0.1 Problem 4

Reiterating Euler's equations of motion,

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y \end{bmatrix}$$

0.1.1 Part a

Assuming no torque-free motion,

$$M_x = 0 \quad , \quad M_y = 0 \quad , \quad M_z = 0$$

Assuming the symmetry conditions, $I_x = I_y = I_t$,

$$I_y - I_x = 0$$

In the z-direction,

$$0 = I_z\dot{\omega}_z$$

Since $I_z \neq 0$, then

$$0 = \dot{\omega}_z$$

This means that ω_z is a constant in time, let $\omega_z = \Omega$, at time $t = 0$. Then it must follow,

$$\omega_z = \Omega$$

for all time t. Let $I_z = I_a$. Substituting these assumptions and findings into the x and y direction of Euler's equations of motion,

$$0 = I_t\dot{\omega}_x + (I_a - I_t)\Omega\omega_y \quad , \quad 0 = I_t\dot{\omega}_y + (I_t - I_a)\omega_x\Omega$$

Re-arranging the equations,

$$0 = \dot{\omega}_x + \frac{(I_a - I_t)}{I_t}\Omega\omega_y \quad , \quad 0 = \dot{\omega}_y + \frac{(I_t - I_a)}{I_t}\omega_x\Omega$$

Differentiating both equations with respect to time,

$$0 = \ddot{\omega_x} + \frac{(I_a - I_t)}{I_t}\Omega\dot{\omega_y} \quad , \quad 0 = \ddot{\omega_y} + \frac{(I_t - I_a)}{I_t}\dot{\omega_x}\Omega$$

Substituting the second order derivative ω_x into the first order derivative ω_y,

$$\dot{\omega_y} = -\frac{(I_t - I_a)}{I_t}\omega_x\Omega$$

$$0 = \ddot{\omega_x} + \frac{(I_a - I_t)}{I_t}\Omega\dot{\omega_y} = \ddot{\omega_x} - \frac{(I_a - I_t)\Omega}{I_t}\frac{(I_t - I_a)\Omega}{I_t}\omega_x = \ddot{\omega_x} + \frac{(I_t - I_a)^2\Omega^2}{I_t^2}\omega_x = \ddot{\omega_x} + \left[\frac{(I_t - I_a)\Omega}{I_t}\right]^2\omega_x$$

Therefore, the characteristic equation to the differential equation above,

$$0 = r^2 + \left[\frac{(I_t - I_a)\Omega}{I_t}\right]^2$$

$$r^2 = -\left[\frac{(I_t - I_a)\Omega}{I_t}\right]^2$$

$$r = \pm\left[\frac{(I_t - I_a)\Omega}{I_t}\right]i$$

Let $\lambda = \frac{(I_t - I_a)\Omega}{I_t}$, then $r = \pm\lambda i$. By some choice of arbitrary constants, the solution to the differential equation,

$$\omega_x = A\cos(\lambda t) + B\sin(\lambda t)$$

When $t = 0$, $\omega_x = \omega_{x,o}$

$$\omega_{x,o} = A$$

Taking the derivative of ω_x,

$$\dot{\omega_x} = A\frac{d}{dt}[\cos(\lambda t)] + B\frac{d}{dt}[\sin(\lambda t)] = A\times -\sin(\lambda t)\times\lambda + B\cos(\lambda t)\times\lambda = -A\lambda\sin(\lambda t) + B\lambda\cos(\lambda t)$$

When $t = 0$, $\dot{\omega_x} = \dot{w_{x,o}}$. Substituting,

$$\dot{\omega_{x,o}} = B\lambda$$

$$B = \frac{\dot{\omega_{x,o}}}{\lambda}$$

Substituting for A and B to form the solution ω_x,

$$\omega_x = A\cos(\lambda t) + B\sin(\lambda t) = \omega_{x,o}\cos(\lambda t) + \frac{\dot{\omega_{x,o}}}{\lambda}\sin(\lambda t)$$

Re-writing Euler's equations of motion in terms of λ for ω_y,

$$0 = \dot{\omega_x} + \frac{(I_a - I_t)}{I_t}\Omega\omega_y$$

$$\dot{\omega_x} = -\frac{(I_a - I_t)}{I_t}\Omega\omega_y = \frac{(I_t - I_a)\Omega}{I_t}\omega_y = \lambda\omega_y$$

$$\omega_y = \frac{\dot{\omega_x}}{\lambda} = \frac{1}{\lambda}\frac{d}{dt}\left[\omega_{x,o}\cos(\lambda t) + \frac{\dot{\omega_{x,o}}}{\lambda}\sin(\lambda t)\right] = \frac{1}{\lambda}\left[-\omega_{x,o}\lambda\sin(\lambda t) + \dot{\omega_{x,o}}\cos(\lambda t)\right]$$

$$\omega_y = -\omega_{x,o}\sin(\lambda t) + \frac{\dot{\omega_{x,o}}}{\lambda}\cos(\lambda t)$$

Since $\dot{\omega_x} = \lambda\omega_y$, $\dot{\omega_{x,o}} = \lambda\omega_{y,o}$. Manpulating, $\frac{\dot{\omega_{x,o}}}{\lambda} = \omega_{y,o}$. Substituting, into the rotation ω_x,

$$\omega_x = \omega_{x,o}\cos(\lambda t) + \omega_{y,o}\sin(\lambda t)$$

Substituting into the rotation ω_y,

$$\omega_y = -\omega_{x,o}\sin(\lambda t) + \omega_{y,o}\cos(\lambda t)$$

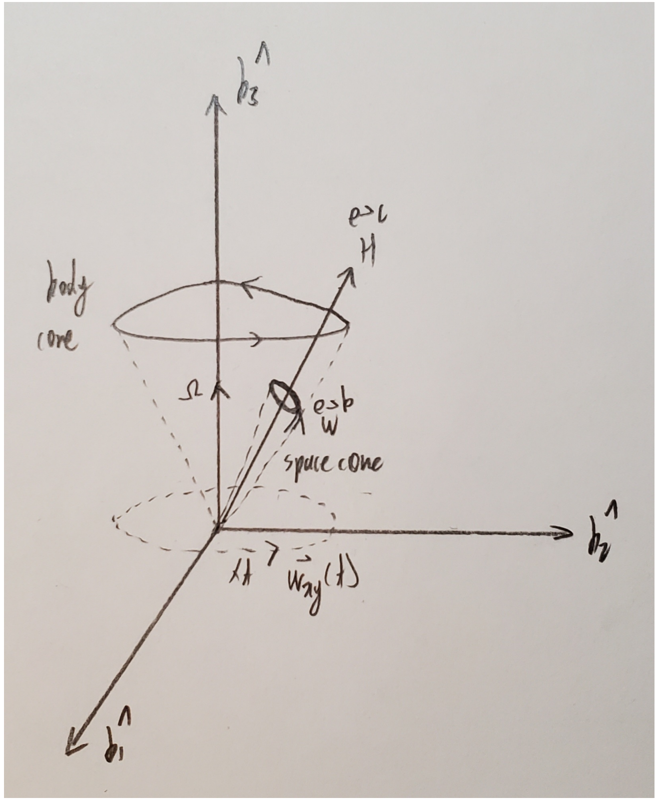
Here the results differ slightly due to the slightly different formulation of λ. To reconcile the results to the lecture results, let $\lambda_l = -\lambda$, wherein λ_l is the λ defined in the lecture, substituting,

$$\omega_x = \omega_{x,o}\cos(-\lambda_l t) + \omega_{y,o}\sin(-\lambda_l t) \quad , \quad \omega_y = -\omega_{x,o}\sin(-\lambda_l t) + \omega_{y,o}\cos(-\lambda_l t)$$

$$\boxed{\omega_x = \omega_{x,o}\cos(\lambda_l t) - \omega_{y,o}\sin(\lambda_l t) \quad , \quad \omega_y = \omega_{x,o}\sin(\lambda_l t) + \omega_{y,o}\cos(\lambda_l t)}$$

0.1.2 Part b

The diagram of the ${}^e\omega^b$ as well as the body cone is shown below,



0.1.3 Part c

The angular velocity vector,

$$\bar{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$$

Substituting, for the results from the previous part a,

$$\bar{\omega} = \begin{bmatrix} \omega_{x,o}\cos(\lambda_l t) - \omega_{y,o}\sin(\lambda_l t) \\ \omega_{x,o}\sin(\lambda_l t) + \omega_{y,o}\cos(\lambda_l t) \\ \Omega \end{bmatrix}$$

This could be re-arranged into the following matrix equation,

$$\bar{\omega} = A\bar{I}_c = \begin{bmatrix} \cos(\lambda_l t) & -\sin(\lambda_l t) & 0 \\ \sin(\lambda_l t) & \cos(\lambda_l t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_{x,o} \\ \omega_{y,o} \\ \Omega \end{bmatrix}$$

wherein the matrix A represents the rotation matrix about axis z and $\bar{I}_c$ represent the initial conditions of the problem. Typically the rotation matrix accepts arguments θ wherein θ represents the angle of rotation in the counter-clockwise direction. As shown above, the arguments of the trigonometric function is instead $\lambda_l t$. As long as $\lambda_l > 0$, then the angular velocity vector would rotate counter-clockwise. If $\lambda_l < 0$, then the angular velocity vector would rotate in the clockwise direction.

By re-formulating the angular velocity vector as a rotation matrix, it is proven that the angular velocity vector rotates about the z-axis of the body frame in the counter clockwise direction at a rate λ_l