0.1 Direction Cosine Matrix

The dot product can be thought as some form of projection of one vector onto another vector. Consider the dot product of two vectors $\bar{a}$ and $\bar{b}$. Based on the definition of dot product,

$$\bar{a} \cdot \bar{b} = |a||b| \cos\theta$$

wherein θ is the angle between the vectors $\bar{a}$ and $\bar{b}$. Let an arbitrary vector $\bar{r}$ be represented in basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Let these basis vectors be of mangitude 1. Let another set of basis vectors of magnitude 1 be defined as $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$. The arbitrary vector $\bar{r}$ could be expressed as a linear combination of these basis vectors,

$$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'$$

The vector component $\bar{r}$ in the $\hat{\imath}'$ direction is the summation of all the weighted basis vector components on the $\hat{\imath}'$ direction. Since it was established earlier that this would mean taking the dot product,

$$x' = x(\hat{\imath} \cdot \hat{\imath}') + y(\hat{\jmath} \cdot \hat{\imath}') + z(\hat{k} \cdot \hat{\imath}')$$

Let $\theta_{fg'}$ represent the angle between the f axis and the g' axis,

$$\hat{\imath} \cdot \hat{\imath}' = |\hat{\imath}||\hat{\imath}'| \cos\theta_{ii'} \quad , \quad \hat{\jmath} \cdot \hat{\imath}' = |\hat{\jmath}||\hat{\imath}'| \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{\imath}' = |\hat{k}||\hat{\imath}'| \cos\theta_{ki'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{\imath} \cdot \hat{\imath}' = \cos\theta_{ii'} \quad , \quad \hat{\jmath} \cdot \hat{\imath}' = \cos\theta_{ji'} \quad , \quad \hat{k} \cdot \hat{\imath}' = \cos\theta_{ki'}$$

Substituting the basis vector projections,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

Repeating similar operations for the $\hat{\jmath}'$ direction,

$$y' = x(\hat{\imath} \cdot \hat{\jmath}') + y(\hat{\jmath} \cdot \hat{\jmath}') + z(\hat{k} \cdot \hat{\jmath}')$$

$$\hat{\imath} \cdot \hat{\jmath}' = |\hat{\imath}||\hat{\jmath}'| \cos\theta_{ij'} \quad , \quad \hat{\jmath} \cdot \hat{\jmath}' = |\hat{\jmath}||\hat{\jmath}'| \cos\theta_{jj'} \quad , \quad \hat{k} \cdot \hat{\jmath}' = |\hat{k}||\hat{\jmath}'| \cos\theta_{kj'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{\imath} \cdot \hat{\jmath}' = \cos\theta_{ij'} \quad , \quad \hat{\jmath} \cdot \hat{\jmath}' = \cos\theta_{jj'} \quad , \quad \hat{k} \cdot \hat{\jmath}' = \cos\theta_{kj'}$$

Substituting the basis vector projections,

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

Repeating similar operations for the $\hat{k}'$ direction,

$$z' = x(\hat{i} \cdot \hat{k}') + y(\hat{j} \cdot \hat{k}') + z(\hat{k} \cdot \hat{k}')$$

$$\hat{i} \cdot \hat{k}' = |\hat{i}||\hat{k}'| \cos\theta_{ik'} \quad , \quad \hat{j} \cdot \hat{k}' = |\hat{j}||\hat{k}'| \cos\theta_{jk'} \quad , \quad \hat{k} \cdot \hat{k}' = |\hat{k}||\hat{k}'| \cos\theta_{kk'}$$

Using the earlier assumption that the magnitude of the basis vectors are all 1,

$$\hat{i} \cdot \hat{k}' = \cos\theta_{ik'} \quad , \quad \hat{j} \cdot \hat{k}' = \cos\theta_{jk'} \quad , \quad \hat{k} \cdot \hat{k}' = \cos\theta_{kk'}$$

Substituting the basis vector projections,

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Collecting the various expressions together,

$$x' = x\cos\theta_{ii'} + y\cos\theta_{ji'} + z\cos\theta_{ki'}$$

$$y' = x\cos\theta_{ij'} + y\cos\theta_{jj'} + z\cos\theta_{kj'}$$

$$z' = x\cos\theta_{ik'} + y\cos\theta_{jk'} + z\cos\theta_{kk'}$$

Re-arranging the expressions into matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence, based on the problem definition,

$$\bar{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad , \quad \bar{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad , \quad l = \begin{bmatrix} \cos\theta_{ii'} & \cos\theta_{ji'} & \cos\theta_{ki'} \\ \cos\theta_{ij'} & \cos\theta_{jj'} & \cos\theta_{kj'} \\ \cos\theta_{ik'} & \cos\theta_{jk'} & \cos\theta_{kk'} \end{bmatrix}$$

$$\bar{r}' = l\bar{r}$$

The figure below shows some of the angles referenced in the l matrix,