Chapter 1

Rigid Body Kinematics

1.1 Direction Cosine Matrix

The dot product can be thought as some form of projection of one vector onto another vector. Consider the dot product of two vectors $\bar{a}$ and $\bar{b}$. Based on the definition of dot product,

$$\bar{a} \cdot \bar{b} = |a||b| \cos\theta$$

wherein θ is the angle between the vectors $\bar{a}$ and $\bar{b}$. Let an arbitrary vector $\bar{r}$ be represented in basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$. Let these basis vectors be of mangitude 1. Let another set of basis vectors of magnitude 1 be defined as $\hat{i}'$, $\hat{j}'$, and $\hat{k}'$. The arbitrary vector $\bar{r}$ could be expressed as a linear combination of these basis vectors,

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$

The vector component $\bar{r}$ in the $\hat{i}'$ direction is the summation of all the weighted basis vector components on the $\hat{i}'$ direction. Since it was established earlier that this would mean taking the dot product,

$$x' = x(\hat{i} \cdot \hat{i}') + y(\hat{j} \cdot \hat{i}') + z(\hat{k} \cdot \hat{i}')$$

Let $\theta_{fg'}$ represent the angle between the f axis and the g' axis,

$$\hat{i} \cdot \hat{i}' = |\hat{i}||\hat{i}'| \cos\theta_{ii'} \quad , \quad \hat{j} \cdot \hat{i}' = |\hat{j}||\hat{i}'| \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{i}' = |\hat{k}||\hat{i}'| \cos\theta_{ki'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{i} \cdot \hat{i}' = \cos\theta_{ii'} \quad , \quad \hat{j} \cdot \hat{i}' = \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{i}' = \cos\theta_{ki'}$$

Substituting the basis vector projections,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

Repeating similar operations for the $\hat{j}'$ direction,

$$y' = x(\hat{i}\cdot\hat{j}') + y(\hat{j}\cdot\hat{j}') + z(\hat{k}\cdot\hat{j}')$$

$$\hat{i}\cdot\hat{j}' = |\hat{i}||\hat{j}'|\cos\theta_{ij'} \quad , \quad \hat{j}\cdot\hat{j}' = |\hat{j}||\hat{j}'|\cos\theta_{jj'} \quad , \quad \hat{k}\cdot\hat{j}' = |\hat{k}||\hat{j}'|\cos\theta_{kj'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{i}\cdot\hat{j}' = \cos\theta_{ij'} \quad , \quad \hat{j}\cdot\hat{j}' = \cos\theta_{jj'} \quad , \quad \hat{k}\cdot\hat{j}' = \cos\theta_{kj'}$$

Substituting the basis vector projections,

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

Repeating similar operations for the $\hat{k}'$ direction,

$$z' = x(\hat{i}\cdot\hat{k}') + y(\hat{j}\cdot\hat{k}') + z(\hat{k}\cdot\hat{k}')$$

$$\hat{i}\cdot\hat{k}' = |\hat{i}||\hat{k}'|\cos\theta_{ik'} \quad , \quad \hat{j}\cdot\hat{k}' = |\hat{j}||\hat{k}'|\cos\theta_{jk'} \quad , \quad \hat{k}\cdot\hat{k}' = |\hat{k}||\hat{k}'|\cos\theta_{kk'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{i}\cdot\hat{k}' = \cos\theta_{ik'} \quad , \quad \hat{j}\cdot\hat{k}' = \cos\theta_{jk'} \quad , \quad \hat{k}\cdot\hat{k}' = \cos\theta_{kk'}$$

Substituting the basis vector projections,

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Collecting the various expressions together,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Re-arranging the expressions into matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence, based on the problem definition,

$$\bar{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad , \quad \bar{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad , \quad l = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix}$$

$$\bar{r}' = l\bar{r}$$

The figure below shows some of the angles referenced in the l matrix,

1.2 Euler Angles

In 2-dimensions, the rotation matrix r is typically defined below,

$$r = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

The matrix r rotates a set of coordinate points by angle α in the counter-clockwise direction. Typically, a coordinate system would have its axes rotated in the counter-clockwise direction. As a result, all coordinates are now perceived in the rotated coordinate system to have been rotated in the clockwise direction. Suppose the ange $\alpha = -\theta$, the rotation matrix would take the form,

$$r = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

The eulerian angles represent parameters in successive rotation transformation to determine orientation. These successive rotation transformations are non-commutative. For the specific order of transformation eulerian angles $3-1-3$, the coordinate system is typically rotated in the z-axis by angle ϕ, then rotated in the x-axis by angle θ, before rotated in the z-axis again by angle ψ.

To express coordinates in an inertial coordinate system in terms of a body-fitted coordinate system, it is possible to apply successive rotation transformations on the coordinates in the inertial coordinate system to determine how the same coordinates are perceived in the body-fitted coordinate system. For this solution, the eulerian angle sequence $3-1-3$ is chosen.

For rotation in the z-axis by angle ϕ, the z-coordinate is held constant. Therefore,

$$A_3(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For rotation in the x-axis by angle θ, the x-coordinate is held constant. Therefore,

$$A_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

For rotation in the z-axis by angle ψ, the z-coordinate is held constant. Therefore,

$$A_3(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying the transformations in $3-1-3$ order,

$$A_{313}(\psi, \theta, \phi) = A_3(\psi)A_1(\theta)A_3(\phi)$$

The resulting matrix $A_{313}(\psi, \theta, \phi)$ would be the direction cosine matrix that would have the properties,

$$\bar{v_b} = A_{313}(\psi, \theta, \phi)\bar{v_i}$$

wherein $\bar{v_b}$ represents the coordinates in the body-fitted coordinate system and $\bar{v_i}$ represents the coordinates in the inertial coordinate system. Computing the direction matrix,

$$A_1(\theta)A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As a short-hand to make notation easier, let $\sin(\theta) = s_\theta$, $\cos(\theta) = c_\theta$. Substituting,

$$A_1(\theta)A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

$$A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

Therefore,

$$A_{313}(\psi, \theta, \phi) = A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

For the first transformation, ϕ represents the rotation angle along the z-axis in the inertial coordinate system. Let $\dot{\vec{\phi}}$ represent the time derivative of this transformation. Let $\dot{\vec{\phi}}_i$ represent the vector expressed in inertial coordinates and $\dot{\vec{\phi}}_b$ represent the vector expressed in body-fitted coordinates,

$$\dot{\vec{\phi}}_i = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T$$

To express the time derivative rotation vector in terms of body-fitted coordinates,

$$\dot{\vec{\phi}}_b = A_{313}(\psi, \theta, \phi)\dot{\vec{\phi}}_i$$

Substituting for the relevant terms,

$$\dot{\vec{\phi}}_b = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} s_\psi s_\theta \dot{\phi} \\ c_\psi s_\theta \dot{\phi} \\ c_\theta \dot{\phi} \end{bmatrix}$$

Expressing the vector $\dot{\vec{\phi}}_b$ verbosely,

$$\dot{\vec{\phi}}_b = f_1 \hat{b_1} + f_2 \hat{b_2} + f_3 \hat{b_3} = s_\psi s_\theta \dot{\phi} \hat{b_1} + c_\psi s_\theta \dot{\phi} \hat{b_2} + c_\theta \dot{\phi} \hat{b_3}$$

By comparing the terms,

$$f_1 = \sin(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_2 = \cos(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_3 = \cos(\theta)\dot{\phi}$$