Chapter 1

Dynamical Systems: Eigenvalues and Eigenvectors

Let A represent a $n \times n$ matrix (a matrix with n rows and n columns), x represent a column vector of n variables and x' represent the derivative of the column vector x. The system below is known as a dynamical system:

$$x' = Ax$$

Consider the dynamical system $x' = kx$ wherein k is some arbitrary constant. Therefore,

$$\frac{dx}{dt} = kx$$

$$dt = \frac{1}{kx}dx$$

$$\int dt = \int \frac{1}{kx}dx$$

$$t = \frac{1}{k}\ln x + C$$

$$\ln x = kt + C$$

$$x = Ce^{kt}$$

Wherein C is a constant determined by the initial conditions.

1.1 Non-Repeated Real Eigenvalues of n $\times$ n Case

The previous working gives the conjecture that the general solution set $x(t)$ to the dynamical system $x' = Ax$ is the linear combination of exponential functions analogous to the example shown above. Consider the possibility that one solution to the dynamical system takes the form below:

$$x(t) = \bar{v_i}e^{\lambda_i t}$$

wherein $\bar{v_i}$ represents a vector and λ_i represents a constant. By taking derivative of the solution,

$$x'(t) = \lambda_i\bar{v_i}e^{\lambda_i t}$$

$$Ax(t) = A\bar{v_i}e^{\lambda_i t}$$

By considering that $x(t)$ represents a solution to the dynamical system, $x' = Ax$

$$\lambda_i\bar{v_i}e^{\lambda_i t} = A\bar{v_i}e^{\lambda_i t}$$

Since $e^{\lambda_i t} \neq 0$ for all values of t,

$$A\bar{v_i} = \lambda_i\bar{v_i}$$

This is a familiar equation for eigenvalues and eigenvectors. This shows that each eigenvalue-eigenvector pairs of the matrix A represents a solution set. Therefore, the general solution set is:

$$x(t) = span[\bar{v_1}e^{\alpha_1 t}, \bar{v_2}e^{\alpha_2 t}, \ldots \bar{v_n}e^{\alpha_n t}]$$

$$x(t) = \sum_{i=1}^{n}\left[c_i\bar{v_i}e^{\lambda_i t}\right]$$

wherein c_i are constants determined by the initial value of the problem.

1.2 Non-Repeated Complex Eigenvalues of 2 $\times$ 2 Case

Consider the special case wherein the matrix A is a 2×2 matrix and that the eigenvalues are complex, by conjecture,

$$x(t) = c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = k_1 Re[\bar{v_1}e^{\lambda_1 t}] + k_2 Im[\bar{v_1}e^{\lambda_1 t}]$$

wherein c_1 and c_2 are complex values meanwhile k_1 and k_2 are real values. There must always be some choice of complex values c_1 and c_2 such that the expression above is true. The proof is shown below,

Let

$$\bar{v_1} = \bar{v_r} + i\bar{v_i} \qquad \lambda_1 = a + bi$$

$$x(t) = (\bar{v_r} + i\bar{v_i})e^{(a+bi)t}$$

$$x(t) = e^{at}(\bar{v_r} + i\bar{v_i})[\cos{(bt)} + i\sin{(bt)}]$$

$$x(t) = e^{at}[\bar{v_r}\cos{(bt)} + i\bar{v_r}\sin{(bt)} + i\bar{v_i}\cos{(bt)} - \bar{v_i}\sin{(bt)}]$$

$$x(t) = e^{at}[\bar{v_r}\cos{(bt)} - \bar{v_i}\sin{(bt)}] + ie^{at}[\bar{v_r}\sin{(bt)} + \bar{v_i}\cos{(bt)}]$$

$$Re[\bar{v_1}e^{\lambda_1 t}] = e^{at}[\bar{v_r}\cos{(bt)} - \bar{v_i}\sin{(bt)}]$$

$$Im[\bar{v_1}e^{\lambda_1 t}] = e^{at}[\bar{v_r}\sin{(bt)} + \bar{v_i}\cos{(bt)}]$$

$$LHS = k_1 Re[\bar{v_1}e^{\lambda_1 t}] + k_2 Im[\bar{v_1}e^{\lambda_1 t}]$$

$$LHS = k_1 e^{at}[\bar{v_r}\cos{(bt)} - \bar{v_i}\sin{(bt)}] + k_2 e^{at}[\bar{v_r}\sin{(bt)} + \bar{v_i}\cos{(bt)}]$$

$$LHS = e^{at}[k_1\bar{v_r}\cos{(bt)} - k_1\bar{v_i}\sin{(bt)} + k_2\bar{v_r}\sin{(bt)} + k_2\bar{v_i}\cos{(bt)}]$$

$$LHS = e^{at}\{[k_1\bar{v_r} + k_2\bar{v_i}]\cos{(bt)} + [k_2\bar{v_r} - k_1\bar{v_i}]\sin{(bt)}\}$$

$$LHS = e^{at}[k_1\bar{v_r} + k_2\bar{v_i}]\cos{(bt)} + e^{at}[k_2\bar{v_r} - k_1\bar{v_i}]\sin{(bt)}$$

It is important to note that eigenvalues and their corresponding eigenvectors occur in conjugate pairs. Therefore, if $\lambda_1 = a + bi$, then $\lambda_2 = \lambda_1^* = a - bi$ and if the eigenvector $\bar{v_1} = \bar{v_r} + i\bar{v_i}$, then $\bar{v_2} = \bar{v_1}^* = \bar{v_r} - i\bar{v_i}$.

Let

$$c_1 = f_1 + g_1 i \qquad c_2 = f_2 + g_2 i$$

$$c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = (f_1 + g_1 i)(\bar{v_r} + i\bar{v_i})e^{(a+bi)t} + (f_2 + g_2 i)(\bar{v_r} - i\bar{v_i})e^{(a-bi)t}$$

For ease of notation,

$$A(t) = (f_1 + g_1 i)(\bar{v_r} + i\bar{v_i})e^{(a+bi)t} \qquad B(t) = (f_2 + g_2 i)(\bar{v_r} - i\bar{v_i})e^{(a-bi)t}$$

$$c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = A(t) + B(t)$$

$$A(t) = e^{at}(f_1 + g_1 i)(\bar{v_r} + i\bar{v_i})\left[\cos{(bt)} + i\sin{(bt)}\right]$$

$$A(t) = e^{at}(f_1\bar{v_r} + if_1\bar{v_i} + ig_1\bar{v_r} - g_1\bar{v_i})\left[\cos{(bt)} + i\sin{(bt)}\right]$$

$$A(t) = e^{at}[f_1\bar{v_r} - g_1\bar{v_i} + i(f_1\bar{v_i} + g_1\bar{v_r})]\left[\cos{(bt)} + i\sin{(bt)}\right]$$

$$A(t) = e^{at}[(f_1\bar{v_r}-g_1\bar{v_i})\cos(bt)+i(f_1\bar{v_i}+g_1\bar{v_r})\cos(bt)+i(f_1\bar{v_r}-g_1\bar{v_i})\sin(bt)-(f_1\bar{v_i}+g_1\bar{v_r})\sin(bt)]$$

$$B(t) = (f_2 + g_2 i)(\bar{v_r} - i\bar{v_i})e^{(a-bi)t}$$

$$B(t) = e^{at}(f_2\bar{v_r} - if_2\bar{v_i} + ig_2\bar{v_r} + g_2\bar{v_i})\left[\cos(-bt) + i\sin(-bt)\right]$$

$$B(t) = e^{at}[(f_2\bar{v_r} + g_2\bar{v_i}) + i(g_2\bar{v_r} - f_2\bar{v_i})]\left[\cos(bt) - i\sin(bt)\right]$$

$$B(t) = e^{at}[(f_2\bar{v_r}+g_2\bar{v_i})\cos(bt)+i(g_2\bar{v_r}-f_2\bar{v_i})\cos(bt)+i(-f_2\bar{v_r}-g_2\bar{v_i})\sin(bt)+(g_2\bar{v_r}-f_2\bar{v_i})\sin(bt)]$$

$$c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = Re[A(t)] + Re[B(t)] + i\{Im[A(t)] + Im[B(t)]\}$$

$$0 = Im[A(t)] + Im[B(t)]$$

$$0 = (f_1\bar{v_i}+g_1\bar{v_r})\cos(bt)+(f_1\bar{v_r}-g_1\bar{v_i})\sin(bt)+(g_2\bar{v_r}-f_2\bar{v_i})\cos(bt)-(f_2\bar{v_r}+g_2\bar{v_i})\sin(bt)$$

$$0 = (f_1\bar{v_i} + g_1\bar{v_r} + g_2\bar{v_r} - f_2\bar{v_i})\cos(bt) + (f_1\bar{v_r} - g_1\bar{v_i} - f_2\bar{v_r} - g_2\bar{v_i})\sin bt$$

$$0 = [(g_1 + g_2)\bar{v_r} + (f_1 - f_2)\bar{v_i}]\cos(bt) + [(f_1 - f_2)\bar{v_r} - (g_1 + g_2)\bar{v_i}]\sin(bt)$$

For as long as the condition below is met, the imaginary component of $A(t) + B(t)$ is negligible.

$$g_1 = -g_2 \qquad f_1 = f_2$$

$$c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = Re[A(t)] + Re[B(t)]$$

$$\begin{aligned} c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} =& e^{at}(f_1\bar{v_r} - g_1\bar{v_i})\cos(bt) - (f_1\bar{v_i} + g_1\bar{v_r})\sin(bt) \\ &+ (f_2\bar{v_r} + g_2\bar{v_i})\cos(bt) + (g_2\bar{v_r} - f_2\bar{v_i})\sin(bt) \end{aligned}$$

$$c_1\bar{v_1}e^{\lambda_1 t}+c_2\bar{v_2}e^{\lambda_2 t} = e^{at}(f_1\bar{v_r}-g_1\bar{v_i}+f_2\bar{v_r}+g_2\bar{v_i})\cos(bt)+(g_2\bar{v_r}-f_2\bar{v_i}-f_1\bar{v_i}-g_1\bar{v_r})\sin(bt)$$

$$c_1\bar{v_1}e^{\lambda_1 t}+c_2\bar{v_2}e^{\lambda_2 t} = e^{at}[(f_1+f_2)\bar{v_r}+(g_2-g_1)\bar{v_i}]\cos(bt)+[(g_2-g_1)\bar{v_r}-(f_1+f_2)\bar{v_i}]\sin(bt)$$

$$RHS = c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t}$$

$$RHS = e^{at}[(f_1 + f_2)\bar{v_r} + (g_2 - g_1)\bar{v_i}]\cos(bt) + e^{at}[(g_2 - g_1)\bar{v_r} - (f_1 + f_2)\bar{v_i}]\sin(bt)$$

$$LHS = e^{at}[k_1\bar{v_r} + k_2\bar{v_i}]\cos(bt) + e^{at}[k_2\bar{v_r} - k_1\bar{v_i}]\sin(bt)$$

If the conditions below are met, therefore $LHS = RHS$ and the statement $x(t) = c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = k_1 Re[\bar{v_1}e^{\lambda_1 t}] + k_2 Im[\bar{v_1}e^{\lambda_1 t}]$ is true.

$$g_1 + g_2 = 0 \qquad f_1 + f_2 - k_1 = 0 \qquad f_1 - f_2 = 0 \qquad g_2 - g_1 - k_2 = 0$$

The corresponding augmented matrix of the following conditions is

$$\begin{array}{c} \begin{matrix} f_1 & f_2 & g_1 & g_2 & k_1 & k_2 & C \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The row-reduced echelon form of the corresponding augmented matrix is

$$\begin{array}{ccccccc} f_1 & f_2 & g_1 & g_2 & k_1 & k_2 & C \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

The row-reduced echelon form is unique and is consistent, therefore the system has a consistent solution. This proves that for some special choice of c_1 and c_2, the expression below is correct.

$$x(t) = c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = k_1 Re[\bar{v}_1 e^{\lambda_1 t}] + k_2 Im[\bar{v}_1 e^{\lambda_1 t}]$$

A restatement of the general real solution set is:

$$x(t) = k_1 e^{at}[\bar{v}_r \cos(bt) - \bar{v}_i \sin(bt)] + k_2 e^{at}[\bar{v}_r \sin(bt) + \bar{v}_i \cos(bt)]$$

The solution set for all real numbers could be better expressed as a matrix multiplication

$$x(t) = e^{at}\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} k_2 \\ k_1 \end{pmatrix}$$

The real and imaginary components of the eigenvector v_1 form a linearly independent set. Therfore, the matrix $\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}$ must be invertible. Through the invertible matrix theorem, the matrix $\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}$ must have a suitable inverse.

$$x(t) = e^{at}\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}^{-1}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$x(t) = e^{at}\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}^{-1} x_0$$

$$\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}^{-1} x(t) = e^{at}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}^{-1} x_0$$

By considering the substitution $y = \begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}^{-1} x(t)$ and $y_0 = \begin{pmatrix} \bar{v}_i & \bar{v}_r \end{pmatrix}^{-1} x_0$,

$$y = e^{at}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix} y_0$$

wherein e^{at} represents a scaling transformation and $\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}$ represents a rotation. Therefore, for a suitable substitution, the general real solution set of the dynamical system $x' = Ax$ will form a rotation with a scaling component. The rotation is sometimes known as the "hidden rotation". Some possibilities of the solution set may be ellipses, circles, and spirals.

1.3 Non-Repeated Complex Eigenvalues of 3×3 Case

Consider the case wherein $n = 3$

$$x(t) = \sum_{i=1}^{3} \left[c_i \bar{v_i} e^{\lambda_i t} \right]$$

$$x(t) = c_1 \bar{v_1} e^{\lambda_1 t} + c_2 \bar{v_2} e^{\lambda_2 t} + c_3 \bar{v_3} e^{\lambda_3 t}$$

Complex eigenvalues occure in conjugate pairs. When A is a 3×3 matrix, 2 of the eigenvalues will be complex conjugate pairs and the third one will be a real value. Therefore, two of the eigenvectors must be complex vectors with the third eigenvector being a real vector. Therefore, through the similar argument and proof written above,

$$x(t) = k_1 Re \left[\bar{v_1} e^{\lambda_1 t} \right] + k_2 Re \left[\bar{v_1} e^{\lambda_1 t} \right] + k_3 \bar{v_3} e^{\lambda_3 t}$$

$$x(t) = k_1 e^{at} [\bar{v_r} \cos(bt) - \bar{v_i} \sin(bt)] + k_2 e^{at} [\bar{v_r} \sin(bt) + \bar{v_i} \cos(bt)] + k_3 \bar{v_3} e^{\lambda_3 t}$$

The following solution set could be factorised as matrix multiplications

$$x(t) = e^{at} \begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix} \begin{pmatrix} \cos(bt) & -\sin(bt) & 0 \\ \sin(bt) & \cos(bt) & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} \begin{pmatrix} k_2 \\ k_1 \\ k_3 \end{pmatrix}$$

The vectors $\bar{v_i}, \bar{v_r}, \bar{v_3}$ form a linearly independent set, therefore, the matrix $\begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix}$ is invertible and its inverse must exist.

$$\text{Let } y_0 = \begin{pmatrix} k_2 \\ k_1 \\ k_3 \end{pmatrix}$$

$$\begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix}^{-1} x(t) = e^{at} \begin{pmatrix} \cos(bt) & -\sin(bt) & 0 \\ \sin(bt) & \cos(bt) & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} y_0$$

$$\text{Let } y(t) = \begin{pmatrix} \bar{v_i} & \bar{v_r} & \bar{v_3} \end{pmatrix}^{-1} x(t)$$

$$y(t) = e^{at} \begin{pmatrix} \cos{(bt)} & -\sin{(bt)} & 0 \\ \sin{(bt)} & \cos{(bt)} & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} y_0$$

y_0 is dependent on the system's initial conditions. This shows that for some suitable substitution, the general solution set forms a helix. The geometrical implication of the solution set is a spiral around the z-axis while it is moving away from the xy plane. The substitution back into the conventional axis x_1, x_2, x_3 could be considered as a transformation that "distorts" the helix.

1.4 Repeated Eigenvalues

Given the matrix A in the system $x' = Ax$ is a matrix with repeated eigenvalues with multiplicity k, a reasonable conjecture is the solution to the system is similar in form to the repeated roots case in the linear differential equation. By conjecture,

$$x(t) = \sum_{i=0}^{k-1} \left[\bar{v_i} t^{k-1-i} e^{\lambda t} \right]$$

$$x'(t) = \sum_{i=0}^{k-1} \left[\bar{v_i} \frac{d}{dt} \left[t^{k-1-i} e^{\lambda t} \right] \right]$$

$$\frac{d}{dt} \left[t^{k-1-i} e^{\lambda t} \right] = (k-1-i) t^{k-2-i} e^{\lambda t} + \lambda t^{k-1-i} e^{\lambda t}$$

$$x'(t) = \sum_{i=0}^{k-1} \left[(k-1-i) t^{k-2-i} \bar{v_i} e^{\lambda t} + \lambda t^{k-1-i} \bar{v_i} e^{\lambda t} \right]$$

Remembering $x'(t) = Ax(t)$,

$$\sum_{i=0}^{k-1} \left[A \bar{v_i} t^{k-1-i} e^{\lambda t} \right] = \sum_{i=0}^{k-1} \left[(k-1-i) t^{k-2-i} \bar{v_i} e^{\lambda t} + \lambda t^{k-1-i} \bar{v_i} e^{\lambda t} \right]$$

Considering that $e^{\lambda t} \neq 0$, therefore,

$$\sum_{i=0}^{k-1} \left[A \bar{v_i} t^{k-1-i} \right] = \sum_{i=0}^{k-1} \left[\lambda t^{k-1-i} \bar{v_i} + (k-1-i) t^{k-2-i} \bar{v_i} \right]$$

For the 0^{th} element,

$$A\bar{v_0}t^{k-1} = \lambda t^{k-1}\bar{v_0}$$

Considering that $t^{k-1} \neq 0$ for as long as $t \neq 0$,

$$A\bar{v_0} = \lambda\bar{v_0}$$

For the α^{th} element,

$$A\bar{v_\alpha}t^{k-1-\alpha} = \lambda t^{k-1-\alpha}\bar{v_\alpha} + \left[k-1-(\alpha-1)\right] t^{k-2-(\alpha-1)}\bar{v_{\alpha-1}}$$

$$A\bar{v_\alpha}t^{k-1-\alpha} = \lambda t^{k-1-\alpha}\bar{v_\alpha} + [k-\alpha]\, t^{k-1-\alpha}\bar{v_{\alpha-1}}$$

For as long as $t \neq 0$, $t^{k-1-\alpha} \neq 0$. Therefore,

$$A\bar{v_\alpha} = \lambda\bar{v_\alpha} + [k-\alpha]\,\bar{v_{\alpha-1}}$$

$$\frac{1}{[k-\alpha]}(A-\lambda I)\bar{v_\alpha} = \bar{v_{\alpha-1}}$$

By applying definition recursively,

$$\frac{1}{\prod_{i=0}^{j-1}[k-i]}(A-\lambda I)^j\bar{v_\alpha} = \bar{v_{\alpha-j}}$$

For when $j = \alpha$,

$$\frac{1}{\prod_{i=0}^{\alpha-1}[k-i]}(A-\lambda I)^\alpha\bar{v_\alpha} = \bar{v_0}$$

1.5 Simple First Order Non-Homogenous System

Suppose, for a non-homogeneous dynamical system, $x' = Ax + k$. The non-homogenous dynamical system could be reduced to a homogenous dynamical system, $y' = Ay$ by an appropriate substitution shown below:

$$y_1 = x_1 + c_1 \qquad y_2 = x_2 + c_2 \qquad \ldots \qquad y_n = x_n + c_n$$

wherein $c_1, c_2, c_3 \ldots c_n$ are constants

$$y_1' = x_1' \qquad y_2' = x_2' \qquad \ldots \qquad y_n' = x_n'$$

Let the columns of matrix A be denoted as $a_1, a_2, a_3, \dots a_n$

$$A = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix}$$

$$Ay = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Ay = \sum_{i=1}^{n} [\bar{a}_i y_i]$$

$$Ay = \sum_{i=1}^{n} \big[\bar{a}_i (x_i + c_i)\big]$$

$$Ay = \sum_{i=1}^{n} [\bar{a}_i x_i + \bar{a}_i c_i]$$

$$Ay = \sum_{i=1}^{n} [\bar{a}_i x_i] + \sum_{i=1}^{n} [\bar{a}_i c_i]$$

$$Ax + k = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$Ax + k = \sum_{i=1}^{n} [\bar{a}_i x_i] + k$$

$$Ay = Ax + k$$

$$\sum_{i=1}^{n} [\bar{a}_i x_i] + \sum_{i=1}^{n} [\bar{a}_i c_i] = \sum_{i=1}^{n} [\bar{a}_i x_i] + k$$

$$\sum_{i=1}^{n} [\bar{a}_i c_i] = k$$

The system above is equivalent to an augmented matrix whose first column until nth column is the columns of the matrix A and its last column is the column vector k. Therefore, the augmented matrix is written below:

$$c_1 \quad c_2 \quad \dots \quad c_n \quad K$$

$$\begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} & k \end{bmatrix}$$

The solution to the augmented matrix will be the values for the constants $c_1, c_2, \dots, c_n$ that would be used in the substitution process in transforming the non-homogenous dynamical system into a homogenous dynamical system. The augmented matrix above would only have a solution for all k in $\mathbb{R}^n$ if the matrix A is invertible. If the matrix A is non-invertible , then k must be in $col[A]$, otherwise, then the augmented system forms an inconsistent system. In otherwords, a substitution with the above methods may not exist for an aribtrary choice of $n \times n$ matrix A and arbitrary column vector k.

1.6 Simple Higher Order System

Suppose the dynamical system follows the expression $\overset{m}{x} = Ax$, a similar technique with eigenvalues and eigenvectors may be employed along with the roots of unity. By conjecture, the partial solution to the dynamical system $\overset{m}{x} = Ax$ follows

$$x_p = \bar{v}_i e^{\alpha_i t}$$

$$\dot{x_p} = \alpha_i \bar{v}_i e^{\alpha_i t}$$
$$\ddot{x_p} = \alpha_i^2 \bar{v}_i e^{\alpha_i t}$$
$$\overset{m}{x_p} = \alpha_i^m \bar{v}_i e^{\alpha_i t}$$
$$Ax_p = \overset{m}{x_p}$$
$$A\bar{v}_i e^{\alpha_i t} = \alpha_i^m \bar{v}_i e^{\alpha_i t}$$
$$A\bar{v}_i = \alpha_i^m \bar{v}_i$$

Since $A\bar{v}_i = \alpha_i^m \bar{v}_i$ wherein λ_i are eigenvalues of A, then $\lambda_i = \alpha_i^m$. Since λ_i may be a complex number, α_i must be the roots of unity to the complex number λ_i. If $\lambda_i = a + bi$

$$\alpha_n = (a^2 + b^2)^{\frac{1}{2m}} cis \left[\frac{1}{m} arctan \left(\frac{b}{a} \right) + \frac{2\pi n}{m} \right]$$

The general solution to the problem must be the linear combination of the partial solutions $\sum_{i=1}^{m} \left[c_i \bar{v}_i e^{\alpha_{in} t} \right]$ wherein c_i are constants determined by the initial conditions and α_{in} represents the n^{th} root of unity of the i^{th} eigenvalue albeit complex or real.

1.7 Simple n^{th} Order Homogenous System

Suppose the differential equation follows the expression:

$$0 = \sum_{i=0}^{m} [A_i \overset{i}{x}] = A_0 x + A_1 \dot{x} + A_2 \ddot{x} + \cdots + A_{i-1} \overset{i-1}{x} + A_i \overset{i}{x}$$

The general solution to the system above is a linear combination of the partial solutions, $x(t) = \sum_{j=1}^{n} [c_j \bar{v_j} e^{\lambda_j t}]$ wherein partial solutions are defined as $x_{partial}(t) = c_j \bar{v_j} e^{\lambda_j t}$ and $c_1, c_2 \ldots c_n$ are constants determined by the initial value of the problem.

$$x_p(t) = c_j \bar{v_j} e^{\lambda_j t}$$

$$\overset{k}{x_p}(t) = c_j \bar{v_j} \lambda_j^k e^{\lambda_j t}$$

$$0 = \sum_{i=0}^{m} [A_i \bar{v_j} c_j \lambda_j^i e^{\lambda_j t}] = A_0 \bar{v_j} c_j e^{\lambda_j t} + A_1 \bar{v_j} c_j \lambda_j e^{\lambda_j t} + \cdots + A_m \bar{v_j} c_j \lambda_j^m e^{\lambda_j t}$$

For the non-trivial solutions to the homoegenous system of differential equations, $c_j, \bar{v_j}, \lambda_j \neq 0$. The function $e^{\lambda_j t} \neq 0$ for all time. Therefore,

$$0 = \left\{ \sum_{i=0}^{m} [A_i \lambda_j^i] \right\} \bar{v_j}$$

For $\bar{v_j} \neq 0$, the matrix $\sum_{i=0}^{m} [A_i \lambda_j^i]$ must be non-invertible. Therefore,

$$det \left\{ \sum_{i=0}^{m} [A_i \lambda_j^i] \right\} = 0$$

The expressions for λ_j^i could be substituted to the expression $A_i \lambda_j^i \bar{v_j} = 0$ to express vector $\bar{v_j}$ explicitly.