Math115A 1/12 notes

Vincent

2023-01-13

3.1 Definition
A field F is a set on which one has two operations +, $*$, called addition and multiplication, are defined so that for each x,y $\in$ F corresponds a unique element in F denoted x+y and a unique element denoted $x * y$ such that the following properties are satisfied for all elements a, b, c $\in$F:
(F1) $a + b = b + a$, $a * b = b * a$

(F2) $(a + b) + c = a + (b + c); (a * b) * c = a * (b * c)$

(F3) There exist distinct elements 0 and 1 in F such that 0+a=a and 1$*$a = a, $\forall$a $\in$F

(F4) For each a$\in$F and each b$\in$F, b!=0 there exist elements c$\in$F, d$\in$F such that a+c=0, b*d=1

(F5) $a * (b + c) = a * b + a * c$ distributivity of multiplication.

The element x+y called the sum of x&y x$*$y called the product of x&y
The element 0 is called the identity element for addition
The element 1 called the identity element for multiplication
The element c in (F4) with property a + c = 0 called the addition inverse of a.
The element d in (F4) with property a$*$d = 1 called the multiplication inverse of c

Examples:
1. The set R of all real numbers with the usual +, $*$ is a field.
2. The set Q of rational numbers with usual +, $*$ is a field
indeed, because the sum, product and inverses of rational numbes are rational numbers.
3. The set Z of integers with the usual +, $*$ operations is not a field: properties (F1), (F2), (F3), (F5) are satisfied and also exsitence of additive inverse in (F4) but not the existence of multiplicative inverse: for instance z$\in Z$ such that 2$*$d = 1
4. DEnote by Z_2 the set with two elements 0 and 1 on which we define the operations + and $*$ as follows:
0+0=0,0+1=1,1+0=1,1+1=0,0\$$0 = 0, 01 = 0, 10 = 0, 11 = 1$
$Thenoneclearlyhas(F1) - (F5)satisfied!So($Z_2\$, +, $*$) is a field. It is called the field with two elements.
Note: the additive inverse of 1 is 1 itself because 1+1=0.
One can show that Z_2 is the unique field with two elements.

3.4 Theorem(cancellation law in a field)
let(F,+,$*$) be a field. For any a,b,c $\in$ F we have:
(1) if a+b =c+b Then a=c
(2) if ab=cb and b!=0, then a=c.

Proof:
(1). By (F4) There exists d $\in$ F such that b+d=0. Since a+b =c+b, we can add to both sides the element to obtain:
(a+b)+d = (c+b)+d
So by (F2) a+(b+d) = c+(b+d) so a+0 = c+0 Thus a=c.
(2) has similar proof

3.5 Theorem:
The element 0 and 1 in a field are unique. Also the additive inverse of an element and the multiplicative inverse of a != 0 element are unique
proof:
if 0'$\in$F is another element with the property that 0'+a=a,$\forall$a$\in$F, then we have 0'+0=0, Since + is commutative, 0+0'= 0'+0 and since 0 is identity for addition we also have 0+0'=0' thus 0'=0 similarly for multiplication if 1'$\in$F satisfies 1' * a=a $\forall$ a$\in$F then 1'*1=1*
For uniqueness of addition ad multiply inverse use cancellation thus.

3.6 Theorem
If (F,+,) is a field then we have:
(1) a*0=0*a=0, $\forall$ a $\in$F
(2)(-a)*b* = *a*(-b) = -(a*b)* $\forall$ *a,b* $\in$*F*
(3)(-a)(-b) = a*b

Proof:
(1) we have a*0=a*(0+0) =a*0+a*0. So by cancellation theorem, a*0 =0. Some for 0*a=0
(2) Showing that (-a)*b=-(a*b) amount to showing that (-a)*b is the additive inverse of a*b(because of uniqueness of additive inverse in Theorem 3.5).
we have (-a)*b+a*b = (-a+a)*b* = *0b = 0
Some for a*(-b)+ab=0*
*(3) Note that by (2) above we have(-1)*a=-a=a(-1). Thus, by using(F1) we have (-a)*(-b) = ((-1)a)(-b)= a((-1)(-b)). Where we have used (-1)(-b)=b which in turn follows from the fact that (-1)(-1)=1.*
Conclusion from now on, we can just write in a field(F,+,) -a for the additive inverse of a$\in$F

1/a or a^-1 for the multiplicative inverse of a!=0

Vector Space
Definition: A vector space V over a field F consists of a set V on which two operations(called addition and scalar multiplication) are defined, so that for each x,y $\in$V, we have a unique element x+y in V and for each x$\in$V and a$\in$F we have a unique element ax$\in$V(scalar) such that the following conditions hold:
(VS1) x+y=y+x, $\forall$ x,y$\in$V
(VS2) (x+y)+z=x+(y+z), $\forall$ x,y,z$\in$V
(VS3) There exists on element in V denoted 0 such that x+0=x, $\forall$x$\in$V
(VS4) For each x$\in$V have exist y$\in$V such that x+y=0
(VS5) For each x$\in$V we have 1x=x
(VS6) For each x$\in$V, a,b$\in$F we have (ab)x=a(bx)
(VS7) For each x,y$\in$V, a$\in$F we have a(x+y) =ax+ay

(VS8) For each $x \in V$, $a,b \in F$ we have $(a+b)x = ay + bx$