Fluid Dynamics Archives

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Contents

Chapter 1

Basic Definitions

1.1 Dynamic Variables

Name	Symbollic Representation	Units	Description
Lift	L	N	Upward force experienced by the aircraft
Drag	D	N	Backward force experienced by the aircraft

1.2 Geometrical variables

Name	Symbollic Representation	Units	Description
Angle of attack	α	rad	How pitched up or down the wing or horizontal stabilizer is usually, could represent more than just wings or horizontal stabilizers though
Leading Edge	-	-	The front-most edge of the airfoil
Trailing Edge	-	-	The back-most edge of the airfoil
Chord length		m	Length of the chord line, wherein chord line is a line joining the leading edge and trailing edge
Span length		m	The sideways length of the wing. The distance between one wing tip to another wing tip
Mean Camber line			
Chord line			

1.3 Processed Geometry

Name	Symbollic Representation	Units	Description
Aerodynamic Center		-	A specific point in the airfoil wherein the moments acting on the airfoil due to fluid pressures is unchanging with angle of attack
Center of Pressure		-	A specific point in the airfoil wherein the airfoil experiences no resultant moment about this point
Neutral Point			
Aspect Ratio			

1.4 Dimensionless Coefficients

Name	Symbollic Representation	Units	Description
Coefficient of Lift			
Coefficient of Drag			
Coefficient of Moments			

1.5 Definition of Processes

Name	Symbollic Representation	Description
Isothermal	it	Constant temperature
Isobaric	ib	Constant pressure
Isochoric	ic	Constant volume
Adiabatic	ad	No heat exchange with external system
Reversible	rev	No dissipative phenomena, no mass diffusion, no thermal conducitvity, no viscosity
Isentropic	ise	Both Adiabatic and Reversible

Chapter 2

Reynold's Transport Theorem

One variation of Liebniz Rule applicable for volumetric integrals is shown below. for the variable T wherein T may represent a time dependent scalar, vector, or tensor.

$$\frac{d}{dt}\iiint_{R(t)} T\,dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T]dV_o + \iint_{S(t)} T\bar{v}_s\bar{n}dS$$

wherein $R(t)$ represents an arbitray region of space, V_o represents volume, $S(t)$ represents the surface of the region defined by $R(t)$, $\bar{v}_s$ represents the velocity of the moving surface, $\bar{n}$ represents normal vector of the surface. Depending on the variable type T, the operation $T\bar{v}_s\bar{n}$ would depend on a case to case basis.

2.1 Substantive Derivative

Suppose a quantity b is dependent on the the variable time t and the typical cartesian coordinates x, y, z. Taking the derivative of variable a with respect to time yields the following based on chain rule,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \frac{\partial}{\partial x}[b] \times \frac{\partial}{\partial t}[x] + \frac{\partial}{\partial y}[b] \times \frac{\partial}{\partial t}[y] + \frac{\partial}{\partial z}[b] \times \frac{\partial}{\partial t}[z]$$

Taking note that the partial derivatives of the cartesian coordinates defines velocity in the cartesian coordinates. Therefore,

$$\frac{\partial}{\partial t}[x] = u \quad , \quad \frac{\partial}{\partial t}[y] = v \quad , \quad \frac{\partial}{\partial t}[z] = w$$

wherein u, v, and w typically represents velocity in the x, y, and z directions respectively. Therefore, the derivative of y with respect to time t would take the form,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

If the ∇ operator is defined as

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T$$

Therefore, the derivative of y with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

Let the velocity vector be defined as

$$\bar{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

It follows that the derivative of b with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \bar{v} \cdot \nabla b$$

2.2 Divergence Theorem

The Divergence Theorem is stated below. The variable $\bar{F}$ must represent a vector in R^3

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

Alternately,

$$\iiint_{R(t)} \frac{\partial}{\partial x}[\bar{F}] + \frac{\partial}{\partial y}[\bar{F}] + \frac{\partial}{\partial z}[\bar{F}]\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

wherein dV_o represents an infinitesmially small volume. $S(t)$ is the surface encapsulating the region $R(t)$. $\bar{n}$ is the normal vector of the control volume, and dS is an infinitesmial area of surface $S(t)$.

Chapter 3

Governing Equations

3.1 Governing Equation: Continuum Equation

The Governing Continuum Equation in its differential form:

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v_f})$$

The Governing Continuum Equation in its integral form:

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

A more useful alternate form:

$$\frac{d}{dt}M(t) = \iint_{S(t)} \rho(\bar{v}_s - \bar{v_f}) \cdot \bar{n} dS$$

wherein $M(t)$ represent mass contained in a control volume, $\bar{v_f}$ represent the velocity of the fluid and $\bar{v}_s$ represent the velocity of the deforming control volume $R(t)$. $S(t)$ represents the surface that is encapsuating the control volume $R(t)$.

3.1.1 Differential Continuity Proof

Starting with the definition of mass contained in the arbitrary control volume $R(t)$,

$$M(t) = \iiint_{R(t)} \rho \, dV_o$$

Taking the derivative of the mass contained within the control volume with respect to time,

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o$$

By application of Liebniz rule, substituting T with ρ,

$$\frac{d}{dt}\iiint_{R(t)} T \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T] dV_o + \iint_{S(t)} T \bar{v}_s \bar{n} dS$$

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

If the velocity of the surface expanding is equivalent to the velocity of the fluid at the boundary of the control volume ($\bar{v}_s = \bar{v}_f$), then the amount of mass within the control volume must remain constant.

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho]dV_o + \iint_{S(t)} \rho\bar{v}_f \cdot \bar{n}dS$$

The second term of the expression above could be converted into a volumetric integral based on the divergence theorem by substituting F with $\rho\bar{v}_f$.

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n}dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho\bar{v}_f)\, dV_o = \iint_{S(t)} (\rho\bar{v}_f) \cdot \bar{n}dS$$

Therefore,

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho]dV_o + \iiint_{R(t)} \nabla \cdot (\rho\bar{v}_f)\, dV_o$$

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho\bar{v}_f)\, dV_o$$

Since the integration is zero for an arbitrary region the integrand must be zero everywhere. To prove this, simply choose the arbitray region to be infinitesmially small at all points in R^3 and it could be seen that the integrand is always zero everywhere.

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho\bar{v}_f)$$

3.1.2 Integral Continuity Proof

To prove the integral form for the Governing Continuum Equation, consider the time rate of change of a mass enclosed within the control volume:

$$\frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho]dV_o + \iint_{S(t)} \rho\bar{v}_s \cdot \bar{n}dS$$

From the differential form of the Governing Continuum Equation,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho\bar{v}_f)$$

Therefore,

$$\frac{\partial}{\partial t}[\rho] = -\nabla \cdot (\rho\bar{v}_f)$$

Therefore,

$$\frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho]dV_o + \iint_{S(t)} \rho\bar{v}_s \cdot \bar{n}dS = -\iiint_{R(t)} \nabla \cdot (\rho\bar{v}_f)\, dV_o + \iint_{S(t)} \rho\bar{v}_s \cdot \bar{n}dS$$

By applying the divergence theorem to convert the first term volumetric integral into a surface integral,

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n}dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho \bar{v_f}) \, dV_o = \iint_{S(t)} \rho \bar{v_f} \cdot \bar{n} dS$$

$$\frac{d}{dt} M(t) = - \iint_{S(t)} \rho \bar{v_f} \cdot \bar{n} dS + \iint_{S(t)} \rho \bar{v_s} \cdot \bar{n} dS = \iint_{S(t)} \rho (\bar{v_s} - \bar{v_f}) \cdot \bar{n} dS$$

A more familiar form would yield,

$$0 = \frac{d}{dt} M(t) + \iint_{S(t)} \rho (\bar{v_f} - \bar{v_s}) \cdot \bar{n} dS$$

$$0 = \frac{d}{dt} \iiint_{R(t)} \rho \, dV_o + \iint_{S(t)} \rho (\bar{v_f} - \bar{v_s}) \cdot \bar{n} dS$$

3.2 Governing Equation: Momentum Equation

The Governing Momentum Equation in its differential form:

$$\frac{\partial}{\partial t}(\rho\bar{v_f}) + \nabla \cdot (\rho\bar{v_f}\bar{v_f}) = -\nabla P_r + \nabla \cdot \tau + \rho\bar{F_b}$$

wherein ρ represents density, $\bar{v_f}$ represents fluid velocity vector, P_r represents fluid pressure at a particular point, τ represents viscous forces, $\bar{F_b}$ represents body force experienced by the fluid inside the control volume. The Governing Momentum Equation in its integral form:

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o = \iiint_{R(t)} \rho\bar{F_b}\, dV_o + \iint_{S(t)} \bar{F_s} \cdot \bar{n} dS = \iiint_{R(t)} \frac{\partial}{\partial t}(\rho\bar{v_f}) + \nabla \cdot (\rho\bar{v_f}\bar{v_s}) dV_o$$

wherein $\bar{F_s}$ represents surface forces. Like in the previous proof, $S(t)$ represents the surface binding the control volume region $R(t)$. An alternate form to the momentum governing equation exists. It is shown below,

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o = \iint_{S(t)} \rho\bar{v_f}[(\bar{v_s} - \bar{v_f}) \cdot \bar{n}] dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho\bar{F_b}\, dV_o$$

3.2.1 Differential Momentum Proof

The total Momentum $\bar{P_m}$ contained in a control volume,

$$\bar{P_m} = \iiint_{R(t)} \rho\bar{v_f}\, dV_o$$

The derivative of momentum with respect to time,

$$\frac{d}{dt}\bar{P_m} = \frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o$$

By applying Liebniz's rule, substituting T with $\rho\bar{v_f}$

$$\frac{d}{dt}\iiint_{R(t)} T\, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T] dV_o + \iint_{S(t)} T\bar{v_s}\bar{n} dS$$

$$\frac{d}{dt}\bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}]\, dV_o + \iint_{S(t)} \rho\bar{v_f}\bar{v_s}\bar{n} dS$$

By applying Divergence Theorem substituting F with $\rho\bar{v_f}\bar{v_s}$

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho\bar{v_f}\bar{v_s})\, dV_o = \iint_{S(t)} \rho\bar{v_f}\bar{v_s} \cdot \bar{n} dS$$

By substituting the terms to the derivative of momentum with respect to time,

$$\frac{d}{dt}\bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}]\, dV_o + \iiint_{R(t)} \nabla \cdot (\rho\bar{v_f}\bar{v_s})\, dV_o$$

$$\frac{d}{dt}\bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}] + \nabla \cdot (\rho\bar{v_f}\bar{v_s})\, dV_o$$

Since the derivative of momentum with respect to time is the total force applied to the control volume,

$$\iiint_{R(t)} \rho \bar{F}_b \, dV_o + \iint_{S(t)} \bar{F}_s \cdot \bar{n} dS = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho \bar{v_f}] + \nabla \cdot (\rho \bar{v_f} \bar{v_s}) \, dV$$

The first term in the expression above represents the total body force acting on the control volume meanwhile the second term in the expression represents the total surface force acting on the control volume. When the velocity of the surface is identical to the velocity of the fluid flow, $\bar{v_s} = \bar{v_f}$, the total force acting on the specific volume of region $R_s(t)$,

$$\iiint_{R_s(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F}_b \, dV_o = \iiint_{R_s(t)} \frac{\partial}{\partial t}[\rho \bar{v_f}] + \nabla \cdot (\rho \bar{v_f} \bar{v_f}) \, dV$$

Since the equation above is always true under the constraint that the surface velocity of the region is identical to the velocity of the fluid flow, one can simply choose smaller and smaller regions until $R_s(t)$ is an infinitesmially small region This process can be applied everywhere in the fluid. Therefore,

$$-\nabla P_r + \nabla \cdot \tau + \rho \bar{F}_b = \frac{\partial}{\partial t}[\rho \bar{v_f}] + \nabla \cdot (\rho \bar{v_f} \bar{v_f})$$

The equation above is the differential formulation and is always true all throughout the fluid.

3.2.2 Integral Momentum Proof

To prove the alternate form of the momentum governing equation in integral form, the differential formulation of the momentum equation would be vital. Rearranging for the $\frac{\partial}{\partial t}(\rho \bar{v_f})$ term,

$$\frac{\partial}{\partial t}(\rho \bar{v_f}) + \nabla \cdot (\rho \bar{v_f} \bar{v_f}) = -\nabla P_r + \nabla \cdot \tau + \rho \bar{F}_b$$

$$\frac{\partial}{\partial t}(\rho \bar{v_f}) = -\nabla \cdot (\rho \bar{v_f} \bar{v_f}) - \nabla P_r + \nabla \cdot \tau + \rho \bar{F}_b$$

Referencing the previous equation for derivative of momentum within an arbitrary region $R(t)$ with respect to time,

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f} \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho \bar{v_f}] \, dV_o + \iint_{S(t)} \rho \bar{v_f} (\bar{v_s} \cdot \bar{n}) dS$$

Substituting the term, $\frac{\partial}{\partial t}(\rho \bar{v_f})$,

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f} \, dV_o = \iiint_{R(t)} -\nabla \cdot (\rho \bar{v_f} \bar{v_f}) - \nabla P_r + \nabla \cdot \tau + \rho \bar{F}_b \, dV_o + \iint_{S(t)} \rho \bar{v_f} (\bar{v_s} \cdot \bar{n}) dS$$

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f} \, dV_o = -\iiint_{R(t)} \nabla \cdot (\rho \bar{v_f} \bar{v_f}) \, dV_o + \iint_{S(t)} \rho \bar{v_f} (\bar{v_s} \cdot \bar{n}) dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F}_b \, dV_o$$

The applying the Divergence Theorem substituting F with $\rho \bar{v_f} \bar{v_f}$

$$\iiint_{R(t)} \nabla \cdot \bar{F} \, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot \rho \bar{v_f} \bar{v_f} \, dV_o = \iint_{S(t)} \rho \bar{v_f} (\bar{v_f} \cdot \bar{n}) dS$$

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f} \, dV_o = - \iint_{S(t)} \rho \bar{v_f} (\bar{v_f} \cdot \bar{n}) dS + \iint_{S(t)} \rho \bar{v_f} (\bar{v_s} \cdot \bar{n}) dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b} \, dV_o$$

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f} \, dV_o = \iint_{S(t)} \rho \bar{v_f} [(\bar{v_s} - \bar{v_f}) \cdot \bar{n}] dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b} \, dV_o$$

3.3 Governing Equation: Energy Equation

Chapter 4

Navier-Stokes Equations

The full set of the Navier-Stokes Equations are shown below. The three equations below correspond to the differential continuity, momentum, energy laws.

$$\frac{\partial}{\partial t}[\rho] + \bar{v_f} \cdot \nabla\rho = -\rho\nabla \cdot \bar{v_f}$$

$$\rho\left[\frac{\partial}{\partial t}[\bar{v_f}] + (\bar{v_f} \cdot \nabla)\bar{v_f}\right] = -\nabla P_r + \rho\bar{F_b} + \mu\nabla^2\bar{v_f} = -\nabla P_r + \rho\bar{F_b} - \frac{2}{3}\nabla(\mu\nabla \cdot \bar{v_f}) + 2\nabla \cdot (\mu S)$$

$$\rho c_p\left[\frac{\partial}{\partial t}(T) + \bar{v_f} \cdot \nabla T\right] = \nabla \cdot (k\nabla T) - \frac{2}{3}\mu(\nabla \cdot \bar{v_f})^2 + 2\mu S : S + \beta T\frac{D}{Dt}[P_r]$$

4.0.1 Part b

The continuity governing equation for incompressible fluids in cylindrical coordinates,

$$0 = \nabla \cdot \bar{v} = \frac{1}{r}\frac{\partial}{\partial r}[rv_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] + \frac{\partial}{\partial z}[v_z]$$

In 3-dimensions, the fluid velocity has a radial component, an azimuthal (swirl) component and a z component. For this derivation, it is assumed that the fluid does not have swirl and that the flow is axis-symmetric. All quantities are not expected to vary in the θ direction. The flow is also assumed to be steady. Let the fluid velocity vector $\bar{v}$ be defined below,

$$\bar{v} = \begin{bmatrix} v_r & v_\theta & v_z \end{bmatrix}^T$$

wherein the v_r, v_θ and v_z correspond to the fluid velocity components in the radial, azimuthal and z-direction respectively. The radial component of velocity for axis-symmetric potential flow is defined to be,

$$v_{r,i} = cr$$

Reiterating the continuity expression,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[rv_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] + \frac{\partial}{\partial z}[v_z]$$

Due to the axis-symmetric assumption, $\frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] = 0$. Substituting,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[rv_r] + \frac{\partial}{\partial z}[v_z]$$

Substituting the radial velocity for the inviscid flow into the continuity expression,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[r \times cr] + \frac{\partial}{\partial z}[v_z]$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[cr^2] + \frac{\partial}{\partial z}[v_z]$$

$$0 = \frac{1}{r} \times 2cr + \frac{\partial}{\partial z}[v_z]$$

$$0 = 2c + \frac{\partial}{\partial z}[v_z]$$

$$-2c = \frac{\partial}{\partial z}[v_z]$$

$$-2cz + k = v_z$$

wherein k is some constant to satisfy some boundary condition. Stagnation point is defined to be a point in the fluid where there is no velocity. Since a stagnation point occurs at $z = 0$, then the v_z velocity must also be zero at said point. Substituting,

$$0 + k = 0$$

From applying the boundary condition of the stagnation point, k can be safely neglected. Hence, the fluid velocity in the z-direction for inviscid flows,

$$-2cz = v_{z,i}$$

To make an "intelligent" guess on the modification performed for the fluid flow, the stream function for the inviscid flow must first be determined. Assume that the $\bar{A}$ is the velocity potential vector, whose components,

$$\bar{A} = \begin{bmatrix} A_r & A_\theta & A_z \end{bmatrix}^T$$

The relationship of the velocity vector field to the velocity potential vector,

$$\bar{v} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} = \nabla \times \bar{A} = \begin{bmatrix} \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \\ \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \end{bmatrix}$$

Since the problem is assumed to be axis-symmetric, and also no swirl, then

$$0 = \frac{\partial A_r}{\partial z} \quad , \quad 0 = \frac{\partial A_z}{\partial r}$$

Therefore,

$$0 = A_r \quad , \quad 0 = A_z$$

Let $A_\theta = \psi$ which would represent the stream function of this axis-symmetric fluid flow. Substituting for the simplification,

$$v_r = -\frac{\partial A_\theta}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

Susbtituting for ψ,

$$v_r = -\frac{\partial \psi}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

If ψ_i represents the stream function to the inviscid fluid flow,

$$v_{r,i} = -\frac{\partial \psi_i}{\partial z} \quad , \quad v_{z,i} = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

Substituting for the inviscid flow field that was determined earlier,

$$cr = -\frac{\partial \psi_i}{\partial z} \quad , \quad -2cz = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

Analyzing the velocity in the axis-direction,

$$-2cz = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

$$-2czr = \frac{\partial}{\partial r}(r\psi_i)$$

$$-2cz\int r\,dr = \int \frac{\partial}{\partial r}(r\psi_i)\,dr$$

$$-2cz \times \frac{1}{2}r^2 = \int d(r\psi_i)$$

$$-cz \times r^2 = r\psi_i + k(z)$$

$$-czr^2 = r\psi_i + k(z)$$

Analyzing the velocity in the radial direction,

$$cr = -\frac{\partial \psi_i}{\partial z}$$

$$-cr = \frac{\partial \psi_i}{\partial z}$$

Deriving the previous expression with respect to z,

$$\frac{\partial}{\partial z}[-czr^2] = \frac{\partial}{\partial z}[r\psi_i] + \frac{\partial}{\partial z}[k(z)]$$

$$-cr^2 = r\frac{\partial}{\partial z}[\psi_i] + k'(z)$$

Substituting for the derivative of ψ_i with respect to z,

$$-cr^2 = r \times -cr + k'(z)$$

$$-cr^2 = -cr^2 + k'(z)$$

$$k'(z) = 0$$

Hence, $k(z) = constant$. This shows that some constant can be added to the stream function and the resulting fluid flow would still be identical. Let $k(z) = 0$ purely for convenience,

$$-czr^2 = r\psi_i + k(z) = r\psi_i$$

$$-czr = \psi_i$$

This is the stream function for the inviscid flow. The stream function should be altered in order to satisfy the no-slip boundary condition on the walls. Making the minimum changes necessary to allow for this, z in the stream function of the inviscid flow is replaced with an arbitrar function $f(z)$. f is a function purely in z. Let ψ be the stream function to a fluid flow that satisfies the Navier-Stokes equations near a stagnation point,

$$\psi = -crf(z)$$

To simplify notation,

$$\psi = -crf$$

Finding the velocities with the modified stream function,

$$v_r = -\frac{\partial \psi}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

$$v_r = -\frac{\partial}{\partial z}(\psi) \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

$$v_r = -\frac{\partial}{\partial z}(-crf) \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(-cr^2 f)$$

$$v_r = crf' \quad , \quad v_z = \frac{1}{r} \times (-2crf)$$

$$v_r = crf' \quad , \quad v_z = -2cf$$

Although the fluid velocity fiield that obeys the Navier-Stokes equation would be different than the inviscid flow field near the wall, the velocity field should be identical to the inviscid flow field very far from the wall. Hence,

$$\lim_{z\to\infty}\left[v_{r,i}\right] = \lim_{z\to\infty}\left[crf'\right]$$

$$\lim_{z\to\infty}\left[cr\right] = \lim_{z\to\infty}\left[cr\right]\lim_{z\to\infty}\left[f'\right]$$

$$1 = \lim_{z\to\infty}\left[f'\right]$$

$$\lim_{z\to\infty}\left[v_{z,i}\right] = \lim_{z\to\infty}\left[-2cf\right]$$

$$\lim_{z\to\infty}\left[-2cz\right] = \lim_{z\to\infty}\left[-2cf\right]$$

$$\lim_{z\to\infty}\left[-2c\right]\lim_{z\to\infty}\left[z\right] = \lim_{z\to\infty}\left[-2c\right]\lim_{z\to\infty}\left[f\right]$$

$$\lim_{z\to\infty}\left[z\right] = \lim_{z\to\infty}\left[f\right]$$

To summarize the stream function of the flow field which is an exact-solution to the Navier-Stokes equation,

$$\psi = -crf$$

The resulting velocity components of the flow field with such a stream function,

$$v_r = crf' \quad , \quad v_z = -2cf$$

The limits of f such that the boundary conditions of the flow field matching with that of the invscid fluid flow very far away from the wall,

$$1 = \lim_{z\to\infty}\left[f'\right] \quad , \quad \lim_{z\to\infty}\left[z\right] = \lim_{z\to\infty}\left[f\right]$$

The laplacian of an arbitrary vector v in polar coordinates,

$$\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2}$$

The momentum equation in the radial direction,

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

Due to the steady state assumption, $\frac{\partial v_r}{\partial t} = 0$. Due to the axis-symmetric assumption, $\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} = 0$. Assuming the flow field does not have any swirl, $\frac{v_\theta^2}{r} = 0$ and $-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} = 0$. Neglecting any body force, $\rho g_r = 0$. Substituting,

$$\rho\left[v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2}\right]$$

Substituting the definition of v_r from the stream-function analysis,

$$\rho\left[crf'\frac{\partial}{\partial r}(crf') + v_z\frac{\partial}{\partial z}(crf')\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{crf'}{r^2}\right]$$

$$\rho\left[crf' \times cf' + v_z \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

$$\rho\left[c^2 rf'^2 + v_z \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

Substituting for the velocity v_z,

$$\rho\left[c^2 rf'^2 - 2cf \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

$$\rho\left[c^2 rf'^2 - 2c^2 rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

The laplacian of the radial velocity,

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_r}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2}$$

Substituting for the radial velocity v_r.

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial crf'}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 crf'}{\partial \theta^2} + \frac{\partial^2 crf'}{\partial z^2}$$

Due to the axis-symmetric assumption

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial crf'}{\partial r}\right] + \frac{\partial^2 crf'}{\partial z^2}$$

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[crf'\right] + \frac{\partial^2 crf'}{\partial z^2}$$

$$\nabla^2 v_r = \frac{1}{r} \times cf' + crf'''$$

$$\nabla^2 v_r = \frac{cf'}{r} + crf'''$$

Reiterating where we left off with the radial momentum,

$$\rho\left[c^2 rf'^2 - 2c^2 rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

Substituting the laplacian of radial velocity into the momentum equation in the radial direction,

$$\rho\left[c^2 rf'^2 - 2c^2 rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\frac{cf'}{r} + crf''' - \frac{cf'}{r}\right]$$

$$\rho\left[c^2 rf'^2 - 2c^2 rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[crf'''\right]$$

The above is the first ordinary differential equation which describes f. Unfortunately, the term $\frac{\partial p}{\partial r}$ cannot be ignored when considering the exact case because there is significant pressure changes in the fluid flow. If the exact solution of the Navier-Stokes mimics potential flow far away, and the inviscid solution does allow for significant pressure gradients, then pressure gradients must also be taken into account when considering the exact solution to the Navier-Stokes.

The momentum equation in the axial direction,

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

Just like before, due to the steady state assumption, $\frac{\partial v_z}{\partial t} = 0$. Due to the axis-symmetric assumption, $\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} = 0$. Neglecting any body force in the axial direction, $\rho g_z = 0$.

Substituting for these simplifications,

$$\rho\left[v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

Substituting for axial velocity v_z in terms of f,

$$v_z = -2cf$$

$$\rho\left[v_r\frac{\partial}{\partial r}(-2cf) - 2cf\frac{\partial}{\partial z}(-2cf)\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

Substituting for radial velocity,

$$v_r = crf'$$

$$\rho\left[crf'\frac{\partial}{\partial r}(-2cf) - 2cf\frac{\partial}{\partial z}(-2cf)\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

$$\rho\left[-2cf \times -2cf'\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

$$\rho\left[4c^2 ff'\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

The laplacian of the axial velocity,

$$\nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}$$

Due to the axis-symmetric assumption, $\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} = 0$

$$\nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \frac{\partial^2 v_z}{\partial z^2}$$

Substituting for radial velocity v_r and axial velocity v_z,

$$\nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(-2cf)\right] + \frac{\partial^2}{\partial z^2}(-2cf)$$

$$\nabla^2 v_z = -2cf''$$

Substituting laplacian of axial velocity into the axial momentum equation,

$$\rho\left[4c^2 ff'\right] = -\frac{\partial p}{\partial z} + \mu \times -2cf''$$

$$4\rho c^2 ff' = -\frac{\partial p}{\partial z} - 2\mu cf''$$

We now have 2 ordinary differential equations that describe f it is now possible to potentially solve the expressions by using differential equation 2 to determine pressure in terms of f and substituting into differential equation 1 to solve for f. Integrating differential equation 2,

$$4\rho c^2 \int ff'\, dz = -\int \frac{\partial p}{\partial z}\, dz - 2\mu c \int f''\, dz$$

$$4\rho c^2 \int f\frac{df}{dz}\, dz = -\int dp - 2\mu c \int f''\, dz$$

$$4\rho c^2 \int f\, df = -\int dp - 2\mu c \int f''\, dz$$

$$4\rho c^2 \times \frac{1}{2}f^2 = -p - 2\mu cf' + k(r)$$

wherein $k(r)$ is a function that is purely in terms of r. This function appears as a consequence of integrating with respect to z.

$$2\rho c^2 f^2 = -p - 2\mu cf' + k(r)$$

Making pressure the subject of the expression,

$$p = -2\mu cf' - 2\rho c^2 f^2 + k(r)$$

The exact solution to the Navier-Stokes equation should match the inviscid solution very far away from the wall. Implying that the pressure for the exact solution and the inviscid solution must match infinitely far away from the wall. Taking the limits as z approaches ∞,

$$\lim_{z\to\infty}[p] = -2\mu c \lim_{z\to\infty}\left[f'\right] - 2\rho c^2 \lim_{z\to\infty}\left[f^2\right] + \lim_{z\to\infty}\left[k(r)\right]$$

Susbtituting the limits for f which was determined earlier,

$$1 = \lim_{z\to\infty} \left[f'\right] \quad , \quad \lim_{z\to\infty} [z] = \lim_{z\to\infty} [f]$$

$$\lim_{z\to\infty} [p] = -2\mu c - 2\rho c^2 \lim_{z\to\infty} \left[z^2\right] + \lim_{z\to\infty} \left[k(r)\right]$$

For values of $z \to \infty$,

$$p = -2\mu c - 2\rho c^2 z^2 + k(r)$$

For the inviscid fluid flow, bernoulli's equation can be used to determine pressure,

$$p_0 = p + \frac{1}{2}\rho|\bar{v}_i|^2$$

wherein $\bar{v}_i$ represents the local fluid velocity for inviscid flow. Considering that there is only radial and axial velocity in an axis-symetric problem,

$$|\bar{v}_i|^2 = v_{r,i}^2 + v_{z,i}^2$$

Substituting,

$$p_0 = p + \frac{1}{2}\rho \left[v_{r,i}^2 + v_{z,i}^2\right]$$

Making pressure the subject of the expression,

$$p = p_0 - \frac{1}{2}\rho \left[v_{r,i}^2 + v_{z,i}^2\right]$$

Substituting for the inviscid radial and axial velocities,

$$p = p_0 - \frac{1}{2}\rho \left[(cr)^2 + (-2cz)^2\right]$$

$$p = p_0 - \frac{1}{2}\rho \left[c^2 r^2 + 4c^2 z^2\right]$$

$$p = p_0 - \frac{1}{2}\rho c^2 r^2 - 2\rho c^2 z^2$$

Matching the exact solution's pressure to the inviscid solution's pressure far away from the wall,

$$p = -2\mu c - 2\rho c^2 z^2 + k(r) = p_0 - \frac{1}{2}\rho c^2 r^2 - 2\rho c^2 z^2$$

$$-2\mu c + k(r) = p_0 - \frac{1}{2}\rho c^2 r^2$$

$$k(r) = p_0 - \frac{1}{2}\rho c^2 r^2 + 2\mu c$$

Substituting the function k into the expression for pressure in the exact solution,

$$p = -2\mu c - 2\rho c^2 z^2 + p_0 - \frac{1}{2}\rho c^2 r^2 + 2\mu c$$

$$p = -2\rho c^2 z^2 + p_0 - \frac{1}{2}\rho c^2 r^2$$

Taking the derivative with respect to r,

$$\frac{\partial p}{\partial r} = -2\rho c^2 \frac{\partial}{\partial r}(z^2) + \frac{\partial p_0}{\partial r} - \frac{1}{2}\rho \frac{\partial}{\partial r}(c^2 r^2)$$

$$\frac{\partial p}{\partial r} = -\frac{1}{2}\rho \frac{\partial}{\partial r}(c^2 r^2)$$

$$\frac{\partial p}{\partial r} = -\frac{1}{2}\rho \times 2c^2 r$$

$$\frac{\partial p}{\partial r} = -\rho c^2 r$$

$$-\frac{\partial p}{\partial r} = \rho c^2 r$$

Substituting the gradient of pressure in r into differential equation 1,

$$\rho \left[c^2 r f'^2 - 2c^2 r f f''\right] = -\frac{\partial p}{\partial r} + \mu \left[c r f'''\right]$$

$$\rho \left[c^2 r f'^2 - 2c^2 r f f''\right] = \rho c^2 r + \mu c r f'''$$

$$\left[c^2 r f'^2 - 2c^2 r f f''\right] = c^2 r + \frac{\mu}{\rho} c r f'''$$

$$r f'^2 - 2 r f f'' = r + \frac{\mu}{\rho}\frac{1}{c} r f'''$$

The relation between kinematic and dynamic viscosity is shown below,

$$\nu = \frac{\mu}{\rho}$$

Substituting,

$$r f'^2 - 2 r f f'' = r + \nu \frac{1}{c} r f'''$$

$$f'^2 - 2 f f'' = 1 + \nu \frac{1}{c} f'''$$

$$-\frac{\nu}{c} f''' + f'^2 - 2 f f'' - 1 = 0$$

Let the function ϕ and variable η be defined below,

$$f = \left(\frac{2c}{\nu}\right)^{-1/2} \phi \quad , \quad \eta = \left(\frac{2c}{\nu}\right)^{1/2} z$$

By conjecture,

$$\frac{d^n}{dz^n} = \left[\left(\frac{2c}{\nu}\right)^{1/2}\right]^n \frac{d^n}{d\eta^n}$$

$$\frac{d^n}{dz^n} = 2^{n/2} \left(\frac{c}{\nu}\right)^{n/2} \frac{d^n}{d\eta^n}$$

Consider when $n = 1$,

$$\frac{d}{dz} = \frac{d}{d\eta} \times \frac{d\eta}{dz}$$

$$\frac{d}{dz} = \frac{d}{d\eta} \times \frac{d}{dz}\left[\left(\frac{2c}{\nu}\right)^{1/2} z\right]$$

$$\frac{d}{dz} = 2^{1/2}\left(\frac{c}{\nu}\right)^{1/2} \frac{d}{d\eta}$$

Let $n = k+1$

$$\frac{d}{dz}\left[\frac{d^k}{dz^k}\right] = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2} \frac{d^{k+1}}{d\eta^{k+1}}$$

Let LHS and RHS be defined,

$$LHS = \frac{d}{dz}\left[\frac{d^k}{dz^k}\right] \quad , \quad RHS = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2} \frac{d^{k+1}}{d\eta^{k+1}}$$

$$LHS = \frac{d}{dz}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2} \frac{d^k}{d\eta^k}\right]$$

$$LHS = \frac{d}{d\eta}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2} \frac{d^k}{d\eta^k}\right] \times \frac{d\eta}{dz}$$

$$LHS = \frac{d}{d\eta}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2} \frac{d^k}{d\eta^k}\right] \times 2^{1/2}\left(\frac{c}{\nu}\right)^{1/2}$$

$$LHS = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2} \frac{d}{d\eta}\left[\frac{d^k}{d\eta^k}\right]$$

Since $LHS = RHS$, by principle of mathematical induction, the formula is true. Applying to the exact solution,

$$-\frac{\nu}{c} f''' + f'^2 - 2ff'' - 1 = 0$$

$$-\frac{\nu}{c} f''' = -\frac{\nu}{c} \times 2^{3/2}\left(\frac{c}{\nu}\right)^{3/2} \frac{d^3}{d\eta^3}\left[2^{-1/2}\left(\frac{c}{\nu}\right)^{-1/2} \phi\right]$$

$$-\frac{\nu}{c} f''' = -2^{3/2-1/2}\left(\frac{c}{\nu}\right)^{3/2-1-1/2} \frac{d^3}{d\eta^3}\left[\phi\right]$$

$$-\frac{\nu}{c} f''' = -2\frac{d^3}{d\eta^3}\left[\phi\right]$$

$$-\frac{\nu}{c} f''' = -2\phi'''$$

$$f'^2 = \left\{2^{1/2}\left(\frac{c}{\nu}\right)^{1/2} \frac{d}{d\eta}\left[2^{-1/2}\left(\frac{c}{\nu}\right)^{-1/2} \phi\right]\right\}^2$$

$$f'^2 = \left\{\frac{d}{d\eta}\left[\phi\right]\right\}^2$$

$$f'^2 = \{\phi'\}^2$$
$$f'^2 = \phi'^2$$

$$-2ff'' = -(2)2^{-1/2}\left[\left(\frac{c}{\nu}\right)^{-1/2}\phi\right]2\left(\frac{c}{\nu}\right)\frac{d^2}{d\eta^2}\left[2^{-1/2}\left(\frac{c}{\nu}\right)^{-1/2}\phi\right]$$

$$-2ff'' = -2^{1-1/2+1-1/2}\left[\left(\frac{c}{\nu}\right)^{-1/2+1-1/2}\phi\right]\frac{d^2}{d\eta^2}[\phi]$$

$$-2ff'' = -2\phi\phi''$$

Substituting all the terms together,

$$-2\phi''' + \phi'^2 - 2\phi\phi'' - 1 = 0$$

$$-2\phi''' - 2\phi\phi'' + \phi'^2 - 1 = 0$$

$$\phi''' + \phi\phi'' - \frac{1}{2}\phi'^2 + \frac{1}{2} = 0$$

Reiterating the radial and axial velocities based on the modified stream function,

$$v_r = crf' \quad , \quad v_z = -2cf$$

At the stagntion point $z = 0$ and since η is a linear function of z, $z = 0$. The stagnation point is defined as a point in the fluid where fluid velocity is zero. Therefore,

$$0 = crf'(0) \quad , \quad 0 = -2cf(0)$$

$$0 = f'(0) \quad , \quad 0 = -f(0)$$

Both derivative and function f have linear scaling with ϕ' and ϕ respectively. Therefore,

$$\phi(0) = 0 \quad , \quad \phi'(0) = 0$$

Infinitely far away,

$$v_r = cr = crf'$$

$$1 = f'(\infty)$$

Taking the derivative of f,

$$\frac{df}{dz} = \left(\frac{2c}{\nu}\right)^{1/2}\frac{d}{d\eta}\left[\left(\frac{2c}{\nu}\right)^{-1/2}\phi\right]$$

$$\frac{df}{dz} = \phi'(\eta)$$

Substituting for infinite distance away,

$$\frac{df}{dz}_{z=\infty} = \phi'(\eta = \infty)$$

$$1 = \phi'(\infty)$$

Chapter 5

Potential Flows

Potential and inviscid flows are flows wherein the effects of viscosity is neglected. The degeneracy from the general Navier Stokes equation is shown below,
Consider the continuity differential governing equation,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v_f})$$

For a steady-state flow, $\frac{\partial}{\partial t}[\rho] = 0$. Substituting to the continuity differential governing equation,

$$0 = \nabla \cdot (\rho \bar{v_f})$$

5.1 Compressible Potential Flow

For the compressible potential flow, let the potential function ψ_c be defined as,

$$\rho \bar{v_f} = \nabla \psi_c$$

Substituting the definition of the potential function into the steady state continuity differential governing equation,

$$0 = \nabla \cdot (\nabla \psi_c) = \nabla^2 \psi_c$$

In cartesian coordinates, this yields,

$$0 = \frac{\partial^2}{\partial x^2}[\psi_c] + \frac{\partial^2}{\partial y^2}[\psi_c] + \frac{\partial^2}{\partial z^2}[\psi_c]$$

5.2 Incompressible Potential Flow

Due to the nature of the fluid, the density could be considered a scalar constant. The $\frac{\partial}{\partial t}[\rho]$ term in this case is evaluates to zero under two conditions: steady state means that the gradient is unchanging, but also since density is non-changing, this particular term also evaluates to zero. Therefore, for for incompressible flows, the potential flow function would still be applicable for non-steady fluid states.

$$0 = \nabla \cdot (\rho \bar{v_f}) = \rho \nabla \cdot (\bar{v_f})$$

Division of both sides by ρ,

$$0 = \nabla \cdot (\bar{v_f})$$

For the incompressible potential flow, let the potential function ψ_i be defined as,

$$\bar{v_f} = \nabla \psi_i$$

By substituting the fluid velocity field into the continuity degenerate differential form,

$$0 = \nabla \cdot (\nabla \psi_i) = \nabla^2 \psi_i = \frac{\partial^2}{\partial x^2}[\psi_i] + \frac{\partial^2}{\partial y^2}[\psi_i] + \frac{\partial^2}{\partial z^2}[\psi_i]$$

Chapter 6

Numerical Methods: Potential Flows

6.1 Gauss-Siedel Grid Method

The Taylor expansion series for an arbitray function $f(t)$ is defined as the following,

$$f(t) = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [f(a)](x-a)^n \right\}$$

Let the control volume be split up into infinitesmially small grids. Each grid will have horizontal width of dx and a vertical height of dy. Let $\psi_{i,j}$ represent the i^{th} column and the j^{th} row value of the stream function. Columns are defined as the vertical edges of the infinitesmially small grids meanwhile rows are defined as the horizontal edges of the infinitesmailly small grids. Therefore, the analysis performed occurs at the edges of the infinitesmially small grids. Indexing starts from the bottom left corner of the control volume, at the origin of the declared coordinate system. Indices start at 0 and progress by increments of 1 along the x and y axes. Analyzing the Taylor Expansion series horizontally in the x-direction,

$$\psi_{i+1,j} = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](\Delta x)^n \right\}$$

For the $i-1^{th}$ term,

$$\psi_{i-1,j} = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](-\Delta x)^n \right\}$$

Taking the second order approximation, and neglecting the higher order terms,

$$\sum_{n=3}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](\Delta x)^n \right\} \approx \sum_{n=3}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](-\Delta x)^n \right\} \approx 0$$

Therefore, for the $i+1^{th}$ term and the $i-1^{th}$ term respectively,

$$\psi_{i+1,j} \approx \psi_{i,j} + \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 \quad , \quad \psi_{i-1,j} \approx \psi_{i,j} - \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

Adding the terms together and manipulating the equation to isolate the $\frac{d^2\psi_{i,j}}{dx^2}$ term,

$$\psi_{i+1,j} + \psi_{i-1,j} \approx \psi_{i,j} + \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 + \psi_{i,j} - \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

$$\psi_{i+1,j} + \psi_{i-1,j} \approx 2\psi_{i,j} + \frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

$$\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 \approx \psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}$$

$$\frac{d^2\psi_{i,j}}{dx^2} \approx \frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2}$$

Using the same process in the y-direction,

$$\frac{d^2\psi_{i,j}}{dy^2} \approx \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2}$$

Substituting the relevant terms into the governing equation,

$$\frac{\partial^2}{\partial x^2}[\psi] + \frac{\partial^2}{\partial y^2}[\psi] = 0$$

$$\frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2} = 0$$

Making $\psi_{i,j}$ the subject of the equation above,

$$(\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j})(\Delta x)^2 = 0$$

$$(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2 = 2\psi_{i,j}(\Delta y)^2 + 2\psi_{i,j}(\Delta x)^2$$

$$(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2 = 2[(\Delta y)^2 + (\Delta x)^2](\psi_{i,j})$$

$$\psi_{i,j} = \frac{(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2}{2[(\Delta y)^2 + (\Delta x)^2]}$$

The relative error ϵ of the stream function for the mesh corners are defined as

$$\left| \frac{\overset{p+1}{\psi_{i,j}} - \overset{p}{\psi_{i,j}}}{\overset{p+1}{\psi_{i,j}}} \right| = \epsilon$$

wherein $\overset{p+1}{\psi_{i,j}}$ represents the $p+1^{th}$ iteration of $\psi_{i,j}$ formulated through the Gauss-Seidel method.The relative error for the control volume would likewise be defined as

$$\frac{1}{k}\sqrt{\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\left(\frac{\overset{p+1}{\psi_{i,j}} - \overset{p}{\psi_{i,j}}}{\overset{p+1}{\psi_{i,j}}}\right)^2} = \epsilon_t \quad , \quad k = (n-1)(m-1)$$

The value of the error ϵ would represent the error tolerance and a direct numerical simulation of the potential flow algorithm should keep running until the error tolerance would be low enough. The values of n and m would be shown analytically below,

$$m = \frac{H}{dy} \quad , \quad n = \frac{L}{dx}$$

Chapter 7

Inviscid Incompressible Flow

7.1 Thin Airfoil Theory

An infintely long wing could be considered a 2-dimensional problem. The thin airfoil theory also incorporates all vortices generated by the viscous boundary layer as a line of free-stream vortices on the mean chamber line of the airfoil. The vortices extend parallel in the spanwise direction. The thin airfoil theory differentiates between different geometries of an airfoil.

7.1.1 Derivation and Theorem-Specific Coefficients

A coordinate change is used in this theorem to express x in terms of θ_d wherein x represents the horizontal distance from the leading edge of the airfoil and θ_d is the change of coordinates variable that is used to represnt x. Here, c represents the chord length of the airfoil. The relationship between x and θ_d is shown below,

$$x = \frac{c}{2}[1 - \cos(\theta_d)] \quad , \quad 0 \leq \theta_d \leq \pi$$

Conversely θ_d in terms of x,

$$\frac{2x}{c} = 1 - \cos(\theta_d)$$

$$\cos(\theta_d) = 1 - \frac{2x}{c}$$

$$\theta_d = \arccos\left[1 - \frac{2x}{c}\right] \quad , \quad 0 \leq x \leq c$$

The coefficients A_0 and A_n are shown below,

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx} d\theta_d \quad , \quad A_n = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos(n\theta_d)\, d\theta_d$$

wherein α represents angle of attack of the specific airfoil.

7.1.2 Lift

The equation for the coefficient of lift C_L is shown below,

$$C_L = 2\pi\left(A_0 + \frac{A_1}{2}\right) = 2\pi\left(\alpha - \alpha_0\right)$$

wherein α_0 represents the angle of attack for the airfoil when lift is zero.

7.1.3 Moments

The moments about the aerodynamic center of the airfoil,

$$C_{M,ac} = C_{M,c/4} = \frac{\pi}{4}(A_2 - A_1)$$

wherein C_M represent the moments coefficient and the subscript ac represents at the aerodynamic center. Since the aerodynamic center is assumed to be quarter chord of the airfoil, the coefficients of moments at the aerodynamic center is identical to the coefficient of moments at the quarter chord of the airfoil. The coefficients A_1 and a_2 are obtainable via the equations above.

7.1.4 Limitations

The thin airfoil theori fails at thicker airfols, also, could not predict drag. The theory also fails to predict stall since the airflow is assumed to be inviscid. The theorem can only be used to find lift and pitching coefficient.

7.1.5 Other Implications

7.2 Lifting Line Theory

An early attempt at describing the behaviours of a finite wing. Uniquely the lifting line theory incorporates free-stream vortices in a 'u'-shape. Firstly in the configuration of a horse-shoe vortex , and in more advanced interpretation,s as a bunch of horse-shoe vortices superimposed on top of each other along the 'lifting-line'. The lifting line theory differentiates could only take into account the total effect of airfoil twist, geometry and perhaps deployed flaps instead of each individual effect. The total factor of those effects affect the free-stream vortex distribution along the 'lifting-line'.

7.2.1 Lift

Based on the Kutta-Joukowski theorem,

$$\frac{dL}{dy} = \rho_\infty v_\infty \Gamma(y)$$

wherein ρ_∞ represents the free stream density of the airflow, v_∞ represents free stream velocity and $\Gamma(y)$ represents the vortex strength at some point along the wing. Integrating the expression above,

$$L = \rho_\infty v_\infty \int_{-b/2}^{b/2} \Gamma(y)\, dy$$

wherein b represents the full wingspan. The coordinate as per usual is nested on the root of the wing at the leading edge of the main wing. The x-direction is prallel to the chord line, the y-direction is parallel ot the span length, and z-direction is vertical.

7.2.2 Drag

Since the fluid is inviscid, there is no drag due to viscosity, or flow seperation. In the lifting line theory, then the drag is purely induced drag. Induced drag is caused by the "downwash" of the wing causing a slightly altered angle of attack. This causes the lift force to tilt backwards slightly, contributing to drag. Since this phenomenon only exists due to the existence of downwash which in turn is due to lift generation, induced drag is also known as lift-induced drag. The formula for induced drag is below,

$$D_i = \rho_\infty v_\infty \int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y)\, dy$$

wherein the induced angle of attack α_i is determined by the downwash and free stream velocity. The downash in turn is determined by integrating the vortex distribution via biot-savart law.

$$\alpha_i(y) = \arctan\left[\frac{-w(y)}{v_\infty}\right]$$

Due to the small angle approximation or taking the first order taylor approximation, $\lim_{x\to 0}[\tan(x)] = x$. By the small angle, approximation, then,

$$\alpha_i(y) \approx -\frac{w(y)}{v_\infty}$$

Substituting for the definition of downwash $w(y)$,

$$w(y) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y - y_d}\frac{d\Gamma}{dy}\, dy_d$$

wherein y_d represents a dummy variable that is used for integration purposes only.

$$\alpha_i(y) \approx \frac{1}{4\pi v_\infty}\int_{-b/2}^{b/2} \frac{1}{y - y_d}\frac{d\Gamma}{dy}\, dy_d$$

7.2.3 Elliptical Vortex Distribution Wing

By using the vortex distribution,

$$\Gamma(y) = \Gamma_0\sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

The results of performing the steps above and performing the relevant substitutions,

$$\alpha_i = \frac{\Gamma_0}{2bv_\infty} \quad , \quad C_L = \alpha_i \pi A_R \quad , \quad C_{Di} = \frac{C_L^2}{\pi A_R}$$

wherein A_R represents aspect ratio, and b represents total wingspan.

7.3 Static Longitudinal Stability

The coefficient of moments about the center of mass $C_{M,cm}$ is shown below,

$$C_{M,cm} = C_{M,0} + C_{M,\alpha}\alpha_a$$

wherein $C_{M,0}$ represents the coefficeint of moments about the center of mass when the absolute angle of attack is zero. α_a represents the absolute angle of attack, which is the angle of attack of the airfoil starting at zero from the airfoil producing zero lift. $C_{M,\alpha}$ represents the derivative of $C_{M,cm}$ with respect to the absolute angle of attack α_a. There are only 2 conditions for static longitudinal stability:

1. $C_{M,0}$ must be a positive value
2. $C_{M,\alpha}$ must be a negative value

Chapter 8

Inviscid Compressible Flow

8.1 Thermodynamic Relations

For an ideal gas,

$$p = \rho RT$$

wherein p represents pressure, ρ represents density, and T represents temperature in Kelvins.To compute the c_v and c_p constants,

$$c_p = \frac{\gamma R}{\gamma - 1} \quad , \quad c_v = \frac{R}{\gamma - 1}$$

wherein $\gamma = \dfrac{c_p}{c_v}$. The expression for enthalphy h,

$$h = e + pv$$

Under the assumption that c_v and c_p as are constants,

$$e = c_v T \quad , \quad h = c_p T$$

Assuming no entropy generation via diffusion and that the coefficients c_v and c_p are constant,

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \quad , \quad s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right)$$

wherein the subscript denotes the state of the fluid,a nd v represents specific volume, the reiprocral of density. Only for isentropic processes, the following is true,

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

For flow that is steady, adiabatic, and inviscid, the following equation holds true,

$$h_0 = h + \frac{1}{2}v_f^2$$

wherein h_0 represents stagnation enthalphy, h represents current enthalphy, and v_f represents fluid velocity. The equation above holds true for any two points in a single streamline that follows the conditions stated above.

8.2 Shock Relations

8.2.1 Normal Shocks

For normal shocks, there are 3 governing equations and 2 thermodynamical relationships that is applicable. For continuity, momentum and energy,

$$\rho_1 u_1 = \rho_2 u_2 \quad , \quad p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad , \quad h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

The thermodynamic relations,

$$h_2 = c_p T_2 \quad , \quad p_2 = \rho_2 R T_2$$

The speed of sound a as well as the Mach number M is given by the following equations,

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T} \quad , \quad M = \frac{v_f}{a}$$

The 0 subscript is often used to symbolize stagnation conditions, wherein the small element of fluid is brought to rest adiabatically, the $*$ subscript is used to represent sonic conditions, wherein the small element of fluid is brought to sonic conditions. The equation relating speed of sound and the relationship between speed of sound and sonic speed of sound, as well as Mach and sonic Mach number,

$$\frac{a^2}{\gamma - 1} + \frac{1}{2}u^2 = \frac{\gamma + 1}{2(\gamma - 1)}a_*^2 \quad , \quad M_*^2 = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2}$$

For calorically perfect gases, which is the assumption that c_v and c_p remain as constants, the general relations between temperature, pressure and density to their respective stagnation conditions,

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2 \quad , \quad \frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2}M^2\right]^{\gamma/(\gamma - 1)} \quad , \quad \frac{\rho_0}{\rho} = \left[1 + \frac{\gamma - 1}{2}M^2\right]^{1/(\gamma - 1)}$$

The temperature, pressure and density relations between sonic conditions and stagnant conditions,

$$\frac{T_*}{T_0} = \frac{2}{\gamma + 1} \quad , \quad \frac{p_*}{p_0} = \left[\frac{2}{\gamma + 1}\right]^{\gamma/(\gamma - 1)} \quad , \quad \frac{\rho_*}{\rho_0} = \left[\frac{2}{\gamma + 1}\right]^{1/(\gamma - 1)}$$

The Mach number before and after shock,

$$M_2^2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}$$

The density, velocity, and pressure relations before and after shock,

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \quad , \quad \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

wherein the subscript 1 is used to represent quantities before the shock and 2 is after the shock. The relations for temperature and enthalphy,

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\left[\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right]$$

For the stagnation temperature and pressure before and after the shock,

$$T_{0,1} = T_{0,2} \quad , \quad \frac{p_{0,2}}{p_{0,1}} = e^{-(s_2-s_1)/R} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2-1)\right]^{-1/(\gamma-1)} \left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right]^{\gamma/(\gamma-1)}$$

The ratio of nozzle cross-section at any arbitrary point of the nozzle to the nozzle cross-section at sonic conditions as a function of Mach number,

$$\left(\frac{A}{A_*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}$$

For choked flows, the exit Mach number M_e with or without shock,

$$M_e^2 = \frac{1}{(\gamma-1)}\left\{-1 + \left[1 + 2(\gamma-1)\left(\frac{2}{\gamma+1}\right)^{\dfrac{\gamma+1}{\gamma-1}}\left(\frac{p_{0,1}A_t}{p_e A_e}\right)\right]^{1/2}\right\}$$

8.2.2 Oblique Shocks

The mathematical expression used relating Mach number, θ and β is shown below,

$$\tan(\theta) = 2\cot(\beta)\left\{\frac{M_1^2\sin^2(\beta)-1}{M_1^2[\gamma+\cos(2\beta)]+2}\right\}$$

Reiterating the relation shown in the previous problem,

$$M_{n,1} = M_1\sin\beta \quad , \quad M_{n,2} = M_2\sin(\beta-\theta)$$

The relationship of the quantities between at $M_{n,1}$ and $M_{n,2}$ could be found using the relations at the normal shock relations. The only minor modification is that the stagnant relations for the normal shocks are no longer applicable to oblique shocks due to the tangential velocity of the fluid being conserved across the oblique shock.

8.3 Prantdl-Meyer Expansion Theory

An interesting property of the Prandtl Meyer function $\nu(M)$ wherein M represents Mach number,

$$\theta = \nu(M_2) - \nu(M_1)$$

wherein M_2 represents Mach number after expansion meanwhile M_1 represents Mach number before expansion. Manipulating this relation for convenience,

$$\nu(M_2) = \theta + \nu(M_1)$$

8.4 Shock-Expansion Theory

The coefficient of lift c_L and coefficient of drag c_D in terms of coefficient of pressures c_{p3} and c_{p2} is shown below,

$$c_L = (c_{p3} - c_{p2}) \cos\alpha \quad , \quad c_D = (c_{p3} - c_{p2}) \sin\alpha$$

The coefficient of pressure by definition,

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2(p - p_\infty)}{\rho_\infty U_\infty^2} = \frac{2p_\infty}{\rho_\infty U_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

wherein U_∞ is used to indicate velocity infinitely far away from the wing. Other relations for Mach number and speed of sound,

$$M = \frac{U}{a} \quad , \quad a = \sqrt{\gamma R T} \quad , \quad p = \rho R T$$

Manipulating the Mach number relation and the ideal gas relation,

$$M_\infty a_\infty = U_\infty \quad , \quad \frac{p_\infty}{R T_\infty} = \rho_\infty$$

Substituting both relations into each other,

$$\rho_\infty U_\infty^2 = \frac{p_\infty}{R T_\infty} \times M_\infty^2 a_\infty^2$$

Substituting for the speed of sound a_∞ in terms of temperature and ratio of specific heats,

$$\rho_\infty U_\infty^2 = \frac{p_\infty}{R T_\infty} \times M_\infty^2 \gamma R T_\infty = p_\infty M_\infty^2 \gamma$$

Substituting into the expression for coefficient of pressures,

$$c_p = \frac{2p_\infty}{p_\infty \gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

8.5 Linear Theory

8.5.1 Supersonic

The coefficient of pressure in supersonic linear theory is shown below,

$$c_{p,u} = -\frac{2\theta}{\sqrt{M_\infty^2 - 1}} \quad , \quad c_{p,l} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$

The important equations for the coefficients in supersonic linear theory are listed below,

$$c_L = \frac{4(\alpha + \Delta\alpha)}{\sqrt{M_\infty^2 - 1}} \quad , \quad c_D = \frac{4}{\sqrt{M_\infty^2 - 1}}[(\alpha + \Delta\alpha)^2 + K_2 + K_3]$$

$$c_{m,le} = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[-\frac{1}{2}\alpha + K_1\right] \quad , \quad c_{m,ac} = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[K_1 + \frac{\Delta\alpha}{2}\right]$$

wherein the various constants K as well as $\Delta\alpha$,

$$K_1 = \int_0^1 \left(\frac{d\hat{y}_c}{d\hat{x}}\right) \hat{x}\, d\hat{x} \quad , \quad K_2 = \int_0^1 \left(\frac{d\hat{y}_c}{d\hat{x}}\right)^2 d\hat{x} \quad , \quad K_3 = \int_0^1 \left(\frac{d\hat{y}_t}{d\hat{x}}\right)^2 d\hat{x}$$

$$\Delta\alpha = -\int_0^1 \frac{d\hat{y}_c}{d\hat{x}}\, d\hat{x} = -(\hat{y_{te}} - \hat{y_{le}})$$

wherein the various y-values are further expressed below,

$$y_c = \frac{1}{2}[y_u(x) + y_l(x)] \quad , \quad y_t = \frac{1}{2}[y_u(x) - y_l(x)]$$

The areodynamic centers and variables of integration are shown below,

$$\hat{x} = \frac{x}{c} \quad , \quad \hat{y} = \frac{y}{c} \quad , \quad \frac{x_{ac}}{x} = \frac{1}{2}$$

8.5.2 Subsonic

Based on the Prandtl-Glauert Rule

$$c_L = \frac{c_{L,0}}{\beta} \quad , \quad c_{m,le} = \frac{c_{m,le,0}}{\beta} \quad , \quad c_D = 0$$

wherein the variable $\beta = \sqrt{1 - M_\infty^2}$. Let c_p represet coefficient of pressure in compressible flow meanwhile c_{p0} represent the corresponding coefficient of pressure in the incompressible flow. The Prandtl-Glauert rule relating c_P to c_{p0},

$$c_p = \frac{c_{p0}}{\sqrt{1 - M^2}}$$

The Karman-Tsien rule relating c_P to c_{p0},

$$c_p = \frac{c_{p0}}{\sqrt{1 - M^2} + \left(\frac{M^2}{1 + \sqrt{1 - M^2}}\right)\left(\frac{c_{p0}}{2}\right)}$$

The Laitone's rule relating c_P to c_{p0},

$$c_p = \frac{c_{p0}}{\sqrt{1-M^2} + c_{p0}M^2 \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right] \left[\frac{\sqrt{1-M^2}}{2}\right]}$$

The equation for critical Mach number M_{cr} is shown below,

$$c_{p,cr} = \frac{2}{\gamma M_{cr}^2} \left\{ \left[\frac{1 + \left(\frac{\gamma-1}{2}\right) M_{cr}^2}{\frac{\gamma+1}{2}} \right]^{\gamma/(\gamma-1)} - 1 \right\}$$

The minimum coefficient of pressure at critical Mach number could then be expressed by the Prandtl-Glauert rule shown below,

$$c_{p,cr} = \frac{c_{p0,min}}{\sqrt{1-M_{cr}^2}}$$

Other rules could potentially be used such as the Laitone or the Karman-Tsien for the expression of $c_{p,cr}$. Equating $c_{p,cr}$ would produce an equation whose solution of M_{cr} represents the critical Mach number.

Chapter 9

Creeping Flows

Creeping Flows are the opposite of Potential Flows. The effects of viscosity are important but the inertial effects of the fluid is negligible. The degeneracy from the general Navier Stokes equation is shown below,

9.1 Cartesian Parallel Flows

The continuum governing equation for incompressible fluids in cartesian coordinates are shown below,

$$0 = \nabla \cdot \bar{v_f} = \frac{\partial}{\partial x}[\bar{v_f}] + \frac{\partial}{\partial y}[\bar{v_f}] + \frac{\partial}{\partial z}[\bar{v_f}]$$

Ignoring the y-direction degenerates this problem to a 2-dimensional case and evaluates the term $\frac{\partial}{\partial y}[\bar{v_f}]$ to zero. It is known that this case is a parallel flow case, therefore, the velocity in the z-direction is also non-existent. Therefore, the term $\frac{\partial}{\partial z}[\bar{v_f}]$ evaulates to zero as well.

$$0 = \frac{\partial}{\partial x}[\bar{v_f}]$$

Since this is a 2-dimensional problem, then the velocity is also independent in the thickness y-direction. This also means that the fluid velocity is identical to just the horizontal velocity $\bar{v_f} = \bar{v_x}$. Therefore,

$$0 = \frac{\partial}{\partial x}[\bar{v_f}] = \frac{\partial}{\partial x}[\bar{v_x}]$$

This equation shows that the velocity is independent in the x-direction. An alternate version of the Navier-Stokes momentum equation,

$$\rho\left[\frac{\partial \bar{v_f}}{\partial t} + (\bar{v_f} \cdot \nabla)\bar{v_f}\right] = \rho\bar{g} - \nabla P_r + \mu\nabla^2\bar{v_f}$$

wherein $\nabla^2\bar{v_f} = (\nabla \cdot \nabla)\bar{v_f}$ in this case. Evaluating the Navier-Stokes momentum equation in the x-direction,

$$\rho\left[\frac{\partial \bar{v_x}}{\partial t} + \bar{v_x}\frac{\partial}{\partial x}[\bar{v_x}] + \bar{v_y}\frac{\partial}{\partial y}[\bar{v_x}] + \bar{v_z}\frac{\partial}{\partial z}[\bar{v_x}]\right] = -\frac{\partial}{\partial x}[P_r] + \mu\left[\frac{\partial^2}{\partial x^2}[\bar{v_x}] + \frac{\partial^2}{\partial y^2}[\bar{v_x}] + \frac{\partial^2}{\partial z^2}[\bar{v_x}]\right]$$

If analysis is performed on a steady state fluid flow, the term $\frac{\partial \bar{v_x}}{\partial t}$ evaluates to zero. Using the previous finding after applying the continuity governing equation, the term $\bar{v_x}\frac{\partial}{\partial x}[\bar{v_x}]$ evaluates to zero. Due to the 2-dimensional nature of the problem, the term $\bar{v_y}\frac{\partial}{\partial y}[\bar{v_x}]$ evaluates to zero. Since there is no velocity in the veritcal z-direction due to the parallel nature of the flow, the term $\bar{v_z}\frac{\partial}{\partial z}[\bar{v_x}]$ evaluates to zero. Consistent with the previous findings, since the velocity is not a function of horizontal displacement x, the term $\frac{\partial^2}{\partial x^2}[\bar{v_x}]$ evaluates to zero. Due to the 2-dimensional nature of the problem, the term $\frac{\partial^2}{\partial y^2}[\bar{v_x}]$ evaluates to zero. Therefore,

$$0 = -\frac{\partial}{\partial x}[P_r] + \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

$$\frac{\partial}{\partial x}[P_r] = \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

Evaluating the Navier-Stokes momentum equation in the z-direction,

$$\rho\left[\frac{\partial \bar{v_z}}{\partial t} + \bar{v_x}\frac{\partial}{\partial x}[\bar{v_z}] + \bar{v_y}\frac{\partial}{\partial y}[\bar{v_z}] + \bar{v_z}\frac{\partial}{\partial z}[\bar{v_z}]\right] = -\rho g - \frac{\partial}{\partial z}[P_r] + \mu\left[\frac{\partial^2}{\partial x^2}[\bar{v_z}] + \frac{\partial^2}{\partial y^2}[\bar{v_z}] + \frac{\partial^2}{\partial z^2}[\bar{v_z}]\right]$$

Since the the vertical velocities are zero, then,

$$\frac{\partial \bar{v_z}}{\partial t} = \bar{v_x}\frac{\partial}{\partial x}[\bar{v_z}] = \bar{v_y}\frac{\partial}{\partial y}[\bar{v_z}] = \bar{v_z}\frac{\partial}{\partial z}[\bar{v_z}] = \frac{\partial^2}{\partial x^2}[\bar{v_z}] = \frac{\partial^2}{\partial y^2}[\bar{v_z}] = \frac{\partial^2}{\partial z^2}[\bar{v_z}] = 0$$

Therefore,

$$0 = -\rho g - \frac{\partial}{\partial z}[P_r]$$

Suppose one were to attempt to find the pressure function,

$$-\rho g\int dz = \int dP_r$$

$$-\rho g z + f(x) = P_r$$

wherein $f(x)$ is a function purely in terms of x. Taking the second conclusion we obtained from applying the Navier-Stokes momentum equation in the x-direction,

$$\frac{\partial}{\partial x}[-\rho g z + f(x)] = \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

$$f'(x) = \mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

The velocity $\bar{v_x}$ is only dependent on the z-direction. Therefore, the second order partial derivative term, $\mu\frac{\partial^2}{\partial z^2}[\bar{v_x}]$ must also be fully dependent on the z-direction. This means that the function $f'(x)$ is a constant function $f'(x) = c_1$, therefore, $f(x)$. Firstly this computes the pressure function,

$$P_r = -\rho g z + c_1 x + \alpha$$

wherein both c_1 and α are arbitrary constsants. Secondly, the implications for the velocity profile,

$$f'(x) = c_1 = \mu \frac{\partial^2}{\partial z^2}[\bar{v_x}]$$

$$\frac{1}{\mu}\int c_1 \, dz = \frac{c_1}{\mu}z + c_2 = \frac{\partial}{\partial z}[\bar{v_x}]$$

$$\frac{c_1}{\mu}\int z \, dz + \int c_2 \, dz = \bar{v_x}$$

$$\frac{c_1}{2\mu}z^2 + c_2 z + c_3 = \bar{v_x}$$

9.2 Radial Parallel Flows

The continuum governing equation is shown below,

$$0 = \nabla \cdot \bar{v_f} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

The Navier-Stokes Equation is shown below,

$$\rho \left[\frac{\partial \bar{v_f}}{\partial t} + (\bar{v_f} \cdot \nabla)\bar{v_f}\right] = \rho \bar{g} - \nabla P + \mu \nabla^2 \bar{v_f}$$

9.2.1 Velocity Profile

To find the velocity profile of the fluid in the radial direction, the continuum governing equation is applied first in polar coordiantes.

$$0 = \nabla \cdot \bar{v_f} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Ignoring the z-axis,

$$0 = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

By inspection it could be seen that the radial velocity is zero, and $v_r = 0$. Therefore,

$$0 = \frac{\partial v_\theta}{\partial \theta} \quad , \quad 0 = \frac{\partial^2 v_\theta}{\partial \theta^2}$$

This shows that the tangential velocity is purely a function of radius and not a function of θ. It is Assumed that gravity acts on the z-direction and is largely ignored. Evaluating the expression in the radial direction,

$$\rho \left[\frac{dv_r}{dt} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right] = -\frac{\partial P}{\partial r} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

Since radial velocity is always zero, $\frac{dv_r}{dt} = v_r \frac{\partial v_r}{\partial r} = v_\theta \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} = \frac{v_r}{r^2} = 0$. Since tangential velocity is not a function of θ, $\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} = 0$

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial P}{\partial r}$$

$$\frac{\rho v_\theta^2}{r} = \frac{\partial P}{\partial r}$$

Evaluating the expression in the tangential direction,

$$\rho \left[\frac{dv_\theta}{dt} + v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}\right] = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

$\frac{dv_\theta}{dt} = 0$ because it is assumed that the system is already in a steady state.

$v_r \frac{\partial v_\theta}{\partial r} = \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} = \frac{v_r v_\theta}{r} = 0$ because $v_r = 0$ as the previous assumption. Since tangential velocity is purely a function of radius, $\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} = v_\theta \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = 0$. Therefore, the equation degenerates into,

$$0 = -\frac{1}{r}\frac{\partial P}{\partial\theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}\right]$$

By a seperate mathematical proof,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

Therefore,

$$0 = -\frac{1}{r}\frac{\partial P}{\partial\theta} + \mu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

By the relation found by applying the Incompressible Navier-Stokes on the radial direction, it could be seen that pressure P is dependent on tangential velocity v_θ and r. However, since tangential velocity is only dependent on r from the relation found by applying continuity, then it follows that pressure must only be dependent on r. Therefore, $\frac{\partial P}{\partial r} = 0$. Therefore, the Navier-Stokes applied on the tangential direction further degenerates into,

$$0 = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

Solving for tangential velocity as a function of radius,

$$c_\alpha = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)$$

$$\int_{r=R_1}^{r} rc_\alpha\, dr = \int_{r=R_1,v_\theta=\omega_1 R_1}^{r,v} drv_\theta$$

$$\frac{1}{2}c_\alpha\left[r^2\right]_{r=R_1}^{r} = \left[rv_\theta\right]_{r=R_1,v_\theta=\omega_1 R_1}^{r,v_\theta}$$

$$\frac{1}{2}c_\alpha\left[r^2 - R_1^2\right] = \left[rv_\theta - \omega_1 R_1^2\right]$$

When $r = R_2$, $v_\theta = \omega_2 R_2$. Therefore,

$$\frac{1}{2}c_\alpha\left[R_2^2 - R_1^2\right] = \left[R_2\omega_2 R_2 - \omega_1 R_1^2\right]$$

$$c_\alpha\left[R_2^2 - R_1^2\right] = 2\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]$$

$$c_\alpha = \frac{2\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}$$

By substituting the newly found definition for the constant c_α,

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r^2 - R_1^2\right] = rv_\theta - \omega_1 R_1^2$$

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r - \frac{R_1^2}{r}\right] + \frac{\omega_1 R_1^2}{r} = v_\theta$$

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r - \frac{R_1^2}{r}\right] + \frac{\omega_1 R_1^2\left[R_2^2 - R_1^2\right]}{r\left[R_2^2 - R_1^2\right]} = v_\theta$$

$$\frac{\left[\omega_2 R_2^2 - \omega_1 R_1^2\right]}{\left[R_2^2 - R_1^2\right]}\left[r - \frac{R_1^2}{r}\right] + \frac{\left[\omega_1 R_1^2 R_2^2 - \omega_1 R_1^4\right]}{r\left[R_2^2 - R_1^2\right]} = v_\theta$$

$$\frac{1}{\left[R_2^2 - R_1^2\right]}\left[r\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \frac{R_1^2\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \left[\omega_1 R_1^2 R_2^2 - \omega_1 R_1^4\right]}{r}\right] = v_\theta$$

$$\frac{1}{\left[R_2^2 - R_1^2\right]}\left[r\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \frac{\omega_2 R_1^2 R_2^2 - \omega_1 R_1^4 - \omega_1 R_1^2 R_2^2 + \omega_1 R_1^4}{r}\right] = v_\theta$$

$$v_\theta = \frac{1}{\left[R_2^2 - R_1^2\right]}\left[r\left(\omega_2 R_2^2 - \omega_1 R_1^2\right) - \frac{R_1^2 R_2^2\left[\omega_2 - \omega_1\right]}{r}\right]$$

9.2.2 Scalar Pressure Field

To find the pressure field of the fluid, the two relations that are obtained from applying the incompressible Navier-Stokes equation on the radial and tangential direction is used,

$$\frac{\rho v_\theta^2}{r} = \frac{\partial P}{\partial r} \quad , \quad 0 = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right]$$

The second relation when further operated on gives the function of tangential velocity v_θ with respect to radius r. Integrating the first relation with respect to r and applying the boundary conditions yields,

$$\begin{aligned} P =& P_1 + \frac{\rho}{(R_2^2 - R_1^2)^2}\left[(\omega_2 R_2^2 - \omega_1 R_1^2)^2\left(\frac{r^2 - R_1^2}{2}\right)\right. \\ &\left. - 2R_1^2 R_2^2(\omega_2 - \omega_1)(\omega_2 R_2^2 - \omega_1 R_1^2)\ln\frac{r}{R_1} - \frac{R_1^4 R_2^4}{2}(\omega_2 - \omega_1)^2\left(\frac{1}{r^2} - \frac{1}{R_1^2}\right)\right] \end{aligned}$$

9.2.3 Induced Torque

Suppose the outer cylinder is rotating and the inner cylinder is held at rest, the equation for torque exerted on the inner cylinder must be

$$\Gamma = 2\pi\mu\omega_2 R_1^2\left[\frac{R_2 H}{R_2 - R_1} + \frac{R_1^2}{4b}\right]$$

wherein b represents the gap between the two cylindrical surfaces and Γ is the torque exerted on the inner cylinder. Since this is typically ignored,

$$\Gamma = 2\pi\mu\omega_2 R_1^2\left[\frac{R_2 H}{R_2 - R_1}\right]$$

The torque per unit width length,

$$\Gamma_H = 2\pi\mu\omega_2 R_1^2\left[\frac{R_2}{R_2 - R_1}\right]$$

This is very useful to test for the viscosity of a specific fluid.