Assign3

Haojie Liu

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Section 1.3: 10,11,13,17,20,22,23,25,30

Section 1.4: 12,13,14

Section 1.5: 6,10,19

1.3

10. Prove that W_1 $=\{(a_1, a_2, ..., a_n) \in F^n : a_1 + a_2 + ... + a_n = 0\}$ is a subspace of F^n, but W_2 $=\{(a_1, a_2, ..., a_n) \in F^n : a_1 + a_2 + ... + a_n = 1\}$ is not.

Proof:

First of all, we want to check that if $0 \in W_1$ and W_2.
in W_1, we have $0 + 0 + ... + 0 = 0$, $0 \in W_1$
in W_2, we have $0 + 0 + ... + 0 = 1$ where $0 \notin W_2$ Therefore, W_2 is not a subspace of F^n
Also, in W_1, $c(u + v) = 0, \forall u, v$ where $(u + v) \in W_1$ and $cu \in W_1$. Therefore, W_1 is a subset of F^n

11. Is the set W $=\{f(x) \in P(F) : f(x) = 0$ or $f(x)$ has degree n$\}$ a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

Proof:

For any $f(x)$ has a degree ≥ 1, we all have $f(x) = 0$ where $0 \in P(F)$
To test the closure under addition: we have $f(x) = x, g(x) = -x + 1$ where $f(x), g(x) \in W$, However, $f(x) + g(x) = x - x - 1 = -1$ where we have that $f(x) + g(x) \neq 0$ and it does not has a degree of n. Therefore, W is not a subspace of $P(F)$ if $n \geq 1$

13. Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$,$\{f \in F(S, F) : f(s_0) = 0\}$, is a subspace of F(S,F)

Proof:

We know that $f(s_0) = 0, \forall s_0 \in S$, means that $0 \in\{f \in F(S, F) : f(s_0) = 0\}$
To test the closure under addition: we assume $f(s_0), g(s_0) \in\{f \in F(S, F) : f(s_0) = 0\}$, we have that

$f(s_0) + g(s_0) = 0 + 0 = 0$ where $f(s_0) + g(s_0) = 0 \in\{f \in F(S,F) : f(s_0) = 0\}$
To test the closure under multiplication: $f(s_0) \in\{f \in F(S,F) : f(s_0) = 0\}$ and $c \in F$. we have that $cf(s_0) = c(0) = 0 \in \{f \in F(S,F) : f(s_0) = 0\}$
Therefore, $\{f \in F(S,F) : f(s_0) = 0\}$ is a subspace of F(S,F)

17. Prove that a subset W of a vector space V is a subspace of V is and only if $W \neq \emptyset$ and, whenever $a \in F$ and $x, y \in W$ then $ax \in W$ and $x + y \in W$

Proof:

We already know that W is closure under addition and multiplication. In order to become a subspace of V, we need to have $0 \in W$. If $W = \emptyset$, then $0 \notin W$ and it is not consider as a subspace of V. by contradiction. If $W \neq \emptyset$, and $a \in F$ we can have that $0x = 0 \in W$ which can satisify all three conditions. Therefore, W will consider as a subspace of V, if and only if $W \neq \emptyset$ and, whenever $a \in F$ and $x, y \in W$ then $ax \in W$ and $x + y \in W$.

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, ..., w_n$ are in W, then $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$ for any scalar $a_1, a_2, ..., a_n$.

Proof:

As long as W is a subspace of V, we can summarize that$0 \in W$, for any element $w \in W, w_1 + w_2 \in W$ and $aw \in W, a \in F$. Therefore, $\forall a_1, a_2, ...a_n \in F$, and $w_1, w_2, ...w_n \in W$. We can conclude that $a_1w_1 + a_2w_2 + ...a_nw_n \in W$.

22. Let F_1 and F_2 be fields. A function $g \in F(F_1, F_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in F_1$ and is called an odd function if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even function in $F(F_1, F_2)$ and the set of all odd functions in $F(F_1, F_2)$ are subspace of $F(F_1, F_2)$

Proof:
When F_1 and F_2 are fields, $0 \in F_1, F_2$. For F_1:
closure under addition: $g(-t) + f(-t) = g(t) + f(t) = (g + f)(t) \in$ even function
closure under multiplication: $cg(-t) = cg(t) \in$ even function
For F_2:
closure under addition: $g(-t) + f(-t) = -g(t) - f(t) = (-g - f)(t) = -(g + f)(t) \in$ odd function
closure under multiplication: $cg(-t) = -cg(t) = (-c)g(t) \in$ odd function
Therefore, both the even and odd function are subspace for $F(F_1, F_2)$

23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2
(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$

Proof:

(a) $0 \in W_1, W_2$, we have $0 = 0 + 0 \in W_1 + W_2$
closure under addition: $u_1 + u_2 \in W_1$ and $u_1, u_2 \in W_1$, $v_1 + v_2 \in W_2$ and $v_1, v_2 \in W_2$ where

$(u_1 + u_2) + (v_1 + v_2) \in W_1 + W_2$
closure under multiplication: $cu \in W_1, \forall c \in F, u \in W_1$, and $cv \in W_2, \forall c \in F, v \in W_2$ where $c(u)+c(v) = c(u+v) \in W_1 + W_2$
$W_1 + 0 = W_1 \in W_1 + W_2$ and $0 + W_2 = W_2 \in W_1 + W_2$ Therefore $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2

(b) We know that $W_1, W_2 \in V$ and $W_1, W_2 \in W_1 + W_2$ Therefore, we will have that $W_1 + W_2 \in V$, any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$

25. Let W_1 denote the set of all polynomials $f(x)$ in $P(F)$ such that in the representation $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... a_1 x + a_0$, we have $a_i = 0$ whenever i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $P(F)$ such that in the representation $g(x) = b_n x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0$, we have $b_i = 0$ whenever i is odd. Prove that $P(F) = W_1 \bigoplus W_2$.

Proof:

for $f(x)$ we have that $a_i = 0$ whenever i is even. which means that $f(x)$ is a function with only odd coefficient and odd degree polynomial.
for $g(x)$ we have that $b_i = 0$ whenever i is odd. which means that $g(x)$ is a function with only even coefficient and even degree polynomial.
i.e. we have $f(x) = \Sigma_1^n a_{2n-1}x^{2n-1}$ and $g(x) = \Sigma_0^n a_{2n}x^{2n}$
$f(x) \bigoplus g(x) = \Sigma_1^n a_{2n-1}x^{2n-1} + \Sigma_0^n b_{2n}x^{2n} = b_0 + \Sigma_1^n a_{2n-1}x^{2n-1} + b_{2n}x^{2n} = P(F)$
We do have all degree of terms polynomials in P(F) generate by direct sum of f(x) and g(x)

30. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$

Proof:

We can assume that there is a different way to represent $v \in V$, in other word v is not unique. Then we can have that $v = x_1 + x_2 = y_1 + y_2, y_1 \in W_1, y_2 \in W_2$ where $y_1 + y_2$ is another representation of v. Base on the equation we can have that $x_1 - y_1 = x_2 - y_2$. To have this condition, the only vector belong to both side of the equation is $0 \in W_1, W_2$. Which means that each vector in V can be uniquely written as $x_1 + x_2$.

1.4

12. Show that a subset W of a vector space V is a subspace of V if and only if $span(W) = W$

Proof:

=> When W is a subspace of V, we will have $0 \in W$ and it is closure under addition and multiplication. As long as W is a subspace of V, the all of the linear combination have to be inside of the subspace W. Where $span(W) = W$
<= If W contains all of its linear combination, where $span(W) = W$, and we can have that $0u = 0, 0 \in F, u \in W$ where W contains the zero vector. Therefore, subset W of a vector space V is a subspace of V if and only if $span(W) = W$

13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $span(S_1) \subseteq span(S_2)$. In particular, if $S_1 \subseteq S_2$ and $span(S_1) = V$ deduce that $span(S_2) = V$.

Proof:

$span(S_1)$ represent all linear combination of S_1 in vector space V. if $S_1 \subseteq S_2$,in other word element belong in S_1 also belong in S_2, we can say that $S_2 \subseteq S_1, span(S_1) = span(S_2)$ where $span(S_2)$ represents all of the linear combination of S_2 in vector space V.
If $span(S_1) = V$, where subset S_1 generates V, and $S_1 \subseteq S_2$. we can alternatively say that S_2 also generates V, where $span(S_2) = V$.

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space v, then $span(S_1 \cup S_2) = span(S_1) + span(S_2)$.

Proof:

$S_1, S_2 \subset v$, and $S_1 \neq S_2$. $span(S_1)$ represents all of the linear combination of S_1 where$span(S_1) \in v$ and similarly for $S_2 \in v$.
LHS: $span(S_2 \cup S_2) = span(S_1) + span(S_2)$, lets assume that vector $x \in span(S_2 \cup S_2)$ or we can say that $x \in span(S_1) \cup span(S_2)$ which means that x can be produced in either $span(S_1)$ or $span(S_2)$. In other word, $x \in span(S_1) + span(S_2)$
RHS: $span(S_2 \cup S_2) = span(S_1) + span(S_2)$, lets assume that vector $x \in span(x_1) + span(x_2)$ which tells us that x can be generate from the sum of two vectors. In other word, $x \in span(x_1 \cup x_2)$. Therefore, $span(S_2 \cup S_2) = span(S_1) + span(S_2)$.

1.5

6. In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Proof:

With a matrix $A_{n\times m} = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ | & & & | \\ 1 & 1 & ... & 1 \end{pmatrix}$ the only way to make $A = 0$ is to using scalar $cA = 0, c = 0$.
Therefore, the matrix E^{ij} is linear independent.

10. Given a example of three linearly dependent vector in R^3 such that non of the three is a multiple of another.

Answer:

$\{(1, 0, 1), (0, 0, -1), (1, 0, 0)\}$

19. Prove that if $\{A_1, A_2, ...A_k\}$ is a linearly independent subset of $M_{m\times n}(F)$ then $\{A_1^t, A_2^t, ...A_k^t\}$ is also linearly independent.

Proof:

We say that $\{A_1, A_2, ...A_k\}$ is a linearly independent when the only way to make $cA = 0$ is to have all of the scalar $c = 0$. for any A^t we can write it in form of $A^{t-1} * A$, in other word, if we want $cA^t = c(A^{t-1})A = (cA^{t-1})A = 0$, we can either have $c = 0, A = 0$ or $A^{t-1} = 0$, As we assume that A is not a zero vector, then the only solution is that $c = 0$
Therefore, $\{A_1^t, A_2^t, ...A_k^t\}$ is also linearly independent.