115A/4, Winter 2023

Lecture 7

Wed, Jan 25

• Last time we saw that "a few" vectors $v_1, \dots, v_m$ in a vector space V could generate (span) the entire V, i.e. any other vector $v \in V$ can be written as a linear combination of $v_1, \dots, v_m$. For instance, any matrix $A \in M_{2\times 2}(\mathbb{R})$ is a linear combination of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

• It is important to find "small" sets of vectors in V that span V, in other words "optimal", "most economical" ways to generate V.

Linear dependence & linear independence of vectors (§1.5)

7.1 Definition A subset of vectors S in a vector space V is linearly dependent if there exist finitely many distinct vectors $v_1, \dots, v_n \in S$ and scalars $c_1, \dots, c_n \in F$ not all of them 0, such that $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$

In other words: if one can express the vector 0 as a linear combination of distinct vectors in S with non-zero coefficients.

7.2 Example. The set

$$S = \{\underbrace{(-1,1,0)}_{v_1}, \underbrace{(1,-3,2)}_{v_2}, \underbrace{(0,1,-1)}_{v_3}\}$$

in $\mathbb{R}^3$ is linearly dependent because $v_1 + v_2 + 2v_3 = 0$

Indeed $(-1,1,0) + (1,-3,2) + 2(0,1,-1)$

$= (-1,1,0) + (1,-3,2) + (0,2,-2)$

$= (1-1, 1-3+2, 0+2-2) = (0,0,0)$

7.3. Definition A subset S of a vector space V is linearly independent if it is not linearly dependent.
We then also say that the vectors in V are linearly independent.

7.4. Theorem. The set $S \overset{\neq\emptyset}{\subset} V$ is linearly independent iff a linear combination $c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ of distinct vectors $v_1, \dots, v_n \in S$ with $c_1, \dots, c_n \in F$ is equal to the vector 0 only when all coefficients $c_1, \dots, c_n$ are zero, i.e. $\sum_{i=1}^{n} c_i v_i = 0$ implies $c_i = 0$ $\forall i$

Proof By definition, S linearly independent means it is NOT linearly dependent. In other words, The only way to write the vector 0 as a linear combination of some distinct vectors $v_1, \dots, v_n \in S$ is if we take all coefficients $c_1, \dots, c_n \in F$ equal to 0.

7.5 Example Let $S = \{\overbrace{(1,0,0,-1)}^{v_1}, \overbrace{(0,1,0,-1)}^{v_2}, \overbrace{(0,0,1,-1)}^{v_3}, \overbrace{(0,0,0,1)}^{v_4}\} < \mathbb{R}^4$.

Show that S is linearly independent.

Solution By Thm 7.4, we need to show that the only linear combination of v_1, v_2, v_3, v_4 that equals 0 is the one in which all coefficients are 0, i.e. to prove that if

$$(*)\begin{cases} c_1(1,0,0,-1)+c_2(0,1,0,-1)+c_3(0,0,1,-1) \\ \quad + c_4(0,0,0,1) = 0 \end{cases}$$

Then $c_1 = c_2 = c_3 = c_4 = 0$.

indeed $(*) = (c_1, 0, 0, -c_1)$
$+ (0, c_2, 0, -c_2) + (0, 0, c_3, -c_3)$
$+ (0, 0, 0, c_4) = (c_1, c_2, c_3, -c_1-c_2-c_3+c_4)$

so $(*) = 0$ means
$(c_1, c_2, c_3, -c_1-c_2-c_3+c_4) = (0,0,0,0)$

i.e. $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $-c_1-c_2-c_3+c_4 = 0$

thus $c_4 = 0$ as well.

so all coefficients c_1, c_2, c_3, c_4 must be equal to 0, showing that indeed the set S is linearly independent.

7.6 Example A set S consisting of just one non-zero vector,

$S = \{v\}$ with $v \neq 0$, is always linearly independent, because the only possible linear combination with vectors in S is

cv with $c \in F$, and if

$cv = 0$ then $c = 0$. Indeed, for if $c \neq 0$ then $cv = 0$ implies

$$c^{-1}(cv) = 0$$

$$\shortparallel$$

$$(c^{-1}c)v = 0 \quad \text{so} \quad 1 \cdot v = v = 0,$$ contradiction

7.7. Theorem. Let V be a vector space and $S_1 \subset S_2 \subset V$ subsets of V.

(a) if S_2 is linearly independent then S_1 is linearly independent.

(b) if S_1 is linearly dependent then S_2 is linearly dependent.

$\boxed{\text{Proof}}$. We only need to prove (b) because (a) is logically equivalent to (b).

If S_1 is lin. dependent then there exist distinct vectors $v_1, \dots, v_m \in S_1$ and non-zero scalars $c_1, \dots, c_m \in F$ such that $c_1 v_1 + c_2 v_2 \dots + c_m v_m = 0$.

But because $S_1 \subset S_2$, the vectors $v_1, \dots, v_m$ are in S_2 as well, so in S_2 we have $c_1 v_1 + \dots + c_m v_m = 0$ with $c_i \neq 0$ and v_i distinct, thus S_2 lin. dependent

- The above Thm says that any subset of a lin. independent set is lin. independent

7.8 Theorem Let V be a vector space and $S \subset V$ a subset. Then S is linearly independent iff for any (strictly) smaller subset $S' \subsetneq S$, we have $\mathrm{span}(S') \neq \mathrm{span}(S)$

Proof. "$\Rightarrow$" Assume S lin. indep. and let $S' \subset S$ be a subset, $S' \neq S$. Let $v \in S \setminus S'$. If by contradiction we assume $\mathrm{span}(S') = \mathrm{span}(S)$, then there exist $v_1, \dots, v_m \in S'$ distinct and $c_1, \dots, c_m \in F$ such that $v = \sum_{i=1}^{m} c_i v_i$. Thus,

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m - 1 \cdot v = 0$$

with $v_1, \dots, v_m, v \in S$ distinct vectors and $c_1, c_2, \dots, c_m, -1$ not all $= 0$ contradicting the fact that S is lin. indep.

"$\Leftarrow$" Assume that $\forall S' \subsetneq S$ we have $\mathrm{span}(S') \neq \mathrm{span}(S)$. If S would be lin. dependent (by contrad) then $\exists v_1, \dots, v_m \in S$ distinct and $c_1, \dots, c_m \in F$, $\neq 0$, such that

$c_1 v_1 + \dots + c_m v_m = 0$. By 7.6

we know that we must have $n \geq 2$.

so $c_1 v_1 = -c_2 v_2 - \dots - c_n v_n$

and multiplying both sides by c_1^{-1} we get

$$v_1 = -\frac{c_2}{c_1} v_2 - \dots - \frac{c_n}{c_1} v_n$$

Thus, if we take $S' = S \setminus \{v_1\}$

then v_1 is in the linear span of $\{v_2, \dots, v_n\} \subset S'$, thus

$v_1 \in \operatorname{span}(S')$

so $\operatorname{span}(S') = \operatorname{span}(S)$,

contradiction $\boxtimes$

• Another way to state the Thm 7.8 is this:

7.8' Let V be a vector space and $S \subset V$ a linearly independent subset. Let $v \in V$ be a vector that's not in S. Then $S \cup \{v\}$ is linearly dependent iff $v \in \operatorname{span}(S)$

and $S \cup \{v\}$ is linearly independent iff $v \notin \mathrm{span}(S)$.

$\boxed{\text{7.9 Exercise}}$. Label the following statements as true / false with justific.

(a). Any set $S \subset V$ with $0 \in S$ is linearly dependent

Answer: $\boxed{\text{YES}}$, because for any distinct $v_1 = 0, v_2, \dots, v_n \in S$ we can take $c_1 = 1, c_2 = c_3 = \dots = c_n = 0$ and get $1 \cdot 0 + 0 \cdot v_1 + \dots + 0 \cdot v_n = 0$

(b). Subsets of linearly dependent sets are lin. dependent

Answer: $\boxed{\text{NO}}$ For instance

$S = \{\underbrace{(1,0)}_{v_1}, \underbrace{(0,1)}_{v_2}, \underbrace{(-1,-1)}_{v_3}\} \subset \mathbb{R}^2$ is linearly dependent set, because $v_1 + v_2 + v_3 = 0$. But $S' = \{(1,0), (0,1)\}$

is a linearly independent subset of S

(c) Subsets of lin. ind. sets are lin. indep.

<u>Answer</u>: $\boxed{\text{YES}}$ this is part Thm 7.7

7.10 Exercise. Show that

$$S = \left\{ \underset{v_1}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{v_5}{\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}} \right\}$$

$\subset M_{3\times 2}(\mathbb{R})$ is linearly dependent

<u>solution</u> In general, to check that $v_1, \dots, v_5$ are lin. dep/indep one has to solve the system of equations resulting from

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + c_5 v_5 = 0$$

with the unknowns $c_1, c_2, \dots, c_5$.

If one gets that the only solution is when $c_1 = c_2 = \dots = c_5 = 0$ then $\{v_1, \dots, v_5\}$ lin. indep.

If one gets some other solutions, where some $c_i \neq 0$, then lin. dep.

In our case though one sees right away that

$$v_1 + v_2 + v_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and also

$$v_4 + v_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus

$$v_1 + v_2 + v_3 = v_4 + v_5,$$

in other words

$$v_1 + v_2 + v_3 - v_4 - v_5 = 0$$

so $\{v_1, \dots, v_5\}$ lin. indep.