Chapter 1

Radial Fields

1.1 Curl and Circulation

Let the general equation denoting force produced by a radial field be: $\bar{F} = kr^n\hat{r}$ wherein k and n are arbitrary constants. Since $\hat{r}$ represents a unit vector in the radial direction,

$$\hat{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$r = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$$

$$r^n = \left(x^2+y^2+z^2\right)^{\frac{n}{2}}$$

$$\bar{F} = \frac{k\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\bar{F} = k(x^2+y^2+z^2)^{\frac{1}{2}(n-1)}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\bar{F} = \begin{pmatrix} kx(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} \\ ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} \\ kz(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} \end{pmatrix}$$

Stokes theorem states: $\oint_R \bar{F}\cdot d\bar{r} = \oiint_S \nabla\times\bar{F}\cdot\hat{n}dS$ wgereub R represents a closed path in $\mathbb{R}^3$ and S represents the corresponding surface bounded by the closed path similarly in $\mathbb{R}^3$. In conservative vector field, the circulation, $\oint_R \bar{F} \cdot \bar{dr} = 0$, therefore, the curl, $\nabla \times \bar{F} = 0$.

$$\text{Let } \bar{F} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$\nabla \times \bar{F} = \begin{pmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\frac{\partial h}{\partial y} = kyz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial g}{\partial z} = kyz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} = 0$$

$$\frac{\partial f}{\partial z} = kxz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial h}{\partial x} = kxz(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial g}{\partial x} = kxy(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial f}{\partial y} = kxy(n-1)(x^2+y^2+z^2)^{\frac{1}{2}(n-3)}$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

Therefore, $\nabla \times \bar{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Any radial vector field with the following expression $\bar{F} = kr^n\hat{r}$ is conservative. Since radial vector fields are considered as conservative force fields, a potential function must exist with the following conditions:

$$f = \frac{\partial p}{\partial x} \qquad g = \frac{\partial p}{\partial y} \qquad h = \frac{\partial p}{\partial z}$$

$$p = \int kx(x^2+y^2+z^2)^{\frac{1}{2}(n-1)}dx$$

$$p = \frac{k}{n+1}(x^2+y^2+z^2)^{\frac{1}{2}(n+1)} + C(y,z) + k_1$$

$C(y,z)$ represents functions strictly in terms of y and z and k_1 represents some arbitrary numerical integration constant.

$$\frac{\partial p}{\partial y} = ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} + C'(y,z)$$

The potential function's partial derivative with respect to y represents y component of the force field. Therefore,

$$ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} + C'(y,z) = ky(x^2+y^2+z^2)^{\frac{1}{2}(n-1)}$$

$$C'(y,z) = 0$$

$$\int dC(y,z) = 0 \times \int dy$$

$$C(y,z) = C_2(z) + k_2$$

$$p = \frac{k}{n+1}(x^2+y^2+z^2)^{\frac{1}{2}(n+1)} + C_2(z) + k_1 + k_2$$

$$\frac{\partial p}{\partial z} = kz(x^2+y^2+z^2)^{\frac{1}{2}(n-1)} + C_2'(z) = kz(x^2+y^2+z^2)^{\frac{1}{2}(n-1)}$$

$$C_2'(z) = 0$$

$$C_2(z) = k_3$$

$$p = \frac{k}{n+1}(x^2+y^2+z^2)^{\frac{1}{2}(n+1)} + k_1 + k_2 + k_3$$

By convention, the potential of a force field is defined as 0 infinitely away from the origin. Therefore,

$$0 = p = \frac{k}{n+1}\lim_{(x,y,z)\to\infty}\left[(x^2+y^2+z^2)^{\frac{1}{2}(n+1)}\right] + k_1 + k_2 + k_3$$

$$0 = k_1 + k_2 + k_3$$

Therefore, the potential function for any given radial force field is given as:

$$p = \frac{k}{n+1}(x^2+y^2+z^2)^{\frac{1}{2}(n+1)}$$