Chapter 1

Governing Equations

1.1 Governing Equation: Continuum Equation

The Governing Continuum Equation in its differential form:

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v_f})$$

The Governing Continuum Equation in its integral form:

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

A more useful alternate form:

$$\frac{d}{dt}M(t) = \iint_{S(t)} \rho(\bar{v}_s - \bar{v_f}) \cdot \bar{n} dS$$

wherein $M(t)$ represent mass contained in a control volume, $\bar{v_f}$ represent the velocity of the fluid and $\bar{v}_s$ represent the velocity of the deforming control volume $R(t)$. $S(t)$ represents the surface that is encapsuating the control volume $R(t)$.

1.1.1 Differential Continuity Proof

Starting with the definition of mass contained in the arbitrary control volume $R(t)$,

$$M(t) = \iiint_{R(t)} \rho \, dV_o$$

Taking the derivative of the mass contained within the control volume with respect to time,

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o$$

By application of Liebniz rule, substituting T with ρ,

$$\frac{d}{dt}\iiint_{R(t)} T \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T] dV_o + \iint_{S(t)} T \bar{v}_s \bar{n} dS$$

$$\frac{d}{dt}M(t) = \frac{d}{dt}\iiint_{R(t)} \rho \, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

If the velocity of the surface expanding is equivalent to the velocity of the fluid at the boundary of the control volume ($\bar{v}_s = \bar{v}_f$), then the amount of mass within the control volume must remain constant.

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_f \cdot \bar{n} dS$$

The second term of the expression above could be converted into a volumetric integral based on the divergence theorem by substituting F with $\rho \bar{v}_f$.

$$\iiint_{R(t)} \nabla \cdot \bar{F} \, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho \bar{v}_f) \, dV_o = \iint_{S(t)} (\rho \bar{v}_f) \cdot \bar{n} dS$$

Therefore,

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iiint_{R(t)} \nabla \cdot (\rho \bar{v}_f) \, dV_o$$

$$0 = \frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v}_f) \, dV_o$$

Since the integration is zero for an arbitrary region the integrand must be zero everywhere. To prove this, simply choose the arbitray region to be infinitesmially small at all points in R^3 and it could be seen that the integrand is always zero everywhere.

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v}_f)$$

1.1.2 Integral Continuity Proof

To prove the integral form for the Governing Continuum Equation, consider the time rate of change of a mass enclosed within the control volume:

$$\frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

From the differential form of the Governing Continuum Equation,

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot (\rho \bar{v}_f)$$

Therefore,

$$\frac{\partial}{\partial t}[\rho] = -\nabla \cdot (\rho \bar{v}_f)$$

Therefore,

$$\frac{d}{dt}M(t) = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho] dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS = -\iiint_{R(t)} \nabla \cdot (\rho \bar{v}_f) \, dV_o + \iint_{S(t)} \rho \bar{v}_s \cdot \bar{n} dS$$

By applying the divergence theorem to convert the first term volumetric integral into a surface integral,

$$\iiint_{R(t)} \nabla \cdot \bar{F} \, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho \bar{v_f})\, dV_o = \iint_{S(t)} \rho \bar{v_f} \cdot \bar{n} dS$$

$$\frac{d}{dt} M(t) = -\iint_{S(t)} \rho \bar{v_f} \cdot \bar{n} dS + \iint_{S(t)} \rho \bar{v_s} \cdot \bar{n} dS = \iint_{S(t)} \rho (\bar{v_s} - \bar{v_f}) \cdot \bar{n} dS$$

A more familiar form would yield,

$$0 = \frac{d}{dt} M(t) + \iint_{S(t)} \rho (\bar{v_f} - \bar{v_s}) \cdot \bar{n} dS$$

$$0 = \frac{d}{dt} \iiint_{R(t)} \rho\, dV_o + \iint_{S(t)} \rho (\bar{v_f} - \bar{v_s}) \cdot \bar{n} dS$$

1.2 Governing Equation: Momentum Equation

The Governing Momentum Equation in its differential form:

$$\frac{\partial}{\partial t}(\rho \bar{v_f}) + \nabla \cdot (\rho \bar{v_f} \bar{v_f}) = -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}$$

wherein ρ represents density, $\bar{v_f}$ represents fluid velocity vector, P_r represents fluid pressure at a particular point, τ represents viscous forces, $\bar{F_b}$ represents body force experienced by the fluid inside the control volume. The Governing Momentum Equation in its integral form:

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f}\, dV_o = \iiint_{R(t)} \rho \bar{F_b}\, dV_o + \iint_{S(t)} \bar{F_s} \cdot \bar{n} dS = \iiint_{R(t)} \frac{\partial}{\partial t}(\rho \bar{v_f}) + \nabla \cdot (\rho \bar{v_f} \bar{v_s}) dV_o$$

wherein $\bar{F_s}$ represents surface forces. Like in the previous proof, $S(t)$ represents the surface binding the control volume region $R(t)$. An alternate form to the momentum governing equation exists. It is shown below,

$$\frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f}\, dV_o = \iint_{S(t)} \rho \bar{v_f} [(\bar{v_s} - \bar{v_f}) \cdot \bar{n}] dS + \iiint_{R(t)} -\nabla P_r + \nabla \cdot \tau + \rho \bar{F_b}\, dV_o$$

1.2.1 Differential Momentum Proof

The total Momentum $\bar{P_m}$ contained in a control volume,

$$\bar{P_m} = \iiint_{R(t)} \rho \bar{v_f}\, dV_o$$

The derivative of momentum with respect to time,

$$\frac{d}{dt} \bar{P_m} = \frac{d}{dt} \iiint_{R(t)} \rho \bar{v_f}\, dV_o$$

By applying Liebniz's rule, substituting T with $\rho \bar{v_f}$

$$\frac{d}{dt} \iiint_{R(t)} T\, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T] dV_o + \iint_{S(t)} T \bar{v_s} \bar{n} dS$$

$$\frac{d}{dt} \bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho \bar{v_f}]\, dV_o + \iint_{S(t)} \rho \bar{v_f} \bar{v_s} \bar{n} dS$$

By applying Divergence Theorem substituting F with $\rho\bar{v_f}\bar{v_s}$

$$\iiint_{R(t)} \nabla \cdot \bar{F}\, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

$$\iiint_{R(t)} \nabla \cdot (\rho\bar{v_f}\bar{v_s})\, dV_o = \iint_{S(t)} \rho\bar{v_f}\bar{v_s} \cdot \bar{n} dS$$

By substituting the terms to the derivative of momentum with respect to time,

$$\frac{d}{dt}\bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}]\, dV_o + \iiint_{R(t)} \nabla \cdot (\rho\bar{v_f}\bar{v_s})\, dV_o$$

$$\frac{d}{dt}\bar{P_m} = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}] + \nabla \cdot (\rho\bar{v_f}\bar{v_s})\, dV_o$$

Since the derivative of momentum with respect to time is the total force applied to the control volume,

$$\iiint_{R(t)} \rho\bar{F_b}\, dV_o + \iint_{S(t)} \bar{F_s} \cdot \bar{n} dS = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}] + \nabla \cdot (\rho\bar{v_f}\bar{v_s})\, dV$$

The first term in the expression above represents the total body force acting on the control volume meanwhile the second term in the expression represents the total surface force acting on the control volume. When the velocity of the surface is identical to the velocity of the fluid flow, $\bar{v_s} = \bar{v_f}$, the total force acting on the specific volume of region $R_s(t)$,

$$\iiint_{R_s(t)} -\nabla P_r + \nabla \cdot \tau + \rho\bar{F_b}\, dV_o = \iiint_{R_s(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}] + \nabla \cdot (\rho\bar{v_f}\bar{v_f})\, dV$$

Since the equation above is always true under the constraint that the surface velocity of the region is identical to the velocity of the fluid flow, one can simply choose smaller and smaller regions until $R_s(t)$ is an infinitesmially small region This process can be applied everywhere in the fluid. Therefore,

$$-\nabla P_r + \nabla \cdot \tau + \rho\bar{F_b} = \frac{\partial}{\partial t}[\rho\bar{v_f}] + \nabla \cdot (\rho\bar{v_f}\bar{v_f})$$

The equation above is the differential formulation and is always true all throughout the fluid.

1.2.2 Integral Momentum Proof

To prove the alternate form of the momentum governing equation in integral form, the differential formulation of the momentum equation would be vital. Rearranging for the $\frac{\partial}{\partial t}(\rho\bar{v_f})$ term,

$$\frac{\partial}{\partial t}(\rho\bar{v_f}) + \nabla \cdot (\rho\bar{v_f}\bar{v_f}) = -\nabla P_r + \nabla \cdot \tau + \rho\bar{F_b}$$

$$\frac{\partial}{\partial t}(\rho\bar{v_f}) = -\nabla \cdot (\rho\bar{v_f}\bar{v_f}) - \nabla P_r + \nabla \cdot \tau + \rho\bar{F_b}$$

Referencing the previous equation for derivative of momentum within an arbitrary region $R(t)$ with respect to time,

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[\rho\bar{v_f}]\, dV_o + \iint_{S(t)} \rho\bar{v_f}(\bar{v_s} \cdot \bar{n}) dS$$

Substituting the term, $\frac{\partial}{\partial t}(\rho\bar{v_f})$,

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o = \iiint_{R(t)} -\nabla\cdot(\rho\bar{v_f}\bar{v_f}) - \nabla P_r + \nabla\cdot\tau + \rho\bar{F_b}\, dV_o + \iint_{S(t)} \rho\bar{v_f}(\bar{v_s}\cdot\bar{n})dS$$

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o = -\iiint_{R(t)} \nabla\cdot(\rho\bar{v_f}\bar{v_f})\, dV_o + \iint_{S(t)} \rho\bar{v_f}(\bar{v_s}\cdot\bar{n})dS + \iiint_{R(t)} -\nabla P_r + \nabla\cdot\tau + \rho\bar{F_b}\, dV_o$$

The applying the Divergence Theorem substituting F with $\rho\bar{v_f}\bar{v_f}$

$$\iiint_{R(t)} \nabla\cdot\bar{F}\, dV_o = \iint_{S(t)} \bar{F}\cdot\bar{n}dS$$

$$\iiint_{R(t)} \nabla\cdot\rho\bar{v_f}\bar{v_f}\, dV_o = \iint_{S(t)} \rho\bar{v_f}(\bar{v_f}\cdot\bar{n})dS$$

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o = -\iint_{S(t)} \rho\bar{v_f}(\bar{v_f}\cdot\bar{n})dS + \iint_{S(t)} \rho\bar{v_f}(\bar{v_s}\cdot\bar{n})dS + \iiint_{R(t)} -\nabla P_r + \nabla\cdot\tau + \rho\bar{F_b}\, dV_o$$

$$\frac{d}{dt}\iiint_{R(t)} \rho\bar{v_f}\, dV_o = \iint_{S(t)} \rho\bar{v_f}[(\bar{v_s}-\bar{v_f})\cdot\bar{n}]dS + \iiint_{R(t)} -\nabla P_r + \nabla\cdot\tau + \rho\bar{F_b}\, dV_o$$

1.3 Governing Equation: Energy Equation