

Math115A 1/27 notes

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2023-01-27

We talked about how important it is to identify subsets S of a vector space V that generate (span) V and are in some scalar “minimal” (or most efficient) with this property.

Today we’ll study more in depth such sets, moving that such minimal generating sets” S are automatically linearly independent, and have the remarkable property that any vector in V can be uniquely written as a linear combination of vectors in S . We’ll call such S , basis for V

8.1 Definition

A linearly independent subset S of a vector space V that generates (spans) V is called a basis for V .

8.2 Example

Let $V \in \mathbb{R}^3$ and $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ Then S is a basis for V .

Indeed: we already showed that $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ are linearly independent vectors in \mathbb{R}^3 . And if $v = (a, b, c) \in \mathbb{R}^3$ is an arbitrary vector, then $v = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$, Thus, S spans V as well

8.3 Exercise

Let $V = M_{2 \times 2}(R)$ and consider the set $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Show that S is a basis for V .

Solution:

we already showed in an exercise on Monday (6.13) That S spans V if we would have $a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} +$

$$c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

Then this entails $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $a=0, b=0, c=0, d=0$ showing that S is linear independent.

8.4 Example

if F is a field and $V = F[X]$ is the vector space over F of all polynomials in undetermined X and coefficients in F then $S = \{1, X, X^2, X^3, \dots\}$ is a basis for V .

indeed, we already showed that $\text{span}(S) = V$ if $a_0, a_1, \dots \in F$ so that $a_0 1 + a_1 X + a_2 X^2 + \dots + a_n X^n = 0$ then by the definition of the polynomials we must have $a_0 = 0, a_1 = 0, \dots, a_n = 0$ showing that S is linear independent as well

8.5 Theorem

Let V be a vector space. A set $S = \{v_1, \dots, v_n\} \in V$ is a basis for V iff any vector $v \in V$ can be uniquely expressed as a linear combination of elements in S , i.e., there exist unique scalars $c_1, c_2, \dots, c_n \in F$ such that $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

Proof:

if $S = \{v_1, \dots, v_n\}$ is a basis for V , $\text{span}(S) = V$, so if $v \in V$ is an arbitrary vector in V , then there exist $c_1, \dots, c_n \in F$ such that $v = \sum_{i=1}^n c_i v_i$ if $a_1, \dots, a_n \in F$ are other scalars such that $v = \sum_{i=1}^n a_i v_i$ as well then $\sum_{i=1}^n c_i v_i = \sum_{i=1}^n a_i v_i$ so by cancellation then we get $\sum_{i=1}^n c_i v_i - \sum_{i=1}^n a_i v_i = 0$ which using additive & commutativity of addition + distributivity of scalar multiplication gives $(c_1 - a_1)v_1 + (c_2 - a_2)v_2 + \dots + (c_n - a_n)v_n = 0$ But since $S = \{v_1, \dots, v_n\}$ is a basis, the vectors v_1, \dots, v_n are linear independent, so this implies all coefficients $c_i - a_i$ in $(c_1 - a_1)v_1 + (c_2 - a_2)v_2 + \dots + (c_n - a_n)v_n = 0$. So we showed v can be expressed in only one way as a linear combination of v_1, \dots, v_n

If any $v \in V$ can be expressed in a unique way as linear combination of $S = \{v_1, \dots, v_n\}$, Then in particular $\text{span}(S) = V$ if we would have $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ for some $c_1, \dots, c_n \in F$, Then by uniqueness, since we also have $0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n = 0$ it follows that $c_1 = 0, c_2 = 0, \dots, c_n = 0$ thus $\{v_1, \dots, v_n\}$ linearly independent so S is a basis.

8.6 Theorem

Let S be a finite subset of the vector space V . if $\text{span}(S) = V$ (i.e. if S generate V) then there exist a subset $S' \in S$ such that S' is a basis for V

Proof:

If $S = \{0\}$ then $\text{span}(S) = \{0\}$ so $V = \{0\}$ and S is a basis for V

If S contains at least one non-zero element, say $u_1 \neq 0$, then $\{u_1\}$ is linearly independent we then continue to choose u_2, \dots, u_k in S so that u_1, \dots, u_k are linearly independent, write this is no longer possible (Note that this must be the case, because S is finite)

This happens if either we have exhausted all S , i.e. if

(a) $\{u_1, \dots, u_k\}$, or if

(b) any $u \in S$ that's not among u_1, \dots, u_k is so that u_1, \dots, u_k, u is linearly dependent

in case we have (a), it means $S = \{u_1, \dots, u_k\}$ is linear independent and since we also have $\text{span}(S) = V$, it follows that S itself is a basis for V and we are done.

In case we have (b), it means there exist scalars c_1, c_2, \dots, c_k, c , not all equal to 0, such that $c_1 u_1 + c_2 u_2 + \dots + c_k u_k + c u = 0$ if $c = 0$, then it would follow that $c_1 u_1 + \dots + c_k u_k = 0$ with c_1, c_2, \dots, c_k not all equal to 0, contradicting the fact that $\{u_1, \dots, u_k\}$ is linear independent thus, $c \neq 0$

and then from $c_1u_1 + c_2u_2 + \dots + c_ku_k + cu = 0$ we deduce $u = -\frac{c_1}{c}u_1 - \frac{c_2}{c}u_2 - \dots - \frac{c_k}{c}u_k$ showing that $u \in \text{span}(\{u_1, \dots, u_k\})$. Thus, in case (b), we showed that the set $S' = \{u_1, \dots, u_k\} \in S$ is linear independent and any $u \in S - u_1, \dots, u_k$ is in $\text{span}(S')$ so $\text{span}(S')$ contains u_1, \dots, u_k and call $S - \{u_1, \dots, u_k\}$, so $S \in \text{span}(S')$. Since $\text{span}(S) = V$, by (Theom 6.6 or Theom 7.8), it follows that $\text{span}(S')$ contains all $\text{span}(S)$ thus $\text{span}(S') = V$ (because $\text{span}(S) = V$) so S' is linear independent & $\text{span}(S') = V$ so $S' \in S$ is a basis for V .

8.7 Corollary

If V contains a finite subset $S \in V$ that generates V , i.e. $\text{span}(S) = V$, then V has a finite basis

8.8 Example

Here is a concrete example showing how the method of finalizing a basis S' as a subset of a generating set $S \in V$ works:

Let $S = \{(1, -1), (-1, 1), (0, 2), (3, 0)\} \in \mathbb{R}^2$. Show that there exists $S' \in S$ such that S' is a basis for \mathbb{R}^2

Solution

Since S contains non-zero vectors we can state by choosing $n_1 = (1, -1) \in S$. Then we look at the 2'nd vector $(-1, 1)$ in S . We see that $(-1, 1) = -1 * (1, -1)$, i.e. $(-1, 1) = -n_1$ so $(-1, 1)$ is not linear independent of u_1 . We then take the 3'rd vector in S , $(0, 2)$. If $a(1, -1) + b(0, 2) = 0$. Then $(a, -a + 2b) = (0, 0)$ $a = 0$ and $-0 + 2b = 0$ so $b = 0$ as well. Thus $u_2 = (0, 2) \in S$ is linear independent of u_1 . So we can take add u_2 to our linear independent subset S' of S . So by now we have $u_1 = (1, -1), u_2 = (0, 2) \in S'$. We see that in fact $\text{span}\{u_1, u_2\}$ and we can stop and conclude that $S' = \{u_1, u_2\} = \{(1, -1), (0, 2)\} \in S$ is a basis for \mathbb{R}^2

8.9 Theorem (the so-called replacement theorem)

Let V be a vector space. Assume $G \in V$ is a subset with n vectors that generates V , i.e. $\text{span}(G) = V$ if $L \in V$ is a linearly independent subset of V with m vectors, then $m \leq n$ and there exist a subset $H \in G$ containing $n - m$ vectors such that $L \cup H$ generate V

Proof

We prove this by induction over m (i.e. over the $\#$ of elements in then linear independent set L)

if $m = 0$, this means L has 0 many elements, i.e. $L \neq 0$ and we can just take $H = G$, which satisfies the required conditions.

Suppose now that the statement holds true for some $m \geq 0$. We then want to show that the statement holds true for $m + 1$ as well.

So let $L = \{v_1, \dots, v_{m+1}\}$ be linear independent subset of V . By Theom 7.7 any subset of L is linear independent, so $\{v_1, \dots, v_m\}$ is linear independent by induction, since we have that the statement of thn is true for m , it follows that $m \leq n$ and that there exist $\{u_1, \dots, u_{n-m}\} \in G$ such that $\{v_1, \dots, v_m\} \cup \{u_1, \dots, u_{n-m}\}$ spaces V . So in particular, v_{m+n} can be expressed as a linear combination

$v_{m+1} = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_{n-m}u_{n-m}$ Now notice that in fact we must have $n - m \geq 1$, or else we would have $v_{m+n} = a_1v_1 + \dots + a_mv_m + 0$ which contradicts the fact that $L = \{v_1, \dots, v_{m+1}\}$ is linear independent. In other words, we must have $n \geq m + 1$. Also, In $v_{m+1} = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_{n-m}u_{n-m}$ we must have that some b_i are non-zero. say $b_1 \neq 0$, which allows us to solve in $v_{m+1} = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_{n-m}u_{n-m}$ for $u_1: u_1 = (-\frac{a_1}{b_1})v_1 + (-\frac{a_2}{b_1})v_2 + \dots + (-\frac{a_m}{b_1})v_m + \frac{1}{b_1}v_{m+1} + (-\frac{b_2}{b_1})u_2 + \dots + (-\frac{b_{n-m}}{b_1})u_{n-m}$. Thus, if we take $H = \{u_2, \dots, u_{n-m}\}$ then $u_1 \in \text{span}(L \cup H)$ by $u_1 = (-\frac{a_1}{b_1})v_1 + (-\frac{a_2}{b_1})v_2 + \dots + (-\frac{a_m}{b_1})v_m + \frac{1}{b_1}v_{m+1} +$

$(-\frac{b_2}{b_1})u_2 + \dots + (-\frac{b_{n-m}}{b_n})u_{n-m}$. and since $v_1, \dots, v_m, u_2, \dots, u_{n-m}$ are obviously in $\text{span}(L \cup H)$, we actually have that $\{v_1, \dots, v_m, u_1, \dots, u_{n-m}\} \in \text{span}(L \cup H)$. Since $\{v_1, \dots, v_m, u_1, u_2, \dots, u_{n-m}\}$ generates V , it follows that $\text{span}(L \cup H) = V$ with $H \in G$ being a subset that contains $(n - m) - 1 = n - (m + 1)$ elements, showing that statement of this holds true for $m + 1$