Math115 1/23 notes

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6.1 Definition

If S_1, S_2 are nonempty subsets of a vector space V then the sum of S_1 and S_2, denoted $S_1 + S_2$ is the set $\{x + y : x \in S_1, y \in S_2\}$

6.2 Definition

Let W_1, W_2 be subspaces of the vector space V. We say that V is the direct sum of W_1 and W_2 if $W_1 \cap W_2 =$\{0\} and $W_1 + W_2 = V$, and we then write $V = W_1 + W_2$

6.3 Exercise

Show that $V = \mathbb{R}^2$ is the direct sum of $W_1 =$\{$(x, x) : x \in \mathbb{R}$\} and $W_2 =$\{$(y, -y) : y] \in \mathbb{R}$\}
Solution
First note that W_1, W_2 are indeed vector subspace of $\mathbb{R}^2$ indeed, if we take two elements $(x, x) \in W_1, (Z, Z) \in W_2$, then $(x, x) + (z, z) = (x + z, x + z) \in W_1$. Also, if $c \in \mathbb{R}$ is a scalar, then $c(x, x) = (cx, cx) \in W_1$
Similarly for W_2
Also, we see that $0 = (0, 0) \in W_1$ and $0 = (0, -0) \in W_2$. So W_1, W_2 satisfy the conditions in Theom 5.2 so they are subspace of $V = \mathbb{R}^2$
We want show that:
(a) $W_1 \cap W_2 =$\{0\}
(b) $W_1 + W_2 = V$
To prove (a) assume $v = (v_1, v_2 \in R^2)$ is both in W_1 and in W_2. Since $(V_1, V_2) \in W_1$, we must have $V_2 = -V_1$. Thus $V_2 = V_1, V_2 = -V_1$ so $V_1 = -V_1$, Thus $2V_1 = 0$ which for $V_1 \in \mathbb{R}$ implies $V_1 = 0$. So $V_2 = V_1 = 0$ as well.
To prove (b), let $v = (v_1, v_2) \in R^2$ we want to find $(x, x) \in W_1, (y, -y) \in W_2$ such that $(x, x) + (y, -y) = (V_1, V_2)$. This means $(x + y, x - y) = (V_1, V_2)$

So $x + y = V_1, x - y = V_2$
Thus, to find x,y satisfying these two conditions we need to solve this system of two equation with two unknown x,y in real numbers.
From 2nd equation, we get $x = y + V_2$ and replacing in the 1st equation
$(y + V_2) + y = V_1$
So $2y = V_1 - V_2, y = \frac{V_1 - V_2}{2}$
and so $x = y + V_2 = \frac{V_1 - V_2}{2} + V_2 = \frac{V_1 + V_2}{2}$
Thus $(V_1, V_2) = (\frac{V_1 + V_2}{2}, \frac{V_1 + V_2}{2}) + (\frac{V_1 - V_2}{2}, \frac{V_2 - V_1}{2})$ and so we denoted that $W_1 + W_2 = V$

Linear Combination of Vectors

6.4 Definition

Let V be a vector space over a field F and $S \in V$ a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors in S. If there exist a finite number of vectors $u_1, ..., u_n \in S$ and scalars $c_1, ..., c_n \in F$ such that $v = c_1u_1 + c_2u_2... + c_nu_n$

We then also say that V is a linear combination of $u_1, ...u_n$. The scalars $c_1, ...c_n$ are called the coefficients of the linear combination. because $O_v = O_F * V, \forall V \in S \neq 0$
Note: The vector $0 \in V$ is a linear combination of any $S \in V$

6.5 Example

Denote by V the set of polynomial of degree at most n with coefficients in R, i.e. expressions of the form
$P(X) = a_0 + a_1X + a_2X^2 + ... + a_nX^n$
where $a_0, a_1, ..., a_n \in \mathbb{R}$, with the usual addition and multiplication by scalars in $\mathbb{R}$:
$(a_0, a_1X + ... + a_nX^n) + (b_0 + b_1X + ... + b_nX^n) = (a_0 + b_0) + (a_1 + b_1)X + ... + (a_n + b_n)X^n$ and $c(a_0 + a_1X + ... + a_nX^n) = ca_0 + ca_1X + ... + ca_nX^n$
Show that any polynomial in V is a linear combination of the "monomials" $1, X, X^2, ..., X^n$ indeed, if $P(X) = a_0 + a_1X + ... + a_nX_n \in V$ then $a_0, a_1, ..., a_n \in \mathbb{R}$ are scalars and we have $P(X) = a_0 * 1 + a_1 * X + ... + a_nX^n = a_0u_o + a_1u_1 + ... + a_nu_n$

6.6 Definition

If V is a vector space and $S \neq 0 \in V$, Then the span of S, denoted *span*(S), is the set of all linear combinations of vectors in S.
i.e. *span*(S) =$\{\Sigma_{i=1}^{n} c_iu_i : u_1, ...u_n \in S, c_1, ...c_n \in F, u \geq 1\}$

6.7 Example

If we take $V = \mathbb{R}^3$ and S=$\{(1, 0, 0), (0, 1, 0)\}$ Then *span*(S) is the set of all vectors in $\mathbb{R}^3$ of the form $au_1 + bu_2 = a(1, 0, 0) + b(0, 1, 0) = (a, 0, 0) + (0, b, 0) = (a, b, 0)$, with $a, b \in \mathbb{R}$ arbitrary scalars in $\mathbb{R}$
Thus *span*(S) =$\{(a, b, 0) : a, b \in \mathbb{R}\}$ which we recognize to be the xy plane in the xyz 4-dimensional Euclidean space.

6.8 Example

If we take V to be the vector space of polynomials in X of degree $\leq n$ with coefficients in $\mathbb{R}$ as in Example 6.5 and we let S =$\{1, X, X^2, ..., X^n\}$ then *span*$(S) = V$

6.9 Example

Given a Field F and denotes by F[X] the set of all polynomials in "undeterminate" X over the field F, i.e. expressions of the form $P(X) = a_0 + a_1 X + ... + a_n X^n$ for some $n \geq 0$ and $a_0, a_1, ...a_n \in F$ with the "usual" addition and scalar multiplication
The degree of $P(X)$ is the largest n such that $a_n \neq 0$.
(a) show that if $S = \{1, X, X^2, ...\}$ then span(S)=F[X].
(b) Denote $F_{odd}[X]$ the set of all polynomials with coefficients in F that have only odd coefficients possibly $\neq 0$ and by $F_{even}[X]$ the set of all polynomial with coefficient in F that have only even coefficients possibly $\neq 0$, i.e.$F_{odd}[X] = \{P(X) \in F[X] : P(X) = a_1 X + a_3 X^3 + a_5 X^5 + ... + a_{2n+1} X^{2n+1}, a_1, a_3, ..., a_{2n+1} \in F, n \geq 0\}$

$F_{even}[X] = \{P(X) \in F[X] : P(X) = a_0 + a_2 X^2 + a_4 X^4 + ... + a_{2n} X^{2n}, a_0, a_2, a_4, ..., a_{2n} \in F, n \geq 0\}$
Show that $W_1 = F_{odd}[X], W = F_{even}[X]$ are subspaces of $F[X]$ and that $F[X] = W_1 + W_2$
Proof:(exercise)

(a) As long as set S contains all polynomials of X which degree $\leq n$, and F[X] is the set of all polynomials in "undeterminate" X over the field F. F[X] can be written as a linear combination of S, where $F[X] = aS$. where $a = \{a_1, a_2....a_n\} \in F$

(b) For $F_{odd}[X]$, assume $A(X)$ and $B(X)$ are subspace of $F_{odd}[X]$ where $A(X) = \{ca_1 X + ca_3 X^3 + ca_5 X^5 + ...\}$ and $B(X) = da_1 X + da_3 X^3 + da_5 X^5 + ...$ and $A(X) + B(X) = (c+d)a_1 X + (c+d)a_3 X^3 + (c+d)a_5 X^5 + ... \in F_{odd}[X]$. Also $rA(X) = rca_1 X + rca_3 X^3 + rca_5 X^5 + ... \in F_{odd}[X]$. This denote that W_1 is a subspace of $F[X]$
Similar with W_2
To show that a set V is equal to the sum of two subspaces W1 and W2, we must show that every element of V can be written as the sum of an element of W1 and an element of W2. Let P(X) be any polynomial in F[X]. We can write P(X) as $P(X) = P_o(X) + P_e(X)$ where $P_o(X) = a1X + a3X3 + ... + a2n+1X2n+1$ and $P_e(X) = a0 + a2X2 + ... + a2nX2n$. P_o(X) has only odd coefficients, and P_e(X) has only even coefficients. Thus, $P_o(X)$ is in $W1 = F_{odd}[X]$, and P_e(X) is in $W2 = F_{even}[X]$. Therefore, $P(X) = P_o(X) + P_e(X)$ can be written as the sum of an element of W1 and an element of W2, and thus every element of $F[X]$ can be written as such a sum. Therefore, $F[X] = W1 + W2$

6.10 Theorem

The span of any subset S of a vector space V is a subspace of V. Any subspace of V that contains S must contain $span(S)$
(I.e. if $W \in V$ subspace with $S \in W$ then $span(S) \in W$)
Proof:
We have to prove that if $x, y \in span(S)$ then $x + y \in span(S)$ and $cx \in span(S), \forall c \in F$
Since $x, y \in span(S)$, there exist $u_1, ..., u_n \in S, v_1, ..., v_n \in S$ and scalars $a_1, ...a_n \in F, b_1, ..., b_n \in F$ such that
$x = a_1 u_1 + ... + a_m u_m$
$y = b_1 v_1 + ...b_n v_n$
But then $x + y = a_1 u_1 + ... + a_m u_m + b_1 v_1 + ... + b_n v_n$
So $x + y$ is itself a linear combination of $u_1, ...u_m, v_1, ...v_n \in S$, Thus $x + y \in span(S)$.
Also, $cx = c(a_1 u_1 + ...a_m u_m) = (ca_1)u_1 + (ca_2)u_2 + ... + (ca_m)n_m \in span(S)$

For the last part of Theom: if$W \in V$ is a subspace that contains S and $W \in span(S)$, then there exist $u_1, ..., u_m \in S$ and $a_1, ..., a_m \in F$ such that $W = a_1 u_1 + ... + a_m u_m$. Since W is a subspace and

$u_1 + ... + u_m \in S \in W$, we have $a_1u_1 + ... + a_mu_m \in W$. Thus $w \in W$ showing that $span(S) \in W$.

6.11 Definition

We say that a subset S of a vector space V generates (or spans) V if $span(S) = V$.

6.12 Example

1). If we take $V = \mathbb{R}^2$ like in Exercise 6.3 then S ={(1,1),(1,-1)} generate (span) V, because we showed in that exercise that any $v \in V$ is of the form $v = au_1 + bu_2$ for some scalars $a, b \in \mathbb{R}^2$ 2). If we take $V = F[X]$ and S =$\{1, X, X^2, ...\}$ then $span(S) = F[X]$

6.13 Exercise

Show that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
generate $M_{2\times 2}(F)$
Solution. Any matrix in $M_{2\times 2}(F)$ is of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in F$ But then $a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ So
A is indeed a linear combination of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$