HW8

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Section 2.5: 6; 10(a)

Section 4.4: 2; 3(a),(b)(c)(d); 4(a)(b)c)(d);

Section 5.1: 3(a); 4(a); 10; 11; 13(a).

2.5

6.

(a) **Answer:** $Q^{-1}AQ = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ -1 & -4 \end{pmatrix}$

(b) **Answer:** $Q^{-1}AQ = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

(c) **Answer:** $Q^{-1}AQ = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$

(d) **Answer:** $Q^{-1}AQ = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$

10(a). Let A and B be $n \times n$ matrices such that $AB = I_n$. Use Exercise 9 to conclude that A and B are invertible

Proof:
Since $AB = I_n$ (the identity matrix of size n), we can conclude that both A and B are square matrices of size n. Therefore, by Exercise 9, we know that A and B are invertible and their inverses are given by $B = A^{-1}$ and $A = B^{-1}$.

To see why this is the case, we can use the fact that if $AB = I_n$, then A^{-1} exists and is given by $A^{-1} = B$. Similarly, since $AB = I_n$, we know that B^{-1} exists and is given by $B^{-1} = A$. Therefore, both A and B are invertible with inverses given by $B = A^{-1}$ and $A = B^{-1}$.

Thus, we have shown that if $AB = I_n$ for two $n \times n$ matrices A and B, then A and B are invertible.

4.4

2.

(a) **Answer:** $det(A) = 22$

(b) **Answer:** $det(A) =$ -29

(c) **Answer:** $det(A) =$ 2-4i

(d) **Answer:** $det(A) =$ 6i-24

3

(a) **Answer:** $det(A) =$ -12

(b) **Answer:** $det(A) =$ -13

(c) **Answer:** $det(A) =$ -12

(d) **Answer:** $det(A) =$ -13

4

(a) **Answer:** $det(A) = 0$

(b) **Answer:** $det(A) = 36$

(c) **Answer:** $det(A) =$ -49

(d) **Answer:** $det(A) = 10$

5.1

3(a) For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

$V = R^2, T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta =\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\}$

Answer:
$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$
$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$
$\begin{pmatrix} -2 \\ -3 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$\begin{pmatrix} -2 \\ -4 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$[T]_\beta = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$

4(a)

i)
$det(A - \lambda I) = (1-\lambda)(2-\lambda) - 6 = (\lambda - 4)(\lambda + 1)$
Eigenvalue = 1&4

ii)
$(A - 4I) = \begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$3x_1 = 2x_2, x_1 = \frac{2}{3}x_2, x = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$

$(A + I) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x_1 = -x_2, x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

iii)

Linearly independent, basis = $\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \}$

iv)

$Q = \begin{pmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

10. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Proof:
When M is a 1×1 matrix, it has only one diagonal entry, say m_{11}. The characteristic equation of M is given by $|M - \lambda I| = m_{11} - \lambda = 0$, which has the unique solution $\lambda = m_{11}$. Therefore, the only eigenvalue of M is its diagonal entry.
Let λ be an eigenvalue of M and let v be a corresponding eigenvector. Then we have: $Mv = \lambda v$
Since M is upper triangular, the n-th entry of the vector Mv is given by the n-th diagonal entry of M multiplied by the n-th entry of v. Similarly, the n-th entry of the vector λv is given by λ multiplied by the n-th entry of v. Therefore, we have: $m_{nn}v_n + \Sigma_{i=1}^{n-1} m_{ni}v_i = \lambda v_n$
Since v is an eigenvector, it is nonzero, so $v_n \neq 0$. Therefore, we can solve for λ:$\lambda = m_{nn} - \frac{Sigma_{i=1}^{n-1} m_{ni}v_i}{v_n}$
Note that the right-hand side of this equation depends only on the diagonal entries and the entries in the first $n-1$ rows of M, and the entries of v in the first $n-1$ rows. In other words, the value of λ depends only on the upper left $(n-1) \times (n-1)$ submatrix of M and the first $n-1$ entries of v. By the induction hypothesis, the eigenvalues of the upper left $(n-1) \times (n-1)$ submatrix of M are its diagonal entries. Therefore, λ must be equal to one of the diagonal entries of M.
Since λ was an arbitrary eigenvalue of M, we have shown that all eigenvalues of M are diagonal entries of M. This completes the proof by induction.

11.Let V be a finite-dimensional vector space, and let λ be any scalar.

(a)For any ordered basis β for V, prove that $[\lambda I_V]_\beta = \lambda I$. Proof:
Let $\beta = v_1, v_2, \ldots, v_n$ be an ordered basis for V, where n is the dimension of V. Then the matrix of the identity transformation I_V with respect to β is the $n \times n$ identity matrix I_n. Therefore, the matrix of λI_V with respect to β is the $n \times n$ matrix $[\lambda I_n]$.

To see this, note that for any vector $v \in V$, we have: $(\lambda I_V)(v) = a_1(\lambda v_1) + a_2(\lambda v_2) + ...a_n(\lambda v_n)$
where $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ is the coordinate vector of v with respect to β. Therefore, the matrix of (λI_V) with respect to β is the matrix $[\lambda I_n]$, which has λ as its diagonal entries and 0 as its off-diagonal entries.
Therefore, we have shown that $[\lambda I_V]_\beta = [\lambda I_n]$, as desired.

(b) Compute the characteristic polynomial of λI_V Proof:
$|\lambda I_V - \lambda I| = |(\lambda - \lambda)I_V| = 0$ where n is the dimension of V. Therefore, the characteristic polynomial of λI_V is λ^n.

Note that the fact that the characteristic polynomial is λ^n implies that the only eigenvalue of λI_V is 0 (with algebraic multiplicity n), since the roots of the characteristic polynomial are precisely the eigenvalues of the matrix. In other words, every vector in V is an eigenvector of λI_V with eigenvalue 0.

(c) Show that λI_V is diagonalizable and has only one eigenvalue. Proof:
We have already shown in the previous question that the characteristic polynomial of λI_V is λ^n, which has only one root, namely $\lambda = 0$. Therefore, the only eigenvalue of λI_V is 0, with algebraic multiplicity n.

To show that λI_V is diagonalizable, we need to show that its geometric multiplicity (i.e., the dimension of its eigenspace corresponding to the eigenvalue 0) is also n.

Since every vector in V is an eigenvector of λI_V with eigenvalue 0, the eigenspace corresponding to the eigenvalue 0 is the entire vector space V. Therefore, the geometric multiplicity of 0 as an eigenvalue of λI_V is n, which is equal to its algebraic multiplicity.

Since the algebraic and geometric multiplicities of every eigenvalue of a matrix are equal if and only if the matrix is diagonalizable, we conclude that λI_V is diagonalizable. Moreover, since λI_V has only one eigenvalue, namely 0, it follows that every matrix that is a scalar multiple of the identity matrix (with respect to some basis) is diagonalizable and has only one eigenvalue, which is equal to the scalar multiple.

13(a) Prove that similar matrices have the same characteristic polynomial.

Proof:

Let A and B be two $n \times n$ matrices that are similar, meaning there exists an invertible matrix P such that $B = P^{-1}AP$.

The characteristic polynomial of A is defined as $\det(A - \lambda I)$, where I is the identity matrix and λ is a scalar variable. Let $C = A - \lambda I$.

Then, $B = P^{-1}AP$ implies that $PBP^{-1} = A$, so we can also write $B = PAP^{-1}$.

Substituting this expression for A into our definition of C, we get:

$$C = B - \lambda I = PAP^{-1} - \lambda I = P(A - \lambda I)P^{-1} = PCP^{-1}$$

Now, we take the determinant of both sides:

$$\det(C) = \det(PCP^{-1})$$

Using the property that $\det(AB) = \det(A)\det(B)$ for any matrices A and B, we can simplify this expression as:

$$\det(C) = \det(P)\det(C)\det(P^{-1}) = \det(P)\det(P^{-1})\det(C)$$

Since P is invertible, we know that $\det(P)\det(P^{-1}) = \det(I) = 1$, so we can simplify further:

$$\det(C) = \det(P)\det(P^{-1})\det(C) = \det(P^{-1})\det(P)\det(C) = \det(B - \lambda I)$$

Therefore, we have shown that $\det(A - \lambda I) = \det(B - \lambda I)$, which means that A and B have the same characteristic polynomial.