0.1 Grid Transformations

0.1.1 Transformation Definitions

Let the grid points in physical space have its positions be expressed in x, y, and z. Let the grid points transformed into a uniform mesh have its positione expressed in ξ, η, and ζ coordinates. The transformation is assumed to be smooth and invertible. Invertibility implies that each grid point in x, y, and z have a single corresponding point in ξ, η, and ζ and vice versa. Under the grid transformation,

$$x = x(\xi, \eta, \zeta) \quad , \quad y = y(\xi, \eta, \zeta) \quad , \quad z = z(\xi, \eta, \zeta)$$

Due to invertibility, the reverse would be,

$$\xi = \xi(x, y, z) \quad , \quad \eta = \eta(x, y, z) \quad , \quad \zeta = \zeta(x, y, z)$$

0.1.2 Transformation Properties

Based on chain rule for the curvilinear coordinates,

$$dx = x_\xi d\xi + x_\eta d\eta + x_\zeta d\zeta \quad , \quad dy = y_\xi d\xi + y_\eta d\eta + y_\zeta d\zeta \quad , \quad dz = z_\xi d\xi + z_\eta d\eta + z_\zeta d\zeta$$

Putting into matrix form,

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} \tag{1}$$

Based on chain rule for the uniform coordinates,

$$d\xi = \xi_x dx + \xi_y dy + \xi_z dz \quad , \quad d\eta = \eta_x dx + \eta_y dy + \eta_z dz \quad , \quad d\zeta = \zeta_x dx + \zeta_y dy + \zeta_z dz$$

Putting into matrix form,

$$\begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{2}$$

Substituting equation **??** into equation **??**,

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

The only way the expression above is true is if

$$\begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let

$$A_1 = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \quad , \quad A_2 = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix}$$

The determinant of the identity matrix is 1. Therefore,

$$det(A_1) \times det(A_2) = 1$$

$$det(A_1) = \frac{1}{det(A_2)}$$

Let the jacobian J be defined as,

$$J = det(A_2)$$

Therefore,

$$det(A_1) = \frac{1}{J}$$

$$\frac{1}{det(A_1)} = J$$

The matrix equation above is actually 3 systems of 3×3 linear systems. Parsing out for the partial derivatives in x,

$$\begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} \xi_x \\ \eta_x \\ \zeta_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Crammer rule could be used to solve the matrix above,

$$\xi_x = \frac{\begin{vmatrix} 1 & x_\eta & x_\zeta \\ 0 & y_\eta & y_\zeta \\ 0 & z_\eta & z_\zeta \end{vmatrix}}{\begin{vmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{vmatrix}} \begin{vmatrix} 1 & x_\eta & x_\zeta \\ 0 & y_\eta & y_\zeta \\ 0 & z_\eta & z_\zeta \end{vmatrix} = \frac{1}{det(A_1)} \begin{vmatrix} 1 & x_\eta & x_\zeta \\ 0 & y_\eta & y_\zeta \\ 0 & z_\eta & z_\zeta \end{vmatrix}$$

$$\xi_x = J(y_\eta z_\zeta - y_\zeta z_\eta)$$

Parsing out for the partial derivatives in y,

$$\begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} \xi_y \\ \eta_y \\ \zeta_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Parsing out for the partial derivatives in z,

$$\begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} \xi_z \\ \eta_z \\ \zeta_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

0.1.3 Curvilinear to Uniform Transformation

The Navier-Stokes equations can be expressed in the following vector form,

$$Q = \frac{\partial D}{\partial t} + \frac{\partial B}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial M}{\partial z} \tag{3}$$

wherein D, B, H and M are all column vectors with the same number of elements inside them. To solve the Navier Stokes equation in the uniform grid space ξ, η, and ζ, the derivatives in x, y, and z must be transformed. Using chain rule for derivative in x,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \times \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \times \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \zeta} \times \frac{\partial \zeta}{\partial x}$$

Since the basic multiplication operation is commutative, the expression above can be rewritten as,

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}$$

Using subscripts to imply partial derivative operations,

$$\frac{\partial}{\partial x} = \xi_x\frac{\partial}{\partial \xi} + \eta_x\frac{\partial}{\partial \eta} + \zeta_x\frac{\partial}{\partial \zeta}$$

Using chain rule for derivative in y, and applying the same treatments as for derivative in x,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial y}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}$$

$$\frac{\partial}{\partial y} = \xi_y\frac{\partial}{\partial \xi} + \eta_y\frac{\partial}{\partial \eta} + \zeta_y\frac{\partial}{\partial \zeta}$$

Using chain rule for derivative in z, and applying the same treatments as for derivative in x,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial z}$$

$$\frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}$$

$$\frac{\partial}{\partial z} = \xi_z\frac{\partial}{\partial \xi} + \eta_z\frac{\partial}{\partial \eta} + \zeta_z\frac{\partial}{\partial \zeta}$$

Substituting all of the partial derivatives into equation **??**

$$Q = \frac{\partial D}{\partial t} + \frac{\partial B}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial M}{\partial z}$$

$$Q = \frac{\partial D}{\partial t} + \xi_x\frac{\partial B}{\partial \xi} + \eta_x\frac{\partial B}{\partial \eta} + \zeta_x\frac{\partial B}{\partial \zeta}$$
$$+ \xi_y\frac{\partial H}{\partial \xi} + \eta_y\frac{\partial H}{\partial \eta} + \zeta_y\frac{\partial H}{\partial \zeta}$$
$$+ \xi_z\frac{\partial M}{\partial \xi} + \eta_z\frac{\partial M}{\partial \eta} + \zeta_z\frac{\partial M}{\partial \zeta}$$

$$Q = \frac{\partial D}{\partial t} + \xi_x \frac{\partial B}{\partial \xi} + \xi_y \frac{\partial H}{\partial \xi}$$
$$+ \eta_x \frac{\partial B}{\partial \eta} + \eta_y \frac{\partial H}{\partial \eta} + \eta_z \frac{\partial M}{\partial \eta} + \zeta_y \frac{\partial H}{\partial \zeta}$$
$$+ \xi_z \frac{\partial M}{\partial \xi} + \zeta_z \frac{\partial M}{\partial \zeta} + \zeta_x \frac{\partial B}{\partial \zeta}$$

$$Q = \frac{\partial D}{\partial t} + \xi_x \frac{\partial B}{\partial \xi} + \xi_y \frac{\partial H}{\partial \xi} + \xi_z \frac{\partial M}{\partial \xi}$$
$$+ \eta_x \frac{\partial B}{\partial \eta} + \eta_y \frac{\partial H}{\partial \eta} + \eta_z \frac{\partial M}{\partial \eta}$$
$$+ \zeta_y \frac{\partial H}{\partial \zeta} + \zeta_z \frac{\partial M}{\partial \zeta} + \zeta_x \frac{\partial B}{\partial \zeta}$$

$$Q = \frac{\partial D}{\partial t} + \xi_x \frac{\partial B}{\partial \xi} + \xi_y \frac{\partial H}{\partial \xi} + \xi_z \frac{\partial M}{\partial \xi}$$
$$+ \eta_x \frac{\partial B}{\partial \eta} + \eta_y \frac{\partial H}{\partial \eta} + \eta_z \frac{\partial M}{\partial \eta}$$
$$+ \zeta_x \frac{\partial B}{\partial \zeta} + \zeta_y \frac{\partial H}{\partial \zeta} + \zeta_z \frac{\partial M}{\partial \zeta}$$

Since ξ, η, and ζ are all in terms of x, y, and z, then the partial derivatives ξ_x, η_y and other similar terms must also be purely in terms of x, y, and z. Therefore,

$$Q = \frac{\partial D}{\partial t} + \frac{\partial}{\partial \xi} \left[\xi_x B + \xi_y H + \xi_z M \right]$$
$$+ \frac{\partial}{\partial \eta} \left[\eta_x B + \eta_y H + \eta_z M \right]$$
$$+ \frac{\partial}{\partial \zeta} \left[\zeta_x B + \zeta_y H + \zeta_z M \right]$$

The Jacobian J can be defined in terms of x, y, and z or using ξ, η, and ζ depending on whether the Jacobian is expressed using matrix A_1 or matrix A_2. Let J_{ax} be the Jacobian expressed in terms of x, y, and z.

$$\frac{Q}{J_{ax}} = \frac{\partial}{\partial t} \left[\frac{D}{J_{ax}} \right] + \frac{\partial}{\partial \xi} \left[\frac{1}{J_{ax}} \left(\xi_x B + \xi_y H + \xi_z M \right) \right]$$
$$+ \frac{\partial}{\partial \eta} \left[\frac{1}{J_{ax}} \left(\eta_x B + \eta_y H + \eta_z M \right) \right]$$
$$+ \frac{\partial}{\partial \zeta} \left[\frac{1}{J_{ax}} \left(\zeta_x B + \zeta_y H + \zeta_z M \right) \right]$$

The jacobian J_{ax} expressed in terms of x, y, and z, can "move" in and out of the partial derivative operations because the partial derivative operations are purely in terms of ξ, η and ζ. Let a change of variables be defined as shown below,

$$\hat{Q} = \frac{Q}{J_{ax}} \quad , \quad \hat{D} = \frac{D}{J_{ax}} \quad , \quad \hat{B} = \frac{1}{J_{ax}} \left(\xi_x B + \xi_y H + \xi_z M \right) \tag{4}$$

$$\hat{H} = \frac{1}{J_{ax}} \left(\eta_x B + \eta_y H + \eta_z M \right) \quad , \quad \hat{M} = \frac{1}{J_{ax}} \left(\zeta_x B + \zeta_y H + \zeta_z M \right) \tag{5}$$

Substituting,

$$\hat{Q} = \frac{\partial}{\partial t}\left[\hat{D}\right] + \frac{\partial}{\partial \xi}\left[\hat{B}\right] + \frac{\partial}{\partial \eta}\left[\hat{H}\right] + \frac{\partial}{\partial \zeta}\left[\hat{M}\right]$$

Therefore, the Navier-Stokes equation in vector form on the uniform grid can be rewritten as shown below,

$$\hat{Q} = \frac{\partial \hat{D}}{\partial t} + \frac{\partial \hat{B}}{\partial \xi} + \frac{\partial \hat{H}}{\partial \eta} + \frac{\partial \hat{M}}{\partial \zeta} \tag{6}$$

wherein the transformations necessary to convert equation **??** into equation **??** is equation **??** and equation **??**.

0.1.4 Uniform to Curvilinear Transformation

Reiterating the definition for the transformed variables $\hat{Q}$, $\hat{D}$, $\hat{B}$, $\hat{H}$, $\hat{M}$,

$$\hat{Q} = \frac{Q}{J_{ax}} \quad , \quad \hat{D} = \frac{D}{J_{ax}} \quad , \quad \hat{B} = \frac{1}{J_{ax}}\left(\xi_x B + \xi_y H + \xi_z M\right)$$

$$\hat{H} = \frac{1}{J_{ax}}\left(\eta_x B + \eta_y H + \eta_z M\right) \quad , \quad \hat{M} = \frac{1}{J_{ax}}\left(\zeta_x B + \zeta_y H + \zeta_z M\right)$$

Transforming from uniform grid quantity $\hat{Q}$ and $\hat{D}$ is simple,

$$\hat{Q} = \frac{Q}{J_{ax}} \quad , \quad \hat{D} = \frac{D}{J_{ax}}$$

$$J_{ax}\hat{Q} = Q \quad , \quad J_{ax}\hat{D} = D \tag{7}$$

The other quantities can be re-arranged into a matrix equation shown below,

$$\begin{bmatrix} \hat{B} \\ \hat{H} \\ \hat{M} \end{bmatrix} = \frac{1}{J_{ax}} \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} B \\ H \\ M \end{bmatrix}$$

The 3×3 system could be solved using Crammer's rule to expess B, H, and M in terms of $\hat{B}$, $\hat{H}$ and $\hat{M}$.