Math115A 1/20 notes

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2023-01-21

Sub spaces of vector spaces

5.1 Definition

A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Note that V itself is a subspace of V. Also, the set $\{0\}$, consisting of the vector $0 \in V$, is a subspace of V it is called the zero subspace
If W is a subset of a vector space V over a field F, and we are as lead to check if W is a subspace, then we don't want to check that all conditions VS1-VS8 are satisfied: indeed we do know addition & scalar multiplication check the multiplication check the properties for all vectors in V, so all we need is in fact to check that addition of $x, y \in W$ abd scalar multiplication ax for $x \in W$ keep us within the set W.

5.2 Theorem

Let V be a vector space over field F and $W < V$ a subset. Then W is a subspace of V if and only if the following conditions field.
(a) $0 \in W$ (i.e. the zero vector of V belong to W)
(b) if $x, y \in W$ then $x + y \in W$(i.e. the addition of any two vectors in W stays in W)
(c) if $x \in W$ and $c \in F$ then $cx \in W$(i.e. the multiplication by any scalor of a vector in x stays in W)

Proof
if W is a subspace of V then by definition 5.1 it means that W with the operations of addition and scalar multiplication it "inherits" from V is a vector space. Thus (b)&(c) are satisfied. Also, W has by the definition of vector space a "zero element", say $0' \in W$ satisfying $x + 0' = x, \forall x \in W$. But we also have $x + 0 = x$ where 0 is here the zero element in V By Theom 4.7(a) thus implies $0'0 = 0$, so we have (a) as well "<=" if (a),(b),(c) hold.
Then W with the addition and scalar multiplication of its elements viewed as elements in V satisfies VS1, VS2, and VS5-8 because we already know all these properties are satisfied in V.
All we need to show is that W has a zero vector (thus is VS3) and any $x \in W$ has an additive inverse in W (an "opposite")
but condition (a) tells us that $0 \in W$ (no zero element of V belongs to W), and this element does satisfy $x + 0 = x, \forall x \in W$(because this holds that in fact for all elements x in V in particular for all elements in W)
Also, by condition (c) we love that $\forall x \in W$, the element $y = (-1)x$ is the opposite of x in V. So $x \in W$ does have an additive inverse in W(what we also called an "opposite"). So VS3 & VS4 are satisfied as well.

5.3 Definitions

Given a matrix $A = (A_{ij}) \in M_{m\times n}(F)$ its transpose is the $n \times m$ matrix $A^t \in M_{n\times m}(F)$obtained from A by interchouging the row with the columns, that is $(A^t)_{ij} = A_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$

For example:/newline 1. if $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -4 & 0 \end{pmatrix} \in M_{3\times 2}(R)$

then $A^t \in M_{2\times 3}(R)$ is $A^t = \begin{pmatrix} 0 & 2 & -4 \\ -1 & 3 & 0 \end{pmatrix}$

2. $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}^t = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -3 \end{pmatrix}^t = \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 2 & -3 \end{pmatrix}$

   A symmetric matrix A is a matrix such that $A^t = A$
   Note that such matrices necessary have same # of rows and colums
   i.e. $A \in M_{n\times m}(F)$
   Examples:
   1.$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ so it is symmetric

4. Note that $0 = \begin{pmatrix} 0 & ... & 0 \\ | & & | \\ 2 & ... & 0 \end{pmatrix} \in M_{m\times n}(F)$

5.4 Exercise

show that the set of symmetric matrices in $M_{m\times n}$ is a subspace of $M_{m\times n}(F)$
Solution
We need to check (b)&(c) of theom 5.2(we already noticed that the 0 matrix in $M_{m\times n}(F)$ is symmetric as we have (a) of theom 5.2)

If $A, B \in M_{m\times n}(F)$ are symmetric then $A = (A_{ij})$ and $B = (B_{ij})$ satisfy $A = A^t$, $B = B^t$ so $A_{ij} = A_{ji}, B_{ij} = B_{ji}$ for all $1 \leq i, j \leq n$
But then $(A+B)_{ij} = A_{ij} + B_{ij}$ so $((A+B)^t)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A_{ij} + B_{ij}$, Showing that $A+B$ is symmetric
Similarly, if $c \in F$ is a scalar and $A = A^t$ Then $(cA)_{ij} = cA_{ij}, ((cA)^t)_{ij} = (cA)_{ji} = cA_{ij}$ so cA is symmetric

5.5 Definitions

A matrix $A \in M_{n\times m} \in F$ is called a diagonal matrix
If $A_{ij} = 0, \forall i \neq j$, i.e. if all entries off the diagonal of A are equal to 0

Example:/newline $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_{3\times 3}(R)$ is a diagonal matrix

$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in M_{4\times4}(R)$ is a diagonal matrix
Note the 0 matrix in $M_{m\times n}(F)$ is a diagonal matrix, $\forall n$

Exercise
Show that the set of diagonal matrices $D_{m\times n}(F)$ in $M_{m\times n}(F)$ is a subspace.
So indeed, we noticed that $0 \in D_{m\times n}(F)$
Also the sum of two diagnal matrices $A.B \in D_{m\times n}(F)$ satisfies $(A+B)_{ij} = A_{ij} + B_{ij} = 0$if $i \neq j$, because both $A_{ij} = 0, B_{ij} = 0$ if$i \neq j$ (Since A, B are diagonal)
Similarity if $A \in D_{n\times m}(F)$ and $c \in F$ then $(cA)_{ij} = cA_{ij} = 0$ if $i \neq j$ because $A_{ij} = 0, \forall i \neq j$

5.6 Definition

Given a $n \times m$ matrix $A \in M_{m\times n}(F)$, the trace of A, denoted tr(A) is the sum of the diagonal entries of A, i.e.
$tr(A) = \sum_{i=1}^{n} A_{ii} \in F$
example if $A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & -2 & 0 \\ 1 & 2 & 2 \end{pmatrix}$ then $tr(A) = -1 + (-2) + 2 = -1$

5.7 Exercise

(a) show that the set of matrices $A \in M_{m\times n}(F)$ with $tr(A) = 0$ is a subspace of $M_{m\times n}(F)$
Solution:
Closure under addition: If $A, B \in M_{m\times n}(F)$ with $tr(A) = tr(B) = 0$, then $tr(A+B) = tr(A)+tr(B) = 0 + 0 = 0$. Therefore, the set is closed under matrix addition.
Closure under scalar multiplication: If $A \in M_{m\times n}(F)$ with $tr(A) = 0$ and $k \in F$, then $tr(kA) = ktr(A) = k0 = 0$. Therefore, the set is closed under scalar multiplication.
Containing the zero vector: The zero matrix, denoted by O, is an mxn matrix whose entries are all 0. And since the trace of the zero matrix is the sum of its diagonal entries, which are all zero, we have $tr(O) = 0$. Therefore, the zero matrix is an element of the set.

(b) Show that the set of matrices $A \in M_{m\times n}(F)$ with $tr(A) = 1$ is not a subspace of $M_{n\times m}(F)$
Solution:
Closure under addition: if $A, B \in M_{m\times n}(F)$ with $tr(A) = tr(B) = 1$, Then $tr(A+B) = tr(A)+tr(B) = 1 + 1 = 2$. However $1 \neq 2$. Failed by condition of closure under condition.

(c) Show that the set of matrices $A \in M_{m\times n}(F)$ with $tr(A) \leq 0$ is not a subspace

Proof:
Closure under addition: if $A, B \in M_{m\times n}(F)$ with $tr(A) = tr(B) \leq 0$, Then $tr(A+B) = tr(A)+tr(B) \leq 0$. Therefore, the set is closed under matrix addition.
Closure under multiplication: if $A \in M_{m\times n}(F)$ with $tr(A) \leq 0$ and $k \in F$, Then $ktr(A)$ will less or equal to zero when k is greater or equal than zero. Not all condition. Therefore, failed by closure under multiplication.

5.8 Theorem

If W_1, W_2 are subspace of a vector space V over a field F, then $W_1 \cap W_2$ is a subspace of V as well more generally, if $W_i, i \in I$, is a collection of subspace of V then $\cap_{i \in I} W_i$ is a subspace of V.

Proof
Since both W_1, W_2 are subspace, we have $0 \in W_1, 0 \in W_2$ so $0 \in W_1 \in W_2$
Also, if $x, y \in W_1 \cap W_2$
Then $x, y \in W_1$, so $x + y \in W_1$ and $x, y \in W_2$, so $x + y \in W_2$
Thus, $x + y \in W_1 \cap W_2$
Similarly, if$x \in W_1 \cap W_2$ and $c \in F$
Then $x \in W_1$, so $cx \in W_1$ and $x \in W_2$, so $cx \in W_2$
Thus $cx \in W_1 \cap W_2$, and we showed that (a),(b) and (c) of theom 5.2 are all satisfied for $W_1 \cap W_2$

General case is similar