Chapter 1

Reynold's Transport Theorem

One variation of Liebniz Rule applicable for volumetric integrals is shown below. for the variable T wherein T may represent a time dependent scalar, vector, or tensor.

$$\frac{d}{dt}\iiint_{R(t)} T\, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T] dV_o + \iint_{S(t)} T\bar{v}_s\bar{n} dS$$

wherein $R(t)$ represents an arbitray region of space, V_o represents volume, $S(t)$ represents the surface of the region defined by $R(t)$, $\bar{v}_s$ represents the velocity of the moving surface, $\bar{n}$ represents normal vector of the surface. Depending on the variable type T, the operation $T\bar{v}_s\bar{n}$ would depend on a case to case basis.

Using a change of variables,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \frac{d}{dt}\iiint_{\Gamma} \phi J\, dV_{o,i}$$

wherein $V_s(t)$ represents a control mass region, ϕ may represent some time changing scalar variable, but in general, could represent the elements of an arbitrary tensor as well. dV_o represents infinitesmial volume. Since the vector field is evolving with time, all the points inside $V_s(t)$ is at some place initially at time $t = 0$. The region that contains all the points inside $V_s(t)$ at time $t = 0$ is considered to be Γ. Since we are considering the general case wherein volume may expand or contract, we declare $dV_{o,i}$ to represent infinitesmial volume initially at time $t = 0$. The relationship between infinitesmial volume at the present time dV_o and infinitesmial volume initially,

$$dV_o = J\, dV_{o,i}$$

wherein J represents the Jacobian, which is the determinant of the velocity gradient tensor (more on this later). From all these information, the change of variables could be performed as shown above. ϵ is a region that is not varying with time t. Since the bounds of integration is now unchanging with time, the time derivative operation is now commutative with the volumetric integral. Therefore,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{\Gamma} \frac{d}{dt}[\phi J]\, dV_{o,i}$$

Without loss of generality, assuming that ϕ changes with coordinates x_i and time, the time derivative is equivalent to the substantive or material derivative. Therefore,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi\, dV_o = \iiint_{\Gamma} \frac{D}{Dt}[\phi J]\, dV_{o,i}$$

Using product rule,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{\Gamma} \phi \frac{D}{Dt}[J] + J\frac{D}{Dt}[\phi]\, dV_{o,i}$$

By a tedious mathematical proof,

$$\frac{D}{Dt}[J] = (\nabla \cdot \bar{v}_s)J$$

wherein $\bar{v}_s$ represents the velocity of the moving surface. $\bar{v}_s$ is not to be confused with V_s. V_s represents the control mass region earlier meanwhile $\bar{v}_s$ represents the velocity of the moving boundaries of V_s. Substituting,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{\Gamma} \phi(\nabla \cdot \bar{v}_s)J + J\frac{D}{Dt}[\phi]\, dV_{o,i}$$

Making a change of variables once again to revert back to the region $V_s(t)$ from the initial positions Γ,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{\Gamma} \left\{ \phi(\nabla \cdot \bar{v}_s) + \frac{D}{Dt}[\phi] \right\} J dV_{o,i}$$

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{V_s(t)} \phi(\nabla \cdot \bar{v}_s) + \frac{D}{Dt}[\phi]\, dV_o$$

Expanding the substantive derivative of ϕ as $\frac{D}{Dt}[\phi] = \frac{\partial}{\partial t}[\phi] + \bar{v}_s \cdot \nabla\phi$,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{V_s(t)} \phi(\nabla \cdot \bar{v}_s) + \frac{\partial}{\partial t}[\phi] + \bar{v}_s \cdot \nabla\phi \, dV_o$$

Using the divergence of scalar vector product identity,

$$\nabla \cdot (\phi\bar{v}_s) = \phi(\nabla \cdot \bar{v}_s) + \bar{v}_s \cdot \nabla\phi$$

Substituting,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{V_s(t)} \frac{\partial}{\partial t}[\phi] + \nabla \cdot (\phi\bar{v}_s)\, dV_o$$

Parsing out the integral,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{V_s(t)} \frac{\partial}{\partial t}[\phi]\, dV_o + \iiint_{V_s(t)} \nabla \cdot (\phi\bar{v}_s)\, dV_o$$

Using divergence theorem, $\iiint_{V_s(t)} \nabla \cdot (\phi\bar{v}_s)\, dV_o = \iint_{S_s(t)} \phi\bar{v}_s \cdot \hat{n} dS$. Substituting,

$$\frac{d}{dt}\iiint_{V_s(t)} \phi \, dV_o = \iiint_{V_s(t)} \frac{\partial}{\partial t}[\phi]\, dV_o + \iint_{S_s(t)} \phi\bar{v}_s \cdot \hat{n} dS$$

1.1 Substantive Derivative

Suppose a quantity b is dependent on the the variable time t and the typical cartesian coordinates x, y, z. Taking the derivative of variable b with respect to time yields the following based on chain rule,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \frac{\partial}{\partial x}[b] \times \frac{\partial}{\partial t}[x] + \frac{\partial}{\partial y}[b] \times \frac{\partial}{\partial t}[y] + \frac{\partial}{\partial z}[b] \times \frac{\partial}{\partial t}[z]$$

Taking note that the partial derivatives of the cartesian coordinates defines velocity in the cartesian coordinates. Therefore,

$$\frac{\partial}{\partial t}[x] = u \quad , \quad \frac{\partial}{\partial t}[y] = v \quad , \quad \frac{\partial}{\partial t}[z] = w$$

wherein u, v, and w typically represents velocity in the x, y, and z directions respectively. Therefore, the derivative of y with respect to time t would take the form,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

If the ∇ operator is defined as

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T$$

Therefore, the derivative of y with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

Let the velocity vector be defined as

$$\bar{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

It follows that the derivative of b with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \bar{v} \cdot \nabla b$$

1.2 Divergence Theorem

The Divergence Theorem is stated below. The variable $\bar{F}$ must represent a vector in R^3

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

Alternately,

$$\iiint_{R(t)} \frac{\partial}{\partial x}[\bar{F}] + \frac{\partial}{\partial y}[\bar{F}] + \frac{\partial}{\partial z}[\bar{F}]\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

wherein dV_o represents an infinitesmially small volume. $S(t)$ is the surface encapsulating the region $R(t)$. $\bar{n}$ is the normal vector of the control volume, and dS is an infinitesmial area of surface $S(t)$.