115A, Winter 2023

Lecture 13

Wed, Feb 8

from Sec 2.1:

- Go over exercises ① (some of them), on blackboard

④ $T : M_{2\times 3}(F) \to M_{2\times 2}(F)$

$$T\begin{pmatrix} a_{11} & \cdots \\ \vdots & \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

basis for $N(T)$, $R(T)$ compute nullity & rank then check dim. function

$\ker(T)$ $\qquad 2a_{11} = a_{12} \Rightarrow a_{11} = \frac{a_{12}}{2}$

$\qquad a_{13} + 2a_{12} = 0 \Rightarrow a_{13} = -2a_{12}$

$$= \left\{ \begin{pmatrix} a_{12}/2 & a_{12} & -2a_{12} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : a_{12}, a_{21}, a_{22}, a_{23} \in \mathbb{R} \right\}$$

etc. ...

⑫ . —

## Matrix representation of a linear transformation (§2.2)

- Up to now, we considered basis of a fin. dim. vector space $V$ as a set of elements $S = \{v_1, \dots, v_n\}$, where the order in which we take the vectors $v_1, \dots, v_n$ didn't matter. But we now want to take into consideration the order.

**13.1 Definition.** Let $V$ be a fin. dim. vector space. An ordered basis for $V$ is a basis of $V$ with a specified order of its elements, i.e. it is a finite sequence $v_1, \dots, v_n$ of linearly independent vectors in $V$ that span $V$.

We will still write it $\{v_1, \dots, v_n\}$ but we will specify ordered basis

**13.2 Example** in $V = \mathbb{R}^n$

we usually take basis $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $\dots$, $e_n = (0, 0, \dots, 0, 1)$

We then say $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$

As a set, we can take it to be $\{e_2, e_1, e_4, e_3, e_5, \dots, e_n\}$ as well, or any other re-ordering. It is still the same set, the same basis.

But as ordered basis, they are

everytime different, so for instance in $\mathbb{R}^3$ we have ordered basis $\{e_1, e_2, e_3\}$ $\{e_2, e_1, e_3\}$ $\{e_3, e_2, e_1\}$ ... etc which are different as ordered basis.

Obs. You can use the alternate notation $[v_1, \dots, v_n]$ for an ordered basis, to distinguish it from $\{v_1, \dots, v_n\}$ as a set.

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**13.3 Definition** Let $\beta = \{v_1, \dots, v_n\}$ be an ordered basis for a fin. dim. vec. space $V$. Let $v \in V$ and $c_1, \dots, c_n \in F$ be the unique scalars such that (that's because $\{v_i\}_i$ is basis!)

$$v = \sum_{i=1}^{n} c_i v_i$$

We define the coordinate vector of $v$ relative to the ordered basis $\beta$ by

$$[v]_\beta = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in F^n$$

so it is a column $n$-tuple (a column vector in $F^n$)

• **Obs.** If we define $T : V \to F^n$ by $T(v) = [v]_\beta$ then $T$ is clearly a linear transformation, that's one to one and onto (exercise! ~ but we'll get back to this later)

**13.4 Example.** Let $V = P_2(\mathbb{R})$ the vector space of polynomials of degree $\leq 2$. Then we know $\{1, x, x^2\} \subset P_2(\mathbb{R})$ is a basis, and we consider it as an <u>ordered basis</u> $\beta = \{1, x, x^2\}$

if $f(x) = 2 - 3x + 6x^2 \in P_2(\mathbb{R})$

then $[f]_\beta = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

**13.5. Definition** Let $V, W$ be finite dim. vec. spaces with ordered basis $\beta = \{v_1, \ldots, v_n\}$ for $V$

$\gamma = \{w_1, \ldots, w_m\}$ for $W$

Let $T : V \to W$ be linear.
Notice that for each $j = 1, 2, \dots, n$
there exist unique scalars $a_{ij} \in F$, $i = 1, \dots, m$
such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad , \quad j = 1, \dots, n$$

(because $T(v_j) \in W$ and $\{w_1, \dots, w_m\}$ is ordered basis for $W$)

We call the matrix $A \in M_{m \times n}(F)$ defined by $(A)_{ij} = a_{ij}$ the matrix representation of $T$ with respect to the ordered basis $\beta$ and $\gamma$.
We write $A = [T]_\beta^\gamma$

Thus: The $j$'th column of $A$ is $[T(v_j)]_\gamma$, in the previous notation of Def 13.3!

$\boxed{13.6 \text{ Example}}$, write the matrix representation

Find the linear transformation

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

defined by $T(a_1, a_2) = (a_1 + 2a_2, 0, 2a_1 - a_2)$

with respect to standard ordered basis

$\beta = \{(1,0), (0,1)\}$ for $\mathbb{R}^2$

$\gamma = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$ for $\mathbb{R}^3$

Solution We have to write

$T(1,0)$
$T(0,1)$ in ordered basis $\gamma$ for $\mathbb{R}^3$

$$T(1,0) = (1 + 2\cdot 0, 0, 2\cdot 1 - 0) = (1, 0, 2)$$

$$= 1e_1 + 0e_2 + 2e_3$$

$$T(0,1) = (0 + 2\cdot 1, 0, 2\cdot 0 - 1) = (2, 0, -1)$$

$$= 2e_1 + 0\cdot e_2 + (-1)e_3$$

so $$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 2 & -1 \end{pmatrix}$$

Other examples on black board if time allows!