0.1 Problem 1

0.1.1 Part a

The Joukowski transformation is shown below,

$$z = \zeta + \frac{c^2}{\zeta}$$

wherein ζ represents complex numbers. Analyzing the complex numbers purely in terms of of its real and imaginary components, let

$$\zeta = x + iy$$

Substituting ζ,

$$z = x + iy + \frac{c^2}{x+iy} = x + iy + \frac{c^2}{x+iy} \times \frac{x-iy}{x-iy} = x + iy + \frac{c^2(x-iy)}{(x+iy)(x-iy)}$$

$$z = x + iy + \frac{c^2x - ic^2y}{x^2-(iy)^2} = x + iy + \frac{c^2x - ic^2y}{x^2+y^2} = x + iy + \frac{c^2}{x^2+y^2}x - \frac{c^2}{x^2+y^2}iy$$

$$z = x + \frac{c^2}{x^2+y^2}x + iy - \frac{c^2}{x^2+y^2}iy = \left[1 + \frac{c^2}{x^2+y^2}\right]x + \left[1 - \frac{c^2}{x^2+y^2}\right]yi$$

wherein x and y represents the real and imaginary components respectively of the input to the Joukowski transformation. Manipulating the Joukowski transformation to non-dimensionalize x and y,

$$z = \left[1 + \frac{1}{(x/c)^2+(y/c)^2}\right]x + \left[1 - \frac{1}{(x/c)^2+(y/c)^2}\right]yi$$

$$z/c = \left[1 + \frac{1}{(x/c)^2+(y/c)^2}\right]x/c + \left[1 - \frac{1}{(x/c)^2+(y/c)^2}\right]iy/c$$

The circle in the ζ plane is defined below,

$$\left(\frac{x}{c}\right)^2 + \left(\frac{y}{c} - \frac{y_s}{c}\right)^2 = \left(\frac{r}{c}\right)^2$$

wherein $\frac{y_s}{c}$ and $\frac{r}{c}$ is defined as, $\frac{y_s}{c} = 0.08$ and $\frac{r}{c} = 1.1$ The relation between x and y in the ζ plane are expressed in non-dimensionalized length scales with respect to c. Since the points in the circle of ζ plane correspond to the inputs to the Joukowski transformation to produce the airfoil in the z plane, the non-dimensional lengths x/c and y/c in the expression for the circle in the ζ plane is identical to the non-dimensional lengths in the Joukowski transformation expression. Let the change of notation be denoted below,

$$x_c = \frac{x}{c} \quad , \quad y_c = \frac{y}{c} \quad , \quad z_c = \frac{z}{c}$$

The leading and trailing edges of the airfoil in the z-plane is defined to be the where the airfoil crosses the real axis in the z-plane. To find the leading and trailing edge of the airfoil, the imaginary component in the Joukowski transformation is set to zero.

$$Im[z_c] = \left[1 - \frac{1}{(x/c)^2 + (y/c)^2}\right] y/c = 0$$

Either both of these statements hence must be true,

$$0 = 1 - \frac{1}{(x/c)^2 + (y/c)^2} \quad , \quad 0 = y/c$$

Based on the problem definition,

$$0 = y_c$$

Reiterating the circle in the ζ plane,

$$x_c^2 + (y_c - y_{sc})^2 = r_c^2$$

To find the x_c in the ζ plane corresponding to the leading and trailing edge of the airfoil in the z plane, the value for y_c is substituted into the circle expression,

$$x_c^2 + (-y_{sc})^2 = r_c^2$$

$$x_c^2 = r_c^2 - (y_{sc})^2$$

$$x_c = \pm\sqrt{r_c^2 - y_{sc}^2}$$

On the trailing and leading edge, substituting $0 = y_c$ into the expression for the Joukowski transformation leads the imaginary component of z_c to be zero. The substituted expression is shown below,

$$z/c = \left[1 + \frac{1}{(x/c)^2 + (y/c)^2}\right] x/c$$

$$z/c = \left[1 + \frac{1}{(x/c)^2}\right] x/c$$

$$z_c = \left[1 + \frac{1}{(x_c)^2}\right] x_c = \left[x_c + \frac{x_c}{(x_c)^2}\right]$$

$$z_c = x_c + \frac{1}{x_c}$$

Let z_{cl} represent the complex number at the leading edge and z_{ct} represent the complex number at the trailing edge. Since both z_{cl} and z_{ct} are non-dimensionalized by c, the non-dimensional chord length could be expressed as,

$$l_c = z_{ct} - z_{cl}$$

For the leading edge,

$$x_{cl} = -\sqrt{r_c^2 - y_{sc}^2}$$

$$z_{cl} = x_{cl} + \frac{1}{x_{cl}}$$

Substituting x_{cl},

$$z_{cl} = -\sqrt{r_c^2 - y_{sc}^2} - \frac{1}{\sqrt{r_c^2 - y_{sc}^2}}$$

For the trailing edge,

$$x_{ct} = \sqrt{r_c^2 - y_{sc}^2}$$

$$z_{ct} = x_{ct} + \frac{1}{x_{ct}}$$

Substituting x_{ct},

$$z_{ct} = \sqrt{r_c^2 - y_{sc}^2} + \frac{1}{\sqrt{r_c^2 - y_{sc}^2}}$$

Susbtituting to obtain l_c,

$$l_c = z_{ct} - z_{cl} = \sqrt{r_c^2 - y_{sc}^2} + \frac{1}{\sqrt{r_c^2 - y_{sc}^2}} - \left[-\sqrt{r_c^2 - y_{sc}^2} - \frac{1}{\sqrt{r_c^2 - y_{sc}^2}}\right]$$

$$l_c = \sqrt{r_c^2 - y_{sc}^2} + \frac{1}{\sqrt{r_c^2 - y_{sc}^2}} + \sqrt{r_c^2 - y_{sc}^2} + \frac{1}{\sqrt{r_c^2 - y_{sc}^2}} = 2\sqrt{r_c^2 - y_{sc}^2} + \frac{2}{\sqrt{r_c^2 - y_{sc}^2}}$$

$$l_c = 2\left[\sqrt{r_c^2 - y_{sc}^2} + \frac{1}{\sqrt{r_c^2 - y_{sc}^2}}\right]$$

Substituting for values $r_c = 1.1$ and $y_{sc} = 0.08$,

$$l_c = 4.01718349638$$

0.1.2 Part b

It is important to note that in the previous part, assuming that the leading edge and the trailing edge is defined as the point where the airfoil crosses the real axis of the z-plane, the corresponding points in the ζ plane is the point where the circle crosses the real axis.The diagram below shows potential flow over the cylinder in the complex ζ plane.

The free-stream velocity is u_∞ and comes at the cylinder at an angle of attack α. The leading and trailing edges of the airfoil in the z plane correspond to the points labelled x_{c1} and x_{c2} respectively. The points a_1 and b_1 represent the stagnation points on the circle when there is no circulation about the center of the circle. The points a_2 and b_2 represent the stagnation points on the circle when there is circulation about the center of the circle.

The circulation strength is set such that one of the stagnation points on the circle would coincide with point x_{c2}. The point x_{c2} in the ζ complex plane corresponds to the trailing edge of the airfoil in the z-plane. By setting the circulation such that a stagnation point would occur at the point of the trailing edge adjusts the potential flow to satisfy the Kutta condition. By observing the diagram, one can find the definition of β in terms of α and γ,

$$\beta = \alpha + \gamma$$

We can then express γ in terms of the known variables r_c and $y_s c$,

$$r_c \sin\gamma = y_{sc}$$

Manipulating to make γ subject of the expression,

$$\sin\gamma = \frac{y_{sc}}{r_c}$$

$$\gamma = \arcsin\left(\frac{y_{sc}}{r_c}\right)$$

Substituting γ into the expression for β,

$$\beta = \alpha + \gamma = \alpha + \arcsin\left(\frac{y_{sc}}{r_c}\right)$$

Applying the Milne-Thompson circle theorem on angled uniform free-stream flow, and then adding a free-stream vortex to the origin of the circle to set the stagnation points away from their original positions yield the following expression,

$$\sin\beta = \frac{\Gamma_a}{4\pi r U_\infty}$$

wherein Γ_a represents the strength of the free-stream vortex and r represents the radius of the circle when applying the Milne-Thompson circle theorem. The strength of the free-stream vortex will be important in determining lift. Manipulating the expression above to make Γ_a subject of the expression,

$$4\pi r U_\infty \sin\beta = \Gamma_a$$

According to the Kutta-Joukowski theorem, the lift per unit span L_s of an airfoil is given by the following expression,

$$L_s = \rho U_\infty \Gamma_a$$

The 2-dimensional lift coefficient is defined as,

$$c_L = \frac{L_s}{\frac{1}{2}\rho U_\infty^2 l}$$

wherein l represents the chord length of the airfoil. Substituting for L_s in terms of the strength of the free-stream vortex Γ_a,

$$c_L = \frac{\rho U_\infty \Gamma_a}{\frac{1}{2}\rho U_\infty^2 l} = \frac{\Gamma_a}{\frac{1}{2}U_\infty l} = \frac{2\Gamma_a}{U_\infty l}$$

Substituting the circulation of the free-stream vortex in terms of $\sin\beta$,

$$c_L = \frac{2\Gamma_a}{U_\infty l} = \frac{8\pi r U_\infty \sin\beta}{U_\infty l} = \frac{8\pi r \sin\beta}{l}$$

Non-dimensionalizing the radius of the cylinder and the airfoil to our previous work compatible,

$$c_L = \frac{8\pi(r/c)\sin\beta}{(l/c)} = \frac{8\pi r_c}{l_c}\sin\beta$$

Susbtituting the expression for β in terms of the location of the non-dimensional circle center y_{sc} and non-dimensional circle radius r_c in the ζ plane,

$$c_L = \frac{8\pi r_c}{l_c}\sin\left[\alpha + \arcsin\left(\frac{y_{sc}}{r_c}\right)\right]$$

wherein l_c was the non-dimensional chord length which was determined earlier. Substituting numerical values, $r_c = 1.1$, $l_c = 4.01718349638$, and $y_{sc} = 0.08$,

$$c_L = (6.8819398906)\sin\left[\alpha + 0.072791538003\right]$$

0.1.3 Part c

The coefficient of lift is zero when,

$$c_L = 0 = \sin(0)$$

Therefore,

$$0 = \alpha_{zero} + \arcsin\left(\frac{y_{sc}}{r_c}\right)$$

$$\alpha_{zero} = -\arcsin\left(\frac{y_{sc}}{r_c}\right)$$

Substituting for y_{sc} and r_c,

$$\alpha_{zero} = -0.072791538003\,rad$$

0.1.4 Part d

Reiterating the definition for the angle β,

$$\beta = \alpha + \arcsin\left(\frac{y_{sc}}{r_c}\right)$$

Let ζ_{s1} represent the front stagnation point around the cylinder and ζ_{s2} represent the back stagnation point around the cylinder. When there is no free-stream vortex nested at the center of the circle, the stagnation point ζ_{s1},

$$\zeta_{s1} = r_c e^{i(\pi+\alpha)} + iy_{sc}$$

When there is a free-stream vortex at the center of the circle of strength Γ_a, the stagnation point ζ_{s1} has shifted by an angle β in the counter-clockwise direction,

$$\zeta_{s1} = r_c e^{i(\pi+\alpha+\beta)} + iy_{sc}$$

Using the polar definition of complex numbers, it is simple to prove the following is true,

$$e^{\pi i} = -1$$

Substituting,

$$\zeta_{s1} = r_c e^{\pi i} e^{i(\alpha+\beta)} + iy_{sc} = -r_c e^{i(\alpha+\beta)} + iy_{sc}$$

Expressing ζ_{s1} in the cartesian definition of complex numbers,

$$\zeta_{s1} = -r_c \cos[\alpha+\beta] - r_c i \sin[\alpha+\beta] + iy_{sc}$$

$$\zeta_{s1} = -r_c \cos[\alpha+\beta] + i\left\{y_{sc} - r_c \sin[\alpha+\beta]\right\}$$

Substituting for the definition of β,

$$\zeta_{s1} = -r_c \cos\left[\alpha + \alpha + \arcsin\left(\frac{y_{sc}}{r_c}\right)\right] + i\left\{y_{sc} - r_c \sin\left[\alpha + \alpha + \arcsin\left(\frac{y_{sc}}{r_c}\right)\right]\right\}$$

$$\zeta_{s1} = -r_c \cos\left[2\alpha + \arcsin\left(\frac{y_{sc}}{r_c}\right)\right] + i\left\{y_{sc} - r_c \sin\left[2\alpha + \arcsin\left(\frac{y_{sc}}{r_c}\right)\right]\right\}$$

Substituting the for the angle $\alpha = 0.0872664626\,rad$, and the other known variables, $r_c = 1.1$, $l_c = 4.01718349638$, and $y_{sc} = 0.08$,

$$\zeta_{s1} = -1.06652798048 - 0.189291787589i$$

Applying the Kutta Joukowski transformation to ζ_{s1},

$$z_{cs1} = \left[1 + \frac{1}{(x/c)^2 + (y/c)^2}\right] x/c + \left[1 - \frac{1}{(x/c)^2 + (y/c)^2}\right] iy/c$$

$$z_{cs1} = -1.97551620021 - 0.02796080691i$$

0.2 Problem 2

0.2.1 Part a

The airfoil in the z-plane is shown below,

0.2.2 Part b

The definition for the angle β for this case,

$$\beta = \alpha + \arctan\left(\frac{\eta_0 - \eta_t}{\xi_t - \xi_0}\right)$$

The definition for circulation given angle β from the previous case is shown below,

$$\Gamma_a = 4\pi r U_\infty \sin\beta$$

Substituting the angle β,

$$\Gamma_a = 4\pi r U_\infty \sin\left[\alpha + \arctan\left(\frac{\eta_0 - \eta_t}{\xi_t - \xi_0}\right)\right]$$

$$\frac{\Gamma_a}{U_\infty} = 4\pi r \sin\left[\alpha + \arctan\left(\frac{\eta_0 - \eta_t}{\xi_t - \xi_0}\right)\right]$$

0.2.3 Part c

If the influence of gravity is assumed to be negligible within the scope of the fluid in the problem, then Bernoulli's equation,

$$p_\infty + \frac{1}{2}\rho U_\infty^2 = p + \frac{1}{2}\rho |W|^2$$

Re-arranging Bernoulli's equation,

$$\frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho |W|^2 = p - p_\infty$$

wherein $|W|$ is the magnitude of the local fluid velocity. Coefficient of pressure is typically defined as,

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

Substituting for $p - p\infty$ and assuming that the fluid is incompressible,

$$c_p = \frac{\frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho |W|^2}{\frac{1}{2}\rho U_\infty^2}$$

Simplifying,

$$c_p = \frac{\frac{1}{2}\rho U_\infty^2}{\frac{1}{2}\rho U_\infty^2} - \frac{\frac{1}{2}\rho |W|^2}{\frac{1}{2}\rho U_\infty^2}$$

$$c_p = 1 - \frac{|W|^2}{U_\infty^2}$$

$$c_p = 1 - \left[\frac{|W|}{U_\infty}\right]^2$$

The coefficient of pressure in the z-plane,

$$c_{p,z} = 1 - \left[\frac{|W_z|}{U_\infty}\right]^2$$

Reiterating the complex velocity of the fluid flow around the circle,

$$W_\zeta = \frac{dF_\zeta}{dz} = U_\infty\left(1 - \frac{r^2}{z^2}\right) + i\frac{\Gamma_a}{2\pi z}$$

Non-dimensionalizing the fluid complex velocity to the free-stream velocity,

$$\frac{W_\zeta}{U_\infty} = 1 - \frac{r^2}{z^2} + i\frac{1}{2\pi z}\left(\frac{\Gamma_a}{U_\infty}\right)$$

Here z represents an arbitrary complex number. To apply the expression into the problem, the arbitrary complex number z is a complex number in the ζ plane. Hence,

$$\frac{W_\zeta}{U_\infty} = 1 - \frac{r^2}{\zeta^2} + i\frac{1}{2\pi\zeta}\left(\frac{\Gamma_a}{U_\infty}\right)$$

Now the flow above represents flow over a cylinder in the ζ plane that is nested in the origin of the coordinate system. To shift the origin of the cylinder in the ζ plane,

$$\frac{W_\zeta}{U_\infty} = 1 - \frac{r^2}{(\zeta - \zeta_0)^2} + i\frac{1}{2\pi(\zeta - \zeta_0)}\left(\frac{\Gamma_a}{U_\infty}\right)$$

wherein ζ_0 represents the origin of the cylinder in the ζ plane. For clarity's sake,

$$\zeta_0 = \xi_0 + i\eta_0$$

The relation between the fluid velocities in the ζ plane to the z plane is shown below,

$$W_z = \frac{W_\zeta}{\left(\frac{d\zeta'}{d\zeta}\right)\left(\frac{dz}{d\zeta'}\right)}$$

Non-dimensionalizing fluid velocity by free-stream velocity to make the conformal mapping relations compatible to the complex velocity relations,

$$\frac{W_z}{U_\infty} = \frac{1}{\left(\frac{d\zeta'}{d\zeta}\right)\left(\frac{dz}{d\zeta'}\right)} \times \frac{W_\zeta}{U_\infty}$$

The 2 conformal mappings relating points in the ζ plane to the ζ' plane and from the ζ' plane to the z plane is shown below,

$$\zeta' = \zeta - \frac{\epsilon}{\zeta - d} \quad , \quad z = \zeta' + \frac{1}{\zeta'}$$

Computing the derivatives,

$$\frac{d\zeta'}{d\zeta} = \frac{d}{d\zeta}\left(\zeta\right) - \frac{d}{d\zeta}\left(\frac{\epsilon}{\zeta - d}\right) \quad , \quad \frac{dz}{d\zeta'} = \frac{d}{d\zeta'}\left(\zeta'\right) + \frac{d}{d\zeta'}\left(\frac{1}{\zeta'}\right)$$

$$\frac{d\zeta'}{d\zeta} = 1 + \frac{\epsilon}{\left(\zeta - d\right)^2} \quad , \quad \frac{dz}{d\zeta'} = 1 - \frac{1}{\zeta'^2}$$