115 A , Winter 2023

Lecture 16

Wed, Feb 15

The most important things we did lately:

- Given vector spaces V, W over field F we introduced the vector space $/_F$ of linear transformations $T: V \to W$, denoted $\mathcal{L}(V, W)$, endowed with its natural addition and scalar multiplication operations.

- When V, W fin. dim. with ordered basis $\beta = \{v_1, \dots, v_n\} \subset V$, $\gamma = \{w_1, \dots, w_m\} \subset W$, for each $T \in \mathcal{L}(V, W)$ we defined its matrix representation w.r.to β, γ by $[T]_\beta^\gamma = (a_{ij})_{\substack{j=1,\dots,n \\ i=1,\dots,m}} \in M_{m \times n}(F)$ by $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$

- We proved that the function
$$T \longmapsto [T]_\beta^\gamma \in M_{m\times n}(F), \quad T \in \mathcal{L}(V,W)$$
is a linear transformation from vector space $\mathcal{L}(V, W)$ to vector space $M_{m\times n}(F)$

[Obs: it is actually also onto!]

(both over field F) and that it is one to one

($\Leftarrow\Rightarrow$ if $[T]_\beta^\gamma = 0 \in M_{m\times n}(F)$ then $T = 0$ as linear transf.)

- We proved that if V, W, Z are fin. dim. vec spaces$/_F$ with ordered basis α, β, γ respectively and

$${}_\alpha V \xrightarrow{T} W_\beta \xrightarrow{U} Z_\gamma$$

linear transf. then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta$$

this is matrix multiplic.

- We introduced matrix multiplication: for $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$ the product $AB \in M_{m\times p}(F)$ defined by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, p$

- In particular, if we take $V = W$ (ordered basis β) and denote $\mathcal{L}(V, V) = \mathcal{L}(V)$ for simplicity

Then $\mathcal{L}(V) \ni T \mapsto [T]_\beta = [T]_\beta^\beta$
$\in M_{n\times n}(\mathbb{F})$

gives an identification between
$\mathcal{L}(V)$ with its vec. space structure
and composition operation

and $M_{n\times n}(\mathbb{F})$ with its vec. space
structure and multiplication
operation

- We showed that composition of linear transf., resp. matrix multipl., satisfy associativity and distributivity (w.r. to addition) properties.

$S\circ(T\circ U) = (S\circ T)\circ U$ for compos.

$(AB)C = A(BC)$ matrix mult.

$$T\circ\left(\sum c_i U_i\right) = \sum_i c_i\, T\circ U_i$$

$$\left(\sum c_i U_i\right) T = \sum_i c_i\, U_i T$$

$$A\left(\sum c_i B_i\right) = \sum_i c_i\, A B_i$$

$(\sum_i c_i B_i)A = \sum_i c_i B_i A$

- Multiplication by identity matrix

$I_m A = A$, $A I_n = A$, $\forall A \in M_{m\times n}(F)$

For composition, I_n corresponds to the identity transformation $I: V \to V$ where $\dim(V) = n$.

16.1 Theorem Let V, W be f. dim vector spaces $/_F$ with ordered basis $\beta \subset V$, $\gamma \subset W$ and let $T: V \to W$ linear. Then for any $u \in V$ we have

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

$\boxed{\text{Proof}}$. Let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_m\}$

Let $[u]_\beta = \sum_{j=1}^n c_j v_j$ be the coordinate vector of u in basis $\beta = \{v_1, \dots, v_n\}$

Then $T(u) = T(\sum_{j=1}^n c_j v_j)$

$$\overset{\text{because } T \text{ linear}}{=} \sum_{j=1}^n c_j \overline{T(v_j)} = \sum_{j=1}^n c_j \underbrace{\left(\sum_{i=1}^m a_{ij} w_i\right)}$$

$T''(v_j)$ by definition

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \, c_j \right) w_i$$

So $[T(u)]_\gamma$ is given by the column matrix $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

where $b_i = \sum_{j=1}^{n} a_{ij} \, c_j$

While by the definition of matrix multiplication we have:

$$\underbrace{[T]_\beta^\gamma}_{= (a_{ij})} \; \underbrace{[u]_\beta}_{= \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j} c_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} c_j \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

16.2. Definition Let $A \in M_{m\times n}(\mathbb{F})$

We denote by $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ where $\mathbb{F}^n = M_{n\times 1}(\mathbb{F})$ and $\mathbb{F}^m = M_{m\times 1}(\mathbb{F})$

the map (function) defined by

$L_A(x) = Ax$, where Ax is here matrix multiplication, viewing $x \in \mathbb{F}^n = M_{n\times 1}(\mathbb{F})$, so x as a column matrix, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

We call L_A the left multiplication transformation

- Due to properties of matrix multiplication (e.g. associativity & distributivity w.r.t. $+$) that we already established, we get:

16.3. Corollary. Let $A \in M_{m\times n}(\mathbb{F})$

Then $L_A : F^n \to F^m$ is linear, i.e.

$L_A \in \mathcal{L}(F^n, F^m)$. Also, if $B \in M_{m\times n}(F)$ and β, γ are the usual basis for F^n, F^m then we have:

(a) $[L_A]_\beta^\gamma = A$

(b) $L_A = L_B$ iff $A = B$

(c) $L_{A+B} = L_A + L_B$, $L_{cA} = cL_A, \forall c \in F$

(d) If $T \in \mathcal{L}(F^n, F^m)$ then $C = [T]_\beta^\gamma$ satisfies $L_C = T$ and it is the unique linear transf $: F^n \to F^m$ satisfying this property

(e) If $E \in M_{n\times p}(F)$ then $L_{AE} = L_A L_E$

(f) If $m = n$ then $L_{I_n} = I_{F^n}$

$\boxed{\text{Proof}}$ These are all clear from Thms we have done $\square$

Do examples & exercises on blackboard if time remains!