Math115A 1/20 notes

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2023-01-21

Sub spaces of vector spaces

5.1 Definition

A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Note that V itself is a subspace of V. Also, the set $\{0\}$, consisting of the vector $0 \in V$, is a subspace of V it is called the zero subspace
If W is a subset of a vector space V over a field F, and we are as lead to check if W is a subspace, then we don't want to check that all conditions VS1-VS8 are satisfied: indeed we do know addition & scalar multiplication check the multiplication check the properties for all vectors in V, so all we need is in fact to check that addition of $x, y \in W$ abd scalar multiplication ax for $x \in W$ keep us within the set W.

5.2 Theorem

Let V be a vector space over field F and $W < V$ a subset. Then W is a subspace of V if and only if the following conditions field.
(a) $0 \in W$ (i.e. the zero vector of V belong to W)
(b) if $x, y \in W$ then $x + y \in W$(i.e. the addition of any two vectors in W stays in W)
(c) if $x \in W$ and $c \in F$ then $cx \in W$(i.e. the multiplication by any scalor of a vector in x stays in W)

Proof
if W is a subspace of V then by definition 5.1 it means that W with the operations of addition and scalar multiplication it "inherits" from V is a vector space. Thus (b)&(c) are satisfied. Also, W has by the definition of vector space a "zero element", say $0' \in W$ satisfying $x + 0' = x, \forall x \in W$. But we also have $x + 0 = x$ where 0 is here the zero element in V By Theom 4.7(a) thus implies $0'0 = 0$, so we have (a) as well "<=" if (a),(b),(c) hold.
Then W with the addition and scalar multiplication of its elements viewed as elements in V satisfies VS1, VS2, and VS5-8 because we already know all these properties are satisfied in V.
All we need to show is that W has a zero vector (thus is VS3) and any $x \in W$ has an additive inverse in W (an "opposite")
but condition (a) tells us that $0 \in W$ (no zero element of V belongs to W), and this element does satisfy $x + 0 = x, \forall x \in W$(because this holds that in fact for all elements x in V in particular for all elements in W)
Also, by condition (c) we love that $\forall x \in W$, the element $y = (-1)x$ is the opposite of x in V. So $x \in W$ does have an additive inverse in W(what we also called an "opposite"). So VS3 & VS4 are satisfied as well.

5.3 Definitions

Given a matrix $A = (A_{ij}) \in M_{m\times n}(F)$ its transpose is the $n \times m$ matrix $A^t \in M_{n\times m}(F)$obtained from A by interchouging the row with the columns, that is $(A^t)_{ij} = A_{ij}$ for all $1 \le i \le n$ and $1 \le j \le m$

For example:/newline 1. if $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -4 & 0 \end{pmatrix} \in M_{3\times 2}(R)$

then $A^t \in M_{2\times 3}(R)$ is $A^t = \begin{pmatrix} 0 & 2 & -4 \\ -1 & 3 & 0 \end{pmatrix}$

2. $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}^t = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -3 \end{pmatrix}^t = \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 2 & -3 \end{pmatrix}$

A symmetric matrix A is a matrix such that $A^t = A$
Note that such matrices necessary have same # of rows and colums
i.e. $A \in M_{n\times m}(F)$
Examples:
1.$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ so it is symmetric

4. Note that $0 = \begin{pmatrix} 0 & ... & 0 \\ | & & | \\ 2 & ... & 0 \end{pmatrix} \in M_{m\times n}(F)$

5.4 Exercise

show that the set of symmetric matrices in $M_{m\times n}$ is a subspace of $M_{m\times n}(F)$
Solution
We need to check (b)&(c) of theom 5.2(we already noticed that the 0 matrix in $M_{m\times n}(F)$ is symmetric as we have (a) of theom 5.2)

If $A, B \in M_{m\times n}(F)$ are symmetric then $A = (A_{ij})$ and $B = (B_{ij})$ satisfy $A = A^t$, $B = B^t$ so $A_{ij} = A_{ji}, B_{ij} = B_{ji}$ for all $1 \le i, j \le n$
But then $(A+B)_{ij} = A_{ij} + B_{ij}$ so $((A+B)^t)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A_{ij} + B_{ij}$, Showing that $A+B$ is symmetric
Similarly, if $c \in F$ is a scalar and $A = A^t$ Then $(cA)_{ij} = cA_{ij}, ((cA)^t)_{ij} = (cA)_{ji} = cA_{ij}$ so cA is symmetric

5.5 Definitions

A matrix $A \in M_{n\times m} \in F$ is called a diagonal matrix
If $A_{ij} = 0, \forall i \ne j$, i.e. if all entries off the diagonal of A are equal to 0

Example:/newline $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_{3\times 3}(R)$ is a diagonal matrix

$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in M_{4\times 4}(R)$ is a diagonal matrix
Note the 0 matrix in $M_{m\times n}(F)$ is a diagonal matrix, $\forall n$

Exercise
Show that the set of diagonal matrices $D_{m\times n}(F)$ in $M_{m\times n}(F)$ is a subspace.
So indeed, we noticed that $0 \in D_{m\times n}(F)$
Also the sum of two diagnal matrices $A.B \in D_{m\times n}(F)$ satisfies $(A+B)_{ij} = A_{ij} + B_{ij} = 0$if $i \neq j$, because both $A_{ij} = 0, B_{ij} = 0$ if$i \neq j$ (Since A, B are diagonal)
Similarity if $A \in D_{n\times m}(F)$ and $c \in F$ then $(cA)_{ij} = cA_{ij} = 0$ if $i \neq j$ because $A_{ij} = 0, \forall i \neq j$

5.6 Definition

Given a $n \times m$ matrix $A \in M_{m\times n}(F)$, the trace of A, denoted tr(A) is the sum of the diagonal entries of A, i.e.
$tr(A) = \sum_{i=1}^{n} A_{ii} \in F$
example if $A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & -2 & 0 \\ 1 & 2 & 2 \end{pmatrix}$ then $tr(A) = -1 + (-2) + 2 = -1$

5.7 Exercise

(a) show that the set of matrices $A \in M_{m\times n}(F)$ with $tr(A) = 0$ is a subspace of $M_{m\times n}(F)$

(b) Show that the set of matrices $A \in M_{m\times n}(F)$ with $tr(A) = 1$ is not a subspace of $M_{n\times m}(F)$

(c) Show that the set of matrices $A \in M_{m\times n}(F)$ with $tr(A) \leq 0$ is not a subspace

5.8 Theorem

If W_1, W_2 are subspace of a vector space V over a field F, then $W_1 \cap W_2$ is a subspace of V as well more generally, if $W_i, i \in I$, is a collection of subspace of V then $\cap_{i\in I} W_i$ is a subspace of V.

Proof
Since both W_1, W_2 are subspace, we have $0 \in W_1, 0 \in W_2$ so $0 \in W_1 \in W_2$
Also, if $x, y \in W_1 \cap W_2$
Then $x, y \in W_1$, so $x + y \in W_1$ and $x, y \in W_2$, so $x + y \in W_2$
Thus, $x + y \in W_1 \cap W_2$
Similarly, if$x \in W_1 \cap W_2$ and $c \in F$
Then $x \in W_1$, so $cx \in W_1$ and $x \in W_2$, so $cx \in W_2$
Thus $cx \in W_1 \cap W_2$, and we showed that (a),(b) and (c) of theom 5.2 are all satisfied for $W_1 \cap W_2$

General case is similar