115A, Winter 2023

Lecture 4

Wed, Jan 18

Recall from last time:

$\boxed{\text{Definition}}$ A vector space (or a linear space) over a field F is a set V on which two operations are defined, called addition and scalar multiplication, so that for each $x, y \in V$ we have a unique element $x + y \in V$ (x plus y) and for each $x \in V$, $a \in F$ we have a unique element $ax \in V$ (a times x) such that the following conditions are satisfied:

$\boxed{\text{VS1}}$ $x + y = y + x$, $\forall\, x, y \in V$ (commutativity of addition)

$\boxed{\text{VS2}}$ $(x+y)+z = x+(y+z)$, $\forall\, x, y, z \in V$ (associativity of addition)

$\boxed{\text{VS3}}$ There exists an element in V denoted 0 such that $x + 0 = x$, $\forall\, x \in V$ (the "zero" or "neutral" element in V)

$\boxed{\text{VS4}}$ For each $x \in V$ there exists $y \in V$ such that $x + y = 0$ (the additive inverse in V, or opposite of x)

$\boxed{\text{VS5}}$ For each $x \in V$, we have $1x = x$

VS 6 For each $a, b \in F$, $x \in V$, we have

$$(ab)x = a(bx)$$

VS 7 For each $a \in F$ and $x, y \in V$ we have

$$a(x+y) = ax + ay$$

VS 8 For each $a, b \in F$ and $x \in V$ we have

$$(a+b)x = ax + by$$

The elements in V are called <u>vectors</u> and the elements in F are called <u>scalars</u>

The most common case of Field considered by us will be $F = \mathbb{R}$ (the field of real numbers with its usual $+$ and $\cdot$ operations)

4.1. <u>Example</u> Given a field F and some $n \geq 1$ the set of all n-tuples $v = (v_1, \dots, v_n)$ with entries $v_1, \dots, v_n \in F$ is denoted F^n. It is a vector space over F with respect to the operations $+$ of addition "coordinate by coordinate":

if $v = \overbrace{(v_1, \dots, v_n)}^{\in F^n}$, $u = \overbrace{(u_1, \dots, u_n)}^{\in F^n}$ then

$$v + u = (v_1 + u_1, v_2 + u_2, \dots, v_n + u_n)$$

and scalar multiplication $\overset{\in F}{a} \overbrace{(v_1, \dots, v_n)}^{\in F^n}$

$$= (a v_1, a v_2, \dots, a v_n) \in F^n$$

It is immediate to check that $\underline{VS1-8}$ are all satisfied, as a consequence of the properties satisfied by $+$ and $\cdot$ in the field F.

Take for instance $F = \mathbb{R}$ and $n = 3$, then $v = (-1, 2, -4)$, $u = (\frac{1}{2}, -1, 3) \in \mathbb{R}^3 = V$ and we have $v + u$

$$= (-1, 2, -4) + (\tfrac{1}{2}, -1, 3) = (-\tfrac{1}{2}, 1, -1)$$

Also, if $a = -1$ then $av = (1, -2, 4)$

4.2 Example If F is a field and $m, n \geq 1$ integers, then the set of all $m \times n$ matrices over F, denoted $M_{m \times n}(F)$, is a vector space over F with respect to the following operations of addition and scalar multiplication:

If $A, B \in M_{m\times n}(F)$ have entries

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & & & \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{pmatrix}$$

then

$$A + B = \begin{pmatrix} A_{11}+B_{11}, & A_{12}+B_{12}, \cdots & A_{1n}+B_{1n} \\ A_{21}+B_{21}, & A_{22}+B_{22}, \cdots & A_{2n}+B_{2n} \\ \vdots & & \\ A_{m1}+B_{m1}, & A_{m2}+B_{m2}, \cdots & A_{mn}+B_{mn} \end{pmatrix}$$

and if $\alpha \in F$ then

$$\alpha A = \begin{pmatrix} \alpha A_{11}, & \alpha A_{12}, \cdots & \alpha A_{1n} \\ \alpha A_{21}, & \alpha A_{22}, \cdots & \alpha A_{2n} \\ \\ \alpha A_{m1}, & \alpha A_{m2} & \alpha A_{mn} \end{pmatrix}$$

For instance, take $F = \mathbb{R}$ and $M_{2\times 3}(\mathbb{R})$ the vector space of 2×3 matrices over $\mathbb{R}$

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -\frac{1}{2} & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & -2 \\ 0 & \frac{1}{2} & -2 \end{pmatrix}$$

then $A + B = \begin{pmatrix} -1+3 & 2+0 & 3-2 \\ 0+0 & -\frac{1}{2}+\frac{1}{2} & 2-2 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The vector $0 \in M_{2\times 3}(\mathbb{R})$ is the 2×3 matrix with all entries $= 0$

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4.3 Example If F is a field and S is a non-empty set, then we denote $\mathcal{F}(S, F)$ the set of all functions $f : S \to F$ with domain S and codomain F, with addition given by:

If $f : S \to F$ and $g : S \to F$ then $(f+g) : S \to F$ is the function on S with values in F

$$\text{defined by} \quad (f+g)(s) = \underbrace{f(s) + g(s)}^{\text{this is summation in } F} \in F$$

and if $\alpha \in F$ then scalar multipl. by α is $\alpha f : S \to F$ defined by

$$(\alpha f)(s) = \underbrace{\alpha \cdot f(s)}_{\text{this is multiplication in } F} \in F$$

Remark. If we take $S = \{1, 2, \dots, n\}$ a finite set with $n \geq 1$ elements, then $\mathcal{F}(\{1, \dots, n\}, F)$ is the same as F^n.

4.4. Example. A sequence with elements in F is a function $f : \{1, 2, \dots\} \to F$ defined on the positive integers with values (codomain) in F. If $f(n) = a_n \in F$, $n = 1, 2, \dots$ is an enumeration of the values of f then one also writes the sequence f as $(a_n)_n$.

The set of sequences in F is

a particular case of the Example 4.3 where $S = \{1, 2, \ldots\}$, and so it is a vector space V with respect to "entry by entry" addition of sequences and scalar multiplication given by:

$$(a_n)_n + (b_n)_n = (a_n + b_n)_n$$

if $c \in F$ then $c(a_n)_n = (ca_n)_n$

4.5 Example Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ and for $(a_1, a_2), (b_1, b_2) \in V$, $c \in \mathbb{R}$ define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$$

$$c(a_1, a_2) = (ca_1, ca_2)$$

Note: V as a set is the same as $\mathbb{R}^2$ but we modified the definition of $+$ (but scalar multiplication is same as in $\mathbb{R}^2$)

is V a vector space with this $+$ and $\cdot$?

NO because $+$ fails to satisfy VS 1

i.e. + as defined is not commutative

$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$

while $(b_1, b_2) + (a_1, a_2) = (b_1 + 2a_1, b_2 + 3a_2)$

For instance we take

$(1, 2) + (2, 1) = (1+4, 2+3) = (5,5)$

but $(2, 1) + (1, 2) = (2+2, 1+6) = (4, 8)$ $\neq$

Note Also VS 8 is not satisfied because

$(1+1)(a_1, a_2) = (a_1, a_2) + (a_1, a_2) = (3a_1, 4a_2)$

but $2(a_1, a_2) = (2a_1, 2a_2)$

4.6. Theorem (cancellation rule for vector addition)

If V is vector space over the field F and $x, y, z \in V$ are so that

$x + z = y + z$ then $x = y$

$\boxed{\text{Proof}}$ By the axioms of $(V, +)$, there exists $v \in V$ such that $z + v = 0$.

Thus, we get that

$x + z = y + z$ implies $(x+z) + v = (y+z) + v$

which by $\boxed{VS\ 2}$ implies

$$x + \overbrace{(z+v)}^{=0} = y + \overbrace{(z+v)}^{=0}$$

so $x + 0 = y + 0$ which by $\boxed{VS3}$ implies $x = y$ $\boxed{VS}$

4.7. Theorem. Let V be vector space over field F

(a) The vector $0 \in V$ is unique with the property $\boxed{VS3}$ i.e. if $0' \in V$ satisfies $x + 0' = x$, $\forall x \in V$ then $0' = 0$

(b) if $x \in V$ then there exists a unique $y \in V$ with $x + y = 0$, i.e. if $y' \in V$ satisfies $x + y' = 0$ then $y = y'$

$\boxed{\text{Proof}}$ (a). if $0'$ satisfies $x + 0' = x$, $\forall x \in V$ then

$$0 + 0' = 0$$

But by $\boxed{VS1}$ we have $0 + 0' = 0' + 0$ an by $\boxed{VS3}$ the element 0 satisfies itself the property $x + 0 = x$, $\forall x \in V$ which applied to $x = 0'$ gives

$0' + 0 = 0'$. Thus

$0 + 0' = 0$

$= \{ \; 0' + 0 = 0'$ so $0 = 0'$

(b) if $x + y = 0$

$x + y' = 0$

Then by $\boxed{VS1}$ we have

$y + x = y' + x$ and by Cancellation Thm 4.6

This implies $y = y'$

4.8 Theorem if V vector space over field F then we have

(a). $0x = 0$, $\forall x \in V$

(b) $(-a)x = -(ax) = a(-x)$, $\forall x \in V, a \in F$

(c) $a0 = 0$ $\forall a \in F$

$\boxed{\text{Proof}}$ (a) we have $0 + 0 = 0$ so

$0x = (0+0)x \overset{VS8}{=} 0x + 0x$

by Thm 4.6 cancellation rule

$0 = 0x$

(b) exercise

(c). $a0 = a(0+0) \overset{\text{by VS8}}{=} a0 + a0$

so by Thm 4.6 (cancellation law)

$0 = a0$

$\boxtimes$