Math115A 1/25 notes

Vincent

2023-01-25

Last time we saw that "a few" vectors $v_1, ..., v_n$ in a vector space V could generate (span) the entire V, i.e. any other vector $v \in V$ can be written as a linear combination of $v_1, ..., v_n$. For instance, any matrix $A \in M_{2\times 2}(\mathbb{R})$ is a linear combination of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
It is important to find "small" sets of vectors in V that span V in other word "optimal", "most economical" ways to generate V.
Linear dependence & linear independence of vectors

7.1 Definition

A subset of vectors since vector space V is linearly dependent if there exist finitely many distinct vectors $v_1, .., v_n \in S$ and scalars $c_1, ..., c_n \in F$ not all of there 0, such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$
In other word: if one can express the vector 0 as a linear combination of distinct vectors in S with non-zero coefficients.

7.2 Example

The set $S = \{(-1, 1, 0), (1, -3, 2), (0, 1, -1)\}$ in $\mathbb{R}^3$ is linearly dependent because $v_1 + v_2 + 2v_3 = 0$ indeed $(-1, 1, 0) + (1, -3, 2) + 2(0, 1, -1) = (-1, 1, 0) + (1, -3, 2) + (0, 2, -2) = (1 - 1, 1 - 3 + 2, 0 + 2 - 2) = (0, 0, 0)$

7.3 Definition

A subset S of a vector space V is linearly independent if it is not linearly dependent.
We then also say that the vectors in V are linearly independent.

7.4 Theorem

The set $S \neq 0 \in V$ is linearly independent iff a linear combination $c_1v_1 + c_2v_2 + ... + c_nv_n$ of distinct vectors $v_1, ..., v_n \in S$ with $c_1, ..., c_n \in F$ is equal to the vector 0 only when all coefficients $c_1, .., c_n$ are zero. i.e. $\Sigma_{i=1}^{n} c_iv_i = 0$ implies $c_i = 0, \forall i$
Proof:
By definition, S linearly independent means it is not linearly dependent. In other words, the only way to

write the vector 0 as a linear combination of some distinct vector $v_1, ...v_n \in S$ is if we take all coefficients $c_1, ..., c_n \in F$ equal to 0.

7.5 Example

Let $S = \{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\} \in \mathbb{R}^4$. Show that S is linearly independent.
Solution: By theom 7.4, we need to show that the only linear combination of v_1, v_2, v_3, v_4 that equals 0 is the one in all coefficients are 0, i.e. to prove that if $c_1(1,0,0,-1)+c_2(0,1,0,-1)+c_3(0,0,1,-1)+c_4(0,0,0,1) = 0$ Then $c_1 = c_2 = c_3 = c_4 = 0$.
indeed $c_1(1,0,0,-1) + c_2(0,1,0,-1) + c_3(0,0,1,-1) + c_4(0,0,0,1) = (c_1,0,0,-c_1) + (0,c_2,0,-c_2) + (0,0,c_3,-c_3) + (0,0,0,c_4) = (c_1,c_2,c_3,-c_1-c_2-c_3+c_4)$ and $(c_1,c_2,c_3,-c_1-c_2-c_3+c_4) = (0,0,0,0)$ i.e. $c_1 = 0, c_2 = 0, c_3 = 0, -c_1-c_2-c_3+c_4 = 0$ thus $c_4 = 0$ as well.
so all coefficients c_1, c_2, c_3, c_4 must be equal to 0, showing that indeed the set S is linearly independent.

7.6 Example

A set S consisting of just one non-zero vector, S=$\{$v$\}$ with $v \neq 0$, is always linearly independent, because the only possible linear combination with vectors in S is cv with $c \in F$, and if $cv = 0$ then $c = 0$. indeed, for if $c \neq 0$ then $cv = 0$. implies $c^{-1}(cv) = 0, (c^{-1}c)v = 0$ so $1 * v = v = 0$, contradiction.

7.7 Theorem

Let V be a vector space and $S_1 \in S_2 \in V$ subsets of V
(a) if S_2 is linearly independent then S_1 is linearly independent
(b) if S_1 is linearly dependent then S_2 is linearly dependent.
Proof:
We only need to prove (b) because (a) is logically equivalent to (b).
if S_1 is linear dependent then there exist distinct vectors $v_1, ..., v_n \in S_1$ and non-zero scalars $c_1, ..., c_n \in F$ such that $c_1v_1 + c_2v_2... + c_nv_n = 0$. But because $S_1 \in S_2$, the vectors $v_1, ..., v_n$ are in S_2 as well, so in S_2 we have $c_1v_1 + ... + c_nv_n == 0$ with $c_i \neq 0$ and v_i distinct, thus S_2 linear dependent.
The above Theorem says that only subset of a linear independent set is linear independent.

7.8 Theorem

Let V be a vector space and $S \in V$ a subset. Then S is linearly independent iff for any strictly smaller subset $S' \leq\neq S$, we have $span(S') \neq span(S)$.
Proof:
Assume S linear independence and let $S' \in S$ be a subset, $S' \neq S$. Let $v \in S - S'$. if by contradiction we assume $span(S') = span(S)$, then there exist $v_1, ...v_n \in S'$ distinct and $c_1, ..., c_n \in F$ such that $v = \Sigma_{i=1}^{n} c_i v_i$. Thus, $c_1v_1 + c_2v_2 + ... + c_nv_n - 1 * v = 0$ with $v_1, ..., v_n, v \in S$ distinct vectors and $c_1, c_2, ...c_n, -1$ not all $= 0$ contradicting the fact that S is linear independent.
Assume that $\forall S' \leq\neq S$ we have $span(S') \neq span(S)$. if S would be linear dependent(by contradiction) then

$v_1,...,v_n \in S$ distinct and $c_1,...,c_n \in F \neq 0$, such that $c_1v_1 + ...c_nv_n = 0$. By 7.6 we know that we must have $v \geq 2$ So $c_1v_1 = -c_2v_2 - ... - c_nv_n$ and multiplying both sides by c_1^{-1} we get $v_1 = \frac{-c_2}{c_1}v_2 - ... - \frac{-c_n}{c_1}v_n$. Thus, if we take $S' = S-\{v_1\}$ then v_1 is in the linear span of $\{v_2,...,v_n\}\in S'$, thus $v_1 \in span(S')$ So $span(S') = span(S)$, contradiction.
Another way to satate the theorem 7.8 is this:
Let V be a vector space and $S \in V$ a linearly independent subset. Let $v \in V$ be a vector that's not in S. Then $s\cup$\{v\} is linearly dependent iff $v \in span(S)$ and $S\cup$\{v\} is linearly independent iff $v \notin span(S)$

7.9 Exercise

Label the following statement as true/false with testifier.
(a) Any set $S \in V$ with $0 \in S$ is linearly dependent
Answer Yes, because for any distinct $v_1 = 0, v_2,...,v_n \in S$ we can take $c_1 = 1, c_2 = c_3 = ... = c_n = 0$ and set $1 * 0 + 0 * v_1 + ... + 0 * v_n = 0$
(B)Subsets of linearly dependent sets are linear dependent
Answer No, For instance $S =\{(1,0),(0,1),(-1,-1)\}\in \mathbb{R}^{\not{2}}$ is linearly dependent set, because $v_1 + v_2 + v_3 = 0$ But S'=$\{(1,0),(0,1)\}$ is a linearly independence subset of S
(c) subsets of linear independence sets are linear independent
Answer Yes this is state in Theom 7.7

7.10 Exercise

Show that $S =\{ \begin{pmatrix}1&1\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix} \}\in \mathbb{M}_{3\times 2}(\mathbb{R})$ is linearly dependent
solution:
In general, to state that $v_1,...,v_5$ are linear dependent/independent are has to solve the system of equation resulting from $c_1v_1 + c_2v_2 + c_2v_3 + c_4v_4 + c_5v_5 = 0$ with the unknowns $c_1, c_2, ..., c_5$. if we gets that the only solution is when $c_1 = c_2 = ... = c_5 = 0$ then $\{v_1,...,v_5\}$ linear independent if we gets solve other solutions where solve $c_i \neq 0$, then linear dependent
In our case though we solve right away that $v_1 + v_2 + v_3 = \begin{pmatrix}1&1\\0&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\1&1\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix} = \begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix}$ Thus $v_1 + v_2 + v_3 = v_4 + v_5$, in other words $v_1 + v_2 + v_3 - v_4 - v_5 = 0$ so $\{v_1,...,v_5\}$ linear independent