Chapter 1

Numerical Methods: Potential Flows

1.1 Potential Flow

1.1.1 Gauss-Siedel Grid Method

The Taylor expansion series for an arbitray function $f(t)$ is defined as the following,

$$f(t) = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [f(a)](x-a)^n \right\}$$

Let the control volume be split up into infinitesmially small grids. Each grid will have horizontal width of dx and a vertical height of dy. Let $\psi_{i,j}$ represent the i^{th} column and the j^{th} row value of the stream function. Columns are defined as the vertical edges of the infinitesmially small grids meanwhile rows are defined as the horizontal edges of the infinitesmailly small grids. Therefore, the analysis performed occurs at the edges of the infinitesmially small grids. Indexing starts from the bottom left corner of the control volume, at the origin of the declared coordinate system. Indices start at 0 and progress by increments of 1 along the x and y axes. Analyzing the Taylor Expansion series horizontally in the x-direction,

$$\psi_{i+1,j} = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](\Delta x)^n \right\}$$

For the $i-1^{th}$ term,

$$\psi_{i-1,j} = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](-\Delta x)^n \right\}$$

Taking the second order approximation, and neglecting the higher order terms,

$$\sum_{n=3}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](\Delta x)^n \right\} \approx \sum_{n=3}^{\infty} \left\{ \frac{1}{n!} \frac{d^n}{dx^n} [\psi_{i,j}](-\Delta x)^n \right\} \approx 0$$

Therefore, for the $i+1^{th}$ term and the $i-1^{th}$ term respectively,

$$\psi_{i+1,j} \approx \psi_{i,j} + \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 \quad , \quad \psi_{i-1,j} \approx \psi_{i,j} - \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

Adding the terms together and manipulating the equation to isolate the $\dfrac{d^2\psi_{i,j}}{dx^2}$ term,

$$\psi_{i+1,j} + \psi_{i-1,j} \approx \psi_{i,j} + \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 + \psi_{i,j} - \frac{d\psi_{i,j}}{dx}\Delta x + \frac{1}{2}\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

$$\psi_{i+1,j} + \psi_{i-1,j} \approx 2\psi_{i,j} + \frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2$$

$$\frac{d^2\psi_{i,j}}{dx^2}(\Delta x)^2 \approx \psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}$$

$$\frac{d^2\psi_{i,j}}{dx^2} \approx \frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2}$$

Using the same process in the y-direction,

$$\frac{d^2\psi_{i,j}}{dy^2} \approx \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2}$$

Substituting the relevant terms into the governing equation,

$$\frac{\partial^2}{\partial x^2}[\psi] + \frac{\partial^2}{\partial y^2}[\psi] = 0$$

$$\frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2} = 0$$

Making $\psi_{i,j}$ the subject of the equation above,

$$(\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j})(\Delta x)^2 = 0$$

$$(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2 = 2\psi_{i,j}(\Delta y)^2 + 2\psi_{i,j}(\Delta x)^2$$

$$(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2 = 2[(\Delta y)^2 + (\Delta x)^2](\psi_{i,j})$$

$$\psi_{i,j} = \frac{(\psi_{i+1,j} + \psi_{i-1,j})(\Delta y)^2 + (\psi_{i,j+1} + \psi_{i,j-1})(\Delta x)^2}{2[(\Delta y)^2 + (\Delta x)^2]}$$

The relative error ϵ of the stream function for the mesh corners are defined as

$$\left| \frac{\overset{p+1}{\psi_{i,j}} - \overset{p}{\psi_{i,j}}}{\overset{p+1}{\psi_{i,j}}} \right| = \epsilon$$

wherein $\overset{p+1}{\psi_{i,j}}$ represents the $p+1^{th}$ iteration of $\psi_{i,j}$ formulated through the Gauss-Seidel method.The relative error for the control volume would likewise be defined as

$$\frac{1}{k}\sqrt{\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\left(\frac{\overset{p+1}{\psi_{i,j}} - \overset{p}{\psi_{i,j}}}{\overset{p+1}{\psi_{i,j}}}\right)^2} = \epsilon_t \quad , \quad k = (n-1)(m-1)$$

The value of the error ϵ would represent the error tolerance and a direct numerical simulation of the potential flow algorithm should keep running until the error tolerance would be low enough. The values of n and m would be shown analytically below,

$$m = \frac{H}{dy} \quad , \quad n = \frac{L}{dx}$$

1.2 Unsteady Diffusion

Consider the grid definition shown below,

Consider the general form the of the unsteady diffusion problem shown below,

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla \cdot (\rho u \phi) = \nabla \cdot (\Gamma \nabla \phi) + S_\phi$$

$\frac{\partial}{\partial t}(\rho\phi)$ represents the unsteady term, $\nabla \cdot (\rho u \phi)$ represents the convection term, $\nabla \cdot (\Gamma \nabla \phi)$ represents the diffusion term and S_ϕ represents the source term. If the convection term is neglected, $\nabla \cdot (\rho u \phi) = 0$. Substituting for this,

$$\frac{\partial}{\partial t}(\rho\phi) = \nabla \cdot (\Gamma \nabla \phi) + S_\phi$$

Integrating the unsteady diffusion equation over a small time step Δt as well as over the control volume CV,

$$\int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial t}(\rho\phi)\, dV_o\, dt = \int_t^{t+\Delta t} \int_{CV} \nabla \cdot (\Gamma \nabla \phi) + S_\phi\, dV_o\, dt$$

For the 1-dimensional case $\nabla \phi = \frac{\partial}{\partial x}(\phi)$. Substituting for the gradient of ϕ, $\nabla \cdot \nabla \phi = \frac{\partial^2}{\partial x^2}(\phi)$. Substituting for this,

$$\int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial t}(\rho\phi)\, dV_o\, dt = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] + S_\phi\, dV_o\, dt$$

The order of integration is commutative for the independent spatial and time variables. Therefore,

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] + S_\phi\, dV_o\, dt$$

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] dV_o\, dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

The differential for volume $dV_o = Adx$. Substituting for this,

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left[\Gamma \frac{\partial}{\partial x}(\phi)\right] A\, dx\, dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

Due to the exact differential simplification for dx,

$$\int_{CV} \int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV_o = \int_t^{t+\Delta t} \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w^e dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

Due to the exact differential simplification for dt,

$$\int_{CV} [\rho\phi]_t^{t+\Delta t}\, dV_o = \int_t^{t+\Delta t} \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w dt + \int_t^{t+\Delta t} \int_{CV} S_\phi\, dV_o\, dt$$

$$\int_{CV} [\rho\phi]_{t+\Delta t} - [\rho\phi]_t \, dV_o = \int_t^{t+\Delta t} \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w dt + \int_t^{t+\Delta t} \int_{CV} S_\phi \, dV_o \, dt$$

Assuming that the value of ϕ_P is the value of ϕ throughout the entirety of a single cell, then ϕ would be a constant with respect to the volumetric integral. Substituting for this,

$$\left\{[\rho\phi]_{t+\Delta t} - [\rho\phi]_t\right\} \Delta V_o = \int_t^{t+\Delta t} \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w dt + \int_t^{t+\Delta t} \int_{CV} S_\phi \, dV_o \, dt$$

Assuming uniformity throughout the grid, that ρ, A and Γ is constant from one cell to the next,

$$\left\{[\rho\phi]_{t+\Delta t} - [\rho\phi]_t\right\} \Delta V_o = \rho \left[\phi_{t+\Delta t} - \phi_t\right] \Delta V_o$$

With a slight abuse of notation, let ϕ_t be represented as $\phi_{t,o}$ and $\phi_{t+\Delta t}$ be represented as ϕ_t. Substituting for this,

$$\left\{[\rho\phi]_{t+\Delta t} - [\rho\phi]_t\right\} \Delta V_o = \rho \Delta V_o \left[\phi_t - \phi_{t,o}\right]$$

With the uniformity assumption,

$$\left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_e - \left[\Gamma A \frac{\partial}{\partial x}(\phi)\right]_w = \Gamma A \left[\frac{\partial}{\partial x}(\phi)\right]_e - \Gamma A \left[\frac{\partial}{\partial x}(\phi)\right]_w = \Gamma A \left\{\left[\frac{\partial}{\partial x}(\phi)\right]_e - \left[\frac{\partial}{\partial x}(\phi)\right]_w\right\}$$

The source term is assumed to be uniform throughout a single grid much like ϕ and is then simplified to,

$$\int_t^{t+\Delta t} \int_{CV} S_\phi \, dV_o \, dt = \int_t^{t+\Delta t} S_\phi \Delta V_o \, dt$$

Substituting for all of these simplifications,

$$\rho \left[\phi_{t+\Delta t} - \phi_t\right] \Delta V_o = \int_t^{t+\Delta t} \Gamma A \left\{\left[\frac{\partial}{\partial x}(\phi)\right]_e - \left[\frac{\partial}{\partial x}(\phi)\right]_w\right\} dt + \int_t^{t+\Delta t} S_\phi \Delta V_o \, dt$$

$$\rho \left[\phi_{t+\Delta t} - \phi_t\right] \Delta V_o = \Gamma A \int_t^{t+\Delta t} \left[\frac{\partial}{\partial x}(\phi)\right]_e - \left[\frac{\partial}{\partial x}(\phi)\right]_w dt + \Delta V_o \int_t^{t+\Delta t} S_\phi \, dt$$

1.2.0.1 Interior Cells

For the interior cells, the spatial derivative $\frac{\partial}{\partial x}(\phi)$ is computed using central differencing. Therefore,

$$\left[\frac{\partial}{\partial x}(\phi)\right]_e = \frac{\phi_E - \phi_P}{\delta x_{EP}} \quad , \quad \left[\frac{\partial}{\partial x}(\phi)\right]_w = \frac{\phi_P - \phi_W}{\delta x_{PW}}$$

Substituting for the definitions of these spatial derivatives,

$$\rho \left[\phi_{t+\Delta t} - \phi_t\right] \Delta V_o = \Gamma A \int_t^{t+\Delta t} \frac{\phi_E - \phi_P}{\delta x_{EP}} - \frac{\phi_P - \phi_W}{\delta x_{PW}} dt + \Delta V_o \int_t^{t+\Delta t} S_\phi \, dt$$

Re-arranging the equation,

$$\rho\left[\phi_{t+\Delta t}-\phi_t\right]\Delta V_o=\Gamma A\int_t^{t+\Delta t}\frac{1}{\delta x_{EP}}\phi_E-\frac{1}{\delta x_{EP}}\phi_P-\frac{1}{\delta x_{PW}}\phi_P+\frac{1}{\delta x_{PW}}\phi_W\,dt+\Delta V_o\int_t^{t+\Delta t}S_\phi\,dt$$

$$\rho\left[\phi_{t+\Delta t}-\phi_t\right]\Delta V_o=\Gamma A\int_t^{t+\Delta t}\frac{1}{\delta x_{EP}}\phi_E-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right]\phi_P+\frac{1}{\delta x_{PW}}\phi_W\,dt+\Delta V_o\int_t^{t+\Delta t}S_\phi\,dt$$

The time integration approximation for ϕ with respect to time is shown below,

$$\int_t^{t+\Delta t}\phi\,dt=[\theta\phi_{t+\Delta t}+(1-\theta)\phi_t]\Delta t$$

Using a change of notation,

$$\int_t^{t+\Delta t}\phi_i\,dt=[\theta\phi_i+(1-\theta)\phi_{i,o}]\Delta t$$

wherein the subscript i would represent some point in the grids, ϕ_i represents the value of ϕ at the new time step meanwhile $\phi_{i,o}$ represents the value of ϕ at the old time step. Substituting for the integration scheme,

$$\begin{aligned}\rho\left[\phi_t-\phi_{t,o}\right]\Delta V_o=&\Gamma A\Delta t\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+\Delta V_o\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

The finite volume could be expressed in terms of area A and grid size, $\Delta V_o=A\delta_{ew}$. Substituting for this,

$$\begin{aligned}\rho\left[\phi_t-\phi_{t,o}\right]A\delta_{ew}=&\Gamma A\Delta t\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+A\delta_{ew}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Dividing both sides by $A\Delta t$,

$$\begin{aligned}\rho\left[\phi_t-\phi_{t,o}\right]\frac{\delta_{ew}}{\Delta t}=&\Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Re-arranging the equation,

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_t-\frac{\rho\delta_{ew}}{\Delta t}\phi_{t,o}=&\Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]-\left[\frac{1}{\delta x_{EP}}+\frac{1}{\delta x_{PW}}\right][\theta\phi_P+(1-\theta)\phi_{P,o}]\right.\\&\left.+\frac{1}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]\right\}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_t - \frac{\rho\delta_{ew}}{\Delta t}\phi_{t,o} + \left[\frac{\Gamma}{\delta x_{EP}} + \frac{\Gamma}{\delta x_{PW}}\right][\theta\phi_P] =& \Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}]\right.\\ & - \left[\frac{1}{\delta x_{EP}} + \frac{1}{\delta x_{PW}}\right][(1-\theta)\phi_{P,o}]\\ & \left.+ \frac{1}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}]\right\} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

Here ϕ_t represents ϕ_P, because the value of ϕ is assumed to be uniform throughout the cell.

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_P + \left[\frac{\Gamma}{\delta x_{EP}} + \frac{\Gamma}{\delta x_{PW}}\right][\theta\phi_P] =& \Gamma\left\{\frac{1}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}]\right.\\ & - \left[\frac{1}{\delta x_{EP}} + \frac{1}{\delta x_{PW}}\right][(1-\theta)\phi_{P,o}]\\ & \left.+ \frac{1}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}]\right\} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt + \frac{\rho\delta_{ew}}{\Delta t}\phi_{P,o}\end{aligned}$$

$$\begin{aligned}\frac{\rho\delta_{ew}}{\Delta t}\phi_P + \left[\frac{\Gamma}{\delta x_{EP}} + \frac{\Gamma}{\delta x_{PW}}\right][\theta\phi_P] =& \left\{\frac{\Gamma}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}]\right.\\ & - \left[\frac{\Gamma}{\delta x_{EP}} + \frac{\Gamma}{\delta x_{PW}}\right][(1-\theta)\phi_{P,o}]\\ & \left.+ \frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}]\right\} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt + \frac{\rho\delta_{ew}}{\Delta t}\phi_{P,o}\end{aligned}$$

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{EP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] =& \frac{\Gamma}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}]\\ & + \left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{EP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}]\\ & + \frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

Let

$$a_W = \frac{\Gamma}{\delta x_{PW}} \quad , \quad a_E = \frac{\Gamma}{\delta x_{EP}} \quad , \quad a_{P,0} = \frac{\rho\delta_{ew}}{\Delta t} \quad , \quad a_P = a_{P,0} + a_E\theta + a_W\theta$$

Under the substitutions above, the expression becomes,

$$\begin{aligned}a_P\phi_P =& a_E[\theta\phi_E + (1-\theta)\phi_{E,o}] + \left[a_{P,0} - a_E(1-\theta) - a_W(1-\theta)\right][\phi_{P,o}]\\ & + a_W[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

When evaluating ϕ at the new time step, the old time step would be considered a known quantity. Re-arranging the equation to place all of the known variables on LHS and the unknown variables at the RHS,

$$\begin{aligned}a_P\phi_P =& a_E\theta\phi_E + a_E(1-\theta)\phi_{E,o} + \left[a_{P,0} - a_E(1-\theta) - a_W(1-\theta)\right][\phi_{P,o}]\\ & + a_W\theta\phi_W + a_W(1-\theta)\phi_{W,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

$$\begin{aligned}-a_E\theta\phi_E + a_P\phi_P - a_W\theta\phi_W =& a_E(1-\theta)\phi_{E,o} + \left[a_{P,0} - a_E(1-\theta) - a_W(1-\theta)\right][\phi_{P,o}]\\ & + a_W(1-\theta)\phi_{W,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\, dt\end{aligned}$$

1.2.0.2 Dirichlet Western Boundary Conditions

The expression for the interior cells before the declaration of definitions of a_W, a_E and substitution is shown below,

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P]=&\frac{\Gamma}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{PW}}[\theta\phi_W+(1-\theta)\phi_{W,o}]+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

When applying dirichlet boundary conditions to the cell that is closest to the origin of the coordinate system, several modifications must be made to the expression for the interior cells. Finite difference instead of central differencing is used at the left boundary of the first cell. This would mean the following change of variables to allow the formulation to be suitable at the boundaries,

$$\delta x_{PW}\rightarrow\delta x_{Pw}\quad,\quad\phi_W\rightarrow\phi_A\quad,\quad\phi_{W,o}\rightarrow\phi_{A,o}$$

The first change is to accomodate for the one-sided finite differencing at the west face, the second change is to accomodate for the boundary condition of ϕ_A at the wall at the new time step and the third change is to accomodate for the boundary condition of $\phi_{A,o}$ at the wall for the old time step. Substituting these changes into the expression,

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{Pw}}\theta\right][\phi_P]=&\frac{\Gamma}{\delta x_{EP}}[\theta\phi_E+(1-\theta)\phi_{E,o}]\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{Pw}}[\theta\phi_A+(1-\theta)\phi_{A,o}]+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Placing the unknwon variables to the LHS and the known variables to RHS,

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{Pw}}\theta\right][\phi_P]=&\frac{\Gamma}{\delta x_{EP}}\theta\phi_E+\frac{\Gamma}{\delta x_{EP}}(1-\theta)\phi_{E,o}\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{Pw}}\theta\phi_A+\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\phi_{A,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{EP}}\theta+\frac{\Gamma}{\delta x_{Pw}}\theta\right][\phi_P]-\frac{\Gamma}{\delta x_{EP}}\theta\phi_E=&\frac{\Gamma}{\delta x_{EP}}(1-\theta)\phi_{E,o}\\&+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{EP}}(1-\theta)-\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\right][\phi_{P,o}]\\&+\frac{\Gamma}{\delta x_{Pw}}\theta\phi_A+\frac{\Gamma}{\delta x_{Pw}}(1-\theta)\phi_{A,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t}S_\phi\,dt\end{aligned}$$

Let

$$a_W = 0 \quad , \quad a_E = \frac{\Gamma}{\delta x_{EP}} \quad , \quad a_{P,0} = \frac{\rho \delta_{ew}}{\Delta t} \quad , \quad S_P = -\frac{\Gamma}{\delta x_{Pw}} \quad , \quad a_P = a_{P,0} + \theta(a_E + a_W - S_P)$$

Substituting these variables into the expression above,

$$\begin{aligned} -a_E\theta\phi_E + a_P\phi_P - a_W\phi_W =& a_E(1-\theta)\phi_{E,o} + \left[a_{P,0} - a_E(1-\theta) + S_P(1-\theta)\right]\phi_{P,o} \\ & - S_P\theta\phi_A - S_P(1-\theta)\phi_{A,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi \, dt \end{aligned}$$

1.2.0.3 Dirichlet Eastern Boundary Conditions

The expression for the interior cells before the declaration of definitions of a_W, a_E and substitution is shown below,

$$\begin{aligned} \left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{EP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] =& \frac{\Gamma}{\delta x_{EP}}[\theta\phi_E + (1-\theta)\phi_{E,o}] \\ & + \left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{EP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}] \\ & + \frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi \, dt \end{aligned}$$

Much like in determining the eastern boundary condition, one-sided finite difference is employed to find the approximate derivative of ϕ with respect to displacement x at the boundary. For the eastern boundary, the unknown boundary would be at the east face of the last cell. The change of variables would then be,

$$\delta x_{EP} \rightarrow \delta x_{eP} \quad , \quad \phi_E = \phi_B \quad , \quad \phi_{E,o} = \phi_{B,o}$$

The first change of variable is to accomodate for the one-sided finite difference wherein the distance between the node point P and the eastern face is δx_{eP} instead of the previous case wherein the distance between the node point P and the eastern node is δx_{EP}. The second change of variable is to accomodate for the boundary condition wherein the value of ϕ at the right face of the last cell is ϕ_B. ϕ_B represents the value of ϕ at the new time step meanwhile $\phi_{B,o}$ represents the value of ϕ at the old time step at the right boundary of the last cell. Substituting these change of variables to modify the interior cell formulation for the right boundary condition,

$$\begin{aligned} \left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{eP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] =& \frac{\Gamma}{\delta x_{eP}}[\theta\phi_B + (1-\theta)\phi_{B,o}] \\ & + \left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{eP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}] \\ & + \frac{\Gamma}{\delta x_{PW}}[\theta\phi_W + (1-\theta)\phi_{W,o}] + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi \, dt \end{aligned}$$

$$\begin{aligned} \left[\frac{\rho\delta_{ew}}{\Delta t} + \frac{\Gamma}{\delta x_{eP}}\theta + \frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P] =& \frac{\Gamma}{\delta x_{eP}}[\theta\phi_B + (1-\theta)\phi_{B,o}] \\ & + \left[\frac{\rho\delta_{ew}}{\Delta t} - \frac{\Gamma}{\delta x_{eP}}(1-\theta) - \frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}] \\ & + \frac{\Gamma}{\delta x_{PW}}\theta\phi_W + \frac{\Gamma}{\delta x_{PW}}(1-\theta)\phi_{W,o} + \frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi \, dt \end{aligned}$$

Placing the unknown variables at LHS and the known variables at RHS,

$$\begin{aligned}\left[\frac{\rho\delta_{ew}}{\Delta t}+\frac{\Gamma}{\delta x_{eP}}\theta+\frac{\Gamma}{\delta x_{PW}}\theta\right][\phi_P]-\frac{\Gamma}{\delta x_{PW}}\theta\phi_W =& \frac{\Gamma}{\delta x_{eP}}[\theta\phi_B+(1-\theta)\phi_{B,o}]\\ &+\left[\frac{\rho\delta_{ew}}{\Delta t}-\frac{\Gamma}{\delta x_{eP}}(1-\theta)-\frac{\Gamma}{\delta x_{PW}}(1-\theta)\right][\phi_{P,o}]\\ &+\frac{\Gamma}{\delta x_{PW}}(1-\theta)\phi_{W,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\,dt\end{aligned}$$

Let

$$a_W=\frac{\Gamma}{\delta x_{PW}}\quad,\quad a_E=0\quad,\quad a_{P,0}=\frac{\rho\delta_{ew}}{\Delta t}\quad,\quad S_P=-\frac{\Gamma}{\delta x_{eP}}\quad,\quad a_P=a_{P,0}+\theta(a_E+a_W-S_P)$$

$$\begin{aligned}-a_E\theta\phi_E+a_P\phi_P-a_W\theta\phi_W =& -S_P[\theta\phi_B+(1-\theta)\phi_{B,o}]+\left[a_{P,0}+S_P(1-\theta)-a_W(1-\theta)\right][\phi_{P,o}]\\ &+a_W(1-\theta)\phi_{W,o}+\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi\,dt\end{aligned}$$

1.2.1 Part c

The workings in the previous section is performed with the arbitray scalar function ϕ, and the arbitrary variables ρ and Γ. The source term $\int_t^{t+\Delta t} S_\phi\,dt$ has not been specified. In this section, the arbitrary variables will be mapped to the problem variables to apply all the formulations to the fortran program.

Geometrically, the variable x could be re-interpreted in this problem to be vertical displacement y. This would set the western boundary condition of the arbitray ϕ case to be the bottom boundary condition in the momentum diffusion problem. The eastern boundary condition of the arbitrary ϕ case would be set to the top boundary condition in the momentum diffusion problem. Let positive direction of y then also be interpreted going "up" in the vertical direction. The top wall of the channel would be at positive H y-value. The unsteady momentum governing equation for this problem is shown below,

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

wherein ρ represents density and ν represents viscosity. The unsteady diffusion equation that is the basis of the previous formulation is shown below,

$$\frac{\partial}{\partial t}(\rho\phi)=\nabla\cdot(\Gamma\nabla\phi)+S_\phi$$

Under the 1-dimensional assumption and uniformity of ρ as well as Γ,

$$\rho\frac{\partial}{\partial t}(\phi)=\nabla\cdot(\Gamma\nabla\phi)+S_\phi=\frac{\partial}{\partial x}\left[\Gamma\frac{\partial}{\partial x}(\phi)\right]+S_\phi=\Gamma\frac{\partial^2}{\partial x^2}(\phi)+S_\phi$$

$$\frac{\partial}{\partial t}(\phi)=\frac{\Gamma}{\rho}\frac{\partial^2}{\partial x^2}(\phi)+\frac{S_\phi}{\rho}$$

By comparing the differential forms of the unsteady diffusion equation and unsteady momentum governing equation,

$$\phi = u \quad , \quad x = y \quad , \quad \frac{\Gamma}{\rho_\phi} = \nu \quad , \quad \frac{S_\phi}{\rho_\phi} = -\frac{1}{\rho_{prob}}\frac{\partial p}{\partial x}$$

Distinction has been made between ρ_ϕ and ρ_{prob}. ρ_ϕ represents the value of the arbitrary variable ρ in the general unsteady diffusion equation meanwhile ρ_{prob} represents the fluid density in the momentum diffusion problem. Since ϕ is a sclar field, and velocity u is typically a sclar, here u is assumed to be the magnitude of the horizontal velocity of the fluid, so that it could be represented by ϕ. In the momentum diffusion problem, the density of the fluid is assumed to remain constant and the derivative of pressure along the horizontal distance of the channel is assumed to be constant. Therefore, S_ϕ must also be a constant,

$$\frac{S_\phi}{\rho_\phi} = -\frac{1}{\rho_{prob}}\frac{\partial p}{\partial x}$$

$$S_\phi = -\frac{\rho_\phi}{\rho_{prob}}\frac{\partial p}{\partial x}$$

Given that S_ϕ is some constant that could be determined by the momentum diffusion problem definition,

$$\frac{\delta_{ew}}{\Delta t}\int_t^{t+\Delta t} S_\phi \, dt = \frac{\delta_{ew}}{\Delta t} S_\phi \, [t]_t^{t+\Delta t} = \frac{\delta_{ew}}{\Delta t} S_\phi \left[(t+\Delta t) - (t)\right] = \frac{\delta_{ew}}{\Delta t} S_\phi \Delta t = \delta_{ew} S_\phi$$

There exists a non-unique way to choose the variable mappings for ρ_ϕ, Γ. For a simple implementation of the general diffusion problem into the momentum diffusion, let $\rho_\phi = 1$. Therefore,

$$\phi = u \quad , \quad x = y \quad , \quad \Gamma = \nu \quad , \quad S_\phi = -\frac{1}{\rho_{prob}}\frac{\partial p}{\partial x}$$

The code was modified to simulate unsteady diffusion of momentum and given a few additional features. The modified fortran code is shown below,

1.3 Fourier Stability Analysis

1.3.1 Discretization of Wave Equation

The one-dimensional wave equation is shown below,

$$0 = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x}$$

Integrating with time and space,

$$0 = \int_t^{t+\Delta t}\int_{CV}\frac{\partial \phi}{\partial t}\, dV\, dt + u\int_t^{t+\Delta t}\int_{CV}\frac{\partial \phi}{\partial x}\, dV\, dt$$

Switching the order of integration in order to form exact differentials,

$$0 = \int_{CV} \int_{t}^{t+\Delta t} \frac{\partial \phi}{\partial t} \, dt \, dV + u \int_{t}^{t+\Delta t} \int_{CV} \frac{\partial \phi}{\partial x} \, dV \, dt$$

$$0 = \int_{CV} [\phi]_{t}^{t+\Delta t} \, dV + u \int_{t}^{t+\Delta t} A \, [\phi]_{CV} \, dt$$

$$0 = [\phi]_{t}^{t+\Delta t} \, \Delta V + u \int_{t}^{t+\Delta t} A\phi_e - A\phi_w \, dt$$

Substituting for the infinitesmially small volume $\Delta V = Adx$,

$$0 = [\phi]_{t}^{t+\Delta t} \, Adx + u \int_{t}^{t+\Delta t} A\phi_e - A\phi_w \, dt$$

Assuming uniform area throughout the grids,

$$0 = [\phi]_{t}^{t+\Delta t} \, dx + u \int_{t}^{t+\Delta t} \phi_e - \phi_w \, dt$$

Assuming that the value of ϕ is constant within a single cell,

$$0 = \left[\phi_P - \phi_{P,o}\right] dx + u \int_{t}^{t+\Delta t} \phi_e - \phi_w \, dt$$

Using the generalized time integration scheme,

$$\int_{t}^{t+\Delta t} \phi \, dt = [\theta \phi_{t+\Delta t} + (1-\theta)\phi_t] \Delta t$$

Substituting the time integration scheme,

$$0 = \left[\phi_P - \phi_{P,o}\right] dx + u \left\{\theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}\right\} dt$$

Factoring for the CFL number,

$$0 = \left[\phi_P - \phi_{P,o}\right] \frac{dx}{u \, dt} + \left\{\theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}\right\}$$

$$0 = \left[\phi_P - \phi_{P,o}\right] \frac{\Delta x}{u \, \Delta t} + \theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}$$

Let

$$\gamma = \frac{1}{CFL} = \frac{\Delta x}{u \, \Delta t}$$

$$0 = \gamma\phi_P - \gamma\phi_{P,o} + \theta\phi_e + (1-\theta)\phi_{e,o} - \theta\phi_w - (1-\theta)\phi_{w,o}$$

Placing the new time step quantities on LHS and the old time step quantities on RHS,

$$-\gamma\phi_P + \theta\phi_w - \theta\phi_e = -\gamma\phi_{P,o} + (1-\theta)\phi_{e,o} - (1-\theta)\phi_{w,o}$$

1.3.2 Cubic Polynomial Spatial Scheme

The discretization of the pure convection problem with a generalized time advancement scheme and a cubic polynomial spatial advancement scheme is shown below,

$$
\begin{aligned}
0 = \frac{\phi_P - \phi_{P,o}}{\Delta t} + \frac{u}{\Delta x}\Bigg[&\theta\left(\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E\right) \\
&+ (1-\theta)\left(\frac{1}{16}\phi_{WW,o} - \frac{5}{16}\phi_{W,o} + \frac{15}{16}\phi_{P,o} + \frac{5}{16}\phi_{E,o}\right) \\
&- \theta\left(\frac{1}{16}\phi_{WWW} - \frac{5}{16}\phi_{WW} + \frac{15}{16}\phi_W + \frac{5}{16}\phi_P\right) \\
&-(1-\theta)\left(\frac{1}{16}\phi_{WWW,o} - \frac{5}{16}\phi_{WW,o} + \frac{15}{16}\phi_{W,o} + \frac{5}{16}\phi_{P,o}\right)\Bigg]
\end{aligned}
$$

Assuming that the solution $\phi(x,t)$ is a product of a time component $\hat{\phi}(t)$ and a spatial component e^{ikx}, the solution $\phi(x,t)$ could be rewritten as,

$$\phi(x,t) = \hat{\phi}(t)e^{ikx}$$

Points at the future time step would be given the time component shown below,

$$\hat{\phi}(t+\Delta t) = \hat{\phi}$$

Points at the old time step would be given the time component shown below,

$$\hat{\phi}(t) = \hat{\phi}_o$$

Following convention, the coordinate system considers eastern direction of the grid points to be positive x and western direction of the grid points to be negative x. The point of interest P would be given a spatial component e^{ikx}. Following the coordinate system definition, the west point would be given spatial component $e^{ik(x-\Delta x)}$. The east point would be given spatial component $e^{ik(x+\Delta x)}$. The diagram below summarizes the coordinate system used in this derivation,Uniform grid spacing is assumed in the following derivation. Parsing the original expression into several parts. For the discrete derivative of ϕ with respect to time,

$$\frac{\phi_P - \phi_{P,o}}{\Delta t} = \frac{1}{\Delta t}\left[\phi_P - \phi_{P,o}\right] = \frac{1}{\Delta t}\left[\hat{\phi}e^{ikx} - \hat{\phi}_o e^{ikx}\right] = \frac{e^{ikx}}{\Delta t}\left[\hat{\phi} - \hat{\phi}_o\right]$$

For the first part of RHS,

$$
\begin{aligned}
\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E =& \frac{1}{16}\hat{\phi}e^{ik(x-2\Delta x)} - \frac{5}{16}\hat{\phi}e^{ik(x-\Delta x)} \\
&+ \frac{15}{16}\hat{\phi}e^{ikx} + \frac{5}{16}\hat{\phi}e^{ik(x+\Delta x)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{16}\phi_{WW} - \frac{5}{16}\phi_W + \frac{15}{16}\phi_P + \frac{5}{16}\phi_E =& \frac{1}{16}\hat{\phi}e^{ikx-2ik\Delta x} - \frac{5}{16}\hat{\phi}e^{ikx-ik\Delta x} \\
&+ \frac{15}{16}\hat{\phi}e^{ikx} + \frac{5}{16}\hat{\phi}e^{ikx+ik\Delta x}
\end{aligned}
$$

$$\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_W+\frac{15}{16}\phi_P+\frac{5}{16}\phi_E=\frac{1}{16}\hat{\phi}e^{ikx}e^{-2ik\Delta x}-\frac{5}{16}\hat{\phi}e^{ikx}e^{-ik\Delta x}+\frac{15}{16}\hat{\phi}e^{ikx}+\frac{5}{16}\hat{\phi}e^{ikx}e^{ik\Delta x}$$

$$\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_W+\frac{15}{16}\phi_P+\frac{5}{16}\phi_E=\hat{\phi}e^{ikx}\left[\frac{1}{16}e^{-2ik\Delta x}-\frac{5}{16}e^{-ik\Delta x}+\frac{15}{16}+\frac{5}{16}e^{ik\Delta x}\right]$$

Performing similar opeartions for the second part of RHS,

$$\frac{1}{16}\phi_{WW,o}-\frac{5}{16}\phi_{W,o}+\frac{15}{16}\phi_{P,o}+\frac{5}{16}\phi_{E,o}=\hat{\phi}_oe^{ikx}\left[\frac{1}{16}e^{-2ik\Delta x}-\frac{5}{16}e^{-ik\Delta x}+\frac{15}{16}+\frac{5}{16}e^{ik\Delta x}\right]$$

Performing similar opeartions for the third part of RHS,

$$\frac{1}{16}\phi_{WWW}-\frac{5}{16}\phi_{WW}+\frac{15}{16}\phi_W+\frac{5}{16}\phi_P=\hat{\phi}e^{ikx}\left[\frac{1}{16}e^{-3ik\Delta x}-\frac{5}{16}e^{-2ik\Delta x}+\frac{15}{16}e^{-ik\Delta x}+\frac{5}{16}\right]$$

Performing similar opeartions for the fourth part of RHS,

$$\frac{1}{16}\phi_{WWW,o}-\frac{5}{16}\phi_{WW,o}+\frac{15}{16}\phi_{W,o}+\frac{5}{16}\phi_{P,o}=\hat{\phi}_oe^{ikx}\left[\frac{1}{16}e^{-3ik\Delta x}-\frac{5}{16}e^{-2ik\Delta x}+\frac{15}{16}e^{-ik\Delta x}+\frac{5}{16}\right]$$

Let

$$A=\frac{1}{16}e^{-2ik\Delta x}-\frac{5}{16}e^{-ik\Delta x}+\frac{15}{16}+\frac{5}{16}e^{ik\Delta x}\quad,\quad B=\frac{1}{16}e^{-3ik\Delta x}-\frac{5}{16}e^{-2ik\Delta x}+\frac{15}{16}e^{-ik\Delta x}+\frac{5}{16}$$

Substituting for A and B shortens the expressions,

$$\frac{1}{16}\phi_{WW}-\frac{5}{16}\phi_W+\frac{15}{16}\phi_P+\frac{5}{16}\phi_E=\hat{\phi}e^{ikx}A$$

$$\frac{1}{16}\phi_{WW,o}-\frac{5}{16}\phi_{W,o}+\frac{15}{16}\phi_{P,o}+\frac{5}{16}\phi_{E,o}=\hat{\phi}_oe^{ikx}A$$

$$\frac{1}{16}\phi_{WWW}-\frac{5}{16}\phi_{WW}+\frac{15}{16}\phi_W+\frac{5}{16}\phi_P=\hat{\phi}e^{ikx}B$$

$$\frac{1}{16}\phi_{WWW,o}-\frac{5}{16}\phi_{WW,o}+\frac{15}{16}\phi_{W,o}+\frac{5}{16}\phi_{P,o}=\hat{\phi}_oe^{ikx}B$$

Susbtituting the simplified parts into the original expression,

$$0=\frac{e^{ikx}}{\Delta t}\left[\hat{\phi}-\hat{\phi}_o\right]+\frac{u}{\Delta x}\left[\theta\left(\hat{\phi}e^{ikx}A\right)+(1-\theta)\left(\hat{\phi}_oe^{ikx}A\right)-\theta\left(\hat{\phi}e^{ikx}B\right)-(1-\theta)\left(\hat{\phi}_oe^{ikx}B\right)\right]$$

$$0=\frac{e^{ikx}}{\Delta t}\left[\hat{\phi}-\hat{\phi}_o\right]+\frac{u}{\Delta x}\left[\theta\hat{\phi}e^{ikx}A+(1-\theta)\hat{\phi}_oe^{ikx}A-\theta\hat{\phi}e^{ikx}B-(1-\theta)\hat{\phi}_oe^{ikx}B\right]$$

Since $e^{ikx}\neq 0$,

$$0=\frac{1}{\Delta t}\left[\hat{\phi}-\hat{\phi}_o\right]+\frac{u}{\Delta x}\left[\theta\hat{\phi}A+(1-\theta)\hat{\phi}_oA-\theta\hat{\phi}B-(1-\theta)\hat{\phi}_oB\right]$$

$$0=\frac{\Delta x}{u\,\Delta t}\left[\hat{\phi}-\hat{\phi}_o\right]+\theta\hat{\phi}A+(1-\theta)\hat{\phi}_oA-\theta\hat{\phi}B-(1-\theta)\hat{\phi}_oB$$

Let

$$\gamma = \frac{1}{CFL} = \frac{\Delta x}{u\,\Delta t}$$

Substituting for the parameter γ,

$$0 = \gamma\hat{\phi} - \gamma\hat{\phi}_o + \theta\hat{\phi}A + (1-\theta)\hat{\phi}_oA - \theta\hat{\phi}B - (1-\theta)\hat{\phi}_oB$$

$$\gamma\hat{\phi}_o - (1-\theta)\hat{\phi}_oA + (1-\theta)\hat{\phi}_oB = \gamma\hat{\phi} + \theta\hat{\phi}A - \theta\hat{\phi}B$$

$$\hat{\phi}\left[\gamma + \theta A - \theta B\right] = \hat{\phi}_o\left[\gamma - (1-\theta)A + (1-\theta)B\right]$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left[\gamma - (1-\theta)A + (1-\theta)B\right]}{\left[\gamma + \theta A - \theta B\right]} = \frac{\gamma + (\theta-1)A + (1-\theta)B}{\gamma + \theta A - \theta B} = \frac{\gamma + \theta A - A + B - \theta B}{\gamma + \theta A - \theta B}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - A + B + \theta A - \theta B}{\gamma + \theta A - \theta B} = \frac{\gamma - (A-B) + \theta(A-B)}{\gamma + \theta(A-B)}$$

Let $\alpha = A - B$. Simplifying,

$$\alpha = \frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x} - \left(\frac{1}{16}e^{-3ik\Delta x} - \frac{5}{16}e^{-2ik\Delta x} + \frac{15}{16}e^{-ik\Delta x} + \frac{5}{16}\right)$$

$$\alpha = \frac{1}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x} - \frac{1}{16}e^{-3ik\Delta x} + \frac{5}{16}e^{-2ik\Delta x} - \frac{15}{16}e^{-ik\Delta x} - \frac{5}{16}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \frac{1}{16}e^{-2ik\Delta x} + \frac{5}{16}e^{-2ik\Delta x} - \frac{5}{16}e^{-ik\Delta x} - \frac{15}{16}e^{-ik\Delta x} - \frac{5}{16} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \left[\frac{1}{16} + \frac{5}{16}\right]e^{-2ik\Delta x} - \left[\frac{5}{16} + \frac{15}{16}\right]e^{-ik\Delta x} - \frac{5}{16} + \frac{15}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \left[\frac{6}{16}\right]e^{-2ik\Delta x} - \left[\frac{20}{16}\right]e^{-ik\Delta x} + \frac{10}{16} + \frac{5}{16}e^{ik\Delta x}$$

$$\alpha = -\frac{1}{16}e^{-3ik\Delta x} + \frac{6}{16}e^{-2ik\Delta x} - \frac{20}{16}e^{-ik\Delta x} + \frac{10}{16} + \frac{5}{16}e^{ik\Delta x}$$

Substituting for α,

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma - \alpha + \theta\alpha}{\gamma + \theta\alpha}$$

1.3.3 Upwind Differencing Scheme

The theoretical section shows that discretizing the 1-dimensional wave equation yields the following expression,

$$-\gamma\phi_P + \theta\phi_w - \theta\phi_e = -\gamma\phi_{P,o} + (1-\theta)\phi_{e,o} - (1-\theta)\phi_{w,o}$$

Assuming upwing differencing,

$$\phi_w = \phi_W \quad , \quad \phi_e = \phi_P$$

Substituting for this,

$$-\gamma\phi_P + \theta\phi_W - \theta\phi_P = -\gamma\phi_{P,o} + (1-\theta)\phi_{P,o} - (1-\theta)\phi_{W,o}$$

Simplifying,

$$\phi_P\left[-\gamma-\theta\right] + \theta\phi_W = \phi_{P,o}\left[-\gamma+(1-\theta)\right] - (1-\theta)\phi_{W,o}$$

Substituting the Fourier representaiton of the solution, $\phi = \hat{\phi}e^{ikx}$,

$$\hat{\phi}e^{ikx}\left[-\gamma-\theta\right] + \theta\hat{\phi}e^{ik(x-\Delta x)} = \hat{\phi}_o e^{ikx}\left[-\gamma+(1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ik(x-\Delta x)}$$

$$\hat{\phi}e^{ikx}\left[-\gamma-\theta\right] + \theta\hat{\phi}e^{ikx-ik\Delta x} = \hat{\phi}_o e^{ikx}\left[-\gamma+(1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ikx-ik\Delta x}$$

$$\hat{\phi}e^{ikx}\left[-\gamma-\theta\right] + \theta\hat{\phi}e^{ikx}e^{-ik\Delta x} = \hat{\phi}_o e^{ikx}\left[-\gamma+(1-\theta)\right] - (1-\theta)\hat{\phi}_o e^{ikx}e^{-ik\Delta x}$$

$$\hat{\phi}e^{ikx}\left\{\left[-\gamma-\theta\right] + \theta e^{-ik\Delta x}\right\} = \hat{\phi}_o e^{ikx}\left\{\left[-\gamma+(1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}$$

Since $e^{ikx} \neq 0$ for all values of x,

$$\hat{\phi}\left\{\left[-\gamma-\theta\right] + \theta e^{-ik\Delta x}\right\} = \hat{\phi}_o\left\{\left[-\gamma+(1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\left[-\gamma+(1-\theta)\right] - (1-\theta)e^{-ik\Delta x}\right\}}{\left\{\left[-\gamma-\theta\right] + \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\left[\gamma-(1-\theta)\right] + (1-\theta)e^{-ik\Delta x}\right\}}{\left\{\left[\gamma+\theta\right] - \theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\left\{\gamma-1+\theta+(1-\theta)e^{-ik\Delta x}\right\}}{\left\{\gamma+\theta-\theta e^{-ik\Delta x}\right\}}$$

$$\frac{\hat{\phi}}{\hat{\phi}_o} = \frac{\gamma-1+\theta+(1-\theta)e^{-ik\Delta x}}{\gamma+\theta\left(1-e^{-ik\Delta x}\right)}$$

The amplification factor for the polynomial interpolation as well as the upwind differencing is implemented into the Matlab script. The matlab script was modified to accomodate multiple plots in a single figure and write the maximum amplification factor to a file. The matlab script is shown below,