Math115A 1/30 notes

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Recall that on Friday we proved the following
"Replacement Theorem"

Let V be a vector space. Let $G \subset V, L \subset V$ be finite subsets of V such that:
a) G has n vectors & it generates V
b) L has m vectors and is linearly independent.
The $m \leq n$ and there exists a subset $H \subset G$ with $n - m$ vectors such that $L \cup H$ generate V.

Recall also that we introduced the following important concept

Definition: A subset S of a vector space V that's linearly independent and generates V is called a basis of the vector space V
Today we'll use the replacement Theom and the concept of basis of a vector space V to introduce the notion of dimension of vector space V

9.1 Corollary

Let V be a vector space having a finite basis (i.e. $\exists$ subset $S \subset V$ with S finite, linear independent and generating/spanning V)
Then any other basis for V contains the some number of vectors
Proof:
Let $S' \subset V$ be another basis for V, i.e. S' linear independent and $span(S') = V$
Denote by n the number of vectors in S. Assume S' contains n' vectors, with $n' > n$. Since S' is linearly independent and $span(S) = V$, the replacement theom tells us that $n' \leq n$.
Thus, S' must be finite and the number n' of elements in S' must be $n' \leq n$.
Reversing the role of S, S'(which are both basis for V!) we obtain $n \leq n'$ as well, thus $n' = n$

The above corollary stats that if a vector space V has a finite basis, then the number of elements in that basis is an imtrihisic property of V. Thus allocating hte following:

9.2 Definition

A vector space V is said to be finite dimensional if it has a basis consisting of a finite number of elements. The unique integer n (confirm for 9.1!) such that any basis of V has exactly n elements is called the dimension of V, denoted $dim(V)$.
A vector space that does not have a finite basis is called infinite dimensional.

9.3 Examples

1) The vector space $\{0\}$ consisting of just the 0 vector has dimension 0, $dim(\{0\}) = 0$.

2) $dim(R^n) = n$, more generally if F is an arbitary field, then $dim(F^n) = n$.
Indeed, we have shown that the set of vector $S = \{(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 0, 1)\}$ is linearly independent is F^n and it generates F^n, so it is a basis, and we see S has exactly n vectors

3) $dim(M_{m\times n}(F)) = mn$ because $E_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{m\times n}(F)$ are linear independent & generate $M_{m\times n}(F)$ so it is a basis, and it has $m \times n$ vectors

4) The vector space $V = F[X]$ of all polynomials with coefficients in a field F is infinite-dimensional

   Indeed, we know that $S = \{1, X, X^2, ...\}$ are linearly independent if V would be finite dimensional, then it can be generated by a finite set of n elements and so by replacement Theom it would follow that any finite subset of S has $\leq n$ many elements, te=hus S would be finite, contradiction. (because S has infinite many vectors.)

9.4 Exercise

Do the polynomials $P_1 = x^3 - 2x^2 + 1, P_2 = 3x - 2, P_3 = 4x^2 - x + 3$ in $V = P_3(\mathbb{R})$(i.e. the vector space of all polynomials of degree ≤ 3 with coefficients in $\mathbb{R}$). generate $P_3(\mathbb{R})$?
Solution No, they don't. Because the vector space $P_3(\mathbb{R})$ has the polynomials $1, X, X^2, X^3 \in P_3(\mathbb{R})$ which are linearly independent & span V. Thus, by replacement Theom, any basis for $P_3(\mathbb{R})$ must have exactly 4 vectors in it, and $\{P_1, P_2, P_3\}$ has any 3 vectors.

9.5 Exercise

is the set $S = \{(1, 4, -6), (1, 5, 8), (2, 1, 1), (0, 1, 0)\} \subset \mathbb{R}^3$ a linear independent subset of $\mathbb{R}^3$
Solution: No because by Replacement Theom, if $S \subset V$ linearly independent anad has n elements then $n \leq dim(V)$ But $dim(V) = 3$, and $4 > 3$ contradiction

Related to the above exercises, let us repeat are more time that conditions in the replacement theom and Corollary 9.1:
if a vector space V is spanned (generated) by a subset of n vectors, then $dim(V) \leq n$ and if are takes any set S of linearly independent vectors in V has $\leq n$ many elements in it, i.e. if # elements in S is m, then $m \leq n$
Another consequence of the these results is that if $S \subset V$ is a set with m elements and $m < dim(V)$ then S cannot generate V, in particular S cannot be a basis for V.

##**9.6 Exercise**

Let W_1, W_2 be subspaces of the vector space V and assume $dim(W_1) = m, dim(W_2) = n$. Where m, n are finite integers.
Prove that $dim(W_1 + W_2) \leq m + n$. $W_1 + W_2 =\{x + y : x \in W_1, y \in W_2\}$
Solution: By the definition of dimension, the assumptions imply that there exist sets $S_1 =\{v_1, ..., v_m\}\subset W_1$ and $S_12 =\{v_1, ..., v_m\}\subset W_2$ such that:
S_1 is linear independent & $span(S_1) = W_1$
S_2 is linear independent & $span(S_2) = W_2$
(i.e., S_1 is a basis for W_1, S_2 is a basis for W_2)
But then $span(S_1 \cup S_2) = span(S_1) + span(S_2) = W_1 + W_2$
Thus, by replace Theom, since $S_1 \cup S_2$ has at most $m + n$ elements, we have $m + n \geq dim(W_1 + W_2)$

9.7 Exercise

Let V be the subset of $M_{m\times n}(F)$ upper triangular
i.e. $A = \begin{pmatrix} 0 & - & - \\ 0 & 0 & - \\ 0 & 0 & 0 \end{pmatrix} = (A_{ij})_j$ then $A_{ij} = 0, , \forall i < j$
Show that V vector subspace of $M_{m\times n}(F)$ find a basis and calculate $dim(V)$
Solution: First note that the set S of all matrices $E_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (A_{ij})_j$ having 1 on entry ij. and all the other entries $= 0$, with $j = i$ is a basis for V
Indeed, we already show that $E =\{E_{ij} : 1 \leq j, j \leq m\}$ is a linear independent and space the entrie vector space $M_{m\times n}(F)$ Thus, its subset S is still linearly independent, and it clearly space V.
To count the number of elements in S note there are n^2 many elements in the large set E, form which we substract the number of matrices E_{ij} in E that have some entry ij equal to 1 under the diagnal