Chapter 1

Partial Differential Equations

The conventional gradient operator in cartesian coordinates is typically defined as,

$$\nabla_{xyz} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T \quad , \quad \nabla^n_{xyz} = \begin{pmatrix} \frac{\partial^n}{\partial x^n} & \frac{\partial^n}{\partial y^n} & \frac{\partial^n}{\partial z^n} \end{pmatrix}^T$$

Let the modified gradient operator $({}_m\nabla)$ in cartesian coordinates be defined as,

$${}_m\nabla_{xyz} = \begin{pmatrix} \alpha_1\frac{\partial}{\partial x} & \alpha_2\frac{\partial}{\partial y} & \alpha_3\frac{\partial}{\partial z} \end{pmatrix}^T \quad , \quad {}_m\nabla^n_{xyz} = \begin{pmatrix} \beta_1\frac{\partial^n}{\partial x^n} & \beta_2\frac{\partial^n}{\partial y^n} & \beta_3\frac{\partial^n}{\partial z^n} \end{pmatrix}^T$$

This modification allows the gradient operator to be more general. The modified gradient operator for m dimensional cartesian coordinates,

$${}_m\nabla_{xyz} = \begin{pmatrix} \alpha_1\frac{\partial}{\partial x_1} & \alpha_2\frac{\partial}{\partial x_2} & \dots & \alpha_n\frac{\partial}{\partial x_m} \end{pmatrix}^T \quad , \quad {}_m\nabla^n_{xyz} = \begin{pmatrix} \beta_1\frac{\partial^n}{\partial x_1^n} & \beta_2\frac{\partial^n}{\partial x_2^n} & \dots & \beta_3\frac{\partial^n}{\partial x_m^n} \end{pmatrix}^T$$

1.1 Methods in Generalized Cartesian Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{x_1\dots x_q}(u) + {}_{m_2}\nabla_{x_1\dots x_q}(u) = \sum_{i=1}^{q}\left[b_i\,\frac{\partial^2 u}{\partial x_i^2} + c_i\frac{\partial u}{\partial x_i}\right]$$

$$\text{wherein } {}_{m_1}\nabla^2_{x_1\dots x_q}(u) = {}_{m_1}\nabla_{x_1\dots x_q}\cdot\nabla_{x_1\dots x_q}u$$

1.2 Methods in Cylindrical Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{r\theta z}(u) + {}_{m_2}\nabla_{r\theta z}\cdot(u) =$$

1.3 Methods in Spherical Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{r\theta\phi}(u) + {}_{m_2}\nabla_{r\theta\phi}(u) =$$