# Mathematics Archives

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# Chapter 1

# Linear Differential Equations

## 1.1 Definition of Differential Operator

Let the differential operator be defined as the following:

$$L = \sum_{i=0}^{n} \left[ a_i \frac{d^{n-i}}{dx^{n-i}} \right] = a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1} \frac{d}{dx} + a_n$$

Let the following operator $D$ be defined:

$$D = \frac{d}{dx} \qquad D^k = \frac{d^k}{dx^k}$$

Therefore, the linear differential operator could be defined as

$$L = \sum_{i=0}^{n} \left[ a_i D^{n-i} \right] = a_0 \prod_{i=1}^{n} [D - r_i]$$

wherein $a_i$ and $r_i$ are constants albeit complex or real. The notation above is always true because an fundamental theorem of algebra states that $n^{th}$ order polynomial must have $n$ roots. Since linear differential operator could be expressed as a polynomial in terms of the operator $D$, therefore the notation above would always be true regardless of the choice of $a_i$ and $n$. The linear differential operator have certain properties associated to them discussed in the propositions,

### 1.1.1 Proposition 1: Operation on 0

The linear differential operator when operated on a 0 will yield 0:

$$L[0] = \sum_{i=0}^{n} \left[ a_i D^{n-i} \right] 0 = a_0 \frac{d^n}{dx^n} 0 + a_1 \frac{d^{n-1}}{dx^{n-1}} 0 + \cdots + a_{n-1} \frac{d}{dx} 0 + a_n 0$$

It is given that $\frac{d}{dx}(0) = 0$, and by reapplying recursively, $\frac{d^k}{dx^k}(0) = 0$. Therefore,

$$L[0] = 0$$

### 1.1.2 Proposition 2: Operation on Constants

The linear differential operator when operated on a constant will yield some constant provided that $a_n \neq 0$:

$$L[c] = \sum_{i=0}^{n} \left[ a_i D^{n-i} \right] c = a_0 \frac{d^n}{dx^n} c + a_1 \frac{d^{n-1}}{dx^{n-1}} c + \cdots + a_{n-1} \frac{d}{dx} c + a_n c$$

$$\text{Considering } \frac{d}{dx} c = \frac{d^k}{dx^k} c = 0,$$

$$L[c] = \sum_{i=0}^{n} \left[ a_i D^{n-i} \right] c = a_n c$$

### 1.1.3 Proposition 3: Operation Commutativity

If there exist two linearly independent differential operators $L_1$ and $L_2$, then the solution of the system $0 = L_1 L_2[y]$ must be a linear combination of the solution to the system $0 = L_1[y]$ and $0 = L_2[y]$:

$$L_1 = \prod_{i=0}^{m} [D - \alpha_i] \quad , \quad L_2 = \prod_{j=0}^{n} \left[ D - \beta_j \right]$$

Let $y_1(x)$ and $y_2(x)$ be such that:

$$0 = L_1[y_1(x)] = \prod_{i=0}^{m} [D - \alpha_i] \, y_1(x) \quad , \quad 0 = L_2[y_2(x)] = \prod_{j=0}^{n} \left[ D - \beta_j \right] y_2(x)$$

Let $T_i$ be the transformation defined as $T_i : f(x) \to g(x) \quad , \quad T_i[f(x)] = (D - r_i) f(x)$, wherein $f(x)$ is some arbitrary continuous function over some interval. Indeed the transformation $T_1$ is linear:

$$T_i[cu(x)] = (D - r_i) cu(x)$$

$$T_i[cu(x)] = c(D - r_i) u(x)$$

$$cT_i[u(x)] = c(D - r_i) u(x)$$

Therefore, $T_i[cu(x)] = cT_i[u(x)]$ wherein $c$ is some arbitrary constant.

$$T_i[u(x) + v(x)] = (D - r_i)[u(x) + v(x)]$$

$$T_i[u(x) + v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

$$T_i[u(x)] + T_i[v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

Therefore, $T_i[u(x) + v(x)] = T_i[u(x)] + T_i[v(x)]$, and $T_i$ must be a linear transformation. Linear transformations applied compositely form a linear transformation:

$$T_0(u + v) = T_0(u) + T_0(v)$$

$$T_1[T_0(u + v)] = T_1[T_0(u)] + T_1[T_0(v)]$$

$$\prod_{i=0}^{\alpha} [T_i] \, (u + v) = \prod_{i=0}^{\alpha} [T_i] \, (u) + \prod_{i=0}^{\alpha} [T_i] \, (v)$$

Since $L_1$ and $L_2$ is only a specific case of the transformation described as $T_1$ it can be considered that the differential operators of $L_1$ and $L_2$ are linear. Therefore, it can be said that $L_1[L_2]$ must be linear.

$$0 = null = L_1[y_1(x)] = \prod_{i=0}^{m} [D - \alpha_i]\, y_1(x) \quad , \quad 0 = null = L_2[y_2(x)] = \prod_{j=0}^{n} \left[D - \beta_j\right] y_2(x)$$

For $y_1(x)$ and $y_2(x)$:

$$0 = null = L_2\{L_1[y_1(x)]\} = \prod_{j=0}^{n} \left[D - \beta_j\right] \prod_{i=0}^{m} [D - \alpha_i]\, y_1(x)$$

$$0 = null = L_1\{L_2[y_2(x)]\} = \prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] y_2(x)$$

$$\prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] = \prod_{j=0}^{n} \left[D - \beta_j\right] \prod_{i=0}^{m} [D - \alpha_i] = L_1[L_2] = L_2[L_1]$$

It follows by definition of linear transformation that, $L_1[k_1 y_1(x)] = k_1 L_1[y_1(x)] = null$ and that $L_2[k_2 y_2(x)] = k_2 L_2[y2(x)] = null$:

$$0 = null = L_2\{L_1[k_1 y_1(x)]\} = \prod_{j=0}^{n} \left[D - \beta_j\right] \prod_{i=0}^{m} [D - \alpha_i]\, k_1 y_1(x)$$

$$0 = null = L_1\{L_2[k_2 y_2(x)]\} = \prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] k_2 y_2(x)$$

$$0 = L_2\{L_1[k_1 y_1(x)]\} + L_2\{L_1[k_2 y_2(x)]\} = L_2\{L_1[k_1 y_1(x) + k_2 y_2(x)]\}$$

$$0 = \prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] [k_1 y_1(x) + k_2 y_2(x)]$$

## 1.2 Homogenous Differential Equation Cases

Consider the following homogenous differential equation:

$$0 = \sum_{i=0}^{n} \left[a_i \overset{n-i}{y}\right] = a_0 \overset{n}{y} + a_1 \overset{n-1}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$

$$0 = L[y] = \sum_{i=0}^{n} \left[a_i D^{n-i}\right] y = \prod_{i=1}^{n} [D - r_i]\, y$$

### 1.2.1 Non-Repeated Roots

By principle of superposition verified by proposition 1 and 3, the general solution to the homogenous $n^{th}$ order differential equation,

$$y_c = \sum_{i=1}^{n} [c_i y_i]$$

Wherein $c_i$ represents either complex or real constants, $y_i$ represents solutions to the $0 = [D - r_i]y_i$ system. Considering the partial system,

$$0 = [D - r_i]y_i$$

$$0 = Dy_i - r_i y_i$$

$$\frac{d}{dx}(y_i) = r_i y_i$$

$$\int \frac{1}{y_i} dy_i = \int r_i dx$$

$$\ln y_i = r_i x + C$$

$$y_i = e^{r_i x + C} = c_i e^{r_i x}$$

Therefore for as long as there are no roots with multiplicity greater than 1, the following is true, for some choice of constants,

$$y_c = \sum_{i=1}^{n} \left[c_i e^{r_i x}\right]$$

### 1.2.2 Repeated Roots

Suppose the $\alpha^{th}$ root has a multiplicity of $k$,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^{n} [D - r_i] (D - r_\alpha)^k y$$

By proposition 3, the general solution to the system must be the linear combination:

$$y_g(x) = c_1 y_1(x) + c_2 y_2(x)$$

wherein $y_1(x)$ is the solution to the system $0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^{n} [D - r_i]\, y$ and $y_2(x)$ is the solution to the system $0 = (D - r_\alpha)^k y$. By conjecture, it is suspected that the $y_2(x) = u(x)e^{r_\alpha x}$ wherein $u(x)$ is some function to be determined.

$$0 = (D - r_\alpha)u(x)e^{r_\alpha x}$$

$$0 = \frac{d}{dx}\left[u(x)e^{r_\alpha x}\right] - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = \overset{1}{u(x)}e^{r_\alpha x} + r_\alpha u(x)e^{r_\alpha x} - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = \overset{1}{u(x)}e^{r_\alpha x}$$

By reapplying the linear differential operator recursively:

$$(D - r_\alpha)^k u(x)e^{r_\alpha x} = \overset{k}{u(x)}e^{r_a x}$$

Therefore, the system would follow:

$$0 = (D - r_\alpha)^k u(x)e^{r_\alpha x} = \overset{k}{u(x)}e^{r_a x}$$

$$0 \neq e^{r_a x} \text{ for all } x$$

$$0 = \overset{k}{u}(x)$$

A function that satisfies the following condition must be a polynomial with at most degree $k-1$. Therefore,

$$u(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right]$$

The general solution $y_{rr}(x)$ to the system $0 = (D - r_\alpha)^k y$:

$$y_{rr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right] e^{r_\alpha x}$$

### 1.2.3 Complex Roots

Suppose the $\alpha^{th}$ root is a complex root, by the fundamental theorem of algebra, some other root must be its complex conjugate. Let the complex conjugate root of the $\alpha^{th}$ root be ordered next to the $\alpha^{th}$ root in the product notation. Therefore,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{i=\alpha+2}^{n} [D - r_i\ ] [D - r_\alpha][D - r_{\alpha+1}]y$$

Let $y_{cr}$ represent the complex root corresponding to the system $0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$ . By principle of superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2} [c_i e^{r_i x}] + y_{cr}$$

$$0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$$

By the principles presented earlier,

$$y_{cr}(x) = c_\alpha e^{(a+bi)x} + c_{\alpha+1} e^{(a-bi)x}$$

$$y_{cr}(x) = e^{ax} \left[c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}\right]$$

By De Moivre's theorem,

$$e^{bxi} = \cos(bx) + i\sin(bx) \quad , \quad e^{-bxi} = \cos(bx) - i\sin(bx)$$

Suppose the constants $c_\alpha$ and $c_{\alpha+1}$ are complex numbers,

$$c_\alpha = f_1 + g_1 i \quad , \quad c_{\alpha+1} = f_2 + g_2 i$$

By substituting to the expression for complex solution,

$$y_{cr}(x) = e^{ax} \left[(f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}\right]$$

Let $y_{cr} = e^{ax} y_{co}$,

$$y_{co} = c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}$$

$$y_{co}(x) = (f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}$$
$$y_{co}(x) = (f_1 + g_1 i)[\cos(bx) + i\sin(bx)] + (f_2 + g_2 i)[\cos(bx) - i\sin(bx)]$$

Let

$$A(x) = (f_1 + g_1 i)[\cos(bx) + i\sin(bx)] \quad , \quad B(x) = (f_2 + g_2 i)[\cos(bx) - i\sin(bx)]$$
$$A(x) = f_1\cos(bx) - g_1\sin(bx) + i[f_1\sin(bx) + g_1\cos(bx)]$$
$$B(x) = f_2\cos(bx) + g_2\sin(bx) + i[-f_2\sin(bx) + g_2\cos(bx)]$$
$$y_{co}(x) = A(x) + B(x)$$
$$y_{co}(x) = (f_1 + f_2)\cos(bx) + (g_2 - g_1)\sin(bx) + i[(f_1 - f_2)\sin(bx) + (g_1 + g_2)\cos(bx)]$$

For the complex root $y_{cr}(x)$ to be real, the imaginary component of $y_{cr}(x)$ must be equals to 0. Therefore, the following must hold true,

$$f_1 = f_2 \quad , \quad g_1 = -g_2$$

For as long as the condition above hold true, the two constants $c_\alpha$ and$c_{\alpha+1}$ must be complex conjugates. Considering the case wherein $c_\alpha$ and$c_{\alpha+1}$ as complex conjugates,

$$y_{co}(x) = 2f_1\cos(bx) + 2g_2\sin(bx)$$
$$y_{cr} = 2e^{ax}[f_1\cos(bx) + g_2\sin(bx)]$$

Therefore, the following is true for each complex root and conjugate pair,

$$y_c = \sum_{i=1}^{n-2}[c_i e^{r_i x}] + 2e^{ax}[f_1\cos(bx) + g_2\sin(bx)]$$

### 1.2.4 Repeated Complex Roots

Suppose the $\alpha^{th}$ root is a complex root with a multiplicity of $k$. Let its complex conjugate be placed adjacent after said complex root,

$$0 = \prod_{i=1}^{\alpha-1}[D - r_i] \prod_{i=\alpha+2k}^{n}[D - r_i]\,[D - r_\alpha]^k\,[D - \bar{r_\alpha}]^k\, y$$

Let $y_{crr}$ be considered as the solution to the system $0 = [D - r_\alpha]^k\,[D - \bar{r_\alpha}]^k\, y_{crr}$ . Based on superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2k}[c_i e^{r_i x}] + y_{crr}$$

Based on the previous work on repeated roots with multiplicity greater than 1,

$$y_{crr}(x) = \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right] C_1 e^{r_\alpha x} + \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right] C_2 e^{\bar{r_\alpha} x}$$

Let $r_\alpha = a + bi$, and $\bar{r_\alpha} = a - bi$,

$$y_{crr}(x) = \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right][C_1 e^{bxi} + C_2 e^{-bxi}]e^{ax}$$

Let $c_1 = f_1 + g_1 i$ and $c_2 = f_2 + g_2 i$ . Based on previous work on complex roots,

$$C_1 e^{bxi} + C_2 e^{-bxi} = 2f_1\cos(bx) + 2g_2\sin(bx)$$

Therefore,

$$y_{crr}(x) = \sum_{i=0}^{k-1}\left[c_i x^{k-1-i}\right][2f_1\cos(bx) + 2g_2\sin(bx)]e^{ax}$$

## 1.3 General Solutions to Homogenous Differential Equations

Therefore, if an $n^{th}$ order homogeneous differential equation with $a$ real non-repeated roots, $b$ complex root pairs, $c$ real repeated roots with multiplicity $\gamma$, and $d$ complex repeated root pairs with multiplicity $\beta$

$$\begin{aligned} y_c = &\sum_{l=1}^{a} \left[c_{1,l} e^{r_{1,l} x}\right] \\ &+ \sum_{j=1}^{b} [c_{2,j,1} \cos{(b_{1,j} x)} + c_{2,j,2} \sin{(b_{1,j} x)}] e^{a_{1,j} x} \\ &+ \sum_{k=1}^{c} \left[ \sum_{m=0}^{\gamma_k - 1} \left[c_{3,m,k} x^{\gamma_k - 1 - m}\right] e^{r_k x} \right] \\ &+ \sum_{i=1}^{d} \left[ \sum_{p=0}^{\beta_p - 1} \left[c_{4,p,i} x^{\beta_p - 1 - p}\right] [k_{4,i,1} \cos{(b_{2,i})} x + k_{4,i,2} \sin{(b_{2,i})} x] e^{r_i x} \right] \end{aligned}$$

The variables $a$, $b$, $c$, $d$, $\gamma$, $\beta$, and $n$ are related by the following expression,

$$n = a + 2b + \sum_{i=1}^{c} [\gamma_i] + \sum_{j=1}^{d} [2\beta_j]$$

## 1.4 Non-Homogenous Differential Equations

Consider the following system:

$$\sum_{i=0}^{n} \left[a_i \overset{n-i}{y}\right] = \sum_{j=0}^{m} \left[c_i f_i(x)\right]$$

wherein $f_i(x)$ represents the $i^{th}$ arbitrary function, and $a_i$ represents the $i^{th}$ arbitrary constant. The following function could be rewritten in terms of the linear differential operator $L$:

$$L[y] = \sum_{j=0}^{m} \left[c_i f_i(x)\right]$$

Let $y_j$ represent the general solution to the $j^{th}$ system:

$$L[y_j] = c_i f_j(x)$$

By taking the summations of the various solutions to the various sytems:

$$L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j] = c_0 f_0(x) + c_1 f_1(x) + \cdots + c_{j-1} f_{j-1}(x) + c_j f_j(x)$$

Since the differential operator $L$ is linear, as shown in proposition 3:

$$L\left[\sum_{j=0}^{m} (y_j)\right] = L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j]$$

$$L\left[\sum_{j=0}^{m}(y_j)\right]=\sum_{j=0}^{m}\left[c_if_i(x)\right]$$

Therefore, a solution to the non-homogenous differential equation:

$$y_p(x)=\sum_{j=0}^{m}(y_j)$$

An $m^{th}$ dimensional subspace spanned by $m$ functions must always contain a null element, in this case, a zero function. Let the $y_c$ represent the general solution to the homogenous differential equation $Ly = null = 0$. Then the general solution must follow:

$$y_g(x)=\sum_{j=0}^{m}(y_j)+y_c$$

# Chapter 2

# Dynamical Systems: Eigenvalues and Eigenvectors

Let A represent a $n \times n$ matrix (a matrix with n rows and n columns), $x$ represent a column vector of $n$ variables and $x'$ represent the derivative of the column vector $x$. The system below is known as a dynamical system:

$$x' = Ax$$

Consider the dynamical system $x' = kx$ wherein k is some arbitrary constant. Therefore,

$$\frac{dx}{dt} = kx$$

$$dt = \frac{1}{kx} dx$$

$$\int dt = \int \frac{1}{kx} dx$$

$$t = \frac{1}{k} \ln x + C$$

$$\ln x = kt + C$$

$$x = Ce^{kt}$$

Wherein C is a constant determined by the initial conditions.

## 2.1 Non-Repeated Real Eigenvalues of n × n Case

The previous working gives the conjecture that the general solution set $x(t)$ to the dynamical system $x' = Ax$ is the linear combination of exponential functions analogous to the example shown above. Consider the possibility that one solution to the dynamical system takes the form below:

$$x(t) = \bar{v}_i e^{\lambda_i t}$$

wherein $\bar{v}_i$ represents a vector and $\lambda_i$ represents a constant. By taking derivative of the solution,

$$x'(t) = \lambda_i \bar{v}_i e^{\lambda_i t}$$

$$Ax(t) = A\bar{v}_i e^{\lambda_i t}$$

By considering that $x(t)$ represents a solution to the dynamical system, $x' = Ax$

$$\lambda_i \bar{v_i} e^{\lambda_i t} = A \bar{v_i} e^{\lambda_i t}$$

Since $e^{\lambda_i t} \neq 0$ for all values of $t$,

$$A\bar{v_i} = \lambda_i \bar{v_i}$$

This is a familiar equation for eigenvalues and eigenvectors. This shows that each eigenvalue-eigenvector pairs of the matrix $A$ represents a solution set. Therefore, the general solution set is:

$$x(t) = span[\bar{v_1} e^{\alpha_1 t}, \bar{v_2} e^{\alpha_2 t}, \dots \bar{v_n} e^{\alpha_n t}]$$

$$x(t) = \sum_{i=1}^{n} \left[ c_i \bar{v_i} e^{\lambda_i t} \right]$$

wherein $c_i$ are constants determined by the initial value of the problem.

## 2.2 Non-Repeated Complex Eigenvalues of 2 × 2 Case

Consider the special case wherein the matrix A is a $2 \times 2$ matrix and that the eigenvalues are complex, by conjecture,

$$x(t) = c_1 \bar{v_1} e^{\lambda_1 t} + c_2 \bar{v_2} e^{\lambda_2 t} = k_1 Re[\bar{v_1} e^{\lambda_1 t}] + k_2 Im[\bar{v_1} e^{\lambda_1 t}]$$

wherein $c_1$ and $c_2$ are complex values meanwhile $k_1$ and $k_2$ are real values. There must always be some choice of complex values $c_1$ and $c_2$ such that the expression above is true. The proof is shown below,

Let

$$\bar{v_1} = \bar{v_r} + i\bar{v_i} \qquad \lambda_1 = a + bi$$

$$x(t) = (\bar{v_r} + i\bar{v_i}) e^{(a+bi)t}$$

$$x(t) = e^{at}(\bar{v_r} + i\bar{v_i})[\cos(bt) + i\sin(bt)]$$

$$x(t) = e^{at}[\bar{v_r}\cos(bt) + i\bar{v_r}\sin(bt) + i\bar{v_i}\cos(bt) - \bar{v_i}\sin(bt)]$$

$$x(t) = e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)] + ie^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

$$Re[\bar{v_1} e^{\lambda_1 t}] = e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)]$$

$$Im[\bar{v_1} e^{\lambda_1 t}] = e^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

$$LHS = k_1 Re[\bar{v_1} e^{\lambda_1 t}] + k_2 Im[\bar{v_1} e^{\lambda_1 t}]$$

$$LHS = k_1 e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)] + k_2 e^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

$$LHS = e^{at}[k_1\bar{v_r}\cos(bt) - k_1\bar{v_i}\sin(bt) + k_2\bar{v_r}\sin(bt) + k_2\bar{v_i}\cos(bt)]$$

$$LHS = e^{at}\{[k_1\bar{v_r} + k_2\bar{v_i}]\cos(bt) + [k_2\bar{v_r} - k_1\bar{v_i}]\sin(bt)\}$$

$$LHS = e^{at}[k_1\bar{v_r} + k_2\bar{v_i}]\cos(bt) + e^{at}[k_2\bar{v_r} - k_1\bar{v_i}]\sin(bt)$$

It is important to note that eigenvalues and their corresponding eigenvectors occur in conjugate pairs. Therefore, if $\lambda_1 = a + bi$, then $\lambda_2 = \lambda_1^* = a - bi$ and if the eigenvector $\bar{v_1} = \bar{v_r} + i\bar{v_i}$, then $\bar{v_2} = \bar{v_1}^* = \bar{v_r} - i\bar{v_i}$.

Let

$$c_1 = f_1 + g_1 i \qquad c_2 = f_2 + g_2 i$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = (f_1 + g_1 i)(\bar{v}_r + i\bar{v}_i)e^{(a+bi)t} + (f_2 + g_2 i)(\bar{v}_r - i\bar{v}_i)e^{(a-bi)t}$$

For ease of notation,

$$A(t) = (f_1 + g_1 i)(\bar{v}_r + i\bar{v}_i)e^{(a+bi)t} \qquad B(t) = (f_2 + g_2 i)(\bar{v}_r - i\bar{v}_i)e^{(a-bi)t}$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = A(t) + B(t)$$

$$A(t) = e^{at}(f_1 + g_1 i)(\bar{v}_r + i\bar{v}_i)\left[\cos(bt) + i\sin(bt)\right]$$

$$A(t) = e^{at}(f_1\bar{v}_r + if_1\bar{v}_i + ig_1\bar{v}_r - g_1\bar{v}_i)\left[\cos(bt) + i\sin(bt)\right]$$

$$A(t) = e^{at}[f_1\bar{v}_r - g_1\bar{v}_i + i(f_1\bar{v}_i + g_1\bar{v}_r)]\left[\cos(bt) + i\sin(bt)\right]$$

$$A(t) = e^{at}[(f_1\bar{v}_r - g_1\bar{v}_i)\cos(bt) + i(f_1\bar{v}_i + g_1\bar{v}_r)\cos(bt) + i(f_1\bar{v}_r - g_1\bar{v}_i)\sin(bt) - (f_1\bar{v}_i + g_1\bar{v}_r)\sin(bt)]$$

$$B(t) = (f_2 + g_2 i)(\bar{v}_r - i\bar{v}_i)e^{(a-bi)t}$$

$$B(t) = e^{at}(f_2\bar{v}_r - if_2\bar{v}_i + ig_2\bar{v}_r + g_2\bar{v}_i)\left[\cos(-bt) + i\sin(-bt)\right]$$

$$B(t) = e^{at}[(f_2\bar{v}_r + g_2\bar{v}_i) + i(g_2\bar{v}_r - f_2\bar{v}_i)]\left[\cos(bt) - i\sin(bt)\right]$$

$$B(t) = e^{at}[(f_2\bar{v}_r + g_2\bar{v}_i)\cos(bt) + i(g_2\bar{v}_r - f_2\bar{v}_i)\cos(bt) + i(-f_2\bar{v}_r - g_2\bar{v}_i)\sin(bt) + (g_2\bar{v}_r - f_2\bar{v}_i)\sin(bt)]$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = Re[A(t)] + Re[B(t)] + i\{Im[A(t)] + Im[B(t)]\}$$

$$0 = Im[A(t)] + Im[B(t)]$$

$$0 = (f_1\bar{v}_i + g_1\bar{v}_r)\cos(bt) + (f_1\bar{v}_r - g_1\bar{v}_i)\sin(bt) + (g_2\bar{v}_r - f_2\bar{v}_i)\cos(bt) - (f_2\bar{v}_r + g_2\bar{v}_i)\sin(bt)$$

$$0 = (f_1\bar{v}_i + g_1\bar{v}_r + g_2\bar{v}_r - f_2\bar{v}_i)\cos(bt) + (f_1\bar{v}_r - g_1\bar{v}_i - f_2\bar{v}_r - g_2\bar{v}_i)\sin bt$$

$$0 = [(g_1 + g_2)\bar{v}_r + (f_1 - f_2)\bar{v}_i]\cos(bt) + [(f_1 - f_2)\bar{v}_r - (g_1 + g_2)\bar{v}_i]\sin(bt)$$

For as long as the condition below is met, the imaginary component of $A(t) + B(t)$ is negligible.

$$g_1 = -g_2 \qquad f_1 = f_2$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = Re[A(t)] + Re[B(t)]$$

$$\begin{aligned} c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} =& e^{at}(f_1\bar{v}_r - g_1\bar{v}_i)\cos(bt) - (f_1\bar{v}_i + g_1\bar{v}_r)\sin(bt) \\ &+ (f_2\bar{v}_r + g_2\bar{v}_i)\cos(bt) + (g_2\bar{v}_r - f_2\bar{v}_i)\sin(bt) \end{aligned}$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = e^{at}(f_1\bar{v}_r - g_1\bar{v}_i + f_2\bar{v}_r + g_2\bar{v}_i)\cos(bt) + (g_2\bar{v}_r - f_2\bar{v}_i - f_1\bar{v}_i - g_1\bar{v}_r)\sin(bt)$$

$$c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = e^{at}[(f_1 + f_2)\bar{v}_r + (g_2 - g_1)\bar{v}_i]\cos(bt) + [(g_2 - g_1)\bar{v}_r - (f_1 + f_2)\bar{v}_i]\sin(bt)$$

$$RHS = c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t}$$

$$RHS = e^{at}[(f_1 + f_2)\bar{v}_r + (g_2 - g_1)\bar{v}_i]\cos(bt) + e^{at}[(g_2 - g_1)\bar{v}_r - (f_1 + f_2)\bar{v}_i]\sin(bt)$$

$$LHS = e^{at}[k_1\bar{v}_r + k_2\bar{v}_i]\cos(bt) + e^{at}[k_2\bar{v}_r - k_1\bar{v}_i]\sin(bt)$$

If the conditions below are met, therefore $LHS = RHS$ and the statement $x(t) = c_1\bar{v}_1 e^{\lambda_1 t} + c_2\bar{v}_2 e^{\lambda_2 t} = k_1 Re[\bar{v}_1 e^{\lambda_1 t}] + k_2 Im[\bar{v}_1 e^{\lambda_1 t}]$ is true.

$$g_1 + g_2 = 0 \qquad f_1 + f_2 - k_1 = 0 \qquad f_1 - f_2 = 0 \qquad g_2 - g_1 - k_2 = 0$$

The corresponding augmented matrix of the following conditions is

$$\begin{array}{ccccccc} f_1 & f_2 & g_1 & g_2 & k_1 & k_2 & C \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The row-reduced echelon form of the corresponding augmented matrix is

$$\begin{array}{ccccccc} f_1 & f_2 & g_1 & g_2 & k_1 & k_2 & C \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

The row-reduced echelon form is unique and is consistent, therefore the system has a consistent solution. This proves that for some special choice of $c_1$ and $c_2$, the expression below is correct.

$$x(t) = c_1\bar{v_1}e^{\lambda_1 t} + c_2\bar{v_2}e^{\lambda_2 t} = k_1 Re[\bar{v_1}e^{\lambda_1 t}] + k_2 Im[\bar{v_1}e^{\lambda_1 t}]$$

A restatement of the general real solution set is:

$$x(t) = k_1 e^{at}[\bar{v_r}\cos(bt) - \bar{v_i}\sin(bt)] + k_2 e^{at}[\bar{v_r}\sin(bt) + \bar{v_i}\cos(bt)]$$

The solution set for all real numbers could be better expressed as a matrix multiplication

$$x(t) = e^{at}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} k_2 \\ k_1 \end{pmatrix}$$

The real and imaginary components of the eigenvector $v_1$ form a linearly independent set. Therfore, the matrix $\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}$ must be invertible. Through the invertible matrix theorem, the matrix $\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}$ must have a suitable inverse.

$$x(t) = e^{at}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$x(t) = e^{at}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x_0$$

$$\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x(t) = e^{at}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}\begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x_0$$

By considering the substitution $y = \begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x(t)$ and $y_0 = \begin{pmatrix} \bar{v_i} & \bar{v_r} \end{pmatrix}^{-1} x_0$,

$$y = e^{at}\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix} y_0$$

wherein $e^{at}$ represents a scaling transformation and $\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}$ represents a rotation. Therefore, for a suitable substitution, the general real solution set of the dynamical system $x' = Ax$ will form a rotation with a scaling component. The rotation is sometimes known as the "hidden rotation". Some possibilities of the solution set may be ellipses, circles, and spirals.

## 2.3 Non-Repeated Complex Eigenvalues of 3 $\times$ 3 Case

Consider the case wherein $n = 3$

$$x(t) = \sum_{i=1}^{3} \left[ c_i \bar{v}_i e^{\lambda_i t} \right]$$

$$x(t) = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} + c_3 \bar{v}_3 e^{\lambda_3 t}$$

Complex eigenvalues occure in conjugate pairs. When A is a $3 \times 3$ matrix, 2 of the eigenvalues will be complex conjugate pairs and the third one will be a real value. Therefore, two of the eigenvectors must be complex vectors with the third eigenvector being a real vector. Therefore, through the similar argument and proof written above,

$$x(t) = k_1 Re \left[ \bar{v}_1 e^{\lambda_1 t} \right] + k_2 Re \left[ \bar{v}_1 e^{\lambda_1 t} \right] + k_3 \bar{v}_3 e^{\lambda_3 t}$$

$$x(t) = k_1 e^{at} [\bar{v}_r \cos{(bt)} - \bar{v}_i \sin{(bt)}] + k_2 e^{at} [\bar{v}_r \sin{(bt)} + \bar{v}_i \cos{(bt)}] + k_3 \bar{v}_3 e^{\lambda_3 t}$$

The following solution set could be factorised as matrix multiplications

$$x(t) = e^{at} \begin{pmatrix} \bar{v}_i & \bar{v}_r & \bar{v}_3 \end{pmatrix} \begin{pmatrix} \cos{(bt)} & -\sin{(bt)} & 0 \\ \sin{(bt)} & \cos{(bt)} & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} \begin{pmatrix} k_2 \\ k_1 \\ k_3 \end{pmatrix}$$

The vectors $\bar{v}_i, \bar{v}_r, \bar{v}_3$ form a linearly independent set, therefore, the matrix $\begin{pmatrix} \bar{v}_i & \bar{v}_r & \bar{v}_3 \end{pmatrix}$ is invertible and its inverse must exist.

$$\text{Let } y_0 = \begin{pmatrix} k_2 \\ k_1 \\ k_3 \end{pmatrix}$$

$$\begin{pmatrix} \bar{v}_i & \bar{v}_r & \bar{v}_3 \end{pmatrix}^{-1} x(t) = e^{at} \begin{pmatrix} \cos{(bt)} & -\sin{(bt)} & 0 \\ \sin{(bt)} & \cos{(bt)} & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} y_0$$

$$\text{Let } y(t) = \begin{pmatrix} \bar{v}_i & \bar{v}_r & \bar{v}_3 \end{pmatrix}^{-1} x(t)$$

$$y(t) = e^{at} \begin{pmatrix} \cos{(bt)} & -\sin{(bt)} & 0 \\ \sin{(bt)} & \cos{(bt)} & 0 \\ 0 & 0 & e^{(\lambda_3 - a)t} \end{pmatrix} y_0$$

$y_0$ is dependent on the system's initial conditions. This shows that for some suitable substitution, the general solution set forms a helix. The geometrical implication of the solution set is a spiral around the z-axis while it is moving away from the xy plane. The substitution back into the conventional axis $x_1, x_2, x_3$ could be considered as a transformation that "distorts" the helix.

## 2.4 Repeated Eigenvalues

Given the matrix $A$ in the system $x' = Ax$ is a matrix with repeated eigenvalues with multiplicity k, a reasonable conjecture is the solution to the system is similar in form to the repeated roots case in the linear differential equation. By conjecture,

$$x(t) = \sum_{i=0}^{k-1} \left[ \bar{v_i} t^{k-1-i} e^{\lambda t} \right]$$

$$x'(t) = \sum_{i=0}^{k-1} \left[ \bar{v_i} \frac{d}{dt} \left[ t^{k-1-i} e^{\lambda t} \right] \right]$$

$$\frac{d}{dt} \left[ t^{k-1-i} e^{\lambda t} \right] = (k-1-i) t^{k-2-i} e^{\lambda t} + \lambda t^{k-1-i} e^{\lambda t}$$

$$x'(t) = \sum_{i=0}^{k-1} \left[ (k-1-i) t^{k-2-i} \bar{v_i} e^{\lambda t} + \lambda t^{k-1-i} \bar{v_i} e^{\lambda t} \right]$$

Remembering $x'(t) = Ax(t)$,

$$\sum_{i=0}^{k-1} \left[ A\bar{v_i} t^{k-1-i} e^{\lambda t} \right] = \sum_{i=0}^{k-1} \left[ (k-1-i) t^{k-2-i} \bar{v_i} e^{\lambda t} + \lambda t^{k-1-i} \bar{v_i} e^{\lambda t} \right]$$

Considering that $e^{\lambda t} \neq 0$, therefore,

$$\sum_{i=0}^{k-1} \left[ A\bar{v_i} t^{k-1-i} \right] = \sum_{i=0}^{k-1} \left[ \lambda t^{k-1-i} \bar{v_i} + (k-1-i) t^{k-2-i} \bar{v_i} \right]$$

For the $0^{th}$ element,

$$A\bar{v_0} t^{k-1} = \lambda t^{k-1} \bar{v_0}$$

Considering that $t^{k-1} \neq 0$ for as long as $t \neq 0$,

$$A\bar{v_0} = \lambda \bar{v_0}$$

For the $\alpha^{th}$ element,

$$A\bar{v_\alpha} t^{k-1-\alpha} = \lambda t^{k-1-\alpha} \bar{v_\alpha} + \left[ k-1-(\alpha-1) \right] t^{k-2-(\alpha-1)} \bar{v_{\alpha-1}}$$

$$A\bar{v_\alpha} t^{k-1-\alpha} = \lambda t^{k-1-\alpha} \bar{v_\alpha} + [k-\alpha]\, t^{k-1-\alpha} \bar{v_{\alpha-1}}$$

For as long as $t \neq 0$, $t^{k-1-\alpha} \neq 0$. Therefore,

$$A\bar{v_\alpha} = \lambda \bar{v_\alpha} + [k-\alpha]\, \bar{v_{\alpha-1}}$$

$$\frac{1}{[k-\alpha]} (A - \lambda I) \bar{v_\alpha} = \bar{v_{\alpha-1}}$$

By applying definition recursively,

$$\frac{1}{\prod_{i=0}^{j-1} [k-i]} (A - \lambda I)^j \bar{v_\alpha} = \bar{v_{\alpha-j}}$$

For when $j = \alpha$,

$$\frac{1}{\prod_{i=0}^{\alpha-1} [k-i]} (A - \lambda I)^\alpha \bar{v_\alpha} = \bar{v_0}$$

## 2.5 Simple First Order Non-Homogenous System

Suppose, for a non-homogeneous dynamical system, $x' = Ax + k$. The non-homogenous dynamical system could be reduced to a homogenous dynamical system, $y' = Ay$ by an appropriate substitution shown below:

$$y_1 = x_1 + c_1 \qquad y_2 = x_2 + c_2 \qquad \dots \qquad y_n = x_n + c_n$$

wherein $c_1, c_2, c_3 \dots c_n$ are constants

$$y_1' = x_1' \qquad y_2' = x_2' \qquad \dots \qquad y_n' = x_n'$$

Let the columns of matrix $A$ be denoted as $a_1, a_2, a_3, \dots a_n$

$$A = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix}$$

$$Ay = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Ay = \sum_{i=1}^{n} [\bar{a}_i y_i]$$

$$Ay = \sum_{i=1}^{n} \left[\bar{a}_i (x_i + c_i)\right]$$

$$Ay = \sum_{i=1}^{n} [\bar{a}_i x_i + \bar{a}_i c_i]$$

$$Ay = \sum_{i=1}^{n} [\bar{a}_i x_i] + \sum_{i=1}^{n} [\bar{a}_i c_i]$$

$$Ax + k = \begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$Ax + k = \sum_{i=1}^{n} [\bar{a}_i x_i] + k$$

$$Ay = Ax + k$$

$$\sum_{i=1}^{n} [\bar{a}_i x_i] + \sum_{i=1}^{n} [\bar{a}_i c_i] = \sum_{i=1}^{n} [\bar{a}_i x_i] + k$$

$$\sum_{i=1}^{n} [\bar{a}_i c_i] = k$$

The system above is equivalent to an augmented matrix whose first column until nth column is the columns of the matrix A and its last column is the column vector k. Therefore, the augmented matrix is written below:

$$c_1 \quad c_2 \quad \dots \quad c_n \quad K$$

$$\begin{bmatrix} \bar{a_1} & \bar{a_2} & \dots & \bar{a_n} & k \end{bmatrix}$$

The solution to the augmented matrix will be the values for the constants $c_1, c_2, \dots, c_n$ that would be used in the substitution process in transforming the non-homogenous dynamical system into a homogenous dynamical system. The augmented matrix above would only have a solution for all k in $\mathbb{R}^n$ if the matrix A is invertible. If the matrix A is non-invertible , then k must be in $col[A]$, otherwise, then the augmented system forms an inconsistent system. In otherwords, a substitution with the above methods may not exist for an aribtrary choice of $n \times n$ matrix A and arbitrary column vector k.

## 2.6 Simple Higher Order System

Suppose the dynamical system follows the expression $\overset{m}{x} = Ax$, a similar technique with eigenvalues and eigenvectors may be employed along with the roots of unity. By conjecture, the partial solution to the dynamical system $\overset{m}{x} = Ax$ follows

$$x_p = \bar{v}_i e^{\alpha_i t}$$

$$\dot{x_p} = \alpha_i \bar{v}_i e^{\alpha_i t}$$
$$\ddot{x_p} = \alpha_i^2 \bar{v}_i e^{\alpha_i t}$$
$$\overset{m}{x_p} = \alpha_i^m \bar{v}_i e^{\alpha_i t}$$
$$Ax_p = \overset{m}{x_p}$$
$$A\bar{v}_i e^{\alpha_i t} = \alpha_i^m \bar{v}_i e^{\alpha_i t}$$
$$A\bar{v}_i = \alpha_i^m \bar{v}_i$$

Since $A\bar{v}_i = \alpha_i^m \bar{v}_i$ wherein $\lambda_i$ are eigenvalues of $A$, then $\lambda_i = \alpha_i^m$. Since $\lambda_i$ may be a complex number, $\alpha_i$ must be the roots of unity to the complex number $\lambda_i$. If $\lambda_i = a + bi$

$$\alpha_n = (a^2 + b^2)^{\frac{1}{2m}} cis \left[ \frac{1}{m} arctan \left( \frac{b}{a} \right) + \frac{2\pi n}{m} \right]$$

The general solution to the problem must be the linear combination of the partial solutions $\sum_{i=1}^{m} \left[ c_i \bar{v}_i e^{\alpha_{in} t} \right]$ wherein $c_i$ are constants determined by the initial conditions and $\alpha_{in}$ represents the $n^{th}$ root of unity of the $i^{th}$ eigenvalue albeit complex or real.

## 2.7 Simple $n^{th}$ Order Homogenous System

Suppose the differential equation follows the expression:

$$0 = \sum_{i=0}^{m} [A_i \overset{i}{x}] = A_0 x + A_1 \dot{x} + A_2 \ddot{x} + \dots + A_{i-1} \overset{i-1}{x} + A_i \overset{i}{x}$$

The general solution to the system above is a linear combination of the partial solutions, $x(t) = \sum_{j=1}^{n} [c_j \bar{v}_j e^{\lambda_j t}]$ wherein partial solutions are defined as $x_{partial}(t) = c_j \bar{v}_j e^{\lambda_j t}$ and $c_1, c_2 \dots c_n$ are constants determined by the initial value of the problem.

$$x_p(t) = c_j \bar{v}_j e^{\lambda_j t}$$

$$\overset{k}{x}_p(t) = c_j \bar{v}_j \lambda_j^k e^{\lambda_j t}$$

$$0 = \sum_{i=0}^{m} [A_i \bar{v}_j c_j \lambda_j^i e^{\lambda_j t}] = A_0 \bar{v}_j c_j e^{\lambda_j t} + A_1 \bar{v}_j c_j \lambda_j e^{\lambda_j t} + \cdots + A_m \bar{v}_j c_j \lambda_j^m e^{\lambda_j t}$$

For the non-trivial solutions to the homoegenous system of differential equations, $c_j, \bar{v}_j, \lambda_j \neq 0$. The function $e^{\lambda_j t} \neq 0$ for all time. Therefore,

$$0 = \left\{ \sum_{i=0}^{m} [A_i \lambda_j^i] \right\} \bar{v}_j$$

For $\bar{v}_j \neq 0$, the matrix $\displaystyle\sum_{i=0}^{m} [A_i \lambda_j^i]$ must be non-invertible. Therefore, $det \left\{ \displaystyle\sum_{i=0}^{m} [A_i \lambda_j^i] \right\} = 0$
The expressions for $\lambda_j^i$ could be substituted to the expression $A_i \lambda_j^i \bar{v}_j = 0$ to express vector $\bar{v}_j$ explicitly.

# Chapter 3

# Fourier Series

## 3.1 Definition of Inner Product

Let f and h be complex valued vectors,

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad , \quad h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

The inner product is formally defined as:

$$(f, h) = \sum_{k=1}^{n} \left[ f_k \bar{h_k} \right]$$

wherein $\bar{h_k}$ represents the complex conjugate of the $k^{th}$ element of the vector $h$. There are some properties of the inner product:

$$(f, h) = \overline{(h, f)} \quad , \quad (\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h) \quad , \quad (f, f) \geq 0$$

$(f, f) = 0$ if and only if $f_k = 0$ for all $k$ of the vector elements. The magnitude of $n^{th}$ dimensional vectors:

$$\|f\| = \left( \sum_{k=1}^{n} |f_k|^2 \right)^{\frac{1}{2}}$$

## 3.2 Definition of Lebesgue Space

A function would be in Lebesgue space if

$$\int_0^\tau |f(t)|^2 dt < \infty$$

The inner product of a function on the Lebesgue space $L^2(0, \tau)$:

$$(f, g) = \frac{1}{\tau} \int_0^\tau f(t) \overline{g(t)} dt$$

The norm of a function in Lebesgue space:

$$\|f\| = \left[\frac{1}{\tau}\int_0^\tau |f(t)|^2 dt\right]^{\frac{1}{2}}$$

Therefore, it follows that

$$\|f\|^2 = (f, f) = \frac{1}{\tau}\int_0^\tau |f(t)|^2 dt$$

Distance between two functions defined in Lebesgue space:

$$\|f - g\| = \left[\frac{1}{\tau}\int_0^\tau |f(t) - g(t)|^2 dt\right]^{\frac{1}{2}}$$

## 3.3 Exponential Fourier Series

Fourier series in exponential form:

$$f(t) = \sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0 t}\right]$$

wherein $k$ represent integers and the coefficients $a_k$ could be found by,

$$a_k = \frac{1}{\tau}\int_0^\tau e^{ik\omega_0 t} f(t) dt$$

wherein $\omega_0 = 2\pi/\tau$. An orthonormal set in Lebesgue space is defined as a collection of functions that are orthonormal to each other in Lebesgue space and have a magnitude of one. The proof below shows that the complex exponential $e^{ik\omega_0 t}$ forms an orthonormal set. If two functions are orthogonal, then their inner products in Lebesgue space must be zero.

$$(f, g) = \frac{1}{\tau}\int_0^\tau f(t)\overline{g(t)} dt$$

Substituting for the complex exponential functions, $f(t) = e^{-ik_1\omega_0 t} \quad , \quad g(t) = e^{-ik_2\omega_0 t}$,

$$(f, g) = \frac{1}{\tau}\int_0^\tau e^{-ik_1\omega_0 t} e^{ik_2\omega_0 t} dt = \frac{1}{\tau}\int_0^\tau e^{i(k_2 - k_1)\omega_0 t} dt$$

For the case wherein $k_2 = k_1$,

$$(f, g) = \frac{1}{\tau}\int_0^\tau 1 dt = \frac{1}{\tau}[t]_0^\tau = \frac{1}{\tau}(\tau) = 1$$

Using the previous definition of function magnitudes in Lebesgue space, $\|f\|^2 = (f, f) \quad , \quad \|f\| = \sqrt{(f, f)}$. Therefore, $\|f\| = 1$ which shows that the complex exponential function has a magnitude of 1 in Lebesgue space. For the case wherein $k_2 \neq k_1$, the subtraction of the two integers yields another non-zero integer.

$$(f, g) = \frac{1}{\tau}\int_0^\tau \cos\left[(k_2 - k_1)\omega_0 t\right] + i\sin\left[(k_2 - k_1)\omega_0 t\right] dt$$

$$(f, g) = \frac{1}{\tau(k_2 - k_1)\omega_0}\left\{\sin\left[(k_2 - k_1)\omega_0 t\right] - i\cos\left[(k_2 - k_1)\omega_0 t\right]\right\}_{t=0}^{t=\tau}$$

Substituting for $\omega_0 = \dfrac{2\pi}{\tau}$

$$\left\{\sin\left[\frac{2\pi(k_2-k_1)t}{\tau}\right]\right\}_{t=0}^{t=\tau} = \sin\left[2\pi(k_2-k_1)\right] - \sin\left[0\right] = 0$$

$$\left\{\cos\left[\frac{2\pi(k_2-k_1)t}{\tau}\right]\right\}_{t=0}^{t=\tau} = \cos\left[2\pi(k_2-k_1)\right] - \cos\left[0\right] = 1 - 1 = 0$$

Therefore, for the case wherein $k_2 \neq k_1$, $(f, g) = 0$. This shows that complex exponentials are form an orthogonal set in Lebesgue space. Since $e^{ik\omega_0 t}$ forms an orthonormal set, $e^{ik\omega_0 t}$ could be used as a basis to represent any function that is in Lebesgue space. The proof below shows the method to find the complex coefficients $a_k$ for an arbitrary function $f(t)$ in Lebesgue space.

$$f(t) = \sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0 t}\right]$$

For some particular integer $k_2$,

$$\frac{1}{\tau}\int_0^{\tau} e^{ik_2\omega_0 t} f(t)dt = \frac{1}{\tau}\int_0^{\tau} e^{ik_2\omega_0 t}\sum_{k=-\infty}^{\infty}\left[a_k e^{-ik\omega_0 t}\right]dt = \sum_{k=-\infty}^{\infty}\left[\frac{a_k}{\tau}\int_0^{\tau} e^{-ik\omega_0 t}e^{ik_2\omega_0 t}dt\right]$$

The above working is true due to the integral operation being a linear operation. Linear operations are discussed earlier in this document. From the previous findings,

$$(f, g) = \frac{1}{\tau}\int_0^{\tau} e^{-ik_1\omega_0 t}e^{ik_2\omega_0 t}dt = \frac{1}{\tau}\int_0^{\tau} e^{i(k_2-k_1)\omega_0 t}dt$$

$(f, g) = 1$ only when $f$ and $g$ are identical to each other. For all other cases, $(f, g) = 0$. Following this, $\dfrac{1}{\tau}\displaystyle\int_0^{\tau} e^{-ik_1\omega_0 t}e^{ik_2\omega_0 t}dt = 1$ only when $k = k_2$, otherwise, $\dfrac{1}{\tau}\displaystyle\int_0^{\tau} e^{-ik_1\omega_0 t}e^{ik_2\omega_0 t}dt = 0$. Therefore,

$$\frac{1}{\tau}\int_0^{\tau} e^{ik_2\omega_0 t} f(t)dt = \sum_{k=-\infty}^{\infty}\left[a_k \times \frac{1}{\tau}\int_0^{\tau} e^{-ik\omega_0 t}e^{ik_2\omega_0 t}dt\right] = a_k$$

## 3.4 Trigonometric Fourier Series

Any arbitrary function $f(t)$ with a periodicity of $L$ could be expressed as a linear combination of sinusoids of varying frequencie. $a_n$ and $b_n$, but $n$ are integers $n = 1, 2, 3, 4 \ldots$. The abritrary function $f(t)$ expressed as a linear combination of trigonometric functions:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)\right]$$

The three equations below is correct and serves as a method to find coefficients $a_0$, $a_n$ and $b_n$,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

The proof of each of the three equations is showb. Below is written the list of trigonometric identities relating multiplication of trigonometric functions of differing frequencies that will be important for the proof:

$$2\cos(\theta)\cos(\phi) = \cos(\theta-\phi)+\cos(\theta+\phi)$$
$$2\sin(\theta)\sin(\phi) = \cos(\theta-\phi)-\cos(\theta+\phi)$$
$$2\sin(\theta)\cos(\phi) = \sin(\theta+\phi)+\sin(\theta-\phi)$$

For coefficient $a_0$,

$$\int_{-L}^{L} f(t)dt = \frac{1}{2}\int_{-L}^{L} a_0 dt + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt\right]$$

$$\frac{1}{2}\int_{-L}^{L} a_0 dt = \frac{1}{2}a_0 \times 2L$$
$$\frac{1}{2}\int_{-L}^{L} a_0 dt = a_0 L$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt = \frac{L}{n\pi}\left[\sin\left(\frac{n\pi}{L}t\right)\right]_{t=-L}^{t=L}$$
$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt = \frac{L}{n\pi}\left[\sin(n\pi)-\sin(-n\pi)\right]$$

Considering that $n$ is an integer, $\sin(n\pi)=\sin(-n\pi)=0$. Therefore,

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt = 0$$

By similar reasoning, it could be seen that $\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)=0$, but the integral is evaluated below anyways,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{L}t\right)\right]_{t=-L}^{t=L}$$
$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = -\frac{L}{n\pi}\left[\cos(n\pi)-\cos(-n\pi)\right]$$

By the even property of the cosine function, $\cos(\theta)=\cos(-\theta)$. Therefore,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = -\frac{L}{n\pi}\left[\cos(n\pi)-\cos(n\pi)\right]$$
$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt = 0$$

By reiterating the integration of $f(t)$ from $t=-L$ until $t=L$

$$\int_{-L}^{L} f(t)dt = a_0 L + \sum_{n=1}^{\infty} [a_n \times 0 + b_n \times 0]$$

$$\int_{-L}^{L} f(t)dt = a_0 L$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

For coefficient $a_n$, let $m$ be some particular integer, either 1, 2, or 3. For some of the terms of the Fourier Series, $m=n$. However, for all other terms, $m \neq n$.

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt = \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt + b_n \int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt\right]$$
$$+\frac{1}{2}a_0\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt$$

$$2\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right) = \cos\left[\frac{(n-m)\pi}{L}t\right] + \cos\left[\frac{(n+m)\pi}{L}t\right]$$

$$\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right) = \frac{1}{2}\cos\left[\frac{(n-m)\pi}{L}t\right] + \frac{1}{2}\cos\left[\frac{(n+m)\pi}{L}t\right]$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = \frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(n-m)\pi}{L}t\right]dt + \frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(n+m)\pi}{L}t\right]dt$$

Consider the case wherein $m=n$,

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{2n\pi}{L}t\right) + 1dt$$

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}\left[\frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}t\right) + t\right]_{t=-L}^{t=L}$$

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}\left[\frac{L}{2n\pi}\Big(\sin(2n\pi) - \sin(-2n\pi)\Big) + 2L\right]$$

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}t\right)dt = \frac{1}{2}[2L] = L$$

Consider the case wherein $m \neq n$,

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = \frac{L}{2(n-m)\pi}\left\{\sin\left[\frac{(n-m)\pi}{L}t\right]\right\}_{t=-L}^{t=L}$$
$$+\frac{L}{2(n+m)\pi}\left\{\sin\left[\frac{(n+m)\pi}{L}t\right]\right\}_{t=-L}^{t=L}$$

By similar argument mentioned previously that sine of a multiple of $\pi$ yields 0, then

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = 0$$

For the second term in the summation notation,

$$2\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right) = \sin\left[\frac{(n+m)\pi}{L}t\right] + \sin\left[\frac{(n-m)\pi}{L}t\right]$$

$$\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right) = \frac{1}{2}\sin\left[\frac{(n+m)\pi}{L}t\right] + \frac{1}{2}\sin\left[\frac{(n-m)\pi}{L}t\right]$$

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = \frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(n+m)\pi}{L}t\right]dt + \frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(n-m)\pi}{L}t\right]dt$$

For the case wherein $m \neq n$,

$$\begin{aligned}\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = &-\frac{L}{2(n+m)\pi}\left\{\cos\left[\frac{(n+m)\pi}{L}t\right]\right\}_{t=-L}^{t=L} \\ &-\frac{L}{2(n-m)\pi}\left\{\cos\left[\frac{(n-m)\pi}{L}t\right]\right\}_{t=-L}^{t=L}\end{aligned}$$

Since $\cos(x)$ is an even function, the integral evaliuates to 0. Therefore,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = 0$$

For the case wherein $m = n$, the second sine function is irrelevant because $\sin(0) = 0$, due to $n - m = 0$. The following is just a degenerate case of the case wherein $m \neq n$.

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = \frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(n+m)\pi}{L}t\right]dt$$

Since the integral above is just a degenerate case of $m \neq n$, then the integral just evaluates to 0. Therefore,

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right)dt = 0$$

For coefficient $b_n$,

$$\int_{-L}^{L}f(t)dt = \frac{1}{2}\int_{-L}^{L}a_0 dt + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)dt + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)dt\right]$$

For the final term in the expression describing the integral of $f(t)\cos\left(\frac{n\pi}{L}t\right)$,

$$\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt = \frac{L}{m\pi}\left[\sin\left(\frac{m\pi}{L}t\right)\right]_{t=-L}^{t=L}$$

Since $m$ is an integer,

$$\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt = 0$$

A reiteration of the integral of $f(t)\cos\left(\frac{n\pi}{L}t\right)$,

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt = \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt\right]$$
$$+\frac{1}{2}a_0\int_{-L}^{L}\cos\left(\frac{m\pi}{L}t\right)dt$$

By substituing all the known parts from the previous workings,

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt = a_m L$$

wherein $m = n$. Therefore,

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt$$

for $n = 1, 2, 3\ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

The proof for the coefficient $b_n$ could be done in a similar way as for the coefficient $a_n$. These two coefficients are analogous to each other.

## 3.5 Discrete Fourier Transform

The usage of the Discrete Fourier Transform Matrix is given below.

$$\begin{bmatrix} p(t_0) \\ p(t_1) \\ p(t_2) \\ \vdots \\ p(t_{v-3}) \\ p(t_{v-2}) \\ p(t_{v-1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{-3} & \lambda_1^{-2} & \lambda_1^{-1} \\ 1 & \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \cdots & \lambda_2^{-3} & \lambda_2^{-2} & \lambda_2^{-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{v-3}^1 & \lambda_{v-3}^2 & \lambda_{v-3}^3 & \cdots & \lambda_{v-3}^{-3} & \lambda_{v-3}^{-2} & \lambda_{v-3}^{-1} \\ 1 & \lambda_{v-2}^1 & \lambda_{v-2}^2 & \lambda_{v-2}^3 & \cdots & \lambda_{v-2}^{-3} & \lambda_{v-2}^{-2} & \lambda_{v-2}^{-1} \\ 1 & \lambda_{v-1}^1 & \lambda_{v-1}^2 & \lambda_{v-1}^3 & \cdots & \lambda_{v-1}^{-3} & \lambda_{v-1}^{-2} & \lambda_{v-1}^{-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{-3} \\ a_{-2} \\ a_{-1} \end{bmatrix}$$

## 3.6 Higher-Dimensional Fourier Series

The Fourier Series in complex exponential form,

$$f(t) = \sum_{k_i=-\infty}^{\infty}\left[a_{k_i}e^{-ik_i\omega_{k_i}t}\right] \quad , \quad \omega_{k_i} = 2\pi/\tau_{k_i}$$

wherein $k_i$ represent integers. Let the two-dimensional Fourier Series be defined as the total expression of a Fourier Series nested in the coefficients of another Fourier Series,

$$f_2(x_1, x_2) = \sum_{k_1=-\infty}^{\infty}\left[\sum_{k_2=-\infty}^{\infty}\left(a_{k_1,k_2}e^{-ik_2\omega_{k_2}x_2}\right)e^{-ik_1\omega_{k_1}x_1}\right] = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\left(a_{k_1,k_2}e^{-ik_2\omega_{k_2}x_2}e^{-ik_1\omega_{k_1}x_1}\right)$$

$$f_2(x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \left[ a_{k_1,\, k_2} e^{-i(k_1 \omega_{k_1} x_1 + k_2 \omega_{k_2} x_2)} \right]$$

wherein the natural frequency $\omega$ are defined as,

$$\omega_{k_1} = 2\pi/\tau_{k_1} \quad , \quad \omega_{k_2} = 2\pi/\tau_{k_2}$$

wherein $\tau_{k_1}$ represent the outer Fourier interval, and $\tau_{k_2}$ represent the inner Fourier interval. Therefore, generalizing to $n$ dimensions,

$$f_m(x_1, x_2, \ldots, x_n) = \prod_{m=1}^{n} \left\{ \sum_{k_m=-\infty}^{\infty} \left[ a_{k_1,\, k_2, \ldots, k_n} e^{-i\left[\sum_{r=1}^{n} \left(k_r \omega_{k_r} x_r\right)\right]} \right] \right\}$$

wherein $\prod_{m=1}^{n} \left[ \sum_{k_m=-\infty}^{\infty} (obj) \right]$ represents $n$ summations of mathematiical objects nested in each other. The expression above does not represent consecutive multiplications of the product notation. The product notation is just used to represent summation notations placed side by side and implemented consecutively in any order.

# Chapter 4

# Laplace Transform

## 4.1 Definition of Laplace Transform

Laplace Transform is defined as the following,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

The Laplace Transform is a linear transform since the integral and product operations are both linear operations as well.

$$\mathcal{L}[\alpha f(t)] = \alpha \mathcal{L}[f(t)] \quad , \quad \mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$$

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

wherein $\alpha$, $\beta$ represent constants and $f(t)$, $g(t)$ represent functions of t.

## 4.2 Transforms of Derivatives

In General form,

$$\mathcal{L}[\overset{n}{f}(t)] = s^n \mathcal{L}[f(t)] - \sum_{k=0}^{n-1} \left[ s^{n-1-k} \overset{k}{f}(0) \right]$$

wherein $\overset{n}{f}(t)$ represents the $n^{th}$ derivative of the function $f(t)$. Proof is shown below,

$$\mathcal{L}[\overset{n}{f}(t)] = \int_0^\infty e^{-st} \overset{n}{f}(t)\, dt$$

$$\int uv'\, dt = uv - \int u'v\, dt$$

$$u = e^{-st} \quad , \quad u' = -se^{-st} \quad , \quad v' = \overset{n}{f}(t) \quad , \quad v = \overset{n-1}{f}(t)$$

$$\mathcal{L}[\overset{n}{f}(t)] = \int_0^\infty e^{-st} \overset{n}{f}(t) dt = \left[ e^{-st} \overset{n-1}{f}(t) \right]_0^\infty - \int_0^\infty -se^{-st} \overset{n-1}{f}(t)\, dt = -\overset{n-1}{f}(0) + s \int_0^\infty e^{-st} \overset{n-1}{f}(t)\, dt$$

Generalizing for the second integral term,

$$\int_0^\infty e^{-st} \overset{n-i}{f}(t)\, dt = \left[ e^{-st} \overset{n-1-i}{f}(t) \right]_0^\infty + s \int_0^\infty e^{-st} \overset{n-1-i}{f}(t)\, dt = -\overset{n-1-i}{f}(0) + s \int_0^\infty e^{-st} \overset{n-1-i}{f}(t)\, dt$$

By applying substitution recursively,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{k}\left[s^i \overset{n-1-i}{f}(0)\right] + \prod_{i=0}^{k}[s]\int_0^\infty e^{-st}\overset{n-1-k}{f}(t)\,dt = -\sum_{i=0}^{k}\left[s^i\overset{n-1-i}{f}(0)\right] + s^{k+1}\int_0^\infty e^{-st}\overset{n-1-k}{f}(t)\,dt$$

Substituting the value for $k = n-1$,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{n-1}\left[s^i\overset{n-1-i}{f}(0)\right] + s^n\int_0^\infty e^{-st}f(t)\,dt$$

A few things should be noted,

$$\int_0^\infty e^{-st}f(t)\,dt = \mathcal{L}[f(t)] \quad , \quad \sum_{i=0}^{n-1}\left[s^i\overset{n-1-i}{f}(0)\right] = \sum_{i=0}^{n-1}\left[s^{n-1-i}\overset{i}{f}(0)\right]$$

By substitution of the counting variable $i$ with $k$,

$$\mathcal{L}[\overset{n}{f}(t)] = -\sum_{i=0}^{n-1}\left[s^i\overset{n-1-i}{f}(0)\right] + s^n\int_0^\infty e^{-st}f(t)\,dt = s^n\mathcal{L}[f(t)] + \sum_{k=0}^{n-1}\left[s^{n-1-k}\overset{k}{f}(0)\right]$$

## 4.3 Transforms of Integrals

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}[f(t)]$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt$$

$$\int uv'\,dt = uv - \int u'v\,dt$$

$$u = \int_0^t f(\tau)d\tau \quad , \quad u' = f(t) \quad , \quad v' = e^{-st} \quad , \quad v = -\frac{1}{s}e^{-st}$$

$$\int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt = -\left[\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\right]_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}f(t)\,dt$$

It should be noted that since the function $f(t)$ is in exponential order,

$$\left[\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\right]_0^\infty = 0$$

Substituting the $uv$ term with zero,

$$\int_0^\infty e^{-st}\int_0^t f(\tau)d\tau\,dt = \int_0^\infty \frac{1}{s}e^{-st}f(t)\,dt = \frac{1}{s}\int_0^\infty e^{-st}f(t)\,dt = \frac{1}{s}\mathcal{L}[f(t)]$$

## 4.4 Derivative of Transforms

$$\mathcal{L}[t^n f(t)] = (-1)^n \overset{n}{F}(s) = (-1)^n \frac{d^n}{ds^n} \left\{\mathcal{L}[f(t)]\right\}$$

wherein $F(s)$ represents the laplace transform of the function $f(t)$. By the definition of Laplace Transforms discussed earlier,

$$\mathcal{L}[t^n f(t)] = \int_0^\infty e^{-st} t^n f(t)\, dt \quad , \quad F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

Differentiating the Laplace Transform of $f(t)$ with respect to $s$ iteratively $n$ times,

$$\overset{n}{F}(s) = \frac{d^n}{ds^n}\{\mathcal{L}[f(t)]\} = \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\, dt = (-1)^n \int_0^\infty e^{-st} t^n f(t)\, dt$$

$$(-1)^n \overset{n}{F}(s) = \int_0^\infty e^{-st} t^n f(t)\, dt$$

By substituting the definition for the Laplace Transform of $t^n f(t)$,

$$(-1)^n \overset{n}{F}(s) = \mathcal{L}[t^n f(t)]$$

## 4.5 Integration of Transforms

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tau)\, d\tau$$

wherein $F(s)$ represents the laplace transform of the function $f(t)$.

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_0^\infty \frac{e^{-st} f(t)}{t}\, dt \quad , \quad F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\int_s^\infty F(\tau)\, d\tau = \int_s^\infty \int_0^\infty e^{-\tau t} f(t)\, dt\, d\tau = \int_0^\infty \int_s^\infty e^{-\tau t} f(t)\, d\tau\, dt = \int_0^\infty \left[-\frac{1}{t} e^{-\tau t} f(t)\right]_{\tau=s}^{\tau=\infty} dt$$

$$\int_s^\infty F(\tau)\, d\tau = -\int_0^\infty \frac{f(t)}{t} \left[e^{-\tau t}\right]_{\tau=s}^{\tau=\infty} dt = -\int_0^\infty \frac{f(t)}{t} \left[\lim_{\tau\to\infty} (e^{-\tau t}) - e^{-st}\right] dt$$

Taking into account that, $\lim_{\tau\to\infty} (e^{-\tau t}) = 0$,

$$\int_s^\infty F(\tau)\, d\tau = -\int_0^\infty \frac{f(t)}{t} \left[-e^{-st}\right] dt = \int_0^\infty \frac{e^{-st} f(t)}{t}\, dt$$

By substituting the definition of the laplace transform of $\frac{f(t)}{t}$,

$$\int_s^\infty F(\tau)\, d\tau = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

## 4.6 Translation of Transforms

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\mathcal{L}\left[f(t+c)\right]$$

By definition of Laplace Transform,

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_0^\infty e^{-st}u(t-c)f(t)\,dt = \int_c^\infty e^{-st}f(t)\,dt + \int_0^c e^{-st}\times 0\,dt$$

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_c^\infty e^{-st}f(t)\,dt$$

Using the substitution $t = \tau + c$. When $t = \infty \quad , \quad \tau = \infty \quad$ and when $t = c \quad , \quad \tau = 0$. Therefore,

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_{t=c}^{t=\infty} e^{-s(\tau+c)}f(\tau+c)\,dt = \int_{\tau=0}^{\tau=\infty} e^{-s(\tau+c)}f(\tau+c)\,d\tau$$

$$\mathcal{L}\left[u(t-c)f(t)\right] = \int_{\tau=0}^{\tau=\infty} e^{-s\tau-sc}f(\tau+c)\,d\tau = \int_{\tau=0}^{\tau=\infty} e^{-cs}e^{-s\tau}f(\tau+c)\,d\tau$$

Since variables $s$ and $c$ are not changing with time, the term $e^{-cs}$ could be treated as some form of constant. Therefore,

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\int_0^\infty e^{-s\tau}f(\tau+c)\,d\tau$$

It should be noted that the change of variables allows,

$$\mathcal{L}\left[f(t+c)\right] = \int_0^\infty e^{-st}f(t+c)\,dt = \int_0^\infty e^{-s\tau}f(\tau+c)\,d\tau$$

By substitution,

$$\mathcal{L}\left[u(t-c)f(t)\right] = e^{-cs}\mathcal{L}\left[f(t+c)\right]$$

## 4.7 Transforms of Translated Functions

$$\mathcal{L}[e^{ct}f(t)] = F(s-c)$$

Reiterating the definition of laplace transforms,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt = F(s)$$

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{ct}e^{-st}f(t)\,dt = \int_0^\infty e^{-st+ct}f(t)\,dt$$

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c)$$

## 4.8 Convolution

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

The convolution is a commutative transformation. Therefore,

$$f * g(t) = g * f(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

One useful property of the convolution function,

$$\mathcal{L}\left[f * g(t)\right] = \mathcal{L}\left[f(t)\right] \times \mathcal{L}\left[g(t)\right]$$

wherein

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \quad , \quad G(s) = \mathcal{L}[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

By a substitution of variables $t = u$ it could be re-written,

$$F(s) = \mathcal{L}[f(u)] = \int_0^\infty e^{-su} f(u)\,du \quad , \quad G(s) = \mathcal{L}[g(u)] = \int_0^\infty e^{-su} g(u)\,du$$

Examining the Lapalce Transform of $g(u)$, and making the substitution $u = t - \tau$

$$\mathcal{L}[g(t-\tau)] = \int_{u=0}^{u=\infty} e^{-s(t-\tau)} g(t-\tau)\,dt$$

When $u = \infty \quad , \quad t = \infty$ and when $u = 0 \quad , \quad t = \tau$. Therefore,

$$\mathcal{L}[g(t-\tau)] = \int_{t=\tau}^{t=\infty} e^{-s(t-\tau)} g(t-\tau)\,dt$$

The $e^{\tau s}$ term could be isolated because both variables $\tau$ and $s$ in this case are non-changing with $t$. The next form is identical to the laplace transform at the Translation of Transforms section,

$$\int_{\tau=0}^{\tau=\infty} e^{-s(\tau+c)} f(\tau+c)\,d\tau = e^{-cs} \int_0^\infty e^{-s\tau} f(\tau+c)\,d\tau$$

By substituting $\tau$ in the Translation of Transforms section with $t$, substituting $c$ with $-\tau$, and substituting the arbitrary function $g$ with the arbitrary function $f$,

$$\int_{t=0}^{t=\infty} e^{-s(t-\tau)} g(t-\tau)\,dt = e^{\tau s} \int_0^\infty e^{-st} g(t-\tau)\,dt$$

Therefore,

$$\mathcal{L}[g(t-\tau)] = G(s) = e^{\tau s} \int_0^\infty e^{-st} g(t-\tau)\,dt$$

Proving the Convolution Property by first examining the product of the two Laplace Transforms,

$$F(s) \times G(s) = G(s) \int_0^\infty e^{-su} f(u)\,du = \int_0^\infty e^{-su} G(s) f(u)\,du$$

The above would be perfectly legal operations because $G(s)$ is a function in terms of $s$ and is unchanging with respect to variable $t$. Therefore, the function $G(s)$ could be treated as a constant that can be place inside and outside of the integral.

$$F(s) \times G(s) = \int_0^\infty e^{-s\tau} f(\tau) \times e^{\tau s} \int_0^\infty e^{-st} g(t-\tau)\, dt\, d\tau$$

$$F(s) \times G(s) = \int_0^\infty \int_0^\infty e^{-st} f(\tau) g(t-\tau)\, dt\, d\tau$$

By chaging the order of integration,

$$F(s) \times G(s) = \int_0^\infty e^{-st} \int_0^\infty f(\tau) g(t-\tau)\, d\tau\, dt = \mathcal{L}\left[\int_0^\infty f(\tau) g(t-\tau)\, d\tau\right]$$

$$F(s) \times G(s) = \mathcal{L}\left[f * g(t)\right]$$

# Chapter 5

# Gradient Operators

Given the the arbitrary fucntion $f$,

## 5.1 Cartesian Coordinates

## 5.2 Cylindrical Coordinates

## 5.3 Spherical Coordinates

# Chapter 6

# Partial Differential Equations

The conventional gradient operator in cartesian coordinates is typically defined as,

$$\nabla_{xyz} = \left( \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right)^T \quad , \quad \nabla_{xyz}^n = \left( \frac{\partial^n}{\partial x^n} \quad \frac{\partial^n}{\partial y^n} \quad \frac{\partial^n}{\partial z^n} \right)^T$$

Let the modified gradient operator (${}_m\nabla$) in cartesian coordinates be defined as,

$${}_m\nabla_{xyz} = \left( \alpha_1 \frac{\partial}{\partial x} \quad \alpha_2 \frac{\partial}{\partial y} \quad \alpha_3 \frac{\partial}{\partial z} \right)^T \quad , \quad {}_m\nabla_{xyz}^n = \left( \beta_1 \frac{\partial^n}{\partial x^n} \quad \beta_2 \frac{\partial^n}{\partial y^n} \quad \beta_3 \frac{\partial^n}{\partial z^n} \right)^T$$

This modification allows the gradient operator to be more general. The modified gradient operator for $m$ dimensional cartesian coordinates,

$${}_m\nabla_{xyz} = \left( \alpha_1 \frac{\partial}{\partial x_1} \quad \alpha_2 \frac{\partial}{\partial x_2} \quad \dots \quad \alpha_n \frac{\partial}{\partial x_m} \right)^T \quad , \quad {}_m\nabla_{xyz}^n = \left( \beta_1 \frac{\partial^n}{\partial x_1^n} \quad \beta_2 \frac{\partial^n}{\partial x_2^n} \quad \dots \quad \beta_3 \frac{\partial^n}{\partial x_m^n} \right)^T$$

## 6.1 Methods in Generalized Cartesian Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{x_1 \dots x_q}(u) + {}_{m_2}\nabla_{x_1 \dots x_q}(u) = \sum_{i=1}^{q} \left[ b_i \frac{\partial^2 u}{\partial x_i^2} + c_i \frac{\partial u}{\partial x_i} \right]$$

$$\text{wherein } {}_{m_1}\nabla^2_{x_1 \dots x_q}(u) = {}_{m_1}\nabla_{x_1 \dots x_q} \cdot \nabla_{x_1 \dots x_q} u$$

## 6.2 Methods in Cylindrical Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{r\theta z}(u) + {}_{m_2}\nabla_{r\theta z} \cdot (u) =$$

## 6.3 Methods in Spherical Coordinates

$$\frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t} = {}_{m_1}\nabla^2_{r\theta\phi}(u) + {}_{m_2}\nabla_{r\theta\phi}(u) =$$

# Chapter 7

# Temperature Distribution of Cartesian Slabs

$$\frac{\partial u}{\partial t} = k\nabla^2_{xy}(u) = k\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

# Chapter 8

# Temperature Distribution of Polar Slabs

$$\frac{\partial u}{\partial t} = k\nabla_r^2(u) = k\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

# Chapter 9

# Longitudinal Structural Bar Vibrations

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

# Chapter 10

# Transverse Structural Bar Vibrations

$$\frac{\partial^2 y}{\partial t^2} = -a^4 \frac{\partial^4 y}{\partial x^4}$$

# Chapter 11

# Natural Frequencies of Beams

$$\frac{\partial^2 y}{\partial t^2} = -a^4 \frac{\partial^4 y}{\partial x^4}$$

# Chapter 12

# Two-Dimensional Wave Equation in Cartesian Coordinates

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2_{xy}(u) = c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

# Chapter 13

# Two-Dimensional Wave Equation in Polar Coordinates

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2_{r\theta}(u) = c^2 \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right]$$

# Chapter 14

# Spherical Harmonics and Ocean Waves

$$\frac{\partial^2 u}{\partial t^2} = b^2 \nabla^2_{\phi\theta}(u) = b^2 \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} \left[ \sin(\phi) \frac{\partial u}{\partial \phi} \right] + \frac{1}{\sin^2(\phi)} \frac{\partial^2 u}{\partial \theta^2} \right\}$$

# Chapter 15

# Sturm-Liouville Problems

## 15.1 Definition of Stum-Liouville Problems

A Sturm-Liouville Problem is a problem that satisfies the following equation with the following boundary conditions,

$$0 = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y$$

Alternately,

$$0 = \frac{dy}{dx}\frac{d}{dx}\left[p(x)\right] + p(x)\frac{d^2y}{dx^2} - q(x)y + \lambda r(x)y$$

$$0 = p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} - q(x)y + \lambda r(x)y$$

The initial conditions are shown below,

$$0 = \alpha_1 y(a) - \alpha_2 y'(a) \quad , \quad 0 = \beta_1 y(b) + \beta_2 y'(b)$$

wherein neither $\alpha_1$ and $\alpha_2$ both zero nor $\beta_1$ and $\beta_2$ both zero. The parameter $\lambda$ is the "eigenvalue" whose possible (constant) values are sought usually via the application of the boundary conditions.

## 15.2 Eigenvalue Theorem of Sturm-Liouville Problems

Suppose that the functions $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous on the closed interval $[a, b]$ and that $p(x) > 0$ and $r(x) > 0$ at each point of $[a, b]$. Then the eigenvalues of the Sturm–Liouville problem constitute an increasing sequence,

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_\infty$$

of real numbers with

$$\lim_{n\to\infty}[\lambda_n] = \infty$$

To within a constant factor, only a single eigenfunction $y_n(x)$ is associated with each eigenvalue $\lambda_n$. Moreover, if $q(x) \geq 0$ on the closed interval $[a, b]$ and the coefficients $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ are all non-negative, then the eigenvalues are all non-negative.

## 15.3 Eigenvalues-Eigenfunctions Series

If the functions $p(x)$, $q(x)$ and $r(x)$ of the Sturm-Liouville problem satisifes the Eigenvalue Theorem, then eigenfunctions $y_i(x)$ and $y_j(x)$ corresponding to eigenvalues $\lambda_i$ and $\lambda_j$ wherein $j \neq 1$ are orthogonal with respect to each other relative to the function $r(x)$,

$$0 = \int_a^b y_i(x) y_j(x) r(x) dx$$

For a sturm-liouville problem with infinite eigenvalues, it is possible to represent an arbitrary function $f(x)$ as the infinite sum of the eigenvalues,

$$f(x) = \sum_{m=1}^{\infty} \left[ c_m y_m(x) \right]$$

wherein $y_m(x)$ represents the $m^{th}$ eigenfunction of the $m^{th}$ eigenvalue $\lambda_m$. Taking the integral in both sides with the product to the eigenfunction $y_n(x)$ relative to the function $r(x)$,

$$\int_a^b f(x) y_n(x) r(x) dx = \int_a^b \sum_{m=1}^{\infty} \left[ c_m y_m(x) \right] y_n(x) r(x) dx$$

Using the assumption, $0 = \int_a^b y_i(x) y_j(x) r(x) dx$,

$$\int_a^b f(x) y_n(x) r(x) dx = \int_a^b c_n \left[ y_n(x) \right]^2 r(x) dx = c_n \int_a^b \left[ y_n(x) \right]^2 r(x) dx$$

Therefore, the constant $c_n$ could be obtained by,

$$c_n = \frac{\int_a^b f(x) y_n(x) r(x) dx}{\int_a^b \left[ y_n(x) \right]^2 r(x) dx}$$

This particular theorem could be used to prove under certain reasonable conditions that there exists an infinite series that would allow the boundary conditions to be implemented analytically into the partial differential equations problems.

# Chapter 16

# Temperature Distribution of a Heated Rod

The function of temperature $u$ at some distance $x$ from the origin of a one-dimensional heated rod of uniform material is governed by the equation below,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

wherein $k$ is some constant related to thermal conductivity. Using the substitution, $u(x,t) = f(x)g(t)$ wherein $f(x)$ is a function purely in $x$ and $g(t)$ is a function purely in terms of $t$,

$$\frac{\partial[f(x)g(t)]}{\partial t} = k\frac{\partial^2[f(x)g(t)]}{\partial x^2}$$

$$f(x)\frac{\partial[g(t)]}{\partial t} = kg(t)\frac{\partial^2[f(x)]}{\partial x^2}$$

$$f(x)g'(t) = kg(t)f''(x)$$

wherein $g'(t)$ represents the first order derivative of $g(t)$ with respect to time $t$, and $f''(x)$ represents second order derivative of $f(x)$ with respect to distance $x$. Further manipulation to yield a left hand side completely in terms of time $t$ and a right hand side completely in terms of displacement $x$,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = \lambda$$

wherein $\lambda$ is some constant. The reason why $\lambda$ is a constant is because $x$ and $t$ are independent variables, therefore a change in one of the values should not affect the other variable. Since the left hand side is and right hand side are in represented completely in independent variables, $\lambda$ must be a constant if $x$ and $t$ are independent variables. $\lambda$ is the eigenvalue of the problem whose values are often sought and is inferred from the boundary conditions. This is as far as analysis can go without specifying the boundary conditions.

## 16.1 Zero-Endpoint Temperatures

Suppose the rod is of length $L$ and that the initial temperature distribution is known. The boundary conditions,

$$u(0,t) = u(L,t) = 0 \quad , \quad u(x,0) = m(x)$$

The first boundary condition is the zero endpoint condition and the second condition is the initial temperature distribution. Reiterating the first boundary condition and substituting $u(x,t)$ as the product of two single variable functions,

$$u(0,t) = u(L,t) = 0 \quad , \quad f(0)g(t) = f(L)g(t) = 0$$

The function $g(t)$ is not trivial, and therefore, $g(t) \neq 0$. Therefore, it follows that,

$$f(0) = f(L) = 0$$

Because the endpoint conditions, it is convenient that the function $f(x)$ be made into a trigonometric function. Consider the eigenvalue to be negative,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = -\lambda$$

Two ordinary differential equation problems can be obtained from this,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = -\lambda \quad , \quad \frac{f''(x)}{f(x)} = -\lambda$$

$$g'(t) = -\lambda k g(t) \quad , \quad f''(x) = -\lambda f(x)$$

The second ordinary differential equation, $f''(x) = -\lambda f(x)$ has the solution,

$$f(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

To satisfy the endpoint condition $f(0) = f(L) = 0$, $c_1 = 0$ and $\sqrt{\lambda}L = n\pi$, wherein $n$ are integers starting from zero. Therefore, $\lambda = n^2\pi^2/L^2$. Substituting the eigenvalues and arbitrary constants $c_1$,

$$f(x) = c_2 \sin\left(\frac{n\pi}{L}x\right)$$

Substituting the eigenvalue and solving the second ordinary differential equations problem,

$$g'(t) = -\frac{n^2\pi^2k}{L^2}g(t)$$

$$\int \frac{1}{g(t)}dg(t) = -\frac{n^2\pi^2k}{L^2}\int dt$$

$$\ln[g(t)] = -\frac{n^2\pi^2k}{L^2}t + c$$

$$g(t) = Ce^{-\left(\frac{n^2\pi^2k}{L^2}\right)t}$$

Substituting the two equations together,

$$u(x,t) = C_n e^{-\left(\frac{n^2\pi^2k}{L^2}\right)t} \sin\left(\frac{n\pi}{L}x\right)$$

Due to the partial differential operator being a linear operator, the superposition principle holds true. Therefore, the general solution to the partial differential equation must be the linear combination of its linearly independent solutions,

$$u_g(x,t) = \sum_{n=1}^{\infty}\left[C_n e^{-\left(\frac{n^2\pi^2k}{L^2}\right)t} \sin\left(\frac{n\pi}{L}x\right)\right]$$

## 16.2 Insulated Ends

With the same length of rod $L$ and known initial temperature distribution $m(x)$, the boundary conditions,

$$\frac{\partial}{\partial x}\left[u(0,t)\right] = \frac{\partial}{\partial x}\left[u(L,t)\right] = 0 \quad , \quad u(x,0) = m(x)$$

Substituting the boundary conditions with the definition of $u$ as the product of two single variable functions,

$$f'(0)g(t) = f'(L)g(t) = 0$$

Similarly to the previous case, since $g(t)$ is not the trivial zero function,

$$f'(0) = f'(L) = 0$$

Just in the previous part, it is convenient to choose the eigenvalues to be negative in the two ordinary differential equation problems,

$$\frac{1}{k}\frac{g'(t)}{g(t)} = -\lambda \quad , \quad \frac{f''(x)}{f(x)} = -\lambda$$

This is advantageous because $f$ will take the form of a linear combination of trigonometric functions, at which we can simply choose the cosine series to satisfy the boundary condition above. The general form of $f(x)$,

$$f(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

$$f'(x) = -c_1 \sin\left(\sqrt{\lambda}x\right) + c_2 \cos\left(\sqrt{\lambda}x\right)$$

The only conditions that would satisfy the end-point boundary conditions, $c_2 = 0$, $\sqrt{\lambda}L = n\pi$, $\lambda = n^2\pi^2/L^2$. Substituting the eigenvalues and arbitrary constants would yield,

$$f(x) = c_1 \cos\left(\frac{n\pi}{L}x\right)$$

Solving for $g(t)$ yields,

$$g'(t) = -\lambda k g(t)$$

Familiarly,

$$g(t) = e^{-\lambda k t} = Ce^{\left(-\frac{n^2\pi^2 k}{L^2}t\right)}$$

Similarly to the previous chapter and by principle of superposition,

$$u_g(x,t) = \sum_{n=1}^{\infty}\left[C_n e^{-\left(\frac{n^2\pi^2 k}{L^2}\right)t} \cos\left(\frac{n\pi}{L}x\right)\right]$$

# Chapter 17

# One-Dimensional Wave Equation

The equation for the one-dimensional wave equation is shown below,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Using the familiar substitution $y(x,t) = u(x) \times r(t)$,

$$\frac{\partial^2}{\partial t^2}\big[u(x)r(t)\big] = a^2 \frac{\partial^2}{\partial x^2}\big[u(x)r(t)\big]$$

$$u(x)r''(t) = a^2 u''(x)r(t)$$

wherein $r''(t)$ and $u''(x)$ represents the second order derivative of $r(t)$ and $u(x)$ respectively. Manipulation of the equation,

$$\frac{1}{a^2}\frac{r''(t)}{r(t)} = \frac{u''(x)}{u(x)}$$

To fit for the somewhat arbitrary conditions,

$$y(0,t) = y(L,t) = 0 \quad , \quad y(x,0) = f(x) \quad , \quad \frac{\partial}{\partial t}\big[y(x,0)\big] = g(x)$$

it is useful to consider the derivative homogenous case (A) and the displacement homogenous case (B) seperately,

$$y_A(0,t) = y_A(L,t) = 0 \quad , \quad y_B(0,t) = y_B(L,t) = 0$$

$$y_A(x,0) = f(x) \quad , \quad y_B(x,0) = 0$$

$$\frac{\partial}{\partial t}[y_A(x,0)] = 0 \quad , \quad \frac{\partial}{\partial t}[y_B(x,0)] = g(x)$$

The somewhat arbitray boundary condition case would be the satisfied by the addition of the derivative homogenous case and the displacement homogenous case. Both function $y_A(x,t)$ and $y_B(x,t)$ satisifes the partial differential equation, therefore, the algebraic addition of them must also satisfy the one dimensional wave equation. When they are added algebraically,

$$y_A(0,t) + y_B(0,t) = y_A(L,t) + y_B(L,t) = 0 + 0$$

$$y_A(x,0) + y_B(x,0) = f(x) + 0 \quad , \quad \frac{\partial}{\partial t}\big[y_A(x,0) + y_B(x,0)\big] = 0 + g(x)$$

Therefore, the algebraic addition of the derivative homogenou and displacement homogenous satisfies the somewhat arbitrary conditions provided earlier.

## 17.1 Derivative Homogenous Case

The boundary conditions for the derivative homogenous case,

$$y(0,t) = y(L,t) = 0 \quad , \quad y(x,0) = f(x) \quad , \quad \frac{\partial}{\partial t}[y(x,0)] = 0$$

To satisfy the first boundary condition listed above, it would be convenient for the eigenvalues to be considered negative,

$$\frac{1}{a^2}\frac{r''(t)}{r(t)} = \frac{u''(x)}{u(x)} = -\lambda$$

Therefore, the two ordinary differential equations,

$$u''(x) = -\lambda u(x) \quad , \quad r''(t) = -\lambda a^2 r(t)$$

The characteristic equation associated to the displacement differential equation,

$$r^2 = -\lambda \quad , \quad r = \sqrt{\lambda} i$$

Therefore, the displacement function $u(x)$ is in terms of the familiar linear combination of trigonometric functions,

$$u(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

The only constants that will satisfy the condition $y(0,t) = y(L,t) = 0$, $c_1 = 0$, and $\sqrt{\lambda} L = n\pi$. Therefore, the eigenvalues are, $\lambda = \dfrac{n^2\pi^2}{L^2}$. Substituting for eigenvalues and arbitray constant $c_1$ into $u(x)$,

$$u(x) = c_2 \sin\left(\frac{n\pi}{L} x\right)$$

The characteristic equation associated to the time dependent differential equation,

$$r^2 = -\lambda a^2 \quad , \quad r = a\sqrt{\lambda} i$$

Therefore, the trigonometric solution to the above characteristic equation,

$$r(t) = k_1 \cos\left(a\sqrt{\lambda} t\right) + k_2 \sin\left(a\sqrt{\lambda} t\right)$$

$$r(t) = k_1 \cos\left(\frac{n\pi a}{L} t\right) + k_2 \sin\left(\frac{n\pi a}{L} t\right)$$

## 17.2 Displacement Homogenous Case

The boundary conditions for the displacement homogenous case,

$$y(0,t) = y(L,t) = 0 \quad , \quad y(x,0) = 0 \quad , \quad \frac{\partial}{\partial t}[y(x,0)] = g(x)$$

# Chapter 18

# Tensors

## 18.1 Tensor Index Notation

Tensors are a generalization of scalars, vectors, and matrices. The order of a tensor represents how many 'axis' the tensor has. For example, a scalar would be a $0^{th}$ order tensor meanwhile a vector would be a $1^{st}$ order tensor and a matrix would be $2^{nd}$ order tensor. Tensors of higher orders are permitted though a visual representation of them is meaningless. One can alternatively imagine tensors as multi-dimensional arrays, much like the case in a programming language.

The tensor index notation comprises of 2 main indices: A free index and a dummy index. A free index corresponds to the positioning of a certain value in a tensor. For example, the $i^{th}$ component of a vector $\bar{v}$ is usually represented as $v_i$. That is an example of a free index usage. A dummy index is an index that is used for summation. Dummy indices occur in pairs and a pair of dummy indices imply summation. For example in the case of a dot product, $A_jB_j$ represents scalar multiplication between the $j^{th}$ components of vectors $\bar{A}$ and $\bar{B}$, added all together for the enirety of the length of vector $\bar{A}$ and vector $\bar{B}$.

Since what specific name one gives to a an index is arbitrary, this leads to index renaming rules. Dummy indices may be renamed within a single term. For example $A_jB_j = A_iB_i$. Free indices however, must be renamed across all algebraically summed terms. For example, $A_iB_pC_p + D_iE_qF_q = A_jB_pC_p + D_jE_qF_q$

## 18.2 Kronecker-Delta & Permutation Tensor

The kronecker-delta is a function that maps 2 integers to a 1 or 0. A mathematical description of the kronecker-delta function $\delta_{ij}$ is shown below,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

The permutation tensor $\epsilon_{ijk}$ a $3^{rd}$ order tensor and is anti-symmetric in any 2 of the indices. The indices can accept a range of integers from 1 until 3. Therefore, $\epsilon_{123}$, $\epsilon_{213}$ are both valid but $\epsilon_{352}$ is not. The permutation tensor has a cyclic property described below,

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

Switching any 2 index of the permutaton tensor makes it negative. This property is described below,

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

The exact value of the permutation tensor,

$$\epsilon_{123} = 1 \quad , \quad \epsilon_{213} = -1$$

The other cases of $i$, $j$, and $k$ are all obtainable by applying the properties above.

## 18.3 Common Vector Operations

Let $\bar{A}$ and $\bar{B}$ be vector fields, and $\phi$, $\psi$ be scalar fields. Let $A_i$ and $B_i$ represent the $i^{th}$ component of the vector $\bar{A}$ and $\bar{B}$ respectively.

### 18.3.1 Scalar Multiplication

Since scalar multiplication simplt multiples all components of a vector by some scalar,

$$\left[\phi \bar{A}\right]_i = \phi A_i$$

wherein the $LHS$ represents the vector notation and the $RHS$ represents the index notation equivalent. Note that the $[]_i$ is used to denote the $i^{th}$ index of the vector notation.

### 18.3.2 Dot Product

Dot products can be represented very elegantly in tensor index notation,

$$\bar{A} \cdot \bar{B} = A_j B_j$$

The repeated index $j$ here makes $j$ a dummy index which is used for counting. A repeated index such as $j$, implies summation. Therefore,

$$A_j B_j = A_1 B_1 + A_2 B_2 + A_3 B_3$$

### 18.3.3 Cross Product

The cross product of 2 vectors is defined with the permutation tensor,

$$\left[\bar{A} \times \bar{B}\right]_i = \epsilon_{ijk} A_j B_k$$

## 18.4 Tensor Index Identities

Let $\bar{A}$ and $\bar{B}$ be vector fields, and $\phi$, $\psi$ be scalar fields. Let $\bar{\bar{\mu}}$ and $\bar{\bar{\gamma}}$ represent second order tensors,

### 18.4.1 Symmetric-Antisymmetric Tensor

Let $\bar{\bar{\mu}}$ be a symmetric tensor and $\bar{\bar{\gamma}}$ be an anti-symmetric tensor. By the properties of the symmetric and anti-symmetric tensors,

$$\mu_{ij} = \mu_{ji} \quad , \quad \gamma_{ij} = -\gamma_{ji}$$

Consider the following,

$$\mu_{ij}\gamma_{ij} = -\mu_{ji}\gamma_{ji}$$

Here, the dummy indices have been switched, and this is true due to the symemtric and anti-symmetric definitions of $\mu$ and $\gamma$. The dummy indices are renamed, $j \to p$, $i \to q$,

$$\mu_{ij}\gamma_{ij} = -\mu_{pq}\gamma_{pq} \tag{18.1}$$

Next, start with $m\bar{\bar{u}}$ and $\bar{\bar{\gamma}}$ again, but this time rename them based on a different set of variable change. $i \to p$ and $j \to q$. This seems illegal, but it is not. Remember, the naming are arbitrary and we have not violated any of the rules. Therefore,

$$\mu_{ij}\gamma_{ij} = \mu_{pq}\gamma_{pq} \tag{18.2}$$

Susbtituting $\mu_{ij}\gamma_{ij}$ out from euqation 18.1 and equation 18.2,

$$\mu_{pq}\gamma_{pq} = -\mu_{pq}\gamma_{pq}$$

Therefore,

$$0 = \mu_{pq}\gamma_{pq}$$

Hence, the element-wise multiplication of a symmetric and anti-symmetric tensor added together for the entire tensor would yield zero.

### 18.4.2 Double Permutation Tensor

Arguably one of the most important identities for tensor indices,

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

### 18.4.3 Kronecker-Delta Renaming

The kronecker-delta function can be used to rename the indices of a tensor,

$$\delta_{ij}A_i = A_j$$

This is because when $i \neq j$, the kronecker-delta function is zero, which means that $\delta_{ij}A_i$ is only non-zero when $i = j$, which renames the dummy variable of $i$ in $A_i$ into $j$.

### 18.4.4 Curl of Scalar Gradient

The curl of a scalar gradient is zero,

$$0 = \nabla \times (\nabla\phi)$$

Let ,

$$LHS = 0 \quad , \quad RHS = \nabla \times (\nabla \phi)$$

Converting $LHS$ and $RHS$ into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[ \frac{\partial \phi}{\partial x_k} \right] = \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_k} (\phi)$$

Since partial derivative operators are commutative, $\frac{\partial^2}{\partial x_j \partial x_k}(\phi)$ is a symmetry tensor. If $i$ is held constant, the permutation tensor $\epsilon_{ijk}$ is anti-symmetric. The element-wise multiplication of a symmetric tensor and anti-symmetric tensor added up together yields zero. Therefore,

$$RHS_i = 0$$

Since $LHS_i = RHS_i$, the claim is proven to be true.

### 18.4.5 Divergence of Vector Curl

The divergence of the curl of a vector field is zero,

$$0 = \nabla \cdot (\nabla \times \bar{A})$$

Let,

$$LHS = 0 \quad , \quad RHS = \nabla \cdot (\nabla \times \bar{A})$$

Converting $LHS$ and $RHS$ into index notation,

$$LHS_i = 0 \quad , \quad RHS_i = \frac{\partial}{\partial x_j} \left[ \epsilon_{jkl} \frac{\partial}{\partial x_k} (A_l) \right] = \epsilon_{jkl} \frac{\partial}{\partial x_j} \left[ \frac{\partial}{\partial x_k} (A_l) \right] = \epsilon_{jkl} \frac{\partial^2}{\partial x_j \partial x_k} (A_l)$$

Since $\epsilon_{jkl}$ is an anti-symmetric tensor and $\frac{\partial^2}{\partial x_j \partial x_k}(A_l)$ is a symmetric tensor, then $RHS_i = 0$. Since $LHS_i = RHS_i$, then the claim is proven to be true.

### 18.4.6 Curl of 2 Vector Cross Products

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Let,

$$LHS = \nabla \times (\bar{A} \times \bar{B}) \quad , \quad RHS = \bar{B} \cdot \nabla \bar{A} + \bar{A} \nabla \cdot \bar{B} - \bar{A} \cdot \nabla \bar{B} - \bar{B} \nabla \cdot \bar{A}$$

Converting $RHS$ into index notation,

$$RHS_i = B_j \frac{\partial}{\partial x_j} (A_i) + A_i \frac{\partial}{\partial x_j} (B_j) - A_j \frac{\partial}{\partial x_j} (B_i) - B_i \frac{\partial}{\partial x_j} (A_j)$$

Converting $LHS$ into index notation,

$$LHS_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[ \epsilon_{klm} A_l B_m \right] = \epsilon_{ijk} \epsilon_{klm} \frac{\partial}{\partial x_j} \left[ A_l B_m \right]$$

Using the cylcic permutation property of the permutation tensor $\epsilon_{ijk} = \epsilon_{kij}$. Therefore,

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm}$$

Using the double permutation tensor identity,

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Substituting into $LHS_i$,

$$LHS_i = \epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}[A_l B_m] = \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right]\frac{\partial}{\partial x_j}[A_l B_m] = \delta_{il}\delta_{jm}\frac{\partial}{\partial x_j}[A_l B_m] - \delta_{im}\delta_{jl}\frac{\partial}{\partial x_j}[A_l B_m]$$

$$LHS_i = \delta_{jm}\frac{\partial}{\partial x_j}[A_i B_m] - \delta_{jl}\frac{\partial}{\partial x_j}[A_l B_i] = \frac{\partial}{\partial x_j}\left[A_i B_j\right] - \frac{\partial}{\partial x_j}\left[A_j B_i\right]$$

Expanding using product rule,

$$LHS_i = A_i\frac{\partial}{\partial x_j}\left[B_j\right] + B_j\frac{\partial}{\partial x_j}[A_i] - \left\{A_j\frac{\partial}{\partial x_j}[B_i] + B_i\frac{\partial}{\partial x_j}\left[A_j\right]\right\}$$

$$LHS_i = A_i\frac{\partial}{\partial x_j}\left[B_j\right] + B_j\frac{\partial}{\partial x_j}[A_i] - A_j\frac{\partial}{\partial x_j}[B_i] - B_i\frac{\partial}{\partial x_j}\left[A_j\right]$$

Since $LHS_i = RHS_i$, the vector identity is proven to be true.

### 18.4.7 Double Curl of Vector

$$\nabla \times \left(\nabla \times \bar{A}\right) = \nabla\left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Let

$$LHS = \nabla \times \left(\nabla \times \bar{A}\right) \quad , \quad RHS = \nabla\left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Converting $LHS$ into index notation,

$$LHS_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{kmn}\frac{\partial}{\partial x_m}A_n$$

Since the permutation tensor $\epsilon_{kmn}$ is a constant in $x_j$ and $x_m$,

$$LHS_i = \epsilon_{ijk}\epsilon_{kmn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}A_n$$

Using the permutation tensor cyclic identity,

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn}$$

Using the double permutation tensor identity,

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

Substituting,

$$LHS_i = [\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}]\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n)$$

$$LHS_i = \delta_{im}\delta_{jn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n) - \delta_{in}\delta_{jm}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}(A_n)$$

Using the renaming identity of the kronecker-delta function,

$$LHS_i = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Since partial derivatives are commutative with one another, $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(A_j) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j)$.

Susbtituting,

$$LHS_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Reiterating $RHS$,

$$RHS = \nabla\left(\nabla \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

Converting $RHS$ into index notation,

$$RHS_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(A_j) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(A_i)$$

Since $LHS_i = RHS_i$, then the identity is proven to be true.

### 18.4.8 Curl of Vector Scalar

$$\nabla \times (\phi\bar{A}) = \phi\nabla \times \bar{A} + (\nabla\phi) \times \bar{A}$$

Let

$$LHS = \nabla \times (\phi\bar{A}) \quad , \quad RHS = \phi\nabla \times \bar{A} + (\nabla\phi) \times \bar{A}$$

Converting $LHS$ into index notation,

$$LHS_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi A_k)$$

Using product rule,

$$LHS_i = \epsilon_{ijk}\left[\phi\frac{\partial}{\partial x_j}(A_k) + A_k\frac{\partial}{\partial x_j}(\phi)\right]$$

$$LHS_i = \epsilon_{ijk}\phi\frac{\partial}{\partial x_j}(A_k) + \epsilon_{ijk}A_k\frac{\partial}{\partial x_j}(\phi)$$

Converting $RHS$ into index notation,

$$RHS_i = \phi\epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k) + \epsilon_{ijk}\left[\frac{\partial}{\partial x_j}(\phi)\right]A_k$$

$$RHS_i = \phi\epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k) + \epsilon_{ijk}A_k\left[\frac{\partial}{\partial x_j}(\phi)\right]$$

Since $LHS_i = RHS_i$, the identity is proven to be true.

### 18.4.9 Triple Curl of Vector

$$\nabla \times [\nabla \times (\nabla \times \bar{A})] = -\nabla^2(\nabla \times \bar{A})$$

Let,

$$LHS = \nabla \times [\nabla \times (\nabla \times \bar{A})] \quad , \quad RHS = -\nabla^2(\nabla \times \bar{A})$$

In index notation,

$$(\nabla \times \bar{A})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k)$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \epsilon_{lmi}\frac{\partial}{\partial x_m}\left[\epsilon_{ijk}\frac{\partial}{\partial x_j}(A_k)\right]$$

Since $\epsilon_{ijk}$ is simply a constant in $x_m$ or $x_j$,

$$[\nabla \times (\nabla \times \bar{A})]_l = \epsilon_{lmi}\epsilon_{ijk}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right]$$

Using the cyclic property of the permutation tensor,

$$\epsilon_{lmi}\epsilon_{ijk} = \epsilon_{ilm}\epsilon_{ijk}$$

Using the double permutation tensor identity,

$$\epsilon_{lmi}\epsilon_{ijk} = \epsilon_{ilm}\epsilon_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm}$$

Substituting for the double permutation tensor identity,

$$[\nabla \times (\nabla \times \bar{A})]_l = [\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm}]\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right]$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \delta_{lj}\delta_{mk}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right] - \delta_{lk}\delta_{jm}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_j}(A_k)\right]$$

$$[\nabla \times (\nabla \times \bar{A})]_l = \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right] - \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]$$

$$\left\{\nabla \times [\nabla \times (\nabla \times \bar{A})]\right\}_p = \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right] - \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

$$\left\{\nabla \times [\nabla \times (\nabla \times \bar{A})]\right\}_p = \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right]\right\} - \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(A_l)\right]\right\}$$

Since partial derivative operations are commutative with one another,

$$\epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(A_k)\right]\right\} = \epsilon_{pql}\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_l}\left[\frac{\partial}{\partial x_k}(A_k)\right]\right\}$$

The permutation tensor is anti-symmetric in any 2 of its indices, and $\frac{\partial}{\partial x_q}\left\{\frac{\partial}{\partial x_l}\left[\frac{\partial}{\partial x_k}(A_k)\right]\right\}$ is symmetric in $q$ and $l$ due to the commutativity of the partial differential operator. Since this would mean a symmetric tensor multiplied by an anti-symmetric element-wise and added together,

$$0 = \epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_k} \left[ \frac{\partial}{\partial x_l} (A_k) \right] \right\}$$

Therefore,

$$\left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_p = -\epsilon_{pql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[ \frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

renaming the free index $p \to i$,

$$\left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_i = -\epsilon_{iql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[ \frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

Since $LHS_i = \left\{ \nabla \times [\nabla \times (\nabla \times \bar{A})] \right\}_i$,

$$LHS_i = -\epsilon_{iql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[ \frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

Reiterating definition of $RHS$,

$$RHS = -\nabla^2 (\nabla \times \bar{A})$$

Converting $RHS$ into index notation,

$$RHS_i = -\frac{\partial}{\partial x_j} \left\{ \frac{\partial}{\partial x_j} \left[ \epsilon_{iql} \frac{\partial}{\partial x_q} (A_l) \right] \right\}$$

Since the permutation tensor is a constant in $x_j$ and $x_q$ and that partial derivative operations are commutative with one another,

$$RHS_i = -\epsilon_{iql} \frac{\partial}{\partial x_q} \left\{ \frac{\partial}{\partial x_j} \left[ \frac{\partial}{\partial x_j} (A_l) \right] \right\}$$

Since $LHS_i = RHS_i$, the identity is proven to be true.

### 18.4.10 Divergence of Vector Scalar

$$\nabla \cdot (\phi \bar{A}) = \phi (\nabla \cdot \bar{A}) + \bar{A} \cdot \nabla \phi$$

Let,

$$LHS = \nabla \cdot (\phi \bar{A}) \quad , \quad RHS = \phi (\nabla \cdot \bar{A}) + \bar{A} \cdot \nabla \phi$$

Converting $RHS$ into index notation,

$$RHS_i = \phi \frac{\partial}{\partial x_j} [A_j] + A_j \frac{\partial}{\partial x_j} [\phi]$$

Converting $LHS$ into index notation,

$$LHS_i = \frac{\partial}{\partial x_j}[\phi A_j]$$

Using product rule,

$$LHS_i = \phi \frac{\partial}{\partial x_j}[A_j] + A_j \frac{\partial}{\partial x_j}[\phi]$$

Since $LHS_i = RHS_i$, the identity is proven to be true.