115A , Winter 2023

Lecture 15

Mo , Feb 13

Summary of last two lectures:

- Given V, W vec. spaces with ordered basis $\beta = \{v_1, \dots, v_n\}$ resp. $\gamma = \{w_1, \dots, w_m\}$: for any $T: V \to W$ linear we defined the matrix representation of T $[T]_\beta^\gamma = (a_{ij})$ where $a_{ij} \in F$ are so that
$$T(v_j) = \sum_i a_{ij} w_i \quad , \forall i, j$$

- We defined the set $\mathcal{L}(V, W)$ of linear transformations $T: V \to W$ and considered an operation of addition & scalar multiplication for linear trans. $T, U \in \mathcal{L}(V,W)$
$$(T + U)(x) = T(x) + U(x), \quad x \in V$$
$$(cT)(x) = c\,T(x) \qquad c \in F$$
- Showed that $\mathcal{L}(V, W)$ with these operations is a vector space $/F$
- Showed that the matrix representation $T \longmapsto [T]_\beta^\gamma$ from $\mathcal{L}(V, W)$ to $M_{m \times n}(F)$ is a linear transformation of vector spaces and that it is one to one (injective)
- If $V = W$ we denote $\boxed{\mathcal{L}(V, V) = \mathcal{L}(V)}$

Composition of linear transformations & matrix multiplication (§2.3)

- Recall that if $f : X \to Y$, $g : Y \to Z$ are functions then their composition is the function $g \circ f : X \to Z$ defined by $g \circ f(x) = g(f(x))$, $x \in X$.

- We study the case when the functions we want to compose are linear transformations between vector spaces

15.1 Theorem. Let V, W, Z be vector spaces $/F$ and $T : V \to W$, $U : W \to Z$ linear transformations. Then their composition $\underbrace{U \circ T}_{\text{for simplicity denoted } UV} : V \to Z$

defined by $UV(x) = U(V(x))$, $x \in V$, is a linear transformation, i.e. $UV \in \mathcal{L}(V, Z)$

$\boxed{\text{Pf}}$. If $x, y \in V$ then $UV(x+y)$

$$= U(V(x+y)) \overset{\substack{\text{because } V \text{ linear} \\ \downarrow}}{=} U(V(x) + V(y))$$

$$\overset{\substack{\text{because } U \text{ linear} \\ \downarrow}}{=} U(V(x)) + U(V(y)).$$

Similarily $UV(cx) = U(V(cx))$
$= U(cV(x)) = cUV(x), \quad \forall c \in F$

$\boxed{}$

- We next establish some easy properties of associativity and distributivity of composition w.r.t. addition in $\mathcal{L}(V) \overset{def}{=} \mathcal{L}(V, V)$

15.2. Theorem. Let V be vector space, $T, U_1, U_2 \in \mathcal{L}(V)$. Then we have:

(a) $T(U_1 + U_2) = TU_1 + TU_2$ and
$(U_1 + U_2)T = U_1T + U_2T$

(b) $T(U_1U_2) = (TU_1)U_2$

(c) $TI = IT$, where $I : V \to V$ is the identity transf. $I(x) = x, \forall x \in V$

(d) $a(U_1U_2) = (aU_1)U_2$
$= U_1(aU_2) \qquad \forall a \in F$

$\boxed{\text{Pf}}$. easy exercise, let's do (a) as a sample:

$T(U_1 + U_2)(x) = T((U_1 + U_2)(x))$
$\underset{\text{def of addition of fct.}}{=} T(U_1(x) + U_2(x)) \underset{\text{linearity of } T}{=} T(U_1(x)) + T(U_2(x))$
$= (TU_1 + TU_2)(x)$ $\boxed{}$

15.3 Notation: if $T \in \mathcal{L}(V)$ and $n = 1, 2, 3, \dots$ then we denote $T^n = \underbrace{T \circ T \circ \dots \circ T}_{n\text{-times}}$

15.4 Remarks The composition of linear transformations in $\mathcal{L}(V)$ defines an operation of multiplication in $\mathcal{L}(V)$ which by Thm 14.2 is associative and distributive w.r.t. addition.

But this multiplication is not (always) commutative, i.e. in general it is not true that if $T, U \in \mathcal{L}(V)$ then $TU = UT$

$T, U \in \mathcal{L}(F^2)$

- Example: let $T : F^2 \to F^2$, for $(a,b) \in F^2$

$T(a,b) \overset{\text{def}}{=} (b, 0)$, $U(a,b) \overset{\text{def}}{=} (0, a)$

Then $TU(a,b) = T(0,a) = (a,0)$

$UT(a,b) = U(b,0) = (0,b)$

$\neq$

Also, note that this T satisfies

$T^2(a,b) = T(T(a,b)) = T(b,0) = (0,0)$

So in $\mathcal{L}(V)$ we can have $T^2 = 0$ although $T \neq 0$.

- We'll now define an operation of multiplication of matrices (you may have seen this before!)

- We will then relate multiplication of matrices with composition of the underlying linear transformation, showing that

$$\boxed{\begin{array}{ccc} \text{Composition of linear transf. } \mathcal{L}(V) & \longleftrightarrow & \text{multiplication of their matrix representation } M_n(F) \end{array}}$$

15.5 Definition. Let $A \in M_{m \times n}(F)$, $B \in M_{n \times p}(F)$. We define the product of A and B, denoted by AB to be the $m \times p$ matrix with entries

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} \quad \text{for } \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, p \end{array}$$

- Note: We need to have A $m\times n$, B $n\times p$ (same size) in order to do product AB

Examples $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \in M_{2\times 3}(\mathbb{R})$

$B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$

Then we cannot do the multiplication AB because A is 2×3, B is 2×2

But we can do $\underset{2\times 2}{B}\ \underset{2\times 3}{A}$

$$= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

15.6. Theorem. Let V, W, Z fin. dim. vec. spaces $/F$ with ordered basis α, β, γ resp.
Let $T: V \to W$, $U: W \to Z$ linear

Then: $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

$\boxed{\text{Pf}}$. This is clear by the definitions of matrix representation & matrix multiplication

Indeed if $\alpha = \{v_1, \dots, v_m\}$, $\beta = \{w_1, \dots, w_n\}$,
$\gamma = \{u_1, \dots, u_p\}$ and we denote

$$[T]_\alpha^\beta = \overbrace{(a_{ij})}^{A \ n\times m \text{ matrix}} \text{ with } T(v_j) = \sum_{i=1}^{n} a_{ij} w_i$$

$$[U]_\beta^\gamma = \underbrace{(b_{ki})}_{B \ p\times n \text{ matrix}} \text{ with } U(w_i) = \sum_{k=1}^{p} b_{ki} u_k$$

Then using the definitions we see that $[UT]_\alpha^\gamma = (c_{kj})$

with $\underbrace{UT(v_j)}_{\shortparallel} = \sum_{k=1}^{p} c_{kj} u_k$

$$U(T(v_j)) = U\left(\sum_{i=1}^{n} a_{ij} w_i\right)$$

$$= \sum_{i=1}^{n} a_{ij} U(w_i) = \sum_{i=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{ki} u_k\right)$$

$$= \sum_{k=1}^{p} \underbrace{\left(\sum_{i=1}^{n} b_{ki} a_{ij}\right)}_{=(BA)_{kj}} u_k$$

$p \times m$ matrix $\boxtimes$

15.7 Corollary. Let $T, U \in \mathcal{L}(V)$ (fin. dim. with ordered basis $\beta = \{v_1, \dots, v_m\}$)

Then $[UT]_\beta = [U]_\beta [T]_\beta$

Recall that if $V = W$ and β is ordered basis for V then $[T]_\beta^\beta$ denoted $[T]_\beta$, $T \in \mathcal{L}(V)$

15.8 Theorem. Let $A \in M_{m\times n}(F)$

$B_1, \dots, B_k \in M_{n\times p}(F)$

$C_1, \dots, C_k \in M_{q\times m}(F)$

and $a_1, \dots, a_k \in F$ then

$$A\left(\sum_{i=1}^{k} a_i B_i\right) = \sum_{i=1}^{k} a_i A B_i$$

$$\left(\sum_{i=1}^{k} a_i C_i\right) A = \sum_{i=1}^{k} a_i C_i A$$

PR. Exercise

Do examples & exercises on blackboard!