

Chapter 1

Linear Differential Equations

1.1 Definition of Differential Operator

Let the differential operator be defined as the following:

$$L = \sum_{i=0}^n \left[a_i \frac{d^{n-i}}{dx^{n-i}} \right] = a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1} \frac{d}{dx} + a_n$$

Let the following operator D be defined:

$$D = \frac{d}{dx} \quad D^k = \frac{d^k}{dx^k}$$

Therefore, the linear differential operator could be defined as

$$L = \sum_{i=0}^n \left[a_i D^{n-i} \right] = a_0 \prod_{i=1}^n [D - r_i]$$

wherein a_i and r_i are constants albeit complex or real. The notation above is always true because an fundamental theorem of algebra states that n^{th} order polynomial must have n roots. Since linear differential operator could be expressed as a polynomial in terms of the operator D , therefore the notation above would always be true regardless of the choice of a_i and n . The linear differential operator have certain properties associated to them discussed in the propositions,

1.1.1 Proposition 1: Operation on 0

The linear differential operator when operated on a 0 will yield 0:

$$L[0] = \sum_{i=0}^n \left[a_i D^{n-i} \right] 0 = a_0 \frac{d^n}{dx^n} 0 + a_1 \frac{d^{n-1}}{dx^{n-1}} 0 + \cdots + a_{n-1} \frac{d}{dx} 0 + a_n 0$$

It is given that $\frac{d}{dx}(0) = 0$, and by reapplying recursively, $\frac{d^k}{dx^k}(0) = 0$. Therefore,

$$L[0] = 0$$

1.1.2 Proposition 2: Operation on Constants

The linear differential operator when operated on a constant will yield some constant provided that $a_n \neq 0$:

$$L[c] = \sum_{i=0}^n \left[a_i D^{n-i} \right] c = a_0 \frac{d^n}{dx^n} c + a_1 \frac{d^{n-1}}{dx^{n-1}} c + \cdots + a_{n-1} \frac{d}{dx} c + a_n c$$

$$\text{Considering } \frac{d}{dx} c = \frac{d^k}{dx^k} c = 0,$$

$$L[c] = \sum_{i=0}^n \left[a_i D^{n-i} \right] c = a_n c$$

1.1.3 Proposition 3: Operation Commutativity

If there exist two linearly independent differential operators L_1 and L_2 , then the solution of the system $0 = L_1 L_2[y]$ must be a linear combination of the solution to the system $0 = L_1[y]$ and $0 = L_2[y]$:

$$L_1 = \prod_{i=0}^m [D - \alpha_i] \quad , \quad L_2 = \prod_{j=0}^n [D - \beta_j]$$

Let $y_1(x)$ and $y_2(x)$ be such that:

$$0 = L_1[y_1(x)] = \prod_{i=0}^m [D - \alpha_i] y_1(x) \quad , \quad 0 = L_2[y_2(x)] = \prod_{j=0}^n [D - \beta_j] y_2(x)$$

Let T_i be the transformation defined as $T_i : f(x) \rightarrow g(x)$, $T_i[f(x)] = (D - r_i)f(x)$, wherein $f(x)$ is some arbitrary continuous function over some interval. Indeed the transformation T_1 is linear:

$$T_i[cu(x)] = (D - r_i)cu(x)$$

$$T_i[cu(x)] = c(D - r_i)u(x)$$

$$cT_i[u(x)] = c(D - r_i)u(x)$$

Therefore, $T_i[cu(x)] = cT_i[u(x)]$ wherein c is some arbitrary constant.

$$T_i[u(x) + v(x)] = (D - r_i)[u(x) + v(x)]$$

$$T_i[u(x) + v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

$$T_i[u(x)] + T_i[v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

Therefore, $T_i[u(x) + v(x)] = T_i[u(x)] + T_i[v(x)]$, and T_i must be a linear transformation. Linear transformations applied compositely form a linear transformation:

$$T_0(u + v) = T_0(u) + T_0(v)$$

$$T_1[T_0(u + v)] = T_1[T_0(u)] + T_1[T_0(v)]$$

$$\prod_{i=0}^{\alpha} [T_i] (u + v) = \prod_{i=0}^{\alpha} [T_i] (u) + \prod_{i=0}^{\alpha} [T_i] (v)$$

Since L_1 and L_2 is only a specific case of the transformation described as T_1 it can be considered that the differential operators of L_1 and L_2 are linear.

Therefore, it can be said that $L_1[L_2]$ must be linear.

$$0 = null = L_1[y_1(x)] = \prod_{i=0}^m [D - \alpha_i] y_1(x) \quad , \quad 0 = null = L_2[y_2(x)] = \prod_{j=0}^n [D - \beta_j] y_2(x)$$

For $y_1(x)$ and $y_2(x)$:

$$0 = null = L_2\{L_1[y_1(x)]\} = \prod_{j=0}^n [D - \beta_j] \prod_{i=0}^m [D - \alpha_i] y_1(x)$$

$$0 = null = L_1\{L_2[y_2(x)]\} = \prod_{i=0}^m [D - \alpha_i] \prod_{j=0}^n [D - \beta_j] y_2(x)$$

$$\prod_{i=0}^m [D - \alpha_i] \prod_{j=0}^n [D - \beta_j] = \prod_{j=0}^n [D - \beta_j] \prod_{i=0}^m [D - \alpha_i] = L_1[L_2] = L_2[L_1]$$

It follows by definition of linear transformation that, $L_1[k_1 y_1(x)] = k_1 L_1[y_1(x)] = null$ and that $L_2[k_2 y_2(x)] = k_2 L_2[y_2(x)] = null$:

$$0 = null = L_2\{L_1[k_1 y_1(x)]\} = \prod_{j=0}^n [D - \beta_j] \prod_{i=0}^m [D - \alpha_i] k_1 y_1(x)$$

$$\begin{aligned}
0 = null &= L_1\{L_2[k_2y_2(x)]\} = \prod_{i=0}^m [D - \alpha_i] \prod_{j=0}^n [D - \beta_j] k_2y_2(x) \\
0 &= L_2\{L_1[k_1y_1(x)]\} + L_2\{L_1[k_2y_2(x)]\} = L_2\{L_1[k_1y_1(x) + k_2y_2(x)]\} \\
0 &= \prod_{i=0}^m [D - \alpha_i] \prod_{j=0}^n [D - \beta_j] [k_1y_1(x) + k_2y_2(x)]
\end{aligned}$$

1.2 Homogenous Differential Equation Cases

Consider the following homogenous differential equation:

$$\begin{aligned}
0 &= \sum_{i=0}^n \left[a_i \frac{d^{n-i}}{dx^{n-i}} y \right] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y \\
0 &= L[y] = \sum_{i=0}^n \left[a_i D^{n-i} \right] y = \prod_{i=1}^n [D - r_i] y
\end{aligned}$$

1.2.1 Non-Repeated Roots

By principle of superposition verified by proposition 1 and 3, the general solution to the homogenous n^{th} order differential equation,

$$y_c = \sum_{i=1}^n [c_i y_i]$$

Wherein c_i represents either complex or real constants, y_i represents solutions to the $0 = [D - r_i]y_i$ system. Considering the partial system,

$$\begin{aligned}
0 &= [D - r_i]y_i \\
0 &= Dy_i - r_i y_i \\
\frac{d}{dx}(y_i) &= r_i y_i \\
\int \frac{1}{y_i} dy_i &= \int r_i dx \\
\ln y_i &= r_i x + C \\
y_i &= e^{r_i x + C} = c_i e^{r_i x}
\end{aligned}$$

Therefore for as long as there are no roots with multiplicity greater than 1, the following is true, for some choice of constants,

$$y_c = \sum_{i=1}^n [c_i e^{r_i x}]$$

1.2.2 Repeated Roots

Suppose the α^{th} root has a multiplicity of k ,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^n [D - r_i] (D - r_\alpha)^k y$$

By proposition 3, the general solution to the system must be the linear combination:

$$y_g(x) = c_1 y_1(x) + c_2 y_2(x)$$

wherein $y_1(x)$ is the solution to the system $0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^n [D - r_i] y$ and $y_2(x)$ is the solution to the system $0 = (D - r_\alpha)^k y$. By conjecture, it is suspected that the $y_2(x) = u(x)e^{r_\alpha x}$ wherein $u(x)$ is some function to be determined.

$$0 = (D - r_\alpha)u(x)e^{r_\alpha x}$$

$$0 = \frac{d}{dx} [u(x)e^{r_\alpha x}] - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = u(x)e^{r_\alpha x} + r_\alpha u(x)e^{r_\alpha x} - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = u(x)e^{r_\alpha x}$$

By reapplying the linear differential operator recursively:

$$(D - r_\alpha)^k u(x)e^{r_\alpha x} = u(x)e^{r_\alpha x}$$

Therefore, the system would follow:

$$0 = (D - r_\alpha)^k u(x)e^{r_\alpha x} = u(x)e^{r_\alpha x}$$

$$0 \neq e^{r_\alpha x} \text{ for all } x$$

$$0 = u(x)$$

A function that satisfies the following condition must be a polynomial with at most degree $k - 1$. Therefore,

$$u(x) = \sum_{i=0}^{k-1} [c_i x^{k-1-i}]$$

The general solution $y_{rr}(x)$ to the system $0 = (D - r_\alpha)^k y$:

$$y_{rr}(x) = \sum_{i=0}^{k-1} [c_i x^{k-1-i}] e^{r_\alpha x}$$

1.2.3 Complex Roots

Suppose the α^{th} root is a complex root, by the fundamental theorem of algebra, some other root must be its complex conjugate. Let the complex conjugate root of the α^{th} root be ordered next to the α^{th} root in the product notation.

Therefore,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{i=\alpha+2}^n [D - r_i] [D - r_\alpha] [D - r_{\alpha+1}] y$$

Let y_{cr} represent the complex root corresponding to the system $0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$. By principle of superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2} [c_i e^{r_i x}] + y_{cr}$$

$$0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$$

By the principles presented earlier,

$$y_{cr}(x) = c_\alpha e^{(a+bi)x} + c_{\alpha+1} e^{(a-bi)x}$$

$$y_{cr}(x) = e^{ax} [c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}]$$

By De Moivre's theorem,

$$e^{bxi} = \cos(bx) + i \sin(bx) \quad , \quad e^{-bxi} = \cos(bx) - i \sin(bx)$$

Suppose the constants c_α and $c_{\alpha+1}$ are complex numbers,

$$c_\alpha = f_1 + g_1 i \quad , \quad c_{\alpha+1} = f_2 + g_2 i$$

By substituting to the expression for complex solution,

$$y_{cr}(x) = e^{ax} [(f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}]$$

$$\text{Let } y_{cr} = e^{ax} y_{co},$$

$$y_{co} = c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}$$

$$y_{co}(x) = (f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}$$

$$y_{co}(x) = (f_1 + g_1 i)[\cos(bx) + i \sin(bx)] + (f_2 + g_2 i)[\cos(bx) - i \sin(bx)]$$

Let

$$A(x) = (f_1 + g_1 i)[\cos(bx) + i \sin(bx)] \quad , \quad B(x) = (f_2 + g_2 i)[\cos(bx) - i \sin(bx)]$$

$$A(x) = f_1 \cos(bx) - g_1 \sin(bx) + i[f_1 \sin(bx) + g_1 \cos(bx)]$$

$$B(x) = f_2 \cos(bx) + g_2 \sin(bx) + i[-f_2 \sin(bx) + g_2 \cos(bx)]$$

$$y_{co}(x) = A(x) + B(x)$$

$$y_{co}(x) = (f_1 + f_2) \cos(bx) + (g_2 - g_1) \sin(bx) + i[(f_1 - f_2) \sin(bx) + (g_1 + g_2) \cos(bx)]$$

For the complex root $y_{cr}(x)$ to be real, the imaginary component of $y_{cr}(x)$ must be equals to 0. Therefore, the following must hold true,

$$f_1 = f_2 \quad , \quad g_1 = -g_2$$

For as long as the condition above hold true, the two constants c_α and $c_{\alpha+1}$ must be complex conjugates. Considering the case wherein c_α and $c_{\alpha+1}$ as complex conjugates,

$$y_{co}(x) = 2f_1 \cos(bx) + 2g_2 \sin(bx)$$

$$y_{cr} = 2e^{ax}[f_1 \cos(bx) + g_2 \sin(bx)]$$

Therefore, the following is true for each complex root and conjugate pair,

$$y_c = \sum_{i=1}^{n-2} [c_i e^{r_i x}] + 2e^{ax}[f_1 \cos(bx) + g_2 \sin(bx)]$$

1.2.4 Repeated Complex Roots

Suppose the α^{th} root is a complex root with a multiplicity of k . Let its complex conjugate be placed adjacent after said complex root,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{i=\alpha+2k}^n [D - r_i] [D - r_\alpha]^k [D - \bar{r}_\alpha]^k y$$

Let y_{crr} be considered as the solution to the system $0 = [D - r_\alpha]^k [D - \bar{r}_\alpha]^k y_{crr}$.

Based on superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2k} [c_i e^{r_i x}] + y_{crr}$$

Based on the previous work on repeated roots with multiplicity greater than 1,

$$y_{crr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i} \right] C_1 e^{r_\alpha x} + \sum_{i=0}^{k-1} \left[c_i x^{k-1-i} \right] C_2 e^{r_\alpha^- x}$$

Let $r_\alpha = a + bi$, and $r_\alpha^- = a - bi$,

$$y_{crr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i} \right] [C_1 e^{bxi} + C_2 e^{-bxi}] e^{ax}$$

Let $c_1 = f_1 + g_1 i$ and $c_2 = f_2 + g_2 i$. Based on previous work on complex roots,

$$C_1 e^{bxi} + C_2 e^{-bxi} = 2f_1 \cos(bx) + 2g_2 \sin(bx)$$

Therefore,

$$y_{crr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i} \right] [2f_1 \cos(bx) + 2g_2 \sin(bx)] e^{ax}$$

1.3 General Solutions to Homogenous Differential Equations

Therefore, if an n^{th} order homogeneous differential equation with a real non-repeated roots, b complex root pairs, c real repeated roots with multiplicity γ , and d complex repeated root pairs with multiplicity β

$$\begin{aligned} y_c = & \sum_{l=1}^a [c_{1,l} e^{r_{1,l} x}] \\ & + \sum_{j=1}^b [c_{2,j,1} \cos(b_{1,j} x) + c_{2,j,2} \sin(b_{1,j} x)] e^{a_{1,j} x} \\ & + \sum_{k=1}^c \left[\sum_{m=0}^{\gamma_k-1} [c_{3,m,k} x^{\gamma_k-1-m}] e^{r_k x} \right] \\ & + \sum_{i=1}^d \left[\sum_{p=0}^{\beta_p-1} [c_{4,p,i} x^{\beta_p-1-p}] [k_{4,i,1} \cos(b_{2,i} x) + k_{4,i,2} \sin(b_{2,i} x)] e^{r_i x} \right] \end{aligned}$$

The variables a , b , c , d , γ , β , and n are related by the following expression,

$$n = a + 2b + \sum_{i=1}^c [\gamma_i] + \sum_{j=1}^d [2\beta_j]$$

1.4 Non-Homogenous Differential Equations

Consider the following system:

$$\sum_{i=0}^n \left[a_i \frac{d^n y}{dx^n} \right] = \sum_{j=0}^m [c_j f_j(x)]$$

wherein $f_i(x)$ represents the i^{th} arbitrary function, and a_i represents the i^{th} arbitrary constant. The following function could be rewritten in terms of the linear differential operator L :

$$L[y] = \sum_{j=0}^m [c_j f_j(x)]$$

Let y_j represent the general solution to the j^{th} system:

$$L[y_j] = c_j f_j(x)$$

By taking the summations of the various solutions to the various systems:

$$L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j] = c_0 f_0(x) + c_1 f_1(x) + \cdots + c_{j-1} f_{j-1}(x) + c_j f_j(x)$$

Since the differential operator L is linear, as shown in proposition 3:

$$L \left[\sum_{j=0}^m (y_j) \right] = L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j]$$

$$L \left[\sum_{j=0}^m (y_j) \right] = \sum_{j=0}^m [c_j f_j(x)]$$

Therefore, a solution to the non-homogenous differential equation:

$$y_p(x) = \sum_{j=0}^m (y_j)$$

An m^{th} dimensional subspace spanned by m functions must always contain a null element, in this case, a zero function. Let the y_c represent the general solution to the homogenous differential equation $Ly = null = 0$. Then the general solution must follow:

$$y_g(x) = \sum_{j=0}^m (y_j) + y_c$$