Math115A 1/27 notes

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We talked about how important it is to identify subsets S of a vector space V that generate (span) V and are in some scalar "minimal" (or most efficient) with this property.
Today we'll study more in depth such sets, moving that such minimal generating sets" S are automation linearly independent, and have the remarkable property that any vector in V can be uniquely written as a linear combination of vectors in S. We'll call such S, basis for V

8.1 Definition

A linearly independent subset S of a vector space V that generates (spans) V is called a basis for V.

8.2 Example

Let $V \in \mathbb{R}^3$ and $S =\{(1,0,0),(0,1,0),(0,0,1)\}$ Then S is a basis for V.

Indeed: we already showed that $(1,0,0),(0,1,0),(0,0,1)$ are linearly independent vectors in $\mathbb{R}$. And if $v = (a,b,c) \in \mathbb{R}^3$ is a arbitary vector, then $v = a(1,0,0) + b(0,1,0) + c(0,0,1)$, Thus, S spans V as well

8.3 Exercise

Let $V = \mathbb{M}_{2\times 2}(R)$ and consider the set $S =\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\}$ Show that S is a basis for V.
Solution:
we already showed in an exercise on Monday (6.13) That S spans V if we would have $a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 0$
Then this entails $\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ so a=0,b=0,c=0,d=0 showing that S is linear independent.

8.4 Example

if F is a field and $V = F[X]$ is the vector space over F of all polynomials in undetermined X and coefficients in F then S =$\{1, X, X^2, X^3, ...\}$ is a basis for V.

indeed, we already showed that $span(S) = V$ if $a_0, a_1, ... \in F$ so that $a_0 1 + a_1 X + a_2 X^2 + ... + a_n X^n = 0$ then by the definition of the polynomials we must have $a_0 = 0, a_1 = 0, ... a_n = 0$ showing that S is linear independent as well

8.5 Theorem

Let V be a vector space. A set S =$\{v_1, ..., v_n\}$$\in V$ is a basis for V iff any vector $v \in V$ can ve uniquely expressed as a linear combination of elements in S, i.e., there exist unique scalars $c_1, c_2, ..., c_n \in F$ such that $v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$

Proof:

if S =$\{v_1, ... v_n\}$ is a basis for V, $span(S) = V$, so if $v \in V$ is an arbitrary vector in V, then there exist $c_1, ... c_n \in F$ such that $v = \Sigma_{i=1}^{n}$ if $a_1, ..., a_n \in F$ are other scalars such that $v = \Sigma_{i=1}^{n} a_i v_i$ as well then $\Sigma_{i=1}^{n} c_i v_i = \Sigma_{i=1}^{n} v_i a_i$ so by cancellation then we get $\Sigma_{i=1}^{n} c_i v_i - \Sigma_{i=1}^{n} a_i v_i = 0$ which using assure & commutativity of addition + distributivity of scalar multiplication gives $(c_1 - a_1)v_1 + (c_2 - a_2)v_2 + ... + (c_n - a_n)v_n = 0$ But since S =$\{v_1, ..., v_n\}$ is a basis, the vectors $v_1, ... v_n$ are linear independent, so this implies all coefficients $c_i - a_i$ in $(c_1 - a_1)v_1 + (c_2 - a_2)v_2 + ... + (c_n - a_n)v_n = 0$. So we showed v can be expressed in only one way as a linear combination of $v_1, ..., v_n$

If any $v \in V$ can expressed in a unique way as linear combination of S =$\{v_1, ... v_n\}$, Then in particular $span(S) = V$ if we would have $c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$ for some $c_1, ..., c_n \in F$, Then by uniqueness, since we also have $0 * v_1 + 0 * v_2 + ... + 0 * v_n = 0$ it follows that $c_1 = 0, c_2 = 0, ... c_n = 0$ thus $\{v_1, ..., v_n\}$ linearly independent so S is a basis.

8.6 Theorem

Let S be a finite subset of the vector space V. if $span(S) = V$(i.e. if S generate V) then there exist a subset $S' \in S$ such that S' is a basis for V

Proof:

If S =$\{0\}$ then $span(S)$ =$\{0\}$ so V =$\{0\}$ and S is a basis for V

IF S contains at least one non-zero element, say $u_1 \neq 0$, then $\{u_1\}$ is linearly independent we then continue to choose $u_2, ..., u_k$ in S so that $u_1, ..., u_k$ are linearly independent, write this is no longer possible (Note that this must be the case, because S is finite)

This happens if either we have exhausted all S, i.e. if

(a) $\{u_1, .., u_k\}$, or if

(b) any $u \in S$ that's not among $u_1, ... u_k$ is so that $u_1, ... u_k$ u is linearly dependent

in case we have (a), it means S =$\{u_1, ... u_k\}$ is linear independent and since we also have $span(S) = V$, it follows that S itself is a basis for V and we are done.

In case we have (b), it means there exis tscalars $c_1, c_2, ... c_k, c$, not all equal to 0, such that $c_1 u_1 + c_2 u_2 + ... + c_k u_k + cu = 0$ if $c = 0$, then it would follow that $c_1 u_1 + ... + c_k u_k = 0$ with $c_1, c_2, ... c_k$ not all equal to 0, contradicting the fact that {u_1,... u_n} is linear independent thus, $c \neq 0$

and then from $c_1u_1 + c_2u_2 + ... + c_ku_k + cu = 0$ we deduce $u = -\frac{c_1}{c}u_1 - \frac{c_2}{c}u_2 - ... - \frac{c_k}{c}u_k$ showing that $u \in span(\{u_1, ...u_k\})$ Thus, in case (b), we showed that the set $S' = \{u_1, ...u_k\} \in S$ is linear independent and any $u \in S - u_1, ..., u_k$ is in $span(S')$ so $span(S')$ contains $u_1, ...u_k$ and call $S - \{u_1, ...u_k\}$, so $S \in span(S')$ Since $span(S) = V$, by (Theom 6.6 or Theom 7.8), it follows that $span(S')$ contains all $span(S)$ thus $span(S') = V$(because $span(S) = V$) so S' is lieanr independent & $span(S') = V$ so $S' \in S$ is a basis for V.

8.7 Corallery

If V contains a finite subset $S \in V$ that generates V, i.e. $span(S) = V$, then V has a finite basis

8.8 Example

Here is a concrete example showing how the method of finalizing a basis S' as a subset of a generating set $S \in V$ works:
Let $S = \{(1, -1), (-1, 1), (0, 2), (3, 0)\} \in \mathbb{R}^2$. Show that there exists $S' \in S$ such that S' is a basis for $\mathbb{R}^2$
Solution
Since S contains non-zero vectors we can state by choosing $n_1 = (1, -1) \in S$. Then we look at the 2'nd vector $(-1, 1)$ in S. We see that $(-1, 1) = -1 * (1, -1)$, i.e.$(-1, 1) = -n_1$ so $(-1, 1)$ is not linear independent of u_1. We then take the 3'rd vector in S, $(0, 2)$. If $a(1, -1) + b(0, 2) = 0$. Then $(a, -a + 2b) = (0, 0)$ $a = 0$ and $-0 + 2b = 0$ so $b = 0$ as well Thus $u_2 = (0, 2) \in S$ is linear independent of u_1. So we can take add u_2 to our linear independent subset S' of S. So by now we have $u_1 = (1, -1), u_2 = (0, 2) \in S'$. We see that in fact $span\{u_1, u_2\}$ and we can stop and canclude that $S' = \{u_1, u_2\} = \{(1, -1), (0, 2)\} \in S$ is a basis for $\mathbb{R}^2$

8.9 Theorem (the so-called replacement theorem)

Let V be a vector space. Assume $G \in V$ is a subset with n vectors that generatesV, i.e. $span(G) = V$ if $L \in V$ is a linearly independent subset of V with m vectors, then $m \leq n$ and there exist a subset $H \in G$ containing $n - m$ vectors such that $L \cup V$ generate V
Proof
We prove this by induction over m (i.e. over the # of elements in then linear independent set L)
if $m = 0$, this means L has 0 many elements, i.e. $L \neq 0$ and we can just take $H = G$, which satisfies the required conditions.
Suppose now that the statement holds true for some $m \geq 0$. We then want to show that the statement holds true for $m + 1$ as well.
So let $L = \{v_1, ...v_{m+1}\}$ be linear independent subset of V. By Theom 7.7 any subset of L is linear independent, so $\{v_1, ...v_m\}$ is linear independent by induction, since we have that the statement of thn is true for m, it follows that $m \leq n$ and that there exist $\{u_1, ...u_{n-m}\} \in G$ such that $\{v_1, ...v_m\} \cup \{u_1, ...u_{n-m}\}$ spaces V. So in particular, v_{m+n} can be expressed as a linear combination
$v_{m+1} = a_1v_1 + ...a_mv_m + b_1u_1 + ... + b_{n-m}u_{n-m}$ Now notice that in fact we must have $n - m \geq 1$, or else we would have $v_{m+n} = a_1v_1 + ... + a_mv_m + 0$ which contradicts the fact that $L = \{v_1, ..., v_m + 1\}$ is linear independent. In other words, we must gave $n \geq m + 1$. Also, In $v_{m+1} = a_1v_1 + ...a_mv_m + b_1u_1 + ... + b_{n-m}u_{n-m}$ we must have that some b_i are non-zero. say $b_1 \neq 0$, which allows us to solve in $v_{m+1} = a_1v_1 + ...a_mv_m + b_1u_1 + ... + b_{n-m}u_{n-m}$ for u_1:$u_1 = (-\frac{a_1}{b_1})v_1 + (-\frac{a_2}{b_1})v_2 + ... + (-\frac{a_m}{b_1})v_m + \frac{1}{b_1}v_{m+n} + (-\frac{b_2}{b_1})u_2 + ... + (-\frac{b_{n-m}}{b_n})u_{n-m}$. Thus, if we take $H = \{u_2, ..., u_{n-m}\}$ then $u_1 \in span(L \cup H)$ by $u_1 = (-\frac{a_1}{b_1})v_1 + (-\frac{a_2}{b_1})v_2 + ... + (-\frac{a_m}{b_1})v_m + \frac{1}{b_1}v_{m+n} +$

$(-\frac{b_2}{b_1})u_2 + ... + (-\frac{b_{n-m}}{b_n})u_{n-m}$. and since $v_1, ...v_m,\ u_2, ..., u_{n-m}$ are obviously in $span(L \cup H)$, we actually have that $\{v_1, ...v_m, u_1, ...u_{n-m}\} \in span(L \cup H)$ Since $\{v_1, ...v_m, u_1, u_2, ..., u_{n-m}\}$ generates V, it follows that $span(L \cup H) = V$ with $H \in G$ being a subset that contains $(n - m) - 1 = n - (m + 1)$ elements, showing that statement of this holds true for $m + 1$