Chapter 1

Sturm-Liouville Problems

1.1 Definition of Stum-Liouville Problems

A Sturm-Liouville Problem is a problem that satisfies the following equation with the following boundary conditions,

$$0 = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y$$

Alternately,

$$0 = \frac{dy}{dx}\frac{d}{dx}\left[p(x)\right] + p(x)\frac{d^2y}{dx^2} - q(x)y + \lambda r(x)y$$

$$0 = p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} - q(x)y + \lambda r(x)y$$

The initial conditions are shown below,

$$0 = \alpha_1 y(a) - \alpha_2 y'(a) \quad , \quad 0 = \beta_1 y(b) + \beta_2 y'(b)$$

wherein neither α_1 and α_2 both zero nor β_1 and β_2 both zero. The parameter λ is the "eigenvalue" whose possible (constant) values are sought usually via the application of the boundary conditions.

1.2 Eigenvalue Theorem of Sturm-Liouville Problems

Suppose that the functions $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous on the closed interval $[a, b]$ and that $p(x) > 0$ and $r(x) > 0$ at each point of $[a, b]$. Then the eigenvalues of the Sturm–Liouville problem constitute an increasing sequence,

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_\infty$$

of real numbers with

$$\lim_{n\to\infty}[\lambda_n] = \infty$$

To within a constant factor, only a single eigenfunction $y_n(x)$ is associated with each eigenvalue λ_n. Moreover, if $q(x) \geq 0$ on the closed interval $[a, b]$ and the coefficients α_1, α_2, β_1, and β_2 are all non-negative, then the eigenvalues are all non-negative.

1.3 Eigenvalues-Eigenfunctions Series

If the functions $p(x)$, $q(x)$ and $r(x)$ of the Sturm-Liouville problem satisifes the Eigenvalue Theorem, then eigenfunctions $y_i(x)$ and $y_j(x)$ corresponding to eigenvalues λ_i and λ_j wherein $j \neq 1$ are orthogonal with respect to each other relative to the function $r(x)$,

$$0 = \int_a^b y_i(x)y_j(x)r(x)dx$$

For a sturm-liouville problem with infinite eigenvalues, it is possible to represent an arbitrary function $f(x)$ as the infinite sum of the eigenvalues,

$$f(x) = \sum_{m=1}^{\infty}\left[c_m y_m(x)\right]$$

wherein $y_m(x)$ represents the m^{th} eigenfunction of the m^{th} eigenvalue λ_m. Taking the integral in both sides with the product to the eigenfunction $y_n(x)$ relative to the function $r(x)$,

$$\int_a^b f(x)y_n(x)r(x)dx = \int_a^b \sum_{m=1}^{\infty}\left[c_m y_m(x)\right] y_n(x)r(x)dx$$

Using the assumption, $0 = \displaystyle\int_a^b y_i(x)y_j(x)r(x)dx$,

$$\int_a^b f(x)y_n(x)r(x)dx = \int_a^b c_n\left[y_n(x)\right]^2 r(x)dx = c_n\int_a^b \left[y_n(x)\right]^2 r(x)dx$$

Therefore, the constant c_n could be obtained by,

$$c_n = \frac{\displaystyle\int_a^b f(x)y_n(x)r(x)dx}{\displaystyle\int_a^b \left[y_n(x)\right]^2 r(x)dx}$$

This particular theorem could be used to prove under certain reasonable conditions that there exists an infinite series that would allow the boundary conditions to be implemented analytically into the partial differential equations problems.