115A, Winter 2023

Lecture 18

Wed, Feb 22

• Last time we proved (Thm 17.6) that if $T \in \mathcal{L}(V, W)$, with V, W fin dim vector spaces, $\dim V = \dim W$, and β, γ ordered basis for V, W then T invertible iff $[T]_\beta^\gamma$ is invertible matrix (in $M_{n\times n}(F)$ where $n = \dim V = \dim W$)

and if this is the case then $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$

• So in particular we have:

<u>18.1 Corollary</u> if V fin. dim. (say $\dim V = n$) vector space with β ordered basis then $T \in \mathcal{L}(V)$ invertible iff $[T]_\beta \in M_{n\times n}(F)$ invertible matrix

and if this is the case then $[T^{-1}]_\beta = ([T]_\beta)^{-1}$

18.2 Corollary. Let $A \in M_{n\times n}(A)$.
Then A inventible iff $L_A \in \mathcal{L}(F^n)$
is inventible and we have
$(L_A)^{-1} = L_{A^{-1}}$

18.3. Definition Let V, W be
vector spaces over same field F. We
say that V is isomorphic to W if
there exists $T \in \mathcal{L}(V, W)$ that's invertible

A linear transformation $T: V \to W$
that's inventible called an isomorphism
from V to W (or between V and W)

18.4. Example Let $V = P_3(R)$
the space of polynomials of deg. ≤ 3
with coefficients in R. Let $W = R^4$

Then $P_3(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$

Indeed, we can define $T : P_3(\mathbb{R}) \to \mathbb{R}^4$

by $T(1) = (1,0,0,0)$

$T(X) = (0,1,0,0)$

$T(X^2) = (0,0,1,0)$

$T(X^3) = (0,0,0,1)$

Then extend T by linearity,

$$T\left(\sum_{i=1}^{4} c_i v_i\right) = \sum_{i=1}^{4} c_i w_i$$

where $\{v_1, v_2, v_3, v_4\} = \{1, X, X^2, X^3\} \subset V$

$\{w_1, w_2, w_3, w_4\} = \{(1,0,0,0), \ldots, (0,0,0,1)\} \subset W$

are the above standard ordered basis.

18.5. Theorem Let V, W be fin. dim. vector spaces $/F$. Then V, W are isomorphic iff $\dim V = \dim W$.

$\boxed{\text{Pf}}$ "$\Rightarrow$" If V, W isomorphic and $T : V \to W$ is an isomorphism with $\beta = \{v_1, \dots, v_n\} \subset V$ an ordered basis for V then $\gamma = \{T(v_1), \dots, T(v_n)\}$ span W (because T is onto) and they are lin. ind. (because T is one to one)

So γ is an basis for W, then

$\dim W$ = number of elements in γ

which is equal to $n = \dim V$.

"$\Leftarrow$" Assume $\dim V = \dim W = n$.

Let $\beta = \{v_1, \dots, v_n\}$ ordered basis for V

$\gamma = \{w_1, \dots, w_n\}$ —— " —— W

Define $T: V \to W$ by letting

$$T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i w_i, \quad c_1, \dots, c_n \in F$$

Since any $x \in V$ has a unique linear span as $x = \sum_{i=1}^{n} c_i v_i$

this is well defined function

and it is linear, it is one to one

because if $T\left(\sum_{i=1}^{n} c_i v_i\right) = 0_W = \sum_{i=1}^{n} c_i w_i$

then we must have $c_1 = c_2 = \dots = c_n = 0_F$

(because $w_1, \dots, w_n$ are lin. ind. in W)

T is also onto, because any $y \in W$ is of the form
$$y = \sum_{i=1}^{n} c_i w_i \quad \text{for some } c_1, \dots, c_n \in F$$
(because $\{w_1, \dots, w_n\}$ is basis for W)
and then $x = \sum_{i=1}^{n} c_i v_i \in V$ satisfies
$$T(x) = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i w_i = y.$$

18.6 Corollary. Let V be a vector space $/F$. Then V is isomorphic to F^n iff $\dim V = n$

We saw that to each linear transf between fin. dim. vec. spaces with ordered basis one can associate a matrix. We next show that this "matrix representation" map is an isomorphism of vec. spaces.

18.8. Theorem. Let V, W be fin. dim. vector spaces of dimension $\dim V = n$, $\dim W = m$, and let $\beta \subset V$, $\gamma \subset W$ be ordered bases. Then the map (function)

$$\Phi_\beta^\gamma : \mathcal{L}(V, W) \to M_{m \times n}(F)$$

defined by $\Phi_\beta^\gamma(T) = [T]_\beta^\gamma$, $\forall T \in \mathcal{L}(V,W)$

is an isomorphism.

Proof. We already proved that Φ_β^γ is linear in Cor 14.6.

To prove that Φ_β^γ is an isomorphism we need to show it is onto and 1 to 1. To this end it is sufficient to show that $\forall A \in M_{m \times n}(F)$ there exists a unique $T \in \mathcal{L}(V,W)$ such that $\Phi_\alpha^\beta(T) = A$.

Let $A = (A_{ij}) \in M_{m\times n}(F)$ be the entries of the matrix A.

Let $\beta = \{v_1, \ldots, v_n\} < V$

$\gamma = \{w_1, \ldots, w_m\} < W$

be ordered bases.

For each $j = 1, 2, \ldots, n$ denote

$$w'_j = \sum_{i=1}^{m} A_{ij} w_i \in W$$

We can then apply Theorem 12.6 to conclude that there exists a unique linear $T : V \to W$ such that $T(v_j) = w'_j$, $j = 1, 2, \ldots, n$

i.e. $T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$, $j = 1, \ldots, n$

But this means $\Phi_\beta^\gamma(T) = [T]_\beta^\gamma$

18.9. Corollary. Let V, W be fin. dim. vec. spaces $/F$ with $\dim V = n$, $\dim W = m$. Then $\mathcal{L}(V, W)$ has dimension $m \cdot n$.

Pf We showed that $\mathcal{L}(V,W)$ is isomorphic to $M_{m \times n}(F)$ (18.8) and we know $M_{m \times n}(F)$ has dimension $m \cdot n$. Thus, by Cor 18.6. ◻

18.10 Definition. Let V be fin. dim vector space $/F$ and

$\beta = \{v_1, \ldots, v_n\}$ ordered basis.

The standard representation of V with respect to β is the isomorphism $\Phi_\beta : V \to F^n$ defined by

$$\Phi_\beta(x) = [x]_\beta$$

i.e., if $x = \sum_{i=1}^{n} c_i v_i$ then $\Phi_\beta(x) = (c_1, \ldots, c_n) \in F^n$

as a line vector or as a column vector

18.11. Theorem. Let V be a fin. dim. vector space $/F$ and β, β' two ordered bases for V

Let $Q = [I_V]_{\beta'}^{\beta}$ be the matrix representation of the identity linear map on V (so $I_V(x) = x$ $\forall x \in V$) from basis β' to basis β. Then:

(a). Q is an invertible matrix in $M_{n \times n}(F)$ (where $n = \dim(V)$)

(b). If $x \in V$ then
$$[x]_\beta = Q[x]_{\beta'}$$

Proof. (a). Since I_V is invertible linear transformation, $Q = [I_V]_{\beta'}^{\beta}$ is an invertible matrix by Thm. 17.6

(b). If $x \in V$ then by Thm 16.1

we have:

$$[x]_\beta = [I_V(x)]_\beta$$

$$= [I_V]_{\beta'}^{\beta} [x]_{\beta'} = Q[x]_{\beta'}$$

$\boxtimes$

18.12. Definition / Terminology

If we give two ordered bases β, β' for the same fin. dim vector space V then the matrix $Q = [I_V]_{\beta'}^{\beta}$ in Thm. 18.11 is called the change of coordinate matrix.

We also say Q changes β'-coordinates to β-coordinates.

• Note that if $\beta = \{v_1, \dots, v_n\}$ $\beta' = \{v_1', \dots, v_n'\}$ and we denote $Q = (Q_{ij})_{i,j}$ the entries of the matrix Q then

$$v_j = \sum_{i=1}^{n} Q_{ij} v_i' \quad , \quad j = 1, 2, \dots, n$$

• Notice that if Q changes β' to β then Q^{-1} changes β to β'

18.13 Theorem. Let V be fin. dim. vec. space with ordered bases β, β'. Let $Q = [I_V]_{\beta'}^{\beta}$ be the matrix that changes coordinates from β' to β as in 18.11, 18.12.

Then for any $T \in \mathcal{L}(V)$ we have $[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$\boxed{\text{Pf}}$, exercise: use that

$$T I_V = I_V T = T$$

and so $[T I_V]^{\beta}_{\beta'}$

$$= [T]^\beta_\beta [I_V]^\beta_{\beta'} = [T]_\beta Q$$

while $[I_V T]^\beta_{\beta'}$

$$= [I_V]^\beta_{\beta'} [T]^{\beta'}_{\beta'} = Q[T]_{\beta'}$$

so we get $[T]_\beta Q = Q[T]_{\beta'}$

and we then multiply from left by Q^{-1}

$\boxtimes$

Applications
& exercises on blackboard if time permits