Math115A 2/06 notes

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Recall that we defined the notion of linear transformation from one vector space V to a vector space W, $T : V \to W$, when it satisfies $T(v_1 + v_2) = T(v_1) + T(v_2), T(cv) = cT(v) \forall v_1, v_2, v \in V, \forall c \in F$
We also defined the null space (or kernel) of a linear $T : V \to W$ by $N(T) = \{v \in V : T(v) = 0_w\}$ and range of T, by $R(T) : \{T(v) : v \in V\}$
We proved in Theom 11.12 that $N(T), R(T)$ are subspace of V respectively W

12.1 Theorem

Let V, W be vector spaces and $T : V \to W$ linear. If $S = \{v_1, ..., v_m\} \subset V$ is a basis for V, then $R(T) = span(T(S)) = span\{T(v_1), T(v_2), ..., T(v_n)\}$
Proof: We have $T(S) \subset T(V) = R(T)$ by the definitions, so $T(v_1), ...T(v_n) \in R(T)$ and since R(T) is a subspace we get $span(\{T(v_1), ..., T(v_n)\}) \subset R(T)$.
If $w \in R(T)$ is a vector in the range of T, then there exists $v \in V$ such that $T(v) = w$. But V is spanned by $\{v_1, ..., v_n\}$ (because $\{v_1, ...v_n\}$ is a basis for V), so $\exists c_1, ..., c_n \in F$ such that $v = \Sigma_{i=1}^n c_i v_i$ Thus $w = T(v) = T(\Sigma_{i=1}^n c_i v_i) = \Sigma_{i=1}^n c_i T(v_i) \in span\{T(v_1), ...T(v_n)\}$

12.2 Definitions

Let V, W be vector spaces and $T : V \to W$ linear if $N(T), R(T)$ are finite dimensional then we define the nullity of T, denoted nullity(T), to be the dimensional of N(T) and the rank of T, denoted rank(T), to be the dimension of R(T)

So nullity(T)=dim(N(T)), rank(T)=dim(R(T))

12.3 Theorem (Dimension Theorem)

Let V, W be vector spaces and $T : V \to W$ linear. If V is finite dimensional then $nullity(T) + rank(T) = dim(V)$
Proof: Let n denote the dimension of V and k then dimensional of $N(T)$, i.e. $n = dim(V), k = dim(N(T))$. Let $\{v_1, ..., v_k\} \subset N(T)$ be a basis for N(T). By corollary 11.3 $\{v_1, ...v_k\}$ can be extended to a basis $S = \{v_1, ...v_k, v_{k+1}, ...v_n\}$ for V

We'll show that $S_0 = \{T(V_{k+1}), T(V_{k+2}), ..., T(V_n)\} \subset R(T) \subset W$ is a basis for R(T)
To see this, we need to show that the set S_0 spans R(T) and is linearly independent
Indeed, since $v_1, ..v_k \in N(T)$ we have $T(v_1) = T(v_2) = ... = T(v_k) = 0$
Thus R(T) = span$\{T(v_1), T(v_2), ..., T(v_n), T(v_{k+1}, ...T(v_n)\}$=span$\{T(v_{k+1}, ...T(v_n))\} = span(S_0)$

To see that S_0 is linear independent assume $\Sigma_{i=k+1}^{n} c_i T(v_i) = 0$ for some $c_i \in F$ using linearly of T we get $T(\Sigma_{i=k+1}^{n} c_i v_i) = 0$ implying that $\Sigma_{i=k+1}^{n} c_i v_i \in N(T)$
Since $\{v_1, ...v_k\} \subset N(T)$ is a basis for N(T) this implies $\exists c_1, ..c_k \in F$ such that $\Sigma_{i=1}^{k} c_i v_i = \Sigma_{i=k+1}^{n} c_i v_i$
Thus $\Sigma_{i=1}^{n}(-c_i)v_i + \Sigma_{i=k+1}^{n} c_i v_i = 0$ Since $S = \{v_1, .., v_k, v_{k+1}, ...v_n\}$ is a basis for V, this implies $c_i = 0$ for all $i = 1, 2, ..., n$ in particular $c_{k+1}, ..., c_n = 0$
Showing that $S_0 = \{T(v_{k+1}), ...T(v_n))\}$ is indeed linearly independent
Thus, S_0 is a basis for $R(T)$ and so $dim(R(T))$ equals the number of elements in S_0, whihc is $n - k$
Thus $dim(V) = k + (n - k) = dim(N(T)) + dim(R(T))$

12.4 Theorem

Let V, W be vector spaces and $T : V \to W$ linear. Then T is one to one iff N(T)={0}
Proof: if T is one to one and $v \in N(T)$, then $T(v) = 0_w$ implies $v = 0_v$, so we get $N(T) = \{0\}$
If conversely $N(T) = \{0\}$ and we have $T(v_1) = T(v_2)$ for some $v_1, v_2 \in V$, then $0_w = T(v_1) - T(v_2) = T(v_1 - v_2)$ so $v_1 - v_2 \in N(T) = \{0\}$, Thus $v_1 - v_2 = 0_v$, implying $v_1 = v_2$, Thus showing T is one to one.

12.5 Theorem

Let V, W be finite dimensional vector space with $dim(V) = dim(W)$ and $T : V \to W$ linear. Then the following conditions are equivalent:
a) T is one to one (T is injective)
b) T is on to (T is surjective)
c) rank(T)=dim(V)
Proof: From dimension Theom 12.3 we have $rank(T) + nullity(T) = dim(V)$ we showed in Theom 12.4 that T is one to one iff N(T)={0} iff $nullity(T) = 0$
Thus, by $rank(T) + nullity(T) = dim(V)$ we get T one to one iff $rank(T) = dum(V)$ so (a) is equivalent to (c)
On the other hand, we have T is onto iff $R(T) = W$ iff $rank(T) = dim(W)$ and from $rank(T) + nullity(T) = dim(V)$ we get $dim(W) = dim(V)$ iff $nullity(T) = 0$ i.e. iff T is one to one
Thus shows that (b) is equivalent to (a), so all together (a)(b)(c) are all equivalent

12.6 Exercise

Show that $T : R^3 \to R^3$ defined by $T(a, b, c) = (a - b, 2c)$ is onto
Solution It is easier to see what the kernel of T is: $N(T) = \{(a, b, c) \in \mathbb{R}^3 : T(a, b, c) = 0\}$ which by the definition of T means $(a, b, c) \in N(T)$ iff $a - b = 0, 2c = 0$ so $c = 0$ and $a = b$ so $N(T) = \{(a, a, 0) : a \in \mathbb{R}\}$. Thus $nullity(N(T)) = 1$ so by 12.3 $3 = dim, \mathbb{R}^3 = dim(N(T)) + dim(R(T)) = 1 + dim(R(T))$. SO $dim(R(T)) = 3 - 1 = 2$