Math115A 2/03 notes

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Recall that over the last two lectures L9&L10 we defined the notice of basis and dimension of a vector space:

Given a vector space V, a basis for V is a subset $S \subset V$ with the properties that S is linearly independent and generates V

We proved that if V is finitely generated (i.e., if $\exists S \subset V$ finite such that $span(S) = V$) then V has a finite basis and that all basis of V have the same number of elements. We called this common number the dimension of V denote $dim(V)$

We now enumerate several more consequences of the replacement theorem

11.1 Corollary

Let V be a finite dimensional vector space with $dim(V) = m$
1. if $S \subset V$ is a set with m element that generate V, then $m \geq n$
2. if $S \subset V$ has n elements and $span(S) = V$, then S is a basis from V
3. if $S \subset V$ has m elements and S is linearly independent and has n elements then S is a basis
4. Any linearly independent subset S of V can be extended to a basis of V, i.e. $\exists S' \subset V$ subset that contains S such that S' is a basis for V.

Proof: We already discussed these consequences of the replacement theom on Monday L9.

11.2 Theorem

Let V be a finite dimensional vector space and $W \subset V$ a subspace. Then W is finite dimensional and $dim(W) < dim(V)$. Moreover, if $dim(W) = dim(V)$ then $W = V$
Proof: Denote $dim(V) = n$ and let $\{w_1, ...w_k\}\subset W$ be a linearly independent subset of W. By 11.1, this set can be extended to a basis of V, thus $k \leq n$. If $span(\{w_1, ...w_k\}) \neq W$, then set $w_{k+1} \in W - span(\{w_1, ...w_k\})$. We claim that $\{w_1, ...w_k, w_k + 1\}$ follows linearly independent indeed, because if $\Sigma_{i=1}^{k+1} c_1 w_1 = 0$ then for sure $c_{k+1} = 0$, or else we get $w_{k+1} = \Sigma_{j=1}^{k}(-\frac{c_j}{c_{k+1}})w_j \in span\{w_1, ..., w_k\}$, contradiction. Thus $\Sigma_{i=1}^{k+1} c_1 w_1 = 0$ implies $\Sigma_{i=1}^{k} c_1 w_1 = 0$, whihc by linear independent of $\{w_1, ..., w_k\}$ implies $c_1 = c_2 = ... = c_k = 0$.
So indeed $\{w_1, ..., w_{k+1}\}$ follows linear independent.

Thus, since any linear independent subset of W has $\leq n$ many elements with n finite, we can continues the process of choosing more and more linear independent vectors $w_1, ..., w_k$ in W. But since $k \leq n$ and n is finite this process must end after finitely many steps, say m steps, assume we got a linear independent $\{w_1, ..., w_m\}$ to span W, and from the above we have $m \leq n$. Since span$\{w_1, ...w_m\}$= W and $\{w_1, ...w_m\}$ linear independent, $\{w_1, ...w_m\}$ is a basis for W. Thus $dim(W) = m \leq n = dim(V)$

if in addition $dim(W) = dim(V)$ then by 11.1.3 we must have that $\{w_1, ...w_m\}$ is also a basis of V thus $W = V$

11.3 Corollary

If V is finite dim vector space and $W \subset V$ subspace then any basis for W can be extended to a basis for V
Proof: By Theom 11.2 we know that $m = dim(W) \leq n$. Let $\{w_1, ...w_m\} \subset W$ be a basis for W in particular they are linear independent and by 11.1.4 we can extend this set V to a basis of V.

Linear Transformations Between Vector Spaces (2.1)

11.4 definition

Let V, W be vector spaces over the field F. A function $T : V \to W$ satisfying to properties
a) $T(x + y) = T(x) + T(y), \forall x, y \in V$
b) $T(cx) - cT(x), \forall x \in V, c \in F$
is called a linear transformation from V to W

We often just use the term linear, as an adjective/noun to call such function $T : V \to W$

11.5 Remarks

if $T : V \to W$ is linear, as above, then $T(0_V) = 0_W$. Indeed, because by (b) above we have $T(0 * 0_v) = 0T(0_V)$
$2^o : T(x - y) = T(x) - T(y), \forall x, y \in V$
indeed $T(x - y) = T(x + (-y)) = T(x) + T(-y) = T(x) + T((-1)y) = T(x) + (-1)T(y) = T(x) - T(y)$ 3^o : $T(\Sigma_{i=1}^{n} c_1 x_i) = \Sigma_{i=1}^{n} c_i T(x_i), \forall x_1, ...x_n \in V, c_1, ...c_n \in F$
Indeed, This is clear by applying (a), (b) repeatly

11.6 Example

Define $T : R^2 \to R^2$ by $T(a, b) = (a, -b)$. Then T is linear, because on the one hand $T((a_1, b_1) + (a_2, b_2)) = T((a_1 + a_2, b_1 + b_2)) = (a_1 + a_2, -b_1 - b_2)$
on the other hand
$T(a_1, b_1) = (a_1, -b_1)$
$T(a_2, b_2) = (a_2, -b_2)$
So $T(a_1, b_1) + T(a_2, b_2) = (a_1, -b_1) + (a_2, -b_2) = (a_1 + a_2, -b_1 - b_2)$
Similarly, $T(c(a, b)) = T(ca, cb) = (ca, -cb) = c(a, -b) = cT(a, b)$

11.7 Example

$T : R^2 \to R^2$ defined by $T(a, b) = (a, 0)$ is linear

Indeed, trivial to check:

The transformation $T : R^2 \to R^2$, $T(a, b) = (a, -b)$ in 12.6 is called the reflection of R^2 almost the x-axis and $T : R^2 \to R^2, T(a, b) = (a, 0)$. 12.7 is called the projection of R^2 on the x-axis

11.8 Example

Let V denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that have derivative of any order. This is obviously a vector space over field $\mathbb{R}$ when endowed with usual addition of functions and multiplication of functions by scalars: if $f, g \in V$ then $(f + g)(t) = f(t) + g(t), t \in \mathbb{R}, (cf)(t) = cf(t)$
Define, $T : V \to V$ by $T(f) = f'$ (the derivative of f)
Then T is a linear transformation because of the linear properties of the derivation of functions:
$(f + g)' = f' + g', (cf)' = cf'.\forall f, g \in V, c \in \mathbb{R}$
Obs one effect denotes this vector space by $V = C^\infty(\mathbb{R})$

11.9 Example

Let $V = C(\mathbb{R})$ the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Then $C(\mathbb{R})$ with the usual addition of functions and multiplication by scalars is a vector space. Let $a, b \in \mathbb{R}$, $a < b$. Then the function $T : C(\mathbb{R}) \to \mathbb{R}$ defined by $T(f) = \int_a^b f(t)dt$ is linear because of the known properties of the integral $\int_a^b (f+g)(t)dt = \int_a^b f(t)dt+\int_a^b g(t)dt$, $\int_a^b (cf)(t)dt = c\int_a^b f(t)dt$

11.10 Definition

Let V, W be vector spaces and $T : V \to W$ linear.
The null space of T (or kernel of T) is the set $N(T) :=\{x \in V : T(x) = 0\}$
The range of T (or image of T) is the set $R(T) :=\{T(x) : x \in V\}$

11.11 Examples

The projection of $\mathbb{R}^2$ denote the x axis theom 12.7 defined by $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(a, b) = (a, 0), \forall(a, b) \in \mathbb{R}^2$
$N(T) =\{(0, b) : b \in \mathbb{R}\}$ i.e. the y-axis
$R(T) =\{a, 0 : a \in \mathbb{R}\}$ i.e. the x axis

11.12 Theorem

Let V, W be vector space and $T : V \to W$ be linear. Then $N(T), R(T)$ are subspaces of V and respectively W
Proof: Since $T(0_V) = 0_W$ we have $0_V \in N(T)$. if $v_1, v_2 \in N(T)$, i.e. $T(v_1) = 0_W$,$T(v_2) = 0_W$ Then $T(v_1 + v_2) = T(v_1) + T(v_2) = 0_w + 0_w = 0_w$, so $v_1 + v_2 \in N(T)$ if $v \in N(T)$ then $T(v) = 0_W$ so if $c \in \mathbb{R}$ scalar then $T(cv) = cT(v) = 0_w$ so $cv \in N(T)$. Similarly for R(T)