since the vector field is evolving with time, all the points inside $V_s(t)$ is at some place initially at time $t = 0$. The region that contains all the points inside $V_s(t)$ at time $t = 0$ is considered to be Γ. Since we are considering the general case wherein volume may expand or contract, we declare $dV_{o,i}$ to represent infinitesmial volume initially at time $t = 0$. The relationship between infinitesmial volume at the present time dV_o and infinitesmial volume initially,

$$dV_o = J\,dV_{o,i}$$

wherein J represents the Jacobian, which is the determinant of the velocity gradient tensor (more on this later). From all these information, the change of variables could be performed as shown above. ϵ is a region that is not varying with time t. Since the bounds of integration is now unchanging with time, the time derivative operation is now commutative with the volumetric integral. Therefore,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\,dV_o = \iiint_{\Gamma} \frac{d}{dt}[\phi J]\,dV_{o,i}$$

Without loss of generality, assuming that ϕ changes with coordinates x_i and time, the time derivative is equivalent to the substantive or material derivative. Therefore,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\,dV_o = \iiint_{\Gamma} \frac{D}{Dt}[\phi J]\,dV_{o,i}$$

Using product rule,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{\Gamma} \phi \frac{D}{Dt}[J] + J\frac{D}{Dt}[\phi]\, dV_{o,i}$$

By a tedious mathematical proof,

$$\frac{D}{Dt}[J] = (\nabla \cdot \bar{v}_s)J$$

wherein $\bar{v}_s$ represents the velocity of the moving surface. $\bar{v}_s$ is not to be confused with V_s. V_s represents the control mass region earlier meanwhile $\bar{v}_s$ represents the velocity of the moving boundaries of V_s. Substituting,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{\Gamma} \phi(\nabla \cdot \bar{v}_s)J + J\frac{D}{Dt}[\phi]\, dV_{o,i}$$

Making a change of variables once again to revert back to the region $V_s(t)$ from the initial positions Γ,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{\Gamma} \left\{\phi(\nabla \cdot \bar{v}_s) + \frac{D}{Dt}[\phi]\right\} J dV_{o,i}$$

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{V_s(t)} \phi(\nabla \cdot \bar{v}_s) + \frac{D}{Dt}[\phi]\, dV_o$$

Expanding the substantive derivative of ϕ as $\dfrac{D}{Dt}[\phi] = \dfrac{\partial}{\partial t}[\phi] + \bar{v}_s \cdot \nabla\phi$,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{V_s(t)} \phi(\nabla \cdot \bar{v}_s) + \frac{\partial}{\partial t}[\phi] + \bar{v}_s \cdot \nabla\phi\, dV_o$$

Using the divergence of scalar vector product identity,

$$\nabla \cdot (\phi\bar{v}_s) = \phi(\nabla \cdot \bar{v}_s) + \bar{v}_s \cdot \nabla\phi$$

Substituting,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{V_s(t)} \frac{\partial}{\partial t}[\phi] + \nabla \cdot (\phi\bar{v}_s)\, dV_o$$

Parsing out the integral,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{V_s(t)} \frac{\partial}{\partial t}[\phi]\, dV_o + \iiint_{V_s(t)} \nabla \cdot (\phi\bar{v}_s)\, dV_o$$

Using divergence theorem, $\displaystyle\iiint_{V_s(t)} \nabla \cdot (\phi\bar{v}_s)\, dV_o = \iint_{S_s(t)} \phi\bar{v}_s \cdot \hat{n} dS$. Substituting,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{V_s(t)} \frac{\partial}{\partial t}[\phi]\, dV_o + \iint_{S_s(t)} \phi\bar{v}_s \cdot \hat{n} dS$$

21.1 Substantive Derivative

Suppose a quantity b is dependent on the the variable time t and the typical cartesian coordinates x, y, z. Taking the derivative of variable b with respect to time yields the following based on chain rule,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \frac{\partial}{\partial x}[b] \times \frac{\partial}{\partial t}[x] + \frac{\partial}{\partial y}[b] \times \frac{\partial}{\partial t}[y] + \frac{\partial}{\partial z}[b] \times \frac{\partial}{\partial t}[z]$$

Taking note that the partial derivatives of the cartesian coordinates defines velocity in the cartesian coordinates. Therefore,

$$\frac{\partial}{\partial t}[x] = u \quad , \quad \frac{\partial}{\partial t}[y] = v \quad , \quad \frac{\partial}{\partial t}[z] = w$$

wherein u, v, and w typically represents velocity in the x, y, and z directions respectively. Therefore, the derivative of y with respect to time t would take the form,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

If the ∇ operator is defined as

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T$$

Therefore, the derivative of y with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

Let the velocity vector be defined as

$$\bar{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

It follows that the derivative of b with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \bar{v} \cdot \nabla b$$

21.2 Divergence Theorem

The Divergence Theorem is stated below. The variable $\bar{F}$ must represent a vector in R^3

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

Alternately,

$$\iiint_{R(t)} \frac{\partial}{\partial x}[\bar{F}] + \frac{\partial}{\partial y}[\bar{F}] + \frac{\partial}{\partial z}[\bar{F}]\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

wherein dV_o represents an infinitesmially small volume. $S(t)$ is the surface encapsulating the region $R(t)$. $\bar{n}$ is the normal vector of the control volume, and dS is an infinitesmial area of surface $S(t)$.

Chapter 22

Governing Equations

22.1 Governing Equation: Continuum Equation

The Governing Continuum Equation in its differential form:

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v_f})$$

The Governing Continuum Equation in its integral form:

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v_s} \cdot \bar{n} dS$$

A more useful alternate form:

$$\frac{d}{dt}M(t) = \iint_{S(t)} \rho(\bar{v_s} - \bar{v_f}) \cdot \bar{n} dS$$

wherein $M(t)$ represent mass contained in a control volume, $\bar{v_f}$ represent the velocity of the fluid and $\bar{v_s}$ represent the velocity of the deforming control volume $R(t)$. $S(t)$ represents the surface that is encapsuating the control volume $R(t)$.

22.1.1 Differential Continuity Proof

Starting with the definition of mass contained in the arbitrary control volume $R(t)$,

$$M(t) = \iiint_{R(t)} \rho \, dV_o$$

Taking the derivative of the mass contained within the control volume with respect to time,

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o$$

By application of Liebniz rule, substituting T with ρ,

$$\frac{d}{dt}\iiint_{R(t)} T \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T] dV_o + \iint_{S(t)} T \bar{v_s} \bar{n} dS$$

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v_s} \cdot \bar{n} dS$$

If the velocity of the surface expanding is equivalent to the velocity of the fluid at the boundary of the control volume ($\bar{v}_s = \bar{v}_f$), then the amount of mass within the control volume must remain constant.

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_f \cdot \bar{n} dS$$

The second term of the expression above could be converted into a volumetric integral based on the divergence theorem by substituting F with $\rho\bar{v}_f$.

$$\iiint_{R(t)} \nabla \cdot \bar{F} \, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho \bar{v}_f) \, dV_o = \iint_{S(t)} (\rho \bar{v}_f) \cdot \bar{n} dS$$

Therefore,

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iiint_{R(t)} \nabla \cdot (\rho \bar{v}_f) \, dV_o$$

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v}_f) \, dV_o$$

Since the integration is zero for an arbitrary region the integrand must be zero everywhere. To prove this, simply choose the arbitray region to be infinitesmially small at all points in R^3 and it could be seen that the integrand is always zero everywhere.

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v}_f)$$

22.1.2 Integral Continuity Proof

To prove the integral form for the Governing Continuum Equation, consider the time rate of change of a mass enclosed within the control volume:

$$\frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

From the differential form of the Governing Continuum Equation,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v}_f)$$

Therefore,

$$\frac{\partial}{\partial t}[\rho] = -\nabla \cdot (\rho \bar{v}_f)$$

Therefore,

$$\frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS = -\iiint_{R(t)} \nabla \cdot (\rho \bar{v}_f) \, dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

By applying the divergence theorem to convert the first term volumetric integral into a surface integral,

$$\iiint_{R(t)} \nabla \cdot \bar{F} \, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho \bar{v_f})\, dV_o = \iint_{S(t)} \rho \bar{v_f} \cdot \bar{n} dS$$

$$\frac{d}{dt} M(t) = -\iint_{S(t)} \rho \bar{v_f} \cdot \bar{n} dS + \iint_{S(t)} \rho \bar{v_s} \cdot \bar{n} dS = \iint_{S(t)} \rho (\bar{v_s} - \bar{v_f}) \cdot \bar{n} dS$$

A more familiar form would yield,

$$0 = \frac{d}{dt} M(t) + \iint_{S(t)} \rho (\bar{v_f} - \bar{v_s}) \cdot \bar{n} dS$$

$$0 = \frac{d}{dt} \iiint_{R(t)} \rho\, dV_o + \iint_{S(t)} \rho (\bar{v_f} - \bar{v_s}) \cdot \bar{n} dS$$

22.2 Governing Equation: Momentum Equation

The Governing Momentum Equation in its differential form:

$$\frac{\partial}{\partial t}(\rho \bar{v_f}) + \nabla \cdot (\rho \bar{v_f} \bar{v_f}) = -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}$$

wherein ρ represents density, $\bar{v_f}$ represents fluid velocity vector, P_r represents fluid pressure at a particular point, τ represents viscous forces, $\bar{F_b}$ represents body force experienced by the fluid inside the control volume. The Governing Momentum Equation in its integral form:

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f}\, dV_o = \iiint_{R(t)} \rho \bar{F_b}\, dV_o + \iint_{S(t)} \bar{F_s} \cdot \bar{n} dS = \iiint_{R(t)} \frac{\partial}{\partial t}(\rho \bar{v_f}) + \nabla \cdot (\rho \bar{v_f} \bar{v_s}) dV_o$$

wherein $\bar{F_s}$ represents surface forces. Like in the previous proof, $S(t)$ represents the surface binding the control volume region $R(t)$. An alternate form to the momentum governing equation exists. It is shown below,

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f}\, dV_o = \iint_{S(t)} \rho \bar{v_f} [(\bar{v_s} - \bar{v_f}) \cdot \bar{n}] dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}\, dV_o$$

22.2.1 Differential Momentum Proof

The total Momentum $\bar{P_m}$ contained in a control volume,

$$\bar{P_m} = \iiint_{R(t)} \rho \bar{v_f}\, dV_o$$

The derivative of momentum with respect to time,

$$\frac{d}{dt} \bar{P_m} = \frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f}\, dV_o$$

By applying Liebniz's rule, substituting T with $\rho \bar{v_f}$

$$\frac{d}{dt} \iiint_{R(t)} T\, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T] dV_o + \iint_{S(t)} T \bar{v_s} \bar{n} dS$$

$$\frac{d}{dt} \bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho \bar{v_f}]\, dV_o + \iint_{S(t)} \rho \bar{v_f} \bar{v_s} \bar{n} dS$$

By applying Divergence Theorem substituting F with $\rho\bar{v}_f\bar{v}_s$

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho\bar{v}_f\bar{v}_s)\, dV_o = \iint_{S(t)} \rho\bar{v}_f\bar{v}_s \cdot \bar{n} dS$$

By substituting the terms to the derivative of momentum with respect to time,

$$\frac{d}{dt}\bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v}_f]\, dV_o + \iiint_{R(t)} \nabla \cdot (\rho\bar{v}_f\bar{v}_s)\, dV_o$$

$$\frac{d}{dt}\bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v}_f] + \nabla \cdot (\rho\bar{v}_f\bar{v}_s)\, dV_o$$

Since the derivative of momentum with respect to time is the total force applied to the control volume,

$$\iiint_{R(t)} \rho\bar{F}_b\, dV_o + \iint_{S(t)} \bar{F}_s \cdot \bar{n} dS = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v}_f] + \nabla \cdot (\rho\bar{v}_f\bar{v}_s)\, dV$$

The first term in the expression above represents the total body force acting on the control volume meanwhile the second term in the expression represents the total surface force acting on the control volume. When the velocity of the surface is identical to the velocity of the fluid flow, $\bar{v}_s = \bar{v}_f$, the total force acting on the specific volume of region $R_s(t)$,

$$\iiint_{R_s(t)} -\nabla P_r + \nabla \cdot \tau + \rho\bar{F}_b\, dV_o = \iiint_{R_s(t)} \frac{\partial}{\partial t}[\rho\bar{v}_f] + \nabla \cdot (\rho\bar{v}_f\bar{v}_f)\, dV$$

Since the equation above is always true under the constraint that the surface velocity of the region is identical to the velocity of the fluid flow, one can simply choose smaller and smaller regions until $R_s(t)$ is an infinitesmially small region This process can be applied everywhere in the fluid. Therefore,

$$-\nabla P_r + \nabla \cdot \tau + \rho\bar{F}_b = \frac{\partial}{\partial t}[\rho\bar{v}_f] + \nabla \cdot (\rho\bar{v}_f\bar{v}_f)$$

The equation above is the differential formulation and is always true all throughout the fluid.

22.2.2 Integral Momentum Proof

To prove the alternate form of the momentum governing equation in integral form, the differential formulation of the momentum equation would be vital. Rearranging for the $\frac{\partial}{\partial t}(\rho\bar{v}_f)$ term,

$$\frac{\partial}{\partial t}(\rho\bar{v}_f) + \nabla \cdot (\rho\bar{v}_f\bar{v}_f) = -\nabla P_r + \nabla \cdot \tau + \rho\bar{F}_b$$

$$\frac{\partial}{\partial t}(\rho\bar{v}_f) = -\nabla \cdot (\rho\bar{v}_f\bar{v}_f) - \nabla P_r + \nabla \cdot \tau + \rho\bar{F}_b$$

Referencing the previous equation for derivative of momentum within an arbitrary region $R(t)$ with respect to time,

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v}_f\, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v}_f]\, dV_o + \iint_{S(t)} \rho\bar{v}_f(\bar{v}_s \cdot \bar{n}) dS$$

Substituting the term, $\frac{\partial}{\partial t}(\rho \bar{v_f})$,

$$\frac{d}{dt}\iiint_{R(t)} \rho \bar{v_f}\, dV_o = \iiint_{R(t)} -\nabla \cdot (\rho \bar{v_f} \bar{v_f}) - \nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}\, dV_o + \iint_{S(t)} \rho \bar{v_f}(\bar{v_s} \cdot \bar{n}) dS$$

$$\frac{d}{dt}\iiint_{R(t)} \rho \bar{v_f}\, dV_o = -\iiint_{R(t)} \nabla \cdot (\rho \bar{v_f} \bar{v_f})\, dV_o + \iint_{S(t)} \rho \bar{v_f}(\bar{v_s} \cdot \bar{n}) dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}\, dV_o$$

The applying the Divergence Theorem substituting F with $\rho \bar{v_f} \bar{v_f}$

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot \rho \bar{v_f} \bar{v_f}\, dV_o = \iint_{S(t)} \rho \bar{v_f}(\bar{v_f} \cdot \bar{n}) dS$$

$$\frac{d}{dt}\iiint_{R(t)} \rho \bar{v_f}\, dV_o = -\iint_{S(t)} \rho \bar{v_f}(\bar{v_f} \cdot \bar{n}) dS + \iint_{S(t)} \rho \bar{v_f}(\bar{v_s} \cdot \bar{n}) dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}\, dV_o$$

$$\frac{d}{dt}\iiint_{R(t)} \rho \bar{v_f}\, dV_o = \iint_{S(t)} \rho \bar{v_f}[(\bar{v_s} - \bar{v_f}) \cdot \bar{n}] dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}\, dV_o$$

22.3 Governing Equation: Energy Equation

Chapter 23

Navier-Stokes Equations

The full set of the Navier-Stokes Equations are shown below. The three equations below correspond to the differential continuity, momentum, energy laws.

$$\frac{\partial}{\partial t}[\rho] + \bar{v_f} \cdot \nabla\rho = -\rho\nabla \cdot \bar{v_f}$$

$$\rho\left[\frac{\partial}{\partial t}[\bar{v_f}] + (\bar{v_f} \cdot \nabla)\bar{v_f}\right] = -\nabla P_r + \rho\bar{F_b} + \mu\nabla^2\bar{v_f} = -\nabla P_r + \rho\bar{F_b} - \frac{2}{3}\nabla(\mu\nabla \cdot \bar{v_f}) + 2\nabla \cdot (\mu S)$$

$$\rho c_p\left[\frac{\partial}{\partial t}(T) + \bar{v_f} \cdot \nabla T\right] = \nabla \cdot (k\nabla T) - \frac{2}{3}\mu(\nabla \cdot \bar{v_f})^2 + 2\mu S : S + \beta T\frac{D}{Dt}[P_r]$$

23.1 Hiemenz Flow

23.1.1 Cartesian Coordinates

23.1.2 Polar Coordinates

The continuity governing equation for incompressible fluids in cylindrical coordinates,

$$0 = \nabla \cdot \bar{v} = \frac{1}{r}\frac{\partial}{\partial r}[rv_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] + \frac{\partial}{\partial z}[v_z]$$

In 3-dimensions, the fluid velocity has a radial component, an azimuthal (swirl) component and a z component. For this derivation, it is assumed that the fluid does not have swirl and that the flow is axis-symmetric. All quantities are not expected to vary in the θ direction. The flow is also assumed to be steady. Let the fluid velocity vector $\bar{v}$ be defined below,

$$\bar{v} = \begin{bmatrix} v_r & v_\theta & v_z \end{bmatrix}^T$$

wherein the v_r, v_θ and v_z correspond to the fluid velocity components in the radial, azimuthal and z-direction respectively. The radial component of velocity for axis-symmetric potential flow is defined to be,

$$v_{r,i} = cr$$

Reiterating the continuity expression,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[rv_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] + \frac{\partial}{\partial z}[v_z]$$

Due to the axis-symmetric assumption, $\frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] = 0$. Substituting,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[rv_r] + \frac{\partial}{\partial z}[v_z]$$

Substituting the radial velocity for the inviscid flow into the continuity expression,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[r \times cr] + \frac{\partial}{\partial z}[v_z]$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[cr^2] + \frac{\partial}{\partial z}[v_z]$$

$$0 = \frac{1}{r} \times 2cr + \frac{\partial}{\partial z}[v_z]$$

$$0 = 2c + \frac{\partial}{\partial z}[v_z]$$

$$-2c = \frac{\partial}{\partial z}[v_z]$$

$$-2cz + k = v_z$$

wherein k is some constant to satisfy some boundary condition. Stagnation point is defined to be a point in the fluid where there is no velocity. Since a stagnation point occurs at $z = 0$, then the v_z velocity must also be zero at said point. Substituting,

$$0 + k = 0$$

From applying the boundary condition of the stagnation point, k can be safely neglected. Hence, the fluid velocity in the z-direction for inviscid flows,

$$-2cz = v_{z,i}$$

To make an "intelligent" guess on the modification performed for the fluid flow, the stream function for the inviscid flow must first be determined. Assume that the $\bar{A}$ is the velocity potential vector, whose components,

$$\bar{A} = \begin{bmatrix} A_r & A_\theta & A_z \end{bmatrix}^T$$

The relationship of the velocity vector field to the velocity potential vector,

$$\bar{v} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} = \nabla \times \bar{A} = \begin{bmatrix} \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \\ \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \end{bmatrix}$$

Since the problem is assumed to be axis-symmetric, and also no swirl, then

$$0 = \frac{\partial A_r}{\partial z} \quad , \quad 0 = \frac{\partial A_z}{\partial r}$$

Therefore,

$$0 = A_r \quad , \quad 0 = A_z$$

Let $A_\theta = \psi$ which would represent the stream function of this axis-symmetric fluid flow. Substituting for the simplification,

$$v_r = -\frac{\partial A_\theta}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

Susbtituting for ψ,

$$v_r = -\frac{\partial \psi}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

If ψ_i represents the stream function to the inviscid fluid flow,

$$v_{r,i} = -\frac{\partial \psi_i}{\partial z} \quad , \quad v_{z,i} = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

Substituting for the inviscid flow field that was determined earlier,

$$cr = -\frac{\partial \psi_i}{\partial z} \quad , \quad -2cz = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

Analyzing the velocity in the axis-direction,

$$-2cz = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

$$-2czr = \frac{\partial}{\partial r}(r\psi_i)$$

$$-2cz \int r\, dr = \int \frac{\partial}{\partial r}(r\psi_i)\, dr$$

$$-2cz \times \frac{1}{2}r^2 = \int d(r\psi_i)$$

$$-cz \times r^2 = r\psi_i + k(z)$$

$$-czr^2 = r\psi_i + k(z)$$

Analyzing the velocity in the radial direction,

$$cr = -\frac{\partial \psi_i}{\partial z}$$

$$-cr = \frac{\partial \psi_i}{\partial z}$$

Deriving the previous expression with respect to z,

$$\frac{\partial}{\partial z}[-czr^2] = \frac{\partial}{\partial z}[r\psi_i] + \frac{\partial}{\partial z}[k(z)]$$

$$-cr^2 = r\frac{\partial}{\partial z}[\psi_i] + k'(z)$$

Substituting for the derivative of ψ_i with respect to z,

$$-cr^2 = r \times -cr + k'(z)$$

$$-cr^2 = -cr^2 + k'(z)$$

$$k'(z) = 0$$

Hence, $k(z) = constant$. This shows that some constant can be added to the stream function and the resulting fluid flow would still be identical. Let $k(z) = 0$ purely for convenience,

$$-czr^2 = r\psi_i + k(z) = r\psi_i$$

$$-czr = \psi_i$$

This is the stream function for the inviscid flow. The stream function should be altered in order to satisfy the no-slip boundary condition on the walls. Making the minimum changes necessary to allow for this, z in the stream function of the inviscid flow is replaced with an arbitrar function $f(z)$. f is a function purely in z. Let ψ be the stream function to a fluid flow that satisfies the Navier-Stokes equations near a stagnation point,

$$\psi = -crf(z)$$

To simplify notation,

$$\psi = -crf$$

Finding the velocities with the modified stream function,

$$v_r = -\frac{\partial \psi}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

$$v_r = -\frac{\partial}{\partial z}(\psi) \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

$$v_r = -\frac{\partial}{\partial z}(-crf) \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(-cr^2 f)$$

$$v_r = crf' \quad , \quad v_z = \frac{1}{r} \times (-2crf)$$

$$v_r = crf' \quad , \quad v_z = -2cf$$

Although the fluid velocity fiield that obeys the Navier-Stokes equation would be different than the inviscid flow field near the wall, the velocity field should be identical to the inviscid flow field very far from the wall. Hence,

$$\lim_{z\to\infty} \left[v_{r,i}\right] = \lim_{z\to\infty} \left[crf'\right]$$

$$\lim_{z\to\infty} [cr] = \lim_{z\to\infty} [cr] \lim_{z\to\infty} \left[f'\right]$$

$$1 = \lim_{z\to\infty} \left[f'\right]$$

$$\lim_{z\to\infty} \left[v_{z,i}\right] = \lim_{z\to\infty} [-2cf]$$

$$\lim_{z\to\infty} [-2cz] = \lim_{z\to\infty} [-2cf]$$

$$\lim_{z\to\infty} [-2c] \lim_{z\to\infty} [z] = \lim_{z\to\infty} [-2c] \lim_{z\to\infty} [f]$$

$$\lim_{z\to\infty} [z] = \lim_{z\to\infty} [f]$$

To summarize the stream function of the flow field which is an exact-solution to the Navier-Stokes equation,

$$\psi = -crf$$

The resulting velocity components of the flow field with such a stream function,

$$v_r = crf' \quad , \quad v_z = -2cf$$

The limits of f such that the boundary conditions of the flow field matching with that of the invscid fluid flow very far away from the wall,

$$1 = \lim_{z \to \infty} \left[f'\right] \quad , \quad \lim_{z \to \infty} [z] = \lim_{z \to \infty} [f]$$

The laplacian of an arbitrary vector v in polar coordinates,

$$\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2}$$

The momentum equation in the radial direction,

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

Due to the steady state assumption, $\frac{\partial v_r}{\partial t} = 0$. Due to the axis-symmetric assumption, $\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} = 0$. Assuming the flow field does not have any swirl, $\frac{v_\theta^2}{r} = 0$ and $-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} = 0$. Neglecting any body force, $\rho g_r = 0$. Substituting,

$$\rho\left[v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2}\right]$$

Substituting the definition of v_r from the stream-function analysis,

$$\rho\left[crf'\frac{\partial}{\partial r}(crf') + v_z\frac{\partial}{\partial z}(crf')\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{crf'}{r^2}\right]$$

$$\rho\left[crf' \times cf' + v_z \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

$$\rho\left[c^2rf'^2 + v_z \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

Substituting for the velocity v_z,

$$\rho\left[c^2rf'^2 - 2cf \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

$$\rho\left[c^2rf'^2 - 2c^2rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

The laplacian of the radial velocity,

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_r}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2}$$

Substituting for the radial velocity v_r.

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial crf'}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 crf'}{\partial \theta^2} + \frac{\partial^2 crf'}{\partial z^2}$$

Due to the axis-symmetric assumption

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial crf'}{\partial r}\right] + \frac{\partial^2 crf'}{\partial z^2}$$

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[crf'\right] + \frac{\partial^2 crf'}{\partial z^2}$$

$$\nabla^2 v_r = \frac{1}{r} \times cf' + crf'''$$

$$\nabla^2 v_r = \frac{cf'}{r} + crf'''$$

Reiterating where we left off with the radial momentum,

$$\rho\left[c^2rf'^2 - 2c^2rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

Substituting the laplacian of radial velocity into the momentum equation in the radial direction,

$$\rho\left[c^2rf'^2 - 2c^2rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\frac{cf'}{r} + crf''' - \frac{cf'}{r}\right]$$

$$\rho\left[c^2rf'^2 - 2c^2rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[crf'''\right]$$

The above is the first ordinary differential equation which describes f. Unfortunately, the term $\frac{\partial p}{\partial r}$ cannot be ignored when considering the exact case because there is significant pressure changes in the fluid flow. If the exact solution of the Navier-Stokes mimics potential flow far away, and the inviscid solution does allow for significant pressure gradients, then pressure gradients must also be taken into account when considering the exact solution to the Navier-Stokes.

The momentum equation in the axial direction,

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

Just like before, due to the steady state assumption, $\frac{\partial v_z}{\partial t} = 0$. Due to the axis-symmetric assumption, $\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} = 0$. Neglecting any body force in the axial direction, $\rho g_z = 0$.

Substituting for these simplifications,

$$\rho\left[v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

Substituting for axial velocity v_z in terms of f,

$$v_z = -2cf$$

$$\rho\left[v_r\frac{\partial}{\partial r}(-2cf)-2cf\frac{\partial}{\partial z}(-2cf)\right]=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

Substituting for radial velocity,

$$v_r = crf'$$

$$\rho\left[crf'\frac{\partial}{\partial r}(-2cf)-2cf\frac{\partial}{\partial z}(-2cf)\right]=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

$$\rho\left[-2cf\times -2cf'\right]=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

$$\rho\left[4c^2ff'\right]=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

The laplacian of the axial velocity,

$$\nabla^2 v_z=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}$$

Due to the axis-symmetric assumption, $\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}=0$

$$\nabla^2 v_z=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]+\frac{\partial^2 v_z}{\partial z^2}$$

Substituting for radial velocity v_r and axial velocity v_z,

$$\nabla^2 v_z=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(-2cf)\right]+\frac{\partial^2}{\partial z^2}(-2cf)$$

$$\nabla^2 v_z=-2cf''$$

Substituting laplacian of axial velocity into the axial momentum equation,

$$\rho\left[4c^2ff'\right]=-\frac{\partial p}{\partial z}+\mu\times -2cf''$$

$$4\rho c^2ff'=-\frac{\partial p}{\partial z}-2\mu cf''$$

We now have 2 ordinary differential equations that describe f it is now possible to potentially solve the expressions by using differential equation 2 to determine pressure in terms of f and substituting into differential equation 1 to solve for f. Integrating differential equation 2,

$$4\rho c^2\int ff'\,dz=-\int\frac{\partial p}{\partial z}\,dz-2\mu c\int f''\,dz$$

$$4\rho c^2\int f\frac{df}{dz}\,dz=-\int\,dp-2\mu c\int f''\,dz$$

$$4\rho c^2\int f\,df=-\int\,dp-2\mu c\int f''\,dz$$

$$4\rho c^2\times\frac{1}{2}f^2=-p-2\mu cf'+k(r)$$

wherein $k(r)$ is a function that is purely in terms of r. This function appears as a consequence of integrating with respect to z.

$$2\rho c^2 f^2=-p-2\mu cf'+k(r)$$

Making pressure the subject of the expression,

$$p = -2\mu c f' - 2\rho c^2 f^2 + k(r)$$

The exact solution to the Navier-Stokes equation should match the inviscid solution very far away from the wall. Implying that the pressure for the exact solution and the inviscid solution must match infinitely far away from the wall. Taking the limits as z approaches ∞,

$$\lim_{z\to\infty}[p] = -2\mu c \lim_{z\to\infty}\left[f'\right] - 2\rho c^2 \lim_{z\to\infty}\left[f^2\right] + \lim_{z\to\infty}\left[k(r)\right]$$

Susbtituting the limits for f which was determined earlier,

$$1 = \lim_{z\to\infty}\left[f'\right] \quad , \quad \lim_{z\to\infty}[z] = \lim_{z\to\infty}[f]$$

$$\lim_{z\to\infty}[p] = -2\mu c - 2\rho c^2 \lim_{z\to\infty}\left[z^2\right] + \lim_{z\to\infty}\left[k(r)\right]$$

For values of $z \to \infty$,

$$p = -2\mu c - 2\rho c^2 z^2 + k(r)$$

For the inviscid fluid flow, bernoulli's equation can be used to determine pressure,

$$p_0 = p + \frac{1}{2}\rho|\bar{v}_i|^2$$

wherein $\bar{v}_i$ represents the local fluid velocity for inviscid flow. Considering that there is only radial and axial velocity in an axis-symetric problem,

$$|\bar{v}_i|^2 = v_{r,i}^2 + v_{z,i}^2$$

Substituting,

$$p_0 = p + \frac{1}{2}\rho\left[v_{r,i}^2 + v_{z,i}^2\right]$$

Making pressure the subject of the expression,

$$p = p_0 - \frac{1}{2}\rho\left[v_{r,i}^2 + v_{z,i}^2\right]$$

Substituting for the inviscid radial and axial velocities,

$$p = p_0 - \frac{1}{2}\rho\left[(cr)^2 + (-2cz)^2\right]$$

$$p = p_0 - \frac{1}{2}\rho\left[c^2r^2 + 4c^2z^2\right]$$

$$p = p_0 - \frac{1}{2}\rho c^2 r^2 - 2\rho c^2 z^2$$

Matching the exact solution's pressure to the inviscid solution's pressure far away from the wall,

$$p = -2\mu c - 2\rho c^2 z^2 + k(r) = p_0 - \frac{1}{2}\rho c^2 r^2 - 2\rho c^2 z^2$$

$$-2\mu c + k(r) = p_0 - \frac{1}{2}\rho c^2 r^2$$

$$k(r) = p_0 - \frac{1}{2}\rho c^2 r^2 + 2\mu c$$

Substituting the function k into the expression for pressure in the exact solution,

$$p = -2\mu c - 2\rho c^2 z^2 + p_0 - \frac{1}{2}\rho c^2 r^2 + 2\mu c$$

$$p = -2\rho c^2 z^2 + p_0 - \frac{1}{2}\rho c^2 r^2$$

Taking the derivative with respect to r,

$$\frac{\partial p}{\partial r} = -2\rho c^2 \frac{\partial}{\partial r}(z^2) + \frac{\partial p_0}{\partial r} - \frac{1}{2}\rho \frac{\partial}{\partial r}(c^2 r^2)$$

$$\frac{\partial p}{\partial r} = -\frac{1}{2}\rho \frac{\partial}{\partial r}(c^2 r^2)$$

$$\frac{\partial p}{\partial r} = -\frac{1}{2}\rho \times 2c^2 r$$

$$\frac{\partial p}{\partial r} = -\rho c^2 r$$

$$-\frac{\partial p}{\partial r} = \rho c^2 r$$

Substituting the gradient of pressure in r into differential equation 1,

$$\rho\left[c^2 r f'^2 - 2c^2 r f f''\right] = -\frac{\partial p}{\partial r} + \mu\left[c r f'''\right]$$

$$\rho\left[c^2 r f'^2 - 2c^2 r f f''\right] = \rho c^2 r + \mu c r f'''$$

$$\left[c^2 r f'^2 - 2c^2 r f f''\right] = c^2 r + \frac{\mu}{\rho} c r f'''$$

$$r f'^2 - 2 r f f'' = r + \frac{\mu}{\rho}\frac{1}{c} r f'''$$

The relation between kinematic and dynamic viscosity is shown below,

$$\nu = \frac{\mu}{\rho}$$

Substituting,

$$r f'^2 - 2 r f f'' = r + \nu \frac{1}{c} r f'''$$

$$f'^2 - 2 f f'' = 1 + \nu \frac{1}{c} f'''$$

$$-\frac{\nu}{c} f''' + f'^2 - 2 f f'' - 1 = 0$$

Let the function ϕ and variable η be defined below,

$$f = \left(\frac{2c}{\nu}\right)^{-1/2} \phi \quad , \quad \eta = \left(\frac{2c}{\nu}\right)^{1/2} z$$

By conjecture,

$$\frac{d^n}{dz^n} = \left[\left(\frac{2c}{\nu}\right)^{1/2}\right]^n \frac{d^n}{d\eta^n}$$

$$\frac{d^n}{dz^n} = 2^{n/2}\left(\frac{c}{\nu}\right)^{n/2}\frac{d^n}{d\eta^n}$$

Consider when $n = 1$,

$$\frac{d}{dz} = \frac{d}{d\eta} \times \frac{d\eta}{dz}$$

$$\frac{d}{dz} = \frac{d}{d\eta} \times \frac{d}{dz}\left[\left(\frac{2c}{\nu}\right)^{1/2} z\right]$$

$$\frac{d}{dz} = 2^{1/2}\left(\frac{c}{\nu}\right)^{1/2}\frac{d}{d\eta}$$

Let $n = k + 1$

$$\frac{d}{dz}\left[\frac{d^k}{dz^k}\right] = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2}\frac{d^{k+1}}{d\eta^{k+1}}$$

Let LHS and RHS be defined,

$$LHS = \frac{d}{dz}\left[\frac{d^k}{dz^k}\right] \quad , \quad RHS = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2}\frac{d^{k+1}}{d\eta^{k+1}}$$

$$LHS = \frac{d}{dz}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2}\frac{d^k}{d\eta^k}\right]$$

$$LHS = \frac{d}{d\eta}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2}\frac{d^k}{d\eta^k}\right] \times \frac{d\eta}{dz}$$

$$LHS = \frac{d}{d\eta}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2}\frac{d^k}{d\eta^k}\right] \times 2^{1/2}\left(\frac{c}{\nu}\right)^{1/2}$$

$$LHS = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2}\frac{d}{d\eta}\left[\frac{d^k}{d\eta^k}\right]$$

Since $LHS = RHS$, by principle of mathematical induction, the formula is true. Applying to the exact solution,

$$-\frac{\nu}{c}f''' + f'^2 - 2ff'' - 1 = 0$$

$$-\frac{\nu}{c}f''' = -\frac{\nu}{c} \times 2^{3/2}\left(\frac{c}{\nu}\right)^{3/2}\frac{d^3}{d\eta^3}\left[2^{-1/2}\left(\frac{c}{\nu}\right)^{-1/2}\phi\right]$$

$$-\frac{\nu}{c}f''' = -2^{3/2-1/2}\left(\frac{c}{\nu}\right)^{3/2-1-1/2}\frac{d^3}{d\eta^3}[\phi]$$

$$-\frac{\nu}{c}f''' = -2\frac{d^3}{d\eta^3}[\phi]$$

$$-\frac{\nu}{c}f''' = -2\phi'''$$

$$f'^2 = \left\{ 2^{1/2} \left(\frac{c}{\nu}\right)^{1/2} \frac{d}{d\eta} \left[2^{-1/2} \left(\frac{c}{\nu}\right)^{-1/2} \phi \right] \right\}^2$$

$$f'^2 = \left\{ \frac{d}{d\eta} [\phi] \right\}^2$$

$$f'^2 = \{\phi'\}^2$$

$$f'^2 = \phi'^2$$

$$-2ff'' = -(2)2^{-1/2} \left[\left(\frac{c}{\nu}\right)^{-1/2} \phi \right] 2 \left(\frac{c}{\nu}\right) \frac{d^2}{d\eta^2} \left[2^{-1/2} \left(\frac{c}{\nu}\right)^{-1/2} \phi \right]$$

$$-2ff'' = -2^{1-1/2+1-1/2} \left[\left(\frac{c}{\nu}\right)^{-1/2+1-1/2} \phi \right] \frac{d^2}{d\eta^2} [\phi]$$

$$-2ff'' = -2\phi\phi''$$

Substituting all the terms together,

$$-2\phi''' + \phi'^2 - 2\phi\phi'' - 1 = 0$$

$$-2\phi''' - 2\phi\phi'' + \phi'^2 - 1 = 0$$

$$\phi''' + \phi\phi'' - \frac{1}{2}\phi'^2 + \frac{1}{2} = 0$$

Reiterating the radial and axial velocities based on the modified stream function,

$$v_r = crf' \quad , \quad v_z = -2cf$$

At the stagntion point $z = 0$ and since η is a linear function of z, $z = 0$. The stagnation point is defined as a point in the fluid where fluid velocity is zero. Therefore,

$$0 = crf'(0) \quad , \quad 0 = -2cf(0)$$

$$0 = f'(0) \quad , \quad 0 = -f(0)$$

Both derivative and function f have linear scaling with ϕ' and ϕ respectively. Therefore,

$$\phi(0) = 0 \quad , \quad \phi'(0) = 0$$

Infinitely far away,

$$v_r = cr = crf'$$

$$1 = f'(\infty)$$

Taking the derivative of f,

$$\frac{df}{dz} = \left(\frac{2c}{\nu}\right)^{1/2} \frac{d}{d\eta} \left[\left(\frac{2c}{\nu}\right)^{-1/2} \phi \right]$$

$$\frac{df}{dz} = \phi'(\eta)$$

Substituting for infinite distance away,

$$\frac{df}{dz}_{z=\infty} = \phi'(\eta = \infty)$$

$$1 = \phi'(\infty)$$

23.2 Alternate Forms

23.2.1 Strain And Rotation

23.2.1.1 Continuity: Index Notation

The continuity and momentum equations in vector form,

$$\nabla \cdot u = 0 \quad , \quad \frac{Du}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u$$

The continuity equation in index form,

$$0 = \frac{\partial u_j}{\partial x_j}$$

Simplifying the momentum equation in vector form,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u$$

Converting the momentum equation into index form,

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(u_i)$$

23.2.1.2 Momentum: Velocity Gradient

Renaming the dummy indices in the index momentum equation $j \to k$,

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(u_i)$$

Taking the derivative of the index momentum equation with respect to x_j,

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}(u_i) + \frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(p) + \nu\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(u_i)$$

Here, the fluid is assumed to be incompressible, hence ρ is a simple known fluid property. Since the partial derivative operator is commutative,

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x_j}(u_i) + \frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p) + \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}(u_i)$$

Substituting $e_{ij} = \dfrac{\partial u_i}{\partial x_j}$,

$$\frac{\partial}{\partial t}e_{ij} + \frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p) + \nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}e_{ij}$$

Simplifying the convective acceleration term by applying chain rule,

$$\frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = u_k\frac{\partial}{\partial x_j}\left[\frac{\partial u_i}{\partial x_k}\right] + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}[u_k]$$

Due to the partial derivative operator being commutative,

$$\frac{\partial}{\partial x_j}\left[u_k\frac{\partial u_i}{\partial x_k}\right] = u_k\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}(u_i) + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}[u_k]$$

Substituting $e_{ij} = \dfrac{\partial u_i}{\partial x_j}$,

$$\frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = u_k \frac{\partial}{\partial x_k} e_{ij} + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}\left[u_k\right]$$

Based on the definition of e_{ij},

$$e_{ik} = \frac{\partial u_i}{\partial x_k} \quad , \quad e_{kj} = \frac{\partial u_k}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = u_k \frac{\partial}{\partial x_k} e_{ij} + e_{ik}e_{kj}$$

Susbtituting the convective acceleration into the momentum equation,

$$\frac{\partial}{\partial t} e_{ij} + u_k \frac{\partial}{\partial x_k} e_{ij} + e_{ik}e_{kj} = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k} e_{ij}$$

Manipulating the equation further,

$$\frac{\partial}{\partial t} e_{ij} + u_k \frac{\partial}{\partial x_k} e_{ij} = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj} + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k} e_{ij}$$

$$\frac{\partial}{\partial t}(e_{ij}) + u_k \frac{\partial}{\partial x_k}(e_{ij}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj}$$

23.2.1.3 Strain Rate Form

Reiterating the momentum equation in index form,

$$\frac{\partial}{\partial t}(e_{ij}) + u_k \frac{\partial}{\partial x_k}(e_{ij}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj}$$

Renaming the indices, $i \to j$, and $j \to i$,

$$\frac{\partial}{\partial t}(e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ji}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_j}(p) - e_{jk}e_{ki}$$

Adding the 2 equations above together and taking into account that $\dfrac{\partial^2}{\partial x_i \partial x_j}(p) = \dfrac{\partial^2}{\partial x_j \partial x_i}(p)$,

$$\frac{\partial}{\partial t}(e_{ij} + e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) = -\frac{2}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj} - e_{jk}e_{ki}$$

$$\frac{\partial}{\partial t}(e_{ij} + e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) = -\frac{2}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - (e_{ik}e_{kj} + e_{jk}e_{ki})$$

Multiplying both sides by half,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(e_{ij} + e_{ji})\right] + u_k \frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij} + e_{ji})\right] - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij} + e_{ji})\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - \frac{1}{2}(e_{ik}e_{kj} + e_{jk}e_{ki})$$

The symmetric strain rate tensor is defined as, $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. Expressing the symmetric strain rate tensor in terms of e_{ij} and e_{ji}, $S_{ij} = \frac{1}{2}\left(e_{ij} + e_{ji}\right)$ Substituting for the symmetric strain rate tensor into the momentum equation,

$$\frac{\partial}{\partial t}\left[S_{ij}\right] + u_k \frac{\partial}{\partial x_k}\left[S_{ij}\right] - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[S_{ij}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - \frac{1}{2}(e_{ik}e_{kj} + e_{jk}e_{ki})$$

Simplifying further,

$$\left\{\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k} - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\right\} S_{ij} = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - \frac{1}{2}(e_{ik}e_{kj} + e_{jk}e_{ki})$$

$\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}$ represents the substantive derivative in index notation meanwhile $\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}$ represents the laplacian in index notation.

23.2.1.4 Rotation Rate Form

Performing the same steps as the previous part but instead of adding 2 equations together, the equations are subtracted off each other,

$$\frac{\partial}{\partial t}(e_{ij} - e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ij} - e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij} - e_{ji}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik}e_{kj} + e_{jk}e_{ki}$$

Since the partial differential operators are commutative, $-\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) = 0$. Substituting,

$$\frac{\partial}{\partial t}(e_{ij} - e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ij} - e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij} - e_{ji}) = -e_{ik}e_{kj} + e_{jk}e_{ki}$$

Multiplying both sides by half,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(e_{ij} - e_{ji})\right] + u_k \frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij} - e_{ji})\right] - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij} - e_{ji})\right] = -\frac{1}{2}\left[e_{ik}e_{kj} - e_{jk}e_{ki}\right]$$

The rotation rate tensor is defined as $\Omega_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})$. Substituting for $e_{ij} = \frac{\partial u_i}{\partial x_j}$ and $e_{ji} = \frac{\partial u_j}{\partial x_i}$, $\Omega_{ij} = \frac{1}{2}(e_{ij} - e_{ji})$ Substituting the rotation rate tensor,

$$\frac{\partial}{\partial t}\left[\Omega_{ij}\right] + u_k \frac{\partial}{\partial x_k}\left[\Omega_{ij}\right] - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\Omega_{ij}\right] = -\frac{1}{2}\left[e_{ik}e_{kj} - e_{jk}e_{ki}\right]$$

$$\left\{\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k} - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\right\} \Omega_{ij} = -\frac{1}{2}\left[e_{ik}e_{kj} - e_{jk}e_{ki}\right]$$

Just as before, $\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}$ represents the substantive derivative in index notation meanwhile $\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}$ represents the laplacian in index notation. Interestingly, here the expression is independent of the pressure gradient tensor sinc ethe pressure gradient tensor is symmetric.

23.2.2 Vorticity Equation

23.2.2.1 Derivation

The substantive derivative of fluid velocity $\bar{v_f}$ appears in the non-conservative form of the momentum equation. The substantive derivative of fluid velocity $\bar{v_f}$ in vector form,

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f}$$

By conjecture,

$$\bar{v_f} \cdot \nabla \bar{v_f} = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Let,

$$LHS = \bar{v_f} \cdot \nabla \bar{v_f} \quad , \quad RHS = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

wherein $\bar{\omega_f}$ represents the fluid vorticity. Fluid vorticity is defined as the curl of fluid velocity, $\bar{\omega_f} = \nabla \times \bar{v_f}$.
Expressing LHS in index notation,

$$LHS_i = v_j \frac{\partial v_i}{\partial x_j}$$

wherein v_i represents the i^{th} component of the velocity vector $\bar{v_f}$.
Expressing RHS in index notation,

$$RHS_i = \epsilon_{ijk} \omega_j v_k + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

wherein ω_j represents the j^{th} component of fluid vorticity vector $\bar{\omega_f}$. Expressing the definition of vorticity as curl of fluid velocity in index notation,

$$\omega_j = \epsilon_{jlm} \frac{\partial v_m}{\partial x_l}$$

Substituting ω_j into RHS_i,

$$RHS_i = \epsilon_{ijk} \epsilon_{jlm} \frac{\partial v_m}{\partial x_l} v_k + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = \epsilon_{ijk} \epsilon_{jlm} v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

Based on the cyclic permutation properties of the permutation tensors ϵ_{ijk}, $\epsilon_{ijk} = \epsilon_{jki}$.
Therefore,

$$\epsilon_{ijk} \epsilon_{jlm} = \epsilon_{jki} \epsilon_{jlm}$$

Based on the double permutation tensor identity,

$$\epsilon_{ijk} \epsilon_{jlm} = \epsilon_{jki} \epsilon_{jlm} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il}$$

Substituting into RHS_i,

$$RHS_i = [\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}] v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = \delta_{kl} \delta_{im} v_k \frac{\partial v_m}{\partial x_l} - \delta_{km} \delta_{il} v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

$$RHS_i = v_l\frac{\partial v_i}{\partial x_l} - v_k\frac{\partial v_k}{\partial x_i} + \frac{\partial}{\partial x_i}\left[\frac{1}{2}v_jv_j\right] = v_l\frac{\partial v_i}{\partial x_l} - v_k\frac{\partial v_k}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial x_i}\left[v_jv_j\right]$$

$$RHS_i = v_l\frac{\partial v_i}{\partial x_l} - v_k\frac{\partial v_k}{\partial x_i} + \frac{1}{2}\left[v_j\frac{\partial}{\partial x_i}\left(v_j\right) + v_j\frac{\partial}{\partial x_i}\left(v_j\right)\right] = v_l\frac{\partial v_i}{\partial x_l} - v_k\frac{\partial v_k}{\partial x_i} + v_j\frac{\partial v_j}{\partial x_i} = v_l\frac{\partial v_i}{\partial x_l}$$

Renaming the dummy index $l \to j$,

$$RHS_i = v_j\frac{\partial v_i}{\partial x_j}$$

Since $LHS_i = RHS_i$, then the conjecture shown below must be true,

$$\bar{v_f}\cdot\nabla\bar{v_f} = \bar{\omega_f}\times\bar{v_f} + \nabla\left(\frac{1}{2}\bar{v_f}\cdot\bar{v_f}\right)$$

Susbtituting into the substantive derivative of fluid velocity,

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial\bar{v_f}}{\partial t} + \bar{v_f}\cdot\nabla\bar{v_f} = \frac{\partial\bar{v_f}}{\partial t} + \bar{\omega_f}\times\bar{v_f} + \nabla\left(\frac{1}{2}\bar{v_f}\cdot\bar{v_f}\right)$$

Taking the curl of the substantive derivative of fluid velocity,

$$\nabla\times\left(\frac{D\bar{v_f}}{Dt}\right) = \nabla\times\left(\frac{\partial\bar{v_f}}{\partial t}\right) + \nabla\times\left(\bar{\omega_f}\times\bar{v_f}\right) + \nabla\times\left[\nabla\left(\frac{1}{2}\bar{v_f}\cdot\bar{v_f}\right)\right]$$

The curl operations and the partial derivative operations are commutative. Therefore,

$$\nabla\times\left(\frac{\partial\bar{v_f}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\nabla\times\bar{v_f}\right)$$

Substituting for the definition of fluid vorticity $\bar{\omega_f} = \nabla\times\bar{v_f}$,

$$\nabla\times\left(\frac{\partial\bar{v_f}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\bar{\omega_f}\right) = \frac{\partial\bar{\omega_f}}{\partial t}$$

Substituting into the curl of fluid velocity substantive derivative,

$$\nabla\times\left(\frac{D\bar{v_f}}{Dt}\right) = \frac{\partial\bar{\omega_f}}{\partial t} + \nabla\times\left(\bar{\omega_f}\times\bar{v_f}\right) + \nabla\times\left[\nabla\left(\frac{1}{2}\bar{v_f}\cdot\bar{v_f}\right)\right]$$

$\bar{v_f}\cdot\bar{v_f}$ is a scalar. The curl of a scalar gradient is zero. Therefore,

$$0 = \nabla\times\left[\nabla\left(\frac{1}{2}\bar{v_f}\cdot\bar{v_f}\right)\right]$$

Negletcting the $\nabla\times\left[\nabla\left(\frac{1}{2}\bar{v_f}\cdot\bar{v_f}\right)\right]$ term,

$$\nabla\times\left(\frac{D\bar{v_f}}{Dt}\right) = \frac{\partial\bar{\omega_f}}{\partial t} + \nabla\times\left(\bar{\omega_f}\times\bar{v_f}\right)$$

This is a vector identity,

$$\nabla\times(\bar{A}\times\bar{B}) = \bar{B}\cdot\nabla\bar{A} + \bar{A}\nabla\cdot\bar{B} - \bar{A}\cdot\nabla\bar{B} - \bar{B}\nabla\cdot\bar{A}$$

Let $\bar{A} = \bar{\omega_f}$ and $\bar{B} = \bar{v_f}$,

$$\nabla \times (\bar{\omega_f} \times \bar{v_f}) = \bar{v_f} \cdot \nabla \bar{\omega_f} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f} - \bar{v_f} \nabla \cdot \bar{\omega_f}$$

Based on the definition of fluid vorticity $\bar{\omega_f} = \nabla \times \bar{v_f}$,

$$\bar{v_f} \nabla \cdot \bar{\omega_f} = \bar{v_f} \nabla \cdot (\nabla \times \bar{v_f})$$

Since the divergence of a vector field curl is zero, $\nabla \cdot (\nabla \times \bar{v_f}) = 0$. Therefore,

$$0 = \bar{v_f} \nabla \cdot \bar{\omega_f}$$

Neglecting the $\bar{v_f} \nabla \cdot \bar{\omega_f}$ term,

$$\nabla \times (\bar{\omega_f} \times \bar{v_f}) = \bar{v_f} \cdot \nabla \bar{\omega_f} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$

Substituting $\nabla \times (\bar{\omega_f} \times \bar{v_f})$, into the substantive derivative of vorticity,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{\partial \bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{\omega_f} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$

The definition of substantive derivative of vorticity is shown below,

$$\frac{D\bar{\omega_f}}{Dt} = \frac{\partial \bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{\omega_f}$$

Substituting for the substantive derivative of vorticity into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$

The symmetric strain rate tensor is defined as,

$$S_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

The symmetric strain rate tensor is symmetric.
The rotation rate tensor is defined as,

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

The rotation rate tensor is anti-symmetric.
The summation of the strain rate tensor and the rotation rate tensor,

$$S_{ij} + \Omega_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j} \right)$$

$$S_{ij} + \Omega_{ij} = \frac{\partial v_i}{\partial x_j}$$

Therefore, this shows that the fluid velocity gradient tensor $\frac{\partial v_i}{\partial x_j}$ can be decomposed into an algebraic sum of a symmetric tensor S_{ij} and an anti-symmetric tensor Ω_{ij}.

The rotation rate tensor is somewhat related to the fluid vorticity vector. Consider i^{th} component of the fluid vorticity vector the i^{th} component of fluid velocity curl,

$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Using the fluid vorticity to contract the permutation tensor on along its third dimension,

$$\epsilon_{lmi}\omega_i = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Due to the permutation cyclic property, $\epsilon_{lmi} = \epsilon_{ilm}$. Therefore,

$$\epsilon_{lmi}\epsilon_{ijk} = \epsilon_{ilm}\epsilon_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$$

Substituting into the permutation tensor contraction,

$$\epsilon_{lmi}\omega_i = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j} = [\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}]\frac{\partial v_k}{\partial x_j} = \delta_{lj}\delta_{mk}\frac{\partial v_k}{\partial x_j} - \delta_{lk}\delta_{mj}\frac{\partial v_k}{\partial x_j} = \delta_{mk}\frac{\partial v_k}{\partial x_l} - \delta_{mj}\frac{\partial v_l}{\partial x_j}$$

$$\epsilon_{lmi}\omega_i = \frac{\partial v_m}{\partial x_l} - \frac{\partial v_l}{\partial x_m}$$

Under an index varible change $l \to i$, $m \to j$, $i \to k$,

$$\epsilon_{ijk}\omega_k = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}$$

This form is already similar to the rotation rate tensor. In essence, contracting the permutation tensor along any dimension would allow the usage of the double permutation tensor identity. The third dimension was chosen in order to obtain the 'alternating' pattern similar to the rotation rate tensor. Minor algebraic manipulatins can then be performed to match $\epsilon_{ijk}\omega_k$ to the rotation rate tensor,

$$-\epsilon_{ijk}\omega_k = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$$

$$-\frac{1}{2}\epsilon_{ijk}\omega_k = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

The RHS matches the rotation rate tensor. Therefore,

$$-\frac{1}{2}\epsilon_{ijk}\omega_k = \Omega_{ij}$$

By conjecture, using the fluid vorticity vector $\bar{\omega_f}$ to contract the rotation rate tensor along the second dimension would yield zero. This claim is expressed in index notation,

$$0 = \omega_j\Omega_{ij}$$

Let

$$LHS_i = 0 \quad , \quad RHS_i = \omega_j\Omega_{ij}$$

Substituting the definition of the rotation rate tensor in terms of the fluid vorticity vector,

$$RHS_i = \omega_j\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k$$

Since ϵ_{ijk} is an anti-symmetric tensor, and $\omega_j\omega_k$ is a symmetric tensor due to multiplication being a commutative operation,

$$RHS_i = \omega_j\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k = 0$$

Therefore, the claim is proven to be true.
Reiterating the last form of the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

Converting the last term in RHS into index notation,

$$\left(\bar{\omega_f} \cdot \nabla\bar{v_f}\right)_i = \omega_j\frac{\partial v_i}{\partial x_j}$$

Expressing the velocity gradient tensor $\frac{\partial v_i}{\partial x_j}$ in terms of its symmetric and anti-symmetric components,

$$\left(\bar{\omega_f} \cdot \nabla\bar{v_f}\right)_i = \omega_j\frac{\partial v_i}{\partial x_j} = \omega_j[S_{ij} + \Omega_{ij}] = \omega_j S_{ij} + \omega_j\Omega_{ij}$$

Based on previous work, the contraction of the rotation tensor on the second index using the fluid vorticity vector yields zero,

$$0 = \omega_j\Omega_{ij}$$

Neglecting the $\omega_j\Omega_{ij}$ term,

$$\left(\bar{\omega_f} \cdot \nabla\bar{v_f}\right)_i = \omega_j S_{ij}$$

Converting into vector notation,

$$\bar{\omega_f} \cdot \nabla\bar{v_f} = \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

wherein $\bar{\bar{S}}_f$ represents the symmetric strain rate tensor for the fluid velocity vector field.
Substituting into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

The differential continuity equation is shown below,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho\bar{v_f})$$

Converting the differential continuity equation into tensor index notation,

$$0 = \frac{\partial}{\partial t}[\rho] + \frac{\partial}{\partial x_j}(\rho v_j)$$

Using chain rule,

$$0 = \frac{\partial}{\partial t}[\rho] + \rho\frac{\partial}{\partial x_j}(v_j) + v_j\frac{\partial}{\partial x_j}(\rho)$$

Manipulating further,

$$\rho\frac{\partial}{\partial x_j}(v_j) = -\frac{\partial}{\partial t}[\rho] - v_j\frac{\partial}{\partial x_j}(\rho)$$

$$\frac{\partial}{\partial x_j}(v_j) = -\frac{1}{\rho}\left[\frac{\partial}{\partial t}(\rho) + v_j\frac{\partial}{\partial x_j}(\rho)\right]$$

The substantive derivative of fluid density ρ in vector notation,

$$\frac{D}{Dt}(\rho) = \frac{\partial}{\partial t}(\rho) + \bar{v_f}\cdot\nabla\rho$$

Converting the substatntive derivative of fluid density into index notation,

$$\left[\frac{D}{Dt}(\rho)\right]_i = \frac{\partial}{\partial t}(\rho) + v_j\frac{\partial}{\partial x_j}(\rho)$$

Substituting,

$$\frac{\partial}{\partial x_j}(v_j) = -\frac{1}{\rho}\left[\frac{D}{Dt}(\rho)\right]_i$$

Converting into vector index notation,

$$\nabla\cdot\bar{v_f} = -\frac{1}{\rho}\frac{D}{Dt}(\rho)$$

Substituting the divergence of fluid velocity $\bar{v_f}$ into the fluid velocity substantive derivative,

$$\nabla\times\left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho) - \bar{\omega_f}\cdot\bar{\bar{S_f}}$$

By conjecture,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho)$$

Let

$$LHS = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] \quad , \quad RHS = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho)$$

Using quotient rule,

$$LHS = \rho\times\frac{1}{\rho^2}\left\{\rho\frac{D}{Dt}\left[\omega_f\right] - \bar{\omega_f}\frac{D}{Dt}\left[\rho\right]\right\} = \frac{1}{\rho}\left\{\rho\frac{D}{Dt}\left[\omega_f\right] - \bar{\omega_f}\frac{D}{Dt}\left[\rho\right]\right\} = \frac{1}{\rho}\rho\frac{D}{Dt}\left[\omega_f\right] - \frac{1}{\rho}\bar{\omega_f}\frac{D}{Dt}\left[\rho\right]$$

$$LHS = \frac{D}{Dt}\left[\omega_f\right] - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}\left[\rho\right]$$

Since $LHS = RHS$, the claim is proven.
Substituting for this simplification,

$$\nabla\times\left(\frac{D\bar{v_f}}{Dt}\right) = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] - \bar{\omega_f}\cdot\bar{\bar{S_f}}$$

The non-conservative form of the momentum equation,

$$\rho\frac{D\bar{v}_f}{Dt} = \nabla\cdot\bar{\bar{T}}_f + \bar{g}_b$$

Making the substantive derivative of fluid velocity the subject of the equation,

$$\frac{D\bar{v}_f}{Dt} = \frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f + \frac{1}{\rho}\bar{g}_b$$

Taking the curl of the resulting expression so that it might be substituted into the main equation,

$$\nabla\times\left(\frac{D\bar{v}_f}{Dt}\right) = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f + \frac{1}{\rho}\bar{g}_b\right] = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

Substituting the complete stress tensor and external acceleration into the main equation,

$$\nabla\times\left(\frac{D\bar{v}_f}{Dt}\right) = \rho\frac{D}{Dt}\left[\frac{\bar{\omega}_f}{\rho}\right] - \bar{\omega}_f\cdot\bar{\bar{S}}_f = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega}_f}{\rho}\right] - \bar{\omega}_f\cdot\bar{\bar{S}}_f = \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

Hence, the 'basic' vorticity equation,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega}_f}{\rho}\right] = \bar{\omega}_f\cdot\bar{\bar{S}}_f + \nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right] + \nabla\times\left[\frac{1}{\rho}\bar{g}_b\right]$$

The complete stress tensor $\bar{\bar{T}}_f$ defined in index notation,

$$T_{ij} = -P_r\delta_{ij} + \mu\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] + \lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}$$

wherein $\left(\bar{\bar{T}}_f\right)_{ij} = T_{ij}$, μ is the dynamic viscosity and λ is the second coefficient of viscosity. The viscous stress tensor $\bar{\bar{\tau}}_f$ has a rank of 2 and its ij component is referred as τ_{ij}. The viscous stress tensor components in index form is defined to be,

$$\tau_{ij} = \mu\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] + \lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}$$

Therefore, the complete stress tensor can be expressed in terms of the viscous stress tensor,

$$T_{ij} = -P_r\delta_{ij} + \tau_{ij}$$

Expressing the complete stress tensor term in index notation,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i = \epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(T_{ml})\right]$$

Renaming the indices $i\rightarrow m$ $j\rightarrow l$,

$$T_{ml} = -P_r\delta_{ml} + \tau_{ml}$$

Substituting for the complete stress tensor,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(-P_r\delta_{ml}+\tau_{ml})\right]$$

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(-P_r\delta_{ml})\right]+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

By applying the kronecker-delta contraction,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=-\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial P_r}{\partial x_l}\right]+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

By applying product rule on the pressure-related term,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=-\epsilon_{ijl}\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\frac{\partial P_r}{\partial x_l}\right]-\epsilon_{ijl}\frac{\partial P_r}{\partial x_l}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\right]+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

Due to the partial derivative operations being commutative $\frac{\partial}{\partial x_j}\left[\frac{\partial P_r}{\partial x_l}\right]$ is a symmetric tensor of rank 2. Since the permuation tensor ϵ_{ijl} is anti-symmetric,

$$0=-\epsilon_{ijl}\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\frac{\partial P_r}{\partial x_l}\right]$$

Neglecting the term,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=-\epsilon_{ijl}\frac{\partial P_r}{\partial x_l}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\right]+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

Applying chain rule, $\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\right]=-\frac{1}{\rho^2}\frac{\partial\rho}{\partial x_j}$. Substituting,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=\epsilon_{ijl}\frac{\partial P_r}{\partial x_l}\frac{1}{\rho^2}\frac{\partial\rho}{\partial x_j}+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]=\epsilon_{ijl}\frac{1}{\rho^2}\frac{\partial\rho}{\partial x_j}\frac{\partial P_r}{\partial x_l}+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

Converting into index notation,

$$\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]=\frac{1}{\rho^2}\nabla\rho\times\nabla P_r+\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{\tau}}_f\right]$$

Substituting into the basic vorticity equation,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right]=\bar{\omega_f}\cdot\bar{\bar{S}}_f+\frac{1}{\rho^2}\nabla\rho\times\nabla P_r+\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{\tau}}_f\right]+\nabla\times\left[\frac{1}{\rho}\bar{g_b}\right]$$

23.2.2.2 Summary

Basic Substantive derivative of fluid velocity

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f}$$

Kinematic Relations

Intermediate derivative kinetic energy form

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f} = \frac{\partial \bar{v_f}}{\partial t} + \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Double $\epsilon\epsilon$ Identity

$$\nabla \times \nabla \phi = 0, \quad \nabla \cdot (\nabla \times v) = 0$$

Vorticity 1^{st} relation

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{\partial \bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{\omega_f} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$

Definition of material derivative

3 term equation part 1

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$

Chapter 24

Potential Flows

Potential and inviscid flows are flows wherein the effects of viscosity is neglected. The degeneracy from the general Navier Stokes equation is shown below,
Consider the continuity differential governing equation,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v_f})$$

For a steady-state flow, $\frac{\partial}{\partial t}[\rho] = 0$. Substituting to the continuity differential governing equation,

$$0 = \nabla \cdot (\rho \bar{v_f})$$

24.1 Compressible Potential Flow

For the compressible potential flow, let the potential function ψ_c be defined as,

$$\rho \bar{v_f} = \nabla \psi_c$$

Substituting the definition of the potential function into the steady state continuity differential governing equation,

$$0 = \nabla \cdot (\nabla \psi_c) = \nabla^2 \psi_c$$

In cartesian coordinates, this yields,

$$0 = \frac{\partial^2}{\partial x^2}[\psi_c] + \frac{\partial^2}{\partial y^2}[\psi_c] + \frac{\partial^2}{\partial z^2}[\psi_c]$$

24.2 Incompressible Potential Flow

Due to the nature of the fluid, the density could be considered a scalar constant. The $\frac{\partial}{\partial t}[\rho]$ term in this case is evaluates to zero under two conditions: steady state means that the gradient is unchanging, but also since density is non-changing, this particular term also evaluates to zero. Therefore, for for incompressible flows, the potential flow function would still be applicable for non-steady fluid states.

$$0 = \nabla \cdot (\rho \bar{v_f}) = \rho \nabla \cdot (\bar{v_f})$$

Division of both sides by ρ,

$$0 = \nabla \cdot (\bar{v_f})$$

For the incompressible potential flow, let the potential function ψ_i be defined as,

$$\bar{v_f} = \nabla \psi_i$$

By substituting the fluid velocity field into the continuity degenerate differential form,

$$0 = \nabla \cdot (\nabla \psi_i) = \nabla^2 \psi_i = \frac{\partial^2}{\partial x^2}[\psi_i] + \frac{\partial^2}{\partial y^2}[\psi_i] + \frac{\partial^2}{\partial z^2}[\psi_i]$$

Chapter 25

Numerical Methods: Potential Flows

25.1 Potential Flow

25.1.1 Gauss-Siedel Grid Method

The Taylor expansion series for an arbitray function $f(t)$ is defined as the following,

$$f(t) = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [f(a)](x-a)^n \right\}$$

Let the control volume be split up into infinitesmially small grids. Each grid will have horizontal width of dx and a vertical height of dy. Let $\psi_{i,j}$ represent the i^{th} column and the j^{th} row value of the stream function. Columns are defined as the vertical edges of the infinitesmially small grids meanwhile rows are defined as the horizontal edges of the infinitesmailly small grids. Therefore, the analysis performed occurs at the edges of the infinitesmially small grids. Indexing starts from the bottom left corner of the control volume, at the origin of the declared coordinate system. Indices start at 0 and progress by increments of 1 along the x and y axes. Analyzing the Taylor Expansion series horizontally in the x-direction,

$$\psi_{i+1,j} = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](\Delta x)^n \right\}$$

For the $i-1^{th}$ term,

$$\psi_{i-1,j} = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](-\Delta x)^n \right\}$$

Taking the second order approximation, and neglecting the higher order terms,

$$\sum_{n=3}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](\Delta x)^n \right\} \approx \sum_{n=3}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](-\Delta x)^n \right\} \approx 0$$

Therefore, for the $i+1^{th}$ term and the $i-1^{th}$ term respectively,

$$\psi_{i+1,j} \approx \psi_{i,j} + \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 \quad , \quad \psi_{i-1,j} \approx \psi_{i,j} - \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

Adding the terms together and manipulating the equation to isolate the $\dfrac{d^2\psi_{i,j}}{dx^2}$ term,

$$\psi_{i+1,j} + \psi_{i-1,j} \approx \psi_{i,j} + \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 + \psi_{i,j} - \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

$$\psi_{i+1,j} + \psi_{i-1,j} \approx 2\psi_{i,j} + \frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

$$\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 \approx \psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}$$

$$\frac{d^2\psi_{i,j}}{dx^2} \approx \frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2}$$

Using the same process in the y-direction,

$$\frac{d^2\psi_{i,j}}{dy^2} \approx \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2}$$

Substituting the relevant terms into the governing equation,

$$\frac{\partial^2}{\partial x^2}[\psi] + \frac{\partial^2}{\partial y^2}[\psi] = 0$$

$$\frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2} = 0$$

Making $\psi_{i,j}$ the subject of the equation above,

$$(\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j})(\Delta x)^2 = 0$$

$$(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2 = 2\psi_{i,j}(\Delta y)^2 + 2\psi_{i,j}(\Delta x)^2$$

$$(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2 = 2[(\Delta y)^2 + (\Delta x)^2](\psi_{i,j})$$

$$\psi_{i,j} = \frac{(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2}{2[(\Delta y)^2 + (\Delta x)^2]}$$

The relative error ϵ of the stream function for the mesh corners are defined as

$$\left| \frac{\overset{p+1}{\psi_{i,j}} - \overset{p}{\psi_{i,j}}}{\overset{p+1}{\psi_{i,j}}} \right| = \epsilon$$

wherein $\overset{p+1}{\psi_{i,j}}$ represents the $p+1^{th}$ iteration of $\psi_{i,j}$ formulated through the Gauss-Seidel method.The relative error for the control volume would likewise be defined as

$$\frac{1}{k}\sqrt{\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\left(\frac{\overset{p+1}{\psi_{i,j}} - \overset{p}{\psi_{i,j}}}{\overset{p+1}{\psi_{i,j}}}\right)^2} = \epsilon_t \quad , \quad k = (n-1)(m-1)$$

The value of the error ϵ would represent the error tolerance and a direct numerical simulation of the potential flow algorithm should keep running until the error tolerance would be low enough. The values of n and m would be shown analytically below,

$$m = \frac{H}{dy} \quad , \quad n = \frac{L}{dx}$$

25.2 Unsteady Diffusion

Consider the grid definition shown below,

Consider the general form the of the unsteady diffusion problem shown below,

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla \cdot (\rho u \phi) = \nabla \cdot (\Gamma \nabla \phi) + S_\phi$$

$\frac{\partial}{\partial t}(\rho\phi)$ represents the unsteady term, $\nabla \cdot (\rho u \phi)$ represents the convection term, $\nabla \cdot (\Gamma \nabla \phi)$ represents the diffusion term and S_ϕ represents the source term. If the convection term is neglected, $\nabla \cdot (\rho u \phi) = 0$. Substituting for this,

$$\frac{\partial}{\partial t}(\rho\phi) = \nabla \cdot (\Gamma \nabla \phi) + S_\phi$$

Integrating the unsteady diffusion equation over a small time step Δt as well as over the control volume CV,

$$\int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial t}(\rho\phi)\, dV_o\, dt = \int_t^{t+\Delta t} \int_{CV} \nabla \cdot (\Gamma \nabla \phi) + S_\phi\, dV_o\, dt$$

For the 1-dimensional case $\nabla\phi = \frac{\partial}{\partial x}(\phi)$. Substituting for the gradient of ϕ, $\nabla \cdot \nabla\phi = \frac{\partial^2}{\partial x^2}(\phi)$. Substituting for this,

$$\int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial t}(\rho\phi)\, dV_o\, dt = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] + S_\phi\, dV_o\, dt$$

The order of integration is commutative for the independent spatial and time variables. Therefore,

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] + S_\phi\, dV_o\, dt$$

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] dV_o\, dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

The differential for volume $dV_o = Adx$. Substituting for this,

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] A\, dx\, dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

Due to the exact differential simplification for dx,

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w^e dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

Due to the exact differential simplification for dt,

$$\int_{CV} [\rho\phi]_t^{t+\Delta t}\, dV_o = \int_t^{t+\Delta t} \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

$$\int_{CV} [\rho\phi]_{t+\Delta t} - [\rho\phi]_t\, dV_o = \int_t^{t+\Delta t} \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

Assuming that the value of ϕ_P is the value of ϕ throughout the entirety of a single cell, then ϕ would be a constant with respect to the volumetric integral. Substituting for this,

$$\left\{[\rho\phi]_{t+\Delta t} - [\rho\phi]_t\right\}\Delta V_o = \int_t^{t+\Delta t}\left[\Gamma A\frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A\frac{\partial}{\partial x}(\phi)\right]_w dt + \int_t^{t+\Delta t}\int_{CV} S_\phi\, dV_o\, dt$$

Assuming uniformity throughout the grid, that ρ, A and Γ is constant from one cell to the next,

$$\left\{[\rho\phi]_{t+\Delta t} - [\rho\phi]_t\right\}\Delta V_o = \rho\left[\phi_{t+\Delta t} - \phi_t\right]\Delta V_o$$

With a slight abuse of notation, let ϕ_t be represented as $\phi_{t,o}$ and $\phi_{t+\Delta t}$ be represented as ϕ_t. Substituting for this,

$$\left\{[\rho\phi]_{t+\Delta t} - [\rho\phi]_t\right\}\Delta V_o = \rho\Delta V_o\left[\phi_t - \phi_{t,o}\right]$$

With the uniformity assumption,

$$\left[\Gamma A\frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A\frac{\partial}{\partial x}(\phi)\right]_w = \Gamma A\left[\frac{\partial}{\partial x}(\phi)\right]_e - \Gamma A\left[\frac{\partial}{\partial x}(\phi)\right]_w = \Gamma A\left\{\left[\frac{\partial}{\partial x}(\phi)\right]_e - \left[\frac{\partial}{\partial x}(\phi)\right]_w\right\}$$

The source term is assumed to be uniform throughout a single grid much like ϕ and is then simplified to,

$$\int_t^{t+\Delta t}\int_{CV} S_\phi\, dV_o\, dt = \int_t^{t+\Delta t} S_\phi\Delta V_o\, dt$$

Substituting for all of these simplifications,

$$\rho\left[\phi_{t+\Delta t} - \phi_t\right]\Delta V_o = \int_t^{t+\Delta t}\Gamma A\left\{\left[\frac{\partial}{\partial x}(\phi)\right]_e - \left[\frac{\partial}{\partial x}(\phi)\right]_w\right\} dt + \int_t^{t+\Delta t} S_\phi\Delta V_o\, dt$$

$$\rho\left[\phi_{t+\Delta t} - \phi_t\right]\Delta V_o = \Gamma A\int_t^{t+\Delta t}\left[\frac{\partial}{\partial x}(\phi)\right]_e - \left[\frac{\partial}{\partial x}(\phi)\right]_w dt + \Delta V_o\int_t^{t+\Delta t} S_\phi\, dt$$

25.2.0.1 Interior Cells

For the interior cells, the spatial derivative $\frac{\partial}{\partial x}(\phi)$ is computed using central differencing. Therefore,

$$\left[\frac{\partial}{\partial x}(\phi)\right]_e = \frac{\phi_E - \phi_P}{\delta x_{EP}} \quad , \quad \left[\frac{\partial}{\partial x}(\phi)\right]_w = \frac{\phi_P - \phi_W}{\delta x_{PW}}$$

Substituting for the definitions of these spatial derivatives,

$$\rho\left[\phi_{t+\Delta t} - \phi_t\right]\Delta V_o = \Gamma A\int_t^{t+\Delta t}\frac{\phi_E - \phi_P}{\delta x_{EP}} - \frac{\phi_P - \phi_W}{\delta x_{PW}}\, dt + \Delta V_o\int_t^{t+\Delta t} S_\phi\, dt$$

Re-arranging the equation,

$$\rho\left[\phi_{t+\Delta t} - \phi_t\right]\Delta V_o = \Gamma A\int_t^{t+\Delta t}\frac{1}{\delta x_{EP}}\phi_E - \frac{1}{\delta x_{EP}}\phi_P - \frac{1}{\delta x_{PW}}\phi_P + \frac{1}{\delta x_{PW}}\phi_W\, dt + \Delta V_o\int_t^{t+\Delta t} S_\phi\, dt$$

$$\rho\left[\phi_{t+\Delta t}-\phi_t\right]\Delta V_o=\Gamma A\int_t^{t+\Delta t}\frac{1}{\delta x_{EP}}\phi_E-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right]\phi_P+\frac{1}{\delta x_{PW}}\phi_W\,dt+\Delta V_o\int_t^{t+\Delta t}S_\phi\,dt$$

The time integration approximation for ϕ with respect to time is shown below,

$$\int_t^{t+\Delta t}\phi\,dt=[\theta\phi_{t+\Delta t}+(1-\theta)\phi_t]\Delta t$$

Using a change of notation,

$$\int_t^{t+\Delta t}\phi_i\,dt=[\theta\phi_i+(1-\theta)\phi_{i,o}]\Delta t$$

wherein the subscript i would represent some point in the grids, ϕ_i represents the value of ϕ at the new time step meanwhile $\phi_{i,o}$ represents the value of ϕ at the old time step. Substituting for the integration scheme,

$$\begin{aligned}\rho\left[\phi_t-\phi_{t,o}\right]\Delta V_o=&\Gamma A\Delta t\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+\Delta V_o\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

The finite volume could be expressed in terms of area A and grid size, $\Delta V_o=A\delta_{ew}$. Substituting for this,

$$\begin{aligned}\rho\left[\phi_t-\phi_{t,o}\right]A\delta_{ew}=&\Gamma A\Delta t\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+A\delta_{ew}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Dividing both sides by $A\Delta t$,

$$\begin{aligned}\rho\left[\phi_t-\phi_{t,o}\right]\frac{\delta_{ew}}{\Delta t}=&\Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Re-arranging the equation,

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_t-\frac{\rho\delta_{ew}}{\Delta t}\phi_{t,o}=&\Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_t-\frac{\rho\delta_{ew}}{\Delta t}\phi_{t,o}+\left[\frac{\Gamma}{\delta x_{EP}}+\frac{\Gamma}{\delta x_{PW}}\right][\theta\phi_P]=&\Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]\right.\\&-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][(1-\theta)\phi_{P,o}]\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Here ϕ_t represents ϕ_P, because the value of ϕ is assumed to be uniform throughout the cell.

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_P + \left[\frac{\Gamma}{\delta x_{EP}} + \frac{\Gamma}{\delta x_{PW}}\right][\theta\phi_P] =&\Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}]\right.\\ &-\left[\frac{1}{\delta x_{EP}} + \frac{1}{\delta x_{PW}}\right][(1-\theta)\phi_{P,o}]\\ &\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}]\right\} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt + \frac{\rho\delta_{ew}}{\Delta t}\phi_{P,o}\end{aligned}$$

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_P + \left[\frac{\Gamma}{\delta x_{EP}} + \frac{\Gamma}{\delta x_{PW}}\right][\theta\phi_P] =&\left\{\frac{\Gamma}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}]\right.\\ &-\left[\frac{\Gamma}{\delta x_{EP}} + \frac{\Gamma}{\delta x_{PW}}\right][(1-\theta)\phi_{P,o}]\\ &\left.+\frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}]\right\} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt + \frac{\rho\delta_{ew}}{\Delta t}\phi_{P,o}\end{aligned}$$

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{EP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] =&\frac{\Gamma}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}]\\ &+\left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{EP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}]\\ &+\frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

Let

$$a_W = \frac{\Gamma}{\delta x_{PW}} \quad , \quad a_E = \frac{\Gamma}{\delta x_{EP}} \quad , \quad a_{P,0} = \frac{\rho\delta_{ew}}{\Delta t} \quad , \quad a_P = a_{P,0} + a_E\theta + a_W\theta$$

Under the substitutions above, the expression becomes,

$$\begin{aligned}a_P\phi_P =&a_E[\theta\phi_E + (1-\theta)\phi_{E,o}] + \big[a_{P,0} - a_E(1-\theta) - a_W(1-\theta)\big][\phi_{P,o}]\\ &+ a_W[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

When evaluating ϕ at the new time step, the old time step would be considered a known quantity. Re-arranging the equation to place all of the known variables on LHS and the unknown variables at the RHS,

$$\begin{aligned}a_P\phi_P =&a_E\theta\phi_E + a_E(1-\theta)\phi_{E,o} + \big[a_{P,0} - a_E(1-\theta) - a_W(1-\theta)\big][\phi_{P,o}]\\ &+ a_W\theta\phi_W + a_W(1-\theta)\phi_{W,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

$$\begin{aligned}-a_E\theta\phi_E + a_P\phi_P - a_W\theta\phi_W =&a_E(1-\theta)\phi_{E,o} + \big[a_{P,0} - a_E(1-\theta) - a_W(1-\theta)\big][\phi_{P,o}]\\ &+ a_W(1-\theta)\phi_{W,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

25.2.0.2 Dirichlet Western Boundary Conditions

The expression for the interior cells before the declaration of definitions of a_W, a_E and substitution is shown below,

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P]=&\frac{\Gamma}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

When applying dirichlet boundary conditions to the cell that is closest to the origin of the coordinate system, several modifications must be made to the expression for the interior cells. Finite difference instead of central differencing is used at the left boundary of the first cell. This would mean the following change of variables to allow the formulation to be suitable at the boundaries,

$$\delta x_{PW}\to\delta x_{Pw}\quad,\quad\phi_W\to\phi_A\quad,\quad\phi_{W,o}\to\phi_{A,o}$$

The first change is to accomodate for the one-sided finite differencing at the west face, the second change is to accomodate for the boundary condition of ϕ_A at the wall at the new time step and the third change is to accomodate for the boundary condition of $\phi_{A,o}$ at the wall for the old time step. Substituting these changes into the expression,

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{Pw}}\theta\right][\phi_P]=&\frac{\Gamma}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{Pw}}[\theta\phi_A+(1-\theta)\phi_{A,o}]+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Placing the unknwon variables to the LHS and the known variables to RHS,

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{Pw}}\theta\right][\phi_P]=&\frac{\Gamma}{\delta x_{EP}}\theta\phi_E+\frac{\Gamma}{\delta x_{EP}}(1-\theta)\phi_{E,o}\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{Pw}}\theta\phi_A+\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\phi_{A,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{Pw}}\theta\right][\phi_P]-\frac{\Gamma}{\delta x_{EP}}\theta\phi_E=&\frac{\Gamma}{\delta x_{EP}}(1-\theta)\phi_{E,o}\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{Pw}}\theta\phi_A+\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\phi_{A,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Let

$$a_W=0\quad,\quad a_E=\frac{\Gamma}{\delta x_{EP}}\quad,\quad a_{P,0}=\frac{\rho\delta_{ew}}{\Delta t}\quad,\quad S_P=-\frac{\Gamma}{\delta x_{Pw}}\quad,\quad a_P=a_{P,0}+\theta(a_E+a_W-S_P)$$

Substituting these variables into the expression above,

$$-a_E\theta\phi_E + a_P\phi_P - a_W\phi_W = a_E(1-\theta)\phi_{E,o} + \left[a_{P,0} - a_E(1-\theta) + S_P(1-\theta)\right]\phi_{P,o} - S_P\theta\phi_A - S_P(1-\theta)\phi_{A,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt$$

25.2.0.3 Dirichlet Eastern Boundary Conditions

The expression for the interior cells before the declaration of definitions of a_W, a_E and substitution is shown below,

$$\left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{EP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] = \frac{\Gamma}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}] + \left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{EP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}] + \frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt$$

Much like in determining the eastern boundary condition, one-sided finite difference is employed to find the approximate derivative of ϕ with respect to displacement x at the boundary. For the eastern boundary, the unknown boundary would be at the east face of the last cell. The change of variables would then be,

$$\delta x_{EP} \to \delta x_{eP} \quad , \quad \phi_E = \phi_B \quad , \quad \phi_{E,o} = \phi_{B,o}$$

The first change of variable is to accomodate for the one-sided finite difference wherein the distance between the node point P and the eastern face is δx_{eP} instead of the previous case wherein the distance between the node point P and the eastern node is δx_{EP}. The second change of variable is to accomodate for the boundary condition wherein the value of ϕ at the right face of the last cell is ϕ_B. ϕ_B represents the value of ϕ at the new time step meanwhile $\phi_{B,o}$ represents the value of ϕ at the old time step at the right boundary of the last cell. Substituting these change of variables to modify the interior cell formulation for the right boundary condition,

$$\left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{eP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] = \frac{\Gamma}{\delta x_{eP}}[\theta\phi_B + (1-\theta)\phi_{B,o}] + \left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{eP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}] + \frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt$$

$$\left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{eP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] = \frac{\Gamma}{\delta x_{eP}}[\theta\phi_B + (1-\theta)\phi_{B,o}] + \left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{eP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}] + \frac{\Gamma}{\delta x_{PW}}\theta\phi_W + \frac{\Gamma}{\delta x_{PW}}(1-\theta)\phi_{W,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt$$

Placing the unknown variables at LHS and the known variables at RHS,

$$\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{eP}}\theta+\frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P]-\frac{\Gamma}{\delta x_{PW}}\theta\phi_W=\frac{\Gamma}{\delta x_{eP}}[\theta\phi_B+(1-\theta)\phi_{B,o}]$$
$$+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{eP}}(1-\theta)-\frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}]$$
$$+\frac{\Gamma}{\delta x_{PW}}(1-\theta)\phi_{W,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt$$

Let

$$a_W=\frac{\Gamma}{\delta x_{PW}}\quad,\quad a_E=0\quad,\quad a_{P,0}=\frac{\rho\delta_{ew}}{\Delta t}\quad,\quad S_P=-\frac{\Gamma}{\delta x_{eP}}\quad,\quad a_P=a_{P,0}+\theta(a_E+a_W-S_P)$$

$$-a_E\theta\phi_E+a_P\phi_P-a_W\theta\phi_W=-S_P[\theta\phi_B+(1-\theta)\phi_{B,o}]+\left[a_{P,0}+S_P(1-\theta)-a_W(1-\theta)\right][\phi_{P,o}]$$
$$+a_W(1-\theta)\phi_{W,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt$$

25.2.1 Part c

The workings in the previous section is performed with the arbitray scalar function ϕ, and the arbitrary variables ρ and Γ. The source term $\int_t^{t+\Delta t}S_\phi\,dt$ has not been specified. In this section, the arbitrary variables will be mapped to the problem variables to apply all the formulations to the fortran program.

Geometrically, the variable x could be re-interpreted in this problem to be vertical displacement y. This would set the western boundary condition of the arbitray ϕ case to be the bottom boundary condition in the momentum diffusion problem. The eastern boundary condition of the arbitrary ϕ case would be set to the top boundary condition in the momentum diffusion problem. Let positive direction of y then also be interpreted going "up" in the vertical direction. The top wall of the channel would be at positive H y-value. The unsteady momentum governing equation for this problem is shown below,

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

wherein ρ represents density and ν represents viscosity. The unsteady diffusion equation that is the basis of the previous formulation is shown below,

$$\frac{\partial}{\partial t}(\rho\phi)=\nabla\cdot(\Gamma\nabla\phi)+S_\phi$$

Under the 1-dimensional assumption and uniformity of ρ as well as Γ,

$$\rho\frac{\partial}{\partial t}(\phi)=\nabla\cdot(\Gamma\nabla\phi)+S_\phi=\frac{\partial}{\partial x}\left[\Gamma\frac{\partial}{\partial x}(\phi)\right]+S_\phi=\Gamma\frac{\partial^2}{\partial x^2}(\phi)+S_\phi$$

$$\frac{\partial}{\partial t}(\phi)=\frac{\Gamma}{\rho}\frac{\partial^2}{\partial x^2}(\phi)+\frac{S_\phi}{\rho}$$

By comparing the differential forms of the unsteady diffusion equation and unsteady momentum governing equation,

$$\phi=u\quad,\quad x=y\quad,\quad\frac{\Gamma}{\rho_\phi}=\nu\quad,\quad\frac{S_\phi}{\rho_\phi}=-\frac{1}{\rho_{prob}}\frac{\partial p}{\partial x}$$

Distinction has been made between ρ_ϕ and ρ_{prob}. ρ_ϕ represents the value of the arbitrary variable ρ in the general unsteady diffusion equation meanwhile ρ_{prob} represents the fluid density in the momentum diffusion problem. Since ϕ is a sclar field, and velocity u is typically a sclar, here u is assumed to be the magnitude of the horizontal velocity of the fluid, so that it could be represented by ϕ. In the momentum diffusion problem, the density of the fluid is assumed to remain constant and the derivative of pressure along the horizontal distance of the channel is assumed to be constant. Therefore, S_ϕ must also be a constant,

$$\frac{S_\phi}{\rho_\phi} = -\frac{1}{\rho_{prob}}\frac{\partial p}{\partial x}$$

$$S_\phi = -\frac{\rho_\phi}{\rho_{prob}}\frac{\partial p}{\partial x}$$

Given that S_ϕ is some constant that could be determined by the momentum diffusion problem definition,

$$\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt = \frac{\delta_{ew}}{\Delta t} S_\phi \left[t\right]_t^{t+\Delta t} = \frac{\delta_{ew}}{\Delta t} S_\phi \big[(t+\Delta t)-(t)\big] = \frac{\delta_{ew}}{\Delta t} S_\phi \Delta t = \delta_{ew} S_\phi$$

There exists a non-unique way to choose the variable mappings for ρ_ϕ, Γ. For a simple implementation of the general diffusion problem into the momentum diffusion, let $\rho_\phi = 1$. Therefore,

$$\phi = u \quad , \quad x = y \quad , \quad \Gamma = \nu \quad , \quad S_\phi = -\frac{1}{\rho_{prob}}\frac{\partial p}{\partial x}$$

The code was modified to simulate unsteady diffusion of momentum and given a few additional features. The modified fortran code is shown below,

25.3 Fourier Stability Analysis

25.3.1 Discretization of Wave Equation

The one-dimensional wave equation is shown below,

$$0 = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x}$$

Integrating with time and space,

$$0 = \int_t^{t+\Delta t}\int_{CV} \frac{\partial \phi}{\partial t}\, dV dt + u\int_t^{t+\Delta t}\int_{CV} \frac{\partial \phi}{\partial x}\, dV dt$$

Switching the order of integration in order to form exact differentials,

$$0 = \int_{CV}\int_t^{t+\Delta t} \frac{\partial \phi}{\partial t}\, dt\, dV + u\int_t^{t+\Delta t}\int_{CV} \frac{\partial \phi}{\partial x}\, dV dt$$

$$0 = \int_{CV} \left[\phi\right]_t^{t+\Delta t}\, dV + u\int_t^{t+\Delta t} A\left[\phi\right]_{CV}\, dt$$

$$0 = \left[\phi\right]_t^{t+\Delta t}\Delta V + u\int_t^{t+\Delta t} A\phi_e - A\phi_w\, dt$$

Substituting for the infinitesmially small volume $\Delta V = Adx$,

$$0 = [\phi]_t^{t+\Delta t} Adx + u \int_t^{t+\Delta t} A\phi_e - A\phi_w \, dt$$

Assuming uniform area throughout the grids,

$$0 = [\phi]_t^{t+\Delta t} dx + u \int_t^{t+\Delta t} \phi_e - \phi_w \, dt$$

Assuming that the value of ϕ is constant within a single cell,

$$0 = \left[\phi_P - \phi_{P,o}\right] dx + u \int_t^{t+\Delta t} \phi_e - \phi_w \, dt$$

Using the generalized time integration scheme,

$$\int_t^{t+\Delta t} \phi \, dt = [\theta\phi_{t+\Delta t} + (1-\theta)\phi_t]\Delta t$$

Substituting the time integration scheme,

$$0 = \left[\phi_P - \phi_{P,o}\right] dx + u \left\{\theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}\right\} dt$$

Factoring for the CFL number,

$$0 = \left[\phi_P - \phi_{P,o}\right] \frac{dx}{u\,dt} + \left\{\theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}\right\}$$

$$0 = \left[\phi_P - \phi_{P,o}\right] \frac{\Delta x}{u\,\Delta t} + \theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}$$

Let

$$\gamma = \frac{1}{CFL} = \frac{\Delta x}{u\,\Delta t}$$

$$0 = \gamma\phi_P - \gamma\phi_{P,o} + \theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}$$

Placing the new time step quantities on LHS and the old time step quantities on RHS,

$$-\gamma\phi_P + \theta\phi_w - \theta\phi_e = -\gamma\phi_{P,o} + (1-\theta)\phi_{e,o} - (1-\theta)\phi_{w,o}$$

25.3.2 Cubic Polynomial Spatial Scheme

The discretization of the pure convection problem with a generalized time advancement scheme and a cubic polynomial spatial advancement scheme is shown below,

$$\begin{aligned}
0 = \frac{\phi_P - \phi_{P,o}}{\Delta t} + \frac{u}{\Delta x}\Bigg[&\theta\left(\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E\right) \\
&+ (1-\theta)\left(\frac{1}{16}\phi_{WW,o} - \frac{5}{16}\phi_{W,o} + \frac{15}{16}\phi_{P,o} + \frac{5}{16}\phi_{E,o}\right) \\
&- \theta\left(\frac{1}{16}\phi_{WWW} - \frac{5}{16}\phi_{WW} + \frac{15}{16}\phi_W + \frac{5}{16}\phi_P\right) \\
&-(1-\theta)\left(\frac{1}{16}\phi_{WWW,o} - \frac{5}{16}\phi_{WW,o} + \frac{15}{16}\phi_{W,o} + \frac{5}{16}\phi_{P,o}\right)\Bigg]
\end{aligned}$$

Assuming that the solution $\phi(x,t)$ is a product of a time component $\hat{\phi}(t)$ and a spatial component e^{ikx}, the solution $\phi(x,t)$ could be rewritten as,

$$\phi(x,t) = \hat{\phi}(t)e^{ikx}$$

Points at the future time step would be given the time component shown below,

$$\hat{\phi}(t+\Delta t) = \hat{\phi}$$

Points at the old time step would be given the time component shown below,

$$\hat{\phi}(t) = \hat{\phi}_o$$

Following convention, the coordinate system considers eastern direction of the grid points to be positive x and western direction of the grid points to be negative x. The point of interest P would be given a spatial component e^{ikx}. Following the coordinate system definition, the west point would be given spatial component $e^{ik(x-\Delta x)}$. The east point would be given spatial component $e^{ik(x+\Delta x)}$. The diagram below summarizes the coordinate system used in this derivation,

Uniform grid spacing is assumed in the following derivation. Parsing the original expression into several parts. For the discrete derivative of ϕ with respect to time,

$$\frac{\phi_P - \phi_{P,o}}{\Delta t} = \frac{1}{\Delta t}\left[\phi_P - \phi_{P,o}\right] = \frac{1}{\Delta t}\left[\hat{\phi}e^{ikx} - \hat{\phi}_o e^{ikx}\right] = \frac{e^{ikx}}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right]$$

For the first part of RHS,

$$\begin{aligned}\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E =& \frac{1}{16}\hat{\phi}e^{ik(x-2\Delta x)} - \frac{5}{16}\hat{\phi}e^{ik(x-\Delta x)} \\ &+ \frac{15}{16}\hat{\phi}e^{ikx} + \frac{5}{16}\hat{\phi}e^{ik(x+\Delta x)}\end{aligned}$$

$$\begin{aligned}\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E =& \frac{1}{16}\hat{\phi}e^{ikx-2ik\Delta x} - \frac{5}{16}\hat{\phi}e^{ikx-ik\Delta x} \\ &+ \frac{15}{16}\hat{\phi}e^{ikx} + \frac{5}{16}\hat{\phi}e^{ikx+ik\Delta x}\end{aligned}$$

$$\begin{aligned}\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E =& \frac{1}{16}\hat{\phi}e^{ikx}e^{-2ik\Delta x} - \frac{5}{16}\hat{\phi}e^{ikx}e^{-ik\Delta x} \\ &+ \frac{15}{16}\hat{\phi}e^{ikx} + \frac{5}{16}\hat{\phi}e^{ikx}e^{ik\Delta x}\end{aligned}$$

$$\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E = \hat{\phi}e^{ikx}\left[\frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}\right]$$

Performing similar opeartions for the second part of RHS,

$$\frac{1}{16}\phi_{WW,o} - \frac{5}{16}\phi_{W,o} + \frac{15}{16}\phi_{P,o} + \frac{5}{16}\phi_{E,o} = \hat{\phi}_o e^{ikx}\left[\frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}\right]$$

Performing similar opeartions for the third part of RHS,

$$\frac{1}{16}\phi_{WWW} - \frac{5}{16}\phi_{WW} + \frac{15}{16}\phi_W + \frac{5}{16}\phi_P = \hat{\phi}e^{ikx}\left[\frac{1}{16}e^{-3ik\Delta x} - \frac{5}{16}e^{-2ik\Delta x} + \frac{15}{16}e^{-ik\Delta x} + \frac{5}{16}\right]$$

Performing similar opeartions for the fourth part of RHS,

$$\frac{1}{16}\phi_{WWW,o} - \frac{5}{16}\phi_{WW,o} + \frac{15}{16}\phi_{W,o} + \frac{5}{16}\phi_{P,o} = \hat{\phi}_o e^{ikx}\left[\frac{1}{16}e^{-3ik\Delta x} - \frac{5}{16}e^{-2ik\Delta x} + \frac{15}{16}e^{-ik\Delta x} + \frac{5}{16}\right]$$

Let

$$A = \frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x} \quad , \quad B = \frac{1}{16}e^{-3ik\Delta x} - \frac{5}{16}e^{-2ik\Delta x} + \frac{15}{16}e^{-ik\Delta x} + \frac{5}{16}$$

Substituting for A and B shortens the expressions,

$$\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_{W} + \frac{15}{16}\phi_{P} + \frac{5}{16}\phi_{E} = \hat{\phi}e^{ikx}A$$

$$\frac{1}{16}\phi_{WW,o} - \frac{5}{16}\phi_{W,o} + \frac{15}{16}\phi_{P,o} + \frac{5}{16}\phi_{E,o} = \hat{\phi}_o e^{ikx}A$$

$$\frac{1}{16}\phi_{WWW} - \frac{5}{16}\phi_{WW} + \frac{15}{16}\phi_{W} + \frac{5}{16}\phi_{P} = \hat{\phi}e^{ikx}B$$

$$\frac{1}{16}\phi_{WWW,o} - \frac{5}{16}\phi_{WW,o} + \frac{15}{16}\phi_{W,o} + \frac{5}{16}\phi_{P,o} = \hat{\phi}_o e^{ikx}B$$

Susbtituting the simplified parts into the original expression,

$$0 = \frac{e^{ikx}}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right] + \frac{u}{\Delta x}\left[\theta\left(\hat{\phi}e^{ikx}A\right) + (1-\theta)\left(\hat{\phi}_o e^{ikx}A\right) - \theta\left(\hat{\phi}e^{ikx}B\right) - (1-\theta)\left(\hat{\phi}_o e^{ikx}B\right)\right]$$

$$0 = \frac{e^{ikx}}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right] + \frac{u}{\Delta x}\left[\theta\hat{\phi}e^{ikx}A + (1-\theta)\hat{\phi}_o e^{ikx}A - \theta\hat{\phi}e^{ikx}B - (1-\theta)\hat{\phi}_o e^{ikx}B\right]$$

Since $e^{ikx} \neq 0$,

$$0 = \frac{1}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right] + \frac{u}{\Delta x}\left[\theta\hat{\phi}A + (1-\theta)\hat{\phi}_o A - \theta\hat{\phi}B - (1-\theta)\hat{\phi}_o B\right]$$

$$0 = \frac{\Delta x}{u\,\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right] + \theta\hat{\phi}A + (1-\theta)\hat{\phi}_o A - \theta\hat{\phi}B - (1-\theta)\hat{\phi}_o B$$

Let

$$\gamma = \frac{1}{CFL} = \frac{\Delta x}{u\,\Delta t}$$

Substituting for the parameter γ,

$$0 = \gamma\hat{\phi} - \gamma\hat{\phi}_o + \theta\hat{\phi}A + (1-\theta)\hat{\phi}_o A - \theta\hat{\phi}B - (1-\theta)\hat{\phi}_o B$$

$$\gamma\hat{\phi}_o - (1-\theta)\hat{\phi}_o A + (1-\theta)\hat{\phi}_o B = \gamma\hat{\phi} + \theta\hat{\phi}A - \theta\hat{\phi}B$$

$$\hat{\phi}\left[\gamma + \theta A - \theta B\right] = \hat{\phi}_o\left[\gamma - (1-\theta)A + (1-\theta)B\right]$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left[\gamma - (1-\theta)A + (1-\theta)B\right]}{\left[\gamma + \theta A - \theta B\right]} = \frac{\gamma + (\theta-1)A + (1-\theta)B}{\gamma + \theta A - \theta B} = \frac{\gamma + \theta A - A + B - \theta B}{\gamma + \theta A - \theta B}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - A + B + \theta A - \theta B}{\gamma + \theta A - \theta B} = \frac{\gamma - (A-B) + \theta(A-B)}{\gamma + \theta(A-B)}$$

Let $\alpha = A - B$. Simplifying,

$$\alpha = \frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x} - \left(\frac{1}{16}e^{-3ik\Delta x} - \frac{5}{16}e^{-2ik\Delta x} + \frac{15}{16}e^{-ik\Delta x} + \frac{5}{16}\right)$$

$$\alpha = \frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x} - \frac{1}{16}e^{-3ik\Delta x} + \frac{5}{16}e^{-2ik\Delta x} - \frac{15}{16}e^{-ik\Delta x} - \frac{5}{16}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \frac{1}{16}e^{-2ik\Delta x} + \frac{5}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} - \frac{15}{16}e^{-ik\Delta x} - \frac{5}{16} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \left[\frac{1}{16} + \frac{5}{16}\right] e^{-2ik\Delta x} - \left[\frac{5}{16} + \frac{15}{16}\right] e^{-ik\Delta x} - \frac{5}{16} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \left[\frac{6}{16}\right] e^{-2ik\Delta x} - \left[\frac{20}{16}\right] e^{-ik\Delta x} + \frac{10}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \frac{6}{16}e^{-2ik\Delta x} - \frac{20}{16}e^{-ik\Delta x} + \frac{10}{16} + \frac{5}{16}e^{ik\Delta x}$$

Substituting for α,

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - \alpha + \theta\alpha}{\gamma + \theta\alpha}$$

25.3.3 Upwind Differencing Scheme

The theoretical section shows that discretizing the 1-dimensional wave equation yields the following expression,

$$-\gamma\phi_P + \theta\phi_w - \theta\phi_e = -\gamma\phi_{P,o} + (1-\theta)\phi_{e,o} - (1-\theta)\phi_{w,o}$$

Assuming upwing differencing,

$$\phi_w = \phi_W \quad , \quad \phi_e = \phi_P$$

Substituting for this,

$$-\gamma\phi_P + \theta\phi_W - \theta\phi_P = -\gamma\phi_{P,o} + (1-\theta)\phi_{P,o} - (1-\theta)\phi_{W,o}$$

Simplifying,

$$\phi_P\left[-\gamma - \theta\right] + \theta\phi_W = \phi_{P,o}\left[-\gamma + (1-\theta)\right] - (1-\theta)\phi_{W,o}$$

Substituting the Fourier representaiton of the solution, $\phi = \hat{\phi}e^{ikx}$,

$$\hat{\phi}e^{ikx}\left[-\gamma - \theta\right] + \theta\hat{\phi}e^{ik(x-\Delta x)} = \hat{\phi}_o e^{ikx}\left[-\gamma + (1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ik(x-\Delta x)}$$

$$\hat{\phi}e^{ikx}\left[-\gamma - \theta\right] + \theta\hat{\phi}e^{ikx-ik\Delta x} = \hat{\phi}_o e^{ikx}\left[-\gamma + (1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ikx-ik\Delta x}$$

$$\hat{\phi}e^{ikx}\left[-\gamma - \theta\right] + \theta\hat{\phi}e^{ikx}e^{-ik\Delta x} = \hat{\phi}_o e^{ikx}\left[-\gamma + (1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ikx}e^{-ik\Delta x}$$

$$\hat{\phi}e^{ikx}\left\{\left[-\gamma - \theta\right] + \theta e^{-ik\Delta x}\right\} = \hat{\phi}_o e^{ikx}\left\{\left[-\gamma + (1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}$$

Since $e^{ikx} \neq 0$ for all values of x,

$$\hat{\phi}\left\{\left[-\gamma - \theta\right] + \theta e^{-ik\Delta x}\right\} = \hat{\phi}_o\left\{\left[-\gamma + (1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\left[-\gamma + (1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}}{\left\{\left[-\gamma - \theta\right] + \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\left[\gamma - (1-\theta)\right] + (1-\theta)e^{-ik\Delta x}\right\}}{\left\{[\gamma + \theta] - \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\gamma - 1 + \theta + (1-\theta)e^{-ik\Delta x}\right\}}{\left\{\gamma + \theta - \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - 1 + \theta + (1-\theta)e^{-ik\Delta x}}{\gamma + \theta\left(1 - e^{-ik\Delta x}\right)}$$

The amplification factor for the polynomial interpolation as well as the upwind differencing is implemented into the Matlab script. The matlab script was modified to accomodate multiple plots in a single figure and write the maximum amplification factor to a file. The matlab script is shown below,

Chapter 26

Inviscid Incompressible Flow

26.1 Thin Airfoil Theory

An infintely long wing could be considered a 2-dimensional problem. The thin airfoil theory also incorporates all vortices generated by the viscous boundary layer as a line of free-stream vortices on the mean chamber line of the airfoil. The vortices extend parallel in the spanwise direction. The thin airfoil theory differentiates between different geometries of an airfoil.

26.1.1 Derivation and Theorem-Specific Coefficients

A coordinate change is used in this theorem to express x in terms of θ_d wherein x represents the horizontal distance from the leading edge of the airfoil and θ_d is the change of coordinates variable that is used to represnt x. Here, c represents the chord length of the airfoil. The relationship between x and θ_d is shown below,

$$x = \frac{c}{2}[1 - \cos(\theta_d)] \quad , \quad 0 \le \theta_d \le \pi$$

Conversely θ_d in terms of x,

$$\frac{2x}{c} = 1 - \cos(\theta_d)$$

$$\cos(\theta_d) = 1 - \frac{2x}{c}$$

$$\theta_d = \arccos\left[1 - \frac{2x}{c}\right] \quad , \quad 0 \le x \le c$$

The coefficients A_0 and A_n are shown below,

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx} d\theta_d \quad , \quad A_n = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos(n\theta_d)\, d\theta_d$$

wherein α represents angle of attack of the specific airfoil.

26.1.2 Lift

The equation for the coefficient of lift C_L is shown below,

$$C_L = 2\pi\left(A_0 + \frac{A_1}{2}\right) = 2\pi\left(\alpha - \alpha_0\right)$$

wherein α_0 represents the angle of attack for the airfoil when lift is zero.

26.1.3 Moments

The moments about the aerodynamic center of the airfoil,

$$C_{M,ac} = C_{M,c/4} = \frac{\pi}{4}(A_2 - A_1)$$

wherein C_M represent the moments coefficient and the subscript ac represents at the aerodynamic center. Since the aerodynamic center is assumed to be quarter chord of the airfoil, the coefficients of moments at the aerodynamic center is identical to the coefficient of moments at the quarter chord of the airfoil. The coefficients A_1 and a_2 are obtainable via the equations above.

26.1.4 Limitations

The thin airfoil theori fails at thicker airfols, also, could not predict drag. The theory also fails to predict stall since the airflow is assumed to be inviscid. The theorem can only be used to find lift and pitching coefficient.

26.1.5 Other Implications

26.2 Lifting Line Theory

An early attempt at describing the behaviours of a finite wing. Uniquely the lifting line theory incorporates free-stream vortices in a 'u'-shape. Firstly in the configuration of a horse-shoe vortex , and in more advanced interpretation,s as a bunch of horse-shoe vortices superimposed on top of each other along the 'lifting-line'. The lifting line theory differentiates could only take into account the total effect of airfoil twist, geometry and perhaps deployed flaps instead of each individual effect. The total factor of those effects affect the free-stream vortex distribution along the 'lifting-line'.

26.2.1 Lift

Based on the Kutta-Joukowski theorem,

$$\frac{dL}{dy} = \rho_\infty v_\infty \Gamma(y)$$

wherein ρ_∞ represents the free stream density of the airflow, v_∞ represents free stream velocity and $\Gamma(y)$ represents the vortex strength at some point along the wing. Integrating the expression above,

$$L = \rho_\infty v_\infty \int_{-b/2}^{b/2} \Gamma(y)\, dy$$

wherein b represents the full wingspan. The coordinate as per usual is nested on the root of the wing at the leading edge of the main wing. The x-direction is prallel to the chord line, the y-direction is parallel ot the span length, and z-direction is vertical.

26.2.2 Drag

Since the fluid is inviscid, there is no drag due to viscosity, or flow seperation. In the lifting line theory, then the drag is purely induced drag. Induced drag is caused by the "downwash" of the wing causing a slightly altered angle of attack. This causes the lift force to tilt backwards slightly, contributing to drag. Since this phenomenon only exists due to the existence of downwash which in turn is due to lift generation, induced drag is also known as lift-induced drag. The formula for induced drag is below,

$$D_i = \rho_\infty v_\infty \int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y)\,dy$$

wherein the induced angle of attack α_i is determined by the downwash and free stream velocity. The downash in turn is determined by integrating the vortex distribution via biot-savart law.

$$\alpha_i(y) = \arctan\left[\frac{-w(y)}{v_\infty}\right]$$

Due to the small angle approximation or taking the first order taylor approximation, $\lim_{x\to 0}[\tan(x)] = x$. By the small angle, approximation, then,

$$\alpha_i(y) \approx -\frac{w(y)}{v_\infty}$$

Substituting for the definition of downwash $w(y)$,

$$w(y) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y-y_d}\frac{d\Gamma}{dy}\,dy_d$$

wherein y_d represents a dummy variable that is used for integration purposes only.

$$\alpha_i(y) \approx \frac{1}{4\pi v_\infty}\int_{-b/2}^{b/2} \frac{1}{y-y_d}\frac{d\Gamma}{dy}\,dy_d$$

26.2.3 Elliptical Vortex Distribution Wing

By using the vortex distribution,

$$\Gamma(y) = \Gamma_0\sqrt{1-\left(\frac{2y}{b}\right)^2}$$

The results of performing the steps above and performing the relevant substitutions,

$$\alpha_i = \frac{\Gamma_0}{2bv_\infty} \quad , \quad C_L = \alpha_i \pi A_R \quad , \quad C_{Di} = \frac{C_L^2}{\pi A_R}$$

wherein A_R represents aspect ratio, and b represents total wingspan.

26.3 Static Longitudinal Stability

The coefficient of moments about the center of mass $C_{M,cm}$ is shown below,

$$C_{M,cm} = C_{M,0} + C_{M,\alpha}\alpha_a$$

wherein $C_{M,0}$ represents the coefficeint of moments about the center of mass when the absolute angle of attack is zero. α_a represents the absolute angle of attack, which is the angle of attack of the airfoil starting at zero from the airfoil producing zero lift. $C_{M,\alpha}$ represents the derivative of $C_{M,cm}$ with respect to the absolute angle of attack α_a. There are only 2 conditions for static longitudinal stability:

1. $C_{M,0}$ must be a positive value
2. $C_{M,\alpha}$ must be a negative value

Chapter 27

Inviscid Compressible Flow

27.1 Thermodynamic Relations

For an ideal gas,

$$p = \rho RT$$

wherein p represents pressure, ρ represents density, and T represents temperature in Kelvins.To compute the c_v and c_p constants,

$$c_p = \frac{\gamma R}{\gamma - 1} \quad , \quad c_v = \frac{R}{\gamma - 1}$$

wherein $\gamma = \dfrac{c_p}{c_v}$. The expression for enthalphy h,

$$h = e + pv$$

Under the assumption that c_v and c_p as are constants,

$$e = c_v T \quad , \quad h = c_p T$$

Assuming no entropy generation via diffusion and that the coefficients c_v and c_p are constant,

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \quad , \quad s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right)$$

wherein the subscript denotes the state of the fluid,a nd v represents specific volume, the reiprocral of density. Only for isentropic processes, the following is true,

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

For flow that is steady, adiabatic, and inviscid, the following equation holds true,

$$h_0 = h + \frac{1}{2}v_f^2$$

wherein h_0 represents stagnation enthalphy, h represents current enthalphy, and v_f represents fluid velocity. The equation above holds true for any two points in a single streamline that follows the conditions stated above.

27.2 Shock Relations

27.2.1 Normal Shocks

For normal shocks, there are 3 governing equations and 2 thermodynamical relationships that is applicable. For continuity, momentum and energy,

$$\rho_1 u_1 = \rho_2 u_2 \quad , \quad p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad , \quad h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

The thermodynamic relations,

$$h_2 = c_p T_2 \quad , \quad p_2 = \rho_2 R T_2$$

The speed of sound a as well as the Mach number M is given by the following equations,

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T} \quad , \quad M = \frac{v_f}{a}$$

The 0 subscript is often used to symbolize stagnation conditions, wherein the small element of fluid is brought to rest adiabatically, the $*$ subscript is used to represent sonic conditions, wherein the small element of fluid is brought to sonic conditions. The equation relating speed of sound and the relationship between speed of sound and sonic speed of sound, as well as Mach and sonic Mach number,

$$\frac{a^2}{\gamma - 1} + \frac{1}{2}u^2 = \frac{\gamma + 1}{2(\gamma - 1)}a_*^2 \quad , \quad M_*^2 = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2}$$

For calorically perfect gases, which is the assumption that c_v and c_p remain as constants, the general relations between temperature, pressure and density to their respective stagnation conditions,

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2 \quad , \quad \frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2}M^2\right]^{\gamma/(\gamma-1)} \quad , \quad \frac{\rho_0}{\rho} = \left[1 + \frac{\gamma - 1}{2}M^2\right]^{1/(\gamma-1)}$$

The temperature, pressure and density relations between sonic conditions and stagnant conditions,

$$\frac{T_*}{T_0} = \frac{2}{\gamma + 1} \quad , \quad \frac{p_*}{p_0} = \left[\frac{2}{\gamma + 1}\right]^{\gamma/(\gamma-1)} \quad , \quad \frac{\rho_*}{\rho_0} = \left[\frac{2}{\gamma + 1}\right]^{1/(\gamma-1)}$$

The Mach number before and after shock,

$$M_2^2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}$$

The density, velocity, and pressure relations before and after shock,

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \quad , \quad \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

wherein the subscript 1 is used to represent quantities before the shock and 2 is after the shock. The relations for temperature and enthalphy,

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\left[\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right]$$

For the stagnation temperature and pressure before and after the shock,

$$T_{0,1} = T_{0,2} \quad , \quad \frac{p_{0,2}}{p_{0,1}} = e^{-(s_2-s_1)/R} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right]^{-1/(\gamma-1)} \left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right]^{\gamma/(\gamma-1)}$$

The ratio of nozzle cross-section at any arbitrary point of the nozzle to the nozzle cross-section at sonic conditions as a function of Mach number,

$$\left(\frac{A}{A_*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}$$

For choked flows, the exit Mach number M_e with or without shock,

$$M_e^2 = \frac{1}{(\gamma-1)}\left\{-1 + \left[1 + 2(\gamma-1)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{p_{0,1}A_t}{p_eA_e}\right)\right]^{1/2}\right\}$$

27.2.2 Oblique Shocks

The mathematical expression used relating Mach number, θ and β is shown below,

$$\tan(\theta) = 2\cot(\beta)\left\{\frac{M_1^2\sin^2(\beta) - 1}{M_1^2[\gamma + \cos(2\beta)] + 2}\right\}$$

Reiterating the relation shown in the previous problem,

$$M_{n,1} = M_1 \sin\beta \quad , \quad M_{n,2} = M_2\sin(\beta - \theta)$$

The relationship of the quantities between at $M_{n,1}$ and $M_{n,2}$ could be found using the relations at the normal shock relations. The only minor modification is that the stagnant relations for the normal shocks are no longer applicable to oblique shocks due to the tangential velocity of the fluid being conserved across the oblique shock.

27.3 Prantdl-Meyer Expansion Theory

An interesting property of the Prandtl Meyer function $\nu(M)$ wherein M represents Mach number,

$$\theta = \nu(M_2) - \nu(M_1)$$

wherein M_2 represents Mach number after expansion meanwhile M_1 represents Mach number before expansion. Manipulating this relation for convenience,

$$\nu(M_2) = \theta + \nu(M_1)$$

27.4 Shock-Expansion Theory

The coefficient of lift c_L and coefficient of drag c_D in terms of coefficient of pressures c_{p3} and c_{p2} is shown below,

$$c_L = (c_{p3} - c_{p2})\cos\alpha \quad , \quad c_D = (c_{p3} - c_{p2})\sin\alpha$$

The coefficient of pressure by definition,

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2(p - p_\infty)}{\rho_\infty U_\infty^2} = \frac{2p_\infty}{\rho_\infty U_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

wherein U_∞ is used to indicate velocity infinitely far away from the wing. Other relations for Mach number and speed of sound,

$$M = \frac{U}{a} \quad , \quad a = \sqrt{\gamma RT} \quad , \quad p = \rho RT$$

Manipulating the Mach number relation and the ideal gas relation,

$$M_\infty a_\infty = U_\infty \quad , \quad \frac{p_\infty}{RT_\infty} = \rho_\infty$$

Substituting both relations into each other,

$$\rho_\infty U_\infty^2 = \frac{p_\infty}{RT_\infty} \times M_\infty^2 a_\infty^2$$

Substituting for the speed of sound a_∞ in terms of temperature and ratio of specific heats,

$$\rho_\infty U_\infty^2 = \frac{p_\infty}{RT_\infty} \times M_\infty^2 \gamma RT_\infty = p_\infty M_\infty^2 \gamma$$

Substituting into the expression for coefficient of pressures,

$$c_p = \frac{2p_\infty}{p_\infty \gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

27.5 Linear Theory

27.5.1 Supersonic

The coefficient of pressure in supersonic linear theory is shown below,

$$c_{p,u} = -\frac{2\theta}{\sqrt{M_\infty^2 - 1}} \quad , \quad c_{p,l} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$

The important equations for the coefficients in supersonic linear theory are listed below,

$$c_L = \frac{4(\alpha + \Delta\alpha)}{\sqrt{M_\infty^2 - 1}} \quad , \quad c_D = \frac{4}{\sqrt{M_\infty^2 - 1}}[(\alpha + \Delta\alpha)^2 + K_2 + K_3]$$

$$c_{m,le} = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[-\frac{1}{2}\alpha + K_1\right] \quad , \quad c_{m,ac} = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[K_1 + \frac{\Delta\alpha}{2}\right]$$

wherein the various constants K as well as $\Delta\alpha$,

$$K_1 = \int_0^1 \left(\frac{d\hat{y_c}}{d\hat{x}}\right)\hat{x}\, d\hat{x} \quad , \quad K_2 = \int_0^1 \left(\frac{d\hat{y_c}}{d\hat{x}}\right)^2 d\hat{x} \quad , \quad K_3 = \int_0^1 \left(\frac{d\hat{y_t}}{d\hat{x}}\right)^2 d\hat{x}$$

$$\Delta\alpha = -\int_0^1 \frac{d\hat{y_c}}{d\hat{x}}\, d\hat{x} = -(\hat{y_{te}} - \hat{y_{le}})$$

wherein the various y-values are further expressed below,

$$y_c = \frac{1}{2}[y_u(x) + y_l(x)] \quad , \quad y_t = \frac{1}{2}[y_u(x) - y_l(x)]$$

The areodynamic centers and variables of integration are shown below,

$$\hat{x} = \frac{x}{c} \quad , \quad \hat{y} = \frac{y}{c} \quad , \quad \frac{x_{ac}}{x} = \frac{1}{2}$$

27.5.2 Subsonic

Based on the Prandtl-Glauert Rule

$$c_L = \frac{c_{L,0}}{\beta} \quad , \quad c_{m,le} = \frac{c_{m,le,0}}{\beta} \quad , \quad c_D = 0$$

wherein the variable $\beta = \sqrt{1 - M_\infty^2}$. Let c_p represet coefficient of pressure in compressible flow meanwhile c_{p0} represent the corresponding coefficient of pressure in the incompressible flow. The Prandtl-Glauert rule relating c_P to c_{p0},

$$c_p = \frac{c_{p0}}{\sqrt{1 - M^2}}$$

The Karman-Tsien rule relating c_P to c_{p0},

$$c_p = \frac{c_{p0}}{\sqrt{1 - M^2} + \left(\frac{M^2}{1 + \sqrt{1 - M^2}}\right)\left(\frac{c_{p0}}{2}\right)}$$

The Laitone's rule relating c_P to c_{p0},

$$c_p = \frac{c_{p0}}{\sqrt{1 - M^2} + c_{p0} M^2 \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right] \left[\frac{\sqrt{1 - M^2}}{2}\right]}$$

The equation for critical Mach number M_{cr} is shown below,

$$c_{p,cr} = \frac{2}{\gamma M_{cr}^2} \left\{ \left[\frac{1 + \left(\frac{\gamma - 1}{2}\right) M_{cr}^2}{\frac{\gamma + 1}{2}} \right]^{\gamma/(\gamma - 1)} - 1 \right\}$$

The minimum coefficient of pressure at critical Mach number could then be expressed by the Prandtl-Glauert rule shown below,

$$c_{p,cr} = \frac{c_{p0,min}}{\sqrt{1 - M_{cr}^2}}$$

Other rules could potentially be used such as the Laitone or the Karman-Tsien for the expression of $c_{p,cr}$. Equating $c_{p,cr}$ would produce an equation whose solution of M_{cr} represents the critical Mach number.

Chapter 28

Creeping Flows

Creeping Flows are the opposite of Potential Flows. The effects of viscosity are important but the inertial effects of the fluid is negligible. The degeneracy from the general Navier Stokes equation is shown below,

28.1 Cartesian Parallel Flows

The continuum governing equation for incompressible fluids in cartesian coordinates are shown below,

$$0 = \nabla \cdot \bar{v_f} = \frac{\partial}{\partial x}[\bar{v_f}] + \frac{\partial}{\partial y}[\bar{v_f}] + \frac{\partial}{\partial z}[\bar{v_f}]$$

Ignoring the y-direction degenerates this problem to a 2-dimensional case and evaluates the term $\frac{\partial}{\partial y}[\bar{v_f}]$ to zero. It is known that this case is a parallel flow case, therefore, the velocity in the z-direction is also non-existent. Therefore, the term $\frac{\partial}{\partial z}[\bar{v_f}]$ evaulates to zero as well.

$$0 = \frac{\partial}{\partial x}[\bar{v_f}]$$

Since this is a 2-dimensional problem, then the velocity is also independent in the thickness y-direction. This also means that the fluid velocity is identical to just the horizontal velocity $\bar{v_f} = \bar{v_x}$. Therefore,

$$0 = \frac{\partial}{\partial x}[\bar{v_f}] = \frac{\partial}{\partial x}[\bar{v_x}]$$

This equation shows that the velocity is independent in the x-direction. An alternate version of the Navier-Stokes momentum equation,

$$\rho\left[\frac{\partial \bar{v_f}}{\partial t} + (\bar{v_f} \cdot \nabla)\bar{v_f}\right] = \rho\bar{g} - \nabla P_r + \mu\nabla^2\bar{v_f}$$

wherein $\nabla^2\bar{v_f} = (\nabla \cdot \nabla)\bar{v_f}$ in this case. Evaluating the Navier-Stokes momentum equation in the x-direction,

$$\rho\left[\frac{\partial \bar{v_x}}{\partial t} + \bar{v_x}\frac{\partial}{\partial x}[\bar{v_x}] + \bar{v_y}\frac{\partial}{\partial y}[\bar{v_x}] + \bar{v_z}\frac{\partial}{\partial z}[\bar{v_x}]\right] = -\frac{\partial}{\partial x}[P_r] + \mu\left[\frac{\partial^2}{\partial x^2}[\bar{v_x}] + \frac{\partial^2}{\partial y^2}[\bar{v_x}] + \frac{\partial^2}{\partial z^2}[\bar{v_x}]\right]$$

If analysis is performed on a steady state fluid flow, the term $\dfrac{\partial \bar{v_x}}{\partial t}$ evaluates to zero. Using the previous finding after applying the continuity governing equation, the term $\bar{v_x}\dfrac{\partial}{\partial x}[\bar{v_x}]$ evaluates to zero. Due to the 2-dimensional nature of the problem, the term $\bar{v_y}\dfrac{\partial}{\partial y}[\bar{v_x}]$ evaluates to zero. Since there is no velocity in the veritcal z-direction due to the parallel nature of the flow, the term $\bar{v_z}\dfrac{\partial}{\partial z}[\bar{v_x}]$ evaluates to zero. Consistent with the previous findings, since the velocity is not a function of horizontal displacement x, the term $\dfrac{\partial^2}{\partial x^2}[\bar{v_x}]$ evaluates to zero. Due to the 2-dimensional nature of the problem, the term $\dfrac{\partial^2}{\partial y^2}[\bar{v_x}]$ evaluates to zero. Therefore,

$$0 = -\frac{\partial}{\partial x}[P_r] + \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

$$\frac{\partial}{\partial x}[P_r] = \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

Evaluating the Navier-Stokes momentum equation in the z-direction,

$$\rho\left[\frac{\partial \bar{v_z}}{\partial t} + \bar{v_x}\frac{\partial}{\partial x}[\bar{v_z}] + \bar{v_y}\frac{\partial}{\partial y}[\bar{v_z}] + \bar{v_z}\frac{\partial}{\partial z}[\bar{v_z}]\right] = -\rho g - \frac{\partial}{\partial z}[P_r] + \mu\left[\frac{\partial^2}{\partial x^2}[\bar{v_z}] + \frac{\partial^2}{\partial y^2}[\bar{v_z}] + \frac{\partial^2}{\partial z^2}[\bar{v_z}]\right]$$

Since the the vertical velocities are zero, then,

$$\frac{\partial \bar{v_z}}{\partial t} = \bar{v_x}\frac{\partial}{\partial x}[\bar{v_z}] = \bar{v_y}\frac{\partial}{\partial y}[\bar{v_z}] = \bar{v_z}\frac{\partial}{\partial z}[\bar{v_z}] = \frac{\partial^2}{\partial x^2}[\bar{v_z}] = \frac{\partial^2}{\partial y^2}[\bar{v_z}] = \frac{\partial^2}{\partial z^2}[\bar{v_z}] = 0$$

Therefore,

$$0 = -\rho g - \frac{\partial}{\partial z}[P_r]$$

Suppose one were to attempt to find the pressure function,

$$-\rho g\int dz = \int dP_r$$

$$-\rho g z + f(x) = P_r$$

wherein $f(x)$ is a function purely in terms of x. Taking the second conclusion we obtained from applying the Navier-Stokes momentum equation in the x-direction,

$$\frac{\partial}{\partial x}[-\rho g z + f(x)] = \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

$$f'(x) = \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

The velocity $\bar{v_x}$ is only dependent on the z-direction. Therefore, the second order partial derivative term, $\mu\dfrac{\partial^2}{\partial z^2}[\bar{v_x}]$ must also be fully dependent on the z-direction. This means that the function $f'(x)$ is a constant function $f'(x) = c_1$, therefore, $f(x)$. Firstly this computes the pressure function,

$$P_r = -\rho g z + c_1 x + \alpha$$

wherein both c_1 and α are arbitrary constsants. Secondly, the implications for the velocity profile,

$$f'(x) = c_1 = \mu \frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

$$\frac{1}{\mu}\int c_1\,dz = \frac{c_1}{\mu}z + c_2 = \frac{\partial}{\partial z}[\bar{v_x}]$$

$$\frac{c_1}{\mu}\int z\,dz + \int c_2\,dz = \bar{v_x}$$

$$\frac{c_1}{2\mu}z^2 + c_2 z + c_3 = \bar{v_x}$$

28.2 Radial Parallel Flows

The continuum governing equation is shown below,

$$0 = \nabla \cdot \bar{v_f} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

The Navier-Stokes Equation is shown below,

$$\rho\left[\frac{\partial \bar{v_f}}{\partial t} + (\bar{v_f}\cdot\nabla)\bar{v_f}\right] = \rho\bar{g} - \nabla P + \mu\nabla^2\bar{v_f}$$

28.2.1 Velocity Profile

To find the velocity profile of the fluid in the radial direction, the continuum governing equation is applied first in polar coordiantes.

$$0 = \nabla \cdot \bar{v_f} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Ignoring the z-axis,

$$0 = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

By inspection it could be seen that the radial velocity is zero, and $v_r = 0$. Therefore,

$$0 = \frac{\partial v_\theta}{\partial \theta} \quad , \quad 0 = \frac{\partial^2 v_\theta}{\partial \theta^2}$$

This shows that the tangential velocity is purely a function of radius and not a function of θ. It is Assumed that gravity acts on the z-direction and is largely ignored. Evaluating the expression in the radial direction,

$$\rho\left[\frac{dv_r}{dt} + v_r\frac{\partial v_r}{\partial r} + v_\theta\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right] = -\frac{\partial P}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

Since radial velocity is always zero, $\frac{dv_r}{dt} = v_r\frac{\partial v_r}{\partial r} = v_\theta\frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} = \frac{v_r}{r^2} = 0$. Since tangential velocity is not a function of θ, $\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} = 0$

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial P}{\partial r}$$

$$\frac{\rho v_\theta^2}{r} = \frac{\partial P}{\partial r}$$

Evaluating the expression in the tangential direction,

$$\rho\left[\frac{dv_\theta}{dt} + v_r\frac{\partial v_\theta}{\partial r} + v_\theta\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}\right] = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

$\frac{dv_\theta}{dt} = 0$ because it is assumed that the system is already in a steady state. $v_r\frac{\partial v_\theta}{\partial r} = \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} = \frac{v_r v_\theta}{r} = 0$ because $v_r = 0$ as the previous assumption. Since tangential velocity is purely a function of radius, $\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} = v_\theta\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$. Therefore, the equation degenerates into,

$$0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}\right]$$

By a seperate mathematical proof,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

Therefore,

$$0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

By the relation found by applying the Incompressible Navier-Stokes on the radial direction, it could be seen that pressure P is dependent on tangential velocity v_θ and r. However, since tangential velocity is only dependent on r from the relation found by applying continuity, then it follows that pressure must only be dependent on r. Therefore, $\frac{\partial P}{\partial r} = 0$. Therefore, the Navier-Stokes applied on the tangential direction further degenerates into,

$$0 = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

Solving for tangential velocity as a function of radius,

$$c_\alpha = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)$$

$$\int_{r=R_1}^{r} rc_\alpha\, dr = \int_{r=R_1, v_\theta=\omega_1 R_1}^{r,v} drv_\theta$$

$$\frac{1}{2}c_\alpha\left[r^2\right]_{r=R_1}^{r} = \left[rv_\theta\right]_{r=R_1, v_\theta=\omega_1 R_1}^{r,v_\theta}$$

$$\frac{1}{2}c_\alpha\left[r^2 - R_1^2\right] = \left[rv_\theta - \omega_1 R_1^2\right]$$

When $r = R_2$, $v_\theta = \omega_2 R_2$. Therefore,

$$\frac{1}{2}c_\alpha\left[R_2^2 - R_1^2\right] = \left[R_2\omega_2 R_2 - \omega_1 R_1^2\right]$$

$$c_\alpha \left[R_2^2 - R_1^2\right] = 2\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]$$
$$c_\alpha = \frac{2\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}$$

By substituting the newly found definition for the constant c_α,

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r^2 - R_1^2\right] = r v_\theta - \omega_1 R_1^2$$

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r - \frac{R_1^2}{r}\right] + \frac{\omega_1 R_1^2}{r} = v_\theta$$

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r - \frac{R_1^2}{r}\right] + \frac{\omega_1 R_1^2\left[R_2^2 - R_1^2\right]}{r\left[R_2^2 - R_1^2\right]} = v_\theta$$

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r - \frac{R_1^2}{r}\right] + \frac{\left[\omega_1 R_1^2 R_2^2 - \omega_1 R_1^4\right]}{r\left[R_2^2 - R_1^2\right]} = v_\theta$$

$$\frac{1}{\left[R_2^2 - R_1^2\right]}\left[r\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \frac{R_1^2\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \left[\omega_1 R_1^2 R_2^2 - \omega_1 R_1^4\right]}{r}\right] = v_\theta$$

$$\frac{1}{\left[R_2^2 - R_1^2\right]}\left[r\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \frac{\omega_2 R_1^2 R_2^2 - \omega_1 R_1^4 - \omega_1 R_1^2 R_2^2 + \omega_1 R_1^4}{r}\right] = v_\theta$$

$$v_\theta = \frac{1}{\left[R_2^2 - R_1^2\right]}\left[r\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \frac{R_1^2 R_2^2\left[\omega_2 - \omega_1\right]}{r}\right]$$

28.2.2 Scalar Pressure Field

To find the pressure field of the fluid, the two relations that are obtained from applying the incompressible Navier-Stokes equation on the radial and tangential direction is used,

$$\frac{\rho v_\theta^2}{r} = \frac{\partial P}{\partial r} \quad , \quad 0 = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right]$$

The second relation when further operated on gives the function of tangential velocity v_θ with respect to radius r. Integrating the first relation with respect to r and applying the boundary conditions yields,

$$\begin{aligned} P = &P_1 + \frac{\rho}{(R_2^2 - R_1^2)^2}\Bigg[(\omega_2 R_2^2 - \omega_1 R_1^2)^2\left(\frac{r^2 - R_1^2}{2}\right) \\ &- 2R_1^2 R_2^2(\omega_2 - \omega_1)(\omega_2 R_2^2 - \omega_1 R_1^2)\ln\frac{r}{R_1} - \frac{R_1^4 R_2^4}{2}(\omega_2 - \omega_1)^2\left(\frac{1}{r^2} - \frac{1}{R_1^2}\right)\Bigg]\end{aligned}$$

28.2.3 Induced Torque

Suppose the outer cylinder is rotating and the inner cylinder is held at rest, the equation for torque exerted on the inner cylinder must be

$$\Gamma = 2\pi\mu\omega_2 R_1^2 \left[\frac{R_2 H}{R_2 - R_1} + \frac{R_1^2}{4b}\right]$$

wherein b represents the gap between the two cylindrical surfaces and Γ is the torque exerted on the inner cylinder. Since this is typically ignored,

$$\Gamma = 2\pi\mu\omega_2 R_1^2 \left[\frac{R_2 H}{R_2 - R_1}\right]$$

The torque per unit width length,

$$\Gamma_H = 2\pi\mu\omega_2 R_1^2 \left[\frac{R_2}{R_2 - R_1}\right]$$

This is very useful to test for the viscosity of a specific fluid.

Part III

Control Systems

Chapter 29

Terminology

29.1 Rationale

When studying the dynamics of motion, variables contain alot of context which defines them uniquely apart. For example, velocity perceived in an inertial frame, would be different than velocity perceived in a non-inertial frame, even when describing the same object. In some of the derivations it might be necessary to reference velocity of object a and velocity of object b in the same equation. If both velocities of object a and object b cannot be distinguished from one another, the equation is ambiguously useless. Therefore, the notation presented in the perceding derivations and workings must be able to express many of the different contexts to tell each variable uniquely apart from one another.

29.2 Specifiers

Variables would typically be represented like shown below,

$$ {}^{A}_{C}\bar{v}^{B}_{D} $$

Here the variable being represented $\bar{v}$ is a vector. However, this notation could potentially be extended to scalars v and tensors of higher orders $\bar{\bar{v}}$ as well. The 'A' specifier represents the top left superscript. The 'B' specifier represents top right superscript. 'C' specifier represent bottom left subscript, and 'D' specifier represent bottom right subscript. This notation form is quite strange, but has alot of expressive power, with 4 possible specifiers. Note that specifiers are used flexibly in this document. 1, 2, all, or none of the specifiers may be used in variable expression. If a specifier is not used, as in the common case of the 'C' specifier, it means that no attributes are linked to that specifier, or it does not matter.

29.3 Variable Context

29.3.1 Laws of Motion

29.3.1.1 Force vectors

$$ {}^{A}_{C}\bar{F}^{B}_{D} $$

- 'A': Frame the force is asociated with. If the specified frame is a non-inertial frame, it implies the addition of fictitious forces. If the specified frame is inertial, fictitious forces excluded.

- 'B': Object experiencing force. This object could be anything from a particle, to a vehicle, to some structural part. This is to settle some of the ambiguity related to force diagrams.
- 'C': Basis vectors. This specifier is to represent which frame or set of vectors is used to represent the force vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

29.3.1.2 Torque vectors

$${}^{A}_{C}\bar{\Gamma}^{B}_{D}$$

- 'A': Frame the torque is asociated with. If specified frame is inertial, no fictitious torque is included. If the specified frame is non-inertial, fictitious torque is included. Fictitious torque is generated as a consequence of fictitious forces.
- 'B': Point of reference. A point in 3-dimensional space that if a force acts on that point, no torque is generated. The further away a particular force is from this point of reference, the larger the magnitude of the torque vector generated.
- 'C': basis vector. The frame or set of vectors used to represent the torque vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

29.3.2 Kinematics

29.3.2.1 Position vectors

$${}^{A}_{C}\bar{r}^{B}_{D}$$

- 'A': none. This specifier is reserved for the frame used in the frame derivative operation.
- 'B': For position vectors, usually this specifier starts with the beginning point of the position vector and ends with the end point of the position vector.
- 'C': basis vector. The frame or set of vectors used to represent the position vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

29.3.2.2 Velocity vectors

$$^{A}_{C}\bar{v}^{B}_{D}$$

- 'A': Frame that is used when taking the frame derivative of position vector.
- 'B': This specifier functions very similarly to its counterpart in the position vectors. This specifier is to represent what position vector this velocity vector is derived from.
- 'C': basis vector. The frame or set of vectors used to represent the velocity vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

29.3.2.3 Acceleration Vectors

$$^{A}_{C}\bar{a}^{B}_{D}$$

- 'A': Frame that is used when taking the second order frame derivative of position vector.
- 'B': This specifier functions very similarly to its counterpart in the position vectors. This specifier is to represent what position vector this acceleration vector is derived from.
- 'C': basis vector. The frame or set of vectors used to represent the velocity vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples: Acceleration vector and momentum vector is similar to the velocity vector notation-wise.

29.3.2.4 Angular velocity vectors

$$^{A}_{C}\bar{\omega}^{B}_{D}$$

- 'A': First frame of reference
- 'B': Second frame of reference
- 'C': basis vector. The frame or set of vectors used to represent the angular velocity vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

29.3.3 Conservation Laws

29.3.3.1 Momentum vectors

$${}^{A}_{C}\bar{p}^{B}_{D}$$

- 'A': The frame the linear momentum is asociated with. Linear momentum is dependent on velocity which is dependent on the frame it is perceived from. This specifier represents which frame the velocity needed for linear momentum is viewed from.
- 'B': Point of reference. This specifier is very similar to the position vector case. Linear momentum is dependent on velocity vector which is dependent on position vector of object of interest. This specifier represents the beginning point and end point of the position vector.
- 'C': none. The set of basis vectors used to represent the momentum vector must be identical to the basis vectors the momentum is taken with respect to.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

29.3.3.2 Angular Momentum vectors

$${}^{A}_{C}\bar{L}^{B}_{D}$$

- 'A': Frame the angular momentum vector is asociated with. Angular momentum is dependent on linear momentum, which is dependent on velocity vector, which varies from frame to frame. Hence this specifier is reserved for the frame velocity hence momentum is perceived from.
- 'B': Point of reference. Functions very similarly to the torque case.
- 'C': basis vector. The frame or set of vectors used to represent the angular momentum vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

29.3.4

Chapter 30

Kinematic Transport Theorem

The Kinematic Transport Theorem (KTT) specifies the relationship between vector time derivatives of different reference frames. Kinematics is the study of motion without cause. The consequences of the derivations below can only tell how some vector time derivative is 'perceived' when viewed in a different way. The derivations below tell nothing of how some object should move. The derivation below only claims, if this is a particular type of movement, this can be 'described' differently when 'viewed' differently.

Reference frames are typically defined with an origin, and a set of basis vectors. Reference frames in 3-dimensions typically have 3 basis vectors to allow general vectors to be expressed. Vectors are typically defined as linear combinations of reference frame basis vectors. Suppose the e reference frame has 3 basis vectors, $\hat{e_1}$, $\hat{e_2}$, and $\hat{e_3}$. If the position of some particle ($\bar{r}^{op}$) relative to the origin of the e reference frame is defined below,

$$\bar{r}^{op} = a\hat{e_1} + b\hat{e_2} + c\hat{e_3}$$

wherein a, b, and c, are scalar coefficients, then the vector $\bar{r}^{op}$ could be expressed as,

$$\bar{r}^{op} = \begin{bmatrix} a & b & c \end{bmatrix}^T$$

Since reference frames are arbitrarily constructed, let reference frame b have its origin in the same position as reference frame e. However, reference frame b has different basis vectors than reference frame e. The basis vectors for reference frame b is defined to be $\hat{b_1}$, $\hat{b_2}$, and $\hat{b_3}$. The vector $\bar{r}^{op}$ can now be defined as,

$$\bar{r}^{op} = d\hat{b_1} + f\hat{b_2} + g\hat{b_3}$$

wherein d, f, and g, are different scalar coefficients than a, b, and c. Just as in the previous case, using the notation rule described earlier, $\bar{r}^{op}$ can be expressed as,

$$\bar{r}^{op} = \begin{bmatrix} d & f & g \end{bmatrix}^T$$

This is very confusing. To distinguish the basis vectors that are used to express $\bar{r}^{op}$, the C specifier is used to differentiate basis vector sets. Hence, we can safely express $\bar{r}^{op}$ below,

$${}_e\bar{r}^{op} = \begin{bmatrix} a & b & c \end{bmatrix}^T \quad , \quad {}_f\bar{r}^{op} = \begin{bmatrix} d & f & g \end{bmatrix}^T$$

Throughout the workings, the attribute 'inertial' and 'non-inertial' are often used. Inertial reference frames are reference frames that are either still in free space or are moving at some constant velocity in a particular direction in free space. Any rotation, and acceleration of the reference frame origins are not allowed in inertial reference frames. If a particular reference frame exhibits rotation in free space or acceleration of its origin, then the reference frame must be non-inertial.

30.1 Frame Derivative

Vectors in R^n can be expressed by an infinite combination of n linearly independent basis vectors. Since the basis vectors used is somewhat arbitrary, then there are infinite ways to express a vector in R^n generally, and more specifically in R^3. The time derivative of a vector is a representation of how the scalar coefficients of the basis vectors change with respect to time. Since the scalar coefficients used to represent a vector is dependent on the basis vectors used, which in turn, is dependent on a declared reference frame, then it is only appropriate to specify time derivative for vectors to a specific reference frame. Taking the time derivative of a vector 'v' with respect to reference frame 'M' is to determine how the scalar coefficients representing vector 'v'as a linear combination of basis vectors for reference frame 'M' changes with respect to time.

Taking the time derivative of a vector, 'with respect to a frame' is defined as a new operation: Frame derivative. The LHS of the equation below represents the frame derivative and the RHS shows the implementation of the frame derivative. We use the same vector $\bar{r}^{op}$ as before.

$$\frac{\overset{e}{\partial}}{\partial t}[r^{\bar{o}p}] = \hat{e_1}\frac{d}{dt}[a] + \hat{e_2}\frac{d}{dt}[b] + \hat{e_3}\frac{d}{dt}[c] \quad , \quad \frac{\overset{f}{\partial}}{\partial t}[\bar{r}^{op}] = \hat{b_1}\frac{d}{dt}[d] + \hat{b_2}\frac{d}{dt}[f] + \hat{b_3}\frac{d}{dt}[g]$$

Note that the time derivative of the scalar quantity must match with the appropriate basis vector and must match with the appropriate refrence frame. More generally,

$$\frac{\overset{e}{\partial^n}}{\partial t^n}[r^{\bar{o}p}] = \hat{e_1}\frac{d^n}{dt^n}[a] + \hat{e_2}\frac{d^n}{dt^n}[b] + \hat{e_3}\frac{d^n}{dt^n}[c] \quad , \quad \frac{\overset{f}{\partial^n}}{\partial t^n}[\bar{r}^{op}] = \hat{b_1}\frac{d^n}{dt^n}[d] + \hat{b_2}\frac{d^n}{dt^n}[f] + \hat{b_3}\frac{d^n}{dt^n}[g]$$

30.2 1^{st} Order Derivative

Let e be an inertial reference frame and b be a non-inertial reference frame. For a vector $\bar{v}$ that starts at the origin of reference frame b,

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{v}] = \frac{\overset{b}{\partial}}{\partial t}[\bar{v}] + {}^e\bar{\omega}^b \times \bar{v} \tag{30.1}$$

Let o_e represent the origin of reference frame e and o_b represent the origin of reference frame b. Let the position vector $\bar{R}^{o_e o_b}$ represent o_b relative to o_e. Let $\bar{r}^{o_b p}$ represent the position of point p with respect to o_b. Then, it must follow,

$$\bar{r}^{o_e p} = \bar{R}^{o_e o_b} + \bar{r}^{o_b p}$$

Taking the time derivative with respect to the inertial frame e

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}] + \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_b p}]$$

Based on equation 30.1,

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_b p}] = \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^e\bar{\omega}^b \times \bar{r}^{o_b p}$$

Substituting,

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}] + \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^e\bar{\omega}^b \times \bar{r}^{o_b p}$$

This is an important equation. LHS represents the first order time derivative with respect to an inertial reference frame of $\bar{r}^{o_e p}$. If vector $\bar{r}^{o_e p}$ represents position vector of some point, then LHS represents the velocity of that point according to the inertial reference frame e. The term $\frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}]$ represents the velocity of non-inertial reference frame b origin relative relative to the origin of inertial reference frame e viewed in the e frame. The term $\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]$ represents the velocity of point p relative to origin of reference frame b viewed in non-inertial reference frame b. The term ${}^e\bar{\omega}^b \times \bar{r}^{o_b p}$ represents additional velocity perceived in the inertial reference frame e point p exhibits due to the rotation of reference frame b relative to reference frame e.

30.3 2^{nd} Order Derivative

Taking the frame derivative of the velocity expression with respect to the inertial frame e,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}]\right\} = \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}]\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b \times \bar{r}^{o_b p}\right\} \tag{30.2}$$

Simplifying to second order frame derivatives,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}]\right\} = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] \quad , \quad \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}]\right\} = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}]$$

Susbtituting into equation 30.2,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] + \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b \times \bar{r}^{o_b p}\right\} \tag{30.3}$$

Applying equation 30.1 to one of the terms in equation 30.2,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} = \frac{\overset{b}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} + {}^e\bar{\omega}^b \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]$$

Simplifying to second order frame derivatives,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} = \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}] + {}^e\bar{\omega}^b \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] \tag{30.4}$$

Using product rule for the last term in equation 30.2,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b \times \bar{r}^{o_b p}\right\} = {}^e\bar{\omega}^b \times \frac{\overset{e}{\partial}}{\partial t}\left\{\bar{r}^{o_b p}\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b\right\} \times \bar{r}^{o_b p} \tag{30.5}$$

By conjecture,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b\right\} = \frac{\overset{b}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b\right\}$$

This claim is rather simple to prove. Simply apply equation 30.1 to the claim,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b}\right\} = \frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}] + {}^{e}\bar{\omega}^{b} \times {}^{e}\bar{\omega}^{b}$$

The cross product of a vector with itself is zero, ${}^{e}\bar{\omega}^{b} \times {}^{e}\bar{\omega}^{b} = 0$. Hence, we can recover our claim,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b}\right\} = \frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]$$

Applying equation 30.1, to $\frac{\overset{e}{\partial}}{\partial t}\left\{\bar{r}^{o_b p}\right\}$,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\bar{r}^{o_b p}\right\} = \frac{\overset{b}{\partial}}{\partial t}\left[\bar{r}^{o_b p}\right] + {}^{e}\bar{\omega}^{b} \times \left[\bar{r}^{o_b p}\right]$$

Susbtituting into equation 30.5,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b} \times \bar{r}^{o_b p}\right\} = {}^{e}\bar{\omega}^{b} \times \left\{\frac{\overset{b}{\partial}}{\partial t}\left[\bar{r}^{o_b p}\right] + {}^{e}\bar{\omega}^{b} \times \left[\bar{r}^{o_b p}\right]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p}$$

Expanding,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b} \times \bar{r}^{o_b p}\right\} = {}^{e}\bar{\omega}^{b} \times \left\{\frac{\overset{b}{\partial}}{\partial t}\left[\bar{r}^{o_b p}\right]\right\} + {}^{e}\bar{\omega}^{b} \times \left\{{}^{e}\bar{\omega}^{b} \times \left[\bar{r}^{o_b p}\right]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p} \tag{30.6}$$

Substituting equation 30.4 and 30.6 into equation 30.3,

$$\begin{aligned}\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] =& \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] + \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \left\{\frac{\overset{b}{\partial}}{\partial t}\left[\bar{r}^{o_b p}\right]\right\} \\ &+ {}^{e}\bar{\omega}^{b} \times \left\{{}^{e}\bar{\omega}^{f} \times \left[\bar{r}^{o_b p}\right]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p}\end{aligned}$$

Simplifying,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] + \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}] + 2{}^{e}\bar{\omega}^{b} \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \left\{{}^{e}\bar{\omega}^{b} \times \left[\bar{r}^{o_b p}\right]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p} \tag{30.7}$$

The expression above is a purely kinematic relation. If $\bar{r}^{o_e p}$ is considered to be the position vector of a particular point p, then $\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$ would represent acceleration of point p relative to the origin of reference frame e and perceived in the inertial reference frame e. $\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}]$

represents the acceleration of the origin of reference frame b relative to the origin of reference frame e perceived in the inertial reference frame e. $\frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}]$ represents the acceleration perceived in non-inertial frame b of point p relative to the origin of frame b. The next few terms have specific formal names.

Equation 30.7 expresses the acceleration perceived in the inertial e frame of point p relative to origin of frame e. It is possible to rewrite the equation 30.7 to instead express the acceleration perceived in the inertial b frame of point p relative to origin of frame b,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] - \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] - 2\,{}^e\bar{\omega}^b \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] - {}^e\bar{\omega}^b \times \left\{{}^e\bar{\omega}^b \times [\bar{r}^{o_b p}]\right\} - \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^e\bar{\omega}^b]\right\} \times \bar{r}^{o_b p} = \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}] \tag{30.8}$$

The $-2\,{}^e\bar{\omega}^b \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]$ term is known as the coriolis acceleration. The $-{}^e\bar{\omega}^b \times \left\{{}^e\bar{\omega}^b \times [\bar{r}^{o_b p}]\right\}$ term is known as centrifugal acceleration, and the $-\left\{\frac{\overset{b}{\partial}}{\partial t}[{}^e\bar{\omega}^b]\right\} \times \bar{r}^{o_b p}$ term is known as the azimuthal acceleration. Apart from the $\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$ term, the rest of the terms in the LHS is often referred to as fictitious acceleration.

The fictitious acceleration is a consequence of the rotational and translational motion of reference frame b. These fictitious accelerations are not a consequence of physical interactions, but rather a consequence of perception, hence their names. If reference frame e and reference frame b are both inertial reference frames,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] = \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}]$$

This would require 2 things to be simultaneously true,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] = 0 \quad , \quad {}^e\bar{\omega}^b = 0$$

Another definition for inertial reference frames are reference frames that exhibit no fictitious acceleration. Non-inertial frames are allowed to exhibit fictitious acceleration but inertial reference frames cannot. Therefore, the constraint at the beginning that inertial reference frames are allowed to move translationally relative to one another at some constant velocity and are not allowed any rotation with respect to one another.

Chapter 31

Laws of Motions

31.1 Newton's 2^{nd} Law

31.1.1 Point Particle

31.1.1.1 Inertial Reference Frame

Newton's 2^{nd} law for a point particle p is shown below,

$$\sum_{i=1}^{n}\left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

LHS represents the summation of n number of forces acting on particle p. The forces here are perceived through an inertial frame e, and hence, represent true forces and not 'fictitious forces'. RHS represents the acceleration of particle p relative to the origin of the inertial frame e and perceived from the inertial frame e. m_p represents the mass of the particle p and $\bar{r}^{o_e p}$ represents the position vector of particle p relative to the origin of the inertial frame e. Note that the left-right sub-superscripts follow the notation convention described in the terminology. $o_e p$ represents the position vector starting from the origin of frame e and ending at the position of particle p.

31.1.1.2 Non-Inertial Reference Frame

31.1.2 Rigid Bodies

31.1.2.1 Inertial Reference Frame

Newton's 2^{nd} law for the i^{th} point particle in a rigid body is shown below,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right] = m_{p_i} \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p_i}]$$

Making the substitutions,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p_i}] = {}^{e}\bar{a}^{o_e p_i} \quad , \quad \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right] = {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right] \tag{31.1}$$

Newton's 2^{nd} law for the i^{th} point particle in a rigid body becomes,

$${}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right] = m_i {}^{e}\bar{a}^{o_e p_i}$$

wherein ${}^{e}\bar{F}^{p_i}_{ext}$ represents the resultant external force. The 'B' specifier represents that the external force is acting on the i^{th} particle. The 'A' specifier represents that the external force referred to are perceived in an inertial reference frame. The external forces expressed here are real and does not contain fictitious elements.

${}^{e}\bar{F}^{p_i}_{int,j}$ represents the forces acting on the i^{th} particle due to intermolecular interaction between the particles in the rigid body. The 'A' specifier represents the intermolecular forces are perceived in an inertial reference frame. The 'B' specifier represents that the intermolecular forces are acting on the i^{th} particle. The 'D' specifier holds the counter variable j that is used to sum over all the intermolecular forces experienced by the i^{th} particle.

m_i represents the mass of the i^{th} particle and n_2 represents the number of adjacent particles that has a force interaction with the i^{th} particle.

${}^{e}\bar{a}^{o_e p_i}$ represents acceleration of the i^{th} particle. The 'A' specifier represents the reference frame that acceleration is perceived in. The 'B' specifier represents that this acceleration is for i^{th} particle taken with reference to the origin of the inertial frame e.

To find the resultant force on a rigid body, one must find the summation of all forces acting on each particle of the rigid body. Taking the summation of Newton's 2^{nd} law for point particles on the entire rigid body,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right] + \sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right] = \sum_{i=1}^{n_2}\left[m_i{}^{e}\bar{a}^{o_e p_i}\right]$$

Here n_2 represents the total number of discrete particles that exist within the rigid body. There are 2 statements that can be made regarding the intermolecular forces:

- A particle cannot exert an intermolecular force on itself. Therefore the intermolecular force a particle exerts on itself is zero.
- By Newton's third law, the force acting on the i^{th} particle by the j^{th} particle is exactly equal and opposite to the force acting on the j^{th} particle by the i^{th} particle.

These 2 statements lead to the summation of intermolecular forces within the rigid body evaluating to zero,

$$\sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right] = 0$$

Simplifying Newton's 2^{nd} law,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right] = \sum_{i=1}^{n_2}\left[m_i{}^{e}\bar{a}^{o_e p_i}\right] \tag{31.2}$$

The center of mass definition for the rigid body is shown below,

$$\bar{r}^{o_e c} = \frac{\sum_{i=1}^{n_2}\left[m_i\bar{r}^{o_e p_i}\right]}{\sum_{i=1}^{n_2}\left[m_i\right]}$$

$\sum_{i=1}^{n_2} [m_i]$ represents the total mass of the rigid body. This is a scalar, and is constant due to the conservation of mass. Manipulating the equation,

$$\left\{ \sum_{i=1}^{n_2} [m_i] \right\} \bar{r}^{o_e c} = \sum_{i=1}^{n_2} [m_i \bar{r}^{o_e p_i}]$$

Taking the frame derivative with respect to the inertial e frame,

$$\frac{\overset{e}{\partial}}{\partial t} \left\{ \left[\sum_{i=1}^{n_2} (m_i) \right] \bar{r}^{o_e c} \right\} = \frac{\overset{e}{\partial}}{\partial t} \left\{ \sum_{i=1}^{n_2} [m_i \bar{r}^{o_e p_i}] \right\}$$

Since m_i is a constant due to conservation of mass,

$$\left[\sum_{i=1}^{n_2} (m_i) \right] \frac{\overset{e}{\partial}}{\partial t} \{\bar{r}^{o_e c}\} = \sum_{i=1}^{n_2} \left[\frac{\overset{e}{\partial}}{\partial t} \{m_i \bar{r}^{o_e p_i}\} \right]$$

$$\left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{v}^{o_e c} = \sum_{i=1}^{n_2} \left[m_i \frac{\overset{e}{\partial}}{\partial t} \{\bar{r}^{o_e p_i}\} \right]$$

Here we introduce velocity of the center of mass perceived in the e-frame and relative to the origin of frame e as ${}^e\bar{v}^{o_e c}$. Simplifying further,

$$\left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{v}^{o_e c} = \sum_{i=1}^{n_2} [m_i {}^e\bar{v}^{o_e p_i}]$$

Taking the frame derivative with respect to the e-frame once more,

$$\frac{\overset{e}{\partial}}{\partial t} \left\{ \left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{v}^{o_e c} \right\} = \frac{\overset{e}{\partial}}{\partial t} \left\{ \sum_{i=1}^{n_2} [m_i {}^e\bar{v}^{o_e p_i}] \right\}$$

Performing similar operations as before,

$$\left[\sum_{i=1}^{n_2} (m_i) \right] \frac{\overset{e}{\partial}}{\partial t} \{{}^e\bar{v}^{o_e c}\} = \sum_{i=1}^{n_2} \left[m_i \frac{\overset{e}{\partial}}{\partial t} \{{}^e\bar{v}^{o_e p_i}\} \right]$$

$$\left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{a}^{o_e c} = \sum_{i=1}^{n_2} [m_i {}^e\bar{a}^{o_e p_i}] \tag{31.3}$$

Here we introduce the acceleration of the center of mass perceived in the e-frame and relative to the origin of frame e as ${}^e\bar{a}^{o_e p_i}$. Substituting equation 31.3 to equation 31.2,

$$\sum_{i=1}^{n_2} \left[{}^e\bar{F}_{ext}^{p_i} \right] = \sum_{i=1}^{n_2} [m_i {}^e\bar{a}^{o_e p_i}] = \left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{a}^{o_e c}$$

Let $M = \left[\sum_{i=1}^{n_2} (m_i)\right]$. Here M represents the total mass of the rigid body. Substituting for M,

$$\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}^{p_i}_{ext}\right] = M{}^{e}\bar{a}^{o_e c}$$

This is the final form for Newton's 2^{nd} law for a rigid body. On the LHS there is the resultant external force acting on the rigid body. On the RHS there is the total mass of the rigid body multiplied by acceleration of the center of mass. Due to Newton's 3^{rd} law, the internal forces of a rigid body can be safely ignored. The rigid body form of Newton's 2^{nd} law replaces forces on individual point particles with the summation of external forces and also replaces acceleration of single particles with the acceleration of the rigid body's center of mass. The rigid body form of Newton's 2^{nd} law replaces mass of the i^{th} particle with the total mass of the rigid body M when compared to Newton's 2^{nd} law for a single particle.

31.1.2.2 Non-Inertial Reference Frame

31.2 Euler's Law

31.2.1 Point Particle

31.2.1.1 Inertial Reference Frame

Newton's 2^{nd} law could be extended to rotation as well. Taking the cross product of Newton's 2^{nd} law with position vector $\bar{r}^{o_e p}$,

$$\bar{r}^{o_e p} \times \sum_{i=1}^{n} \left[{}^{e}\bar{F}^{p}_{i}\right] = m_p \bar{r}^{o_e p} \times \frac{{}^{e}\partial^2}{\partial t^2}[\bar{r}^{o_e p}]$$

The summation $\sum$ operator and the vector cross-product operator are commutative with one another. Therefore,

$$\sum_{i=1}^{n} \left[\bar{r}^{o_e p} \times {}^{e}\bar{F}^{p}_{i}\right] = m_p \bar{r}^{o_e p} \times \frac{{}^{e}\partial^2}{\partial t^2}[\bar{r}^{o_e p}] \tag{31.4}$$

Torque $\bar{\Gamma}$ is defined as shown below,

$${}^{e}\bar{\Gamma}^{o_e}_{p,i} = \bar{r}^{o_e p} \times {}^{e}\bar{F}^{p}_{i} \tag{31.5}$$

The 'A' specifier on torque $\bar{\Gamma}$ represents that the torque is perceived in an inertial reference frame. This means ficitious forces should be exluded. The 'B'specifier on torque $\bar{\Gamma}$ represents that the moment is taken with respect to the origin of the inertial reference frame e. The 'D' specifier is to represent that the i^{th} torque produced from the i^{th} force is acting on particle p. Substituting the definition for torque to equation 31.4,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] = m_p \bar{r}^{o_e p} \times \frac{{}^{e}\partial^2}{\partial t^2}[\bar{r}^{o_e p}] \tag{31.6}$$

Here we make the claim,

$$\frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

Let LHS and RHS be defined below,

$$LHS = \frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] \quad , \quad RHS = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

Epxanding LHS based on product rule,

$$LHS = \bar{r}^{o_e p} \times \frac{\overset{e}{\partial}}{\partial t}\left[m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] + \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)$$

The vector crosss-product with itself is zero. Therefore the term $\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right) = 0$. Ignorinng the second term in the equation above,

$$LHS = \bar{r}^{o_e p} \times \frac{\overset{e}{\partial}}{\partial t}\left[m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right]$$

Scalar multiplication and frame derivative operations are commutative as long as the scalar remains constant with time. Since mass of particle m_p is unchanging in time due to conservation of mass,

$$LHS = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial}}{\partial t}\left[\frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right]$$

Simplifying to 2^{nd} order derivative,

$$LHS = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}\left(\bar{r}^{o_e p}\right)$$

Since $LHS = RHS$, the claim is proven to be true. Since the claim is true, the claim can be substituted into equation 31.6,

$$\sum_{i=1}^{n}\left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] = \frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] \tag{31.7}$$

Let angular momentum $\bar{L}$ be defined below,

$${}^{e}\bar{L}^{o_e}_{p} = \bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)$$

Angular momentum is dependent on $\frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)$. This term is dependent on the frame derivative operation and which frame the time derivative of $\bar{r}^{o_e p}$ is taken with respect to. The frame used for the frame derivative operation when constructing the angular momentum definition is represented in the 'A' specifier. The angular momentum is also dependent on the position vector of particle p, $\bar{r}^{o_e p}$. 2 facts can be deduced from the position vector in this form: the point of reference the particle p is taken with respect to, in this case origin of inertial frame o_e, and the object the position vector is pointing at, particle p. Likewise, the angular momentum vector notation needs to be able to express the point of reference and the particle being referenced. The point of reference is represented in the 'B' specifier meanwhile the object being referenced, particle p is represented in the 'D' specifier. Substituting the definition for angular momentum, into equation 31.7,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i} \right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}^{o_e}_{p} \right]$$

31.2.1.2 Non-Inertial Reference Frame

31.2.2 Rigid Bodies

31.2.2.1 Inertial Reference Frame

EUler's law for a point particle p,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i} \right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}^{o_e}_{p} \right]$$

Since i is just a counting variable, the expression above would still be true if i is renamed to another variable. Let $i \to j$. The expression above assumes that there are n numeber of torque acting on particle p. Let $n \to n_1$. The expression above is true for the point particle p. In a system of rigid bodies however, there are alot of particles. So, let thte equation above hold true for a particular particle p_i. Substituting these changes,

$$\sum_{j=1}^{n_1} \left[{}^{e}\bar{\Gamma}^{o_e}_{p_i,j} \right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}^{o_e}_{p_i} \right] \tag{31.8}$$

Reiterating the definition of torque defined at equation 31.5,

$${}^{e}\bar{\Gamma}^{o_e}_{p,i} = \bar{r}^{o_e p} \times {}^{e}\bar{F}^{p}_{i}$$

Performing the same substitutions, $i \to j$, $n \to n_1$, and $p \to p_1$,

$${}^{e}\bar{\Gamma}^{o_e}_{p_i,j} = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{j}$$

Taking the summation for all the torque acting on the i^{th} particle,

$$\sum_{j=1}^{n_1} \left[{}^{e}\bar{\Gamma}^{o_e}_{p_i,j} \right] = \sum_{j=1}^{n_1} \left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{j} \right]$$

Since $\bar{r}^{o_e p_i}$ does not contain a j index, it can be considered a scalar constant in the summation operation. Therefore,

$$\sum_{j=1}^{n_1} \left[{}^{e}\bar{\Gamma}^{o_e}_{p_i,j} \right] = \bar{r}^{o_e p_i} \times \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{j} \right]$$

Reiterating the expression for the forces experienced by the i^{th} particle described in equation 31.2.2.1,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right] = {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]$$

Substituting for the forces experienced by the i^{th} particle described in equation 31.2.2.1,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times \left\{{}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]\right\}$$

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \bar{r}^{o_e p_i} \times \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]$$

The term $\bar{r}^{o_e p_i}$ is not dependent on the counting variable j. Therefore, it acts as a scalar constant in the summation of j. Therefore,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]$$

Susbtituting $\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right]$ into equation 31.8,

$$\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right] = \sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]$$

This expression is true for the i^{th} particle. Since the entire rigid body is the summation of all particles, summing the equation above for all particles would yield an equation that is true for the rigid body. Summing the equation above for all particles n_2,

$$\sum_{i=1}^{n_2}\left\{\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right]\right\} = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]\right\}$$

$$\sum_{i=1}^{n_2}\left\{\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right]\right\} = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i}\right\} + \sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]$$

Due to a colinear argument, $\sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right] = 0$. Substituting,

$$\sum_{i=1}^{n_2}\left\{\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right]\right\} = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i}\right\}$$

The summation and partial derivative operations as commutative. Therefore,

$$\frac{{}^{e}\partial}{\partial t}\left[\sum_{i=1}^{n_2}\left\{{}^{e}\bar{L}_{p_i}^{o_e}\right\}\right] = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i}\right\}$$

Let ${}^{e}\bar{L}^{o_e}_{rb} = \sum_{i=1}^{n_2} \left\{ {}^{e}\bar{L}^{o_e}_{p_i} \right\}$. Wherein ${}^{e}\bar{L}^{o_e}_{rb}$ represents the total angular momentum of the rigid body rb. Substituting,

$$\frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}\bar{L}^{o_e}_{rb}\right] = \sum_{i=1}^{n_2} \left\{ \bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{ext} \right\}$$

Let ${}^{e}\bar{\Gamma}^{o_e}_{rb} = \sum_{i=1}^{n_2} \left\{ \bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{ext} \right\}$, wherein ${}^{e}\bar{\Gamma}^{o_e}_{rb}$ represents the torque that is acting on the rigid body as a whole. From the colinear argument made earlier, the internal forces do not contribute to the torque of the rigid body as a whole. Note that ${}^{e}\bar{F}^{p_i}_{ext}$ represents the resultant external forces acting on a single particle p_i in the rigid body rb. Substituting,

$$\frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}\bar{L}^{o_e}_{rb}\right] = {}^{e}\bar{\Gamma}^{o_e}_{rb}$$

31.2.2.2 Non-Inertial Reference Frame

Chapter 32

Conservation Laws

32.1 Single-Particle

32.1.1 Linear Momentum

The general form of Newton's 2^{nd} law is shown below for a single particle p,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

Expressing the 2^{nd} order derivative in terms of 1^{st} order derivative,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \frac{\overset{e}{\partial}}{\partial t} \left\{ \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] \right\}$$

Integrating both sides with respect to time t, for time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] \, dt = \int_{t_1}^{t_2} m_p \frac{\overset{e}{\partial}}{\partial t} \left\{ \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] \right\} \, dt$$

Since mass of particle m_p is a scalar constant unchanging with time,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] \, dt = m_p \int_{t_1}^{t_2} \frac{\overset{e}{\partial}}{\partial t} \left\{ \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] \right\} \, dt$$

For ease of notation,

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] = {}^{e}_{e}\bar{v}^{o_e p}$$

Here we make the satement that the frame derivative of particle p position vector is the velocity vector. The 'A' specifier shows represents the velocity vector perceived in the e frame. The 'C' specifier represents the velocity vector expressed as a linear combination of the e frame basis vectors, this would be important. Lastly, the 'B' specifier is used to represent the velocity vector as velocity of point particle p relative to the origin of the e frame. Performing the substitution,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] \, dt = m_p \int_{t_1}^{t_2} \frac{\overset{e}{\partial}}{\partial t} \left\{{}^{e}_{e}\bar{v}^{o_e p}\right\} \, dt$$

It was important to specify velocity expressed in terms of the basis vectors of e because in doing so, the frame derivative operation with respect to frame e, acts the same way as a conventional time derivative on each component of velocity vector $\bar{v}$. If velocity vector $\bar{v}$ was expressed in basis vector other than the basis vectors of the e frame, the first order KTT form must be used for the frame derivative and complicates the problem further. Since the frame derivative with respect to frame e functions the same way as conventional time derivative, the fundamental theorem of calculus can be employed on each of the components of velocity vector $\bar{v}$. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] \, dt = m_p \int_{t_1}^{t_2} \overset{e}{\partial} \left\{{}_{e}^{e}\bar{v}^{o_e p}\right\} = m_p \left[{}_{e}^{e}\bar{v}^{o_e p}\right]_{t_1}^{t_2}$$

The summation operation $\sum$ and the integral operation $\int$ are commutative. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] \, dt = \sum_{i=1}^{n} \left[\int_{t_1}^{t_2} {}^{e}\bar{F}_{i}^{p} \, dt\right] = m_p \left[{}_{e}^{e}\bar{v}^{o_e p}\right]_{t_1}^{t_2}$$

Linear momentum is defined as,

$${}^{e}\bar{P}^{o_e p} = m_p {}_{e}^{e}\bar{v}^{o_e p}$$

Substituting the expression for linear momentum,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] \, dt = \sum_{i=1}^{n} \left[\int_{t_1}^{t_2} {}^{e}\bar{F}_{i}^{p} \, dt\right] = \left[{}^{e}\bar{P}^{o_e p}\right]_{t_1}^{t_2}$$

The statement above shows the conservation of linear momentum. The change in linear momentum is equivalent to the time integral of the resultant force acting on particle p. The change in linear momentum is also equivalent to the summation of the time integral of every single force acting on particle p.

32.1.2 Angular Momentum

Euler's law is shown below,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}_{p}^{o_e}\right]$$

If the angular momentum vector is expressed in terms of the basis vectors of the e frame basis vector, the frame derivative operation $\frac{\overset{e}{\partial}}{\partial t}$ functions the same way as a conventional derivative operation. Therefore,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right] = \frac{d}{dt} \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right]$$

Integrating the equation with respect to time with time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] \, dt = \int_{t_1}^{t_2} \frac{d}{dt} \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right] \, dt$$

Using the fundamental theorem of calculus,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] dt = \int_{t_1}^{t_2} d\left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right]$$

An exact differential $d\left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right]$ is formed. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] dt = \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right]_{t_1}^{t_2}$$

The summation operation and integral operation is commutative. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] dt = \sum_{i=1}^{n} \left[\int_{t_1}^{t_2} {}^{e}\bar{\Gamma}_{p,i}^{o_e}\, dt\right] = \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right]_{t_1}^{t_2}$$

The conservation of angular momentum is similar to the conservation of linear momentum. The change in angular momentum of particle p is equals to the time integral of the resultant torque applied on particle p. The change in angular momentum of particle p is also equivalent to the summation of time integrals for each individual torque acting on particle p.

32.1.3 Kinetic Energy

Reiterating Newton's 2^{nd} law,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2}\left[\bar{r}^{o_e p}\right]$$

If the position vector $\bar{r}^{o_e p}$ is expressed in terms of the basis vectors in the e-frame, the partial derivative operation $\frac{\overset{e}{\partial^2}}{\partial t^2}$ would function the same as the second order conventional derivative. Therefore,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_{e}\bar{r}^{o_e p}\right] = m_p \frac{d^2}{dt^2}\left[{}_{e}\bar{r}^{o_e p}\right] = m_p \frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}\bar{r}^{o_e p}\right)\right]$$

Let infinitesmial distance in the e frame be defined as a vector below,

$$\overset{e}{\bar{dr}} = \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix}^T$$

Taking the dot product of the resultant force with infinitesmial distance in the e frame $\overset{e}{\bar{dr}}$,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] \cdot \overset{e}{\bar{dr}} = m_p \frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}\bar{r}^{o_e p}\right)\right] \cdot \overset{e}{\bar{dr}}$$

Let ${}_{e}r_{i}^{o_e p}$ represent the index notation for the components of ${}_{e}\bar{r}^{o_e p}$. By using chain rule,

$$\left\{\frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}\bar{r}^{o_e p}\right)\right]\right\}_i = \frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}r_{i}^{o_e p}\right)\right] = \frac{d}{d\alpha}\left[\frac{d}{dt}\left({}_{e}r_{i}^{o_e p}\right)\right] \times \frac{d\alpha}{dt}$$

wherein α is some variable. Here the $\times$ represents scalar multiplication, not the cross product operation because the quantities ${}_e r_i^{oep}$ is a scalar quantity. Expressing the dot product operation with infinitesmaily distance $\overset{e}{\bar{dr}}$,

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{\bar{dr}} = \frac{d}{d\alpha}\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{d\alpha}{dt}\times dx_i$$

Since α could be any variable we desire, let $\alpha = x_i$. Then $d\alpha = dx_i$. Substituting,

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{\bar{dr}} = \frac{d}{dx_i}\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{dx_i}{dt}\times dx_i = d\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{dx_i}{dt}$$

Earlier, $\overset{e}{\bar{dr}}$ was defined to be some infinitesmial distance. Now, let us specify that $\overset{e}{\bar{dr}}$ represents infinitesmial distance of the particle p trajectory. If this is the case, then $x_i = {}_e r_i^{oep}$. Therefore,

$$\frac{dx_i}{dt} = \frac{d}{dt}\left({}_e r_i^{oep}\right)$$

Susbtituting,

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{\bar{dr}} = d\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{d}{dt}\left({}_e r_i^{oep}\right) = \frac{d}{dt}\left({}_e r_i^{oep}\right)\times d\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]$$

Here we make the substitution $\frac{d}{dt}\left({}_e r_i^{oep}\right) = {}^e v_i^{oep}$, wherein ${}^e v_i^{oep}$ is the index notation of ${}^e\bar{v}_i^{oep}$. The 'A' specifier of velocity being e is perfectly valid because the frame derivative earlier was taken with respect to the inertial e frame.

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{\bar{dr}} = {}^e v_i^{oep}\times d\left[{}^e v_i^{oep}\right]$$

Substituting into the resultant force equation,

$$\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{\bar{dr}} = m_p\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{\bar{dr}} = m_p{}^e v_i^{oep} d\left[{}^e v_i^{oep}\right]$$

Let the initial position of the particle be denoted with position vector ${}_e\bar{r}_1$, implying this position vector is expressed in terms of the basis vectors of the inertial e frame. Let particle p have some initial velocity ${}_e^e\bar{v}_1$ initially. At the final position, particle p has the position vector ${}_e\bar{r}_2$ and final velocity of ${}_e^e\bar{v}_2$. Performing integration,

$$\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{\bar{dr}} = \int_{{}_e^e\bar{v}_1}^{{}_e^e\bar{v}_2} m_p{}^e v_i^{oep} d\left[{}^e v_i^{oep}\right] = \left[\frac{1}{2}m_p{}^e v_i^{oep}\ {}^e v_i^{oep}\right]_{{}_e^e\bar{v}_1}^{{}_e^e\bar{v}_2}$$

The LHS is in vector notation but the RHS is in index notation. Changing RHS to vector notation,

$$\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{\bar{dr}} = \left[\frac{1}{2}m_p{}^e\bar{v}^{oep}\cdot{}^e\bar{v}^{oep}\right]_{{}_e^e\bar{v}_1}^{{}_e^e\bar{v}_2}$$

The summation operation and the integral operation are both commutative. Therefore,

$$\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{\bar{dr}} = \sum_{i=1}^{n}\left[\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}{}^e\bar{F}_i^p\cdot\overset{e}{\bar{dr}}\right] = \left[\frac{1}{2}m_p{}^e\bar{v}^{oep}\cdot{}^e\bar{v}^{oep}\right]_{{}_e^e\bar{v}_1}^{{}_e^e\bar{v}_2}$$

This is the final form conservation of kinetic energy. Let us consider what has happened. Newton's 2^{nd} law was invoked for particle p. The expression was taken with a dot product to infinitesmial distance vector which is then set to the actual path of particle p. The expression was integrated applying the bounds for the initial and final states of particle p. The states are position and velocity, both perceived in the inertial frame. The final result is the expresion for conservation of kinetic energy. The force spatial integral of the particle is the change of kinetic energy.

32.2 Systems of Particles

32.2.1 Linear Momentum

Newton's 2^{nd} law for a system of particles is shown below,

$$\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}^{p_i}_{ext}\right] = M\,{}^{e}\bar{a}^{o_e c}$$

wherein M represents the total mass of the system of particles, $M = \left[\sum_{i=1}^{n_2} (m_i)\right]$. Multiplying Newton's 2^{nd} law infinitesmial time dt,

$$\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}^{p_i}_{ext}\right]\, dt = M\,{}^{e}\bar{a}^{o_e c}\, dt$$

Although dt represents infinitesmial time, one can simply treat dt like a scalar constant and that a scalar constant multipled by a vector in a vector equation simply implies the scalar constant is multiplied to all of the components of the vectors. ${}^{e}\bar{a}^{o_e c}$ represents acceleration of the center of mass of the system of particles perceived in the inertial frame e.

$${}^{e}\bar{a}^{o_e c} = \frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}\bar{v}^{o_e c}\right]$$

If the velocity of the center of mass is expressed as a linear combination of the basis vectors of inertial frame e,

$${}^{e}_{e}\bar{a}^{o_e c} = \frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}_{e}\bar{v}^{o_e c}\right] = \frac{d}{dt}\left[{}^{e}_{e}\bar{v}^{o_e c}\right] \tag{32.1}$$

This is one of the properties of the frame derivative.
Enforcing that the acceleration terms in Newton's 2^{nd} law be expressed as a linear combination of the basis vectors of inertial frame e,

$$\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}^{p_i}_{ext}\right]\, dt = M\,{}^{e}_{e}\bar{a}^{o_e c}\, dt \tag{32.2}$$

Substituting equation 32.2.1 to 32.2,

$$\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}^{p_i}_{ext}\right]\, dt = M\frac{d}{dt}\left[{}^{e}_{e}\bar{v}^{o_e c}\right]\, dt$$

Integrating for time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^{e}\bar{F}_{ext}^{p_i} \right] \, dt = \int_{t_1}^{t_2} M \frac{d}{dt} \left[{}^{e}_{e}\bar{v}^{o_e c} \right] \, dt$$

By the fundamental theorem of calculus,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^{e}\bar{F}_{ext}^{p_i} \right] \, dt = \int_{t_1}^{t_2} M \, d \left[{}^{e}_{e}\bar{v}^{o_e c} \right]$$

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^{e}\bar{F}_{ext}^{p_i} \right] \, dt = M \left[{}^{e}_{e}\bar{v}^{o_e c} \right]_{t_1}^{t_2}$$

Since the summation and integration operations are commutative with each other,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^{e}\bar{F}_{ext}^{p_i} \right] \, dt = \sum_{i=1}^{n_2} \left[\int_{t_1}^{t_2} {}^{e}\bar{F}_{ext}^{p_i} \, dt \right] = M \left[{}^{e}_{e}\bar{v}^{o_e c} \right]_{t_1}^{t_2}$$

Susbstituting for the total mass M of the system of particles,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^{e}\bar{F}_{ext}^{p_i} \right] \, dt = \sum_{i=1}^{n_2} \left[\int_{t_1}^{t_2} {}^{e}\bar{F}_{ext}^{p_i} \, dt \right] = \left[\sum_{i=1}^{n_2} (m_i) \right] \left[{}^{e}_{e}\bar{v}^{o_e c} \right]_{t_1}^{t_2}$$

This is the conservation of linear momentum for a system of particles. The conservation of linear momentum differs compared to the single particle case in that the mass of a single particle is now replaced with the total mass of the system of particles and that the velocity is now not the velocity of a single particle, but is instead the velocity of the center of mass for the system of particles.

32.2.2 Angular Momentum

Euler's law for a system of particles is shown below,

$$\frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}_{rb}^{o_e} \right] = {}^{e}\bar{\Gamma}_{rb}^{o_e}$$

wherein ${}^{e}\bar{\Gamma}_{rb}^{o_e}$ represents the total torque that the system of particles experiences. Asserting that the angular momentum vector $\frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}_{rb}^{o_e} \right]$ must be expressed in terms of the e-frame basis vector,

$$ {}^{e}\bar{\Gamma}_{rb}^{o_e} = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}_{e}\bar{L}_{rb}^{o_e} \right] \tag{32.3}$$

Since the angular momentum vector is expressed in terms of the e-frame basis vector,

$$\frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}_{e}\bar{L}_{rb}^{o_e} \right] = \frac{d}{dt} \left[{}^{e}_{e}\bar{L}_{rb}^{o_e} \right] \tag{32.4}$$

Substituting equation 32.4 to equation 32.2.2,

$$ {}^{e}\bar{\Gamma}_{rb}^{o_e} = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}_{e}\bar{L}_{rb}^{o_e} \right] = \frac{d}{dt} \left[{}^{e}_{e}\bar{L}_{rb}^{o_e} \right]$$

Integrating the equation for time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2} {}^{e}\bar{\Gamma}_{rb}^{o_e}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\left[{}^{e}_{e}\bar{L}_{rb}^{o_e}\right]\, dt$$

By fundamental theorem of calculus,

$$\int_{t_1}^{t_2} {}^{e}\bar{\Gamma}_{rb}^{o_e}\, dt = \left[{}^{e}_{e}\bar{L}_{rb}^{o_e}\right]_{t_1}^{t_2}$$

32.2.3 Kinetic Energy

Newton's 2^{nd} law for the i^{th} point particle in a system of particles is shown below,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right] = {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right] = m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p_i}] \tag{32.5}$$

Work for the i^{th} particle in a system of particles is defined as,

$$w_i = \int_{\bar{r}_{1,i}}^{\bar{r}_{2,i}} \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right]\cdot d\bar{r}$$

wherein $\bar{r}_{2,i}$ represents the final position and $\bar{r}_{1,i}$ represents the initial position of the i^{th} particle. The work for the i^{th} particle is a line integral following the path that the particle takes. Since $d\bar{r}$ represents a vector of infinitesmial distance that the i^{th} particle takes, then,

$$\frac{d\bar{r}}{dt} = \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

It is important that the position vector $\bar{r}^{o_e p_i}$ here must be expressed in terms of the e-frame basis vectors because otherwise, KTT of the first order must be invoked when $\bar{r}^{o_e p_i}$ is frame-derived with respect to frame e. Manipulating the differentials,

$$d\bar{r} = \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Substituting into the integral expression for the i^{th} particle,

$$w_i = \int_{\bar{r}_{1,i}}^{\bar{r}_{2,i}} \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right]\cdot d\bar{r} = \int_{\bar{r}_{1,i}}^{\bar{r}_{2,i}} \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right]\cdot \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Substituting for equation 32.5,

$$w_i = \int_{t_1}^{t_2}\left\{{}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt = \int_{t_1}^{t_2}\left\{m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p_i}]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Asserting that the vector $\bar{r}^{o_e p_i}$ must be expressed in terms of the e-frame basis vectors for convenience,

$$w_i = \int_{t_1}^{t_2}\left\{{}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt = \int_{t_1}^{t_2}\left\{m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Taking the summation for all particles existing for the system of particles,

$$\sum_{i=1}^{n_2}[w_i] = \sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{{}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]dt = \sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]dt$$

Let LHS and RHS be defined below,

$$LHS = \sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{{}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]dt \quad , \quad RHS = \sum_{i=1}^{n_2}\int_{t_1}^{t_2} m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]dt$$

By conjecture,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\right\} = \frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right] \tag{32.6}$$

Let

$$lhs = \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\right\} \quad , \quad rhs = \frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]$$

By applying product rule,

$$lhs = \frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]$$

Factoring out the scalar constant 1/2 and simplifying to 2^{nd} order frame derivatives,

$$lhs = \frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right] + \frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]$$

$$lhs = \frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right] + \frac{1}{2}\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]$$

Since dot product is a commutative operation,

$$lhs = \frac{1}{2}\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right] + \frac{1}{2}\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]$$

$$lhs = \frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]$$

Since $lhs = rhs$ equation 32.6 is proven to be true. Substituting equation 32.6 into RHS,

$$RHS = \sum_{i=1}^{n_2}\int_{t_1}^{t_2} m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]dt \quad , \quad \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\right\} = \frac{\overset{e}{\partial^2}}{\partial t^2}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]$$

$$RHS = \sum_{i=1}^{n_2}\int_{t_1}^{t_2} m_{p_i}\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\cdot\frac{\overset{e}{\partial}}{\partial t}\left[{}_e\bar{r}^{o_e p_i}\right]\right\}dt$$

The term $\left\{ \frac{1}{2} \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \right\}$ is asserted to be expressed as a linear combination of frame e-basis vectors. If the term is expressed as a linear combination of frame e basis vectors, then

$$\frac{\overset{e}{\partial}}{\partial t} \left\{ \frac{1}{2} \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \right\} = \frac{d}{dt} \left\{ \frac{1}{2} \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \right\}$$

Substituting to RHS,

$$RHS = \sum_{i=1}^{n_2} \int_{t_1}^{t_2} m_{p_i} \frac{d}{dt} \left\{ \frac{1}{2} \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \right\} dt$$

$$RHS = \sum_{i=1}^{n_2} \int_{\bar{v}_{1,i}}^{\bar{v}_{2,i}} \frac{1}{2} m_{p_i}\, d \left\{ \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \right\}$$

$$RHS = \sum_{i=1}^{n_2} \left\{ \frac{1}{2} m_{p_i} \left\{ \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] \right\}_{\bar{v}_{1,i}}^{\bar{v}_{2,i}} \right\}$$

Making the substitution $\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}] = {}^e_e\bar{v}^{o_e p_i}$,

$$RHS = \sum_{i=1}^{n_2} \left\{ \frac{1}{2} m_{p_i} \ [{}^e_e\bar{v}^{o_e p_i} \cdot {}^e_e\bar{v}^{o_e p_i}]_{\bar{v}_{1,i}}^{\bar{v}_{2,i}} \right\}$$

This is the statement for the sum of the individual particles' kinetic energy. Let $\bar{r}^{o_e p_i} = \bar{r}^{o_e a} + \bar{r}^{a p_i}$, wherein a could be some arbitrary point. Substituting into LHS,

$$LHS = \sum_{i=1}^{n_2} \int_{t_1}^{t_2} \left\{ {}^e\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^e\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a} + \bar{r}^{a p_i}] dt$$

Parsing the terms,

$$LHS = \sum_{i=1}^{n_2} \int_{t_1}^{t_2} \left\{ {}^e\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^e\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] + \left\{ {}^e\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^e\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] dt$$

$$LHS = \sum_{i=1}^{n_2} \int_{t_1}^{t_2} \left\{ {}^e\bar{F}^{p_i}_{ext} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] + \left\{ \sum_{j=1}^{n_1} \left[{}^e\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] + \left\{ {}^e\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^e\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] dt$$

$$LHS = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ \left\{ {}^e\bar{F}^{p_i}_{ext} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] \right\} + \sum_{i=1}^{n_2} \left\{ \left\{ \sum_{j=1}^{n_1} \left[{}^e\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] \right\}$$
$$+ \sum_{i=1}^{n_2} \left\{ \left\{ {}^e\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^e\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] \right\} dt$$

$$LHS = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] \right\} + \sum_{i=1}^{n_2} \left\{ \left\{ \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}]$$
$$+ \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] \right\} dt$$

$\sum_{i=1}^{n_2} \left\{ \left\{ \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \right\} = 0$ because of Newton's 3^{rd} law and that a particle cannot exert a force on itself. If one can form a 5^{th} order tensor which the first two indices correspond to i and j, then the main diagonal of this tensor is zero due to the fact a particle cannot exert a force on itself. Due to Newton's 3^{rd} law, the tensor is anti-symmetric. The contraction of an anti-symmetric tensor on its 2 anti-symmetric 'axis' yields zero. Hence, the summation of the internal forces of the system of particles is zero. Simplifying the expression,

$$LHS = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] \right\} + \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] \right\} dt$$

$$LHS = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] + \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] \right\} dt$$

Typically the point a is defined as the center of mass. And in that case, then the term $\sum_{i=1}^{n_2} \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}]$ represents work done in translating the center of mass to some velocity. The $\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}^{p_i}_{ext} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] \right]$ term represents the work done by the external force in rotating and stretching of the system of particles, meanwhile the $\sum_{i=1}^{n_2} \left\{ \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] \right\}$ term represents work done by internal forces in stretching. For rigid bodies, this term is zero. Combining the terms together,

$$\sum_{i=1}^{n_2} [w_i] = LHS = RHS$$

$$\sum_{i=1}^{n_2} [w_i] = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}] + \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}] \right\} dt$$
$$= \sum_{i=1}^{n_2} \left\{ \frac{1}{2} m_{p_i} \left[{}^{e}_{e}\bar{v}^{o_e p_i} \cdot {}^{e}_{e}\bar{v}^{o_e p_i} \right]^{\bar{v}_{2,i}}_{\bar{v}_{1,i}} \right\}$$

This is the final form of the conservation of kinetic energy.

Chapter 33

Rigid Body Dynamics

33.1 Inertia Tensor

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33.2 Problem 1

Reiterating Euler's law,

$$\bar{\Gamma}^c = {}^e\dot{\bar{H}}^c$$

wherein c represents the center of mass of the system of particles or a stationary point in the inertial reference frame. Reiterating the definition of angular momentum H,

$$\bar{H} = I\bar{\omega}$$

Here, the angular momentum $\bar{H}$ is taken with respect to the origin of the body fitted coordinates. The reference for $\bar{H}$ may or may not be the center of mass in the definition of angular momentum based on the inertia matrix. To prove Euler's equations of motion, it is necessary to take the angular momentum with respect to the center of mass of the rigid body. Therefore,

$$\bar{H}^c = I_c\bar{\omega}$$

wherein I_c represents the inertia matrix of the body taken with respect to the center of mass. Applying the kinematic relations for the vector $\bar{H}^c$,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}\left(\bar{H}^c\right) + {}^i\bar{\omega}^n \times \bar{H}^c$$

wherein i represents the inertia reference frame here and n represents the non-inertial reference fram e. Substituting for angular momentum in terms of angular velocity at RHS,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}\left(I_c\bar{\omega}\right) + {}^i\bar{\omega}^n \times \left(I_c\bar{\omega}\right)$$

Considering that the inertia matrix I_c is unchanging with respect to the body-fitted coordinates,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = I_c\frac{\overset{n}{\partial}}{\partial t}\left(\bar{\omega}\right) + {}^i\bar{\omega}^n \times \left(I_c\bar{\omega}\right)$$

Due to the definition of the body-fitted coordinates, the angular velocity $\bar{w}$ would be identical to ${}^{i}\bar{\omega}^{n}$ which is the rotation of the non-inertial body-fitted coordinates n with respect to the inertial coordinates i. Substituting,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^{c}\right) = I_c \frac{\overset{n}{\partial}}{\partial t}\left({}^{i}\bar{\omega}^{n}\right) + {}^{i}\bar{\omega}^{n} \times \left(I_c {}^{i}\bar{\omega}^{n}\right)$$

Let ${}^{i}\bar{\omega}^{n}$ be declared as a linear combination of the basis vectors in the body fitted coordinate system n. If ${}^{i}\bar{\omega}^{n}$ is declared natively in the body fitted coordinate the derivative of ${}^{i}\bar{\omega}^{n}$ with respect to the body fitted coordinate be a trivial case of taking the derivative of the all components of the vector. Let the angular velocity vector have its components be specified below,

$${}^{i}\bar{\omega}^{n} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$$

wherein $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the basis vectors of the body-fitted coordinates,
Analyzing the first term, in the expression above,

$$I_c \frac{\overset{n}{\partial}}{\partial t}\left({}^{i}\bar{\omega}^{n}\right) = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix}$$

Analyzing the second term in the expression above,

$${}^{i}\bar{\omega}^{n} \times \left(I_c {}^{i}\bar{\omega}^{n}\right) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix}$$

Declaring $\hat{i}$, $\hat{j}$ and $\hat{k}$ as dummy variables for the cross-product operations,

$${}^{i}\bar{\omega}^{n} \times \left(I_c {}^{i}\bar{\omega}^{n}\right) = \left| \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix} \right|$$

$${}^{i}\bar{\omega}^{n} \times \left(I_c {}^{i}\bar{\omega}^{n}\right) = \hat{i}|A| - \hat{j}|B| + \hat{k}|C|$$

For the A matrix,

$$\hat{i}|A| = \hat{i} \left| \begin{bmatrix} \omega_y & \omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix} \right|$$

$$\hat{i}|A| = \hat{i}[\omega_y(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)]$$

$$\hat{i}|A| = \hat{i}[\omega_y I_{zx}\omega_x + \omega_y I_{zy}\omega_y + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y - \omega_z I_{yz}\omega_z]$$

Due to the symmetric properties of the inertia tensor,

$$\hat{i}|A| = \hat{i}[\omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y]$$

For the B matrix,

$$\hat{j}|B| = \hat{j} \left| \begin{bmatrix} \omega_x & \omega_z \\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix} \right|$$

$$\hat{j}|B| = \hat{j}[\omega_x(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)]$$
$$-\hat{j}|B| = \hat{j}[-\omega_x(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) + \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)]$$
$$-\hat{j}|B| = \hat{j}[-\omega_x I_{zx}\omega_x - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y + \omega_z I_{xz}\omega_z]$$

Due to the symmetric properties of the inertia tensor,

$$-\hat{j}|B| = \hat{j}[-(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y]$$

For the C matrix,

$$\hat{k}|C| = \hat{k}\left|\begin{bmatrix} \omega_x & \omega_y \\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \end{bmatrix}\right|$$

$$\hat{k}|C| = \hat{k}[\omega_x(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)]$$
$$\hat{k}|C| = \hat{k}[\omega_x I_{yx}\omega_x + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xy}\omega_y - \omega_y I_{xz}\omega_z]$$

Due to the symmetric properties of the inertia tensor,

$$\hat{k}|C| = \hat{k}[(\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z]$$

Substituting the various components together,

$${}^{i}\bar{\omega}^{n} \times (I_c{}^{i}\bar{\omega}^{n}) = \hat{i}|A| - \hat{j}|B| + \hat{k}|C|$$

$${}^{i}\bar{\omega}^{n} \times (I_c{}^{i}\bar{\omega}^{n}) = \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

Substituting the first term and second term to obtain the full form,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

Reiterating Euler's law,

$$\bar{\Gamma}^c = {}^{e}\dot{\bar{H}}^c$$

Since the angular momentum is now taken with respect to center of mass and the time derivative of the angular momentum is taken with respect to an inertial reference frame,

$$\bar{\Gamma}^c = \frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right)$$

Let the moments $\bar{\Gamma}^c$ be defined to have components below,

$$\bar{\Gamma}^c = \frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \begin{bmatrix} M_x & M_y & M_z \end{bmatrix}^T$$

Substituting for the moments and simplifying,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z + \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - (\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z + (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

33.3 Problem 2

Reiterating Euler's law for an arbitrary body-fitted coordinate with origin nested at the center of mass,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + I_{xy}\dot{\omega_y} + I_{xz}\dot{\omega_z} + \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ I_{yx}\dot{\omega_x} + I_{yy}\dot{\omega_y} + I_{yz}\dot{\omega_z} - (\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ I_{zx}\dot{\omega_x} + I_{zy}\dot{\omega_y} + I_{zz}\dot{\omega_z} + (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

For the case of principal axes, the products of inertia evaluate to zero. Substituting,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + \omega_y I_{zz}\omega_z - \omega_z I_{yy}\omega_y \\ I_{yy}\dot{\omega_y} - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x \\ I_{zz}\dot{\omega_z} + \omega_x I_{yy}\omega_y - \omega_y I_{xx}\omega_x \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + \omega_z\omega_y I_{zz} - \omega_z\omega_y I_{yy} \\ I_{yy}\dot{\omega_y} - \omega_x\omega_z I_{zz} + \omega_x\omega_z I_{xx} \\ I_{zz}\dot{\omega_z} + \omega_x\omega_y I_{yy} - \omega_x\omega_y I_{xx} \end{bmatrix}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + (I_{zz} - I_{yy})\omega_z\omega_y \\ I_{yy}\dot{\omega_y} - (I_{zz} - I_{xx})\omega_x\omega_z \\ I_{zz}\dot{\omega_z} + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + (I_{zz} - I_{yy})\omega_z\omega_y \\ I_{yy}\dot{\omega_y} + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega_z} + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}$$

Earlier, the body-fitted coordinates were declared to be arbitrary as long as the origin of the body-fitted coordinates were nested in the center of mass of the rigid body. To simplify Euler's euqations of motions, the body-fitted coordinates' axes were set to the principal axes. Though this is merely one case of the general case, and it would be reasonable to leave the notations as is, for clarity's sake, the notation for the general moments of inertia are changed to specify moments of inertia with respect to the principal axes,

$$I_{xx} \to I_x \quad , \quad I_{yy} \to I_y \quad , \quad I_{zz} \to I_z$$

Hence, Euler's equations of motions with the constraint of the body-fitted coordinates must have an origin at the center of mass and the axes of the body-fitted coordinates must be the principal axes,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega_x} + (I_z - I_y)\omega_z\omega_y \\ I_y\dot{\omega_y} + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega_z} + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

33.4 Problem 7

Euler's law for a fixed inertia tensor was derived earlier. Reiterating the step before assuming the inertia tensor is unchanging with time,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}\left(I_c\bar{\omega}\right) + {}^i\bar{\omega}^n \times \left(I_c\bar{\omega}\right)$$

If the inertia tensor is changing with time,

$$\frac{\overset{n}{\partial}}{\partial t}\left(I_c\bar{\omega}\right) = I_c\frac{\overset{n}{\partial}}{\partial t}\left(\bar{\omega}\right) + \frac{\overset{n}{\partial}}{\partial t}\left(I_c\right)\bar{\omega}$$

Reiterating the definition of the inertia tensor,

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Assuming that the inertia tensor is taken with respect to center of mass, the inertia tensor simplifies to,

$$I_c = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Substituting the inertia tensor along with its derivatives while making the same assumption of $\bar{\omega} = {}^i\bar{\omega}^n$ and is declared natively in n coordinate system,

$$\frac{\overset{n}{\partial}}{\partial t}(I_c\bar{\omega}) = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{\omega_x} \\ \dot{\omega_y} \\ \dot{\omega_z} \end{bmatrix} + \begin{bmatrix} \dot{I}_x & 0 & 0 \\ 0 & \dot{I}_y & 0 \\ 0 & 0 & \dot{I}_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\frac{\overset{n}{\partial}}{\partial t}(I_c\bar{\omega}) = \begin{bmatrix} I_x\dot{\omega_x} \\ I_y\dot{\omega_y} \\ I_z\dot{\omega_z} \end{bmatrix} + \begin{bmatrix} \dot{I}_x\omega_x \\ \dot{I}_y\omega_y \\ \dot{I}_z\omega_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega_x} + \dot{I}_x\omega_x \\ I_y\dot{\omega_y} + \dot{I}_y\omega_y \\ I_z\dot{\omega_z} + \dot{I}_z\omega_z \end{bmatrix}$$

Reiterating the second term that was determined earlier,

$${}^i\bar{\omega}^n \times (I_c{}^i\bar{\omega}^n) = \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

Taking the products of inertia to be zero,

$${}^i\bar{\omega}^n \times (I_c{}^i\bar{\omega}^n) = \begin{bmatrix} \omega_y I_{zz}\omega_z - \omega_z I_{yy}\omega_y \\ -\omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x \\ \omega_x I_{yy}\omega_y - \omega_y I_{xx}\omega_x \end{bmatrix} = \begin{bmatrix} (I_{zz} - I_{yy})\omega_y\omega_z \\ (I_{xx} - I_{zz})\omega_x\omega_z \\ (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}$$

Susbtituting the moment of inertia to their principal notations,

$${}^i\bar{\omega}^n \times (I_c{}^i\bar{\omega}^n) = \begin{bmatrix} (I_z - I_y)\omega_y\omega_z \\ (I_x - I_z)\omega_x\omega_z \\ (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

Substituting the terms together to obtain the full form,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}(I_c\bar{\omega}) + {}^i\bar{\omega}^n \times (I_c\bar{\omega})$$

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \begin{bmatrix} I_x\dot{\omega_x} + \dot{I}_x\omega_x \\ I_y\dot{\omega_y} + \dot{I}_y\omega_y \\ I_z\dot{\omega_z} + \dot{I}_z\omega_z \end{bmatrix} + \begin{bmatrix} (I_z - I_y)\omega_y\omega_z \\ (I_x - I_z)\omega_x\omega_z \\ (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

Just as before, the time derivative of $\bar{H}^c$ is the moment acting on the body. Substituting,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega_x} + \dot{I}_x\omega_x + (I_z - I_y)\omega_y\omega_z \\ I_y\dot{\omega_y} + \dot{I}_y\omega_y + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega_z} + \dot{I}_z\omega_z + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

33.5 Problem 1

Reiterating Euler's equations of motion,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega_x} + (I_z - I_y)\omega_z\omega_y \\ I_y\dot{\omega_y} + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega_z} + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

Parsing out the scalar quantities,

$$M_x = I_x\dot{\omega_x} + (I_z - I_y)\omega_z\omega_y \quad , \quad M_y = I_y\dot{\omega_y} + (I_x - I_z)\omega_x\omega_z \quad , \quad M_z = I_z\dot{\omega_z} + (I_y - I_x)\omega_x\omega_y$$

Making $\dot{\omega_i}$ subject of the equations,

$$M_x - (I_z - I_y)\omega_z\omega_y = I_x\dot{\omega_x} \quad , \quad M_y - (I_x - I_z)\omega_x\omega_z = I_y\dot{\omega_y} \quad , \quad M_z - (I_y - I_x)\omega_x\omega_y = I_z\dot{\omega_z}$$

$$M_x + (I_y - I_z)\omega_z\omega_y = I_x\dot{\omega_x} \quad , \quad M_y + (I_z - I_x)\omega_x\omega_z = I_y\dot{\omega_y} \quad , \quad M_z + (I_x - I_y)\omega_x\omega_y = I_z\dot{\omega_z}$$

$$\frac{1}{I_x}\left[M_x + (I_y - I_z)\omega_z\omega_y\right] = \dot{\omega_x}, \quad \frac{1}{I_y}\left[M_y + (I_z - I_x)\omega_x\omega_z\right] = \dot{\omega_y}, \quad \frac{1}{I_z}\left[M_z + (I_x - I_y)\omega_x\omega_y\right] = \dot{\omega_z}$$

Let $\bar{F_b}$ represent the force vector represented in a body-fitted coordinate system. According to Newton's second law,

$$\bar{F_b} = m\bar{a_b}$$

The direction cosine matrix $A_{313}(\psi, \theta, \phi)$ is defined to have the following property,

$$\bar{a_b} = A_{313}(\psi, \theta, \phi)\bar{a_i}$$

wherein $\bar{a_i}$ represent acceleration perceived in the inertial coordinate system and $\bar{a_b}$ represent acceleration perceived in the body-fitted coordinate system. Making $\bar{a_b}$ subject of Newton's second law,

$$\bar{a_b} = \frac{1}{m}\bar{F_b}$$

The direction cosine matrix $A_{313}(\psi, \theta, \phi)$ was actually formed as a series of rotation transformations in the perceding sections. The rotation matrices are orthogonal matrice. Therefore, the direction cosine matrix $A_{313}(\psi, \theta, \phi)$ is also orthogonal. One property of orthogonal matrices,

$$A_{313}(\psi, \theta, \phi)^{-1} = A_{313}(\psi, \theta, \phi)^T$$

Therefore,

$$\bar{a_i} = A_{313}(\psi, \theta, \phi)^{-1}\bar{a_b} = A_{313}(\psi, \theta, \phi)^T\bar{a_b}$$

Substituting Newton's second law for $\bar{a_b}$,

$$\bar{a_i} = \frac{1}{m}A_{313}(\psi, \theta, \phi)^T\bar{F_b}$$

The script `Main.m` handles the initial condition, the integrator call, the plotting, and other small utilities. `Main.m` is shown below,

Chapter 34

Rigid Body Kinematics

34.1 Direction Cosine Matrix

The dot product can be thought as some form of projection of one vector onto another vector. Consider the dot product of two vectors $\bar{a}$ and $\bar{b}$. Based on the definition of dot product,

$$\bar{a} \cdot \bar{b} = |a||b| \cos\theta$$

wherein θ is the angle between the vectors $\bar{a}$ and $\bar{b}$. Let an arbitrary vector $\bar{r}$ be represented in basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Let these basis vectors be of mangitude 1. Let another set of basis vectors of magnitude 1 be defined as $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$. The arbitrary vector $\bar{r}$ could be expressed as a linear combination of these basis vectors,

$$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'$$

The vector component $\bar{r}$ in the $\hat{\imath}'$ direction is the summation of all the weighted basis vector components on the $\hat{\imath}'$ direction. Since it was established earlier that this would mean taking the dot product,

$$x' = x(\hat{\imath} \cdot \hat{\imath}') + y(\hat{\jmath} \cdot \hat{\imath}') + z(\hat{k} \cdot \hat{\imath}')$$

Let $\theta_{fg'}$ represent the angle between the f axis and the g' axis,

$$\hat{\imath} \cdot \hat{\imath}' = |\hat{\imath}||\hat{\imath}'| \cos\theta_{ii'} \quad , \quad \hat{\jmath} \cdot \hat{\imath}' = |\hat{\jmath}||\hat{\imath}'| \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{\imath}' = |\hat{k}||\hat{\imath}'| \cos\theta_{ki'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{\imath} \cdot \hat{\imath}' = \cos\theta_{ii'} \quad , \quad \hat{\jmath} \cdot \hat{\imath}' = \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{\imath}' = \cos\theta_{ki'}$$

Substituting the basis vector projections,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

Repeating similar operations for the $\hat{\jmath}'$ direction,

$$y' = x(\hat{\imath} \cdot \hat{\jmath}') + y(\hat{\jmath} \cdot \hat{\jmath}') + z(\hat{k} \cdot \hat{\jmath}')$$

$$\hat{\imath} \cdot \hat{\jmath}' = |\hat{\imath}||\hat{\jmath}'| \cos\theta_{ij'} \quad , \quad \hat{\jmath} \cdot \hat{\jmath}' = |\hat{\jmath}||\hat{\jmath}'| \cos\theta_{jj'} \quad , \quad \hat{k} \cdot \hat{\jmath}' = |\hat{k}||\hat{\jmath}'| \cos\theta_{kj'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{\imath} \cdot \hat{\jmath}' = \cos\theta_{ij'} \quad , \quad \hat{\jmath} \cdot \hat{\jmath}' = \cos\theta_{jj'} \quad , \quad \hat{k} \cdot \hat{\jmath}' = \cos\theta_{kj'}$$

Substituting the basis vector projections,

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

Repeating similar operations for the $\hat{k}'$ direction,

$$z' = x(\hat{\imath}\cdot\hat{k}') + y(\hat{\jmath}\cdot\hat{k}') + z(\hat{k}\cdot\hat{k}')$$

$$\hat{\imath}\cdot\hat{k}' = |\hat{\imath}||\hat{k}'|\cos\theta_{ik'} \quad , \quad \hat{\jmath}\cdot\hat{k}' = |\hat{\jmath}||\hat{k}'|\cos\theta_{jk'} \quad , \quad \hat{k}\cdot\hat{k}' = |\hat{k}||\hat{k}'|\cos\theta_{kk'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{\imath}\cdot\hat{k}' = \cos\theta_{ik'} \quad , \quad \hat{\jmath}\cdot\hat{k}' = \cos\theta_{jk'} \quad , \quad \hat{k}\cdot\hat{k}' = \cos\theta_{kk'}$$

Substituting the basis vector projections,

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Collecting the various expressions together,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Re-arranging the expressions into matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence, based on the problem definition,

$$\bar{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad , \quad \bar{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad , \quad l = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix}$$

$$\bar{r}' = l\bar{r}$$

The figure below shows some of the angles referenced in the l matrix,

34.2 Euler Angles

In 2-dimensions, the rotation matrix r is typically defined below,

$$r = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

The matrix r rotates a set of coordinate points by angle α in the counter-clockwise direction. Typically, a coordinate system would have its axes rotated in the counter-clockwise direction. As a result, all coordinates are now perceived in the rotated coordinate system to have been rotated in the clockwise direction. Suppose the ange $\alpha = -\theta$, the rotation matrix would take the form,

$$r = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

The eulerian angles represent parameters in successive rotation transformation to determine orientation. These successive rotation transformations are non-commutative. For the specific order of transformation eulerian angles $3-1-3$, the coordinate system is typically rotated in the z-axis by angle ϕ, then rotated in the x-axis by angle θ, before rotated in the z-axis again by angle ψ.

To express coordinates in an inertial coordinate system in terms of a body-fitted coordinate system, it is possible to apply successive rotation transformations on the coordinates in the inertial coordinate system to determine how the same coordinates are perceived in the body-fitted coordinate system. For this solution, the eulerian angle sequence $3-1-3$ is chosen.

For rotation in the z-axis by angle ϕ, the z-coordinate is held constant. Therefore,

$$A_3(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For rotation in the x-axis by angle θ, the x-coordinate is held constant. Therefore,

$$A_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

For rotation in the z-axis by angle ψ, the z-coordinate is held constant. Therefore,

$$A_3(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying the transformations in $3-1-3$ order,

$$A_{313}(\psi, \theta, \phi) = A_3(\psi) A_1(\theta) A_3(\phi)$$

The resulting matrix $A_{313}(\psi, \theta, \phi)$ would be the direction cosine matrix that would have the properties,

$$\bar{v_b} = A_{313}(\psi, \theta, \phi)\bar{v_i}$$

wherein $\bar{v_b}$ represents the coordinates in the body-fitted coordinate system and $\bar{v_i}$ represents the coordinates in the inertial coordinate system. Computing the direction matrix,

$$A_1(\theta)A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As a short-hand to make notation easier, let $\sin(\theta) = s_\theta$, $\cos(\theta) = c_\theta$. Substituting,

$$A_1(\theta)A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

$$A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

Therefore,

$$A_{313}(\psi, \theta, \phi) = A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

For the first transformation, ϕ represents the rotation angle along the z-axis in the inertial coordinate system. Let $\dot{\bar{\phi}}$ represent the time derivative of this transformation. Let $\dot{\bar{\phi}}_i$ represent the vector expressed in inertial coordinates and $\dot{\bar{\phi}}_b$ represent the vector expressed in body-fitted coordinates,

$$\dot{\bar{\phi}}_i = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T$$

To express the time derivative rotation vector in terms of body-fitted coordinates,

$$\dot{\bar{\phi}}_b = A_{313}(\psi, \theta, \phi)\dot{\bar{\phi}}_i$$

Substituting for the relevant terms,

$$\dot{\bar{\phi}}_b = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} s_\psi s_\theta \dot{\phi} \\ c_\psi s_\theta \dot{\phi} \\ c_\theta \dot{\phi} \end{bmatrix}$$

Expressing the vector $\dot{\bar{\phi}}_b$ verbosely,

$$\dot{\bar{\phi}}_b = f_1\hat{b_1} + f_2\hat{b_2} + f_3\hat{b_3} = s_\psi s_\theta \dot{\phi}\hat{b_1} + c_\psi s_\theta \dot{\phi}\hat{b_2} + c_\theta \dot{\phi}\hat{b_3}$$

By comparing the terms,

$$f_1 = \sin(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_2 = \cos(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_3 = \cos(\theta)\dot{\phi}$$

Chapter 35

MOSFET Devices

35.1 NMOS Properties

35.2 PMOS Properties

Chapter 36

Op-Amplifier Fundamentals

36.1 Design

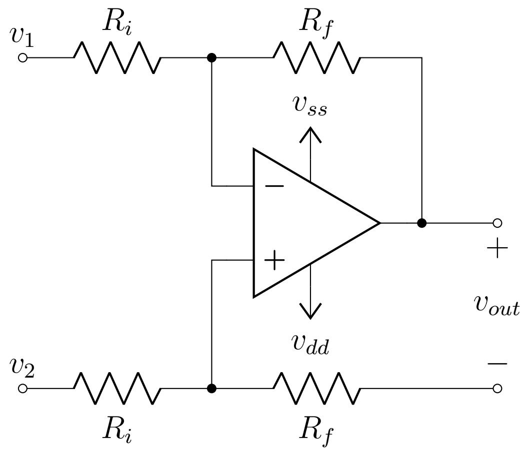
36.2 Properties

Practically, the operational amplifier is a 5-terminal device: source voltage, drain voltage, positive terminal, negative terminal, and output. Both the source voltage and drain voltage are used to power the various nmosfets and pmosfets inside the op amplifier. The ideal operational amplifier is a device with infinite gain. If the voltage at the positive terminal is greater than the voltage at the negative terminal, then the output will be positive infinity. However, practically, the highest output the operational amplifier can output is the source voltage. Therefore, if the voltage at the positive terminal is greater than the negative terminal, then the output will be the source voltage. Likewise, following the same reasoning, if the voltage at the negative terminal is greater than the positive terminal, then the output will be the drain volage. An operational amplifier could act as a comparator, but it is not recommended.

Chapter 37

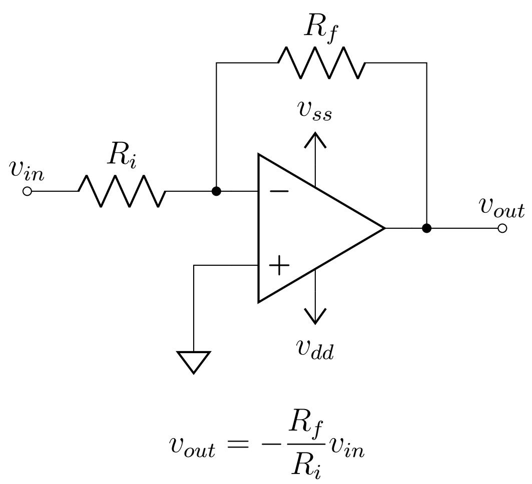
Op-Amplifier Arrangements

37.1 Differential Amplifier



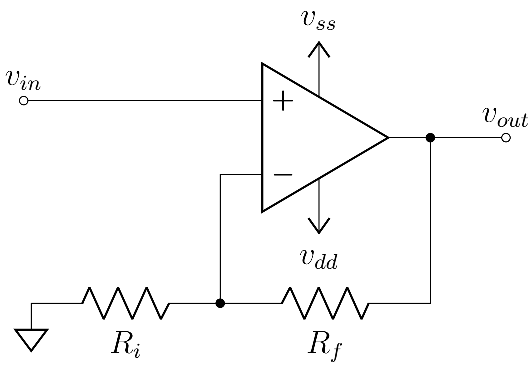
$$v_{out} = \frac{R_f}{R_i}(v_2 - v_1)$$

37.2 Inverting Amplifier



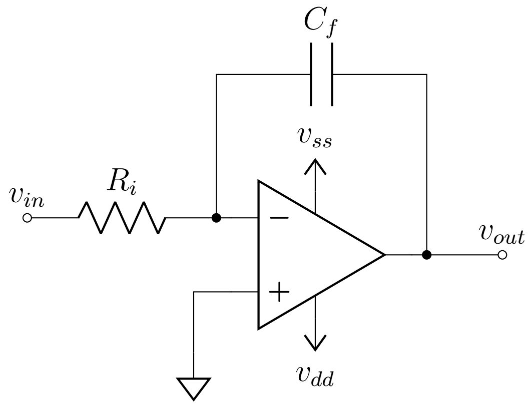
$$v_{out} = -\frac{R_f}{R_i}v_{in}$$

37.3 Non-Inverting Amplifier



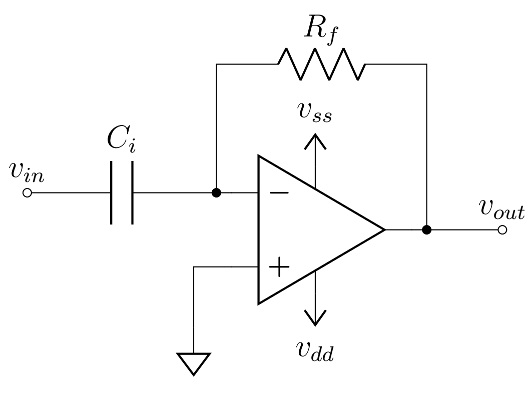
$$v_{out} = \left(1 + \frac{R_f}{R_i}\right) v_{in}$$

37.4 Integrating Amplifier



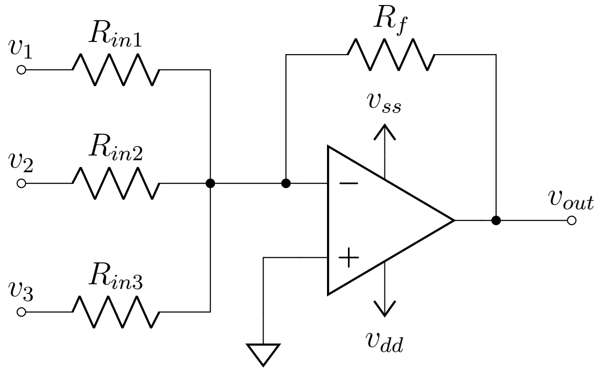
$$\frac{d}{dt}[v_{out}] = -\left(\frac{1}{R_i C_f}\right) v_{in}$$

37.5 Differentiating Amplifier



$$v_{out} = -R_f C_i \frac{d}{dt}[v_{in}]$$

37.6 Summing Amplifier



$$v_{out} = -R_f \left[\frac{v_1}{R_{in1}} + \frac{v_2}{R_{in2}} + \frac{v_3}{R_{in3}} \right]$$

Chapter 38

State-Space Implementations in Circuits

Chapter 39

Long-Term Behaviour of Circuits

39.1 Final Value Theorem

The final value theorem for an arbitrary function $f(t)$ is written as,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

wherein $F(s)$ represents the laplace transform of $f(t)$. The proof of this particular form of the final value theorem is shown below,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{s\to 0}\left\{\int_0^\infty e^{-st}f'(t)\,dt\right\} = \int_0^\infty f'(t)\,dt$$

By fundamental theorem of calculus, and change of variable,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{r\to\infty}\left\{f(r) - f(0)\right\} = \lim_{t\to\infty}\left\{f(t) - f(0)\right\}$$

Taking the approach of the Laplace Transform of derivatives,

$$\mathcal{L}\left[f'(t)\right] = s\mathcal{L}f(t) - f(0) = sF(s) - f(0)$$

Therefore, taking the limits as before,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{s\to 0}\left\{sF(s)\right\}\} - f(0)$$

By equating the $\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\}$ to each other,

$$\lim_{t\to\infty}\left\{f(t)\right\} - f(0) = \lim_{s\to 0}\left\{sF(s)\right\}\} - f(0)$$

This completes the proof,

$$\lim_{t\to\infty}\left\{f(t)\right\} = \lim_{s\to 0}\left\{sF(s)\right\}\}$$

39.2 Impulse Response

The transfer function $G(s)$ is defined as

$$G(s) = \frac{Y(s)}{U(s)}$$

wherein the $Y(s)$ is the output and $U(s)$ is the input, both in the laplace domain. In the laplace domain, the delta-dirac impulse function with arbitrary magnitude μ_0,

$$\mathcal{L}[\mu_0\,\delta(t-c)] = \mu_0 e^{-cs}$$

If the system described by the transfer function $G(s)$ has an input of the dirac delta function of arbitrary magnitude,

$$G(s) = \mu_0 e^{-cs}\,Y(s)$$

Manipulating for $Y(s)$ to be the subject of the equation,

$$Y(s) = \left(\frac{e^{cs}}{\mu_0}\right) G(s)$$

To find the long term behaviour of the output of the system $\lim_{t\to\infty} \big[y(t)\big]$, we should consult the final value theorem. The final value theorem for an arbitrary function $f(t)$,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

By performing the substitution $f(t) = y(t)$ and $F(s) = Y(s)$, wherein $y(t)$ represents the output function in the time domain and $Y(s)$ represents the output function in the laplace domain accordingly,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}[sY(s)]$$

If the input function is the dirac-delta function as shown previously, then $Y(s) = \left(\frac{e^{cs}}{\mu_0}\right) G(s)$. Therefore,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}\left[s\left(\frac{e^{cs}}{\mu_0}\right) G(s)\right]$$

A famous identity for limits of two arbitrary functions $q(t)$ and $p(t)$,

$$\lim_{t\to t_0} \big[p(t)\times q(t)\big] = \lim_{t\to t_0} \big[p(t)\big] \times \lim_{t\to t_0} \big[q(t)\big]$$

By the identity above,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0} \big[sG(s)\big] \times \lim_{s\to 0}\left[\frac{e^{cs}}{\mu_0}\right] = \frac{1}{\mu_0}\lim_{s\to 0} \big[sG(s)\big]$$

39.3 Step Response

Let $G(s)$ represent the transfer function of some system in the laplace domain, $Y(s)$ represent the output of some system in the laplace domain, and $U(s)$ represent the input in the laplace domain.

$$Y(s) = G(s)U(s)$$

Consider the heaviside function $u(t-c)$ wherein c is some arbitrary time the heaviside function "turns on". The laplace transform of the arbitray heaviside function,

$$\mathcal{L}[u(t-c)] = \frac{e^{-}cs}{s}$$

If the input $u(t) = \mu_0$ wherein μ_0 is a constant for all t, then

$$\mu_0 G(0) = \lim_{t\to\infty}[y(t)]$$

39.4 Pure Sinusoid Response

The convolution of two arbitrary functions $f(t)$ and $g(t)$ are defined as

$$f * g(t) = g * f(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

Suppose a system has a transfer function $G(s)$ and the output and input in the laplace domain respectively is $Y(s)$ and $U(s)$.

$$G(s) = \frac{Y(s)}{U(s)}$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\left[G(s)U(s)\right] = \mathcal{L}^{-1}\left[G(s)\right] * \mathcal{L}^{-1}\left[U(s)\right] = g(t) * u(t) = \int_0^t g(\tau)u(t-\tau)\,d\tau$$

wherein $\mathcal{L}^{-1}G(s) = g(t)$, and $\mathcal{L}^{-1}U(s) = u(t)$. Here, the function $u(t)$ does not represent the heaviside unit function. Assuming that the input function is reasonably well-behaved and without loss of generality,

$$u(t) = \sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0 t}\right] \quad , \quad a_k = \frac{1}{\tau}\int_0^{\tau} e^{ik\omega_0 t} f(t)dt \quad , \quad \omega_0 = 2\pi/\tau$$

Substituting the Fourier representation of the input function,

$$y(t) = \int_0^t g(\tau)\sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0(t-\tau)}\right]\,d\tau = \int_0^t g(\tau)\sum_{k=-\infty}^{\infty}\left[a_k e^{ik\omega_0\tau}e^{-ik\omega_0 t}\right]\,d\tau$$

$$y(t) = \sum_{k=-\infty}^{\infty}\left[\int_0^t g(\tau)e^{ik\omega_0\tau}\,d\tau a_k e^{-ik\omega_0 t}\right]$$

Observing the long term-behaviour of the output, $t = \infty$. Therefore,

$$y(t) = \sum_{k=-\infty}^{\infty}\left[\int_0^{\infty} g(\tau)e^{ik\omega_0\tau}\,d\tau a_k e^{-ik\omega_0 t}\right]$$

The term $\int_0^{\infty} g(\tau)e^{ik\omega_0\tau}\,d\tau$ represents a laplace transform with $s = -ik\omega_0$. Therefore,

$$G(-ik\omega_0) = \int_0^{\infty} g(\tau)e^{ik\omega_0\tau}\,d\tau$$

By substitution,

$$y(t) = \sum_{k=-\infty}^{\infty}\left[G(-ik\omega_0)a_k e^{-ik\omega_0 t}\right]$$

For real sinusoidal inputs $u(t) = k\sin(\omega t)$,

$$\lim_{t\to\infty}[y(t)] = k|G(i\omega)|\sin\{\omega t + arg[G(i\omega)]\}$$

Chapter 40

State-Space

Part IV

Newtonian Physics

Chapter 41

Binary Object Systems

41.1 Orbit Trajectory

For the scenario wherein an extraterrestial object of mass m approaches a planet of mass m_1, energy within the system of the planet and the object remains conserved because there is no external force acting on the system. Therefore, the summation of energy, $\sum E$ can be considered a constant in this scenario.

$$\sum E = E_k + E_p$$

$$E_k = \frac{1}{2}m|v|^2$$

By decomposing both radial and tangential components of velocity in polar coordinates:

$$|v|^2 = (\dot{\theta}r)^2 + \dot{r}^2$$

Therefore,

$$E_k = \frac{1}{2}m[(\dot{\theta}r)^2 + \dot{r}^2] = \frac{1}{2}m(\dot{\theta}r)^2 + \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m\dot{\theta}^2r^2 + \frac{1}{2}m\dot{r}^2$$

Since E_p represents gravitational potential energy the object has:

$$E_p = -\frac{Gmm_1}{r}$$

Let $\alpha = -Gmm_1$

$$E_p = \frac{\alpha}{r}$$

By substituting the expressions for kinetic energy and potential energy back into the equation for total energy:

$$\sum E = E_k + E_p = \frac{1}{2}m\dot{\theta}^2r^2 + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r}$$

The expression above is in the form of a differential equation involving two variables, θ and r as a derivative in respect to time. Let L represent angular momentum, and I represent moment inertia:

$$L = I\dot{\theta} \quad , \quad I = mr^2$$

Substituting for the moment of inertia for a single particle,

$$L = mr^2\dot{\theta}$$

making $\dot{\theta}$ subject of the equation,

$$\dot{\theta} = \frac{L}{mr^2}$$

The gravitational forces are always parallel to the radial vector of the object to the planet. Therefore, no net torque is produced on the passing object and angular momentum is conserved for the object. L could be treated as a constant in equation 2. Since L is a constant, equation 2 provides a method to reduce the dual variable problem in equation 1 to a single variable problem in terms of r. By substituting $\dot{\theta}$ from equation 2 to equation 1:

$$\sum E = \frac{1}{2}m\dot{\theta}^2r^2 + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r}$$

Using a change in notation for convenience, let $\sum E = E$,

$$E = \frac{1}{2}m\left(\frac{L}{mr^2}\right)^2 r^2 + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r} = \frac{1}{2}mr^2\frac{L^2}{(mr^2)^2} + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r} = \frac{L^2}{2mr^2} + \frac{1}{2}m\dot{r}^2 + \frac{\alpha}{r}$$

By manipulating the equation to make $\dot{r}$ the subject of the equation:

$$\frac{1}{2}m\dot{r}^2 = E - \frac{L^2}{2mr^2} - \frac{\alpha}{r}$$

$$\dot{r}^2 = \frac{2E}{m} - \frac{L^2}{m^2r^2} - \frac{2\alpha}{mr}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2E}{m} - \frac{L^2}{m^2r^2} - \frac{2\alpha}{mr}$$

The expression above involve r as a derivative in respect to time. Manipulating on this expression further would produce r as a function of time. Since the shape of the path traced by the object getting near to the planet is the subject in question, the derivative of r with respect to time must be converted to the derivative of r in respect to θ.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

Fortunately, equation 2 reduces $\dot{\theta}$ or $\frac{d\theta}{dt}$ into an expression purely in terms of known constants and r.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{L}{mr^2}$$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{d\theta}\frac{L}{mr^2}\right)^2 = \left(\frac{dr}{d\theta}\frac{1}{r^2}\right)^2\frac{L^2}{m^2}$$

By substituting the term $\dot{r}$ from equation 4 into equation 3:

$$\left(\frac{dr}{d\theta}\frac{1}{r^2}\right)^2\frac{L^2}{m^2} = \frac{2E}{m} - \frac{L^2}{m^2r^2} - \frac{2\alpha}{mr}$$

$$\left(\frac{dr}{d\theta}\frac{1}{r^2}\right)^2 = \frac{2E}{m}\frac{m^2}{L^2} - \frac{L^2}{m^2r^2}\frac{m^2}{L^2} - \frac{2\alpha}{mr}\frac{m^2}{L^2} = \frac{2Em}{L^2} - \frac{1}{r^2} - \frac{2\alpha m}{L^2r}$$

On the right hand side, there are lots of terms containing $\frac{1}{r}$ as well as $\frac{1}{r^2}$. For the reason that the left hand side also contain the term $\frac{dr}{d\theta}\frac{1}{r^2}$, let $u = \frac{1}{r}$

$$u = \frac{1}{r}$$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$-\frac{du}{d\theta} = \frac{1}{r^2}\frac{dr}{d\theta}$$

By substituting the term $\frac{dr}{d\theta}\frac{1}{r^2}$ into the main equation,

$$\left(-\frac{du}{d\theta}\right)^2 = \frac{2Em}{L^2} - \frac{1}{r^2} - \frac{2\alpha m}{L^2 r}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2Em}{L^2} - u^2 - \frac{2\alpha m}{L^2}u = \frac{2Em}{L^2} - \left(u^2 + \frac{2\alpha m}{L^2}u\right)$$

The right hand side takes the form of a quadratic in terms of u. The completing the square method should be performed on the right hand side of the expression to simplify the expression.

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2Em}{L^2} - \left[u^2 + \frac{2\alpha m}{L^2}u + \left(\frac{\alpha m}{L^2}\right)^2 - \left(\frac{\alpha m}{L^2}\right)^2\right] = \frac{2Em}{L^2} + \left(\frac{\alpha m}{L^2}\right)^2 - \left[u^2 + \frac{2\alpha m}{L^2}u + \left(\frac{\alpha m}{L^2}\right)^2\right]$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2EmL^2}{L^4} + \frac{\alpha^2 m^2}{L^4} - \left(u + \frac{\alpha m}{L^2}\right)^2 = \frac{2EmL^2 + \alpha^2 m^2}{L^4} - \left(u + \frac{\alpha m}{L^2}\right)^2$$

$$\text{Let } \beta^2 = \frac{2EmL^2 + \alpha^2 m^2}{L^4}$$

$$\left(\frac{du}{d\theta}\right)^2 = \beta^2 - \left(u + \frac{\alpha m}{L^2}\right)^2$$

It can be noticed that within the right hand side, the completing square form contains the variable u in arithmetic addition with some known constants. In order to "get rid" of these known constants to operate the equation further, let $\gamma = u + \frac{\alpha m}{L^2}$

$$\frac{d\gamma}{du} = \frac{d}{du}\left(u + \frac{\alpha m}{L^2}\right)$$

$$\frac{d\gamma}{du} = 1$$

$$d\gamma = du$$

$$\frac{d\gamma}{d\theta} = \frac{du}{d\theta}$$

By substituting γ to the original expression:

$$\left(\frac{d\gamma}{d\theta}\right)^2 = \beta^2 - (\gamma)^2$$

By solving for γ as a function of θ

$$\frac{d\gamma}{d\theta} = \sqrt{\beta^2 - (\gamma)^2}$$

$$\frac{d\theta}{d\gamma} = \frac{1}{\sqrt{\beta^2 - (\gamma)^2}}$$

$$d\theta = \frac{1}{\sqrt{\beta^2 - (\gamma)^2}} d\gamma$$

$$\int d\theta = \int \frac{1}{\sqrt{\beta^2 - (\gamma)^2}} d\gamma$$

By substitution of a inverse trigonometric derivative, let $p = \arcsin\left(\frac{\gamma}{\beta}\right)$

$$\frac{dp}{d\gamma} = \frac{1}{\sqrt{1 - \frac{\gamma^2}{\beta^2}}} \times \frac{1}{\beta} = \frac{1}{\sqrt{\frac{\beta^2 - \gamma^2}{\beta^2}}} \times \frac{1}{\beta}$$

$$\frac{dp}{d\gamma} = \frac{\beta}{\sqrt{\beta^2 - \gamma^2}} \times \frac{1}{\beta}$$

$$d\gamma = \sqrt{\beta^2 - \gamma^2} dp$$

By substuting$d\gamma$ into the original integral:

$$\theta = \int \frac{1}{\sqrt{\beta^2 - \gamma^2}} \sqrt{\beta^2 - \gamma^2} dp$$

$$\theta = p - \theta_0$$

$$\theta = \arcsin\left(\frac{\gamma}{\beta}\right) - \theta_0$$

$$\theta + \theta_0 = \arcsin\left(\frac{\gamma}{\beta}\right)$$

$$\sin(\theta + \theta_0) = \frac{\gamma}{\beta}$$

$$\beta \sin(\theta + \theta_0) = \gamma$$

$$\gamma = u + \frac{\alpha m}{L^2}$$

$$\beta \sin(\theta + \theta_0) = u + \frac{\alpha m}{L^2}$$

$$u = \frac{\beta L^2 sin(\theta + \theta_0) - \alpha m}{L^2}$$

$$u = \frac{1}{r}$$

$$\frac{1}{r} = \frac{\beta L^2 sin(\theta + \theta_0) - \alpha m}{L^2}$$

$$r = \frac{L^2}{\beta L^2 sin(\theta + \theta_0) - \alpha m}$$

$$\beta^2 = \frac{2EmL^2 + \alpha^2 m^2}{L^4}$$

$$\beta = \sqrt{\frac{2EmL^2 + \alpha^2 m^2}{L^4}}$$

$$\beta L^2 = \sqrt{2EmL^2 + \alpha^2 m^2}$$

$$\beta L^2 = \alpha m \sqrt{\frac{2EL^2}{m\alpha^2} + 1}$$

By substituting to the expression for r as a function of θ:

$$r = \frac{L^2}{\alpha m(\sqrt{\frac{2EL^2}{m\alpha^2}} + 1) sin(\theta + \theta_0) - \alpha m}$$

Let choice of θ_0 be such that $\sin(\theta + \theta_0) = \cos(\theta)$. In the real context, the variable θ_0 provides a method to shift the point of reference rotationally. However, since the aim of the derivation is to investigate the matter of the orbit trajectory of a foreign object, the choice of θ_0 does not affect the shape of the trajectory produced. Since we are free to choose any value of θ_0, the choice of θ_0 such that $\sin(\theta + \theta_0) = \cos(\theta)$ simplifies the problem. Let $\epsilon_0 = \sqrt{\frac{2EL^2}{m\alpha^2} + 1}$,

$$r = -\frac{L^2}{m\alpha(1 - \epsilon_0 \cos(\theta))}$$

Since $\frac{L^2}{m\alpha}$ is a constant produced by the expressions of known constants, let $k = -\frac{L^2}{m\alpha}$

$$r = \frac{k}{1 - \epsilon_0 \cos(\theta)}$$

For the values of ϵ_0 to be 0, the orbit trajectory of the object on the planet shall be a circle, for the values of ϵ_0 to be $0 < \epsilon_0 < 1$, the orbit trajectory shall be an ellipse. For values of $\epsilon_0 > 1$, the orbit trajectory shall be a hyperbola. By comparing the expression above to the expression for conic sections in polar coordinates, it can be inferred that ϵ_0 takes the value of the eccentricity of the path the orbit takes.

41.2 Time Dependent Quantities

The previous working shows the trajectory of a binary system. The purpose of this working is to establish a relationship between time and radius r. All variables and symbols written below rerpresent the same quantities as the previous working. A reiteration of the previous equation:

$$\dot{r}^2 = \frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{2\alpha}{mr}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2Emr^2 - L^2 - 2\alpha mr}{m^2 r^2}$$

By taking the sqare root and reciprocral of both sides,

$$\frac{dt}{dr} = \frac{mr}{\sqrt{2Emr^2 - 2\alpha mr - L^2}}$$

Using completing the square method and rearranging the equations,

$$\frac{dt}{dr} = \frac{mr}{\sqrt{2Em\left[r^2 - \frac{\alpha}{E}r + \left(\frac{\alpha}{2E}\right)^2 - \left(\frac{\alpha}{2E}\right)^2 - \frac{L^2}{2Em}\right]}}$$

$$\frac{dt}{dr} = \sqrt{\frac{m}{2E}}\left(\frac{r}{\sqrt{r^2 - \frac{\alpha}{E}r + \left(\frac{\alpha}{2E}\right)^2 - \frac{m\alpha^2}{4E^2m} - \frac{2EL^2}{4E^2m}}}\right)$$

$$\frac{dt}{dr} = \sqrt{\frac{m}{2E}}\left(\frac{r}{\sqrt{\left(r - \frac{\alpha}{2E}\right)^2 - \frac{2EL^2+m\alpha^2}{4E^2m}}}\right)$$

Let

$$k_1 = \sqrt{\frac{m}{2E}} \qquad k_2 = \frac{\alpha}{2E} \qquad k_3 = \frac{2EL^2 + m\alpha^2}{4E^2m}$$

$$\frac{dt}{dr} = k_1\frac{r}{\sqrt{(r-k_2)^2 - k_3}}$$

$$\frac{dt}{dr} = k_1\frac{r-k_2}{\sqrt{(r-k_2)^2 - k_3}} + k_1\frac{k_2}{\sqrt{(r-k_2)^2 - k_3}}$$

$$\int dt = k_1\int \frac{r-k_2}{\sqrt{(r-k_2)^2 - k_3}}dr + k_1\int \frac{k_2}{\sqrt{(r-k_2)^2 - k_3}}dr$$

For the first integrand of the equation above, let $u = r - k_2$

$$\frac{du}{dr} = 1$$

$$\int \frac{r-k_2}{\sqrt{(r-k_2)^2 - k_3}}dr = \int \frac{u}{\sqrt{u^2 - k_3}}du$$

Let $\gamma = \sqrt{u^2 - k_3}$

$$\frac{d\gamma}{du} = \frac{1}{2}\frac{1}{\sqrt{u^2 - k_3}}(2u)$$

$$\int \frac{r-k_2}{\sqrt{(r-k_2)^2 - k_3}}dr = \sqrt{u^2 - k_3} + C$$

$$\int \frac{r-k_2}{\sqrt{(r-k_2)^2 - k_3}}dr = \sqrt{(r-k_2)^2 - k_3} + C$$

For the second integrand of the equation, let $u = r - k_2$

$$\frac{du}{dr} = 1$$

$$\int \frac{k_2}{\sqrt{(r-k_2)^2 - k_3}} = \int \frac{k_2}{\sqrt{u^2 - k_3}}du$$

Using the method of hyperbolic substitution, let $u = \sqrt{k_3}\cosh(\theta)$

$$u^2 = k_3\cosh^2(\theta) = k_3\sinh^2(\theta) + k_3$$

$$\int \frac{k_2}{\sqrt{u^2-k_3}}du = \int \frac{k_2}{\sqrt{k_3\sinh^2(\theta)+k_3-k_3}}$$

$$\int \frac{k_2}{\sqrt{u^2-k_3}}du = \int \frac{k_2}{\sqrt{k_3\sinh^2(\theta)}}du = \int \frac{k_2}{\sqrt{k_3}}\frac{1}{\sinh(\theta)}du$$

Finding and expression of the differential du in terms of θ,

$$u = \sqrt{k_3}\cosh(\theta)$$

$$\frac{du}{d\theta} = \sqrt{k_3}\sinh(\theta)$$

$$du = \sqrt{k_3}\sinh(\theta)d\theta$$

Substituting the definition of the differential du,

$$\int \frac{k_2}{\sqrt{u^2-k_3}}du = \frac{k_2}{\sqrt{k_3}}\int \frac{1}{\sinh(\theta)}\sqrt{k_3}\sinh(\theta)d\theta$$

$$\int \frac{k_2}{\sqrt{u^2-k_3}}du = k_2\theta + C$$

Expressing θ in terms of u,

$$u = \sqrt{k_3}\cosh(\theta)$$

$$\frac{u}{\sqrt{k_3}} = \cosh(\theta)$$

$$\theta = arcosh\left(\frac{u}{\sqrt{k_3}}\right)$$

Substituing the definition of θ

$$\int \frac{k_2}{\sqrt{u^2-k_3}}du = k_2 arcosh\left(\frac{r-k_2}{\sqrt{k_3}}\right) + C$$

Therefore, the second integrand is:

$$\int \frac{k_2}{\sqrt{(r-k_2)^2-k_3}} = k_2 arcosh\left(\frac{r-k_2}{\sqrt{k_3}}\right) + C$$

By placing the two integrands together,

$$\int dt = k_1\sqrt{(r-k_2)^2-k_3} + k_1k_2 arcosh\left(\frac{r-k_2}{\sqrt{k_3}}\right) + C$$

By substituting the expressions for k_1, k_2, and k_3:

$$t = \sqrt{\frac{m}{2E}}\sqrt{(r-\frac{\alpha}{2E})^2 - \frac{2EL^2+m\alpha^2}{4E^2m}} + \sqrt{\frac{m}{2E}}\frac{\alpha}{2E} arcosh\left(\frac{r-\frac{\alpha}{2E}}{\sqrt{\frac{2EL^2+m\alpha^2}{4E^2m}}}\right) + C$$

$$t = \frac{\sqrt{m}}{(2E)^{\frac{3}{2}}}\sqrt{(2Er-\alpha)^2 - \frac{2EL^2+m\alpha^2}{m}} + \frac{\alpha\sqrt{m}}{(2E)^{\frac{3}{2}}} arcosh\left(\frac{\frac{2Er}{2E}-\frac{\alpha}{2E}}{\sqrt{\frac{2EL^2+m\alpha^2}{4E^2m}}}\right) + C$$

$$t = \frac{\sqrt{m}}{(2E)^{\frac{3}{2}}}\sqrt{(2Er-\alpha)^2 - \frac{2EL^2}{m} - \alpha^2} + \frac{\alpha\sqrt{m}}{(2E)^{\frac{3}{2}}} arcosh\left(\frac{2Er-\alpha}{2E}\times\sqrt{\frac{4E^2m}{2EL^2+m\alpha^2}}\right) + C$$

$$t = \frac{\sqrt{m}}{(2E)^{\frac{3}{2}}}\sqrt{(2Er-\alpha)^2 - \frac{2EL^2}{m} - \alpha^2} + \frac{\alpha\sqrt{m}}{(2E)^{\frac{3}{2}}} arcosh\left[(2Er-\alpha)\sqrt{\frac{m}{2EL^2+m\alpha^2}}\right] + C$$

Chapter 42

Gauss Law

42.1 Gravitational Fields

The divergence theorem for any field $\bar{F}$ is denoted as $\iiint \nabla \cdot F \, dV = \oiint F \cdot \bar{n} \, ds$, wherein $div[F] = \nabla \cdot F$

For a spherical mass m_1, the gravitational field generated by the mass m_1 at any point: $\bar{F_{field}} = -\frac{Gm_1}{r^2}\hat{r}$, wherein $\hat{r}$ represents unit vector with direction from mass to object.

$$\hat{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\hat{r} = \begin{pmatrix} x(x^2+y^2+z^2)^{-\frac{1}{2}} \\ y(x^2+y^2+z^2)^{-\frac{1}{2}} \\ z(x^2+y^2+z^2)^{-\frac{1}{2}} \end{pmatrix}$$

Therefore, given that $\quad x \neq 0, \quad y \neq 0, \quad z \neq 0$

$$\bar{F_f} = -Gm_1 \begin{pmatrix} x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ y(x^2+y^2+z^2)^{-\frac{3}{2}} \\ z(x^2+y^2+z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\nabla \cdot \bar{F_f} = -Gm_1 \nabla \cdot \left[\begin{pmatrix} x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ y(x^2+y^2+z^2)^{-\frac{3}{2}} \\ z(x^2+y^2+z^2)^{-\frac{3}{2}} \end{pmatrix} \right]$$

$$\text{Let } k = \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ y(x^2+y^2+z^2)^{-\frac{3}{2}} \\ z(x^2+y^2+z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\nabla \cdot k = f_x + g_y + h_z = \nabla \cdot \begin{pmatrix} x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ y(x^2+y^2+z^2)^{-\frac{3}{2}} \\ z(x^2+y^2+z^2)^{-\frac{3}{2}} \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left[x(x^2+y^2+z^2)^{-\frac{3}{2}} \right]$$

$$\frac{\partial f}{\partial x} = -3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} + (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial g}{\partial y} = -3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} + (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$

$$\frac{\partial f}{\partial z} = -3z^2(x^2+y^2+z^2)^{-\frac{5}{2}} + (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\nabla \cdot k = -3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-\frac{5}{2}} + 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = -3(x^2+y^2+z^2)^{-\frac{3}{2}} + 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\nabla \cdot k = 0$$

$$\nabla \cdot F_f = -Gm_1 \times 0$$

$$\nabla \cdot F_f = 0$$

Therefore, for any point outside of the sphere, the divergence of the gravitational field is nonexistent. Consider a region R which is the space containing empty space and a mass sphere. Divergence in region R is the addition of divergence of free space and divergence of mass sphere. Since $div(F_f) = 0$ outside of the mass sphere as shown in previous example, divergence in R is only divergence of mass sphere.

$$div(F_R) = div(F_{sphere}) + div(F_{freespace})$$

By previous example, $div(F_{freespace}) = 0$

$$div(F_R) = div(F_{sphere})$$

Consider a point mass of mass M of uniform density with radius R:

$$\bar{f}_f = \frac{GM}{r^2}\hat{r}$$

$$\iiint \nabla \cdot \bar{f}_f \, dV = \oiint \bar{f}_f \cdot \bar{n} \, ds$$

By considering a surface to be enveloping the mass sphere,

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi R^2 \bar{f}_f$$

For all points with radius $r = R$ away from the center of the mass sphere, $\bar{f}_f = \dfrac{GM}{r^2}\hat{r}$

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi R^2 \frac{GM}{R^2}$$

$$\iiint \nabla \cdot \bar{f}_f \, dV = 4\pi GM$$

By considering the predetermined property of this mass sphere to have uniform density:

$$(\nabla \cdot \bar{f}_f)V = 4\pi GM$$

$$(\nabla \cdot \bar{f}_f) \lim_{V \to 0}[V] = 4\pi G \lim_{M \to 0}[M]$$

$$(\nabla \cdot \bar{f}_f)dV = 4\pi GdM$$

$$\nabla \cdot \bar{f}_f = 4\pi G \frac{dM}{dV}$$

wherein ρ is density,

$$\nabla \cdot \bar{f}_f = 4\pi G\rho$$

By considering the graviational field of any arbitrary object to be the summation of the gravitational field of small components of the object,

$$\bar{F} = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 + \dots \bar{f}_i$$

$$\bar{F} = \sum_{n=1}^{i} [\bar{f}_n]$$

$\nabla\cdot$ could be considered as a linear transformation because $\nabla \cdot (\bar{a} + \bar{b}) = (\nabla \cdot \bar{a}) + (\nabla \cdot \bar{b})$ and also $\nabla \cdot (c\bar{a}) = c(\nabla \cdot \bar{a})$. Therefore,

$$\nabla \cdot \bar{F} = \sum_{n=1}^{i} [\nabla \cdot \bar{f}_n]$$

$$\nabla \cdot \bar{F} = \sum_{n=1}^{i} [4\pi G\rho_n]$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = \lim_{\Delta V_n \to 0} \left[\sum_{n=1}^{\infty} [4\pi G\rho_n \Delta V_n] \right]$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = 4\pi G \lim_{\Delta V_n \to 0} \left[\sum_{n=1}^{\infty} [\rho_n \Delta V_n] \right]$$

$$\rho_n \Delta V_n = \left(\frac{dM_n}{dV_n} \right) dV_n = dM_n$$

$$\lim_{\Delta V_n \to 0} \left[\sum_{n=1}^{\infty} [\rho_n \Delta V_n] \right] = \int_0^{m_e} dM = m_e$$

$$\iiint_R \nabla \cdot \bar{F}\, dV = 4\pi G m_e$$

wherein m_e represents the mass enclosed by the surface. By reiterating The Divergence Theorem,

$$\iiint \nabla \cdot \bar{F}\, dV = \oiint \bar{F} \cdot \bar{n}\, ds$$

$$4\pi G m_e = \oiint \bar{F} \cdot \bar{n}\, ds$$

For the special case wherein $\rho = f(r)$, the gravitational field would be perpendicular to an imaginary spherical surface at radius r away from the center of the mass sphere. Therefore,

$$\bar{F} \cdot \bar{n} = |\bar{F}|$$

$$\oiint \bar{F} \cdot \bar{n}\, ds = |\bar{F}| \times 4\pi r^2$$

$$4\pi G m_e = |\bar{F}| \times 4\pi r^2$$

$$|\bar{F}| = \frac{G m_e}{r^2}$$

The $|\bar{F}|$ represents gravitational field produced by a spherical mass with varying radial density, $\rho = f(r)$. m_e represents enclosed mass, which in this case is the entirety of the mass sphere. Since the force of gravity experienced by an object with mass m_2 is

$$F_{force} = \bar{F_{field}} \times m_2,$$

$$F_{force} = \frac{G m_e m_2}{r^2} = \frac{G m_e}{r^2} \int_0^R f(r) dr$$

wherein R is the radius of the mass sphere of varying radial density.

Chapter 43

Radial Fields

43.1 Curl and Circulation

Let the general equation denoting force produced by a radial field be: $\bar{F} = kr^n\hat{r}$ wherein k and n are arbitrary constants. Since $\hat{r}$ represents a unit vector in the radial direction,

$$\hat{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$r = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$$

$$r^n = \left(x^2+y^2+z^2\right)^{\frac{n}{2}}$$

$$\bar{F} = \frac{k\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\bar{F} = k(x^2+y^2+z^2)^{\frac{1}{2}(n-1)}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\bar{F} = \begin{pmatrix} kx(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} \\ ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} \\ kz(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} \end{pmatrix}$$

Stokes theorem states: $\oint_R \bar{F}\cdot d\bar{r} = \oiint_S \nabla\times\bar{F}\cdot\hat{n}dS$ wgereub R represents a closed path in $\mathbb{R}^3$ and S represents the corresponding surface bounded by the closed path similarly in $\mathbb{R}^3$. In conservative vector field, the circulation, $\oint_R \bar{F}\cdot d\bar{r} = 0$, therefore, the curl, $\nabla\times\bar{F} = 0$.

$$\text{Let } \bar{F} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$\nabla\times\bar{F} = \begin{pmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\frac{\partial h}{\partial y} = kyz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial g}{\partial z} = kyz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} = 0$$

$$\frac{\partial f}{\partial z} = kxz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial h}{\partial x} = kxz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial g}{\partial x} = kxy(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial f}{\partial y} = kxy(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

Therefore, $\nabla \times \bar{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Any radial vector field with the following expression $\bar{F} = kr^n\hat{r}$ is conservative. Since radial vector fields are considered as conservative force fields, a potential function must exist with the following conditions:

$$f = \frac{\partial p}{\partial x} \qquad g = \frac{\partial p}{\partial y} \qquad h = \frac{\partial p}{\partial z}$$

$$p = \int kx(x^2+y^2+z^2)^{\frac{1}{2}(n-1)}dx$$

$$p = \frac{k}{n+1}(x^2+y^2+z^2)^{\frac{1}{2}(n+1)} + C(y,z) + k_1$$

$C(y,z)$ represents functions strictly in terms of y and z and k_1 represents some arbitrary numerical integration constant.

$$\frac{\partial p}{\partial y} = ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} + C'(y,z)$$

The potential function's partial derivative with respect to y represents y component of the force field. Therefore,

$$ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} + C'(y,z) = ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)}$$

$$C'(y,z) = 0$$

$$\int dC(y,z) = 0 \times \int dy$$

$$C(y,z) = C_2(z) + k_2$$

$$p = \frac{k}{n+1}(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)} + C_2(z) + k_1 + k_2$$

$$\frac{\partial p}{\partial z} = kz(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)} + C_2'(z) = kz(x^2 + y^2 + z^2)^{\frac{1}{2}(n-1)}$$

$$C_2'(z) = 0$$

$$C_2(z) = k_3$$

$$p = \frac{k}{n+1}(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)} + k_1 + k_2 + k_3$$

By convention, the potential of a force field is defined as 0 infinitely away from the origin. Therefore,

$$0 = p = \frac{k}{n+1} \lim_{(x,y,z)\to\infty} \left[(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)}\right] + k_1 + k_2 + k_3$$

$$0 = k_1 + k_2 + k_3$$

Therefore, the potential function for any given radial force field is given as:

$$p = \frac{k}{n+1}(x^2 + y^2 + z^2)^{\frac{1}{2}(n+1)}$$