115 A, Winter 2023

Lecture 14

Fr, Feb 10th

• Recall that last time we introduced the notion of matrix representation of a linear transformation $T: V \to W$ with respect to ordered basis β for V, γ for W, denoted $[T]_\beta^\gamma$. It is defined as follows: if $\beta = \{v_1, \dots, v_n\}$

$\gamma = \{w_1, \dots, w_m\}$

then $[T]_\beta^\gamma$ is the matrix $(a_{ij}) \in M_{m\times n}(F)$ with j'th column given by $[T(v_j)]_\gamma$ (coordinate vector of $T(v_j)$ relative to γ)

so $T(v_j) = \sum_{i=1}^m a_{ij} w_i$, for $j = 1, 2, \dots, n$

• We noticed that $T \rightsquigarrow [T]_\beta^\gamma$ is linear and that if two linear transformations $T, U : V \to W$ have same matrix representations, $[T]_\beta^\gamma = [U]_\beta^\gamma$ then $T = U$

Two important examples:

(1) The matrix representation of the zero linear transformation $0: V \to W$ with respect to (any!) ordered basis β, γ is $[0]_\beta^\gamma = (0)_{ij} \in M_{m\times n}(F)$ the matrix that has all entries $= 0$

(2) if $V = W$ and β is an ordered bases for V then the identity linear transformation

$$I_V : V \to V$$

defined by $I_V(x) = x \quad \forall x \in V$

has matrix rep $[I_V]_\beta^\beta = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{pmatrix} \in M_{n\times n}(F)$

This $n \times n$ matrix is called the identity matrix

- today we continue to study the function $T \longmapsto [T]_\beta^\gamma$

14.1. Definition Let $T, U : V \to W$ be two functions between vector spaces V, W over some field F.
We define $T+U : V \to W$ by
$(T+U)(v) = \underbrace{T(v) + U(v)}_{\text{addition in } W}$, $\forall v \in V$.
Also, if $c \in F$, we define $c \cdot T : V \to W$ by $(c \cdot T)(v) = \underbrace{c \cdot T(v)}_{\text{scalar mult. in } W}$, $\forall v \in V$

14.2. Theorem. Let V, W be vector spaces$/F$ and let $T, U : V \to W$ linear.

(a). $T+U$ and cT are linear, $\forall c \in F$

(b) The set $\mathcal{L}(V, W)$ (curly L!) of linear transformations from V to W, with the operation of addition $T+U$ and scalar multiplication cT defined in 14.1, is a vector space over the field F.

$\boxed{\text{Pf}}$ (a). If $x, y \in V$ then

$$(T+U)(x+y) \underset{\substack{\uparrow \\ \text{by def } 14.1}}{=} T(x+y) + U(x+y)$$

$$\underset{\substack{\uparrow \\ \text{by linearity} \\ \text{of } U, T}}{=} \big(T(x) + T(y)\big) + \big(U(x) + U(y)\big)$$

$$= \big(T(x) + U(x)\big) + \big(T(y) + U(y)\big)$$

$$= (T+U)(x) + (T+U)(y)$$

thus showing that $T+U$ preserves addition

Similarly $(T+U)(cx) \underset{\substack{\uparrow \\ \text{def } 14.1}}{=} T(cx) + U(cx)$

$= c\,T(x) + c\,U(x) = c \cdot (T+U)(x) \quad \forall\, c \in F$

So $T+U : V \to W$ is a linear transf

Similarly one shows that

$c \cdot T : V \to W$ is linear

(b) With the operations of addition and scalar multip. in (a) and defining O to be the zero transformation, that takes any $v \in V$ to O_W, it is very easy to check all VS 1-8 axioms.

14.3 Notation As already mentioned in 14.2 part (b), we denote by $\mathcal{L}(V, W)$ the set of linear transformations from V to W endowed with vector space structure$/_F$ defined in Thm 14.2.

When $V = W$, we denote $\mathcal{L}(V,V) = \mathcal{L}(V)$

14.4 Theorem. Let V, W be fin. dim. vector spaces with ordered basis β, resp γ. If $T, U : V \to W$ are linear, then:

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$, $\forall c \in F$

Pf This is very easy:

Let $\beta = \{v_1, \dots, v_n\} \subset V$
$\gamma = \{w_1, \dots, w_m\} \subset W$ be the respective ordered basis

By the definition, we have $[T]_\beta^\gamma = (a_{ij})$
$[U]_\beta^\gamma = (b_{ij})$ where

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

$$j = 1, 2, \ldots, n$$

$$U(v_j) = \sum_{i=1}^{m} b_{ij} w_i$$

Thus, $(T+U)(v_j) = T(v_j) + U(x_j)$

$$= \left(\sum_{i=1}^{m} a_{ij} w_i\right) + \left(\sum_{i=1}^{m} b_{ij} w_i\right)$$

$$= \sum_{i=1}^{m} (a_{ij} + b_{ij}) w_i$$

so indeed $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

Proof of (b) is similar

Exercises: do exercise 1 from 2.2 page 84

Do calculating exercises showing some concrete $T \mapsto [T]_\beta^\gamma$ from 2.2 pages 85 – 86

$\boxed{\text{14.5 Notation}}$ (Kronecker delta-symbol)

One denotes $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

where $i, j \in \{1, 2, \ldots, n\}$, or more generally $i, j \in J$ (same set J)

For instance, using these symbols, we have

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (\delta_{ij})_{i,j=1}^{n}$$

14.6 Corollary (to thms 14.2 & 14.4).

Let V, W be fin. dim. vector sp. $/F$ and β, γ ordered basis for V, resp W
$\beta = \{v_1, \ldots, v_n\}$, $\gamma = \{w_1, \ldots, w_m\}$

The transformation from $\mathcal{L}(V, W)$ to $M_{m \times n}(F)$ defined by

$$T \longmapsto [T]_\beta^\gamma$$

is linear and one to one.

$\boxed{\text{Pf}}$ We already established linearity of this transformation and showed that if $[T]_\beta^\gamma = [U]_\beta^\gamma$ then $T = U$, so 1to1

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