Chapter 1

Laplace Transform

1.1 Definition of Laplace Transform

Laplace Transform is defined as the following,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

The Laplace Transform is a linear transform since the integral and product operations are both linear operations as well.

$$\mathcal{L}[\alpha f(t)] = \alpha \mathcal{L}[f(t)] \quad , \quad \mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$$

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

wherein α, β represent constants and $f(t)$, $g(t)$ represent functions of t.

1.2 Transforms of Derivatives

In General form,

$$\mathcal{L}[\overset{n}{f}(t)] = s^n \mathcal{L}[f(t)] - \sum_{k=0}^{n-1} \left[s^{n-1-k} \overset{k}{f}(0) \right]$$

wherein $\overset{n}{f}(t)$ represents the n^{th} derivative of the function $f(t)$. Proof is shown below,

$$\mathcal{L}[\overset{n}{f}(t)] = \int_0^\infty e^{-st} \overset{n}{f}(t)\, dt$$

$$\int uv'\,dt = uv - \int u'v\,dt$$

$$u = e^{-st} \quad , \quad u' = -se^{-st} \quad , \quad v' = \overset{n}{f}(t) \quad , \quad v = \overset{n-1}{f}(t)$$

$$\mathcal{L}[\overset{n}{f}(t)] = \int_0^\infty e^{-st}\overset{n}{f}(t)dt = \left[e^{-st}\overset{n-1}{f}(t)\right]_0^\infty - \int_0^\infty -se^{-st}\overset{n-1}{f}(t)\,dt = -\overset{n-1}{f}(0) + s\int_0^\infty e^{-st}\overset{n-1}{f}(t)\,dt$$

Generalizing for the second integral term,

$$\int_0^\infty e^{-st}\overset{n-i}{f}(t)\,dt = \left[e^{-st}\overset{n-1-i}{f}(t)\right]_0^\infty + s\int_0^\infty e^{-st}\overset{n-1-i}{f}(t)\,dt = -\overset{n-1-i}{f}(0) + s\int_0^\infty e^{-st}\overset{n-1-i}{f}(t)\,dt$$

By applying substitution recursively,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{k}\left[s^i\overset{n-1-i}{f}(0)\right] + \prod_{i=0}^{k}[s]\int_0^\infty e^{-st}\overset{n-1-k}{f}(t)\,dt = -\sum_{i=0}^{k}\left[s^i\overset{n-1-i}{f}(0)\right] + s^{k+1}\int_0^\infty e^{-st}\overset{n-1-k}{f}(t)\,dt$$

Substituting the value for $k = n-1$,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{n-1}\left[s^i\overset{n-1-i}{f}(0)\right] + s^n\int_0^\infty e^{-st}f(t)\,dt$$

A few things should be noted,

$$\int_0^\infty e^{-st}f(t)\,dt = \mathcal{L}[f(t)] \quad , \quad \sum_{i=0}^{n-1}\left[s^i\overset{n-1-i}{f}(0)\right] = \sum_{i=0}^{n-1}\left[s^{n-1-i}\overset{i}{f}(0)\right]$$

By substitution of the counting variable i with k,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{n-1}\left[s^i\overset{n-1-i}{f}(0)\right] + s^n\int_0^\infty e^{-st}f(t)\,dt = s^n\mathcal{L}[f(t)] + \sum_{k=0}^{n-1}\left[s^{n-1-k}\overset{k}{f}(0)\right]$$

1.3 Transforms of Integrals

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}[f(t)]$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt$$

$$\int uv'\,dt = uv - \int u'v\,dt$$

$$u = \int_0^t f(\tau)d\tau \quad , \quad u' = f(t) \quad , \quad v' = e^{-st} \quad , \quad v = -\frac{1}{s}e^{-st}$$

$$\int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt = -\left[\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\right]_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}f(t)\,dt$$

It should be noted that since the function $f(t)$ is in exponential order,

$$\left[\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\right]_0^\infty = 0$$

Substituting the uv term with zero,

$$\int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt = \int_0^\infty \frac{1}{s}e^{-st}f(t)\,dt = \frac{1}{s}\int_0^\infty e^{-st}f(t)\,dt = \frac{1}{s}\mathcal{L}[f(t)]$$

1.4 Derivative of Transforms

$$\mathcal{L}[t^n f(t)] = (-1)^n \overset{n}{F}(s) = (-1)^n \frac{d^n}{ds^n}\left\{\mathcal{L}[f(t)]\right\}$$

wherein $F(s)$ represents the laplace transform of the function $f(t)$. By the definition of Laplace Transforms discussed earlier,

$$\mathcal{L}[t^n f(t)] = \int_0^\infty e^{-st}t^n f(t)\,dt \quad , \quad F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt$$

Differentiating the Laplace Transform of $f(t)$ with respect to s iteratively n times,

$$\overset{n}{F}(s) = \frac{d^n}{ds^n}\{\mathcal{L}[f(t)]\} = \frac{d^n}{ds^n}\int_0^\infty e^{-st}f(t)\,dt = (-1)^n\int_0^\infty e^{-st}t^n f(t)\,dt$$

$$(-1)^n\overset{n}{F}(s) = \int_0^\infty e^{-st}t^n f(t)\,dt$$

By substituting the definition for the Laplace Transform of $t^n f(t)$,

$$(-1)^n\overset{n}{F}(s) = \mathcal{L}[t^n f(t)]$$

1.5 Integration of Transforms

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tau)\,d\tau$$

wherein $F(s)$ represents the laplace transform of the function $f(t)$.

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_0^\infty \frac{e^{-st}f(t)}{t}\,dt \quad , \quad F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt$$

$$\int_s^\infty F(\tau)\,d\tau = \int_s^\infty \int_0^\infty e^{-\tau t}f(t)\,dt\,d\tau = \int_0^\infty \int_s^\infty e^{-\tau t}f(t)\,d\tau\,dt = \int_0^\infty \left[-\frac{1}{t}e^{-\tau t}f(t)\right]_{\tau=s}^{\tau=\infty} dt$$

$$\int_s^\infty F(\tau)\,d\tau = -\int_0^\infty \frac{f(t)}{t}\left[e^{-\tau t}\right]_{\tau=s}^{\tau=\infty} dt = -\int_0^\infty \frac{f(t)}{t}\left[\lim_{\tau\to\infty}(e^{-\tau t}) - e^{-st}\right] dt$$

Taking into account that, $\lim_{\tau\to\infty}(e^{-\tau t}) = 0$,

$$\int_s^\infty F(\tau)\,d\tau = -\int_0^\infty \frac{f(t)}{t}\left[-e^{-st}\right] dt = \int_0^\infty \frac{e^{-st}f(t)}{t}\,dt$$

By substituting the definition of the laplace transform of $\dfrac{f(t)}{t}$,

$$\int_s^\infty F(\tau)\,d\tau = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

1.6 Translation of Transforms

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\mathcal{L}\left[f(t+c)\right]$$

By definition of Laplace Transform,

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_0^\infty e^{-st}u(t-c)f(t)\,dt = \int_c^\infty e^{-st}f(t)\,dt + \int_0^c e^{-st}\times 0\,dt$$

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_c^\infty e^{-st}f(t)\,dt$$

Using the substitution $t = \tau + c$. When $t = \infty \quad , \quad \tau = \infty$ and when $t = c \quad , \quad \tau = 0$. Therefore,

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_{t=c}^{t=\infty} e^{-s(\tau+c)}f(\tau+c)\,dt = \int_{\tau=0}^{\tau=\infty} e^{-s(\tau+c)}f(\tau+c)\,d\tau$$

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_{\tau=0}^{\tau=\infty} e^{-s\tau-sc} f(\tau+c)\,d\tau = \int_{\tau=0}^{\tau=\infty} e^{-cs}e^{-s\tau} f(\tau+c)\,d\tau$$

Since variables s and c are not changing with time, the term e^{-cs} could be treated as some form of constant. Therefore,

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\int_0^\infty e^{-s\tau} f(\tau+c)\,d\tau$$

It should be noted that the change of variables allows,

$$\mathcal{L}\left[f(t+c)\right] = \int_0^\infty e^{-st} f(t+c)\,dt = \int_0^\infty e^{-s\tau} f(\tau+c)\,d\tau$$

By substitution,

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\mathcal{L}\left[f(t+c)\right]$$

1.7 Transforms of Translated Functions

$$\mathcal{L}[e^{ct}f(t)] = F(s-c)$$

Reiterating the definition of laplace transforms,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{ct}e^{-st} f(t)\,dt = \int_0^\infty e^{-st+ct} f(t)\,dt$$

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{-(s-c)t} f(t)\,dt = F(s-c)$$

1.8 Convolution

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

The convolution is a commutative transformation. Therefore,

$$f * g(t) = g * f(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

One useful property of the convolution function,

$$\mathcal{L}\left[f * g(t)\right] = \mathcal{L}\left[f(t)\right] \times \mathcal{L}\left[g(t)\right]$$

wherein

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \quad , \quad G(s) = \mathcal{L}[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

By a substitution of variables $t = u$ it could be re-written,

$$F(s) = \mathcal{L}[f(u)] = \int_0^\infty e^{-su} f(u)\,du \quad , \quad G(s) = \mathcal{L}[g(u)] = \int_0^\infty e^{-su} g(u)\,du$$

Examining the Lapalce Transform of $g(u)$, and making the substitution $u = t - \tau$

$$\mathcal{L}[g(t-\tau)] = \int_{u=0}^{u=\infty} e^{-s(t-\tau)} g(t-\tau)\,dt$$

When $u = \infty \quad , \quad t = \infty$ and when $u = 0 \quad , \quad t = \tau$. Therefore,

$$\mathcal{L}[g(t-\tau)] = \int_{t=\tau}^{t=\infty} e^{-s(t-\tau)} g(t-\tau)\,dt$$

The $e^{\tau s}$ term could be isolated because both variables τ and s in this case are non-changing with t. The next form is identical to the laplace transform at the Translation of Transforms section,

$$\int_{\tau=0}^{\tau=\infty} e^{-s(\tau+c)} f(\tau+c)\,d\tau = e^{-cs} \int_0^\infty e^{-s\tau} f(\tau+c)\,d\tau$$

By substituting τ in the Translation of Transforms section with t, substituting c with $-\tau$, and substituting the arbitrary function g with the arbitrary function f,

$$\int_{t=0}^{t=\infty} e^{-s(t-\tau)} g(t-\tau)\,dt = e^{\tau s} \int_0^\infty e^{-st} g(t-\tau)\,dt$$

Therefore,

$$\mathcal{L}[g(t-\tau)] = G(s) = e^{\tau s} \int_0^\infty e^{-st} g(t-\tau)\,dt$$

Proving the Convolution Property by first examining the product of the two Laplace Transforms,

$$F(s) \times G(s) = G(s) \int_0^\infty e^{-su} f(u)\,du = \int_0^\infty e^{-su} G(s) f(u)\,du$$

The above would be perfectly legal operations because $G(s)$ is a function in terms of s and is unchanging with respect to variable t. Therefore, the function

$G(s)$ could be treated as a constant that can be place inside and outside of the integral.

$$F(s) \times G(s) = \int_0^\infty e^{-s\tau} f(\tau) \times e^{\tau s} \int_0^\infty e^{-st} g(t-\tau)\, dt\, d\tau$$

$$F(s) \times G(s) = \int_0^\infty \int_0^\infty e^{-st} f(\tau) g(t-\tau)\, dt\, d\tau$$

By chaging the order of integration,

$$F(s) \times G(s) = \int_0^\infty e^{-st} \int_0^\infty f(\tau) g(t-\tau)\, d\tau\, dt = \mathcal{L}\left[\int_0^\infty f(\tau) g(t-\tau)\, d\tau\right]$$

$$F(s) \times G(s) = \mathcal{L}\left[f * g(t)\right]$$