Chapter 1

Navier-Stokes Equations

The full set of the Navier-Stokes Equations are shown below. The three equations below correspond to the differential continuity, momentum, energy laws.

$$\frac{\partial}{\partial t}[\rho] + \bar{v}_f \cdot \nabla \rho = -\rho \nabla \cdot \bar{v}_f$$

$$\rho \left[\frac{\partial}{\partial t}[\bar{v}_f] + (\bar{v}_f \cdot \nabla)\bar{v}_f \right] = -\nabla P_r + \rho \bar{F}_b + \mu \nabla^2 \bar{v}_f = -\nabla P_r + \rho \bar{F}_b - \frac{2}{3}\nabla(\mu \nabla \cdot \bar{v}_f) + 2\nabla \cdot (\mu S)$$

$$\rho c_p \left[\frac{\partial}{\partial t}(T) + \bar{v}_f \cdot \nabla T \right] = \nabla \cdot (k \nabla T) - \frac{2}{3}\mu(\nabla \cdot \bar{v}_f)^2 + 2\mu S : S + \beta T \frac{D}{Dt}[P_r]$$

1.1 Hiemenz Flow

1.1.1 Cartesian Coordinates

1.1.2 Polar Coordinates

The continuity governing equation for incompressible fluids in cylindrical coordinates,

$$0 = \nabla \cdot \bar{v} = \frac{1}{r}\frac{\partial}{\partial r}[r v_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] + \frac{\partial}{\partial z}[v_z]$$

In 3-dimensions, the fluid velocity has a radial component, an azimuthal (swirl) component and a z component. For this derivation, it is assumed that the fluid does not have swirl and that the flow is axis-symmetric. All quantities are not expected to vary in the θ direction. The flow is also assumed to be steady. Let the fluid velocity vector $\bar{v}$ be defined below,

$$\bar{v} = \begin{bmatrix} v_r & v_\theta & v_z \end{bmatrix}^T$$

wherein the v_r, v_θ and v_z correspond to the fluid velocity components in the radial, azimuthal and z-direction respectively. The radial component of velocity for axis-symmetric potential flow is defined to be,

$$v_{r,i} = cr$$

Reiterating the continuity expression,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[r v_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[v_\theta] + \frac{\partial}{\partial z}[v_z]$$

Due to the axis-symmetric assumption, $\dfrac{1}{r}\dfrac{\partial}{\partial\theta}[v_\theta] = 0$. Substituting,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[rv_r] + \frac{\partial}{\partial z}[v_z]$$

Substituting the radial velocity for the inviscid flow into the continuity expression,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[r \times cr] + \frac{\partial}{\partial z}[v_z]$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}[cr^2] + \frac{\partial}{\partial z}[v_z]$$

$$0 = \frac{1}{r} \times 2cr + \frac{\partial}{\partial z}[v_z]$$

$$0 = 2c + \frac{\partial}{\partial z}[v_z]$$

$$-2c = \frac{\partial}{\partial z}[v_z]$$

$$-2cz + k = v_z$$

wherein k is some constant to satisfy some boundary condition. Stagnation point is defined to be a point in the fluid where there is no velocity. Since a stagnation point occurs at $z = 0$, then the v_z velocity must also be zero at said point. Substituting,

$$0 + k = 0$$

From applying the boundary condition of the stagnation point, k can be safely neglected. Hence, the fluid velocity in the z-direction for inviscid flows,

$$-2cz = v_{z,i}$$

To make an "intelligent" guess on the modification performed for the fluid flow, the stream function for the inviscid flow must first be determined. Assume that the $\bar{A}$ is the velocity potential vector, whose components,

$$\bar{A} = \begin{bmatrix} A_r & A_\theta & A_z \end{bmatrix}^T$$

The relationship of the velocity vector field to the velocity potential vector,

$$\bar{v} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} = \nabla \times \bar{A} = \begin{bmatrix} \dfrac{1}{r}\dfrac{\partial A_z}{\partial\theta} - \dfrac{\partial A_\theta}{\partial z} \\ \dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{1}{r}\dfrac{\partial A_r}{\partial\theta} \end{bmatrix}$$

Since the problem is assumed to be axis-symmetric, and also no swirl, then

$$0 = \frac{\partial A_r}{\partial z} \quad , \quad 0 = \frac{\partial A_z}{\partial r}$$

Therefore,

$$0 = A_r \quad , \quad 0 = A_z$$

Let $A_\theta = \psi$ which would represent the stream function of this axis-symmetric fluid flow. Substituting for the simplification,

$$v_r = -\frac{\partial A_\theta}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

Susbtituting for ψ,

$$v_r = -\frac{\partial \psi}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

If ψ_i represents the stream function to the inviscid fluid flow,

$$v_{r,i} = -\frac{\partial \psi_i}{\partial z} \quad , \quad v_{z,i} = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

Substituting for the inviscid flow field that was determined earlier,

$$cr = -\frac{\partial \psi_i}{\partial z} \quad , \quad -2cz = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

Analyzing the velocity in the axis-direction,

$$-2cz = \frac{1}{r}\frac{\partial}{\partial r}(r\psi_i)$$

$$-2czr = \frac{\partial}{\partial r}(r\psi_i)$$

$$-2cz\int r\,dr = \int \frac{\partial}{\partial r}(r\psi_i)\,dr$$

$$-2cz \times \frac{1}{2}r^2 = \int d(r\psi_i)$$

$$-cz \times r^2 = r\psi_i + k(z)$$

$$-czr^2 = r\psi_i + k(z)$$

Analyzing the velocity in the radial direction,

$$cr = -\frac{\partial \psi_i}{\partial z}$$

$$-cr = \frac{\partial \psi_i}{\partial z}$$

Deriving the previous expression with respect to z,

$$\frac{\partial}{\partial z}[-czr^2] = \frac{\partial}{\partial z}[r\psi_i] + \frac{\partial}{\partial z}[k(z)]$$

$$-cr^2 = r\frac{\partial}{\partial z}[\psi_i] + k'(z)$$

Substituting for the derivative of ψ_i with respect to z,

$$-cr^2 = r \times -cr + k'(z)$$

$$-cr^2 = -cr^2 + k'(z)$$

$$k'(z) = 0$$

Hence, $k(z) = constant$. This shows that some constant can be added to the stream function and the resulting fluid flow would still be identical. Let $k(z) = 0$ purely for convenience,

$$-czr^2 = r\psi_i + k(z) = r\psi_i$$

$$-czr = \psi_i$$

This is the stream function for the inviscid flow. The stream function should be altered in order to satisfy the no-slip boundary condition on the walls. Making the minimum changes necessary to allow for this, z in the stream function of the inviscid flow is replaced with an arbitrar function $f(z)$. f is a function purely in z. Let ψ be the stream function to a fluid flow that satisfies the Navier-Stokes equations near a stagnation point,

$$\psi = -crf(z)$$

To simplify notation,

$$\psi = -crf$$

Finding the velocities with the modified stream function,

$$v_r = -\frac{\partial \psi}{\partial z} \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

$$v_r = -\frac{\partial}{\partial z}(\psi) \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

$$v_r = -\frac{\partial}{\partial z}(-crf) \quad , \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(-cr^2 f)$$

$$v_r = crf' \quad , \quad v_z = \frac{1}{r} \times (-2crf)$$

$$v_r = crf' \quad , \quad v_z = -2cf$$

Although the fluid velocity fiield that obeys the Navier-Stokes equation would be different than the inviscid flow field near the wall, the velocity field should be identical to the inviscid flow field very far from the wall. Hence,

$$\lim_{z\to\infty} [v_{r,i}] = \lim_{z\to\infty} [crf']$$

$$\lim_{z\to\infty} [cr] = \lim_{z\to\infty} [cr] \lim_{z\to\infty} [f']$$

$$1 = \lim_{z\to\infty} [f']$$

$$\lim_{z\to\infty} [v_{z,i}] = \lim_{z\to\infty} [-2cf]$$

$$\lim_{z\to\infty} [-2cz] = \lim_{z\to\infty} [-2cf]$$

$$\lim_{z\to\infty} [-2c] \lim_{z\to\infty} [z] = \lim_{z\to\infty} [-2c] \lim_{z\to\infty} [f]$$

$$\lim_{z\to\infty} [z] = \lim_{z\to\infty} [f]$$

To summarize the stream function of the flow field which is an exact-solution to the Navier-Stokes equation,

$$\psi = -crf$$

The resulting velocity components of the flow field with such a stream function,

$$v_r = crf' \quad , \quad v_z = -2cf$$

The limits of f such that the boundary conditions of the flow field matching with that of the invscid fluid flow very far away from the wall,

$$1 = \lim_{z\to\infty}[f'] \quad , \quad \lim_{z\to\infty}[z] = \lim_{z\to\infty}[f]$$

The laplacian of an arbitrary vector v in polar coordinates,

$$\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2}$$

The momentum equation in the radial direction,

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

Due to the steady state assumption, $\frac{\partial v_r}{\partial t} = 0$. Due to the axis-symmetric assumption, $\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} = 0$. Assuming the flow field does not have any swirl, $\frac{v_\theta^2}{r} = 0$ and $-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} = 0$. Neglecting any body force, $\rho g_r = 0$. Substituting,

$$\rho\left[v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2}\right]$$

Substituting the definition of v_r from the stream-function analysis,

$$\rho\left[crf'\frac{\partial}{\partial r}(crf') + v_z\frac{\partial}{\partial z}(crf')\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{crf'}{r^2}\right]$$

$$\rho\left[crf' \times cf' + v_z \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

$$\rho\left[c^2rf'^2 + v_z \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

Substituting for the velocity v_z,

$$\rho\left[c^2rf'^2 - 2cf \times crf''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

$$\rho\left[c^2rf'^2 - 2c^2rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

The laplacian of the radial velocity,

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_r}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2}$$

Substituting for the radial velocity v_r.

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial crf'}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 crf'}{\partial \theta^2} + \frac{\partial^2 crf'}{\partial z^2}$$

Due to the axis-symmetric assumption

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial crf'}{\partial r}\right] + \frac{\partial^2 crf'}{\partial z^2}$$

$$\nabla^2 v_r = \frac{1}{r}\frac{\partial}{\partial r}\left[crf'\right] + \frac{\partial^2 crf'}{\partial z^2}$$

$$\nabla^2 v_r = \frac{1}{r} \times cf' + crf'''$$

$$\nabla^2 v_r = \frac{cf'}{r} + crf'''$$

Reiterating where we left off with the radial momentum,

$$\rho\left[c^2 rf'^2 - 2c^2 rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{cf'}{r}\right]$$

Substituting the laplacian of radial velocity into the momentum equation in the radial direction,

$$\rho\left[c^2 rf'^2 - 2c^2 rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[\frac{cf'}{r} + crf''' - \frac{cf'}{r}\right]$$

$$\rho\left[c^2 rf'^2 - 2c^2 rff''\right] = -\frac{\partial p}{\partial r} + \mu\left[crf'''\right]$$

The above is the first ordinary differential equation which describes f. Unfortunately, the term $\frac{\partial p}{\partial r}$ cannot be ignored when considering the exact case because there is significant pressure changes in the fluid flow. If the exact solution of the Navier-Stokes mimics potential flow far away, and the inviscid solution does allow for significant pressure gradients, then pressure gradients must also be taken into account when considering the exact solution to the Navier-Stokes.

The momentum equation in the axial direction,

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

Just like before, due to the steady state assumption, $\frac{\partial v_z}{\partial t} = 0$. Due to the axis-symmetric assumption, $\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} = 0$. Neglecting any body force in the axial direction, $\rho g_z = 0$.

Substituting for these simplifications,

$$\rho\left[v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

Substituting for axial velocity v_z in terms of f,

$$v_z = -2cf$$

$$\rho\left[v_r\frac{\partial}{\partial r}(-2cf) - 2cf\frac{\partial}{\partial z}(-2cf)\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

Substituting for radial velocity,

$$v_r = crf'$$

$$\rho\left[crf'\frac{\partial}{\partial r}(-2cf) - 2cf\frac{\partial}{\partial z}(-2cf)\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

$$\rho\left[-2cf \times -2cf'\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

$$\rho\left[4c^2 ff'\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

The laplacian of the axial velocity,

$$\nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}$$

Due to the axis-symmetric assumption, $\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} = 0$

$$\nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \frac{\partial^2 v_z}{\partial z^2}$$

Substituting for radial velocity v_r and axial velocity v_z,

$$\nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(-2cf)\right] + \frac{\partial^2}{\partial z^2}(-2cf)$$

$$\nabla^2 v_z = -2cf''$$

Substituting laplacian of axial velocity into the axial momentum equation,

$$\rho\left[4c^2 ff'\right] = -\frac{\partial p}{\partial z} + \mu \times -2cf''$$

$$4\rho c^2 ff' = -\frac{\partial p}{\partial z} - 2\mu cf''$$

We now have 2 ordinary differential equations that describe f it is now possible to potentially solve the expressions by using differential equation 2 to determine pressure in terms of f and substituting into differential equation 1 to solve for f. Integrating differential equation 2,

$$4\rho c^2 \int ff'\,dz = -\int \frac{\partial p}{\partial z}\,dz - 2\mu c\int f''\,dz$$

$$4\rho c^2 \int f\frac{df}{dz}\,dz = -\int dp - 2\mu c\int f''\,dz$$

$$4\rho c^2 \int f\,df = -\int dp - 2\mu c\int f''\,dz$$

$$4\rho c^2 \times \frac{1}{2}f^2 = -p - 2\mu cf' + k(r)$$

wherein $k(r)$ is a function that is purely in terms of r. This function appears as a consequence of integrating with respect to z.

$$2\rho c^2 f^2 = -p - 2\mu cf' + k(r)$$

Making pressure the subject of the expression,

$$p = -2\mu cf' - 2\rho c^2 f^2 + k(r)$$

The exact solution to the Navier-Stokes equation should match the inviscid solution very far away from the wall. Implying that the pressure for the exact solution and the inviscid solution must match infinitely far away from the wall. Taking the limits as z approaches ∞,

$$\lim_{z\to\infty}[p] = -2\mu c \lim_{z\to\infty}[f'] - 2\rho c^2 \lim_{z\to\infty}\left[f^2\right] + \lim_{z\to\infty}[k(r)]$$

Susbtituting the limits for f which was determined earlier,

$$1 = \lim_{z\to\infty}[f'] \quad , \quad \lim_{z\to\infty}[z] = \lim_{z\to\infty}[f]$$

$$\lim_{z\to\infty}[p] = -2\mu c - 2\rho c^2 \lim_{z\to\infty}\left[z^2\right] + \lim_{z\to\infty}[k(r)]$$

For values of $z \to \infty$,

$$p = -2\mu c - 2\rho c^2 z^2 + k(r)$$

For the inviscid fluid flow, bernoulli's equation can be used to determine pressure,

$$p_0 = p + \frac{1}{2}\rho|\bar{v}_i|^2$$

wherein $\bar{v}_i$ represents the local fluid velocity for inviscid flow. Considering that there is only radial and axial velocity in an axis-symetric problem,

$$|\bar{v}_i|^2 = v_{r,i}^2 + v_{z,i}^2$$

Substituting,

$$p_0 = p + \frac{1}{2}\rho\left[v_{r,i}^2 + v_{z,i}^2\right]$$

Making pressure the subject of the expression,

$$p = p_0 - \frac{1}{2}\rho\left[v_{r,i}^2 + v_{z,i}^2\right]$$

Substituting for the inviscid radial and axial velocities,

$$p = p_0 - \frac{1}{2}\rho\left[(cr)^2 + (-2cz)^2\right]$$

$$p = p_0 - \frac{1}{2}\rho\left[c^2r^2 + 4c^2z^2\right]$$

$$p = p_0 - \frac{1}{2}\rho c^2 r^2 - 2\rho c^2 z^2$$

Matching the exact solution's pressure to the inviscid solution's pressure far away from the wall,

$$p = -2\mu c - 2\rho c^2 z^2 + k(r) = p_0 - \frac{1}{2}\rho c^2 r^2 - 2\rho c^2 z^2$$

$$-2\mu c + k(r) = p_0 - \frac{1}{2}\rho c^2 r^2$$

$$k(r) = p_0 - \frac{1}{2}\rho c^2 r^2 + 2\mu c$$

Substituting the function k into the expression for pressure in the exact solution,

$$p = -2\mu c - 2\rho c^2 z^2 + p_0 - \frac{1}{2}\rho c^2 r^2 + 2\mu c$$

$$p = -2\rho c^2 z^2 + p_0 - \frac{1}{2}\rho c^2 r^2$$

Taking the derivative with respect to r,

$$\frac{\partial p}{\partial r} = -2\rho c^2 \frac{\partial}{\partial r}(z^2) + \frac{\partial p_0}{\partial r} - \frac{1}{2}\rho \frac{\partial}{\partial r}(c^2 r^2)$$

$$\frac{\partial p}{\partial r} = -\frac{1}{2}\rho \frac{\partial}{\partial r}(c^2 r^2)$$

$$\frac{\partial p}{\partial r} = -\frac{1}{2}\rho \times 2c^2 r$$

$$\frac{\partial p}{\partial r} = -\rho c^2 r$$

$$-\frac{\partial p}{\partial r} = \rho c^2 r$$

Substituting the gradient of pressure in r into differential equation 1,

$$\rho\left[c^2 r f'^2 - 2c^2 r f f''\right] = -\frac{\partial p}{\partial r} + \mu\left[c r f'''\right]$$

$$\rho\left[c^2 r f'^2 - 2c^2 r f f''\right] = \rho c^2 r + \mu c r f'''$$

$$\left[c^2 r f'^2 - 2c^2 r f f''\right] = c^2 r + \frac{\mu}{\rho} c r f'''$$

$$r f'^2 - 2r f f'' = r + \frac{\mu}{\rho}\frac{1}{c} r f'''$$

The relation between kinematic and dynamic viscosity is shown below,

$$\nu = \frac{\mu}{\rho}$$

Substituting,

$$r f'^2 - 2r f f'' = r + \nu \frac{1}{c} r f'''$$

$$f'^2 - 2f f'' = 1 + \nu \frac{1}{c} f'''$$

$$-\frac{\nu}{c} f''' + f'^2 - 2f f'' - 1 = 0$$

Let the function ϕ and variable η be defined below,

$$f = \left(\frac{2c}{\nu}\right)^{-1/2} \phi \quad , \quad \eta = \left(\frac{2c}{\nu}\right)^{1/2} z$$

By conjecture,

$$\frac{d^n}{dz^n} = \left[\left(\frac{2c}{\nu}\right)^{1/2}\right]^n \frac{d^n}{d\eta^n}$$

$$\frac{d^n}{dz^n} = 2^{n/2}\left(\frac{c}{\nu}\right)^{n/2}\frac{d^n}{d\eta^n}$$

Consider when $n = 1$,

$$\frac{d}{dz} = \frac{d}{d\eta} \times \frac{d\eta}{dz}$$

$$\frac{d}{dz} = \frac{d}{d\eta} \times \frac{d}{dz}\left[\left(\frac{2c}{\nu}\right)^{1/2} z\right]$$

$$\frac{d}{dz} = 2^{1/2}\left(\frac{c}{\nu}\right)^{1/2}\frac{d}{d\eta}$$

Let $n = k + 1$

$$\frac{d}{dz}\left[\frac{d^k}{dz^k}\right] = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2}\frac{d^{k+1}}{d\eta^{k+1}}$$

Let LHS and RHS be defined,

$$LHS = \frac{d}{dz}\left[\frac{d^k}{dz^k}\right] \quad , \quad RHS = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2}\frac{d^{k+1}}{d\eta^{k+1}}$$

$$LHS = \frac{d}{dz}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2}\frac{d^k}{d\eta^k}\right]$$

$$LHS = \frac{d}{d\eta}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2}\frac{d^k}{d\eta^k}\right] \times \frac{d\eta}{dz}$$

$$LHS = \frac{d}{d\eta}\left[2^{k/2}\left(\frac{c}{\nu}\right)^{k/2}\frac{d^k}{d\eta^k}\right] \times 2^{1/2}\left(\frac{c}{\nu}\right)^{1/2}$$

$$LHS = 2^{(k+1)/2}\left(\frac{c}{\nu}\right)^{(k+1)/2}\frac{d}{d\eta}\left[\frac{d^k}{d\eta^k}\right]$$

Since $LHS = RHS$, by principle of mathematical induction, the formula is true. Applying to the exact solution,

$$-\frac{\nu}{c}f''' + f'^2 - 2ff'' - 1 = 0$$

$$-\frac{\nu}{c}f''' = -\frac{\nu}{c} \times 2^{3/2}\left(\frac{c}{\nu}\right)^{3/2}\frac{d^3}{d\eta^3}\left[2^{-1/2}\left(\frac{c}{\nu}\right)^{-1/2}\phi\right]$$

$$-\frac{\nu}{c}f''' = -2^{3/2-1/2}\left(\frac{c}{\nu}\right)^{3/2-1-1/2}\frac{d^3}{d\eta^3}[\phi]$$

$$-\frac{\nu}{c}f''' = -2\frac{d^3}{d\eta^3}[\phi]$$

$$-\frac{\nu}{c}f''' = -2\phi'''$$

$$f'^2 = \left\{2^{1/2}\left(\frac{c}{\nu}\right)^{1/2}\frac{d}{d\eta}\left[2^{-1/2}\left(\frac{c}{\nu}\right)^{-1/2}\phi\right]\right\}^2$$

$$f'^2 = \left\{ \frac{d}{d\eta} [\phi] \right\}^2$$
$$f'^2 = \{\phi'\}^2$$
$$f'^2 = \phi'^2$$

$$-2ff'' = -(2)2^{-1/2} \left[\left(\frac{c}{\nu}\right)^{-1/2} \phi \right] 2 \left(\frac{c}{\nu}\right) \frac{d^2}{d\eta^2} \left[2^{-1/2} \left(\frac{c}{\nu}\right)^{-1/2} \phi \right]$$
$$-2ff'' = -2^{1-1/2+1-1/2} \left[\left(\frac{c}{\nu}\right)^{-1/2+1-1/2} \phi \right] \frac{d^2}{d\eta^2} [\phi]$$
$$-2ff'' = -2\phi\phi''$$

Substituting all the terms together,

$$-2\phi''' + \phi'^2 - 2\phi\phi'' - 1 = 0$$
$$-2\phi''' - 2\phi\phi'' + \phi'^2 - 1 = 0$$
$$\phi''' + \phi\phi'' - \frac{1}{2}\phi'^2 + \frac{1}{2} = 0$$

Reiterating the radial and axial velocities based on the modified stream function,

$$v_r = crf' \quad , \quad v_z = -2cf$$

At the stagntion point $z = 0$ and since η is a linear function of z, $z = 0$. The stagnation point is defined as a point in the fluid where fluid velocity is zero. Therefore,

$$0 = crf'(0) \quad , \quad 0 = -2cf(0)$$
$$0 = f'(0) \quad , \quad 0 = -f(0)$$

Both derivative and function f have linear scaling with ϕ' and ϕ respectively. Therefore,

$$\phi(0) = 0 \quad , \quad \phi'(0) = 0$$

Infinitely far away,

$$v_r = cr = crf'$$
$$1 = f'(\infty)$$

Taking the derivative of f,

$$\frac{df}{dz} = \left(\frac{2c}{\nu}\right)^{1/2} \frac{d}{d\eta} \left[\left(\frac{2c}{\nu}\right)^{-1/2} \phi \right]$$
$$\frac{df}{dz} = \phi'(\eta)$$

Substituting for infinite distance away,

$$\frac{df}{dz}_{z=\infty} = \phi'(\eta = \infty)$$
$$1 = \phi'(\infty)$$

1.2 Alternate Forms

1.2.1 Strain And Rotation

1.2.1.1 Continuity: Index Notation

The continuity and momentum equations in vector form,

$$\nabla \cdot u = 0 \quad , \quad \frac{Du}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

The continuity equation in index form,

$$0 = \frac{\partial u_j}{\partial x_j}$$

Simplifying the momentum equation in vector form,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

Converting the momentum equation into index form,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(u_i)$$

1.2.1.2 Momentum: Velocity Gradient

Renaming the dummy indices in the index momentum equation $j \to k$,

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(u_i)$$

Taking the derivative of the index momentum equation with respect to x_j,

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}(u_i) + \frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(p) + \nu \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(u_i)$$

Here, the fluid is assumed to be incompressible, hence ρ is a simple known fluid property. Since the partial derivative operator is commutative,

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x_j}(u_i) + \frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}(u_i)$$

Substituting $e_{ij} = \dfrac{\partial u_i}{\partial x_j}$,

$$\frac{\partial}{\partial t} e_{ij} + \frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k} e_{ij}$$

Simplifying the convective acceleration term by applying chain rule,

$$\frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = u_k \frac{\partial}{\partial x_j}\left[\frac{\partial u_i}{\partial x_k}\right] + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}\left[u_k\right]$$

Due to the partial derivative operator being commutative,

$$\frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = u_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}(u_i) + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}\left[u_k\right]$$

Substituting $e_{ij} = \dfrac{\partial u_i}{\partial x_j}$,

$$\frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = u_k \frac{\partial}{\partial x_k} e_{ij} + \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial x_j}\left[u_k\right]$$

Based on the definition of e_{ij},

$$e_{ik} = \frac{\partial u_i}{\partial x_k} \quad , \quad e_{kj} = \frac{\partial u_k}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}\left[u_k \frac{\partial u_i}{\partial x_k}\right] = u_k \frac{\partial}{\partial x_k} e_{ij} + e_{ik} e_{kj}$$

Susbtituting the convective acceleration into the momentum equation,

$$\frac{\partial}{\partial t} e_{ij} + u_k \frac{\partial}{\partial x_k} e_{ij} + e_{ik} e_{kj} = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k} e_{ij}$$

Manipulating the equation further,

$$\frac{\partial}{\partial t} e_{ij} + u_k \frac{\partial}{\partial x_k} e_{ij} = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik} e_{kj} + \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k} e_{ij}$$

$$\frac{\partial}{\partial t}(e_{ij}) + u_k \frac{\partial}{\partial x_k}(e_{ij}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik} e_{kj}$$

1.2.1.3 Strain Rate Form

Reiterating the momentum equation in index form,

$$\frac{\partial}{\partial t}(e_{ij}) + u_k \frac{\partial}{\partial x_k}(e_{ij}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik} e_{kj}$$

Renaming the indices, $i \rightarrow j$, and $j \rightarrow i$,

$$\frac{\partial}{\partial t}(e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ji}) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_j}(p) - e_{jk} e_{ki}$$

Adding the 2 equations above together and taking into account that $\dfrac{\partial^2}{\partial x_i \partial x_j}(p) = \dfrac{\partial^2}{\partial x_j \partial x_i}(p)$,

$$\frac{\partial}{\partial t}(e_{ij} + e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) = -\frac{2}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - e_{ik} e_{kj} - e_{jk} e_{ki}$$

$$\frac{\partial}{\partial t}(e_{ij} + e_{ji}) + u_k \frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij} + e_{ji}) = -\frac{2}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - (e_{ik} e_{kj} + e_{jk} e_{ki})$$

Multiplying both sides by half,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(e_{ij} + e_{ji})\right] + u_k \frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij} + e_{ji})\right] - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij} + e_{ji})\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - \frac{1}{2}(e_{ik} e_{kj} + e_{jk} e_{ki})$$

The symmetric strain rate tensor is defined as, $S_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$. Expressing the symmetric strain rate tensor in terms of e_{ij} and e_{ji}, $S_{ij} = \dfrac{1}{2}\left(e_{ij} + e_{ji}\right)$ Substituting for the symmetric strain rate tensor into the momentum equation,

$$\frac{\partial}{\partial t}\left[S_{ij}\right] + u_k \frac{\partial}{\partial x_k}\left[S_{ij}\right] - \nu \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[S_{ij}\right] = -\frac{1}{\rho}\frac{\partial^2}{\partial x_j \partial x_i}(p) - \frac{1}{2}(e_{ik} e_{kj} + e_{jk} e_{ki})$$

Simplifying further,

$$\left\{\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\right\}S_{ij}=-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)-\frac{1}{2}(e_{ik}e_{kj}+e_{jk}e_{ki})$$

$\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}$ represents the substantive derivative in index notation meanwhile $\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}$ represents the laplacian in index notation.

1.2.1.4 Rotation Rate Form

Performing the same steps as the previous part but instead of adding 2 equations together, the equations are subtracted off each other,

$$\frac{\partial}{\partial t}(e_{ij}-e_{ji})+u_k\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})=-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)+\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)-e_{ik}e_{kj}+e_{jk}e_{ki}$$

Since the partial differential operators are commutative, $-\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)+\frac{1}{\rho}\frac{\partial^2}{\partial x_j\partial x_i}(p)=0$. Substituting,

$$\frac{\partial}{\partial t}(e_{ij}-e_{ji})+u_k\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(e_{ij}-e_{ji})=-e_{ik}e_{kj}+e_{jk}e_{ki}$$

Multiplying both sides by half,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(e_{ij}-e_{ji})\right]+u_k\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij}-e_{ji})\right]-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\frac{1}{2}(e_{ij}-e_{ji})\right]=-\frac{1}{2}\left[e_{ik}e_{kj}-e_{jk}e_{ki}\right]$$

The rotation rate tensor is defined as $\Omega_{ij}=\frac{1}{2}(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i})$. Substituting for $e_{ij}=\frac{\partial u_i}{\partial x_j}$ and $e_{ji}=\frac{\partial u_j}{\partial x_i}$, $\Omega_{ij}=\frac{1}{2}(e_{ij}-e_{ji})$ Substituting the rotation rate tensor,

$$\frac{\partial}{\partial t}\left[\Omega_{ij}\right]+u_k\frac{\partial}{\partial x_k}\left[\Omega_{ij}\right]-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left[\Omega_{ij}\right]=-\frac{1}{2}\left[e_{ik}e_{kj}-e_{jk}e_{ki}\right]$$

$$\left\{\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}-\nu\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\right\}\Omega_{ij}=-\frac{1}{2}\left[e_{ik}e_{kj}-e_{jk}e_{ki}\right]$$

Just as before, $\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}$ represents the substantive derivative in index notation meanwhile $\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}$ represents the laplacian in index notation. Interestingly, here the expression is independent of the pressure gradient tensor sinc ethe pressure gradient tensor is symmetric.

1.2.2 Vorticity Equation

1.2.2.1 Derivation

The substantive derivative of fluid velocity $\bar{v_f}$ appears in the non-conservative form of the momentum equation. The substantive derivative of fluid velocity $\bar{v_f}$ in vector form,

$$\frac{D\bar{v_f}}{Dt}=\frac{\partial\bar{v_f}}{\partial t}+\bar{v_f}\cdot\nabla\bar{v_f}$$

By conjecture,

$$\bar{v_f} \cdot \nabla \bar{v_f} = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Let,

$$LHS = \bar{v_f} \cdot \nabla \bar{v_f} \quad , \quad RHS = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

wherein $\bar{\omega_f}$ represents the fluid vorticity. Fluid vorticity is defined as the curl of fluid velocity, $\bar{\omega_f} = \nabla \times \bar{v_f}$.

Expressing LHS in index notation,

$$LHS_i = v_j \frac{\partial v_i}{\partial x_j}$$

wherein v_i represents the i^{th} component of the velocity vector $\bar{v_f}$.

Expressing RHS in index notation,

$$RHS_i = \epsilon_{ijk} \omega_j v_k + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

wherein ω_j represents the j^{th} component of fluid vorticity vector $\bar{\omega_f}$. Expressing the definition of vorticity as curl of fluid velocity in index notation,

$$\omega_j = \epsilon_{jlm} \frac{\partial v_m}{\partial x_l}$$

Substituting ω_j into RHS_i,

$$RHS_i = \epsilon_{ijk} \epsilon_{jlm} \frac{\partial v_m}{\partial x_l} v_k + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = \epsilon_{ijk} \epsilon_{jlm} v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

Based on the cyclic permutation properties of the permutation tensors ϵ_{ijk}, $\epsilon_{ijk} = \epsilon_{jki}$. Therefore,

$$\epsilon_{ijk} \epsilon_{jlm} = \epsilon_{jki} \epsilon_{jlm}$$

Based on the double permutation tensor identity,

$$\epsilon_{ijk} \epsilon_{jlm} = \epsilon_{jki} \epsilon_{jlm} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il}$$

Substituting into RHS_i,

$$RHS_i = \left[\delta_{kl} \delta_{im} - \delta_{km} \delta_{il} \right] v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = \delta_{kl} \delta_{im} v_k \frac{\partial v_m}{\partial x_l} - \delta_{km} \delta_{il} v_k \frac{\partial v_m}{\partial x_l} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right]$$

$$RHS_i = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + \frac{\partial}{\partial x_i} \left[\frac{1}{2} v_j v_j \right] = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_i} \left[v_j v_j \right]$$

$$RHS_i = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + \frac{1}{2} \left[v_j \frac{\partial}{\partial x_i} (v_j) + v_j \frac{\partial}{\partial x_i} (v_j) \right] = v_l \frac{\partial v_i}{\partial x_l} - v_k \frac{\partial v_k}{\partial x_i} + v_j \frac{\partial v_j}{\partial x_i} = v_l \frac{\partial v_i}{\partial x_l}$$

Renaming the dummy index $l \to j$,

$$RHS_i = v_j \frac{\partial v_i}{\partial x_j}$$

Since $LHS_i = RHS_i$, then the conjecture shown below must be true,

$$\bar{v_f} \cdot \nabla \bar{v_f} = \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Susbtituting into the substantive derivative of fluid velocity,

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f} = \frac{\partial \bar{v_f}}{\partial t} + \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Taking the curl of the substantive derivative of fluid velocity,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \nabla \times \left(\frac{\partial \bar{v_f}}{\partial t} \right) + \nabla \times (\bar{\omega_f} \times \bar{v_f}) + \nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$$

The curl operations and the partial derivative operations are commutative. Therefore,

$$\nabla \times \left(\frac{\partial \bar{v_f}}{\partial t} \right) = \frac{\partial}{\partial t} (\nabla \times \bar{v_f})$$

Substituting for the definition of fluid vorticity $\bar{\omega_f} = \nabla \times \bar{v_f}$,

$$\nabla \times \left(\frac{\partial \bar{v_f}}{\partial t} \right) = \frac{\partial}{\partial t} (\bar{\omega_f}) = \frac{\partial \bar{\omega_f}}{\partial t}$$

Substituting into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{\partial \bar{\omega_f}}{\partial t} + \nabla \times (\bar{\omega_f} \times \bar{v_f}) + \nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$$

$\bar{v_f} \cdot \bar{v_f}$ is a scalar. The curl of a scalar gradient is zero. Therefore,

$$0 = \nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$$

Negletcting the $\nabla \times \left[\nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right) \right]$ term,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{\partial \bar{\omega_f}}{\partial t} + \nabla \times (\bar{\omega_f} \times \bar{v_f})$$

This is a vector identity,

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Let $\bar{A} = \bar{\omega_f}$ and $\bar{B} = \bar{v_f}$,

$$\nabla \times (\bar{\omega_f} \times \bar{v_f}) = \bar{v_f} \cdot \nabla \bar{\omega_f} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f} - \bar{v_f} \nabla \cdot \bar{\omega_f}$$

Based on the definition of fluid vorticity $\bar{\omega_f} = \nabla \times \bar{v_f}$,

$$\bar{v_f} \nabla \cdot \bar{\omega_f} = \bar{v_f} \nabla \cdot (\nabla \times \bar{v_f})$$

Since the divergence of a vector field curl is zero, $\nabla \cdot (\nabla \times \bar{v_f}) = 0$. Therefore,

$$0 = \bar{v_f} \nabla \cdot \bar{\omega_f}$$

Neglecting the $\bar{v_f}\nabla \cdot \bar{\omega_f}$ term,

$$\nabla \times (\bar{\omega_f} \times \bar{v_f}) = \bar{v_f} \cdot \nabla\bar{\omega_f} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

Substituting $\nabla \times (\bar{\omega_f} \times \bar{v_f})$, into the substantive derivative of vorticity,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{\partial\bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla\bar{\omega_f} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

The definition of substantive derivative of vorticity is shown below,

$$\frac{D\bar{\omega_f}}{Dt} = \frac{\partial\bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla\bar{\omega_f}$$

Substituting for the substantive derivative of vorticity into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

The symmetric strain rate tensor is defined as,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

The symmetric strain rate tensor is symmetric.
The rotation rate tensor is defined as,

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

The rotation rate tensor is anti-symmetric.
The summation of the strain rate tensor and the rotation rate tensor,

$$S_{ij}+\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}\right)$$

$$S_{ij} + \Omega_{ij} = \frac{\partial v_i}{\partial x_j}$$

Therefore, this shows that the fluid velocity gradient tensor $\frac{\partial v_i}{\partial x_j}$ can be decomposed into an algebraic sum of a symmetric tensor S_{ij} and an anti-symmetric tensor Ω_{ij}.
The rotation rate tensor is somewhat related to the fluid vorticity vector. Consider i^{th} component of the fluid vorticity vector the i^{th} component of fluid velocity curl,

$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Using the fluid vorticity to contract the permutation tensor on along its third dimension,

$$\epsilon_{lmi}\omega_i = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Due to the permutation cyclic property, $\epsilon_{lmi} = \epsilon_{ilm}$. Therefore,

$$\epsilon_{lmi}\epsilon_{ijk} = \epsilon_{ilm}\epsilon_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$$

Substituting into the permutation tensor contraction,

$$\epsilon_{lmi}\omega_i = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j} = [\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}]\frac{\partial v_k}{\partial x_j} = \delta_{lj}\delta_{mk}\frac{\partial v_k}{\partial x_j} - \delta_{lk}\delta_{mj}\frac{\partial v_k}{\partial x_j} = \delta_{mk}\frac{\partial v_k}{\partial x_l} - \delta_{mj}\frac{\partial v_l}{\partial x_j}$$

$$\epsilon_{lmi}\omega_i = \frac{\partial v_m}{\partial x_l} - \frac{\partial v_l}{\partial x_m}$$

Under an index varible change $l \to i$, $m \to j$, $i \to k$,

$$\epsilon_{ijk}\omega_k = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}$$

This form is already similar to the rotation rate tensor. In essence, contracting the permutation tensor along any dimension would allow the usage of the double permutation tensor identity. The third dimension was chosen in order to obtain the 'alternating' pattern similar to the rotation rate tensor. Minor algebraic manipulatins can then be performed to match $\epsilon_{ijk}\omega_k$ to the rotation rate tensor,

$$-\epsilon_{ijk}\omega_k = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$$

$$-\frac{1}{2}\epsilon_{ijk}\omega_k = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

The RHS matches the rotation rate tensor. Therefore,

$$-\frac{1}{2}\epsilon_{ijk}\omega_k = \Omega_{ij}$$

By conjecture, using the fluid vorticity vector $\bar{\omega_f}$ to contract the rotation rate tensor along the second dimension would yield zero. This claim is expressed in index notation,

$$0 = \omega_j\Omega_{ij}$$

Let

$$LHS_i = 0 \quad , \quad RHS_i = \omega_j\Omega_{ij}$$

Substituting the definition of the rotation rate tensor in terms of the fluid vorticity vector,

$$RHS_i = \omega_j\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k$$

Since ϵ_{ijk} is an anti-symmetric tensor, and $\omega_j\omega_k$ is a symmetric tensor due to multiplication being a commutative operation,

$$RHS_i = \omega_j\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k = 0$$

Therefore, the claim is proven to be true.

Reiterating the last form of the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f}\nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla\bar{v_f}$$

Converting the last term in RHS into index notation,

$$\left(\bar{\omega_f} \cdot \nabla\bar{v_f}\right)_i = \omega_j\frac{\partial v_i}{\partial x_j}$$

Expressing the velocity gradient tensor $\frac{\partial v_i}{\partial x_j}$ in terms of its symmetric and anti-symmetric components,

$$(\bar{\omega_f} \cdot \nabla \bar{v_f})_i = \omega_j \frac{\partial v_i}{\partial x_j} = \omega_j [S_{ij} + \Omega_{ij}] = \omega_j S_{ij} + \omega_j \Omega_{ij}$$

Based on previous work, the contraction of the rotation tensor on the second index using the fluid vorticity vector yields zero,

$$0 = \omega_j \Omega_{ij}$$

Neglecting the $\omega_j \Omega_{ij}$ term,

$$(\bar{\omega_f} \cdot \nabla \bar{v_f})_i = \omega_j S_{ij}$$

Converting into vector notation,

$$\bar{\omega_f} \cdot \nabla \bar{v_f} = \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

wherein $\bar{\bar{S}}_f$ represents the symmetric strain rate tensor for the fluid velocity vector field. Substituting into the curl of fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

The differential continuity equation is shown below,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v_f})$$

Converting the differential continuity equation into tensor index notation,

$$0 = \frac{\partial}{\partial t}[\rho] + \frac{\partial}{\partial x_j}(\rho v_j)$$

Using chain rule,

$$0 = \frac{\partial}{\partial t}[\rho] + \rho \frac{\partial}{\partial x_j}(v_j) + v_j \frac{\partial}{\partial x_j}(\rho)$$

Manipulating further,

$$\rho \frac{\partial}{\partial x_j}(v_j) = -\frac{\partial}{\partial t}[\rho] - v_j \frac{\partial}{\partial x_j}(\rho)$$

$$\frac{\partial}{\partial x_j}(v_j) = -\frac{1}{\rho} \left[\frac{\partial}{\partial t}(\rho) + v_j \frac{\partial}{\partial x_j}(\rho) \right]$$

The substantive derivative of fluid density ρ in vector notation,

$$\frac{D}{Dt}(\rho) = \frac{\partial}{\partial t}(\rho) + \bar{v_f} \cdot \nabla \rho$$

Converting the substatntive derivative of fluid density into index notation,

$$\left[\frac{D}{Dt}(\rho) \right]_i = \frac{\partial}{\partial t}(\rho) + v_j \frac{\partial}{\partial x_j}(\rho)$$

Substituting,

$$\frac{\partial}{\partial x_j}(v_j) = -\frac{1}{\rho}\left[\frac{D}{Dt}(\rho)\right]_i$$

Converting into vector index notation,

$$\nabla \cdot \bar{v_f} = -\frac{1}{\rho}\frac{D}{Dt}(\rho)$$

Substituting the divergence of fluid velocity $\bar{v_f}$ into the fluid velocity substantive derivative,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho) - \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

By conjecture,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho)$$

Let

$$LHS = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] \quad , \quad RHS = \frac{D\bar{\omega_f}}{Dt} - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}(\rho)$$

Using quotient rule,

$$LHS = \rho \times \frac{1}{\rho^2}\left\{\rho\frac{D}{Dt}\left[\omega_f\right] - \bar{\omega_f}\frac{D}{Dt}\left[\rho\right]\right\} = \frac{1}{\rho}\left\{\rho\frac{D}{Dt}\left[\omega_f\right] - \bar{\omega_f}\frac{D}{Dt}\left[\rho\right]\right\} = \frac{1}{\rho}\rho\frac{D}{Dt}\left[\omega_f\right] - \frac{1}{\rho}\bar{\omega_f}\frac{D}{Dt}\left[\rho\right]$$

$$LHS = \frac{D}{Dt}\left[\omega_f\right] - \frac{\bar{\omega_f}}{\rho}\frac{D}{Dt}\left[\rho\right]$$

Since $LHS = RHS$, the claim is proven.
Substituting for this simplification,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] - \bar{\omega_f} \cdot \bar{\bar{S}}_f$$

The non-conservative form of the momentum equation,

$$\rho\frac{D\bar{v_f}}{Dt} = \nabla \cdot \bar{\bar{T}}_f + \bar{g_b}$$

Making the substantive derivative of fluid velocity the subject of the equation,

$$\frac{D\bar{v_f}}{Dt} = \frac{1}{\rho}\nabla \cdot \bar{\bar{T}}_f + \frac{1}{\rho}\bar{g_b}$$

Taking the curl of the resulting expression so that it might be substituted into the main equation,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{T}}_f + \frac{1}{\rho}\bar{g_b}\right] = \nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{T}}_f\right] + \nabla \times \left[\frac{1}{\rho}\bar{g_b}\right]$$

Substituting the complete stress tensor and external acceleration into the main equation,

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt}\right) = \rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] - \bar{\omega_f} \cdot \bar{\bar{S}}_f = \nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{T}}_f\right] + \nabla \times \left[\frac{1}{\rho}\bar{g_b}\right]$$

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right]-\bar{\omega_f}\cdot\bar{\bar{S}}_f=\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]+\nabla\times\left[\frac{1}{\rho}\bar{g_b}\right]$$

Hence, the 'basic' vorticity equation,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right]=\bar{\omega_f}\cdot\bar{\bar{S}}_f+\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]+\nabla\times\left[\frac{1}{\rho}\bar{g_b}\right]$$

The complete stress tensor $\bar{\bar{T}}_f$ defined in index notation,

$$T_{ij}=-P_r\delta_{ij}+\mu\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right]+\lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}$$

wherein $\left(\bar{\bar{T}}_f\right)_{ij}=T_{ij}$, μ is the dynamic viscosity and λ is the second coefficient of viscosity. The viscous stress tensor $\bar{\bar{\tau}}_f$ has a rank of 2 and its ij component is referred as τ_{ij}. The viscous stress tensor components in index form is defined to be,

$$\tau_{ij}=\mu\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right]+\lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}$$

Therefore, the complete stress tensor can be expressed in terms of the viscous stress tensor,

$$T_{ij}=-P_r\delta_{ij}+\tau_{ij}$$

Expressing the complete stress tensor term in index notation,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(T_{ml})\right]$$

Renaming the indices $i\to m$ $j\to l$,

$$T_{ml}=-P_r\delta_{ml}+\tau_{ml}$$

Substituting for the complete stress tensor,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(-P_r\delta_{ml}+\tau_{ml})\right]$$

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(-P_r\delta_{ml})\right]+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

By applying the kronecker-delta contraction,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=-\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial P_r}{\partial x_l}\right]+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

By applying product rule on the pressure-related term,

$$\left\{\nabla\times\left[\frac{1}{\rho}\nabla\cdot\bar{\bar{T}}_f\right]\right\}_i=-\epsilon_{ijl}\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\frac{\partial P_r}{\partial x_l}\right]-\epsilon_{ijl}\frac{\partial P_r}{\partial x_l}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\right]+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

Due to the partial derivative operations being commutative $\frac{\partial}{\partial x_j}\left[\frac{\partial P_r}{\partial x_l}\right]$ is a symmetric tensor of rank 2. Since the permuation tensor ϵ_{ijl} is anti-symmetric,

$$0=-\epsilon_{ijl}\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\frac{\partial P_r}{\partial x_l}\right]$$

Neglecting the term,

$$\left\{\nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{T}}_f\right]\right\}_i = -\epsilon_{ijl}\frac{\partial P_r}{\partial x_l}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\right] + \epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

Applying chain rule, $\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\right] = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial x_j}$. Substituting,

$$\left\{\nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{T}}_f\right]\right\}_i = \epsilon_{ijl}\frac{\partial P_r}{\partial x_l}\frac{1}{\rho^2}\frac{\partial \rho}{\partial x_j}+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right] = \epsilon_{ijl}\frac{1}{\rho^2}\frac{\partial \rho}{\partial x_j}\frac{\partial P_r}{\partial x_l}+\epsilon_{ijl}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\frac{\partial}{\partial x_m}(\tau_{ml})\right]$$

Converting into index notation,

$$\nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{T}}_f\right] = \frac{1}{\rho^2}\nabla\rho \times \nabla P_r + \nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{\tau}}_f\right]$$

Substituting into the basic vorticity equation,

$$\rho\frac{D}{Dt}\left[\frac{\bar{\omega_f}}{\rho}\right] = \bar{\omega_f} \cdot \bar{\bar{S}}_f + \frac{1}{\rho^2}\nabla\rho \times \nabla P_r + \nabla \times \left[\frac{1}{\rho}\nabla \cdot \bar{\bar{\tau}}_f\right] + \nabla \times \left[\frac{1}{\rho}\bar{g}_b\right]$$

1.2.2.2 Summary

Basic Substantive derivative of fluid velocity

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f}$$

Kinematic Relations

Intermediate derivative kinetic energy form

$$\frac{D\bar{v_f}}{Dt} = \frac{\partial \bar{v_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{v_f} = \frac{\partial \bar{v_f}}{\partial t} + \bar{\omega_f} \times \bar{v_f} + \nabla \left(\frac{1}{2} \bar{v_f} \cdot \bar{v_f} \right)$$

Double $\epsilon\epsilon$ Identity

$$\nabla \times \nabla \phi = 0, \quad \nabla \cdot (\nabla \times v) = 0$$

Vorticity 1^{st} relation

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{\partial \bar{\omega_f}}{\partial t} + \bar{v_f} \cdot \nabla \bar{\omega_f} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$

Definition of material derivative

3 term equation part 1

$$\nabla \times \left(\frac{D\bar{v_f}}{Dt} \right) = \frac{D\bar{\omega_f}}{Dt} + \bar{\omega_f} \nabla \cdot \bar{v_f} - \bar{\omega_f} \cdot \nabla \bar{v_f}$$