115A, Winter 2023

Lecture 9

Mo, Jan 30

• Recall that on Friday we proved the following

"Replacement Theorem".

Let V be a vector space. Let $G \subset V$, $L \subset V$ be finite subsets of V such that: (a) G has n vectors & it generates V (b) L has m vectors and is linearly indep.

Then $m \leq n$ and there exists a subset $H \subset G$ with $n-m$ vectors such that $L \cup H$ generates V.

• Recall also that we introduced the following important concept:

Definition A subset S of a vector space V that's linearly independent and generates V is called a basis of the vector sp. V

• Today we'll use the Replacement Thm and

The concept of basis of a vector space V to introduce the notion of dimension of vector space V

9.1 Corollary. Let V be a vector space having a finite basis (i.e. $\exists$ subset $S \subset V$ with S finite, lin. ind., and generating (spaning V)

Then any other basis for V contains the same number of vectors.

$\boxed{\text{Proof}}$. Let $S' \subset V$ be another basis for V, i.e. S' lin. indep and $\operatorname{span}(S') = V$.

Denote by n the number of vectors in S. Assume S' contains n' vectors, with $n' > n$. Since S' is linearly independent and $\operatorname{span}(S) = V$, the replacement thm tells us that $n' \leq n$.

Thus, S' must be finite

and the number n' of elements in S' must be $n' \leq n$.

Reversing the role of S, S' (which are both basis for V!) we obtain $n \leq n'$ as well, thus $n' = n$. ☒

* The above Corollary states that if a vector space V has a finite basis, then the number of elements in that basis is an intrinsic property of V, thus allowing the following:

9.2. Definition A vector space V is said to be finite dimensional if it has a basis consisting of a finite number of elements. The unique integer n (confirm Cor 9.1!) such that any basis of V has exactly n elements is called the dimension of V, denoted $\dim(V)$.

A vector space that does not have a finite basis is called infinite-dimensional.

9.3 Examples. ① The vector space $\{0\}$ consisting of just the 0 vector has dimension 0, $\dim(\{0\})=0$.

② $\dim(\mathbb{R}^n)=n$, more generally if F is an arbitrary field, then $\dim(F^n)=n$. (and $n\geq 1$ integer)

Indeed, we have shown that the set of vectors $S=\{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,0,\dots,0,1)\}$ is linearly independent in F^n and it generates F^n, so it is a basis, and we see S has exactly n vectors

③ $\dim(M_{m\times n}(F))=mn$

because $E_{ij} = \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \in M_{m\times n}(F)$

are lin. indep. & generate $M_{m\times n}(F)$

so it is a basis, and it has $m\cdot n$ vectors

④. The vector space $V = F[X]$ of all polynomials with coefficients in a field F is infinite-dimensional

Indeed, we know that $S = \{1, X, X^2, \ldots\}$ are linearly independent. If V would be finite dimensional, then it can be generated by a finite set of n elements and so by Replacement Thm, it would follow that any finite subset of S has $\leq n$ many elements, thus S would be finite, contradiction (because S has ∞-many vectors).

9.4. Exercise. Do the polynomials
$P_1 = x^3 - 2x^2 + 1$, $P_2 = 3x - 2$, $P_3 = 4x^2 - x + 3$
in $V = P_3(\mathbb{R})$ (i.e. the vector space of all polynomials of degree ≤ 3 with coefficients in $\mathbb{R}$). generate $P_3(\mathbb{R})$?

Solution. No, they don't. Because the vector space $P_3(\mathbb{R})$ has the polynomials $1, x, x^2, x^3 \in P_3(\mathbb{R})$, & span which are linearly independent. Thus, by Replacement Thm, any basis for $P_3(\mathbb{R})$ must have exactly 4 vectors in it, and $\{P_1, P_2, P_3\}$ has only 3 vectors.

9.5 Exercise. Is the set
$$S = \{(1,4,-6), (1,5,8), (2,1,1), (0,1,0)\} \subset \mathbb{R}^3$$
a linearly ind. subset of $\mathbb{R}^3$

Solution. No. Because by

Repl. Thm, if $S \subset V$ lin. ind. and has n elements then $n \leq \dim(V)$. But $\dim(V) = 3$ (see 9.3 ② above), and $\underbrace{4}_{n} > \underbrace{3}_{\dim(\mathbb{R}^3)}$

contradiction.

• Related to the above exercises, let us repeat one more time the conclusions in the Replacement Thm and Corollary 9.1:

• If a vector space V is spanned (generated) by a subset of n vectors, then

$$\boxed{\dim(V) \leq n}$$

and if one takes any set S of linearly independent vectors in V has $\leq n$ many elements in it, i.e. if # elements in S is m, then $m \leq n$

- Another consequence of these results is that if $S \subset V$ is a set with m elements and $m < \dim(V)$ then S cannot generate V, in particular S cannot be a basis for V.

9.6. Exercise. Let W_1, W_2 be subspaces of the vector space V and assume $\dim(W_1) = m$, $\dim(W_2) = n$, where m, n are finite integers.

Prove that $\dim(W_1 + W_2) \leq m + n$.

$W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}$

Solution. By the definition of dimension, the assumptions imply that there exist sets $S_1 = \{v_1, \dots, v_m\} \subset W_1$ and $S_2 = \{w_1, \dots, w_n\} \subset W_2$ such that S_1 is lin. ind. & $\mathrm{span}(S_1) = W_1$

S_2 is lin. ind & $\mathrm{span}(S_2) = W_2$

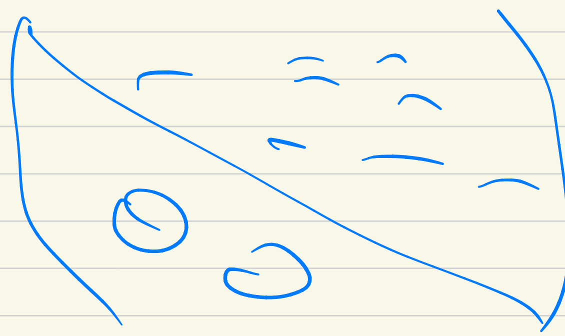
(i.e., S_1 is a basis for W_1
S_2 is a basis for W_2)

But then $\text{span}(S_1 \cup S_2)$
$= \text{span}(S_1) + \text{span}(S_2) = W_1 + W_2$

exercise

Thus, by Repl. Thm, since $S_1 \cup S_2$ has at most $m+n$ elements, we have $m+n \geq \dim(W_1+W_2)$

9.7 Exercise Let V be the subset of $M_{n\times n}(F)$ upper triangular, i.e.

$A =$  $= (A_{ij})$

then $A_{ij} = 0$ for all $i<j$

Show that V vector subspace of $M_{n\times n}(F)$, find a basis and calculate $\dim(V)$

Solution First note that the set

S of all matrices

$$E_{ij} = \begin{array}{c} \\ i \end{array}\begin{pmatrix} 0 & \vdots & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(column j) having 1 on entry ij

and all the other entries $= 0$, with $j \geq i$

is a basis for V.

Indeed, we already know that

$E := \{ E_{ij} : 1 \leq i, j \leq n \}$ is a lin. ind.

and span the entire vector space

$M_{n \times n}(F)$. Thus, its subset

S is still lin. indep., and

it clearly spans V.

To count the # elements in S

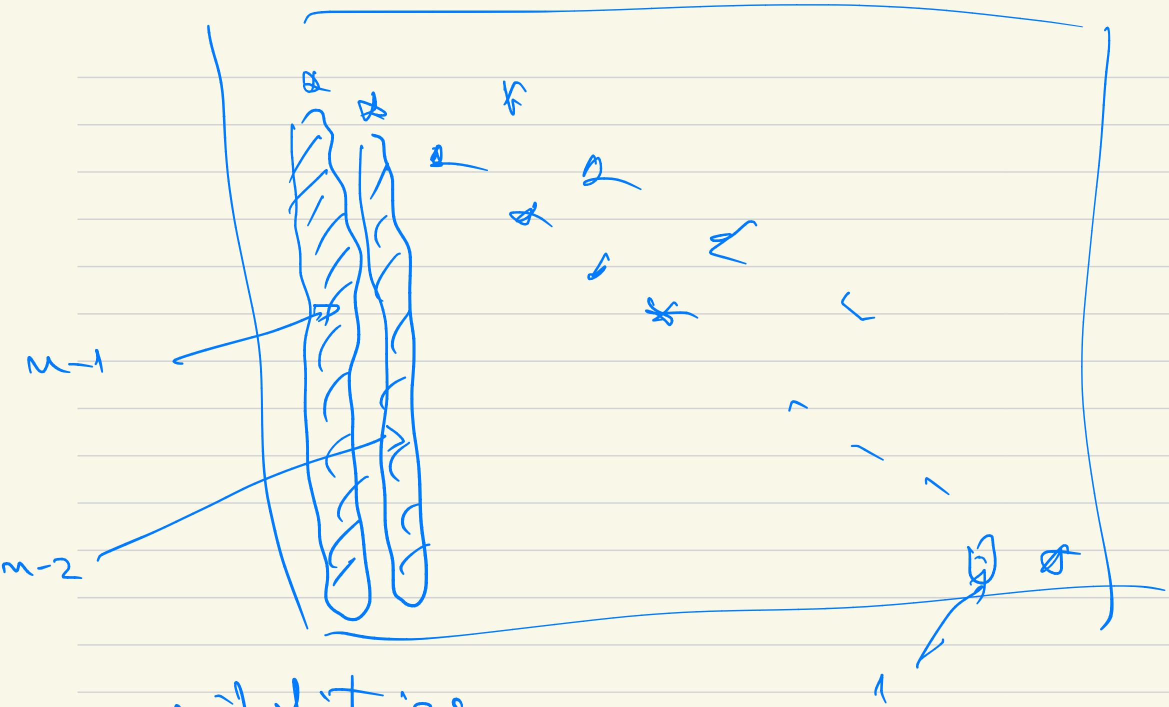
note there are n^2 many elements

in the larger set E, from which

we subtract the number of matrices E_{ij}

in E that have some entry ij equal to 1

under the diagonal



possibilities

so in total we subtract

$$(n-1)+(n-2)+\dots+2+1 = \frac{n(n-1)}{2}$$

Thus we have $n^2 - \frac{n(n-1)}{2}$

$= \frac{2n^2 - n^2 + n}{2} = \frac{n(n+1)}{2}$ many elements in S

so $\dim(V) = \frac{n(n+1)}{2}$