Name: ______________________ PUID: ______________________

Final Exam
AAE 511, Introduction to Fluid Mechanics
Purdue University
Fall 2021

Instructions:

- The time limit is two hours
- Closed book; closed notes
- Equation sheets are provided at the end of the exam
- Only approved calculators are allowed (TI-30Xa or TI-30XIIS)
- Write your solutions on blank sheets of paper, not the exam sheets
- Request additional paper if you need it
- Write legibly

When you are done:

- Put your name on every page of the exam
- Put your solutions to the problems in numerical order
- Number the pages of your solutions
- Staple your exam on top of your solutions

1. (100 points) **Fundamentals.** In this problem, you will derive the nonlinear potential equation for compressible flow.

   The governing equations for compressible, steady, inviscid, homentropic flow are:

   $$\nabla\cdot(\rho\boldsymbol{v}) = 0 \tag{1}$$
   $$\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -a^2\nabla\rho \tag{2}$$

   where $a$ is the speed of sound.

   Equation (1) represents the conservation of mass and Equation (2) represents the conservation of momentum. In Equation (2), the pressure gradient term has been replaced by a density gradient term using the fact that the entropy is constant.

   (a) Write the equations (1)–(2) in tensor index notation.

   (b) Take the dot product (scalar product) of the velocity vector with the momentum equation that you wrote down in Part (a).

   (c) Use the mass conservation equation (1) to eliminate the density gradient term in the equation that you found in Part (b). For your final result, move all the terms to the left-hand-side so that the right hand side is zero. Hint: Expand Equation (1) so that you have a $\boldsymbol{v}\cdot\nabla\rho$ term.

   (d) Assume potential flow, and substitute $\boldsymbol{v} = \nabla\phi$ in the equation you found in Part (c). This is the nonlinear potential equation for compressible flow.

   (e) Divide your result from Part (d) by $a^2$ and take the limit as $a \to \infty$ of your result in Part (d). What is the name of the resulting equation?

   For this problem, do all your work using tensor index notation. Follow the indicated steps exactly and in order. To avoid a common error, remember that each index, say $k$, can appear once (free index) or twice (dummy index) in a term, but not three or more times.

2. (100 points) **Inviscid Flow.** To map flow over a circle to flow over a flat plate, the following transformation can be used:

   $$z = \zeta + \frac{c^2}{\zeta}$$

   where $c$ is real.

   (a) Find the complex potential function $\tilde{F}(\zeta)$ for a flow of a uniform freestream at angle $\alpha$ over a circle of radius $c$ in the $\zeta$-plane. (Take the circulation to be zero.)

   (b) Find the complex velocity $\tilde{W}(\zeta)$.

   (c) Find the complex potential of the flat plate $F(z)$.

   (d) Find the location of the stagnation points in the $\zeta$-plane and the $z$-plane.

   (e) Using the Blasius integral formula, show that the force on the body is zero. Why is this so? Hint: The force in the $\zeta$-plane is the same as that in the $z$-plane.

3. (100 points) **Viscous Flow.** For this problem, we will solve for viscous flow between two rotating cylinders. Take the inner cylinder to have radius $R_1$ and angular velocity zero, and let the outer cylinder have radius $R_2$ and angular velocity $\Omega_2$. Assume that the only non-zero velocity component is $u_\theta$ and that all variables are functions of the radius $R$ only. Neglect gravity.

   (a) Write out the mass conservation equation, the $r$-momentum equation, and the $\theta$-momentum equation. See Equations (22)–(26). Eliminate terms that are zero, noting briefly why each neglected term is zero. Summarize the simplified equations. (Write the non-trivial equations down in a list.)

   (b) Write down the appropriate boundary conditions. Remember that the outer cylinder is rotating.

   (c) Solve for the velocity profile. Try a solution of the form $u_\theta = AR^m$.

Possibly Useful Information

Tensors:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{3}$$

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} \tag{4}$$

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \tag{5}$$

Complex conjugate:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \tag{6}$$

$$\overline{z_1 z_2} = \overline{z_1}\ \ \overline{z_2} \tag{7}$$

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\overline{z}} \tag{8}$$

$$\overline{\log z} = \log \overline{z} \tag{9}$$

Complex potential:

$$F(z) = \phi + i\psi \tag{10}$$

$$W = u - iv = \frac{dF}{dz} \tag{11}$$

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{12}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{13}$$

$$X - iY = i\frac{1}{2}\rho \oint_C W^2\ dz \tag{14}$$

Standard flow patterns:

$$F(z) = Ue^{-i\alpha}z \qquad \text{Uniform flow} \tag{15}$$

$$F(z) = \frac{Q}{2\pi}\log z \qquad \text{Source / sink} \tag{16}$$

$$F(z) = -\frac{i\Gamma}{2\pi}\log z \qquad \text{Vortex} \tag{17}$$

$$F(z) = \frac{\mu}{z} \qquad \text{Doublet} \tag{18}$$

Residues:

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z - z_0)^m f(z)\right] \tag{19}$$

$$\oint_C f(z)\ dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j) \tag{20}$$

Milne-Thomson Circle Theorem:

$$F(z) = F_1(z) + \overline{F_1\left(\frac{a^2}{\overline{z}}\right)} \tag{21}$$

where $F_1(z)$ is a two-dimensional, irrotational, incompressible flow in the $z$-plane with no rigid boundaries, and $a$ is the radius of the circle.

Possibly Useful Information

Here are the equations of motion for incompressible viscous flow in cylindrical coordinates, $(R, \theta, z)$. The conservation of mass is:

$$\frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{22}$$

The conservation of momentum in the $r$-direction is:

$$\rho\left(\frac{\partial u_R}{\partial t} + u_R\frac{\partial u_R}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta^2}{R} + u_z\frac{\partial u_R}{\partial z}\right) = -\frac{\partial p}{\partial R} + \mu\left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\theta}{\partial \theta}\right) + \rho g_R \tag{23}$$

The conservation of momentum in the $\theta$-direction is:

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_R\frac{\partial u_\theta}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_R u_\theta}{R} + u_z\frac{\partial u_\theta}{\partial z}\right) = -\frac{1}{R}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{R^2} + \frac{2}{R^2}\frac{\partial u_R}{\partial \theta}\right) + \rho g_\theta \tag{24}$$

The conservation of momentum in the $z$-direction is:

$$\rho\left(\frac{\partial u_z}{\partial t} + u_R\frac{\partial u_z}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 u_z + \rho g_z \tag{25}$$

where the following pseudo-Laplace operator definition is used for brevity:

$$\nabla^2 u = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \tag{26}$$

Here $u$ stands in for any one of the velocity components, $u_R$, $u_\theta$, or $u_z$.

1.)

$$\nabla \cdot (\rho \vec{v}) = 0$$

$$\rho (\vec{v} \cdot \nabla) \vec{v} = -a^2 \nabla \rho$$

(a) $$\frac{\partial}{\partial x_j}(\rho V_j) = 0$$

$$\rho V_j \frac{\partial V_i}{\partial x_j} = -a^2 \frac{\partial \rho}{\partial x_i}$$

(b) $$\rho V_i V_j \frac{\partial V_i}{\partial x_j} = -a^2 V_i \frac{\partial \rho}{\partial x_i}$$

(c) $$\rho \frac{\partial V_j}{\partial x_j} + V_j \frac{\partial \rho}{\partial x_j} = 0 \Rightarrow V_i \frac{\partial \rho}{\partial x_i} = -\rho \frac{\partial V_k}{\partial x_k}$$

$$\rho V_i V_j \frac{\partial V_i}{\partial x_j} = +a^2 \rho \frac{\partial V_k}{\partial x_k}$$

$$V_i V_j \frac{\partial V_i}{\partial x_j} = a^2 \frac{\partial V_k}{\partial x_k}$$

$$\boxed{V_i V_j \frac{\partial V_i}{\partial x_j} - a^2 \frac{\partial V_k}{\partial x_k} = 0}$$

(d) $$V_i V_j \frac{\partial V_i}{\partial x_j} - a^2 \frac{\partial V_k}{\partial x_k} = 0$$

$$V_i = \frac{\partial \phi}{\partial x_i}$$

$$\boxed{\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} - a^2 \frac{\partial^2 \phi}{\partial x_k \partial x_k} = 0}$$

Another nice form:

$$\boxed{\left[\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} - a^2 \delta_{ij}\right] \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0}$$

(e) $$\frac{1}{a^2} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \frac{\partial^2 \phi}{\partial x_k \partial x_k} = 0$$

In the limit as $a \to \infty$:

$$\boxed{\frac{\partial^2 \phi}{\partial x_k \partial x_k} = 0}$$

This is the Laplace Equation.

2.)

(a) $F_1(\zeta) = U e^{-i\alpha} \zeta$

Milne-Thomson Circle Theorem:

$$F(\zeta) = F_1(\zeta) + \overline{F_1\left(\frac{c^2}{\bar{\zeta}}\right)}$$

$$= U e^{-i\alpha} \zeta + \overline{U e^{-i\alpha}\left(\frac{c^2}{\bar{\zeta}}\right)}$$

$$= U e^{-i\alpha} \zeta + U e^{+i\alpha} \frac{c^2}{\zeta}$$

$$\boxed{F(\zeta) = U\left(e^{-i\alpha} \zeta + e^{i\alpha} \frac{c^2}{\zeta}\right)}$$

(b) $\tilde{W}(\zeta) = \frac{dF}{d\zeta} = U\left(e^{-i\alpha} - e^{i\alpha} \frac{c^2}{\zeta^2}\right)$

$$\boxed{\tilde{W} = U\left(e^{-i\alpha} - e^{+i\alpha} \frac{c^2}{\zeta^2}\right)}$$

(c) $F(z) = F(\zeta(z))$

$$F(\zeta) = U\left(e^{-i\alpha}\zeta + e^{i\alpha}\frac{c^2}{\zeta}\right)$$

$$z = \zeta + \frac{c^2}{\zeta} \Rightarrow \zeta - z + \frac{c^2}{\zeta} = 0$$

$$\zeta^2 - z\zeta + c^2 = 0$$

$$\zeta = \frac{1}{2}\left[z + \sqrt{z^2 - 4c^2}\right]$$

$$\boxed{F(z) = U e^{-i\alpha}\frac{1}{2}\left[z + \sqrt{z^2-4c^2}\right] + U e^{+i\alpha} c^2 \frac{2}{z + \sqrt{z^2-4c^2}}}$$

(d) In the $\zeta$-plane, $\tilde{W} = 0$:

$$U\left(e^{-i\alpha} - e^{+i\alpha}\frac{c^2}{\zeta^2}\right) = 0$$

$$e^{-i\alpha} = e^{i\alpha}\frac{c^2}{\zeta^2} \Rightarrow \zeta^2 = c^2 e^{+i2\alpha}$$

$$\zeta = \pm c e^{i\alpha}$$

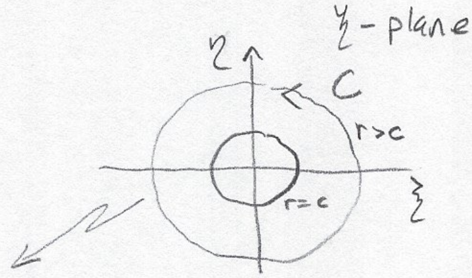
2(d), cont.

$$z = \zeta + \frac{c^2}{\zeta}, \quad \zeta = \pm c e^{i\alpha}$$

$$z = \pm c e^{i\alpha} + \frac{c^2}{\pm c e^{i\alpha}} = \pm c e^{i\alpha} \pm c e^{-i\alpha}$$

$$z = \pm 2c \, \tfrac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right)$$

$$\boxed{z = \pm 2c \cos\alpha}$$




(e) $$\tilde{X} - i\tilde{Y} = i\,\tfrac{1}{2}\rho \oint_C \tilde{W}^2 \, d\zeta$$

$$\tilde{W} = U\left(e^{-i\alpha} - e^{+i\alpha}\frac{c^2}{\zeta^2}\right)$$

$$\tilde{W}^2 = U^2\left(e^{-i\alpha} - e^{+i\alpha}\frac{c^2}{\zeta^2}\right)^2$$

$$\tilde{W}^2 = U^2\left(e^{-i2\alpha} - 2\frac{c^2}{\zeta^2} + e^{+i2\alpha}\frac{c^4}{\zeta^4}\right)$$

$$\tilde{X} - i\tilde{Y} = \frac{i\rho}{2}\oint_C U^2\left(e^{-i2\alpha} - 2\frac{c^2}{\zeta^2} + e^{+i2\alpha}\frac{c^4}{\zeta^4}\right) d\zeta$$

$$\tilde{X} - i\tilde{Y} = i\left(\tfrac{1}{2}\rho U^2\right) e^{-i2\alpha} \cancelto{0}{\oint_C d\zeta} \quad \text{analytic}$$

$$+\, i\left(\tfrac{1}{2}\rho U^2\right)\left(-2c^2\right) \cancelto{0}{\oint_C \frac{d\zeta}{\zeta^2}} \quad \text{residue} = 0$$

$$+\, i\left(\tfrac{1}{2}\rho U^2\right)\left(c^4 e^{+i2\alpha}\right) \cancelto{0}{\oint_C \frac{d\zeta}{\zeta^4}} \quad \text{residue} = 0$$

$$\operatorname{Res}\left(\zeta^{-2}, \zeta_0 = 0\right) = \lim_{\zeta \to 0} \frac{d}{d\zeta}\left(\zeta^2 \frac{1}{\zeta^2}\right) = 0$$

$$\operatorname{Res}\left(\zeta^{-4}, \zeta_0 = 0\right) = \lim_{\zeta \to 0} \frac{1}{3!} \frac{d^3}{d\zeta^3}\left(\zeta^4 \frac{1}{\zeta^4}\right) = 0$$

$$\boxed{\tilde{X} + i\tilde{Y} = 0}$$

| |
|---|
| There is no force without circulation. |

3.)

(a) $$\frac{1}{R}\frac{\partial}{\partial R}\left(R\overset{u_R=0}{\cancelto{0}{u_R}}\right) + \frac{1}{R}\overset{\frac{\partial}{\partial\theta}=0}{\cancelto{0}{\frac{\partial u_\theta}{\partial \theta}}} + \overset{u_z=0}{\cancelto{0}{\frac{\partial u_z}{\partial z}}} = 0$$

$$\rho\left(\overset{u_R=0}{\cancelto{0}{\frac{\partial u_R}{\partial t}}} + \overset{u_R=0}{\cancelto{0}{u_R\frac{\partial u_R}{\partial R}}} + \overset{u_R=0}{\cancelto{0}{\frac{u_\theta}{R}\frac{\partial u_R}{\partial \theta}}} - \frac{u_\theta^2}{R} + \overset{u_R=0}{\cancelto{0}{u_z\frac{\partial u_R}{\partial z}}}\right) = -\frac{\partial p}{\partial R} + \mu\left(\overset{u_R=0}{\cancelto{0}{\nabla^2 u_R}} - \overset{u_R=0}{\cancelto{0}{\frac{u_R}{R^2}}} + \overset{\frac{\partial}{\partial\theta}=0}{\cancelto{0}{\frac{2}{R^2}\frac{\partial u_\theta}{\partial \theta}}}\right) + \overset{\text{neglect}}{\cancelto{0}{\rho g_R}}$$

$$-\rho\frac{u_\theta^2}{R} = -\frac{\partial p}{\partial R} \Rightarrow \frac{\rho u_\theta^2}{R} = \frac{dp}{dR}$$

$$\nabla^2 u_\theta = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_\theta}{\partial R}\right) + \overset{\frac{\partial}{\partial\theta}=0}{\cancelto{0}{\frac{1}{R^2}\frac{\partial^2 u_\theta}{\partial \theta^2}}} + \overset{\frac{\partial}{\partial z}=0}{\cancelto{0}{\frac{\partial^2 u_\theta}{\partial z^2}}} = \frac{1}{R}\frac{d}{dR}\left(R\frac{du_\theta}{dR}\right)$$

$$\rho\left(\overset{\frac{\partial}{\partial t}=0}{\cancelto{0}{\frac{\partial u_\theta}{\partial t}}} + \overset{u_R=0}{\cancelto{0}{u_R\frac{\partial u_\theta}{\partial R}}} + \overset{\frac{\partial}{\partial\theta}=0}{\cancelto{0}{\frac{u_\theta}{R}\frac{\partial u_\theta}{\partial \theta}}} + \overset{u_R=0}{\cancelto{0}{\frac{u_R u_\theta}{R}}} + \overset{u_z=0}{\cancelto{0}{u_z\frac{\partial u_\theta}{\partial z}}}\right) = -\overset{\frac{\partial}{\partial\theta}=0}{\cancelto{0}{\frac{1}{R}\frac{\partial p}{\partial \theta}}} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{R^2} + \overset{\frac{\partial}{\partial\theta}=0}{\cancelto{0}{\frac{2}{R^2}\frac{\partial u_R}{\partial \theta}}}\right) + \overset{\text{neglect}}{\cancelto{0}{\rho g_\theta}}$$

$$0 = \frac{\mu}{R}\frac{d}{dR}\left(R\frac{du_\theta}{dR}\right) - \frac{\mu}{R^2}u_\theta$$

$$\rho\left(\overset{u_z=0}{\cancelto{0}{\frac{\partial u_z}{\partial t}}} + \overset{u_z=0}{\cancelto{0}{u_R\frac{\partial u_z}{\partial R}}} + \overset{u_z=0}{\cancelto{0}{\frac{u_\theta}{R}\frac{\partial u_z}{\partial \theta}}} + \overset{u_z=0}{\cancelto{0}{u_z\frac{\partial u_z}{\partial z}}}\right) = -\overset{\frac{\partial}{\partial z}=0}{\cancelto{0}{\frac{\partial p}{\partial z}}} + \overset{u_z=0}{\cancelto{0}{\mu\nabla^2 u_z}} + \overset{\text{neglect}}{\cancelto{0}{\rho g_z}}$$

Summary:
$$\boxed{\begin{aligned}&\frac{dp}{dR} = \frac{\rho u_\theta^2}{R}\\ &\frac{\mu}{R}\frac{d}{dR}\left(R\frac{du_\theta}{dR}\right) - \frac{\mu}{R^2}u_\theta = 0\end{aligned}}$$

(b) Boundary Conditions:

$$u_\theta(R_1) = 0$$
$$u_\theta(R_2) = R_2\Omega_2$$

$\Big\}$ no-slip

(c) $$\frac{\mu}{R}\frac{d}{dR}\left(R\frac{du_\theta}{dR}\right) - \frac{\mu}{R^2}u_\theta = 0$$

$$\frac{1}{R}\left[R\frac{d^2u_\theta}{dR^2} + \frac{du_\theta}{dR}\right] - \frac{u_\theta}{R^2} = 0$$

$$\frac{d^2u_\theta}{dR^2} + \frac{1}{R}\frac{du_\theta}{dR} - \frac{u_\theta}{R^2} = 0$$

$$R^2u_\theta'' + Ru_\theta' - u_\theta = 0$$

let $u_\theta = AR^m$

$$u_\theta' = mAR^{m-1} \Rightarrow Ru_\theta' = mAR^m$$

$$u_\theta'' = m(m-1)AR^{m-2} \Rightarrow R^2u_\theta'' = m(m-1)AR^m$$

$\therefore$

$$m(m-1) + m - 1 = 0 \Rightarrow m^2 - 1 = 0$$

$$m = \pm 1$$

$$u_\theta = AR + B\frac{1}{R}$$

3 (c), cont.

$$u_\theta(R_1) = 0 = AR_1 + \frac{B}{R_1}$$

$$u_\theta(R_2) = R_2\Omega_2 = AR_2 + \frac{B}{R_2}$$

$$AR_1 + \frac{B}{R_1} = 0 \Rightarrow B = -AR_1^2$$

$$AR_2 + \frac{B}{R_2} = R_2\Omega_2$$

$$AR_2 - \frac{AR_1^2}{R_2} = R_2\Omega_2$$

$$A = \frac{R_2\Omega_2}{R_2 - R_1^2/R_2} = \frac{R_2^2\Omega_2}{R_2^2 - R_1^2}$$

$$B = -\frac{R_1^2R_2^2\Omega_2}{R_2^2 - R_1^2}$$

$$\boxed{u_\theta(R) = \frac{R_2^2\Omega_2}{R_2^2 - R_1^2}R - \frac{R_1^2R_2^2\Omega_2}{R_2^2 - R_1^2}\frac{1}{R}}$$