

Math115A 1/25 notes

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Last time we saw that “a few” vectors v_1, \dots, v_n in a vector space V could generate (span) the entire V , i.e. any other vector $v \in V$ can be written as a linear combination of v_1, \dots, v_n . For instance, any matrix

$A \in M_{2 \times 2}(\mathbb{R})$ is a linear combination of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

It is important to find “small” sets of vectors in V that span V in other word “optimal”, “most economical” ways to generate V .

Linear dependence & linear independence of vectors

7.1 Definition

A subset of vectors since vector space V is linearly dependent if there exist finitely many distinct vectors $v_1, \dots, v_n \in S$ and scalars $c_1, \dots, c_n \in F$ not all of them 0, such that $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$

In other word: if one can express the vector 0 as a linear combination of distinct vectors in S with non-zero coefficients.

7.2 Example

The set $S = \{(-1, 1, 0), (1, -3, 2), (0, 1, -1)\}$ in \mathbb{R}^3 is linearly dependent because $v_1 + v_2 + 2v_3 = 0$ indeed $(-1, 1, 0) + (1, -3, 2) + 2(0, 1, -1) = (-1, 1, 0) + (1, -3, 2) + (0, 2, -2) = (1 - 1, 1 - 3 + 2, 0 + 2 - 2) = (0, 0, 0)$

7.3 Definition

A subset S of a vector space V is linearly independent if it is not linearly dependent.

We then also say that the vectors in V are linearly independent.

7.4 Theorem

The set $S \neq \emptyset \in V$ is linearly independent iff a linear combination $c_1v_1 + c_2v_2 + \dots + c_nv_n$ of distinct vectors $v_1, \dots, v_n \in S$ with $c_1, \dots, c_n \in F$ is equal to the vector 0 only when all coefficients c_1, \dots, c_n are zero. i.e. $\sum_{i=1}^n c_i v_i = 0$ implies $c_i = 0, \forall i$

Proof:

By definition, S linearly independent means it is not linearly dependent. In other words, the only way to

write the vector 0 as a linear combination of some distinct vector $v_1, \dots, v_n \in S$ is if we take all coefficients $c_1, \dots, c_n \in F$ equal to 0.

7.5 Example

Let $S = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\} \in \mathbb{R}^4$. Show that S is linearly independent.

Solution: By theorem 7.4, we need to show that the only linear combination of v_1, v_2, v_3, v_4 that equals 0 is the one in all coefficients are 0, i.e. to prove that if $c_1(1, 0, 0, -1) + c_2(0, 1, 0, -1) + c_3(0, 0, 1, -1) + c_4(0, 0, 0, 1) = 0$ Then $c_1 = c_2 = c_3 = c_4 = 0$.

indeed $c_1(1, 0, 0, -1) + c_2(0, 1, 0, -1) + c_3(0, 0, 1, -1) + c_4(0, 0, 0, 1) = (c_1, 0, 0, -c_1) + (0, c_2, 0, -c_2) + (0, 0, c_3, -c_3) + (0, 0, 0, c_4) = (c_1, c_2, c_3, -c_1 - c_2 - c_3 + c_4)$ and $(c_1, c_2, c_3, -c_1 - c_2 - c_3 + c_4) = (0, 0, 0, 0)$ i.e. $c_1 = 0, c_2 = 0, c_3 = 0, -c_1 - c_2 - c_3 + c_4 = 0$ thus $c_4 = 0$ as well.

so all coefficients c_1, c_2, c_3, c_4 must be equal to 0, showing that indeed the set S is linearly independent.

7.6 Example

A set S consisting of just one non-zero vector, $S = \{v\}$ with $v \neq 0$, is always linearly independent, because the only possible linear combination with vectors in S is cv with $c \in F$, and if $cv = 0$ then $c = 0$. indeed, for if $c \neq 0$ then $cv = 0$. implies $c^{-1}(cv) = 0, (c^{-1}c)v = 0$ so $1 * v = v = 0$, contradiction.

7.7 Theorem

Let V be a vector space and $S_1 \in S_2 \in V$ subsets of V

(a) if S_2 is linearly independent then S_1 is linearly independent

(b) if S_1 is linearly dependent then S_2 is linearly dependent.

Proof:

We only need to prove (b) because (a) is logically equivalent to (b).

if S_1 is linear dependent then there exist distinct vectors $v_1, \dots, v_n \in S_1$ and non-zero scalars $c_1, \dots, c_n \in F$ such that $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$. But because $S_1 \in S_2$, the vectors v_1, \dots, v_n are in S_2 as well, so in S_2 we have $c_1v_1 + \dots + c_nv_n = 0$ with $c_i \neq 0$ and v_i distinct, thus S_2 linear dependent.

The above Theorem says that only subset of a linear independent set is linear independent.

7.8 Theorem

Let V be a vector space and $S \in V$ a subset. Then S is linearly independent iff for any strictly smaller subset $S' \leq S$, we have $\text{span}(S') \neq \text{span}(S)$.

Proof:

Assume S linear independence and let $S' \in S$ be a subset, $S' \neq S$. Let $v \in S - S'$. if by contradiction we assume $\text{span}(S') = \text{span}(S)$, then there exist $v_1, \dots, v_n \in S'$ distinct and $c_1, \dots, c_n \in F$ such that $v = \sum_{i=1}^n c_i v_i$. Thus, $c_1v_1 + c_2v_2 + \dots + c_nv_n - 1 * v = 0$ with $v_1, \dots, v_n, v \in S$ distinct vectors and $c_1, c_2, \dots, c_n, -1$ not all = 0 contradicting the fact that S is linear independent.

Assume that $\forall S' \leq S$ we have $\text{span}(S') \neq \text{span}(S)$. if S would be linear dependent (by contradiction) then

$v_1, \dots, v_n \in S$ distinct and $c_1, \dots, c_n \in F \neq 0$, such that $c_1 v_1 + \dots + c_n v_n = 0$. By 7.6 we know that we must have $n \geq 2$. So $c_1 v_1 = -c_2 v_2 - \dots - c_n v_n$ and multiplying both sides by c_1^{-1} we get $v_1 = \frac{-c_2}{c_1} v_2 - \dots - \frac{-c_n}{c_1} v_n$. Thus, if we take $S' = S - \{v_1\}$ then v_1 is in the linear span of $\{v_2, \dots, v_n\} \in S'$, thus $v_1 \in \text{span}(S')$. So $\text{span}(S') = \text{span}(S)$, contradiction.

Another way to state the theorem 7.8 is this:

Let V be a vector space and $S \in V$ a linearly independent subset. Let $v \in V$ be a vector that's not in S . Then $S \cup \{v\}$ is linearly dependent iff $v \in \text{span}(S)$ and $S \cup \{v\}$ is linearly independent iff $v \notin \text{span}(S)$

7.9 Exercise

Label the following statement as true/false with testifier.

(a) Any set $S \in V$ with $0 \in S$ is linearly dependent

Answer Yes, because for any distinct $v_1 = 0, v_2, \dots, v_n \in S$ we can take $c_1 = 1, c_2 = c_3 = \dots = c_n = 0$ and set $1 * 0 + 0 * v_1 + \dots + 0 * v_n = 0$

(B) Subsets of linearly dependent sets are linear dependent

Answer No, For instance $S = \{(1, 0), (0, 1), (-1, -1)\} \in \mathbb{R}^2$ is linearly dependent set, because $v_1 + v_2 + v_3 = 0$. But $S' = \{(1, 0), (0, 1)\}$ is a linearly independence subset of S

(c) subsets of linear independence sets are linear independent

Answer Yes this is state in Theorem 7.7

7.10 Exercise

Show that $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \in \mathbb{M}_{3 \times 2}(\mathbb{R})$ is linearly dependent

solution:

In general, to state that v_1, \dots, v_5 are linear dependent/independent we have to solve the system of equations resulting from $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + c_5 v_5 = 0$ with the unknowns c_1, c_2, \dots, c_5 . If we get that the only solution is when $c_1 = c_2 = \dots = c_5 = 0$ then $\{v_1, \dots, v_5\}$ linear independent. If we get other solutions where solve $c_i \neq 0$, then linear dependent.

In our case though we solve right away that $v_1 + v_2 + v_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$. Thus $v_1 + v_2 + v_3 = v_4 + v_5$, in other words $v_1 + v_2 + v_3 - v_4 - v_5 = 0$ so $\{v_1, \dots, v_5\}$ linear dependent.