

0.1 Problem 1

Reiterating Euler's law,

$$\bar{\Gamma}^c = {}^e \dot{\bar{H}}^c$$

wherein c represents the center of mass of the system of particles or a stationary point in the inertial reference frame. Reiterating the definition of angular momentum \bar{H} ,

$$\bar{H} = I_c \bar{\omega}$$

Here, the angular momentum \bar{H} is taken with respect to the origin of the body fitted coordinates. The reference for \bar{H} may or may not be the center of mass in the definition of angular momentum based on the inertia matrix. To prove Euler's equations of motion, it is necessary to take the angular momentum with respect to the center of mass of the rigid body. Therefore,

$$\bar{H}^c = I_c \bar{\omega}$$

wherein I_c represents the inertia matrix of the body taken with respect to the center of mass. Applying the kinematic relations for the vector \bar{H}^c ,

$$\frac{{}^i \partial}{\partial t} (\bar{H}^c) = \frac{{}^n \partial}{\partial t} (\bar{H}^c) + {}^i \bar{\omega}^n \times \bar{H}^c$$

wherein i represents the inertia reference frame here and n represents the non-inertial reference frame e . Substituting for angular momentum in terms of angular velocity at RHS ,

$$\frac{{}^i \partial}{\partial t} (\bar{H}^c) = \frac{{}^n \partial}{\partial t} (I_c \bar{\omega}) + {}^i \bar{\omega}^n \times (I_c \bar{\omega})$$

Considering that the inertia matrix I_c is unchanging with respect to the body-fitted coordinates,

$$\frac{{}^i \partial}{\partial t} (\bar{H}^c) = I_c \frac{{}^n \partial}{\partial t} (\bar{\omega}) + {}^i \bar{\omega}^n \times (I_c \bar{\omega})$$

Due to the definition of the body-fitted coordinates, the angular velocity $\bar{\omega}$ would be identical to ${}^i \bar{\omega}^n$ which is the rotation of the non-inertial body-fitted coordinates n with respect to the inertial coordinates i . Substituting,

$$\frac{{}^i \partial}{\partial t} (\bar{H}^c) = I_c \frac{{}^n \partial}{\partial t} ({}^i \bar{\omega}^n) + {}^i \bar{\omega}^n \times (I_c {}^i \bar{\omega}^n)$$

Let ${}^i\bar{\omega}^n$ be declared as a linear combination of the basis vectors in the body fitted coordinate system n . If ${}^i\bar{\omega}^n$ is declared natively in the body fitted coordinate the derivative of ${}^i\bar{\omega}^n$ with respect to the body fitted coordinate be a trivial case of taking the derivative of the all components of the vector. Let the angular velocity vector have its components be specified below,

$${}^i\bar{\omega}^n = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$$

wherein \hat{x} , \hat{y} , and \hat{z} are the basis vectors of the body-fitted coordinates, Analyzing the first term, in the expression above,

$$I_c \frac{\partial}{\partial t} ({}^i\bar{\omega}^n) = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix}$$

Analyzing the second term in the expression above,

$${}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix}$$

Declaring \hat{i} , \hat{j} and \hat{k} as dummy variables for the cross-product operations,

$${}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) = \left| \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix} \right|$$

$${}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) = \hat{i}|A| - \hat{j}|B| + \hat{k}|C|$$

For the A matrix,

$$\hat{i}|A| = \hat{i} \left| \begin{bmatrix} \omega_y & \omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix} \right|$$

$$\hat{i}|A| = \hat{i}[\omega_y(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)]$$

$$\hat{i}|A| = \hat{i}[\omega_y I_{zx}\omega_x + \omega_y I_{zy}\omega_y + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y - \omega_z I_{yz}\omega_z]$$

Due to the symmetric properties of the inertia tensor,

$$\hat{i}|A| = \hat{i}[\omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y]$$

For the B matrix,

$$\begin{aligned}\hat{j}|B| &= \hat{j} \left| \begin{bmatrix} \omega_x & \omega_z \\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix} \right| \\ \hat{j}|B| &= \hat{j}[\omega_x(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)] \\ -\hat{j}|B| &= \hat{j}[-\omega_x(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) + \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)] \\ -\hat{j}|B| &= \hat{j}[-\omega_x I_{zx}\omega_x - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y + \omega_z I_{xz}\omega_z]\end{aligned}$$

Due to the symmetric properties of the inertia tensor,

$$-\hat{j}|B| = \hat{j}[-(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y]$$

For the C matrix,

$$\begin{aligned}\hat{k}|C| &= \hat{k} \left| \begin{bmatrix} \omega_x & \omega_y \\ I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z & I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \end{bmatrix} \right| \\ \hat{k}|C| &= \hat{k}[\omega_x(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)] \\ \hat{k}|C| &= \hat{k}[\omega_x I_{yx}\omega_x + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xy}\omega_y - \omega_y I_{xz}\omega_z]\end{aligned}$$

Due to the symmetric properties of the inertia tensor,

$$\hat{k}|C| = \hat{k}[(\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z]$$

Substituting the various components together,

$$\begin{aligned}{}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) &= \hat{i}|A| - \hat{j}|B| + \hat{k}|C| \\ {}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) &= \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}\end{aligned}$$

Substituting the first term and second term to obtain the full form,

$$\frac{\partial}{\partial t} ({}^i\bar{H}^c) = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ -(\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

Reiterating Euler's law,

$$\bar{\Gamma}^c = {}^e \dot{\bar{H}}^c$$

Since the angular momentum is now taken with respect to center of mass and the time derivative of the angular momentum is taken with respect to an inertial reference frame,

$$\bar{\Gamma}^c = \frac{\overset{i}{\partial}}{\partial t} (\bar{H}^c)$$

Let the moments $\bar{\Gamma}^c$ be defined to have components below,

$$\bar{\Gamma}^c = \frac{\overset{i}{\partial}}{\partial t} (\bar{H}^c) = \begin{bmatrix} M_x & M_y & M_z \end{bmatrix}^T$$

Substituting for the moments and simplifying,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z + \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - (\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z + (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

0.2 Problem 2

Reiterating Euler's law for an arbitrary body-fitted coordinate with origin nested at the center of mass,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z + \omega_y I_{zx}\omega_x + (\omega_y\omega_y - \omega_z\omega_z)I_{yz} + \omega_y I_{zz}\omega_z - \omega_z I_{yx}\omega_x - \omega_z I_{yy}\omega_y \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - (\omega_x\omega_x - \omega_z\omega_z)I_{xz} - \omega_x I_{zy}\omega_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x + \omega_z I_{xy}\omega_y \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z + (\omega_x\omega_x - \omega_y\omega_y)I_{xy} + \omega_x I_{yy}\omega_y + \omega_x I_{yz}\omega_z - \omega_y I_{xx}\omega_x - \omega_y I_{xz}\omega_z \end{bmatrix}$$

For the case of principal axes, the products of inertia evaluate to zero. Substituting,

$$\begin{aligned} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} &= \begin{bmatrix} I_{xx}\dot{\omega}_x + \omega_y I_{zz}\omega_z - \omega_z I_{yy}\omega_y \\ I_{yy}\dot{\omega}_y - \omega_x I_{zz}\omega_z + \omega_z I_{xx}\omega_x \\ I_{zz}\dot{\omega}_z + \omega_x I_{yy}\omega_y - \omega_y I_{xx}\omega_x \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + \omega_z\omega_y I_{zz} - \omega_z\omega_y I_{yy} \\ I_{yy}\dot{\omega}_y - \omega_x\omega_z I_{zz} + \omega_x\omega_z I_{xx} \\ I_{zz}\dot{\omega}_z + \omega_x\omega_y I_{yy} - \omega_x\omega_y I_{xx} \end{bmatrix} \\ \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} &= \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_z\omega_y \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_z\omega_y \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix} \end{aligned}$$

Earlier, the body-fitted coordinates were declared to be arbitrary as long as the origin of the body-fitted coordinates were nested in the center of mass of the rigid body. To simplify Euler's equations of motions, the body-fitted coordinates' axes were set to the principal axes. Though this is merely one case of the general case, and it would be reasonable to leave the notations as is, for clarity's sake, the notation for the general moments of inertia are changed to specify moments of inertia with respect to the principal axes,

$$I_{xx} \rightarrow I_x \quad , \quad I_{yy} \rightarrow I_y \quad , \quad I_{zz} \rightarrow I_z$$

Hence, Euler's equations of motions with the constraint of the body-fitted coordinates must have an origin at the center of mass and the axes of the body-fitted coordinates must be the principal axes,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

0.3 Problem 7

Euler's law for a fixed inertia tensor was derived earlier. Reiterating the step before assuming the inertia tensor is unchanging with time,

$$\frac{\partial}{\partial t} (\bar{H}^c) = \frac{\partial}{\partial t} (I_c \bar{\omega}) + {}^i\bar{\omega}^n \times (I_c \bar{\omega})$$

If the inertia tensor is changing with time,

$$\frac{\partial}{\partial t} (I_c \bar{\omega}) = I_c \frac{\partial}{\partial t} (\bar{\omega}) + \frac{\partial}{\partial t} (I_c) \bar{\omega}$$

Reiterating the definition of the inertia tensor,

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Assuming that the inertia tensor is taken with respect to center of mass, the inertia tensor simplifies to,

$$I_c = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Substituting the inertia tensor along with its derivatives while making the same assumption of $\bar{\omega} = {}^i\bar{\omega}^n$ and is declared natively in n coordinate system,

$$\frac{\partial}{\partial t} (I_c \bar{\omega}) = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \dot{I}_x & 0 & 0 \\ 0 & \dot{I}_y & 0 \\ 0 & 0 & \dot{I}_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\frac{\partial}{\partial t} (I_c \bar{\omega}) = \begin{bmatrix} I_x \dot{\omega}_x \\ I_y \dot{\omega}_y \\ I_z \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \dot{I}_x \omega_x \\ \dot{I}_y \omega_y \\ \dot{I}_z \omega_z \end{bmatrix} = \begin{bmatrix} I_x \dot{\omega}_x + \dot{I}_x \omega_x \\ I_y \dot{\omega}_y + \dot{I}_y \omega_y \\ I_z \dot{\omega}_z + \dot{I}_z \omega_z \end{bmatrix}$$

Reiterating the second term that was determined earlier,

$${}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) = \begin{bmatrix} \omega_y I_{zx} \omega_x + (\omega_y \omega_y - \omega_z \omega_z) I_{yz} + \omega_y I_{zz} \omega_z - \omega_z I_{yx} \omega_x - \omega_z I_{yy} \omega_y \\ -(\omega_x \omega_x - \omega_z \omega_z) I_{xz} - \omega_x I_{zy} \omega_y - \omega_x I_{zz} \omega_z + \omega_z I_{xx} \omega_x + \omega_z I_{xy} \omega_y \\ (\omega_x \omega_x - \omega_y \omega_y) I_{xy} + \omega_x I_{yy} \omega_y + \omega_x I_{yz} \omega_z - \omega_y I_{xx} \omega_x - \omega_y I_{xz} \omega_z \end{bmatrix}$$

Taking the products of inertia to be zero,

$${}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) = \begin{bmatrix} \omega_y I_{zz} \omega_z - \omega_z I_{yy} \omega_y \\ -\omega_x I_{zz} \omega_z + \omega_z I_{xx} \omega_x \\ \omega_x I_{yy} \omega_y - \omega_y I_{xx} \omega_x \end{bmatrix} = \begin{bmatrix} (I_{zz} - I_{yy}) \omega_y \omega_z \\ (I_{xx} - I_{zz}) \omega_x \omega_z \\ (I_{yy} - I_{xx}) \omega_x \omega_y \end{bmatrix}$$

Substituting the moment of inertia to their principal notations,

$${}^i\bar{\omega}^n \times (I_c {}^i\bar{\omega}^n) = \begin{bmatrix} (I_z - I_y) \omega_y \omega_z \\ (I_x - I_z) \omega_x \omega_z \\ (I_y - I_x) \omega_x \omega_y \end{bmatrix}$$

Substituting the terms together to obtain the full form,

$$\frac{\partial}{\partial t} ({}^i\bar{H}^c) = \frac{\partial}{\partial t} (I_c \bar{\omega}) + {}^i\bar{\omega}^n \times (I_c \bar{\omega})$$

$$\frac{\partial}{\partial t} ({}^i\bar{H}^c) = \begin{bmatrix} I_x \dot{\omega}_x + \dot{I}_x \omega_x \\ I_y \dot{\omega}_y + \dot{I}_y \omega_y \\ I_z \dot{\omega}_z + \dot{I}_z \omega_z \end{bmatrix} + \begin{bmatrix} (I_z - I_y) \omega_y \omega_z \\ (I_x - I_z) \omega_x \omega_z \\ (I_y - I_x) \omega_x \omega_y \end{bmatrix}$$

Just as before, the time derivative of \bar{H}^c is the moment acting on the body.

Substituting,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x \dot{\omega}_x + \dot{I}_x \omega_x + (I_z - I_y) \omega_y \omega_z \\ I_y \dot{\omega}_y + \dot{I}_y \omega_y + (I_x - I_z) \omega_x \omega_z \\ I_z \dot{\omega}_z + \dot{I}_z \omega_z + (I_y - I_x) \omega_x \omega_y \end{bmatrix}$$