

# Math115 1/23 notes

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## 6.1 Definition

If  $S_1, S_2$  are nonempty subsets of a vector space  $V$  then the sum of  $S_1$  and  $S_2$ , denoted  $S_1 + S_2$  is the set  $\{x + y : x \in S_1, y \in S_2\}$

## 6.2 Definition

Let  $W_1, W_2$  be subspaces of the vector space  $V$ . We say that  $V$  is the direct sum of  $W_1$  and  $W_2$  if  $W_1 \cap W_2 = \{0\}$  and  $W_1 + W_2 = V$ , and we then write  $V = W_1 + W_2$

## 6.3 Exercise

Show that  $V = \mathbb{R}^2$  is the direct sum of  $W_1 = \{(x, x) : x \in \mathbb{R}\}$  and  $W_2 = \{(y, -y) : y \in \mathbb{R}\}$

### Solution

First note that  $W_1, W_2$  are indeed vector subspaces of  $\mathbb{R}^2$  indeed, if we take two elements  $(x, x) \in W_1, (z, z) \in W_2$ , then  $(x, x) + (z, z) = (x + z, x + z) \in W_1$ . Also, if  $c \in \mathbb{R}$  is a scalar, then  $c(x, x) = (cx, cx) \in W_1$

Similarly for  $W_2$

Also, we see that  $0 = (0, 0) \in W_1$  and  $0 = (0, -0) \in W_2$ . So  $W_1, W_2$  satisfy the conditions in Theorem 5.2 so they are subspaces of  $V = \mathbb{R}^2$

We want to show that:

(a)  $W_1 \cap W_2 = \{0\}$

(b)  $W_1 + W_2 = V$

To prove (a) assume  $v = (v_1, v_2) \in \mathbb{R}^2$  is both in  $W_1$  and in  $W_2$ . Since  $(V_1, V_2) \in W_1$ , we must have  $V_2 = -V_1$ . Thus  $V_2 = V_1, V_2 = -V_1$  so  $V_1 = -V_1$ , Thus  $2V_1 = 0$  which for  $V_1 \in \mathbb{R}$  implies  $V_1 = 0$ . So  $V_2 = V_1 = 0$  as well. To prove (b), let  $v = (v_1, v_2) \in \mathbb{R}^2$  we want to find  $(x, x) \in W_1, (y, -y) \in W_2$  such that  $(x, x) + (y, -y) = (V_1, V_2)$ . This means  $(x + y, x - y) = (V_1, V_2)$

So  $x + y = V_1, x - y = V_2$

Thus, to find  $x, y$  satisfying these two conditions we need to solve this system of two equations with two unknown  $x, y$  in real numbers.

From the 2nd equation, we get  $x = y + V_2$  and replacing in the 1st equation

$(y + V_2) + y = V_1$

So  $2y = V_1 - V_2, y = \frac{V_1 - V_2}{2}$

and so  $x = y + V_2 = \frac{V_1 - V_2}{2} + V_2 = \frac{V_1 + V_2}{2}$

Thus  $(V_1, V_2) = (\frac{V_1 + V_2}{2}, \frac{V_1 + V_2}{2}) + (\frac{V_1 - V_2}{2}, \frac{V_2 - V_1}{2})$  and so we denoted that  $W_1 + W_2 = V$

# Linear Combination of Vectors

## 6.4 Definition

Let  $V$  be a vector space over a field  $F$  and  $S \subseteq V$  a nonempty subset of  $V$ . A vector  $v \in V$  is called a linear combination of vectors in  $S$ . If there exist a finite number of vectors  $u_1, \dots, u_n \in S$  and scalars  $c_1, \dots, c_n \in F$  such that  $v = c_1u_1 + c_2u_2 + \dots + c_nu_n$

We then also say that  $v$  is a linear combination of  $u_1, \dots, u_n$ . The scalars  $c_1, \dots, c_n$  are called the coefficients of the linear combination. because  $O_v = O_F * V, \forall V \in S \neq 0$

**Note:** The vector  $0 \in V$  is a linear combination of any  $S \subseteq V$

## 6.5 Example

Denote by  $V$  the set of polynomial of degree at most  $n$  with coefficients in  $\mathbb{R}$ , i.e. expressions of the form  $P(X) = a_0 + a_1X + a_2X^2 + \dots + a_nX^n$

where  $a_0, a_1, \dots, a_n \in \mathbb{R}$ , with the usual addition and multiplication by scalars in  $\mathbb{R}$ :

$(a_0 + a_1X + \dots + a_nX^n) + (b_0 + b_1X + \dots + b_nX^n) = (a_0 + b_0) + (a_1 + b_1)X + \dots + (a_n + b_n)X^n$  and  $c(a_0 + a_1X + \dots + a_nX^n) = ca_0 + ca_1X + \dots + ca_nX^n$

Show that any polynomial in  $V$  is a linear combination of the “monomials”  $1, X, X^2, \dots, X^n$  indeed, if  $P(X) = a_0 + a_1X + \dots + a_nX^n \in V$  then  $a_0, a_1, \dots, a_n \in \mathbb{R}$  are scalars and we have  $P(X) = a_0 * 1 + a_1 * X + \dots + a_nX^n = a_0u_0 + a_1u_1 + \dots + a_nu_n$

## 6.6 Definition

If  $V$  is a vector space and  $S \neq \emptyset \subseteq V$ , Then the span of  $S$ , denoted  $\text{span}(S)$ , is the set of all linear combinations of vectors in  $S$ .

i.e.  $\text{span}(S) = \{\sum_{i=1}^n c_i u_i : u_1, \dots, u_n \in S, c_1, \dots, c_n \in F, n \geq 1\}$

## 6.7 Example

If we take  $V = \mathbb{R}^3$  and  $S = \{(1, 0, 0), (0, 1, 0)\}$  Then  $\text{span}(S)$  is the set of all vectors in  $\mathbb{R}^3$  of the form  $au_1 + bu_2 = a(1, 0, 0) + b(0, 1, 0) = (a, 0, 0) + (0, b, 0) = (a, b, 0)$ , with  $a, b \in \mathbb{R}$  arbitrary scalars in  $\mathbb{R}$

Thus  $\text{span}(S) = \{(a, b, 0) : a, b \in \mathbb{R}\}$  which we recognize to be the  $xy$  plane in the  $xyz$  3-dimensional Euclidean space.

## 6.8 Example

If we take  $V$  to be the vector space of polynomials in  $X$  of degree  $\leq n$  with coefficients in  $\mathbb{R}$  as in Example 6.5 and we let  $S = \{1, X, X^2, \dots, X^n\}$  then  $\text{span}(S) = V$

## 6.9 Example

Given a Field  $F$  and denotes by  $F[X]$  the set of all polynomials in “undeterminate”  $X$  over the field  $F$ , i.e. expressions of the form  $P(X) = a_0 + a_1X + \dots + a_nX^n$  for some  $n \geq 0$  and  $a_0, a_1, \dots, a_n \in F$  with the “usual” addition and scalar multiplication

The degree of  $P(X)$  is the largest  $n$  such that  $a_n \neq 0$ .

(a) show that if  $S = \{1, X, X^2, \dots\}$  then  $\text{span}(S) = F[X]$ .

(b) Denote  $F_{\text{odd}}[X]$  the set of all polynomials with coefficients in  $F$  that have only odd coefficients possibly  $\neq 0$  and by  $F_{\text{even}}[X]$  the set of all polynomial with coefficient in  $F$  that have only even coefficients possibly  $\neq 0$ , i.e.  $F_{\text{odd}}[X] = \{P(X) \in F[X] : P(X) = a_1X + a_3X^3 + a_5X^5 + \dots + a_{2n+1}X^{2n+1}, a_1, a_3, \dots, a_{2n+1} \in F, n \geq 0\}$

$F_{\text{even}}[X] = \{P(X) \in F[X] : P(X) = a_0 + a_2X^2 + a_4X^4 + \dots + a_{2n}X^{2n}, a_0, a_2, a_4, \dots, a_{2n} \in F, n \geq 0\}$

Show that  $W_1 = F_{\text{odd}}[X], W_2 = F_{\text{even}}[X]$  are subspaces of  $F[X]$  and that  $F[X] = W_1 + W_2$

**Proof:**(exercise)

(a) As long as set  $S$  contains all polynomials of  $X$  which degree  $\leq n$ , and  $F[X]$  is the set of all polynomials in “undeterminate”  $X$  over the field  $F$ .  $F[X]$  can be written as a linear combination of  $S$ , where  $F[X] = aS$ . where  $a = \{a_1, a_2, \dots, a_n\} \in F$

(b) For  $F_{\text{odd}}[X]$ , assume  $A(X)$  and  $B(X)$  are subspace of  $F_{\text{odd}}[X]$  where  $A(X) = \{ca_1X + ca_3X^3 + ca_5X^5 + \dots\}$  and  $B(X) = \{da_1X + da_3X^3 + da_5X^5 + \dots\}$  and  $A(X) + B(X) = (c+d)a_1X + (c+d)a_3X^3 + (c+d)a_5X^5 + \dots \in F_{\text{odd}}[X]$ . Also  $rA(X) = rca_1X + rca_3X^3 + rca_5X^5 + \dots \in F_{\text{odd}}[X]$ . This denote that  $W_1$  is a subspace of  $F[X]$

Similar with  $W_2$

To show that a set  $V$  is equal to the sum of two subspaces  $W_1$  and  $W_2$ , we must show that every element of  $V$  can be written as the sum of an element of  $W_1$  and an element of  $W_2$ . Let  $P(X)$  be any polynomial in  $F[X]$ . We can write  $P(X)$  as  $P(X) = P_o(X) + P_e(X)$  where  $P_o(X) = a_1X + a_3X^3 + \dots + a_{2n+1}X^{2n+1}$  and  $P_e(X) = a_0 + a_2X^2 + \dots + a_{2n}X^{2n}$ .  $P_o(X)$  has only odd coefficients, and  $P_e(X)$  has only even coefficients. Thus,  $P_o(X)$  is in  $W_1 = F_{\text{odd}}[X]$ , and  $P_e(X)$  is in  $W_2 = F_{\text{even}}[X]$ . Therefore,  $P(X) = P_o(X) + P_e(X)$  can be written as the sum of an element of  $W_1$  and an element of  $W_2$ , and thus every element of  $F[X]$  can be written as such a sum. Therefore,  $F[X] = W_1 + W_2$

## 6.10 Theorem

The span of any subset  $S$  of a vector space  $V$  is a subspace of  $V$ . Any subspace of  $V$  that contains  $S$  must contain  $\text{span}(S)$

(I.e. if  $W \in V$  subspace with  $S \in W$  then  $\text{span}(S) \in W$ )

**Proof:**

We have to prove that if  $x, y \in \text{span}(S)$  then  $x + y \in \text{span}(S)$  and  $cx \in \text{span}(S), \forall c \in F$

Since  $x, y \in \text{span}(S)$ , there exist  $u_1, \dots, u_n \in S, v_1, \dots, v_n \in S$  and scalars  $a_1, \dots, a_n \in F, b_1, \dots, b_n \in F$  such that

$$x = a_1u_1 + \dots + a_nu_n$$

$$y = b_1v_1 + \dots + b_nv_n$$

$$\text{But then } x + y = a_1u_1 + \dots + a_nu_n + b_1v_1 + \dots + b_nv_n$$

So  $x + y$  is itself a linear combination of  $u_1, \dots, u_n, v_1, \dots, v_n \in S$ , Thus  $x + y \in \text{span}(S)$ .

Also,  $cx = c(a_1u_1 + \dots + a_nu_n) = (ca_1)u_1 + (ca_2)u_2 + \dots + (ca_n)u_n \in \text{span}(S)$

For the last part of Theom: if  $W \in V$  is a subspace that contains  $S$  and  $W \in \text{span}(S)$ , then there exist  $u_1, \dots, u_m \in S$  and  $a_1, \dots, a_m \in F$  such that  $W = a_1u_1 + \dots + a_mu_m$ . Since  $W$  is a subspace and

$u_1 + \dots + u_m \in S \in W$ , we have  $a_1 u_1 + \dots + a_m u_m \in W$ . Thus  $w \in W$  showing that  $\text{span}(S) \in W$ .

### 6.11 Definition

We say that a subset  $S$  of a vector space  $V$  generates (or spans)  $V$  if  $\text{span}(S) = V$ .

### 6.12 Example

1). If we take  $V = \mathbb{R}^2$  like in Exercise 6.3 then  $S = \{(1,1), (1,-1)\}$  generate (span)  $V$ , because we showed in that exercise that any  $v \in V$  is of the form  $v = au_1 + bu_2$  for some scalars  $a, b \in \mathbb{R}$ . 2). If we take  $V = F[X]$  and  $S = \{1, X, X^2, \dots\}$  then  $\text{span}(S) = F[X]$

### 6.13 Exercise

Show that the matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  generate  $M_{2 \times 2}(F)$

Solution. Any matrix in  $M_{2 \times 2}(F)$  is of the form  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $a, b, c, d \in F$ . But then  $a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . So  $A$  is indeed a linear combination of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$