Math 115A Assign6

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from Section 2.2: exercises 1; 2; 5(c); 8; 14.

from Section 2.3: exercises 1(a),(b),(c),(h); 2; 3; 9; 11; 13.

2.2

1. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $T, U : V \to W$ are linear transformations

a) TRUE

b) TRUE

c) FALSE

d) TRUE

e) TRUE

f) FALSE

2. Let β and γ be the standard order bases for R^n and R^m, respectively for each linear transformation $T : R^n \to R^m$, compute $[T]_\beta^\gamma$

a) $[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

b) $[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

c) $[T]_\beta^\gamma = \begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$

d) $[T]_\beta^\gamma = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{pmatrix}$

e) $[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & ... & 0_{nth} \\ 1 & 0 & ... & 0_{nth} \\ | & & & | \\ 1_{nth} & 0 & ... & 0_{nth} \end{pmatrix}$

f) $[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & ... & 1 \\ 0 & 0 & 1 & 0 \\ | & & & | \\ 1 & 0 & ... & 0 \end{pmatrix}$

g) $[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & ... & 0 & 1_{nth} \end{pmatrix}$

5(c). Let α ={$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$}, β ={$1, x, x^2$}. Define $T : M_{2\times 2}(F) \to R$ by $T(A) = tr(A)$. Compute $[T]_\alpha^\gamma$

Answer:
$tr(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = (1, 0, 0, 1)$ $[T]_\alpha^\gamma = \{1, 0, 0, 1\}$

8. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Proof:
$T(cx) = [cx]_\beta = c[x]_\beta = cT(x)$
$T(x + y) = [x + y]_\beta = [x]_\beta + [y]_\beta = T(x) + T(y)$
T is linear

14. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. if $R(T) \cap R(U)$ ={0}, Prove that {T, U} is a linearly independent subset of $L(V, W)$

Proof:
Let $aT(v)+bU(v) = 0$,(because $(aT+bU)v = 0(v)$) then we will have $T(av)+U(bv) = 0$ and $T(av) = U(-bv)$. Since $R(T)\cap R(U)$ ={0}, we can consider that $a = b = 0$, also conclude that T and U are linearly independent

2.3

1

a) FALSE

b) TRUE

c) FALSE

d) FALSE

2.

a) Let $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$**. Compute** $A(2B+3C)$**,**$(AB)D$ **and** $A(BD)$

$A(2B+3C) = \begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix}$

$(AB)D = A(BD) = \begin{pmatrix} 29 \\ -26 \end{pmatrix}$

b) Let $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}, C = \begin{pmatrix} 4 & 0 & 3 \end{pmatrix}$**. Compute** A^t, A^tB, BC^t, CB **and** CA

$A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$

$A^tB = \begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix}$

$BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$

$CB = \begin{pmatrix} 27 & 7 & 9 \end{pmatrix}$

$CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

3. Let $g(x) = 3 + x$**. Let** $T : P_2(R) \to P_2(R)$ **and** $U : P_2(R) \to R^3$ **be the linear transformations respectively defined by** $T(f(x)) = f'(x)g(x) + 2f(x)$ **and** $U(a + bx + cx^2) = (a + b, c, a - b)$**. Let** β **and** γ **be the standard ordered bases of** $P_2(R)$ **and** R^3**, respectively**

a) Compute $[U]_\beta^\gamma$**,** $[T]_\beta$ **and** $[UT]_\beta^\gamma$ **directly. Then use Theorem 2.11 to verify your result**

Proof:

$T(f(x)) = (b + 2cx)(3 + x) + 2(a + bx + cx^2) = (2a + 3b) + (3b + 6c)x + (4c)x^2$

$UT(1) = (2, 0, 2)$

$UT(x) = (6, 0, 0)$

$UT(x^2) = (6, 4, -6)$

$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

$[U]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 6 & 4 \end{pmatrix}$

$$[U]_\gamma^\beta[T]_\beta = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} = [UT]_\beta^\gamma$$

b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

$[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$[U]_\beta^\gamma[h]_\beta = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = [U(h)]_\gamma$

9. Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$

Proof:

$T(x, y) = (0, 0)$

$U(x, y) = (y, 0)$

Then, we have $UT(x, y) = T(U(x, y)) = T(y, 0) = (0, 0) = T0, andTU(x, y) = U(T(x, y)) = U(0, 0) = (0, 0) = T0$, as required.

To find matrices A and B such that $AB = 0$ but $BA \neq 0$, we can represent T and U as matrices as follows:

T = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

U = $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Then, we have AB = TU = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and BA = UT = $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

$T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, AB = TU = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, BA = UT = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

11. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

Proof:

To prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$, we need to show two implications:
First, assume that $T^2 = T_0$. We want to show that $R(T) \subseteq N(T)$.
Let y be any element in $R(T)$. Then there exists an x in V such that $T(x) = y$. We want to show that $T(y) = T(T(x)) = T^2(x) = T_0(x) = 0$, which means that y is in $N(T)$.
Therefore, we have shown that $R(T) \subseteq N(T)$.
Second, assume that $R(T) \subseteq N(T)$. We want to show that $T^2 = T_0$.

Let x be any element in V. Then we have $T(T(x)) = T^2(x)$ and $T(x)$ is in $R(T)$. Since $R(T) \subseteq N(T)$, we know that $T(T(x)) = T^2(x) = 0$, which means that $T^2 = T_0$.
Therefore, we have shown that $T^2 = T_0$.

13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by $tr(A) = \Sigma_{i=1}^n A_{ii}$. Prove that $tr(AB) = tr(BA)$ and $tr(A) = tr(A^t)$

Proof:

To prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, we can expand both traces using the definition of matrix multiplication and the trace operator:

$\mathrm{tr}(AB) = \sum_{i=1}^n (AB)ii = \sum i = 1^n \sum_{j=1}^n A_{ij}B_{ji}$

$\mathrm{tr}(BA) = \sum_{i=1}^n (BA)ii = \sum i = 1^n \sum_{j=1}^n B_{ij}A_{ji}$

We can then swap the order of summation in the second expression by renaming the indices:

$\mathrm{tr}(BA) = \sum_{j=1}^n \sum_{i=1}^n B_{ji}A_{ij}$

Now, we can see that the two expressions are identical, so we have proven that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

To prove that $\mathrm{tr}(A) = \mathrm{tr}(A^T)$, we can expand both traces using the definition of the trace operator:

$\mathrm{tr}(A) = \sum_{i=1}^n A_{ii}$

$\mathrm{tr}(A^T) = \sum_{i=1}^n (A^T)ii = \sum i = 1^n A_{ii}$

Since the diagonal entries of A are the same as the diagonal entries of A^T, the two expressions are identical, so we have proven that $\mathrm{tr}(A) = \mathrm{tr}(A^T)$.