0.0.1 Discretization of Wave Equation

The one-dimensional wave equation is shown below,

$$0 = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x}$$

Integrating with time and space,

$$0 = \int_t^{t+\Delta t} \int_{CV} \frac{\partial \phi}{\partial t}\, dV\, dt + u \int_t^{t+\Delta t} \int_{CV} \frac{\partial \phi}{\partial x}\, dV\, dt$$

Switching the order of integration in order to form exact differentials,

$$0 = \int_{CV} \int_t^{t+\Delta t} \frac{\partial \phi}{\partial t}\, dt\, dV + u \int_t^{t+\Delta t} \int_{CV} \frac{\partial \phi}{\partial x}\, dV\, dt$$

$$0 = \int_{CV} \left[\phi\right]_t^{t+\Delta t}\, dV + u \int_t^{t+\Delta t} A \left[\phi\right]_{CV}\, dt$$

$$0 = \left[\phi\right]_t^{t+\Delta t} \Delta V + u \int_t^{t+\Delta t} A\phi_e - A\phi_w\, dt$$

Substituting for the infinitesmially small volume $\Delta V = Adx$,

$$0 = \left[\phi\right]_t^{t+\Delta t} Adx + u \int_t^{t+\Delta t} A\phi_e - A\phi_w\, dt$$

Assuming uniform area throughout the grids,

$$0 = \left[\phi\right]_t^{t+\Delta t} dx + u \int_t^{t+\Delta t} \phi_e - \phi_w\, dt$$

Assuming that the value of ϕ is constant within a single cell,

$$0 = \left[\phi_P - \phi_{P,o}\right] dx + u \int_t^{t+\Delta t} \phi_e - \phi_w\, dt$$

Using the generalized time integration scheme,

$$\int_t^{t+\Delta t} \phi\, dt = [\theta\phi_{t+\Delta t} + (1-\theta)\phi_t]\Delta t$$

Substituting the time integration scheme,

$$0 = \left[\phi_P - \phi_{P,o}\right] dx + u \left\{\theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}\right\} dt$$

Factoring for the CFL number,

$$0 = \left[\phi_P - \phi_{P,o}\right] \frac{dx}{u\, dt} + \left\{\theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}\right\}$$

$$0 = \left[\phi_P - \phi_{P,o}\right] \frac{\Delta x}{u\, \Delta t} + \theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}$$

Let

$$\gamma = \frac{1}{CFL} = \frac{\Delta x}{u\, \Delta t}$$

$$0 = \gamma\phi_P - \gamma\phi_{P,o} + \theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}$$

Placing the new time step quantities on LHS and the old time step quantities on RHS,

$$-\gamma\phi_P + \theta\phi_w - \theta\phi_e = -\gamma\phi_{P,o} + (1-\theta)\phi_{e,o} - (1-\theta)\phi_{w,o}$$

0.0.2 Cubic Polynomial Spatial Scheme

The discretization of the pure convection problem with a generalized time advancement scheme and a cubic polynomial spatial advancement scheme is shown below,

$$
\begin{aligned}
0 = \frac{\phi_P - \phi_{P,o}}{\Delta t} + \frac{u}{\Delta x} \Bigg[& \theta \left(\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E \right) \\
& + (1-\theta) \left(\frac{1}{16}\phi_{WW,o} - \frac{5}{16}\phi_{W,o} + \frac{15}{16}\phi_{P,o} + \frac{5}{16}\phi_{E,o} \right) \\
& - \theta \left(\frac{1}{16}\phi_{WWW} - \frac{5}{16}\phi_{WW} + \frac{15}{16}\phi_W + \frac{5}{16}\phi_P \right) \\
& -(1-\theta) \left(\frac{1}{16}\phi_{WWW,o} - \frac{5}{16}\phi_{WW,o} + \frac{15}{16}\phi_{W,o} + \frac{5}{16}\phi_{P,o} \right) \Bigg]
\end{aligned}
$$

Assuming that the solution $\phi(x,t)$ is a product of a time component $\hat{\phi}(t)$ and a spatial component e^{ikx}, the solution $\phi(x,t)$ could be rewritten as,

$$\phi(x,t) = \hat{\phi}(t)e^{ikx}$$

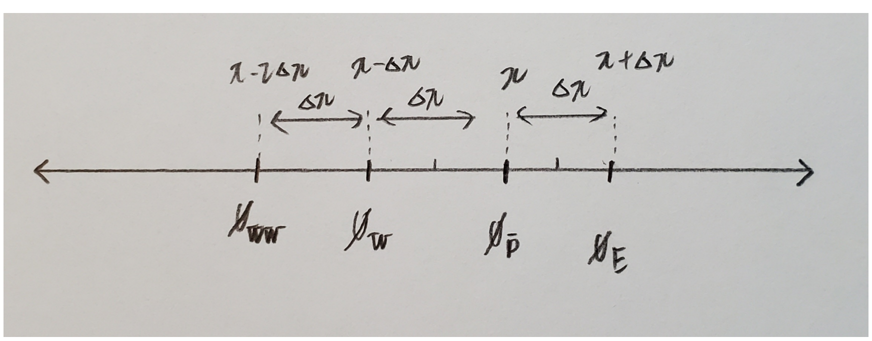
Points at the future time step would be given the time component shown below,

$$\hat{\phi}(t+\Delta t) = \hat{\phi}$$

Points at the old time step would be given the time component shown below,

$$\hat{\phi}(t) = \hat{\phi}_o$$

Following convention, the coordinate system considers eastern direction of the grid points to be positive x and western direction of the grid points to be negative x. The point of interest P would be given a spatial component e^{ikx}. Following the coordinate system definition, the west point would be given spatial component $e^{ik(x-\Delta x)}$. The east point would be given spatial component $e^{ik(x+\Delta x)}$. The diagram below summarizes the coordinate system used in this derivation,



Uniform grid spacing is assumed in the following derivation. Parsing the original expression into several parts.

For the discrete derivative of ϕ with respect to time,

$$\frac{\phi_P - \phi_{P,o}}{\Delta t} = \frac{1}{\Delta t}\left[\phi_P - \phi_{P,o}\right] = \frac{1}{\Delta t}\left[\hat{\phi}e^{ikx} - \hat{\phi}_o e^{ikx}\right] = \frac{e^{ikx}}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right]$$

For the first part of RHS,

$$\begin{aligned}\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_{W}+\frac{15}{16}\phi_{P}+\frac{5}{16}\phi_{E}=&\frac{1}{16}\hat{\phi}e^{ik(x-2\Delta x)}-\frac{5}{16}\hat{\phi}e^{ik(x-\Delta x)}\\&+\frac{15}{16}\hat{\phi}e^{ikx}+\frac{5}{16}\hat{\phi}e^{ik(x+\Delta x)}\end{aligned}$$

$$\begin{aligned}\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_{W}+\frac{15}{16}\phi_{P}+\frac{5}{16}\phi_{E}=&\frac{1}{16}\hat{\phi}e^{ikx-2ik\Delta x}-\frac{5}{16}\hat{\phi}e^{ikx-ik\Delta x}\\&+\frac{15}{16}\hat{\phi}e^{ikx}+\frac{5}{16}\hat{\phi}e^{ikx+ik\Delta x}\end{aligned}$$

$$\begin{aligned}\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_{W}+\frac{15}{16}\phi_{P}+\frac{5}{16}\phi_{E}=&\frac{1}{16}\hat{\phi}e^{ikx}e^{-2ik\Delta x}-\frac{5}{16}\hat{\phi}e^{ikx}e^{-ik\Delta x}\\&+\frac{15}{16}\hat{\phi}e^{ikx}+\frac{5}{16}\hat{\phi}e^{ikx}e^{ik\Delta x}\end{aligned}$$

$$\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_{W}+\frac{15}{16}\phi_{P}+\frac{5}{16}\phi_{E}=\hat{\phi}e^{ikx}\left[\frac{1}{16}e^{-2ik\Delta x}-\frac{5}{16}e^{-ik\Delta x}+\frac{15}{16}+\frac{5}{16}e^{ik\Delta x}\right]$$

Performing similar opeartions for the second part of RHS,

$$\frac{1}{16}\phi_{WW,o}-\frac{5}{16}\phi_{W,o}+\frac{15}{16}\phi_{P,o}+\frac{5}{16}\phi_{E,o}=\hat{\phi}_oe^{ikx}\left[\frac{1}{16}e^{-2ik\Delta x}-\frac{5}{16}e^{-ik\Delta x}+\frac{15}{16}+\frac{5}{16}e^{ik\Delta x}\right]$$

Performing similar opeartions for the third part of RHS,

$$\frac{1}{16}\phi_{WWW}-\frac{5}{16}\phi_{WW}+\frac{15}{16}\phi_{W}+\frac{5}{16}\phi_{P}=\hat{\phi}e^{ikx}\left[\frac{1}{16}e^{-3ik\Delta x}-\frac{5}{16}e^{-2ik\Delta x}+\frac{15}{16}e^{-ik\Delta x}+\frac{5}{16}\right]$$

Performing similar opeartions for the fourth part of RHS,

$$\frac{1}{16}\phi_{WWW,o}-\frac{5}{16}\phi_{WW,o}+\frac{15}{16}\phi_{W,o}+\frac{5}{16}\phi_{P,o}=\hat{\phi}_oe^{ikx}\left[\frac{1}{16}e^{-3ik\Delta x}-\frac{5}{16}e^{-2ik\Delta x}+\frac{15}{16}e^{-ik\Delta x}+\frac{5}{16}\right]$$

Let

$$A=\frac{1}{16}e^{-2ik\Delta x}-\frac{5}{16}e^{-ik\Delta x}+\frac{15}{16}+\frac{5}{16}e^{ik\Delta x}\quad,\quad B=\frac{1}{16}e^{-3ik\Delta x}-\frac{5}{16}e^{-2ik\Delta x}+\frac{15}{16}e^{-ik\Delta x}+\frac{5}{16}$$

Substituting for A and B shortens the expressions,

$$\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_{W}+\frac{15}{16}\phi_{P}+\frac{5}{16}\phi_{E}=\hat{\phi}e^{ikx}A$$

$$\frac{1}{16}\phi_{WW,o}-\frac{5}{16}\phi_{W,o}+\frac{15}{16}\phi_{P,o}+\frac{5}{16}\phi_{E,o}=\hat{\phi}_oe^{ikx}A$$

$$\frac{1}{16}\phi_{WWW}-\frac{5}{16}\phi_{WW}+\frac{15}{16}\phi_{W}+\frac{5}{16}\phi_{P}=\hat{\phi}e^{ikx}B$$

$$\frac{1}{16}\phi_{WWW,o}-\frac{5}{16}\phi_{WW,o}+\frac{15}{16}\phi_{W,o}+\frac{5}{16}\phi_{P,o}=\hat{\phi}_oe^{ikx}B$$

Susbtituting the simplified parts into the original expression,

$$0=\frac{e^{ikx}}{\Delta t}\left[\hat{\phi}-\hat{\phi}_o\right]+\frac{u}{\Delta x}\left[\theta\left(\hat{\phi}e^{ikx}A\right)+(1-\theta)\left(\hat{\phi}_oe^{ikx}A\right)-\theta\left(\hat{\phi}e^{ikx}B\right)-(1-\theta)\left(\hat{\phi}_oe^{ikx}B\right)\right]$$

$$0 = \frac{e^{ikx}}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right] + \frac{u}{\Delta x}\left[\theta\hat{\phi}e^{ikx}A + (1-\theta)\hat{\phi}_o e^{ikx}A - \theta\hat{\phi}e^{ikx}B - (1-\theta)\hat{\phi}_o e^{ikx}B\right]$$

Since $e^{ikx} \neq 0$,

$$0 = \frac{1}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right] + \frac{u}{\Delta x}\left[\theta\hat{\phi}A + (1-\theta)\hat{\phi}_o A - \theta\hat{\phi}B - (1-\theta)\hat{\phi}_o B\right]$$

$$0 = \frac{\Delta x}{u\,\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right] + \theta\hat{\phi}A + (1-\theta)\hat{\phi}_o A - \theta\hat{\phi}B - (1-\theta)\hat{\phi}_o B$$

Let

$$\gamma = \frac{1}{CFL} = \frac{\Delta x}{u\,\Delta t}$$

Substituting for the parameter γ,

$$0 = \gamma\hat{\phi} - \gamma\hat{\phi}_o + \theta\hat{\phi}A + (1-\theta)\hat{\phi}_o A - \theta\hat{\phi}B - (1-\theta)\hat{\phi}_o B$$

$$\gamma\hat{\phi}_o - (1-\theta)\hat{\phi}_o A + (1-\theta)\hat{\phi}_o B = \gamma\hat{\phi} + \theta\hat{\phi}A - \theta\hat{\phi}B$$

$$\hat{\phi}\left[\gamma + \theta A - \theta B\right] = \hat{\phi}_o\left[\gamma - (1-\theta)A + (1-\theta)B\right]$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left[\gamma - (1-\theta)A + (1-\theta)B\right]}{\left[\gamma + \theta A - \theta B\right]} = \frac{\gamma + (\theta-1)A + (1-\theta)B}{\gamma + \theta A - \theta B} = \frac{\gamma + \theta A - A + B - \theta B}{\gamma + \theta A - \theta B}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - A + B + \theta A - \theta B}{\gamma + \theta A - \theta B} = \frac{\gamma - (A-B) + \theta(A-B)}{\gamma + \theta(A-B)}$$

Let $\alpha = A - B$. Simplifying,

$$\alpha = \frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x} - \left(\frac{1}{16}e^{-3ik\Delta x} - \frac{5}{16}e^{-2ik\Delta x} + \frac{15}{16}e^{-ik\Delta x} + \frac{5}{16}\right)$$

$$\alpha = \frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x} - \frac{1}{16}e^{-3ik\Delta x} + \frac{5}{16}e^{-2ik\Delta x} - \frac{15}{16}e^{-ik\Delta x} - \frac{5}{16}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \frac{1}{16}e^{-2ik\Delta x} + \frac{5}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} - \frac{15}{16}e^{-ik\Delta x} - \frac{5}{16} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \left[\frac{1}{16} + \frac{5}{16}\right]e^{-2ik\Delta x} - \left[\frac{5}{16} + \frac{15}{16}\right]e^{-ik\Delta x} - \frac{5}{16} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \left[\frac{6}{16}\right]e^{-2ik\Delta x} - \left[\frac{20}{16}\right]e^{-ik\Delta x} + \frac{10}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \frac{6}{16}e^{-2ik\Delta x} - \frac{20}{16}e^{-ik\Delta x} + \frac{10}{16} + \frac{5}{16}e^{ik\Delta x}$$

Substituting for α,

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - \alpha + \theta\alpha}{\gamma + \theta\alpha}$$

0.0.3 Upwind Differencing Scheme

The theoretical section shows that discretizing the 1-dimensional wave equation yields the following expression,

$$-\gamma\phi_P + \theta\phi_w - \theta\phi_e = -\gamma\phi_{P,o} + (1-\theta)\phi_{e,o} - (1-\theta)\phi_{w,o}$$

Assuming upwing differencing,

$$\phi_w = \phi_W \quad , \quad \phi_e = \phi_P$$

Substituting for this,

$$-\gamma\phi_P + \theta\phi_W - \theta\phi_P = -\gamma\phi_{P,o} + (1-\theta)\phi_{P,o} - (1-\theta)\phi_{W,o}$$

Simplifying,

$$\phi_P\left[-\gamma - \theta\right] + \theta\phi_W = \phi_{P,o}\left[-\gamma + (1-\theta)\right] - (1-\theta)\phi_{W,o}$$

Substituting the Fourier representaiton of the solution, $\phi = \hat{\phi}e^{ikx}$,

$$\hat{\phi}e^{ikx}\left[-\gamma-\theta\right] + \theta\hat{\phi}e^{ik(x-\Delta x)} = \hat{\phi}_o e^{ikx}\left[-\gamma + (1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ik(x-\Delta x)}$$

$$\hat{\phi}e^{ikx}\left[-\gamma-\theta\right] + \theta\hat{\phi}e^{ikx-ik\Delta x} = \hat{\phi}_o e^{ikx}\left[-\gamma + (1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ikx-ik\Delta x}$$

$$\hat{\phi}e^{ikx}\left[-\gamma-\theta\right] + \theta\hat{\phi}e^{ikx}e^{-ik\Delta x} = \hat{\phi}_o e^{ikx}\left[-\gamma + (1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ikx}e^{-ik\Delta x}$$

$$\hat{\phi}e^{ikx}\left\{\left[-\gamma-\theta\right] + \theta e^{-ik\Delta x}\right\} = \hat{\phi}_o e^{ikx}\left\{\left[-\gamma + (1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}$$

Since $e^{ikx} \neq 0$ for all values of x,

$$\hat{\phi}\left\{\left[-\gamma-\theta\right] + \theta e^{-ik\Delta x}\right\} = \hat{\phi}_o\left\{\left[-\gamma + (1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\left[-\gamma + (1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}}{\left\{\left[-\gamma-\theta\right] + \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\left[\gamma - (1-\theta)\right] + (1-\theta)e^{-ik\Delta x}\right\}}{\left\{\left[\gamma+\theta\right] - \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\gamma - 1 + \theta + (1-\theta)e^{-ik\Delta x}\right\}}{\left\{\gamma+\theta - \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - 1 + \theta + (1-\theta)e^{-ik\Delta x}}{\gamma+\theta\left(1 - e^{-ik\Delta x}\right)}$$

The amplification factor for the polynomial interpolation as well as the upwind differencing is implemented into the Matlab script. The matlab script was modified to accomodate multiple plots in a single figure and write the maximum amplification factor to a file. The matlab script is shown below,