Chapter 1

Linear Differential Equations

1.1 Definition of Differential Operator

Let the differential operator be defined as the following:

$$L = \sum_{i=0}^{n} \left[a_i \frac{d^{n-i}}{dx^{n-i}} \right] = a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1} \frac{d}{dx} + a_n$$

Let the following operator D be defined:

$$D = \frac{d}{dx} \qquad D^k = \frac{d^k}{dx^k}$$

Therefore, the linear differential operator could be defined as

$$L = \sum_{i=0}^{n} \left[a_i D^{n-i} \right] = a_0 \prod_{i=1}^{n} [D - r_i]$$

wherein a_i and r_i are constants albeit complex or real. The notation above is always true because an fundamental theorem of algebra states that n^{th} order polynomial must have n roots. Since linear differential operator could be expressed as a polynomial in terms of the operator D, therefore the notation above would always be true regardless of the choice of a_i and n. The linear differential operator have certain properties associated to them discussed in the propositions,

1.1.1 Proposition 1: Operation on 0

The linear differential operator when operated on a 0 will yield 0:

$$L[0] = \sum_{i=0}^{n} \left[a_i D^{n-i}\right] 0 = a_0 \frac{d^n}{dx^n} 0 + a_1 \frac{d^{n-1}}{dx^{n-1}} 0 + \cdots + a_{n-1} \frac{d}{dx} 0 + a_n 0$$

It is given that $\frac{d}{dx}(0) = 0$, and by reapplying recursively, $\frac{d^k}{dx^k}(0) = 0$. Therefore,

$$L[0] = 0$$

1.1.2 Proposition 2: Operation on Constants

The linear differential operator when operated on a constant will yield some constant provided that $a_n \neq 0$:

$$L[c] = \sum_{i=0}^{n} \left[a_i D^{n-i}\right] c = a_0 \frac{d^n}{dx^n} c + a_1 \frac{d^{n-1}}{dx^{n-1}} c + \cdots + a_{n-1} \frac{d}{dx} c + a_n c$$

Considering $\frac{d}{dx} c = \frac{d^k}{dx^k} c = 0$,

$$L[c] = \sum_{i=0}^{n} \left[a_i D^{n-i}\right] c = a_n c$$

1.1.3 Proposition 3: Operation Commutativity

If there exist two linearly independent differential operators L_1 and L_2, then the solution of the system $0 = L_1 L_2[y]$ must be a linear combination of the solution to the system $0 = L_1[y]$ and $0 = L_2[y]$:

$$L_1 = \prod_{i=0}^{m} [D - \alpha_i] \quad , \quad L_2 = \prod_{j=0}^{n} [D - \beta_j]$$

Let $y_1(x)$ and $y_2(x)$ be such that:

$$0 = L_1[y_1(x)] = \prod_{i=0}^{m} [D - \alpha_i]\, y_1(x) \quad , \quad 0 = L_2[y_2(x)] = \prod_{j=0}^{n} [D - \beta_j]\, y_2(x)$$

Let T_i be the transformation defined as
$T_i : f(x) \to g(x) \quad , \quad T_i[f(x)] = (D - r_i)f(x)$, wherein $f(x)$ is some arbitrary continuous function over some interval. Indeed the transformation T_1 is linear:

$$T_i[cu(x)] = (D - r_i)cu(x)$$

$$T_i[cu(x)] = c(D - r_i)u(x)$$

$$cT_i[u(x)] = c(D - r_i)u(x)$$

Therefore, $T_i[cu(x)] = cT_i[u(x)]$ wherein c is some arbitrary constant.

$$T_i[u(x) + v(x)] = (D - r_i)[u(x) + v(x)]$$

$$T_i[u(x) + v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

$$T_i[u(x)] + T_i[v(x)] = (D - r_i)[u(x)] + (D - r_i)[v(x)]$$

Therefore, $T_i[u(x) + v(x)] = T_i[u(x)] + T_i[v(x)]$, and T_i must be a linear transformation. Linear transformations applied compositely form a linear transformation:

$$T_0(u + v) = T_0(u) + T_0(v)$$

$$T_1[T_0(u + v)] = T_1[T_0(u)] + T_1[T_0(v)]$$

$$\prod_{i=0}^{\alpha} [T_i]\,(u + v) = \prod_{i=0}^{\alpha} [T_i]\,(u) + \prod_{i=0}^{\alpha} [T_i]\,(v)$$

Since L_1 and L_2 is only a specific case of the transformation described as T_1 it can be considered that the differential operators of L_1 and L_2 are linear. Therefore, it can be said that $L_1[L_2]$ must be linear.

$$0 = null = L_1[y_1(x)] = \prod_{i=0}^{m} [D - \alpha_i]\, y_1(x) \quad , \quad 0 = null = L_2[y_2(x)] = \prod_{j=0}^{n} \left[D - \beta_j\right] y_2(x)$$

For $y_1(x)$ and $y_2(x)$:

$$0 = null = L_2\{L_1[y_1(x)]\} = \prod_{j=0}^{n} \left[D - \beta_j\right] \prod_{i=0}^{m} [D - \alpha_i]\, y_1(x)$$

$$0 = null = L_1\{L_2[y_2(x)]\} = \prod_{i=0}^{m} [D - \alpha_i] \prod_{j=0}^{n} \left[D - \beta_j\right] y_2(x)$$

$$\prod_{i=0}^{m}[D-\alpha_i]\prod_{j=0}^{n}[D-\beta_j]=\prod_{j=0}^{n}[D-\beta_j]\prod_{i=0}^{m}[D-\alpha_i]=L_1[L_2]=L_2[L_1]$$

It follows by definition of linear transformation that,
$L_1[k_1y_1(x)]=k_1L_1[y_1(x)]=null$ and that $L_2[k_2y_2(x)]=k_2L_2[y2(x)]=null$:

$$0=null=L_2\{L_1[k_1y_1(x)]\}=\prod_{j=0}^{n}[D-\beta_j]\prod_{i=0}^{m}[D-\alpha_i]\,k_1y_1(x)$$

$$0=null=L_1\{L_2[k_2y_2(x)]\}=\prod_{i=0}^{m}[D-\alpha_i]\prod_{j=0}^{n}[D-\beta_j]\,k_2y_2(x)$$

$$0=L_2\{L_1[k_1y_1(x)]\}+L_2\{L_1[k_2y_2(x)]\}=L_2\{L_1[k_1y_1(x)+k_2y_2(x)]\}$$

$$0=\prod_{i=0}^{m}[D-\alpha_i]\prod_{j=0}^{n}[D-\beta_j]\,[k_1y_1(x)+k_2y_2(x)]$$

1.2 Homogenous Differential Equation Cases

Consider the following homogenous differential equation:

$$0 = \sum_{i=0}^{n} \left[a_i \overset{n-i}{y} \right] = a_0 \overset{n}{y} + a_1 \overset{n-1}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$

$$0 = L[y] = \sum_{i=0}^{n} \left[a_i D^{n-i} \right] y = \prod_{i=1}^{n} \left[D - r_i \right] y$$

1.2.1 Non-Repeated Roots

By principle of superposition verified by proposition 1 and 3, the general solution to the homogenous n^{th} order differential equation,

$$y_c = \sum_{i=1}^{n} \left[c_i y_i \right]$$

Wherein c_i represents either complex or real constants, y_i represents solutions to the $0 = [D - r_i]y_i$ system. Considering the partial system,

$$0 = [D - r_i]y_i$$

$$0 = Dy_i - r_i y_i$$

$$\frac{d}{dx}(y_i) = r_i y_i$$

$$\int \frac{1}{y_i} dy_i = \int r_i dx$$

$$\ln y_i = r_i x + C$$

$$y_i = e^{r_i x + C} = c_i e^{r_i x}$$

Therefore for as long as there are no roots with multiplicity greater than 1, the following is true, for some choice of constants,

$$y_c = \sum_{i=1}^{n} \left[c_i e^{r_i x} \right]$$

1.2.2 Repeated Roots

Suppose the α^{th} root has a multiplicity of k,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^{n} [D - r_i] (D - r_\alpha)^k y$$

By proposition 3, the general solution to the system must be the linear combination:

$$y_g(x) = c_1 y_1(x) + c_2 y_2(x)$$

wherein $y_1(x)$ is the solution to the system $0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{j=\alpha+k}^{n} [D - r_i] y$ and $y_2(x)$ is the solution to the system $0 = (D - r_\alpha)^k y$. By conjecture, it is suspected that the $y_2(x) = u(x)e^{r_\alpha x}$ wherein $u(x)$ is some function to be determined.

$$0 = (D - r_\alpha)u(x)e^{r_\alpha x}$$

$$0 = \frac{d}{dx}\left[u(x)e^{r_\alpha x}\right] - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = \overset{1}{u(x)}e^{r_\alpha x} + r_\alpha u(x)e^{r_\alpha x} - r_\alpha u(x)e^{r_\alpha x}$$

$$0 = \overset{1}{u(x)}e^{r_\alpha x}$$

By reapplying the linear differential operator recursively:

$$(D - r_\alpha)^k u(x)e^{r_\alpha x} = \overset{k}{u(x)}e^{r_a x}$$

Therefore, the system would follow:

$$0 = (D - r_\alpha)^k u(x)e^{r_\alpha x} = \overset{k}{u(x)}e^{r_a x}$$

$$0 \neq e^{r_a x} \text{ for all } x$$

$$0 = \overset{k}{u(x)}$$

A function that satisfies the following condition must be a polynomial with at most degree $k-1$. Therefore,

$$u(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right]$$

The general solution $y_{rr}(x)$ to the system $0 = (D - r_\alpha)^k y$:

$$y_{rr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right] e^{r_\alpha x}$$

1.2.3 Complex Roots

Suppose the α^{th} root is a complex root, by the fundamental theorem of algebra, some other root must be its complex conjugate. Let the complex conjugate root of the α^{th} root be ordered next to the α^{th} root in the product notation. Therefore,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{i=\alpha+2}^{n} [D - r_i\ ] [D - r_\alpha][D - r_{\alpha+1}]y$$

Let y_{cr} represent the complex root corresponding to the system $0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$. By principle of superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2} [c_i e^{r_i x}] + y_{cr}$$

$$0 = [D - r_\alpha][D - r_{\alpha+1}]y_{cr}$$

By the principles presented earlier,

$$y_{cr}(x) = c_\alpha e^{(a+bi)x} + c_{\alpha+1} e^{(a-bi)x}$$

$$y_{cr}(x) = e^{ax} \left[c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}\right]$$

By De Moivre's theorem,

$$e^{bxi} = \cos(bx) + i \sin(bx) \quad , \quad e^{-bxi} = \cos(bx) - i \sin(bx)$$

Suppose the constants c_α and $c_{\alpha+1}$ are complex numbers,

$$c_\alpha = f_1 + g_1 i \quad , \quad c_{\alpha+1} = f_2 + g_2 i$$

By substituting to the expression for complex solution,

$$y_{cr}(x) = e^{ax}\left[(f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}\right]$$

Let $y_{cr} = e^{ax} y_{co}$,

$$y_{co} = c_\alpha e^{bxi} + c_{\alpha+1} e^{-bxi}$$

$$y_{co}(x) = (f_1 + g_1 i)e^{bxi} + (f_2 + g_2 i)e^{-bxi}$$

$$y_{co}(x) = (f_1 + g_1 i)[\cos(bx) + i\sin(bx)] + (f_2 + g_2 i)[\cos(bx) - i\sin(bx)]$$

Let

$$A(x) = (f_1 + g_1 i)[\cos(bx) + i\sin(bx)] \quad , \quad B(x) = (f_2 + g_2 i)[\cos(bx) - i\sin(bx)]$$

$$A(x) = f_1\cos(bx) - g_1\sin(bx) + i[f_1\sin(bx) + g_1\cos(bx)]$$

$$B(x) = f_2\cos(bx) + g_2\sin(bx) + i[-f_2\sin(bx) + g_2\cos(bx)]$$

$$y_{co}(x) = A(x) + B(x)$$

$$y_{co}(x) = (f_1 + f_2)\cos(bx) + (g_2 - g_1)\sin(bx) + i[(f_1 - f_2)\sin(bx) + (g_1 + g_2)\cos(bx)]$$

For the complex root $y_{cr}(x)$ to be real, the imaginary component of $y_{cr}(x)$ must be equals to 0. Therefore, the following must hold true,

$$f_1 = f_2 \quad , \quad g_1 = -g_2$$

For as long as the condition above hold true, the two constants c_α and$c_{\alpha+1}$ must be complex conjugates. Considering the case wherein c_α and$c_{\alpha+1}$ as complex conjugates,

$$y_{co}(x) = 2f_1\cos(bx) + 2g_2\sin(bx)$$

$$y_{cr} = 2e^{ax}[f_1\cos(bx) + g_2\sin(bx)]$$

Therefore, the following is true for each complex root and conjugate pair,

$$y_c = \sum_{i=1}^{n-2}[c_i e^{r_i x}] + 2e^{ax}[f_1\cos(bx) + g_2\sin(bx)]$$

1.2.4 Repeated Complex Roots

Suppose the α^{th} root is a complex root with a multiplicity of k. Let its complex conjugate be placed adjacent after said complex root,

$$0 = \prod_{i=1}^{\alpha-1} [D - r_i] \prod_{i=\alpha+2k}^{n} [D - r_i] [D - r_\alpha]^k [D - \bar{r_\alpha}]^k y$$

Let y_{crr} be considered as the solution to the system $0 = [D - r_\alpha]^k [D - \bar{r_\alpha}]^k y_{crr}$. Based on superposition verified by proposition 1 and 3,

$$y_c = \sum_{i=1}^{n-2k} [c_i e^{r_i x}] + y_{crr}$$

Based on the previous work on repeated roots with multiplicity greater than 1,

$$y_{crr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right] C_1 e^{r_\alpha x} + \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right] C_2 e^{\bar{r_\alpha} x}$$

Let $r_\alpha = a + bi$, and $\bar{r_\alpha} = a - bi$,

$$y_{crr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right] [C_1 e^{bxi} + C_2 e^{-bxi}] e^{ax}$$

Let $c_1 = f_1 + g_1 i$ and $c_2 = f_2 + g_2 i$. Based on previous work on complex roots,

$$C_1 e^{bxi} + C_2 e^{-bxi} = 2f_1 \cos(bx) + 2g_2 \sin(bx)$$

Therefore,

$$y_{crr}(x) = \sum_{i=0}^{k-1} \left[c_i x^{k-1-i}\right] [2f_1 \cos(bx) + 2g_2 \sin(bx)] e^{ax}$$

1.3 General Solutions to Homogenous Differential Equations

Therefore, if an n^{th} order homogeneous differential equation with a real non-repeated roots, b complex root pairs, c real repeated roots with multiplicity γ, and d complex repeated root pairs with multiplicity β

$$\begin{aligned} y_c = &\sum_{l=1}^{a} \left[c_{1,l} e^{r_{1,l}x}\right] \\ &+ \sum_{j=1}^{b} [c_{2,j,1} \cos{(b_{1,j}x)} + c_{2,j,2} \sin{(b_{1,j}x)}] e^{a_{1,j}x} \\ &+ \sum_{k=1}^{c} \left[\sum_{m=0}^{\gamma_k - 1} \left[c_{3,m,k} x^{\gamma_k - 1 - m}\right] e^{r_k x}\right] \\ &+ \sum_{i=1}^{d} \left[\sum_{p=0}^{\beta_p - 1} \left[c_{4,p,i} x^{\beta_p - 1 - p}\right] [k_{4,i,1} \cos{(b_{2,i})}x + k_{4,i,2} \sin{(b_{2,i})}x] e^{r_i x}\right] \end{aligned}$$

The variables a, b, c, d, γ, β, and n are related by the following expression,

$$n = a + 2b + \sum_{i=1}^{c} [\gamma_i] + \sum_{j=1}^{d} [2\beta_j]$$

1.4 Non-Homogenous Differential Equations

Consider the following system:

$$\sum_{i=0}^{n}\left[a_i \overset{n-i}{y}\right] = \sum_{j=0}^{m}\left[c_i f_i(x)\right]$$

wherein $f_i(x)$ represents the i^{th} arbitrary function, and a_i represents the i^{th} arbitrary constant. The following function could be rewritten in terms of the linear differential operator L:

$$L[y] = \sum_{j=0}^{m}\left[c_i f_i(x)\right]$$

Let y_j represent the general solution to the j^{th} system:

$$L[y_j] = c_i f_j(x)$$

By taking the summations of the various solutions to the various sytems:

$$L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j] = c_0 f_0(x) + c_1 f_1(x) + \cdots + c_{j-1} f_{j-1}(x) + c_j f_j(x)$$

Since the differential operator L is linear, as shown in proposition 3:

$$L\left[\sum_{j=0}^{m}\left(y_j\right)\right] = L[y_0] + L[y_1] + \cdots + L[y_{j-1}] + L[y_j]$$

$$L\left[\sum_{j=0}^{m}\left(y_j\right)\right] = \sum_{j=0}^{m}\left[c_i f_i(x)\right]$$

Therefore, a solution to the non-homogenous differential equation:

$$y_p(x) = \sum_{j=0}^{m}\left(y_j\right)$$

An m^{th} dimensional subspace spanned by m functions must always contain a null element, in this case, a zero function. Let the y_c represent the general solution to the homogenous differential equation $Ly = null = 0$. Then the general solution must follow:

$$y_g(x) = \sum_{j=0}^{m}\left(y_j\right) + y_c$$