HW5

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from Section 1.6: exercises 30,33

from Section 2.1: exercises 1,3,6,9,10,11,14,17,20,21,22.

1.6

30. Let $V = M_{2\times2}(F), W_1 =$**{**$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V : a, b, c \in F$**}**, $W_2 =$**{**$\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V : a, b \in F$**}**. Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, W_1+W_2 and $W_1 \cap W_2$

Proof:

Base on the expression, we can decompose W_1 into $a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

And, W_2 into $a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

As $a = b = c = 0$, we know that W_1 is linearly independent then we have $dim(W_1) = 3$
As $a = b = 0$, we know that W_2 is also linearly independent then we have $dim(W_2) = 2$

If we put $W_1 + W_2$ we will have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, because $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ can generate by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Therefore, $dim(W_1) + dim(W_2) = dim(W_1 \cap W_2) = 4$

33.

a) Let W__1 and W__2 be subspaces of a vector space V such that $V = W_1 \bigoplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V

Proof:

If that β_1 and β_2 are bases of W_1 and W_2, then they are linear independent to each other. For example, β_1 $=\{1, 0\}$ and β_2 $=\{0, 1\}$. Also, as long as both vector space are direct sum of each other, where $V = W_1 \bigoplus W_2$. Then $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ are basis for V.

b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \bigoplus W_2$.

Proof:
$\beta_1 \cap \beta_2 = \emptyset$ and $span(\beta_1) + span(\beta_2) \subset W_1 \cup W_2$, then we can denote that β_1 and β_2 generate W_1, W_2. because β_1 and $\beta_2 \in V$ then $V = W_1 \bigoplus W_2$.

2.1

1. Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over F), and T is a function from V to W.

a) True
b) False
c) False
d) True
e) False
f) False
g) True
h) False

3. $T : R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

Proof:
$T(c(a_1, a_2, a_3)) = (ca_1(1, 0, 2) + ca_2(1, 0, -1) + ca_3(0, 0, 0)) = c(a_1(1, 0, 2) + a_2(1, 0, -1) + a_3(0, 0, 0)) = cT(a_1, a_2, a_3)$
$T(a_1, a_2, a_3) + T(b_1, b_2, b_3) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3) = (a_1 + b_1)((1, 0, 2) + (1, 0, -1) + (0, 0, 0)) = (a_1(1, 0, 2) + a_2(1, 0, -1) + a_2(0, 0, 0)) + (b_1(1, 0, 2) + b_2(1, 0, -1) + b_3(0, 0, 0)) = T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$
Therefore, T is linear
Then we find that $a_1 + a_2 = 0, 0 = 0, 2a_1 - a_2 = 0$, then $a_1 = -a_2$ and $a_1 = \frac{a_2}{2}$. Base on the expression we can have that $a_1 = a_2 = 0$ and $(a_1, a_2) = (0, 0)$ Then dim of $N(T) = 0$
$T(1, 0) = (1, 0, 2)$ and $T(0, 1) = (1, 0, -1)$ and since the set $\{T(1, 0), T(0, 1)\}$=$\{(1, 0, 2), (1, 0, -1)\}$ is linearly independent. and dim of $R(T) = 2$
Therefore, since $rank(T) + nullity(T) = 2 + 0 = 2 = dim(R^2)$ T is one to one but not onto.

6. $T : M_{n\times n}(F) \to F$ defined by $T(A) = tr(A)$. Recall that $tr(A) = \Sigma_{i=1}^{n} A_{ii}$

Proof:
$T(cA + B) = \Sigma_{i=1}^{n}(cA_{ii} + B_{ii}) = c\Sigma_{i=1}^{n} A_{ii} + \Sigma_{i=1}^{n} B_{ii} = cT(A) + T(B)$ Then T is linear
$N(T) = n \times n - 1 = n^2 - 1$ and $R(T)$ =$\{1\}$, then $nullity(T) + rank(T) = (n^2 - 1) + 1 = n^2 = dim_{n\times n}(F)$.
Therefore, T is not one to one as nullity is greater then one. But T is onto.

9. In this exercise, $T : R^2 \to R^2$ is a function. For each of the following parts, state why T is not linear.

a) $cT(a) + dT(b) = (c + d, ca_2 + db_2)T(ca + db) = T(c(a_1, a_2) + d(b_1, b_2)) = (1, ca_2 + db_2) \neq cT(a) + dT(b)$ not linear

b) $cT(a) + dT(b) = (ca_1 + db_1, (ca_2 + db_2)^2)T(ca + db) = T(c(a_1, a_2) + d(b_1, b_2)) = (ca_1 + db_1, ca_2^2 + db_2^2) \neq cT(a) + dT(b)$ not linear

c) $cT(a) + dT(b) = (sin(ca_1 + db_1), 0)T(ca + db) = T(c(a_1, a_2) + d(b_1, b_2)) = (csin(a_1) + dsin(b_1), 0) \neq cT(a) + dT(b)$ not linear

d) $cT(a) + dT(b) = (|ca_1 + db_1|, ca_2 + db_2)T(ca + db) = T(c(a_1, a_2) + d(b_1, b_2)) = (c|a_1| + d|b_1|, ca_2 + db_2) \neq cT(a) + dT(b)$ not linear

e) $cT(a)+dT(b) = (ca_1+db_1+1, ca_2+db_2)T(ca+db) = T(c(a_1, a_2)+d(b_1, b_2)) = (ca_1+db_1+c+d, ca_2+db_2) \neq cT(a) + dT(b)$ not linear

10. Suppose that $T : R^2 \to R^2$ is linear, $T(1, 0) = (1, 4)$, and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is T one-to-one?

Proof:
$T(2, 3) = aT(1, 0) + bT(1, 1) = -(1, 4) + 3(2, 5) = (5, 11)$ and we can have that $c_1 = -1, c_2 = 3$
$T(1, 0) = T(1, 1) - T(1, 0) = (2 - 1, 5 - 4) = (1, 1)$ and $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ Then $det(A) = 3$ Therefore, A is invertible and T is one to one and onto.

11. Prove that there exists a linear transformation $T : R^2 \to R^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$

Proof:
$T(8, 11) = aT(1, 1) + bT(2, 3) = 5(1, 0, 2) + 3(1, -1, 4) = (8, -3, 22)$

14. Let V and W be vector spaces and $T : V \to W$ be linear

a) Prove that T is one to one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W

Proof:
Suppose T is one-to-one, and let S be a linearly independent subset of V. We want to show that T(S) is a linearly independent subset of W. Suppose that T(S) is linearly dependent, i.e., there exist distinct vectors $w_1, w_2, ..., w_n$ in T(S) and scalars $c_1, c_2, ..., c_n$ not all zero such that $c_1w_1 + c_2w_2 + ... + c_nw_n = 0$. Since each wi is in T(S), we can write wi $= T(v_i)$ for some v_i in S. Then we have: $c_1T(v_1) + c_2T(v_2) + ... + c_nT(v_n) = T(c_1v_1 + c_2v_2 + ... + c_nv_n) = T(0) = 0$ Since T is one-to-one, this implies that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$, which contradicts the assumption that S is linearly independent. Therefore, T(S) must be linearly independent.
Suppose T carries linearly independent subsets of V onto linearly independent subsets of W, and let v_1, v_2 be distinct vectors in V such that $T(v_1) = T(v_2)$. We want to show that $v_1 = v_2$, i.e., that T is one-to-one. Consider the set S $= \{v_1, v_2\}$. Since v_1 and v_2 are distinct, S is linearly independent. By assumption, $T(S) = \{T(v_1), T(v_2)\}$ is linearly independent. Therefore, we must have $c_1T(v_1) + c_2T(v_2) = 0$ only if $c_1 = c_2 = 0$. But we know that $T(v_1) = T(v_2)$, so we have: $c_1T(v_1) + c_2T(v_2) = T(c_1v_1 + c_2v_2) = 0$ This implies that $c_1v_1 + c_2v_2 = 0$, and since S is linearly independent, we must have $c_1 = c_2 = 0$. Therefore, $v_1 = v_2$, and T is one-to-one. Combining the two implications, we conclude that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if T(S) is linearly independent.

Proof:
Suppose T is one-to-one and S is a linearly independent subset of V. We want to show that $T(S)$ is linearly independent. Suppose that T(S) is linearly dependent, i.e., there exist distinct vectors $w_1, w_2, ..., w_n$ in T(S) and scalars $c_1, c_2, ..., c_n$ not all zero such that $c_1w_1+c_2w_2+...+c_nw_n = 0$. Since each wi is in T(S), we can write $w_i = T(v_i)$ for some v_i in S. Then we have: $c_1T(v_1)+c_2T(v_2)+...+c_nT(v_n) = T(c_1v_1+c_2v_2+...+c_nv_n) = 0$ Since T is one-to-one, this implies that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$, which contradicts the assumption that S is linearly independent. Therefore, T(S) must be linearly independent.
Suppose T is one-to-one and T(S) is linearly independent. We want to show that S is linearly independent. Suppose that S is linearly dependent, i.e., there exist distinct vectors $v_1, v_2, ..., v_n$ in S and scalars $c_1, c_2, ..., c_n$ not all zero such that $c_1v_1+c_2v_2+...+c_nv_n = 0$. Then we have: $T(c_1v_1+c_2v_2+...+c_nv_n) = c_1T(v_1)+c_2T(v_2)+...+c_nT(v_n) = 0$. Since T is one-to-one and the vi are distinct, we must have $c_1T(v_1)+c_2T(v_2)+...+c_nT(v_n) = 0$ only if $c_1 = c_2 = ... = c_n = 0$. But we know that T(S) is linearly independent, so this implies that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ only if $c_1 = c_2 = ... = c_n = 0$. Therefore, S is linearly independent. Combining the two implications, we conclude that S is linearly independent if and only if T(S) is linearly independent, when T is a one-to-one linear transformation.

c) Suppose $\beta =${$v_1, v_2, ..., v_n$} is a basis for V and T is one-to-one and onto. Prove that $T(\beta) =${$T(v_1), T(v_2)..., T(v_n)$} is a basis for W.

Proof:
If T is a one-to-one and onto linear transformation from V to W, and $\beta = v1, v2, ..., vn$ is a basis for V, then $T(\beta) = \{T(v1), T(v2), ..., T(vn)\}$ is a basis for W.

17. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear

a) Prove that if $dim(V) < dim(W)$, then T cannot be onto

Proof: Suppose that T is a linear transformation from V to W and $dim(V) < dim(W)$. We will prove that T cannot be onto. Assume, for the sake of contradiction, that T is onto. Then for any w in W, there exists v in V such that $T(v) = w$. In particular, for any basis $\{w_1, w_2, ..., w_d\}$ of W, there exist vectors $v_1, v_2, ..., v_d$ in V such that $T(v_i) = w_i$ for $i = 1, 2, ..., d$. Now consider the set $\{v_1, v_2, ..., v_d\}$. Since $dim(V) < dim(W)$, we have $d > dim(V)$, so this set contains more vectors than the dimension of V. Therefore, this set must be linearly dependent. That is, there exist scalars $c_1, c_2, ..., c_d$, not all zero, such that $c_1v_1 + c_2v_2 + ... + c_dv_d = 0$. Applying T to both sides, we get: $c_1T(v_1) + c_2T(v_2) + ... + c_dT(v_d) = T(c_1v_1 + c_2v_2 + ... + c_dv_d) = T(0) = 0$ But since $T(v_i) = w_i$ for $i = 1, 2, ..., d$, we have $c_1w_1 + c_2w_2 + ... + c_dw_d = 0$, which contradicts the linear independence of the basis $\{w_1, w_2, ..., w_d\}$ of W. Therefore, our assumption that T is onto must be false, and we conclude that if $dim(V) < dim(W)$, then T cannot be onto.

b) Prove that if $dim(V) > dim(W)$, then T cannot be one-to-one

Proof:
Suppose that T is a linear transformation from V to W and $dim(V) > dim(W)$. We will prove that T cannot be one-to-one. Assume, for the sake of contradiction, that T is one-to-one. Then for any two distinct vectors u, v in V, we have $T(u) \neq T(v)$. In particular, for any basis $\{v_1, v_2, ..., v_w\}$ of W, we can extend it to a basis $\{v_1, v_2, ..., v_w, ..., v_n\}$ of V, where n > w. Now consider the set $\{v_1, v_2, ..., v_n\}$. Since $n > w$, this set contains more vectors than the dimension of W. Therefore, this set must be linearly dependent. That is, there exist scalars $c_1, c_2, ..., c_n$, not all zero, such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$. Without loss of generality, assume that $c_1 \neq 0$. Then we can solve for v1 in terms of the other vectors: $v_1 = (-c_2/c_1)v_2 + (-c_3/c_1)v_3 + ... + (-c_n/c_1)v_n$ Now let u be the vector $u = (-c_2/c_1)v_2 + (-c_3/c_1)v_3 + ... + (-c_n/c_1)v_n$. Then u is a non-zero vector in V, and we have $T(u) = T(v1) = 0$, since v_1 can be expressed as a linear combination of the other vectors. This contradicts the assumption that T is one-to-one, since $u \neq 0$ but $T(u) = 0$. Therefore, our assumption that T is one-to-one must be false, and we conclude that if $dim(V) > dim(W)$, then T cannot be one-to-one.

20. Let V and W be vector spaces with subspaces V_1 and W_2 respectively. If $T : V \to W$ is linear, prove that $T(V_1)$ is a subspace of W and that $\{x \in V : T(x) \in W_1\}$ is a subspace of V.

Proof:
To show that $T(V_1)$ is a subspace of W, we need to verify that it satisfies the following three conditions:

It contains the zero vector: Since T is linear, $T(0) = 0$, so $0 \in T(V_1)$. It is closed under addition: Suppose $y_1, y_2 \in T(V_1)$. Then, there exist $x_1, x_2 \in V_1$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. Since V_1 is a subspace of V, we have $x_1 + x_2 \in V_1$. Therefore, $T(x_1 + x_2) = T(x_1) + T(x_2) = y_1 + y_2$. Thus, $y_1 + y_2 \in T(V_1)$. It is closed under scalar multiplication: Suppose $y \in T(V_1)$ and c is a scalar. Then, there exists $x \in V_1$ such that $T(x) = y$. Since V_1 is a subspace of V, we have $cx \in V_1$. Therefore, $T(cx) = cT(x) = cy$. Thus, $cy \in T(V_1)$. Therefore, $T(V_1)$ is a subspace of W.

To show that $x \in V : T(x) \in W_1$ is a subspace of V, we need to verify the following three conditions:

It contains the zero vector: Since $T(0) = 0 \in W_1$, we have $0 \in x \in V : T(x) \in W_1$. It is closed under addition: Suppose $x_1, x_2 \in x \in V : T(x) \in W_1$. Then, $T(x_1), T(x_2) \in W_1$, so $T(x_1 + x_2) = T(x_1) + T(x_2) \in W_1$. Thus, $x_1 + x_2 \in x \in V : T(x) \in W_1$. It is closed under scalar multiplication: Suppose $x \in x \in V : T(x) \in W_1$ and c is a scalar. Then, $T(x) \in W_1$, so $cT(x) \in W_1$. Thus, $T(cx) = cT(x) \in W_1$. Therefore, $cx \in x \in V : T(x) \in W_1$. Therefore, $x \in V : T(x) \in W_1$ is a subspace of V.

21. Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions T, $U : V \to V$ by $T(a_1, a_2, ...) = (a_2, a_3, ...)$ and $U(a_1, a_2, ...) = (0, a_1, a_2, ...)$. T and U are called the left shift and right shift operators on v, respectively.

Prove That T and U are linear

Let a,b be sequences in V, and let c be a scalar in the underlying field. Then we have:
$T(c\mathbf{a} + \mathbf{b}) = T(ca_1 + b_1, ca_2 + b_2, ...) = (ca_2 + b_2, ca_3 + b_3, ...) = c(a_2, a_3, ...) + (b_2, b_3, ...) = cT(\mathbf{a}) + T(\mathbf{b})$
Therefore, T satisfies the additivity and homogeneity properties required for a function to be linear.

Let a,b be sequences in V, and let c be a scalar in the underlying field. Then we have:

$$U(c\mathbf{a}+\mathbf{b}) = U(ca_1+b_1, ca_2+b_2, ...) = (0, ca_1+b_1, ca_2+b_2, ...) = c(0, a_1, a_2, ...)+(0, b_1, b_2, ...) = cU(\mathbf{a})+U(\mathbf{b})$$

Therefore, U satisfies the additivity and homogeneity properties required for a function to be linear.

Prove That T is onto, but not one-to-one.

Let b be an arbitrary sequence in V, and let $a = (0, b_1, b_2, ...)$. Then $T(a) = (b_1, b_2, ...) = b$. Therefore, T is onto.
Consider the sequences $a = (1, 0, 0, ...)$ and $b = (0, 1, 0, ...)$. Then $T(a) = T(b) = (0, 0, ...)$, so T is not one-to-one.

Prove That U is one to one, but not onto.

Suppose $U(a) = U(b)$ for some sequences a,b in V. Then we have $(0, a_1, a_2, ...) = (0, b_1, b_2, ...)$, which implies $a_1 = b_1, a_2 = b_2$, and so on. Therefore, $a = b$, and U is one-to-one.
Let b be the sequence $(1, 0, 0, ...)$, and suppose there exists a sequence a in V such that U(a) = b. Then we have $(0, a_1, a_2, ...) = (1, 0, 0, ...)$, which implies $a_1 = 0$. But then $(0, a_1, a_2, ...) = (0, 0, a_2, ...)$, so $U(a)$ is not equal to b. Therefore, U is not onto.

22. Let $T : R^3 \to R$ be linear. Show that there exist scalars a,b and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in R^3$. can you generalize this result for $T : F \to F$? State and prove an analogous result for $T : F^n \to F^m$

Proof:
Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation. We want to show that there exist scalars a, b, and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$.

Since T is linear, we know that T can be represented by a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. Let $A = [a_1, a_2, a_3]$, where a_1, a_2, and a_3 are the columns of A. Then we have:

$$T(x, y, z) = A \begin{bmatrix} x \ y \ z \end{bmatrix} = x \begin{bmatrix} a_1 \ a_2 \ a_3 \end{bmatrix} \begin{bmatrix} a_1 \ a_2 \ a_3 \end{bmatrix} + z \begin{bmatrix} a_1 \ a_2 \ a_3 \end{bmatrix} = (xa_1 + ya_2 + za_3) = (ax + by + cz)$$

where $a = a_{1,1}$, $b = a_{2,1}$, and $c = a_{3,1}$.

Therefore, we have shown that there exist scalars a, b, and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$.

Let $T : F^n \to F^m$ be a linear transformation. We want to show that there exist matrices $A \in F^{m \times n}$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in F^n$.

Since T is linear, we know that T can be represented by a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in F^n$. Let $A = [a_1, a_2, \ldots, a_n]$, where a_1, a_2, $\ldots$, a_n are the columns of A. Then we have:

$$T(\mathbf{x}) = A \begin{bmatrix} x_1 \ x_2 \ \vdots \ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_1 \ a_2 \ \vdots \ a_m \end{bmatrix} + x_2 \begin{bmatrix} a_1 \ a_2 \ \vdots \ a_m \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_1 \ a_2 \ \vdots \ a_m \end{bmatrix} = (x_1a_1 + x_2a_2 + \cdots + x_na_n)$$