0.1 Euler Angles

In 2-dimensions, the rotation matrix r is typically defined below,

$$r = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

The matrix r rotates a set of coordinate points by angle α in the counter-clockwise direction. Typically, a coordinate system would have its axes rotated in the counter-clockwise direction. As a result, all coordinates are now perceived in the rotated coordinate system to have been rotated in the clockwise direction. Suppose the ange $\alpha = -\theta$, the rotation matrix would take the form,

$$r = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

The eulerian angles represent parameters in successive rotation transformation to determine orientation. These successive rotation transformations are non-commutative. For the specific order of transformation eulerian angles $3-1-3$, the coordinate system is typically rotated in the z-axis by angle ϕ, then rotated in the x-axis by angle θ, before rotated in the z-axis again by angle ψ.

To express coordinates in an inertial coordinate system in terms of a body-fitted coordinate system, it is possible to apply successive rotation transformations on the coordinates in the inertial coordinate system to determine how the same coordinates are perceived in the body-fitted coordinate system. For this solution, the eulerian angle sequence $3-1-3$ is chosen.
For rotation in the z-axis by angle ϕ, the z-coordinate is held constant. Therefore,

$$A_3(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For rotation in the x-axis by angle θ, the x-coordinate is held constant. Therefore,

$$A_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

For rotation in the z-axis by angle ψ, the z-coordinate is held constant. Therefore,

$$A_3(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying the transformations in $3-1-3$ order,

$$A_{313}(\psi, \theta, \phi) = A_3(\psi)A_1(\theta)A_3(\phi)$$

The resulting matrix $A_{313}(\psi, \theta, \phi)$ would be the direction cosine matrix that would have the properties,

$$\bar{v_b} = A_{313}(\psi, \theta, \phi)\bar{v_i}$$

wherein $\bar{v_b}$ represents the coordinates in the body-fitted coordinate system and $\bar{v_i}$ represents the coordinates in the inertial coordinate system. Computing the direction matrix,

$$A_1(\theta)A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As a short-hand to make notation easier, let $\sin(\theta) = s_\theta$, $\cos(\theta) = c_\theta$. Substituting,

$$A_1(\theta)A_3(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

$$A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi & 0 \\ -c_\theta s_\phi & c_\theta c_\phi & s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

Therefore,

$$A_{313}(\psi, \theta, \phi) = A_3(\psi)A_1(\theta)A_3(\phi) = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

For the first transformation, ϕ represents the rotation angle along the z-axis in the inertial coordinate system. Let $\dot{\bar{\phi}}$ represent the time derivative of this transformation. Let $\dot{\bar{\phi}}_i$ represent the vector expressed in inertial coordinates and $\dot{\bar{\phi}}_b$ represent the vector expressed in body-fitted coordinates,

$$\dot{\bar{\phi}}_i = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T$$

To express the time derivative rotation vector in terms of body-fitted coordinates,

$$\dot{\bar{\phi}}_b = A_{313}(\psi, \theta, \phi)\dot{\bar{\phi}}_i$$

Substituting for the relevant terms,

$$\dot{\bar{\phi}}_b = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} s_\psi s_\theta \dot{\phi} \\ c_\psi s_\theta \dot{\phi} \\ c_\theta \dot{\phi} \end{bmatrix}$$

Expressing the vector $\dot{\bar{\phi}}_b$ verbosely,

$$\dot{\bar{\phi}}_b = f_1\hat{b_1} + f_2\hat{b_2} + f_3\hat{b_3} = s_\psi s_\theta \dot{\phi}\hat{b_1} + c_\psi s_\theta \dot{\phi}\hat{b_2} + c_\theta \dot{\phi}\hat{b_3}$$

By comparing the terms,

$$f_1 = \sin(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_2 = \cos(\psi)\sin(\theta)\dot{\phi} \quad , \quad f_3 = \cos(\theta)\dot{\phi}$$