Assign 1 for 115A

Lecture 4 (S. Popa)

Due Wed Jan 18

at 6pm

on Gradescope

(1). Prove in details de Morgan's law I didn't prove in class:
"for any sets X, A, B we have
$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

(2). Let A, B be subsets of the set of real numbers $\mathbb{R}$. For each one of the statements below write its negation:

(a) For all $x \in A$ there exists $b \in B$ such that $b > x$

(b) There is an $x \in A$ such that for all $b \in B$ we have $b > x$

(3). Consider the set $\mathbb{Z}_3 \overset{\text{def}}{=} \{0, 1, 2\}$ and define on it the operations $+$ and $\cdot$ as follows.

For $+$: $0+0=0$; $0+1=1$; $0+2=2$

$1+0=1$; $1+1=2$; $1+2=0$

$2+0=2$; $2+1=0$; $2+2=1$

For $\cdot$: $0\cdot 0=0$; $0\cdot 1=0$; $0\cdot 2=0$

$1\cdot 0=0$; $1\cdot 1=1$; $1\cdot 2=2$

$2\cdot 0=0$; $2\cdot 1=2$; $2\cdot 2=1$

Show that $(\mathbb{Z}_3, +, \cdot)$ is a field

(4). Consider the subset of $\mathbb{C}$:

$$\mathbb{Q}(i) := \{ x+yi : x, y \in \mathbb{Q} \}$$

Prove that $\mathbb{Q}(i)$ with the addition and multiplication inherited from $\mathbb{C}$ (i.e. defined the same way as for all complex numbers) is a field.

(5). Let $(F, +, \cdot)$ be a field, with its operations of addition $+$ and multiplication $\cdot$. Let $E \subset F$ be a subset and assume the following holds:

- for all $a, b \in E$, $a + b \in E$ and $a \cdot b \in E$ (E is closed to addition & multiplication)
- $0 \in E$, $1 \in E$
- if $a \in E$ then $-a \in E$
- if $a \in E$, $a \neq 0$ then $a^{-1} \in E$

Prove that E with the same operations $+$ and $\cdot$ as F, is a field (i.e. with the operations it inherits from F)

6. Let $(F, +, \cdot)$ be a field.

(a). Prove that $(-1) \cdot a = -a$ and $(-2) \cdot a = -(2 \cdot a)$ for any $a \in F$.

(b). Prove that for any $a \in F$, $a \neq 0$ we have $(-a)^{-1} = -a^{-1}$