Chapter 1

Reynold's Transport Theorem

One variation of Liebniz Rule applicable for volumetric integrals is shown below. for the variable T wherein T may represent a time dependent scalar, vector, or tensor.

$$\frac{d}{dt}\iiint_{R(t)} T\,dV_o = \iiint_{R(t)} \frac{\partial}{\partial t}[T]dV_o + \iint_{S(t)} T\bar{v}_s\bar{n}dS$$

wherein $R(t)$ represents an arbitray region of space, V_o represents volume, $S(t)$ represents the surface of the region defined by $R(t)$, $\bar{v}_s$ represents the velocity of the moving surface, $\bar{n}$ represents normal vector of the surface. Depending on the variable type T, the operation $T\bar{v}_s\bar{n}$ would depend on a case to case basis.

1.1 Substantive Derivative

Suppose a quantity b is dependent on the the variable time t and the typical cartesian coordinates x, y, z. Taking the derivative of variable b with respect to time yields the following based on chain rule,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \frac{\partial}{\partial x}[b] \times \frac{\partial}{\partial t}[x] + \frac{\partial}{\partial y}[b] \times \frac{\partial}{\partial t}[y] + \frac{\partial}{\partial z}[b] \times \frac{\partial}{\partial t}[z]$$

Taking note that the partial derivatives of the cartesian coordinates defines velocity in the cartesian coordinates. Therefore,

$$\frac{\partial}{\partial t}[x] = u \quad , \quad \frac{\partial}{\partial t}[y] = v \quad , \quad \frac{\partial}{\partial t}[z] = w$$

wherein u, v, and w typically represents velocity in the x, y, and z directions respectively. Therefore, the derivative of y with respect to time t would take the form,

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

If the ∇ operator is defined as

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T$$

Therefore, the derivative of y with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + u\frac{\partial}{\partial x}[b] + v\frac{\partial}{\partial y}[b] + w\frac{\partial}{\partial z}[b]$$

Let the velocity vector be defined as

$$\bar{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

It follows that the derivative of b with respect to time t would take the form

$$\frac{d}{dt}[b] = \frac{\partial}{\partial t}[b] + \bar{v} \cdot \nabla b$$

1.2 Divergence Theorem

The Divergence Theorem is stated below. The variable $\bar{F}$ must represent a vector in R^3

$$\iiint_{R(t)} \nabla \cdot \bar{F} \, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

Alternately,

$$\iiint_{R(t)} \frac{\partial}{\partial x}[\bar{F}] + \frac{\partial}{\partial y}[\bar{F}] + \frac{\partial}{\partial z}[\bar{F}] \, dV_o = \iint_{S(t)} \bar{F} \cdot \bar{n} dS$$

wherein dV_o represents an infinitesmially small volume. $S(t)$ is the surface encapsulating the region $R(t)$. $\bar{n}$ is the normal vector of the control volume, and dS is an infinitesmial area of surface $S(t)$.