Control Archives

Ginger Gengar

1^{st} June 2021

Contents

Chapter 1

Terminology

1.1 Rationale

When studying the dynamics of motion, variables contain alot of context which defines them uniquely apart. For example, velocity perceived in an inertial frame, would be different than velocity perceived in a non-inertial frame, even when describing the same object. In some of the derivations it might be necessary to reference velocity of object a and velocity of object b in the same equation. If both velocities of object a and object b cannot be distinguished from one another, the equation is ambiguously useless. Therefore, the notation presented in the perceding derivations and workings must be able to express many of the different contexts to tell each variable uniquely apart from one another.

1.2 Specifiers

Variables would typically be represented like shown below,

$$^{A}_{C}\bar{v}^{B}_{D}$$

Here the variable being represented $\bar{v}$ is a vector. However, this notation could potentially be extended to scalars v and tensors of higher orders $\bar{\bar{v}}$ as well. The 'A' specifier represents the top left superscript. The 'B' specifier represents top right superscript. 'C' specifier represent bottom left subscript, and 'D' specifier represent bottom right subscript. This notation form is quite strange, but has alot of expressive power, with 4 possible specifiers. Note that specifiers are used flexibly in this document. 1, 2, all, or none of the specifiers may be used in variable expression. If a specifier is not used, as in the common case of the 'C' specifier, it means that no attributes are linked to that specifier, or it does not matter.

1.3 Variable Context

1.3.1 Laws of Motion

Force vectors

$$^{A}_{C}\bar{F}^{B}_{D}$$

- 'A': Frame the force is asociated with. If the specified frame is a non-inertial frame, it implies the addition of fictitious forces. If the specified frame is inertial, fictitious forces excluded.

- 'B': Object experiencing force. This object could be anything from a particle, to a vehicle, to some structural part. This is to settle some of the ambiguity related to force diagrams.
- 'C': Basis vectors. This specifier is to represent which frame or set of vectors is used to represent the force vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

Torque vectors

$$^{A}_{C}\bar{\Gamma}^{B}_{D}$$

- 'A': Frame the torque is asociated with. If specified frame is inertial, no fictitious torque is included. If the specified frame is non-inertial, fictitious torque is included. Fictitious torque is generated as a consequence of fictitious forces.
- 'B': Point of reference. A point in 3-dimensional space that if a force acts on that point, no torque is generated. The further away a particular force is from this point of reference, the larger the magnitude of the torque vector generated.
- 'C': basis vector. The frame or set of vectors used to represent the torque vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

1.3.2 Kinematics

Position vectors

$$^{A}_{C}\bar{r}^{B}_{D}$$

- 'A': none. This specifier is reserved for the frame used in the frame derivative operation.
- 'B': For position vectors, usually this specifier starts with the beginning point of the position vector and ends with the end point of the position vector.
- 'C': basis vector. The frame or set of vectors used to represent the position vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

Velocity vectors

$$^{A}_{C}\bar{v}^{B}_{D}$$

- 'A': Frame that is used when taking the frame derivative of position vector.
- 'B': This specifier functions very similarly to its counterpart in the position vectors. This specifier is to represent what position vector this velocity vector is derived from.
- 'C': basis vector. The frame or set of vectors used to represent the velocity vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

Acceleration Vectors

$$^{A}_{C}\bar{a}^{B}_{D}$$

- 'A': Frame that is used when taking the second order frame derivative of position vector.
- 'B': This specifier functions very similarly to its counterpart in the position vectors. This specifier is to represent what position vector this acceleration vector is derived from.
- 'C': basis vector. The frame or set of vectors used to represent the velocity vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples: Acceleration vector and momentum vector is similar to the velocity vector notation-wise.

Angular velocity vectors

$$^{A}_{C}\bar{\omega}^{B}_{D}$$

- 'A': First frame of reference
- 'B': Second frame of reference
- 'C': basis vector. The frame or set of vectors used to represent the angular velocity vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

1.3.3 Conservation Laws

Momentum vectors

$${}^{A}_{C}\bar{p}^{B}_{D}$$

- 'A': The frame the linear momentum is asociated with. Linear momentum is dependent on velocity which is dependent on the frame it is perceived from. This specifier represents which frame the velocity needed for linear momentum is viewed from.
- 'B': Point of reference. This specifier is very similar to the position vector case. Linear momentum is dependent on velocity vector which is dependent on position vector of object of interest. This specifier represents the beginning point and end point of the position vector.
- 'C': none. The set of basis vectors used to represent the momentum vector must be identical to the basis vectors the momentum is taken with respect to.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

Angular Momentum vectors

$${}^{A}_{C}\bar{L}^{B}_{D}$$

- 'A': Frame the angular momentum vector is asociated with. Angular momentum is dependent on linear momentum, which is dependent on velocity vector, which varies from frame to frame. Hence this specifier is reserved for the frame velocity hence momentum is perceived from.
- 'B': Point of reference. Functions very similarly to the torque case.
- 'C': basis vector. The frame or set of vectors used to represent the angular momentum vector.
- 'D': Naming, indexing. Additional names, or counting indices such as i or j could be included here.

Examples:

1.3.4

Chapter 2

Kinematic Transport Theorem

The Kinematic Transport Theorem (KTT) specifies the relationship between vector time derivatives of different reference frames. Kinematics is the study of motion without cause. The consequences of the derivations below can only tell how some vector time derivative is 'perceived' when viewed in a different way. The derivations below tell nothing of how some object should move. The derivation below only claims, if this is a particular type of movement, this can be 'described' differently when 'viewed' differently.

Reference frames are typically defined with an origin, and a set of basis vectors. Reference frames in 3-dimensions typically have 3 basis vectors to allow general vectors to be expressed. Vectors are typically defined as linear combinations of reference frame basis vectors. Suppose the e reference frame has 3 basis vectors, $\hat{e_1}$, $\hat{e_2}$, and $\hat{e_3}$. If the position of some particle ($\bar{r}^{op}$) relative to the origin of the e reference frame is defined below,

$$\bar{r}^{op} = a\hat{e_1} + b\hat{e_2} + c\hat{e_3}$$

wherein a, b, and c, are scalar coefficients, then the vector $\bar{r}^{op}$ could be expressed as,

$$\bar{r}^{op} = \begin{bmatrix} a & b & c \end{bmatrix}^T$$

Since reference frames are arbitrarily constructed, let reference frame b have its origin in the same position as reference frame e. However, reference frame b has different basis vectors than reference frame e. The basis vectors for reference frame b is defined to be $\hat{b_1}$, $\hat{b_2}$, and $\hat{b_3}$. The vector $\bar{r}^{op}$ can now be defined as,

$$\bar{r}^{op} = d\hat{b_1} + f\hat{b_2} + g\hat{b_3}$$

wherein d, f, and g, are different scalar coefficients than a, b, and c. Just as in the previous case, using the notation rule described earlier, $\bar{r}^{op}$ can be expressed as,

$$\bar{r}^{op} = \begin{bmatrix} d & f & g \end{bmatrix}^T$$

This is very confusing. To distinguish the basis vectors that are used to express $\bar{r}^{op}$, the C specifier is used to differentiate basis vector sets. Hence, we can safely express $\bar{r}^{op}$ below,

$${}_e\bar{r}^{op} = \begin{bmatrix} a & b & c \end{bmatrix}^T \quad , \quad {}_f\bar{r}^{op} = \begin{bmatrix} d & f & g \end{bmatrix}^T$$

Throughout the workings, the attribute 'inertial' and 'non-inertial' are often used. Inertial reference frames are reference frames that are either still in free space or are moving at some constant velocity in a particular direction in free space. Any rotation, and acceleration of the reference frame origins are not allowed in inertial reference frames. If a particular reference frame exhibits rotation in free space or acceleration of its origin, then the reference frame must be non-inertial.

2.1 Frame Derivative

Vectors in R^n can be expressed by an infinite combination of n linearly independent basis vectors. Since the basis vectors used is somewhat arbitrary, then there are infinite ways to express a vector in R^n generally, and more specifically in R^3. The time derivative of a vector is a representation of how the scalar coefficients of the basis vectors change with respect to time. Since the scalar coefficients used to represent a vector is dependent on the basis vectors used, which in turn, is dependent on a declared reference frame, then it is only appropriate to specify time derivative for vectors to a specific reference frame. Taking the time derivative of a vector 'v' with respect to reference frame 'M' is to determine how the scalar coefficients representing vector 'v'as a linear combination of basis vectors for reference frame 'M' changes with respect to time.

Taking the time derivative of a vector, 'with respect to a frame' is defined as a new operation: Frame derivative. The LHS of the equation below represents the frame derivative and the RHS shows the implementation of the frame derivative. We use the same vector $\bar{r}^{op}$ as before.

$$\frac{\overset{e}{\partial}}{\partial t}[r^{\bar{o}p}] = \hat{e_1}\frac{d}{dt}[a] + \hat{e_2}\frac{d}{dt}[b] + \hat{e_3}\frac{d}{dt}[c] \quad , \quad \frac{\overset{f}{\partial}}{\partial t}[\bar{r}^{op}] = \hat{b_1}\frac{d}{dt}[d] + \hat{b_2}\frac{d}{dt}[f] + \hat{b_3}\frac{d}{dt}[g]$$

Note that the time derivative of the scalar quantity must match with the appropriate basis vector and must match with the appropriate refrence frame. More generally,

$$\frac{\overset{e}{\partial^n}}{\partial t^n}[r^{\bar{o}p}] = \hat{e_1}\frac{d^n}{dt^n}[a] + \hat{e_2}\frac{d^n}{dt^n}[b] + \hat{e_3}\frac{d^n}{dt^n}[c] \quad , \quad \frac{\overset{f}{\partial^n}}{\partial t^n}[\bar{r}^{op}] = \hat{b_1}\frac{d^n}{dt^n}[d] + \hat{b_2}\frac{d^n}{dt^n}[f] + \hat{b_3}\frac{d^n}{dt^n}[g]$$

2.2 1^{st} Order Derivative

Let e be an inertial reference frame and b be a non-inertial reference frame. For a vector $\bar{v}$ that starts at the origin of reference frame b,

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{v}] = \frac{\overset{b}{\partial}}{\partial t}[\bar{v}] + {}^{e}\bar{\omega}^{b} \times \bar{v} \tag{2.1}$$

Let o_e represent the origin of reference frame e and o_b represent the origin of reference frame b. Let the position vector $\bar{R}^{o_e o_b}$ represent o_b relative to o_e. Let $\bar{r}^{o_b p}$ represent the position of point p with respect to o_b. Then, it must follow,

$$\bar{r}^{o_e p} = \bar{R}^{o_e o_b} + \bar{r}^{o_b p}$$

Taking the time derivative with respect to the inertial frame e

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}] + \frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_b p}]$$

Based on equation 2.1,

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_b p}] = \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \bar{r}^{o_b p}$$

Substituting,

$$\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}] + \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \bar{r}^{o_b p}$$

This is an important equation. LHS represents the first order time derivative with respect to an inertial reference frame of $\bar{r}^{o_e p}$. If vector $\bar{r}^{o_e p}$ represents position vector of some point, then LHS represents the velocity of that point according to the inertial reference frame e. The term $\frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}]$ represents the velocity of non-inertial reference frame b origin relative relative to the origin of inertial reference frame e viewed in the e frame. The term $\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]$ represents the velocity of point p relative to origin of reference frame b viewed in non-inertial reference frame b. The term ${}^e\bar{\omega}^b \times \bar{r}^{o_b p}$ represents additional velocity perceived in the inertial reference frame e point p exhibits due to the rotation of reference frame b relative to reference frame e.

2.3 2^{nd} Order Derivative

Taking the frame derivative of the velocity expression with respect to the inertial frame e,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}]\right\} = \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}]\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b \times \bar{r}^{o_b p}\right\} \tag{2.2}$$

Simplifying to second order frame derivatives,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e p}]\right\} = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] \quad , \quad \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[\bar{R}^{o_e o_b}]\right\} = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}]$$

Susbtituting into equation 2.2,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] + \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b \times \bar{r}^{o_b p}\right\} \tag{2.3}$$

Applying equation 2.1 to one of the terms in equation 2.2,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} = \frac{\overset{b}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} + {}^e\bar{\omega}^b \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]$$

Simplifying to second order frame derivatives,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} = \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}] + {}^e\bar{\omega}^b \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] \tag{2.4}$$

Using product rule for the last term in equation 2.2,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b \times \bar{r}^{o_b p}\right\} = {}^e\bar{\omega}^b \times \frac{\overset{e}{\partial}}{\partial t}\left\{\bar{r}^{o_b p}\right\} + \frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b\right\} \times \bar{r}^{o_b p} \tag{2.5}$$

By conjecture,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b\right\} = \frac{\overset{b}{\partial}}{\partial t}\left\{{}^e\bar{\omega}^b\right\}$$

This claim is rather simple to prove. Simply apply equation 2.1 to the claim,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b}\right\} = \frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}] + {}^{e}\bar{\omega}^{b} \times {}^{e}\bar{\omega}^{b}$$

The cross product of a vector with itself is zero, ${}^{e}\bar{\omega}^{b} \times {}^{e}\bar{\omega}^{b} = 0$. Hence, we can recover our claim,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b}\right\} = \frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]$$

Applying equation 2.1, to $\frac{\overset{e}{\partial}}{\partial t}\{\bar{r}^{o_b p}\}$,

$$\frac{\overset{e}{\partial}}{\partial t}\{\bar{r}^{o_b p}\} = \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times [\bar{r}^{o_b p}]$$

Susbtituting into equation 2.5,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b} \times \bar{r}^{o_b p}\right\} = {}^{e}\bar{\omega}^{b} \times \left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times [\bar{r}^{o_b p}]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p}$$

Expanding,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}\bar{\omega}^{b} \times \bar{r}^{o_b p}\right\} = {}^{e}\bar{\omega}^{b} \times \left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\} + {}^{e}\bar{\omega}^{b} \times \left\{{}^{e}\bar{\omega}^{b} \times [\bar{r}^{o_b p}]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p} \tag{2.6}$$

Substituting equation 2.4 and 2.6 into equation 2.3,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] + \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \left\{\frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}]\right\}$$
$$+ {}^{e}\bar{\omega}^{b} \times \left\{{}^{e}\bar{\omega}^{f} \times [\bar{r}^{o_b p}]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p}$$

Simplifying,

$$\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] = \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}] + \frac{\overset{b}{\partial^2}}{\partial t^2}[\bar{r}^{o_b p}] + 2{}^{e}\bar{\omega}^{b} \times \frac{\overset{b}{\partial}}{\partial t}[\bar{r}^{o_b p}] + {}^{e}\bar{\omega}^{b} \times \left\{{}^{e}\bar{\omega}^{b} \times [\bar{r}^{o_b p}]\right\} + \left\{\frac{\overset{b}{\partial}}{\partial t}[{}^{e}\bar{\omega}^{b}]\right\} \times \bar{r}^{o_b p} \tag{2.7}$$

The expression above is a purely kinematic relation. If $\bar{r}^{o_e p}$ is considered to be the position vector of a particular point p, then $\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$ would represent acceleration of point p relative to the origin of reference frame e and perceived in the inertial reference frame e. $\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{R}^{o_e o_b}]$ represents the acceleration of the origin of reference frame b relative to the origin of reference frame e perceived in the inertial reference frame e. $\overset{b}{\frac{\partial^2}{\partial t^2}}[\bar{r}^{o_b p}]$ represents the acceleration perceived in non-inertial frame b of point p relative to the origin of frame b. The next few terms have specific formal names.

Equation 2.7 expresses the acceleration perceived in the inertial e frame of point p relative to origin of frame e. It is possible to rewrite the equation 2.7 to instead express the acceleration perceived in the inertial b frame of point p relative to origin of frame b,

$$\overset{e}{\frac{\partial^2}{\partial t^2}}[\bar{r}^{o_e p}] - \overset{e}{\frac{\partial^2}{\partial t^2}}[\bar{R}^{o_e o_b}] - 2{}^{e}\bar{\omega}^{b} \times \overset{b}{\frac{\partial}{\partial t}}[\bar{r}^{o_b p}] - {}^{e}\bar{\omega}^{b} \times \left\{ {}^{e}\bar{\omega}^{b} \times [\bar{r}^{o_b p}] \right\} - \left\{ \overset{b}{\frac{\partial}{\partial t}}[{}^{e}\bar{\omega}^{b}] \right\} \times \bar{r}^{o_b p} = \overset{b}{\frac{\partial^2}{\partial t^2}}[\bar{r}^{o_b p}] \quad (2.8)$$

The $-2{}^{e}\bar{\omega}^{b} \times \overset{b}{\frac{\partial}{\partial t}}[\bar{r}^{o_b p}]$ term is known as the coriolis acceleration. The $-{}^{e}\bar{\omega}^{b} \times \left\{ {}^{e}\bar{\omega}^{b} \times [\bar{r}^{o_b p}] \right\}$ term is known as centrifugal acceleration, and the $-\left\{ \overset{b}{\frac{\partial}{\partial t}}[{}^{e}\bar{\omega}^{b}] \right\} \times \bar{r}^{o_b p}$ term is known as the azimuthal acceleration. Apart from the $\overset{e}{\frac{\partial^2}{\partial t^2}}[\bar{r}^{o_e p}]$ term, the rest of the terms in the LHS is often referred to as fictitious acceleration.

The fictitious acceleration is a consequence of the rotational and translational motion of reference frame b. These fictitious accelerations are not a consequence of physical interactions, but rather a consequence of perception, hence their names. If reference frame e and reference frame b are both inertial reference frames,

$$\overset{e}{\frac{\partial^2}{\partial t^2}}[\bar{r}^{o_e p}] = \overset{b}{\frac{\partial^2}{\partial t^2}}[\bar{r}^{o_b p}]$$

This would require 2 things to be simultaneously true,

$$\overset{e}{\frac{\partial^2}{\partial t^2}}[\bar{R}^{o_e o_b}] = 0 \quad , \quad {}^{e}\bar{\omega}^{b} = 0$$

Another definition for inertial reference frames are reference frames that exhibit no fictitious acceleration. Non-inertial frames are allowed to exhibit fictitious acceleration but inertial reference frames cannot. Therefore, the constraint at the beginning that inertial reference frames are allowed to move translationally relative to one another at some constant velocity and are not allowed any rotation with respect to one another.

Chapter 3

Laws of Motions

3.1 Newton's 2^{nd} Law

3.1.1 Point Particle

Inertial Reference Frame

Newton's 2^{nd} law for a point particle p is shown below,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p} \right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2} [\bar{r}^{o_e p}]$$

LHS represents the summation of n number of forces acting on particle p. The forces here are perceived through an inertial frame e, and hence, represent true forces and not 'fictitious forces'. RHS represents the acceleration of particle p relative to the origin of the inertial frame e and perceived from the inertial frame e. m_p represents the mass of the particle p and $\bar{r}^{o_e p}$ represents the position vector of particle p relative to the origin of the inertial frame e. Note that the left-right sub-superscripts follow the notation convention described in the terminology. $o_e p$ represents the position vector starting from the origin of frame e and ending at the position of particle p.

Non-Inertial Reference Frame

3.1.2 Rigid Bodies

Inertial Reference Frame

Newton's 2^{nd} law for the i^{th} point particle in a rigid body is shown below,

$$\sum_{j=1}^{n_1} \left[{}^{e}\bar{F}_{j}^{p_i} \right] = m_{p_i} \frac{\overset{e}{\partial^2}}{\partial t^2} [\bar{r}^{o_e p_i}]$$

Making the substitutions,

$$\frac{\overset{e}{\partial^2}}{\partial t^2} [\bar{r}^{o_e p_i}] = {}^{e}\bar{a}^{o_e p_i} \quad , \quad \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}_{j}^{p_i} \right] = {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}_{int,j}^{p_i} \right] \tag{3.1}$$

Newton's 2^{nd} law for the i^{th} point particle in a rigid body becomes,

$${}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}_{int,j}^{p_i} \right] = m_i {}^{e}\bar{a}^{o_e p_i}$$

wherein ${}^{e}\bar{F}^{p_i}_{ext}$ represents the resultant external force. The 'B' specifier represents that the external force is acting on the i^{th} particle. The 'A' specifier represents that the external force referred to are perceived in an inertial reference frame. The external forces expressed here are real and does not contain fictitious elements.

${}^{e}\bar{F}^{p_i}_{int,j}$ represents the forces acting on the i^{th} particle due to intermolecular interaction between the particles in the rigid body. The 'A' specifier represents the intermolecular forces are perceived in an inertial reference frame. The 'B' specifier represents that the intermolecular forces are acting on the i^{th} particle. The 'D' specifier holds the counter variable j that is used to sum over all the intermolecular forces experienced by the i^{th} particle.

m_i represents the mass of the i^{th} particle and n_2 represents the number of adjacent particles that has a force interaction with the i^{th} particle.

${}^{e}\bar{a}^{o_e p_i}$ represents acceleration of the i^{th} particle. The 'A' specifier represents the reference frame that acceleration is perceived in. The 'B' specifier represents that this acceleration is for i^{th} particle taken with reference to the origin of the inertial frame e.

To find the resultant force on a rigid body, one must find the summation of all forces acting on each particle of the rigid body. Taking the summation of Newton's 2^{nd} law for point particles on the entire rigid body,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right]+\sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]=\sum_{i=1}^{n_2}\left[m_i\,{}^{e}\bar{a}^{o_e p_i}\right]$$

Here n_2 represents the total number of discrete particles that exist within the rigid body. There are 2 statements that can be made regarding the intermolecular forces:

- A particle cannot exert an intermolecular force on itself. Therefore the intermolecular force a particle exerts on itself is zero.
- By Newton's third law, the force acting on the i^{th} particle by the j^{th} particle is exactly equal and opposite to the force acting on the j^{th} particle by the i^{th} particle.

These 2 statements lead to the summation of intermolecular forces within the rigid body evaluating to zero,

$$\sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]=0$$

Simplifying Newton's 2^{nd} law,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right]=\sum_{i=1}^{n_2}\left[m_i\,{}^{e}\bar{a}^{o_e p_i}\right] \tag{3.2}$$

The center of mass definition for the rigid body is shown below,

$$\bar{r}^{o_e c}=\frac{\displaystyle\sum_{i=1}^{n_2}\left[m_i\bar{r}^{o_e p_i}\right]}{\displaystyle\sum_{i=1}^{n_2}\left[m_i\right]}$$

$\sum_{i=1}^{n_2} [m_i]$ represents the total mass of the rigid body. This is a scalar, and is constant due to the conservation of mass. Manipulating the equation,

$$\left\{ \sum_{i=1}^{n_2} [m_i] \right\} \bar{r}^{o_e c} = \sum_{i=1}^{n_2} [m_i \bar{r}^{o_e p_i}]$$

Taking the frame derivative with respect to the inertial e frame,

$$\frac{\overset{e}{\partial}}{\partial t} \left\{ \left[\sum_{i=1}^{n_2} (m_i) \right] \bar{r}^{o_e c} \right\} = \frac{\overset{e}{\partial}}{\partial t} \left\{ \sum_{i=1}^{n_2} [m_i \bar{r}^{o_e p_i}] \right\}$$

Since m_i is a constant due to conservation of mass,

$$\left[\sum_{i=1}^{n_2} (m_i) \right] \frac{\overset{e}{\partial}}{\partial t} \{ \bar{r}^{o_e c} \} = \sum_{i=1}^{n_2} \left[\frac{\overset{e}{\partial}}{\partial t} \{ m_i \bar{r}^{o_e p_i} \} \right]$$

$$\left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{v}^{o_e c} = \sum_{i=1}^{n_2} \left[m_i \frac{\overset{e}{\partial}}{\partial t} \{ \bar{r}^{o_e p_i} \} \right]$$

Here we introduce velocity of the center of mass perceived in the e-frame and relative to the origin of frame e as ${}^e\bar{v}^{o_e c}$. Simplifying further,

$$\left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{v}^{o_e c} = \sum_{i=1}^{n_2} [m_i {}^e\bar{v}^{o_e p_i}]$$

Taking the frame derivative with respect to the e-frame once more,

$$\frac{\overset{e}{\partial}}{\partial t} \left\{ \left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{v}^{o_e c} \right\} = \frac{\overset{e}{\partial}}{\partial t} \left\{ \sum_{i=1}^{n_2} [m_i {}^e\bar{v}^{o_e p_i}] \right\}$$

Performing similar operations as before,

$$\left[\sum_{i=1}^{n_2} (m_i) \right] \frac{\overset{e}{\partial}}{\partial t} \{ {}^e\bar{v}^{o_e c} \} = \sum_{i=1}^{n_2} \left[m_i \frac{\overset{e}{\partial}}{\partial t} \{ {}^e\bar{v}^{o_e p_i} \} \right]$$

$$\left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{a}^{o_e c} = \sum_{i=1}^{n_2} [m_i {}^e\bar{a}^{o_e p_i}] \tag{3.3}$$

Here we introduce the acceleration of the center of mass perceived in the e-frame and relative to the origin of frame e as ${}^e\bar{a}^{o_e p_i}$. Substituting equation 3.3 to equation 3.2,

$$\sum_{i=1}^{n_2} \left[{}^e\bar{F}^{p_i}_{ext} \right] = \sum_{i=1}^{n_2} [m_i {}^e\bar{a}^{o_e p_i}] = \left[\sum_{i=1}^{n_2} (m_i) \right] {}^e\bar{a}^{o_e c}$$

Let $M = \left[\sum_{i=1}^{n_2} (m_i)\right]$. Here M represents the total mass of the rigid body. Substituting for M,

$$\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}_{ext}^{p_i}\right] = M^{e}\bar{a}^{o_e c}$$

This is the final form for Newton's 2^{nd} law for a rigid body. On the LHS there is the resultant external force acting on the rigid body. On the RHS there is the total mass of the rigid body multiplied by acceleration of the center of mass. Due to Newton's 3^{rd} law, the internal forces of a rigid body can be safely ignored. The rigid body form of Newton's 2^{nd} law replaces forces on individual point particles with the summation of external forces and also replaces acceleration of single particles with the acceleration of the rigid body's center of mass. The rigid body form of Newton's 2^{nd} law replaces mass of the i^{th} particle with the total mass of the rigid body M when compared to Newton's 2^{nd} law for a single particle.

Non-Inertial Reference Frame

3.2 Euler's Law

3.2.1 Point Particle

Inertial Reference Frame

Newton's 2^{nd} law could be extended to rotation as well. Taking the cross product of Newton's 2^{nd} law with position vector $\bar{r}^{o_e p}$,

$$\bar{r}^{o_e p} \times \sum_{i=1}^{n} \left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

The summation $\sum$ operator and the vector cross-product operator are commutative with one another. Therefore,

$$\sum_{i=1}^{n} \left[\bar{r}^{o_e p} \times {}^{e}\bar{F}_{i}^{p}\right] = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] \tag{3.4}$$

Torque $\bar{\Gamma}$ is defined as shown below,

$${}^{e}\bar{\Gamma}_{p,i}^{o_e} = \bar{r}^{o_e p} \times {}^{e}\bar{F}_{i}^{p} \tag{3.5}$$

The 'A' specifier on torque $\bar{\Gamma}$ represents that the torque is perceived in an inertial reference frame. This means ficitious forces should be exluded. The 'B'specifier on torque $\bar{\Gamma}$ represents that the moment is taken with respect to the origin of the inertial reference frame e. The 'D' specifier is to represent that the i^{th} torque produced from the i^{th} force is acting on particle p. Substituting the definition for torque to equation 3.4,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}] \tag{3.6}$$

Here we make the claim,

$$\frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

Let LHS and RHS be defined below,

$$LHS = \frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] \quad , \quad RHS = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

Epxanding LHS based on product rule,

$$LHS = \bar{r}^{o_e p} \times \frac{\overset{e}{\partial}}{\partial t}\left[m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] + \frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p}\right] \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)$$

The vector crosss-product with itself is zero. Therefore the term $\frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p}\right] \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right) = 0$. Ignorinng the second term in the equation above,

$$LHS = \bar{r}^{o_e p} \times \frac{\overset{e}{\partial}}{\partial t}\left[m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right]$$

Scalar multiplication and frame derivative operations are commutative as long as the scalar remains constant with time. Since mass of particle m_p is unchanging in time due to conservation of mass,

$$LHS = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial}}{\partial t}\left[\frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right]$$

Simplifying to 2^{nd} order derivative,

$$LHS = m_p \bar{r}^{o_e p} \times \frac{\overset{e}{\partial^2}}{\partial t^2}\left(\bar{r}^{o_e p}\right)$$

Since $LHS = RHS$, the claim is proven to be true. Since the claim is true, the claim can be substituted into equation 3.6,

$$\sum_{i=1}^{n}\left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] = \frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)\right] \tag{3.7}$$

Let angular momentum $\bar{L}$ be defined below,

$${}^{e}\bar{L}^{o_e}_{p} = \bar{r}^{o_e p} \times m_p \frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)$$

Angular momentum is dependent on $\frac{\overset{e}{\partial}}{\partial t}\left(\bar{r}^{o_e p}\right)$. This term is dependent on the frame derivative operation and which frame the time derivative of $\bar{r}^{o_e p}$ is taken with respect to. The frame used for the frame derivative operation when constructing the angular momentum definition is represented in the 'A' specifier. The angular momentum is also dependent on the position vector of particle p, $\bar{r}^{o_e p}$. 2 facts can be deduced from the position vector in this form: the point of reference the particle p is taken with respect to, in this case origin of inertial frame o_e, and the object the position vector is pointing at, particle p. Likewise, the angular momentum vector notation needs to be able to express the point of reference and the particle being referenced. The point of reference is represented in the 'B' specifier meanwhile the object being referenced, particle p is represented in the 'D' specifier. Substituting the definition for angular momentum, into equation 3.7,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}^{o_e}_{p}\right]$$

Non-Inertial Reference Frame

3.2.2 Rigid Bodies

Inertial Reference Frame

EUler's law for a point particle p,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}^{o_e}_{p}\right]$$

Since i is just a counting variable, the expression above would still be true if i is renamed to another variable. Let $i \to j$. The expression above assumes that there are n numeber of torque acting on particle p. Let $n \to n_1$. The expression above is true for the point particle p. In a system of rigid bodies however, there are alot of particles. So, let thte equation above hold true for a particular particle p_i. Substituting these changes,

$$\sum_{j=1}^{n_1} \left[{}^{e}\bar{\Gamma}^{o_e}_{p_i,j}\right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}^{o_e}_{p_i}\right] \tag{3.8}$$

Reiterating the definition of torque defined at equation 3.5,

$${}^{e}\bar{\Gamma}^{o_e}_{p,i} = \bar{r}^{o_e p} \times {}^{e}\bar{F}^{p}_{i}$$

Performing the same substitutions, $i \to j$, $n \to n_1$, and $p \to p_1$,

$${}^{e}\bar{\Gamma}^{o_e}_{p_i,j} = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{j}$$

Taking the summation for all the torque acting on the i^{th} particle,

$$\sum_{j=1}^{n_1} \left[{}^{e}\bar{\Gamma}^{o_e}_{p_i,j}\right] = \sum_{j=1}^{n_1} \left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{j}\right]$$

Since $\bar{r}^{o_e p_i}$ does not contain a j index, it can be considered a scalar constant in the summation operation. Therefore,

$$\sum_{j=1}^{n_1} \left[{}^{e}\bar{\Gamma}^{o_e}_{p_i,j}\right] = \bar{r}^{o_e p_i} \times \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{j}\right]$$

Reiterating the expression for the forces experienced by the i^{th} particle described in equation 3.2.2,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{j}^{p_i}\right] = {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]$$

Substituting for the forces experienced by the i^{th} particle described in equation 3.2.2,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times \left\{{}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]\right\}$$

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \bar{r}^{o_e p_i} \times \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}_{int,j}^{p_i}\right]$$

The term $\bar{r}^{o_e p_i}$ is not dependent on the counting variable j. Therefore, it acts as a scalar constant in the summation of j. Therefore,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]$$

Susbtituting $\sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right]$ into equation 3.8,

$$\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right] = \sum_{j=1}^{n_1}\left[{}^{e}\bar{\Gamma}_{p_i,j}^{o_e}\right] = \bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]$$

This expression is true for the i^{th} particle. Since the entire rigid body is the summation of all particles, summing the equation above for all particles would yield an equation that is true for the rigid body. Summing the equation above for all particles n_2,

$$\sum_{i=1}^{n_2}\left\{\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right]\right\} = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i} + \sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]\right\}$$

$$\sum_{i=1}^{n_2}\left\{\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right]\right\} = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i}\right\} + \sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right]$$

Due to a colinear argument, $\sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\left[\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{int,j}^{p_i}\right] = 0$. Substituting,

$$\sum_{i=1}^{n_2}\left\{\frac{{}^{e}\partial}{\partial t}\left[{}^{e}\bar{L}_{p_i}^{o_e}\right]\right\} = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i}\right\}$$

The summation and partial derivative operations as commutative. Therefore,

$$\frac{{}^{e}\partial}{\partial t}\left[\sum_{i=1}^{n_2}\left\{{}^{e}\bar{L}_{p_i}^{o_e}\right\}\right] = \sum_{i=1}^{n_2}\left\{\bar{r}^{o_e p_i} \times {}^{e}\bar{F}_{ext}^{p_i}\right\}$$

Let ${}^{e}\bar{L}^{o_e}_{rb} = \sum_{i=1}^{n_2} \left\{ {}^{e}\bar{L}^{o_e}_{p_i} \right\}$. Wherein ${}^{e}\bar{L}^{o_e}_{rb}$ represents the total angular momentum of the rigid body rb. Substituting,

$$\frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}\bar{L}^{o_e}_{rb}\right] = \sum_{i=1}^{n_2} \left\{ \bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{ext} \right\}$$

Let ${}^{e}\bar{\Gamma}^{o_e}_{rb} = \sum_{i=1}^{n_2} \left\{ \bar{r}^{o_e p_i} \times {}^{e}\bar{F}^{p_i}_{ext} \right\}$, wherein ${}^{e}\bar{\Gamma}^{o_e}_{rb}$ represents the torque that is acting on the rigid body as a whole. From the colinear argument made earlier, the internal forces do not contribute to the torque of the rigid body as a whole. Note that ${}^{e}\bar{F}^{p_i}_{ext}$ represents the resultant external forces acting on a single particle p_i in the rigid body rb. Substituting,

$$\frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}\bar{L}^{o_e}_{rb}\right] = {}^{e}\bar{\Gamma}^{o_e}_{rb}$$

Non-Inertial Reference Frame

Chapter 4

Conservation Laws

4.1 Single-Particle

4.1.1 Linear Momentum

The general form of Newton's 2^{nd} law is shown below for a single particle p,

$$\sum_{i=1}^{n}\left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2}\left[\bar{r}^{o_e p}\right]$$

Expressing the 2^{nd} order derivative in terms of 1^{st} order derivative,

$$\sum_{i=1}^{n}\left[{}^{e}\bar{F}_{i}^{p}\right] = m_p \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p}\right]\right\}$$

Integrating both sides with respect to time t, for time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}\left[{}^{e}\bar{F}_{i}^{p}\right]\,dt = \int_{t_1}^{t_2} m_p \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p}\right]\right\}\,dt$$

Since mass of particle m_p is a scalar constant unchanging with time,

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}\left[{}^{e}\bar{F}_{i}^{p}\right]\,dt = m_p\int_{t_1}^{t_2} \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p}\right]\right\}\,dt$$

For ease of notation,

$$\frac{\overset{e}{\partial}}{\partial t}\left[\bar{r}^{o_e p}\right] = {}^{e}_{e}\bar{v}^{o_e p}$$

Here we make the satement that the frame derivative of particle p position vector is the velocity vector. The 'A' specifier shows represents the velocity vector perceived in the e frame. The 'C' specifier represents the velocity vector expressed as a linear combination of the e frame basis vectors, this would be important. Lastly, the 'B' specifier is used to represent the velocity vector as velocity of point particle p relative to the origin of the e frame. Performing the substitution,

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}\left[{}^{e}\bar{F}_{i}^{p}\right]\,dt = m_p\int_{t_1}^{t_2} \frac{\overset{e}{\partial}}{\partial t}\left\{{}^{e}_{e}\bar{v}^{o_e p}\right\}\,dt$$

It was important to specify velocity expressed in terms of the basis vectors of e because in doing so, the frame derivative operation with respect to frame e, acts the same way as a conventional time derivative on each component of velocity vector $\bar{v}$. If velocity vector $\bar{v}$ was expressed in basis vector other than the basis vectors of the e frame, the first order KTT form must be used for the frame derivative and complicates the problem further. Since the frame derivative with respect to frame e functions the same way as conventional time derivative, the fundamental theorem of calculus can be employed on each of the components of velocity vector $\bar{v}$. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_i^{p}\right] dt = m_p \int_{t_1}^{t_2} \overset{e}{\partial} \left\{{}_{e}^{e}\bar{v}^{o_e p}\right\} = m_p \left[{}_{e}^{e}\bar{v}^{o_e p}\right]_{t_1}^{t_2}$$

The summation operation $\sum$ and the integral operation $\int$ are commutative. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_i^{p}\right] dt = \sum_{i=1}^{n} \left[\int_{t_1}^{t_2} {}^{e}\bar{F}_i^{p}\, dt\right] = m_p \left[{}_{e}^{e}\bar{v}^{o_e p}\right]_{t_1}^{t_2}$$

Linear momentum is defined as,

$${}^{e}\bar{P}^{o_e p} = m_p {}_{e}^{e}\bar{v}^{o_e p}$$

Substituting the expression for linear momentum,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{F}_i^{p}\right] dt = \sum_{i=1}^{n} \left[\int_{t_1}^{t_2} {}^{e}\bar{F}_i^{p}\, dt\right] = \left[{}^{e}\bar{P}^{o_e p}\right]_{t_1}^{t_2}$$

The statement above shows the conservation of linear momentum. The change in linear momentum is equivalent to the time integral of the resultant force acting on particle p. The change in linear momentum is also equivalent to the summation of the time integral of every single force acting on particle p.

4.1.2 Angular Momentum

Euler's law is shown below,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}^{e}\bar{L}_{p}^{o_e}\right]$$

If the angular momentum vector is expressed in terms of the basis vectors of the e frame basis vector, the frame derivative operation $\frac{\overset{e}{\partial}}{\partial t}$ functions the same way as a conventional derivative operation. Therefore,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] = \frac{\overset{e}{\partial}}{\partial t} \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right] = \frac{d}{dt} \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right]$$

Integrating the equation with respect to time with time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}_{p,i}^{o_e}\right] dt = \int_{t_1}^{t_2} \frac{d}{dt} \left[{}_{e}^{e}\bar{L}_{p}^{o_e}\right] dt$$

Using the fundamental theorem of calculus,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] dt = \int_{t_1}^{t_2} d\left[{}^{e}_{e}\bar{L}^{o_e}_{p}\right]$$

An exact differential $d\left[{}^{e}_{e}\bar{L}^{o_e}_{p}\right]$ is formed. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] dt = \left[{}^{e}_{e}\bar{L}^{o_e}_{p}\right]_{t_1}^{t_2}$$

The summation operation and integral operation is commutative. Therefore,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[{}^{e}\bar{\Gamma}^{o_e}_{p,i}\right] dt = \sum_{i=1}^{n} \left[\int_{t_1}^{t_2} {}^{e}\bar{\Gamma}^{o_e}_{p,i}\, dt\right] = \left[{}^{e}_{e}\bar{L}^{o_e}_{p}\right]_{t_1}^{t_2}$$

The conservation of angular momentum is similar to the conservation of linear momentum. The change in angular momentum of particle p is equals to the time integral of the resultant torque applied on particle p. The change in angular momentum of particle p is also equivalent to the summation of time integrals for each individual torque acting on particle p.

4.1.3 Kinetic Energy

Reiterating Newton's 2^{nd} law,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}^{p}_{i}\right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p}]$$

If the position vector $\bar{r}^{o_e p}$ is expressed in terms of the basis vectors in the e-frame, the partial derivative operation $\frac{\overset{e}{\partial^2}}{\partial t^2}$ would function the same as the second order conventional derivative. Therefore,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}^{p}_{i}\right] = m_p \frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p}] = m_p \frac{d^2}{dt^2}[{}_{e}\bar{r}^{o_e p}] = m_p \frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}\bar{r}^{o_e p}\right)\right]$$

Let infinitesmial distance in the e frame be defined as a vector below,

$$\overset{e}{\bar{dr}} = \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix}^T$$

Taking the dot product of the resultant force with infinitesmial distance in the e frame $\overset{e}{\bar{dr}}$,

$$\sum_{i=1}^{n} \left[{}^{e}\bar{F}^{p}_{i}\right] \cdot \overset{e}{\bar{dr}} = m_p \frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}\bar{r}^{o_e p}\right)\right] \cdot \overset{e}{\bar{dr}}$$

Let ${}_{e}r^{o_e p}_{i}$ represent the index notation for the components of ${}_{e}\bar{r}^{o_e p}$. By using chain rule,

$$\left\{\frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}\bar{r}^{o_e p}\right)\right]\right\}_i = \frac{d}{dt}\left[\frac{d}{dt}\left({}_{e}r^{o_e p}_{i}\right)\right] = \frac{d}{d\alpha}\left[\frac{d}{dt}\left({}_{e}r^{o_e p}_{i}\right)\right] \times \frac{d\alpha}{dt}$$

wherein α is some variable. Here the $\times$ represents scalar multiplication, not the cross product operation because the quantities ${}_e r_i^{oep}$ is a scalar quantity. Expressing the dot product operation with infinitesmaily distance $\overset{e}{d\bar{r}}$,

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{d\bar{r}}=\frac{d}{d\alpha}\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{d\alpha}{dt}\times dx_i$$

Since α could be any variable we desire, let $\alpha = x_i$. Then $d\alpha = dx_i$. Substituting,

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{d\bar{r}}=\frac{d}{dx_i}\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{dx_i}{dt}\times dx_i=d\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{dx_i}{dt}$$

Earlier, $\overset{e}{d\bar{r}}$ was defined to be some infinitesmial distance. Now, let us specify that $\overset{e}{d\bar{r}}$ represents infinitesmial distance of the particle p trajectory. If this is the case, then $x_i = {}_e r_i^{oep}$. Therefore,

$$\frac{dx_i}{dt}=\frac{d}{dt}\left({}_e r_i^{oep}\right)$$

Susbtituting,

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{d\bar{r}}=d\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]\times\frac{d}{dt}\left({}_e r_i^{oep}\right)=\frac{d}{dt}\left({}_e r_i^{oep}\right)\times d\left[\frac{d}{dt}\left({}_e r_i^{oep}\right)\right]$$

Here we make the substitution $\frac{d}{dt}\left({}_e r_i^{oep}\right) = {}^e v_i^{oep}$, wherein ${}^e v_i^{oep}$ is the index notation of ${}^e\bar{v}_i^{oep}$. The 'A' specifier of velocity being e is perfectly valid because the frame derivative earlier was taken with respect to the inertial e frame.

$$\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{d\bar{r}}={}^e v_i^{oep}\times d\left[{}^e v_i^{oep}\right]$$

Substituting into the resultant force equation,

$$\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{d\bar{r}}=m_p\frac{d}{dt}\left[\frac{d}{dt}\left({}_e\bar{r}^{oep}\right)\right]\cdot\overset{e}{d\bar{r}}=m_p{}^e v_i^{oep}d\left[{}^e v_i^{oep}\right]$$

Let the initial position of the particle be denoted with position vector ${}_e\bar{r}_1$, implying this position vector is expressed in terms of the basis vectors of the inertial e frame. Let particle p have some initial velocity ${}^e_e\bar{v}_1$ initially. At the final position, particle p has the position vector ${}_e\bar{r}_2$ and final velocity of ${}^e_e\bar{v}_2$. Performing integration,

$$\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{d\bar{r}}=\int_{{}^e_e\bar{v}_1}^{{}^e_e\bar{v}_2}m_p{}^e v_i^{oep}d\left[{}^e v_i^{oep}\right]=\left[\frac{1}{2}m_p{}^e v_i^{oep}\ {}^e v_i^{oep}\right]_{{}^e_e\bar{v}_1}^{{}^e_e\bar{v}_2}$$

The LHS is in vector notation but the RHS is in index notation. Changing RHS to vector notation,

$$\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{d\bar{r}}=\left[\frac{1}{2}m_p{}^e\bar{v}^{oep}\cdot{}^e\bar{v}^{oep}\right]_{{}^e_e\bar{v}_1}^{{}^e_e\bar{v}_2}$$

The summation operation and the integral operation are both commutative. Therefore,

$$\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}\sum_{i=1}^{n}\left[{}^e\bar{F}_i^p\right]\cdot\overset{e}{d\bar{r}}=\sum_{i=1}^{n}\left[\int_{{}_e\bar{r}_1}^{{}_e\bar{r}_2}{}^e\bar{F}_i^p\cdot\overset{e}{d\bar{r}}\right]=\left[\frac{1}{2}m_p{}^e\bar{v}^{oep}\cdot{}^e\bar{v}^{oep}\right]_{{}^e_e\bar{v}_1}^{{}^e_e\bar{v}_2}$$

This is the final form conservation of kinetic energy. Let us consider what has happened. Newton's 2^{nd} law was invoked for particle p. The expression was taken with a dot product to infinitesmial distance vector which is then set to the actual path of particle p. The expression was integrated applying the bounds for the initial and final states of particle p. The states are position and velocity, both perceived in the inertial frame. The final result is the expresion for conservation of kinetic energy. The force spatial integral of the particle is the change of kinetic energy.

4.2 Systems of Particles

4.2.1 Linear Momentum

Newton's 2^{nd} law for a system of particles is shown below,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right] = M{}^{e}\bar{a}^{o_e c}$$

wherein M represents the total mass of the system of particles, $M = \left[\sum_{i=1}^{n_2}(m_i)\right]$. Multiplying Newton's 2^{nd} law infinitesmial time dt,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right]\,dt = M{}^{e}\bar{a}^{o_e c}\,dt$$

Although dt represents infinitesmial time, one can simply treat dt like a scalar constant and that a scalar constant multipled by a vector in a vector equation simply implies the scalar constant is multiplied to all of the components of the vectors. ${}^{e}\bar{a}^{o_e c}$ represents acceleration of the center of mass of the system of particles perceived in the inertial frame e.

$${}^{e}\bar{a}^{o_e c} = \frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}\bar{v}^{o_e c}\right]$$

If the velocity of the center of mass is expressed as a linear combination of the basis vectors of inertial frame e,

$${}^{e}_{e}\bar{a}^{o_e c} = \frac{\overset{e}{\partial}}{\partial t}\left[{}^{e}_{e}\bar{v}^{o_e c}\right] = \frac{d}{dt}\left[{}^{e}_{e}\bar{v}^{o_e c}\right] \tag{4.1}$$

This is one of the properties of the frame derivative.
Enforcing that the acceleration terms in Newton's 2^{nd} law be expressed as a linear combination of the basis vectors of inertial frame e,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right]\,dt = M{}^{e}_{e}\bar{a}^{o_e c}\,dt \tag{4.2}$$

Substituting equation 4.2.1 to 4.2,

$$\sum_{i=1}^{n_2}\left[{}^{e}\bar{F}^{p_i}_{ext}\right]\,dt = M\frac{d}{dt}\left[{}^{e}_{e}\bar{v}^{o_e c}\right]\,dt$$

Integrating for time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^e\bar{F}^{p_i}_{ext}\right] dt = \int_{t_1}^{t_2} M \frac{d}{dt} \left[{}^e_e\bar{v}^{o_e c}\right] dt$$

By the fundamental theorem of calculus,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^e\bar{F}^{p_i}_{ext}\right] dt = \int_{t_1}^{t_2} M \, d \left[{}^e_e\bar{v}^{o_e c}\right]$$

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^e\bar{F}^{p_i}_{ext}\right] dt = M \left[{}^e_e\bar{v}^{o_e c}\right]_{t_1}^{t_2}$$

Since the summation and integration operations are commutative with each other,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^e\bar{F}^{p_i}_{ext}\right] dt = \sum_{i=1}^{n_2} \left[\int_{t_1}^{t_2} {}^e\bar{F}^{p_i}_{ext} \, dt\right] = M \left[{}^e_e\bar{v}^{o_e c}\right]_{t_1}^{t_2}$$

Susbstituting for the total mass M of the system of particles,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left[{}^e\bar{F}^{p_i}_{ext}\right] dt = \sum_{i=1}^{n_2} \left[\int_{t_1}^{t_2} {}^e\bar{F}^{p_i}_{ext} \, dt\right] = \left[\sum_{i=1}^{n_2} (m_i)\right] \left[{}^e_e\bar{v}^{o_e c}\right]_{t_1}^{t_2}$$

This is the conservation of linear momentum for a system of particles. The conservation of linear momentum differs compared to the single particle case in that the mass of a single particle is now replaced with the total mass of the system of particles and that the velocity is now not the velocity of a single particle, but is instead the velocity of the center of mass for the system of particles.

4.2.2 Angular Momentum

Euler's law for a system of particles is shown below,

$$\frac{\overset{e}{\partial}}{\partial t} \left[{}^e\bar{L}^{o_e}_{rb}\right] = {}^e\bar{\Gamma}^{o_e}_{rb}$$

wherein ${}^e\bar{\Gamma}^{o_e}_{rb}$ represents the total torque that the system of particles experiences. Asserting that the angular momentum vector $\frac{\overset{e}{\partial}}{\partial t} \left[{}^e\bar{L}^{o_e}_{rb}\right]$ must be expressed in terms of the e-frame basis vector,

$$ {}^e\bar{\Gamma}^{o_e}_{rb} = \frac{\overset{e}{\partial}}{\partial t} \left[{}^e_e\bar{L}^{o_e}_{rb}\right] \tag{4.3}$$

Since the angular momentum vector is expressed in terms of the e-frame basis vector,

$$\frac{\overset{e}{\partial}}{\partial t} \left[{}^e_e\bar{L}^{o_e}_{rb}\right] = \frac{d}{dt} \left[{}^e_e\bar{L}^{o_e}_{rb}\right] \tag{4.4}$$

Substituting equation 4.4 to equation 4.2.2,

$${}^e\bar{\Gamma}^{o_e}_{rb} = \frac{\overset{e}{\partial}}{\partial t} \left[{}^e_e\bar{L}^{o_e}_{rb}\right] = \frac{d}{dt} \left[{}^e_e\bar{L}^{o_e}_{rb}\right]$$

Integrating the equation for time interval $t_1 \leq t \leq t_2$,

$$\int_{t_1}^{t_2} {}^{e}\bar{\Gamma}^{o_e}_{rb}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\left[{}^{e}_{e}\bar{L}^{o_e}_{rb}\right] dt$$

By fundamental theorem of calculus,

$$\int_{t_1}^{t_2} {}^{e}\bar{\Gamma}^{o_e}_{rb}\, dt = \left[{}^{e}_{e}\bar{L}^{o_e}_{rb}\right]_{t_1}^{t_2}$$

4.2.3 Kinetic Energy

Newton's 2^{nd} law for the i^{th} point particle in a system of particles is shown below,

$$\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{j}\right] = {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right] = m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p_i}] \tag{4.5}$$

Work for the i^{th} particle in a system of particles is defined as,

$$w_i = \int_{\bar{r}_{1,i}}^{\bar{r}_{2,i}} \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{j}\right] \cdot d\bar{r}$$

wherein $\bar{r}_{2,i}$ represents the final position and $\bar{r}_{1,i}$ represents the initial position of the i^{th} particle. The work for the i^{th} particle is a line integral following the path that the particle takes. Since $d\bar{r}$ represents a vector of infinitesmial distance that the i^{th} particle takes, then,

$$\frac{d\bar{r}}{dt} = \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

It is important that the position vector $\bar{r}^{o_e p_i}$ here must be expressed in terms of the e-frame basis vectors because otherwise, KTT of the first order must be invoked when $\bar{r}^{o_e p_i}$ is frame-derived with respect to frame e. Manipulating the differentials,

$$d\bar{r} = \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Substituting into the integral expression for the i^{th} particle,

$$w_i = \int_{\bar{r}_{1,i}}^{\bar{r}_{2,i}} \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{j}\right] \cdot d\bar{r} = \int_{\bar{r}_{1,i}}^{\bar{r}_{2,i}} \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{j}\right] \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Substituting for equation 4.5,

$$w_i = \int_{t_1}^{t_2}\left\{{}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]\right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt = \int_{t_1}^{t_2}\left\{m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[\bar{r}^{o_e p_i}]\right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Asserting that the vector $\bar{r}^{o_e p_i}$ must be expressed in terms of the e-frame basis vectors for convenience,

$$w_i = \int_{t_1}^{t_2}\left\{{}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]\right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt = \int_{t_1}^{t_2}\left\{m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\right\} \cdot \frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Taking the summation for all particles existing for the system of particles,

$$\sum_{i=1}^{n_2}[w_i]=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{{}^{e}\bar{F}^{p_i}_{ext}+\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

Let LHS and RHS be defined below,

$$LHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{{}^{e}\bar{F}^{p_i}_{ext}+\sum_{j=1}^{n_1}\left[{}^{e}\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt \quad , \quad RHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt$$

By conjecture,

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\right\}=\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}] \tag{4.6}$$

Let

$$lhs=\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\right\} \quad , \quad rhs=\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

By applying product rule,

$$lhs=\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\right\}+\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

Factoring out the scalar constant 1/2 and simplifying to 2^{nd} order frame derivatives,

$$lhs=\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]+\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

$$lhs=\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]+\frac{1}{2}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

Since dot product is a commutative operation,

$$lhs=\frac{1}{2}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]+\frac{1}{2}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

$$lhs=\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

Since $lhs=rhs$ equation 4.6 is proven to be true. Substituting equation 4.6 into RHS,

$$RHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}m_{p_i}\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]dt \quad , \quad \frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\right\}=\frac{\overset{e}{\partial^2}}{\partial t^2}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]$$

$$RHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}m_{p_i}\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_{e}\bar{r}^{o_e p_i}]\right\}dt$$

The term $\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\right\}$ is asserted to be expressed as a linear combination of frame e-basis vectors. If the term is expressed as a linear combination of frame e basis vectors, then

$$\frac{\overset{e}{\partial}}{\partial t}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\right\}=\frac{d}{dt}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\right\}$$

Substituting to RHS,

$$RHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2} m_{p_i}\frac{d}{dt}\left\{\frac{1}{2}\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\right\}dt$$

$$RHS=\sum_{i=1}^{n_2}\int_{\bar{v}_{1,i}}^{\bar{v}_{2,i}}\frac{1}{2}m_{p_i}\,d\left\{\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\right\}$$

$$RHS=\sum_{i=1}^{n_2}\left\{\frac{1}{2}m_{p_i}\ \left\{\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\cdot\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]\right\}_{\bar{v}_{1,i}}^{\bar{v}_{2,i}}\right\}$$

Making the substitution $\frac{\overset{e}{\partial}}{\partial t}[{}_e\bar{r}^{o_e p_i}]={}^e_e\bar{v}^{o_e p_i}$,

$$RHS=\sum_{i=1}^{n_2}\left\{\frac{1}{2}m_{p_i}\ [{}^e_e\bar{v}^{o_e p_i}\cdot{}^e_e\bar{v}^{o_e p_i}]_{\bar{v}_{1,i}}^{\bar{v}_{2,i}}\right\}$$

This is the statement for the sum of the individual particles' kinetic energy. Let $\bar{r}^{o_e p_i}=\bar{r}^{o_e a}+\bar{r}^{a p_i}$, wherein a could be some arbitrary point. Substituting into LHS,

$$LHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{{}^e\bar{F}^{p_i}_{ext}+\sum_{j=1}^{n_1}\left[{}^e\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}+\bar{r}^{a p_i}]dt$$

Parsing the terms,

$$LHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}\left\{{}^e\bar{F}^{p_i}_{ext}+\sum_{j=1}^{n_1}\left[{}^e\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}]+\left\{{}^e\bar{F}^{p_i}_{ext}+\sum_{j=1}^{n_1}\left[{}^e\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}]dt$$

$$LHS=\sum_{i=1}^{n_2}\int_{t_1}^{t_2}\{{}^e\bar{F}^{p_i}_{ext}\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}]+\left\{\sum_{j=1}^{n_1}\left[{}^e\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}]+\left\{{}^e\bar{F}^{p_i}_{ext}+\sum_{j=1}^{n_1}\left[{}^e\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}]dt$$

$$LHS=\int_{t_1}^{t_2}\sum_{i=1}^{n_2}\left\{\{{}^e\bar{F}^{p_i}_{ext}\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}]\right\}+\sum_{i=1}^{n_2}\left\{\left\{\sum_{j=1}^{n_1}\left[{}^e\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{o_e a}]\right\}$$
$$+\sum_{i=1}^{n_2}\left\{\left\{{}^e\bar{F}^{p_i}_{ext}+\sum_{j=1}^{n_1}\left[{}^e\bar{F}^{p_i}_{int,j}\right]\right\}\cdot\frac{\overset{e}{\partial}}{\partial t}[\bar{r}^{a p_i}]\right\}dt$$

$$LHS = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{o_e a}] \right\} + \sum_{i=1}^{n_2} \left\{ \left\{ \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{o_e a}]$$
$$+ \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{a p_i}] \right\} dt$$

$\sum_{i=1}^{n_2} \left\{ \left\{ \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \right\} = 0$ because of Newton's 3^{rd} law and that a particle cannot exert a force on itself. If one can form a 5^{th} order tensor which the first two indices correspond to i and j, then the main diagonal of this tensor is zero due to the fact a particle cannot exert a force on itself. Due to Newton's 3^{rd} law, the tensor is anti-symmetric. The contraction of an anti-symmetric tensor on its 2 anti-symmetric 'axis' yields zero. Hence, the summation of the internal forces of the system of particles is zero. Simplifying the expression,

$$LHS = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{o_e a}] \right\} + \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{a p_i}] \right\} dt$$

$$LHS = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{o_e a}] + \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{a p_i}] \right\} dt$$

Typically the point a is defined as the center of mass. And in that case, then the term $\sum_{i=1}^{n_2} \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{o_e a}]$ represents work done in translating the center of mass to some velocity. The $\sum_{i=1}^{n_2} \left[{}^{e}\bar{F}^{p_i}_{ext} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{a p_i}] \right]$ term represents the work done by the external force in rotating and stretching of the system of particles, meanwhile the $\sum_{i=1}^{n_2} \left\{ \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{a p_i}] \right\}$ term represents work done by internal forces in stretching. For rigid bodies, this term is zero. Combining the terms together,

$$\sum_{i=1}^{n_2} [w_i] = LHS = RHS$$

$$\sum_{i=1}^{n_2} [w_i] = \int_{t_1}^{t_2} \sum_{i=1}^{n_2} \left\{ {}^{e}\bar{F}^{p_i}_{ext} \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{o_e a}] + \sum_{i=1}^{n_2} \left\{ \left\{ {}^{e}\bar{F}^{p_i}_{ext} + \sum_{j=1}^{n_1} \left[{}^{e}\bar{F}^{p_i}_{int,j} \right] \right\} \cdot \overset{e}{\frac{\partial}{\partial t}} [\bar{r}^{a p_i}] \right\} dt$$
$$= \sum_{i=1}^{n_2} \left\{ \frac{1}{2} m_{p_i} \left[{}^{e}_{e}\bar{v}^{o_e p_i} \cdot {}^{e}_{e}\bar{v}^{o_e p_i} \right]^{\bar{v}_{2,i}}_{\bar{v}_{1,i}} \right\}$$

This is the final form of the conservation of kinetic energy.

Chapter 5

Rigid Body Dynamics

5.1 Inertia Tensor

awdawdaw

5.2 Problem 1

Reiterating Euler's law,

$$\bar{\Gamma}^c = {}^e\dot{\bar{H}}^c$$

wherein c represents the center of mass of the system of particles or a stationary point in the inertial reference frame. Reiterating the definition of angular momentum H,

$$\bar{H} = I\bar{\omega}$$

Here, the angular momentum $\bar{H}$ is taken with respect to the origin of the body fitted coordinates. The reference for $\bar{H}$ may or may not be the center of mass in the definition of angular momentum based on the inertia matrix. To prove Euler's equations of motion, it is necessary to take the angular momentum with respect to the center of mass of the rigid body. Therefore,

$$\bar{H}^c = I_c\bar{\omega}$$

wherein I_c represents the inertia matrix of the body taken with respect to the center of mass. Applying the kinematic relations for the vector $\bar{H}^c$,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}\left(\bar{H}^c\right) + {}^i\bar{\omega}^n \times \bar{H}^c$$

wherein i represents the inertia reference frame here and n represents the non-inertial reference fram e. Substituting for angular momentum in terms of angular velocity at RHS,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}\left(I_c\bar{\omega}\right) + {}^i\bar{\omega}^n \times \left(I_c\bar{\omega}\right)$$

Considering that the inertia matrix I_c is unchanging with respect to the body-fitted coordinates,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = I_c\frac{\overset{n}{\partial}}{\partial t}\left(\bar{\omega}\right) + {}^i\bar{\omega}^n \times \left(I_c\bar{\omega}\right)$$

Due to the definition of the body-fitted coordinates, the angular velocity $\bar{w}$ would be identical to ${}^i\bar{\omega}^n$ which is the rotation of the non-inertial body-fitted coordinates n with respect to the inertial coordinates i. Substituting,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = I_c\frac{\overset{n}{\partial}}{\partial t}\left({}^i\bar{\omega}^n\right) + {}^i\bar{\omega}^n \times \left(I_c\,{}^i\bar{\omega}^n\right)$$

Let ${}^i\bar{\omega}^n$ be declared as a linear combination of the basis vectors in the body fitted coordinate system n. If ${}^i\bar{\omega}^n$ is declared natively in the body fitted coordinate the derivative of ${}^i\bar{\omega}^n$ with respect to the body fitted coordinate be a trivial case of taking the derivative of the all components of the vector. Let the angular velocity vector have its components be specified below,

$${}^i\bar{\omega}^n = \begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}^T$$

wherein $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the basis vectors of the body-fitted coordinates, Analyzing the first term, in the expression above,

$$I_c\frac{\overset{n}{\partial}}{\partial t}\left({}^i\bar{\omega}^n\right) = \begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix} = \begin{bmatrix}I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z\\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z\\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z\end{bmatrix}$$

Analyzing the second term in the expression above,

$${}^i\bar{\omega}^n \times \left(I_c\,{}^i\bar{\omega}^n\right) = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \times \begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \times \begin{bmatrix}I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{bmatrix}$$

Declaring $\hat{i}$, $\hat{j}$ and $\hat{k}$ as dummy variables for the cross-product operations,

$${}^i\bar{\omega}^n \times \left(I_c\,{}^i\bar{\omega}^n\right) = \left|\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ \omega_x & \omega_y & \omega_z\\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{bmatrix}\right|$$

$${}^i\bar{\omega}^n \times \left(I_c\,{}^i\bar{\omega}^n\right) = \hat{i}|A| - \hat{j}|B| + \hat{k}|C|$$

For the A matrix,

$$\hat{i}|A| = \hat{i}\left|\begin{bmatrix}\omega_y & \omega_z\\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{bmatrix}\right|$$

$$\hat{i}|A| = \hat{i}[\omega_y(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)]$$

$$\hat{i}|A| = \hat{i}[\omega_y I_{zx}\omega_x + \omega_y I_{zy}\omega_y + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y - \omega_z I_{yz}\omega_z]$$

Due to the symmetric properties of the inertia tensor,

$$\hat{i}|A| = \hat{i}[\omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y]$$

For the B matrix,

$$\hat{j}|B| = \hat{j}\left|\begin{bmatrix}\omega_x & \omega_z\\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{bmatrix}\right|$$

$$\hat{j}|B| = \hat{j}[\omega_x(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)]$$
$$-\hat{j}|B| = \hat{j}[-\omega_x(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) + \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)]$$
$$-\hat{j}|B| = \hat{j}[-\omega_x I_{zx}\omega_x - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y + \omega_z I_{xz}\omega_z]$$

Due to the symmetric properties of the inertia tensor,

$$-\hat{j}|B| = \hat{j}[-(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y]$$

For the C matrix,

$$\hat{k}|C| = \hat{k}\left|\begin{bmatrix} \omega_x & \omega_y \\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \end{bmatrix}\right|$$

$$\hat{k}|C| = \hat{k}[\omega_x(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)]$$
$$\hat{k}|C| = \hat{k}[\omega_x I_{yx}\omega_x + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xy}\omega_y - \omega_y I_{xz}\omega_z]$$

Due to the symmetric properties of the inertia tensor,

$$\hat{k}|C| = \hat{k}[(\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z]$$

Substituting the various components together,

$${}^{i}\bar{\omega}^{n} \times (I_c\,{}^{i}\bar{\omega}^{n}) = \hat{i}|A| - \hat{j}|B| + \hat{k}|C|$$

$${}^{i}\bar{\omega}^{n} \times (I_c\,{}^{i}\bar{\omega}^{n}) = \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

Substituting the first term and second term to obtain the full form,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

Reiterating Euler's law,

$$\bar{\Gamma}^c = {}^{e}\dot{\bar{H}}^c$$

Since the angular momentum is now taken with respect to center of mass and the time derivative of the angular momentum is taken with respect to an inertial reference frame,

$$\bar{\Gamma}^c = \frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right)$$

Let the moments $\bar{\Gamma}^c$ be defined to have components below,

$$\bar{\Gamma}^c = \frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \begin{bmatrix} M_x & M_y & M_z \end{bmatrix}^T$$

Substituting for the moments and simplifying,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z + \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - (\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z + (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

5.3 Problem 2

Reiterating Euler's law for an arbitrary body-fitted coordinate with origin nested at the center of mass,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + I_{xy}\dot{\omega_y} + I_{xz}\dot{\omega_z} + \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ I_{yx}\dot{\omega_x} + I_{yy}\dot{\omega_y} + I_{yz}\dot{\omega_z} - (\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ I_{zx}\dot{\omega_x} + I_{zy}\dot{\omega_y} + I_{zz}\dot{\omega_z} + (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

For the case of principal axes, the products of inertia evaluate to zero. Substituting,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + \omega_y I_{zz}\omega_z - \omega_z I_{yy}\omega_y \\ I_{yy}\dot{\omega_y} - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x \\ I_{zz}\dot{\omega_z} + \omega_x I_{yy}\omega_y - \omega_y I_{xx}\omega_x \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + \omega_z\omega_y I_{zz} - \omega_z\omega_y I_{yy} \\ I_{yy}\dot{\omega_y} - \omega_x\omega_z I_{zz} + \omega_x\omega_z I_{xx} \\ I_{zz}\dot{\omega_z} + \omega_x\omega_y I_{yy} - \omega_x\omega_y I_{xx} \end{bmatrix}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + (I_{zz} - I_{yy})\omega_z\omega_y \\ I_{yy}\dot{\omega_y} - (I_{zz} - I_{xx})\omega_x\omega_z \\ I_{zz}\dot{\omega_z} + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega_x} + (I_{zz} - I_{yy})\omega_z\omega_y \\ I_{yy}\dot{\omega_y} + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega_z} + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}$$

Earlier, the body-fitted coordinates were declared to be arbitrary as long as the origin of the body-fitted coordinates were nested in the center of mass of the rigid body. To simplify Euler's euqations of motions, the body-fitted coordinates' axes were set to the principal axes. Though this is merely one case of the general case, and it would be reasonable to leave the notations as is, for clarity's sake, the notation for the general moments of inertia are changed to specify moments of inertia with respect to the principal axes,

$$I_{xx} \to I_x \quad , \quad I_{yy} \to I_y \quad , \quad I_{zz} \to I_z$$

Hence, Euler's equations of motions with the constraint of the body-fitted coordinates must have an origin at the center of mass and the axes of the body-fitted coordinates must be the principal axes,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega_x} + (I_z - I_y)\omega_z\omega_y \\ I_y\dot{\omega_y} + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega_z} + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

5.4 Problem 7

Euler's law for a fixed inertia tensor was derived earlier. Reiterating the step before assuming the inertia tensor is unchanging with time,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}\left(I_c\bar{\omega}\right) + {}^i\bar{\omega}^n \times \left(I_c\bar{\omega}\right)$$

If the inertia tensor is changing with time,

$$\frac{\overset{n}{\partial}}{\partial t}\left(I_c\bar{\omega}\right) = I_c\frac{\overset{n}{\partial}}{\partial t}\left(\bar{\omega}\right) + \frac{\overset{n}{\partial}}{\partial t}\left(I_c\right)\bar{\omega}$$

Reiterating the definition of the inertia tensor,

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Assuming that the inertia tensor is taken with respect to center of mass, the inertia tensor simplifies to,

$$I_c = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Substituting the inertia tensor along with its derivatives while making the same assumption of $\bar{\omega} = {}^i\bar{\omega}^n$ and is declared natively in n coordinate system,

$$\frac{\overset{n}{\partial}}{\partial t}(I_c\bar{\omega}) = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \dot{I}_x & 0 & 0 \\ 0 & \dot{I}_y & 0 \\ 0 & 0 & \dot{I}_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\frac{\overset{n}{\partial}}{\partial t}(I_c\bar{\omega}) = \begin{bmatrix} I_x\dot{\omega}_x \\ I_y\dot{\omega}_y \\ I_z\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \dot{I}_x\omega_x \\ \dot{I}_y\omega_y \\ \dot{I}_z\omega_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega}_x + \dot{I}_x\omega_x \\ I_y\dot{\omega}_y + \dot{I}_y\omega_y \\ I_z\dot{\omega}_z + \dot{I}_z\omega_z \end{bmatrix}$$

Reiterating the second term that was determined earlier,

$${}^i\bar{\omega}^n \times (I_c{}^i\bar{\omega}^n) = \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

Taking the products of inertia to be zero,

$${}^i\bar{\omega}^n \times (I_c{}^i\bar{\omega}^n) = \begin{bmatrix} \omega_y I_{zz}\omega_z - \omega_z I_{yy}\omega_y \\ -\omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x \\ \omega_x I_{yy}\omega_y - \omega_y I_{xx}\omega_x \end{bmatrix} = \begin{bmatrix} (I_{zz} - I_{yy})\omega_y\omega_z \\ (I_{xx} - I_{zz})\omega_x\omega_z \\ (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}$$

Susbtituting the moment of inertia to their principal notations,

$${}^i\bar{\omega}^n \times (I_c{}^i\bar{\omega}^n) = \begin{bmatrix} (I_z - I_y)\omega_y\omega_z \\ (I_x - I_z)\omega_x\omega_z \\ (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

Substituting the terms together to obtain the full form,

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \frac{\overset{n}{\partial}}{\partial t}(I_c\bar{\omega}) + {}^i\bar{\omega}^n \times (I_c\bar{\omega})$$

$$\frac{\overset{i}{\partial}}{\partial t}\left(\bar{H}^c\right) = \begin{bmatrix} I_x\dot{\omega}_x + \dot{I}_x\omega_x \\ I_y\dot{\omega}_y + \dot{I}_y\omega_y \\ I_z\dot{\omega}_z + \dot{I}_z\omega_z \end{bmatrix} + \begin{bmatrix} (I_z - I_y)\omega_y\omega_z \\ (I_x - I_z)\omega_x\omega_z \\ (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

Just as before, the time derivative of $\bar{H}^c$ is the moment acting on the body. Substituting,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega}_x + \dot{I}_x\omega_x + (I_z - I_y)\omega_y\omega_z \\ I_y\dot{\omega}_y + \dot{I}_y\omega_y + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega}_z + \dot{I}_z\omega_z + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

5.5 Problem 1

Reiterating Euler's equations of motion,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega_x} + (I_z - I_y)\omega_z\omega_y \\ I_y\dot{\omega_y} + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega_z} + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

Parsing out the scalar quantities,

$$M_x = I_x\dot{\omega_x} + (I_z - I_y)\omega_z\omega_y \quad , \quad M_y = I_y\dot{\omega_y} + (I_x - I_z)\omega_x\omega_z \quad , \quad M_z = I_z\dot{\omega_z} + (I_y - I_x)\omega_x\omega_y$$

Making $\dot{\omega_i}$ subject of the equations,

$$M_x - (I_z - I_y)\omega_z\omega_y = I_x\dot{\omega_x} \quad , \quad M_y - (I_x - I_z)\omega_x\omega_z = I_y\dot{\omega_y} \quad , \quad M_z - (I_y - I_x)\omega_x\omega_y = I_z\dot{\omega_z}$$

$$M_x + (I_y - I_z)\omega_z\omega_y = I_x\dot{\omega_x} \quad , \quad M_y + (I_z - I_x)\omega_x\omega_z = I_y\dot{\omega_y} \quad , \quad M_z + (I_x - I_y)\omega_x\omega_y = I_z\dot{\omega_z}$$

$$\frac{1}{I_x}\left[M_x + (I_y - I_z)\omega_z\omega_y\right] = \dot{\omega_x}, \quad \frac{1}{I_y}\left[M_y + (I_z - I_x)\omega_x\omega_z\right] = \dot{\omega_y}, \quad \frac{1}{I_z}\left[M_z + (I_x - I_y)\omega_x\omega_y\right] = \dot{\omega_z}$$

Let $\bar{F_b}$ represent the force vector represented in a body-fitted coordinate system. According to Newton's second law,

$$\bar{F_b} = m\bar{a_b}$$

The direction cosine matrix $A_{313}(\psi, \theta, \phi)$ is defined to have the following property,

$$\bar{a_b} = A_{313}(\psi, \theta, \phi)\bar{a_i}$$

wherein $\bar{a_i}$ represent acceleration perceived in the inertial coordinate system and $\bar{a_b}$ represent acceleration perceived in the body-fitted coordinate system. Making $\bar{a_b}$ subject of Newton's second law,

$$\bar{a_b} = \frac{1}{m}\bar{F_b}$$

The direction cosine matrix $A_{313}(\psi, \theta, \phi)$ was actually formed as a series of rotation transformations in the perceding sections. The rotation matrices are orthogonal matrice. Therefore, the direction cosine matrix $A_{313}(\psi, \theta, \phi)$ is also orthogonal. One property of orthogonal matrices,

$$A_{313}(\psi, \theta, \phi)^{-1} = A_{313}(\psi, \theta, \phi)^T$$

Therefore,

$$\bar{a_i} = A_{313}(\psi, \theta, \phi)^{-1}\bar{a_b} = A_{313}(\psi, \theta, \phi)^T\bar{a_b}$$

Substituting Newton's second law for $\bar{a_b}$,

$$\bar{a_i} = \frac{1}{m}A_{313}(\psi, \theta, \phi)^T\bar{F_b}$$

The script `Main.m` handles the initial condition, the integrator call, the plotting, and other small utilities. `Main.m` is shown below,

Chapter 6

Rigid Body Kinematics

6.1 Direction Cosine Matrix

The dot product can be thought as some form of projection of one vector onto another vector. Consider the dot product of two vectors $\bar{a}$ and $\bar{b}$. Based on the definition of dot product,

$$\bar{a} \cdot \bar{b} = |a||b| \cos\theta$$

wherein θ is the angle between the vectors $\bar{a}$ and $\bar{b}$. Let an arbitrary vector $\bar{r}$ be represented in basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Let these basis vectors be of mangitude 1. Let another set of basis vectors of magnitude 1 be defined as $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$. The arbitrary vector $\bar{r}$ could be expressed as a linear combination of these basis vectors,

$$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'$$

The vector component $\bar{r}$ in the $\hat{\imath}'$ direction is the summation of all the weighted basis vector components on the $\hat{\imath}'$ direction. Since it was established earlier that this would mean taking the dot product,

$$x' = x(\hat{\imath} \cdot \hat{\imath}') + y(\hat{\jmath} \cdot \hat{\imath}') + z(\hat{k} \cdot \hat{\imath}')$$

Let $\theta_{fg'}$ represent the angle between the f axis and the g' axis,

$$\hat{\imath} \cdot \hat{\imath}' = |\hat{\imath}||\hat{\imath}'| \cos\theta_{ii'} \quad , \quad \hat{\jmath} \cdot \hat{\imath}' = |\hat{\jmath}||\hat{\imath}'| \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{\imath}' = |\hat{k}||\hat{\imath}'| \cos\theta_{ki'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{\imath} \cdot \hat{\imath}' = \cos\theta_{ii'} \quad , \quad \hat{\jmath} \cdot \hat{\imath}' = \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{\imath}' = \cos\theta_{ki'}$$

Substituting the basis vector projections,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

Repeating similar operations for the $\hat{\jmath}'$ direction,

$$y' = x(\hat{\imath} \cdot \hat{\jmath}') + y(\hat{\jmath} \cdot \hat{\jmath}') + z(\hat{k} \cdot \hat{\jmath}')$$

$$\hat{\imath} \cdot \hat{\jmath}' = |\hat{\imath}||\hat{\jmath}'| \cos\theta_{ij'} \quad , \quad \hat{\jmath} \cdot \hat{\jmath}' = |\hat{\jmath}||\hat{\jmath}'| \cos\theta_{jj'} \quad , \quad \hat{k} \cdot \hat{\jmath}' = |\hat{k}||\hat{\jmath}'| \cos\theta_{kj'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{\imath} \cdot \hat{\jmath}' = \cos\theta_{ij'} \quad , \quad \hat{\jmath} \cdot \hat{\jmath}' = \cos\theta_{jj'} \quad , \quad \hat{k} \cdot \hat{\jmath}' = \cos\theta_{kj'}$$

Substituting the basis vector projections,

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

Repeating similar operations for the $\hat{k}'$ direction,

$$z' = x(\hat{i}\cdot\hat{k}') + y(\hat{j}\cdot\hat{k}') + z(\hat{k}\cdot\hat{k}')$$

$$\hat{i}\cdot\hat{k}' = |\hat{i}||\hat{k}'|\cos\theta_{ik'} \quad , \quad \hat{j}\cdot\hat{k}' = |\hat{j}||\hat{k}'|\cos\theta_{jk'} \quad , \quad \hat{k}\cdot\hat{k}' = |\hat{k}||\hat{k}'|\cos\theta_{kk'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{i}\cdot\hat{k}' = \cos\theta_{ik'} \quad , \quad \hat{j}\cdot\hat{k}' = \cos\theta_{jk'} \quad , \quad \hat{k}\cdot\hat{k}' = \cos\theta_{kk'}$$

Substituting the basis vector projections,

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Collecting the various expressions together,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Re-arranging the expressions into matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence, based on the problem definition,

$$\bar{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad , \quad \bar{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad , \quad l = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix}$$

$$\bar{r}' = l\bar{r}$$

The figure below shows some of the angles referenced in the l matrix,

6.2 Euler Angles

In 2-dimensions, the rotation matrix r is typically defined below,

$$r = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

The matrix r rotates a set of coordinate points by angle α in the counter-clockwise direction. Typically, a coordinate system would have its axes rotated in the counter-clockwise direction. As a result, all coordinates are now perceived in the rotated coordinate system to have been rotated in the clockwise direction. Suppose the ange $\alpha = -\theta$, the rotation matrix would take the form,

$$r = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

The eulerian angles represent parameters in successive rotation transformation to determine orientation. These successive rotation transformations are non-commutative. For the specific order of transformation eulerian angles $3-1-3$, the coordinate system is typically rotated in the z-axis by angle ϕ, then rotated in the x-axis by angle θ, before rotated in the z-axis again by angle ψ.

To express coordinates in an inertial coordinate system in terms of a body-fitted coordinate system, it is possible to apply successive rotation transformations on the coordinates in the inertial coordinate system to determine how the same coordinates are perceived in the body-fitted coordinate system. For this solution, the eulerian angle sequence $3-1-3$ is chosen.

For rotation in the z-axis by angle ϕ, the z-coordinate is held constant. Therefore,

$$A_3(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For rotation in the x-axis by angle θ, the x-coordinate is held constant. Therefore,

$$A_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

For rotation in the z-axis by angle ψ, the z-coordinate is held constant. Therefore,

$$A_3(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying the transformations in $3-1-3$ order,

$$A_{313}(\psi, \theta, \phi) = A_3(\psi) A_1(\theta) A_3(\phi)$$

The resulting matrix $A_{313}(\psi, \theta, \phi)$ would be the direction cosine matrix that would have the properties,

$$\bar{v_b} = A_{313}(\psi, \theta, \phi) \bar{v_i}$$

wherein $\bar{v_b}$ represents the coordinates in the body-fitted coordinate system and $\bar{v_i}$ represents the coordinates in the inertial coordinate system. Computing the direction matrix,

$$A_1(\theta) A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As a short-hand to make notation easier, let $\sin(\theta) = s_\theta$, $\cos(\theta) = c_\theta$. Substituting,

$$A_1(\theta)A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

$$A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

Therefore,

$$A_{313}(\psi,\theta,\phi) = A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

For the first transformation, ϕ represents the rotation angle along the z-axis in the inertial coordinate system. Let $\dot{\bar{\phi}}$ represent the time derivative of this transformation. Let $\dot{\bar{\phi}}_i$ represent the vector expressed in inertial coordinates and $\dot{\bar{\phi}}_b$ represent the vector expressed in body-fitted coordinates,

$$\dot{\bar{\phi}}_i = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T$$

To express the time derivative rotation vector in terms of body-fitted coordinates,

$$\dot{\bar{\phi}}_b = A_{313}(\psi,\theta,\phi)\dot{\bar{\phi}}_i$$

Substituting for the relevant terms,

$$\dot{\bar{\phi}}_b = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} s_\psi s_\theta \dot{\phi} \\ c_\psi s_\theta \dot{\phi} \\ c_\theta \dot{\phi} \end{bmatrix}$$

Expressing the vector $\dot{\bar{\phi}}_b$ verbosely,

$$\dot{\bar{\phi}}_b = f_1\hat{b_1} + f_2\hat{b_2} + f_3\hat{b_3} = s_\psi s_\theta \dot{\phi}\hat{b_1} + c_\psi s_\theta \dot{\phi}\hat{b_2} + c_\theta \dot{\phi}\hat{b_3}$$

By comparing the terms,

$$f_1 = \sin(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_2 = \cos(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_3 = \cos(\theta)\dot{\phi}$$

Chapter 7

MOSFET Devices

7.1 NMOS Properties

7.2 PMOS Properties

Chapter 8

Op-Amplifier Fundamentals

8.1 Design

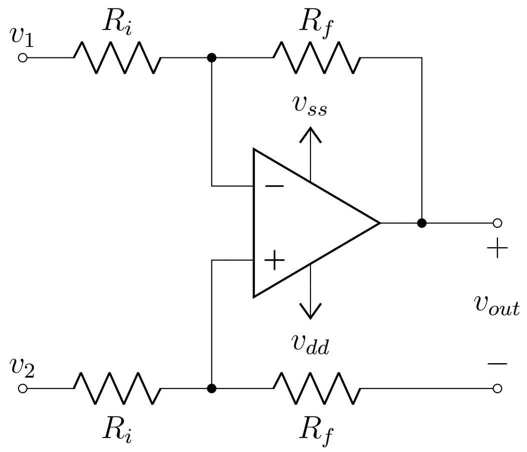
8.2 Properties

Practically, the operational amplifier is a 5-terminal device: source voltage, drain voltage, positive terminal, negative terminal, and output. Both the source voltage and drain voltage are used to power the various nmosfets and pmosfets inside the op amplifier. The ideal operational amplifier is a device with infinite gain. If the voltage at the positive terminal is greater than the voltage at the negative terminal, then the output will be positive infinity. However, practically, the highest output the operational amplifier can output is the source voltage. Therefore, if the voltage at the positive terminal is greater than the negative terminal, then the output will be the source voltage. Likewise, following the same reasoning, if the voltage at the negative terminal is greater than the positive terminal, then the output will be the drain volage. An operational amplifier could act as a comparator, but it is not recommended.

Chapter 9

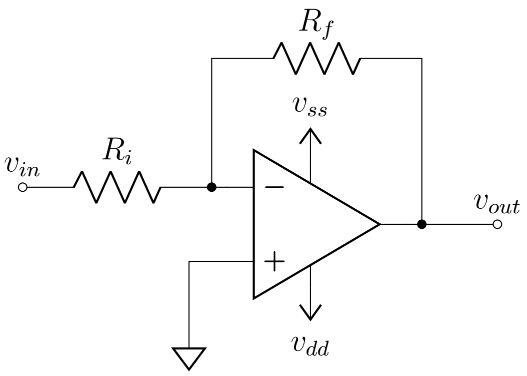
Op-Amplifier Arrangements

9.1 Differential Amplifier



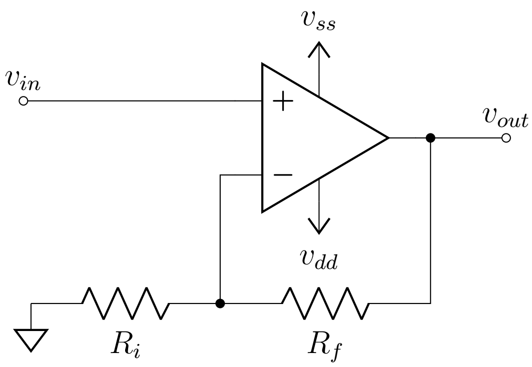
$$v_{out} = \frac{R_f}{R_i}(v_2 - v_1)$$

9.2 Inverting Amplifier



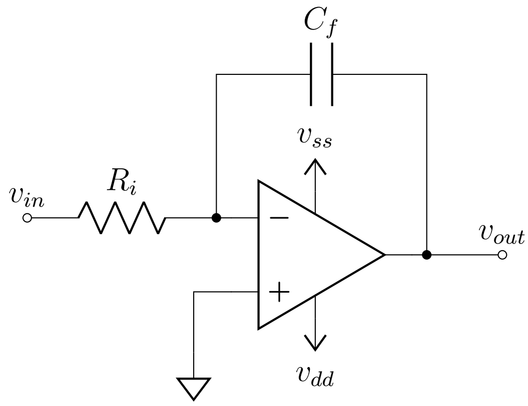
$$v_{out} = -\frac{R_f}{R_i}v_{in}$$

9.3 Non-Inverting Amplifier



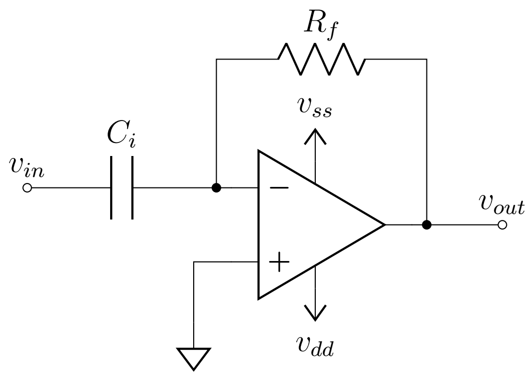
$$v_{out} = \left(1 + \frac{R_f}{R_i}\right) v_{in}$$

9.4 Integrating Amplifier



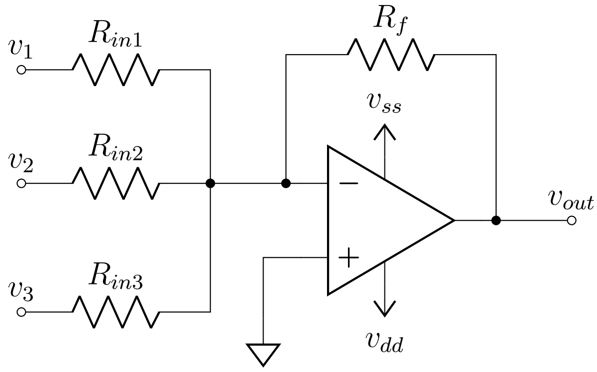
$$\frac{d}{dt}[v_{out}] = -\left(\frac{1}{R_i C_f}\right) v_{in}$$

9.5 Differentiating Amplifier



$$v_{out} = -R_f C_i \frac{d}{dt}[v_{in}]$$

9.6 Summing Amplifier



$$v_{out} = -R_f \left[\frac{v_1}{R_{in1}} + \frac{v_2}{R_{in2}} + \frac{v_3}{R_{in3}} \right]$$

Chapter 10

State-Space Implementations in Circuits

Chapter 11

Long-Term Behaviour of Circuits

11.1 Final Value Theorem

The final value theorem for an arbitrary function $f(t)$ is written as,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

wherein $F(s)$ represents the laplace transform of $f(t)$. The proof of this particular form of the final value theorem is shown below,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{s\to 0}\left\{\int_0^\infty e^{-st}f'(t)\,dt\right\} = \int_0^\infty f'(t)\,dt$$

By fundamental theorem of calculus, and change of variable,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{r\to\infty}\left\{f(r) - f(0)\right\} = \lim_{t\to\infty}\left\{f(t) - f(0)\right\}$$

Taking the approach of the Laplace Transform of derivatives,

$$\mathcal{L}\left[f'(t)\right] = s\mathcal{L}f(t) - f(0) = sF(s) - f(0)$$

Therefore, taking the limits as before,

$$\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\} = \lim_{s\to 0}\left\{sF(s)\right\}\} - f(0)$$

By equating the $\lim_{s\to 0}\left\{\mathcal{L}\left[f'(t)\right]\right\}$ to each other,

$$\lim_{t\to\infty}\left\{f(t)\right\} - f(0) = \lim_{s\to 0}\left\{sF(s)\right\}\} - f(0)$$

This completes the proof,

$$\lim_{t\to\infty}\left\{f(t)\right\} = \lim_{s\to 0}\left\{sF(s)\right\}\}$$

11.2 Impulse Response

The transfer function $G(s)$ is defined as

$$G(s) = \frac{Y(s)}{U(s)}$$

wherein the $Y(s)$ is the output and $U(s)$ is the input, both in the laplace domain. In the laplace domain, the delta-dirac impulse function with arbitrary magnitude μ_0,

$$\mathcal{L}[\mu_0\,\delta(t-c)] = \mu_0 e^{-cs}$$

If the system described by the transfer function $G(s)$ has an input of the dirac delta function of arbitrary magnitude,

$$G(s) = \mu_0 e^{-cs}\,Y(s)$$

Manipulating for $Y(s)$ to be the subject of the equation,

$$Y(s) = \left(\frac{e^{cs}}{\mu_0}\right) G(s)$$

To find the long term behaviour of the output of the system $\lim_{t\to\infty}\big[y(t)\big]$, we should consult the final value theorem. The final value theorem for an arbitrary function $f(t)$,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

By performing the substitution $f(t) = y(t)$ and $F(s) = Y(s)$, wherein $y(t)$ represents the output function in the time domain and $Y(s)$ represents the output function in the laplace domain accordingly,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}[sY(s)]$$

If the input function is the dirac-delta function as shown previously, then $Y(s) = \left(\frac{e^{cs}}{\mu_0}\right) G(s)$. Therefore,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}\left[s\left(\frac{e^{cs}}{\mu_0}\right) G(s)\right]$$

A famous identity for limits of two arbitrary functions $q(t)$ and $p(t)$,

$$\lim_{t\to t_0}\big[p(t)\times q(t)\big] = \lim_{t\to t_0}\big[p(t)\big] \times \lim_{t\to t_0}\big[q(t)\big]$$

By the identity above,

$$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}\big[sG(s)\big] \times \lim_{s\to 0}\left[\frac{e^{cs}}{\mu_0}\right] = \frac{1}{\mu_0}\lim_{s\to 0}\big[sG(s)\big]$$

11.3 Step Response

Let $G(s)$ represent the transfer function of some system in the laplace domain, $Y(s)$ represent the output of some system in the laplace domain, and $U(s)$ represent the input in the laplace domain.

$$Y(s) = G(s)U(s)$$

Consider the heaviside function $u(t-c)$ wherein c is some arbitrary time the heaviside function "turns on". The laplace transform of the arbitray heaviside function,

$$\mathcal{L}[u(t-c)] = \frac{e^{-}cs}{s}$$

If the input $u(t) = \mu_0$ wherein μ_0 is a constant for all t, then

$$\mu_0 G(0) = \lim_{t\to\infty}[y(t)]$$

11.4 Pure Sinusoid Response

The convolution of two arbitrary functions $f(t)$ and $g(t)$ are defined as

$$f * g(t) = g * f(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

Suppose a system has a transfer function $G(s)$ and the output and input in the laplace domain respectively is $Y(s)$ and $U(s)$.

$$G(s) = \frac{Y(s)}{U(s)}$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\left[G(s)U(s)\right] = \mathcal{L}^{-1}\left[G(s)\right] * \mathcal{L}^{-1}\left[U(s)\right] = g(t) * u(t) = \int_0^t g(\tau)u(t-\tau)\,d\tau$$

wherein $\mathcal{L}^{-1}G(s) = g(t)$, and $\mathcal{L}^{-1}U(s) = u(t)$. Here, the function $u(t)$ does not represent the heaviside unit function. Assuming that the input function is reasonably well-behaved and without loss of generality,

$$u(t) = \sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0 t}\right] \quad , \quad a_k = \frac{1}{\tau}\int_0^{\tau} e^{ik\omega_0 t} f(t)dt \quad , \quad \omega_0 = 2\pi/\tau$$

Substituting the Fourier representation of the input function,

$$y(t) = \int_0^t g(\tau)\sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0(t-\tau)}\right]d\tau = \int_0^t g(\tau)\sum_{k=-\infty}^{\infty}\left[a_k e^{ik\omega_0\tau}e^{-ik\omega_0 t}\right]d\tau$$

$$y(t) = \sum_{k=-\infty}^{\infty}\left[\int_0^t g(\tau)e^{ik\omega_0\tau}\,d\tau a_k e^{-ik\omega_0 t}\right]$$

Observing the long term-behaviour of the output, $t = \infty$. Therefore,

$$y(t) = \sum_{k=-\infty}^{\infty}\left[\int_0^\infty g(\tau)e^{ik\omega_0\tau}\,d\tau a_k e^{-ik\omega_0 t}\right]$$

The term $\int_0^\infty g(\tau)e^{ik\omega_0\tau}\,d\tau$ represents a laplace transform with $s = -ik\omega_0$. Therefore,

$$G(-ik\omega_0) = \int_0^\infty g(\tau)e^{ik\omega_0\tau}\,d\tau$$

By substitution,

$$y(t) = \sum_{k=-\infty}^{\infty}\left[G(-ik\omega_0)a_k e^{-ik\omega_0 t}\right]$$

For real sinusoidal inputs $u(t) = k\sin(\omega t)$,

$$\lim_{t\to\infty}[y(t)] = k|G(i\omega)|\sin\{\omega t + arg[G(i\omega)]\}$$

Chapter 12

State-Space