115A/4, Winter 2023

Lecture 6

Mo, Jan 23

6.1. Definition If S_1, S_2 are nonempty subsets of a vector space V then the sum of of S_1 and S_2, denoted $S_1 + S_2$ is the set $\{x+y : x \in S_1, y \in S_2\}$

6.2. Definition Let W_1, W_2 be subspaces of the vector space V.
We say that V is the direct sum of W_1 and W_2 if $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$, and we then write $V = W_1 \oplus W_2$

6.3 Exercise. Show that $V = \mathbb{R}^2$ is the direct sum of $W_1 = \{(x,x) : x \in \mathbb{R}\}$ and $W_2 = \{(y,-y) : y \in \mathbb{R}\}$

Solution. First note that W_1, W_2 are indeed vector subspaces of $\mathbb{R}^2$. indeed, if we take two elements $(x,x) \in W_1$, $(z,z) \in W_2$, then $(x,x) + (z,z) = (x+z, x+z) \in W_1$. Also, if $c \in \mathbb{R}$ is a scalar, then

$c(x,x) = (cx, cx) \in W_1$.

Similarily for W_2.

Also, we see that $0 = (0,0) \in W_1$ and $0 = (0,-0) \in W_2$. So W_1, W_2 satisfy the conditions in Thm. 5.2 so they are subspaces of $V = \mathbb{R}^2$.

We now show that:

(a) $W_1 \cap W_2 = \{0\}$

(b) $W_1 + W_2 = V$

To prove (a) assume $v = (v_1, v_2) \in \mathbb{R}^2$ is both in W_1 and in W_2.

Since $(v_1, v_2) \in W_1$, we must have $v_1 = v_2$. Since $(v_1, v_2) \in W_2$, we must have $v_2 = -v_1$. Thus $v_2 = v_1$, $v_2 = -v_1$ so $v_1 = -v_1$, Thus $2v_1 = 0$ which for $v_1 \in \mathbb{R}$ implies $v_1 = 0$. So $v_2 = v_1 = 0$ as well

i.e. $(v_1, v_2) = (0,0)$

To prove (b), let $v = (v_1, v_2) \in \mathbb{R}^2$. We need to find $(x,x) \in W_1$, $(y,-y) \in W_2$

such that $\underbrace{(x, x) + (y, -y)}_{=(x+y,\ x-y)} = (v_1, v_2)$.
This means

$(x+y,\ x-y) = (v_1, v_2)$

so $\left.\begin{array}{l} x+y = v_1, \quad x-y = v_2 \end{array}\right\} (*)$

Thus, to find x, y satisfying these two conditions we need to solve this system of two eq. with two unknowns x, y in real numbers.

from 2'nd eq in $(*)$ we get

$x = y + v_2$ and replacing in the 1'st one

$(y + v_2) + y = v_1$

so $2y = v_1 - v_2$, $y = \frac{v_1 - v_2}{2}$

and so $x = y + v_2 = \frac{v_1 - v_2}{2} + v_2 = \frac{v_1 + v_2}{2}$

Thus $(v_1, v_2) = \underbrace{\left(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}\right)}_{\in W_1} + \underbrace{\left(\frac{v_1-v_2}{2}, \frac{v_2-v_1}{2}\right)}_{\in W_2}$

and so we checked that $W_1 + W_2 = V$.

i.e (b)

$\boxtimes$

Linear combinations of vectors (§1.4).

6.4. Definition Let V be a vector space over a field F and $S \subset V$ a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors in S if there exist a finite number of vectors $u_1, \ldots, u_n \in S$ and scalars $c_1, \ldots, c_n \in F$ such that $v = \underbrace{c_1 u_1 + c_2 u_2 + \cdots + c_n u_n}_{= \sum_{i=1}^{n} c_i u_i}$

We then also say that v is a linear combination of $u_1, \ldots, u_n$. The scalars $c_1, \ldots, c_n$ are called the coefficients of the linear combination.

because $0_V = 0_F \cdot v$, $\forall v \in S$

Note: the vector $0_V \in V$ is a linear combination of any $S \subset V$! $\neq \emptyset$

6.5. Example. Denote by V the set of polynomials of degree at most n with coefficients in $\mathbb{R}$, i.e. expressions of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, with the usual addition and multiplication by scalars in $\mathbb{R}$:

$$(a_0 + a_1 X + \cdots + a_n X^n) + (b_0 + b_1 X + \cdots + b_n X^n)$$
$$= (a_0 + b_0) + (a_1 + b_1) X + \cdots + (a_n + b_n) X^n$$

and $c(a_0 + a_1 X + \cdots + a_n X^n) = ca_0 + ca_1 X + \cdots + ca_n X^n$

Show that any polynomial in V is a linear combination of the "monomials"

$$\underbrace{1}_{u_0}, \underbrace{X}_{u_1}, \underbrace{X^2}_{u_2}, \ldots, \underbrace{X^n}_{u_n}$$

indeed, if $P(X) = a_0 + a_1 X + \cdots + a_n X^n \in V$ then $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are scalars and we have $P(X) = a_0 \cdot 1 + a_1 \cdot X + \cdots + a_n \cdot X^n$

$$= a_0 u_0 + a_1 u_1 + \cdots + a_n u_n$$

$\boxed{\text{6.6. Definition}}$ if V is a vector space and $S \overset{\neq \emptyset}{\subset} V$, then the span of S, denoted $\mathrm{span}(S)$, is the set of all linear combinations of vectors in S.

i.e. $\mathrm{span}(S) = \left\{ \sum_{i=1}^{n} c_i u_i : u_1, \ldots, u_n \in S, \; c_1, \ldots, c_n \in F, \; n \geq 1 \right\}$

$\boxed{\text{6.7. Example}}$ if we take $V = \mathbb{R}^3$

and $S = \{ \overbrace{(1,0,0)}^{u_1}, \overbrace{(0,1,0)}^{u_2} \}$

then span(S) is the set of all vectors in $\mathbb{R}^3$ of the form $a u_1 + b u_2$

$= a(1,0,0) + b(0,1,0) = (a,0,0) + (0,b,0)$

$= (a,b,0)$, with $a, b \in \mathbb{R}$ arbitrary scalars in $\mathbb{R}$

Thus span$(S) = \{ (a,b,0) : a,b \in \mathbb{R} \}$ which we recognize to be the xy plane in the xyz 3-dimensional Euclidean space.

$\boxed{\text{6.8 Example}}$ If we take V to be the vector space of polynomials in X of degree $\leq n$ with coefficients in $\mathbb{R}$ as in Example 6.5 and we set

$S = \{ 1, X, X^2, \ldots, X^n \}$ then

span$(S) = V$.

$\boxed{\text{6.9. Example}}$ Given a field F one denotes by $F[X]$ the set of all polynomials in "undeterminate" X over the field F, i.e. expressions of the form $P(X) = a_0 + a_1 X + \dots + a_n X^n$

for some $n \geq 0$ and $a_0, a_1, \dots, a_n \in F$

with the "usual" addition and
coefficient by coefficient
scalar multiplication

The degree of $P(x)$ is the largest n such that $a_n \neq 0$.

(a) Show that if $S = \{1, X, X^2, \dots\}$ then $\text{span}(S) = F[X]$

(b) Denote $F_{odd}[X]$ the set of all polynomials with coefficients in F that have only odd coefficients possibly $\neq 0$ and by $F_{even}[X]$ the set of all polyn. with coef. in F that have only even coefficients possibly $\neq 0$,

i.e. $F_{odd}[X] = \{ P(X) \in F[X] : P(X) = a_1 X + a_3 X^3 + a_5 X^5 + \dots + a_{2n+1} X^{2n+1},$
$a_1, a_3, \dots, a_{2n+1} \in F, n \geq 0 \}$

$F_{even}[X] = \{ P(X) \in F[X] :$
$P(X) = a_0 + a_2 X^2 + a_4 X^4 + \dots + a_{2n} X^{2n},$
$a_0, a_2, \dots, a_{2n} \in F, n \geq 0 \}$

Show that $W_1 = F_{odd}[X]$, $W_2 = F_{even}[X]$ are subspaces of $F[X]$
and that $F[X] = W_1 \oplus W_2$

$\boxed{Pf}$ exercise

6.10 Theorem. The span of any subset S of a vector space V is a subspace of V. Any subspace of V that contains S must contain $\text{span}(S)$
(i.e. if $W < V$ subspace with $S \subset W$ then $\text{span}(S) \subset W$)

$\boxed{Pf}$. We have to prove that if $x, y \in \text{span}(S)$ then $x + y \in \text{span}(S)$ and $cx \in \text{span}(S)$ $\forall c \in F$.

Since $x, y \in \text{span}(S)$, there exist $u_1, \dots, u_m \in S$, $v_1, \dots, v_n \in S$ and scalars $a_1, \dots, a_m \in F$, $b_1, \dots, b_n \in F$ such that

$$x = a_1 u_1 + \dots + a_m u_m$$

$$y = b_1 v_1 + \dots + b_n v_n .$$

But then $x + y = a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n$

so $x+y$ is itself a linear combination of $u_1, \dots, u_m, v_1, \dots, v_n \in S$, thus $x+y \in \text{span}(S)$.

Also, $cx = c(a_1 u_1 + \dots + a_m u_m)$

$= (ca_1) u_1 + (ca_2) u_2 + \dots + (ca_m) u_m \in \text{span}(S)$

For the last part of Thm: if $W \subset V$ is a subspace that contains S and $w \in \text{span}(S)$, then there exist $u_1, \dots, u_m \in S$ and $a_1, \dots, a_m \in F$ such that $w = a_1 u_1 + \dots + a_m u_m$. Since W is a subspace and $u_1, \dots, u_m \in S \subset W$, we have $a_1 u_1 + \dots + a_m u_m \in W$. Thus $w \in W$ showing that $\text{span}(S) \subset W$. $\boxtimes$

6.11 Definition We say that a subset S of a vector space V generates (or spans) V if $\text{span}(S) = V$.

6.12. Examples 1° If we take $V = \mathbb{R}^2$ like in Exercise 6.3

then $S = \{ \overbrace{(1,1)}^{u_1}, \overbrace{(1,-1)}^{u_2} \}$ generate (span) V, because we showed in that exercise that any $v \in V$ is of the form $v = a u_1 + b u_2$ for some scalars $a, b \in \mathbb{R}^2$

as in Example 6.9

2° If we take $V = F[X]$ and $S = \{1, X, X^2, \dots\}$ then

$$\operatorname{span}(S) = F[X]$$

6.13 Exercise Show that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ generate $M_{2\times 2}(F)$.

Solution. Any matrix in $M_{2\times 2}(F)$ is of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in F$

But then $a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$+ d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ so

A is indeed a linear combination of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.