Assign2

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Exercises 8, 11, 13, 14, 17, 19, 20, 21 from Section 1.2 (pages 15 an 16) of the book.

Exercises 8

In any vector space, show that$(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$

Proof:
According to VS7 we know that for any $x, y \in V$ and any $a, b \in F$, $(a + b)(x + y) = (a + b)x + (a + b)y$, Then by VS8, we can transfer the equation to $(a + b)x + (a + b)y = ax + bx + ay + by$

Exercise 11

Let V = { 0 } consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar c in F. Prove that V is a Vector space over F.

Proof:
To consider V as a vector space, it has to satisfied all VS1-8.
VS1: We know that V consist a single vector 0, and $0 + 0 = 0 + 0 = 0$. Therefore, it satisfied commutative of addition.
VS2: Same, as V consist a single vector 0, and $(0 + 0) + 0 = 0 + 0 = 0$ and $0 + (0 + 0) = 0 + 0 = 0$ where $(0 + 0) + 0 = 0 + (0 + 0)$. Therefore, satisfied the associative of addition.
VS3: $x + 0 = x.\forall x \in V$, V consist only a single vector, $0 + 0 = 0$. Therefore, satisfied VS3.
VS4: $0 + 0 = 0$, the inverse of the vector 0 is itself. satisfied $x + (-x) = 0, \forall x \in V$
VS5: $1x = x, \forall x \in V$, Where$1(0) = 0$, this also satisfied.
VS6: $(ab)x = a(bx), \forall x \in V$ and $\forall a, b \in F$. We can have scalar b and c, where $b(c0) = b0 = 0$ and $(bc)0 = 0$ (because (bc) is also a scalar in F). Therefore, satisfied
VS7: $c(x + y) = cx + cy$, $\forall x, y \in V$ and $\forall c \in F$, we know that $c(0 + 0) = c(0) = 0$, therefore satisfied.
VS8: $(a + b)x = ax + bx, \forall a, b \in F$ and $\forall x \in V$. we can have two scalars c and d, where $(a + b)0 = 0$ (($a + b$) is also a scalar in F), and $a0 + b0 = 0 + 0 = 0$. which satisfied the rule.
In final, V is a vector space over F.

Exercise 13

Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define:
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$
Is V a vector space over R with these operations? Justify your answer.

Proof:

VS1: $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2b_2)$, and, $(b_1, b_2) + (a_1, a_2) = (b_1 + a_1, a_2b_2)$ $(a_1, a_2) + (b_1, b_2) = (b_1, b_2) + (a_1, a_2)$, Satisfy VS1
VS2: $((a_1, a_2) + (b_1, b_2)) + (c_1, c_2) = (a_1 + b_1, a_2b_2) + (c_1, c_2) = (a_1 + b_1 + c_1, a_2b_2c_2)$, and $(a_1, a_2) + ((b_1, b_2) + (c_1, c_2)) = (a_1, a_2) + (b_1 + c_1, b_2c_2) = (a_1 + b_1 + c_1, a_2b_2c_2)$ $((a_1, a_2) + (b_1, b_2)) + (c_1, c_2) = (a_1, a_2) + ((b_1, b_2) + (c_1, c_2))$, satisfy VS2
VS3: V denote the set of ordered pairs of real numbers which means that V contains 0 as one of it's element. However. $(a_1, a_2) + (0, 0) = (a_1 + 0, a_2(0)) = (a_1, 0)$ which $(a_1, a_2) + 0 \neq (a_1, a_2)$ This failed to satisfy V as a vector space over R.
Therefore, NO, V is not a vector space over R.

Exercise 14

Let V=$\{(a_1, a_2, ...a_n) : a_i \in \mathbb{C}$ for $i = 1, 2, ...n\}$; so V is a vector space over $\mathbb{C}$ by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Proof:
Let V=$\{(a_1, a_2, ...a_n) : a_i \in \mathbb{C}$ for $i = 1, 2, ...n\}$ and let $x =\{x + xi, x + xi, ...x + xi\}$,$y =$\{y+yi,y+yi,. . . y+yi\$\}, c, d. We will have that $cx + dy = c\{x + xi, x + xi, ...x + xi\} + d\{y + yi, y + yi, ...y + yi\} = \{cx + cxi, cx + cxi....cx + cxi\} + \{dy + dyi, dy + dyi, ...dy + dyi\} = \{(cx + dy) + (cx + dy)i, (cx + dy) + (cx + dy)i, ...(cx + dy) + (cx + dy)i\}$ and $(cx + dy) + (cx + dy)i \in \mathbb{C}$. Therefore, V is a vector space over the field of real number.

Exercise 17

Let V = $\{(a_1, a_2) : a_1, a_2 \in F\}$. where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$, define $c(a_1, a_2) = (a_1, 0)$. Is V a vector space over F with these operations? Justify your answer.

Proof:
No, V is not a vector space over F. We have $c(a_1, a_2) = (a_1, 0)$ which does not follow VS5 where $1x = x$ or if V is a vector space over F, then $c(a_1, a_2) = (ca_1, ca_2)$. Therefore, V is not a vector space over F.

Exercise 19

Let V = $\{(a_1, a_2) : a_1, a_2 \in F\}$. Define addition of element of V coordinatewise, and for (a_1, a_2) in V and $c \in \mathbb{R}$ define:

$$c(a_1, a_2) = \begin{array}{c} (0, 0) if\ c = 0 \\ (ca_1, \dfrac{a_2}{c}) if\ c \neq 0 \end{array}.$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Proof:
We have $c(a_1, a_2) = (ca_1, \frac{a_2}{c})$ when $c \neq 0$. In VS8, the vector space needs to satisfy the condition where $(a + b)x = ax + bx$. Apply the equation to set V, for the left side of the equation we have, $(c + b)(a_1, a_2) = ((c + b)a_1, \frac{c_2}{a+b}) = (ca_1 + ba_1, \frac{c_2}{a+b})$. And the right side of the equation we have $c(a_1, a_2) + b(a_1, a_2) = (ca_1, \frac{a_2}{c}) + (ba_1, \frac{a_2}{b}) = (ca_1 + ba_1, \frac{a_2}{c} + \frac{a_2}{b})$. Where $(a + b)x \neq ax + bx$. Therefore, No, V is not a vector space over F.

Exercise 20

Let V denote the set of all real-valued functions f defined on the real line such that $f(1) = 0$. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof:
To satisfied as a vector space with the operations of addition and scalar multiplication, where $(f+g)(s) = f(s) + g(s)$ and $(cf)(s) = cf(s)$ and VS1-8 all satisfied. Consider the first two laws, we have $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0 \in V$ and $(cf)(1) = cf(1) = 0 \in V$.
VS1: $(f+g)(s) = f(s) + g(s) = g(s) + f(s) = (g+f)(s)$.
VS2: $((f+g)+j)(s) = f(s) + g(s) + j(s) = (g + (f+j))(s)$
VS3: $f(1) = 0 \in V$ and $(f+0)(s) = f(s) + 0(s) = f(s) + 0 = f(s)$
VS4: $(f + (-f))(1) = f(1) + (-f)(1) = f(1) - f(1) = 0 - 0 = 0$
VS5: $(1f)(s) = 1f(s) = f(s)$
VS6: $(ab)x = a(bx)$ where $((ab)f)(s) = (ab)f(s) = (a)(b)(f(s)) = (a)(bf(s)) = a((bf)(s))$
VS7: $a(x+y) = ax + ay$ where $a(f+g)(s) = a(f(s) + g(s)) = af(s) + ag(s)$
VS8: $(a+b)x = ax + by$ where $(a+b)f(s) = a(f(s)) + b(f(s)) = af(s) + bf(s)$
Therefore, V is a vector space with the operations of addition and scalar multiplication.

Exercise 21

Let V and W be vector spaces over a field F. Let Z=$\{(v,w) : v \in V,\ and\ w \in W\}$.
Prove that Z is a vector space over F with the operations
$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and $c(v_1, w_1) = (cv_1, cw_1)$

Proof:
We know that V and W are vector spaces over field F, which means that V and W both satisfied VS1-8. When Z=$\{(v,w) : v \in V,\ and\ w \in W\}$ which means that Z is formed by vector space V and W. Therefore, $(v_1) + (v_2) = v_1 + v_2$, $(w_1) + (w_2) = w_1 + w_2$, and $c(v_1) = cv_1$,$c(w_1) = cw_1$, which also lead to $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \in Z$ and $c(v_1, w_1) = (cv_1, cw_1) \in Z$ Therefore, Z is a vector space over F with the operations.