115A, Winter 2023

Linear Algebra

Lecture 2

Wed, Jan 11

Recall from Lecture 1 on Monday Jan 9th:

- A set is a collection of mathematical objects. The members of a set are called elements of the set. We write $a \in A$ to mean: "a is an element of the set A"

- We often use curly brackets for sets whose elements can be "enumerated"

example $\{-1, 2, 3, 7\}$ is the set consisting of the numbers (elements) -1, 2, 3, and 7.

- More often we describe a set as "elements with some properties", like

$$A = \{x : x \text{ has property } P\}$$

in such cases one often mentions that x is from a larger set (most often set of numbers) that was already defined and has an established notation.

- Important fact about sets: one cannot have at the same time $x \in A$ and $x \notin A$!

Example $\{x \in \mathbb{R} : x \geq -2\}$

means "the set of real numbers larger than or equal to -2"

- There are some established notations for several important sets:

→ the empty set is denoted $\emptyset$

↳ set without elements in it

→ the set of natural numbers denoted

$$\mathbb{N} := \{0, 1, 2, 3, 4, \dots\}$$

→ the set of integers

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \dots\}$$

→ the set of rationals

$$\mathbb{Q} := \left\{\frac{m}{n} ; m, n \in \mathbb{Z}, n \neq 0\right\}$$

→ the set of real numbers $\mathbb{R}$

→ the set of complex numbers

$$\mathbb{C} := \{a + bi ; a, b \in \mathbb{R}\}$$

where $i = \sqrt{-1}$, i.e. $i^2 = -1$

- Suppose A, B are sets. We say A is a subset of B and write $A \subset B$ if for any $a \in A$ we have $a \in B$
(every element of A is an element of B)

- For any set A we have
$\emptyset \subset A$ and $A \subset A$
- Two sets A, B are equal if $A \subset B$ and $B \subset A$. We write $A = B$

Operations with sets

Suppose A, B are sets.

- We write $\boxed{A \cup B}$ for the set $\{x : x \in A \text{ or } x \in B\}$
We read $A \cup B$ as "A union B"
(or "the union of A and B")
- We write $\boxed{A \cap B}$ for the set $\{x : x \in A \text{ and } x \in B\}$
We read $A \cap B$ as "A intersect B"
(or "the intersection of A and B")

- We write $\boxed{A \setminus B}$ for the set $\{x : x \in A \text{ and } x \notin B\}$

we read $A \setminus B$ as "the set A minus the set B"

(or "A take away B")

- If $A \subset X$ then we call $X \setminus A$ the complement of A in X

- Let $A, B \subset X$ (so A and B are subsets of a set X)

Then $A \subset B$ if and only if $X \setminus B \subset X \setminus A$

iff or $\Leftrightarrow$

Proof: the statement asks us to prove two things: ① if $A \subset B$ then $X \setminus B \subset X \setminus A$ ("$\Rightarrow$")

② if $X \setminus B \subset X \setminus A$ then $A \subset B$ ("$\Leftarrow$")

Proof of ① (i.e. of "$\Rightarrow$")

Assume $A \subset B$. Let $x \in X \setminus B$. We want to show that $x \in X \setminus A$ i.e. that x does not belong to A. Indeed, for if $x \in A$ then we would have $x \in B$ (because $A \subset B$) giving us $x \in A$ and $x \notin A$ at the same time which is absurd (contradiction)

Proof of ② (i.e. of "$\Leftarrow$")

Assume $X \setminus B \subset X \setminus A$. We want to prove that $A \subset B$. Let $a \in A$.

If we assume by contradiction that $x \notin B$, then $x \in X \setminus B$. But this implies $x \in X \setminus A$, in other words $x \notin A$.

So again we get $a \notin A$, $a \in A$, absurd (contradiction)

- If $A \subset X$ then $X \setminus (X \setminus A) = A$

(the complement of the complement of a set A is the set A itself).

Proof: Need to show $X \setminus (X \setminus A) \subset A$ ①, and $A \subset X \setminus (X \setminus A)$ ②

① Let $x \in X \setminus (X \setminus A)$. This means $x \notin X \setminus A$. We want to show that $x \in A$. Indeed, for if we assume (by contrad.) that $x \notin A$, then $x \in X \setminus A$, so we get $x \in X \setminus A$ and $x \notin X \setminus A$, a contradiction.

② is similar (exercise).

- If A, B, X are arbitrary sets then $A \subset B$ implies $X \setminus B \subset X \setminus A$. Proof: exercise

1.1 Theorem (de Morgan's Laws). If A, B, X are sets then we have

(I) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

(II) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proof (I) "$\subset$" If $x \in X \setminus (A \cup B)$

then $x \in X \setminus A$ (because $A \subset A \cup B$ so $X \setminus (A \cup B) \subset X \setminus A$ by previous fact)

and $x \in X \setminus B$

(because $B \subset A \cup B$)

Thus $x \in (X \setminus A) \cap (X \setminus B)$

(I) "$\supset$" Assume $x \in (X \setminus A) \cap (X \setminus B)$.

This means $x \notin A$ and $x \notin B$.

We want to prove that $x \in X \setminus (A \cup B)$

Assume by contradiction that this is not true. Thus $x \notin X \setminus (A \cup B)$ which by a fact shown before means $x \in A \cup B$. So we have at the same time $x \in A \cup B$ and $x \notin A \cup B$ which is a contradiction

Do proof of (II) as an exercise!

(included as HW1)

For today, we introduce the notion (or concept) of $\boxed{\text{Functions}}$

2.1 Definition. A function is a triple consisting of:

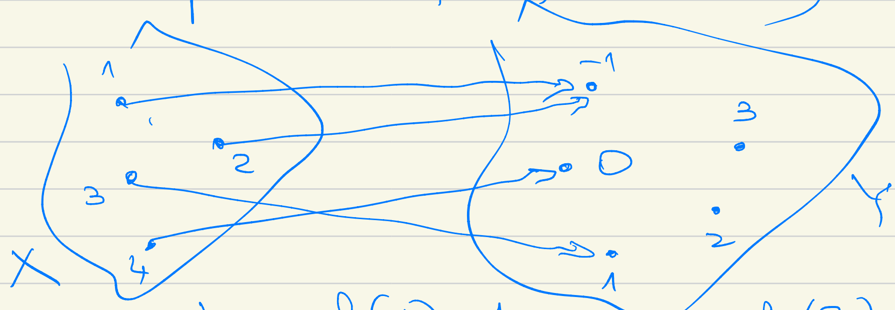
- a set X called the domain of the function
- a set Y called the codomain of the function
- a rule of assigning to each element $x \in X$ a unique element $y \in Y$

(often this "rule" or "assignment" is given by a formula)

We write such a triple $f : X \to Y$ with the y assigned x denoted $f(x)$

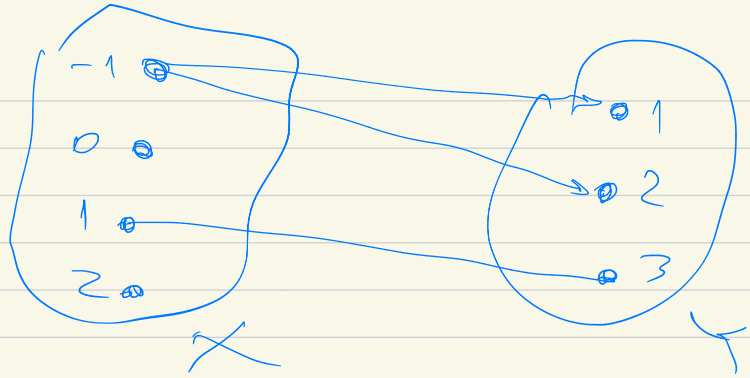
(we often use the notation $x \mapsto f(x)$ to emphasize that the function f assigns $f(x)$ to x; read it "x maps to $f(x)$")

Example



$f : X \to Y$ is defined here by $f(1) = -1$, $f(2) = -1$, $f(3) = 0$, $f(4) = 1$

Example



this triple, consisting of the sets $X = \{-1, 0, 1, 2\}$, $Y = \{1, 2, 3\}$ and assignment shown by the arrows **does not** represent a function, because it assignes to -1 both 1 and 2 and because one assigns no element in Y to $0 \in X$

Example

$$f : \overset{X}{\mathbb{R}} \to \overset{Y}{\mathbb{R}}$$

$$f(x) = x^2$$

the assignment rule (a formula in this case)

to each $x \in X = \mathbb{R}$ one assigns a unique $y \in X = \mathbb{R}$, namely $y = x^2$

2.2. Definitions. Let $f : X \to Y$ be a function

(2.2.1) We say that f is *injective* if whenever $x_1, x_2 \in X$ are so that $f(x_1) = f(x_2)$, it implies $x_1 = x_2$

Equivalently: non-equal elements in X map to non-equal elements in Y, under f
i.e. $x_1 \neq x_2$ in X implies $f(x_1) \neq f(x_2)$

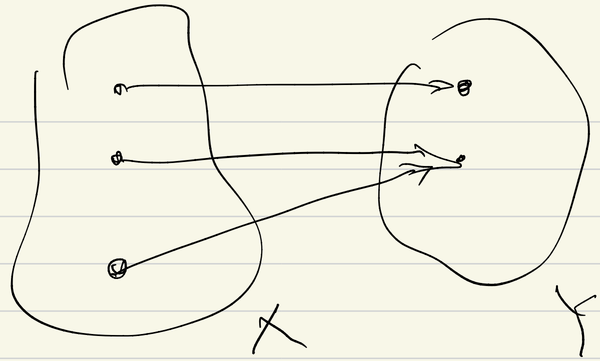
(2.2.2) We say that f is *surjective* if for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

equivalently, any $y \in Y$ is the image of some $x \in X$, under f

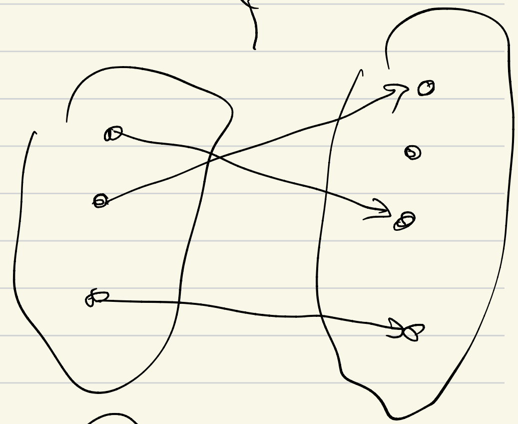
(2.2.3) We say that f is *bijective* if f is both injective and surjective

Not: This is same as saying that any $y \in Y$ is the image of exactly one $x \in X$, under f

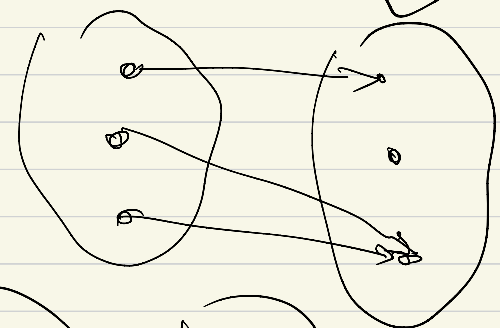
2.3 Example (1)



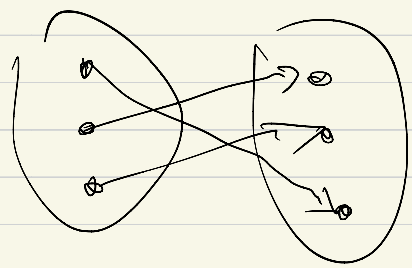
surjective but not injective



(2), injective but not surjective



(3). not inj., not surj.



(4) both inj. & surj. so it is bijective

2.4 Exercises: Let X, Y be finite sets with X having n elements, Y having m elements (given a set A one often denotes $|A|$ the number of elements in A, so above we have $|X| = n$, $|Y| = m$). Let $f: X \to Y$ be a function with domain X, codomain Y

1° Show that if f is surjective, then
$n \geq m$ (i.e. $|X| \geq |Y|$)

2° Show that if f is injective, then
$n \leq m$ ($|X| \leq |Y|$)

3° Show that if f is bijective then
$n = m$ ($|X| = |Y|$)

4° (Pigeonhole Principle) Show that
if $n > m$ ($\underbrace{|Y|}_{m} < \overbrace{|X|}^{n}$) then
there must exist $y \in Y$ so that
$f(x_1) = y$, $f(x_2) = y$ for some $x_1, x_2 \in X$
$x_1 \neq x_2$

5° Assume $m = n$ (so $|X| = |Y|$)
Show that f is injective iff f is surj.
iff f is bijective.

3.5 Related Exercise Let X, Y be finite
sets $|X| = n$, $|Y| = m$.

1° Show that there exist m^n many
functions $f: X \to Y$

2° If $|X| = |Y| = n$ then there exist
$n! = n(n-1)(n-2)\cdots 2\cdot 1$ many bijections
$f: X \to Y$

3°. If $m = |Y| \geq |X| = n$ then there are $m\cdot(m-1)\cdots(m-n+1) = \frac{m!}{(m-n)!}$ many injective functions $f : X \to Y$

4° Let $X = \{1,2,3,4\}$, $Y = \{1,2,3\}$ calculate the number of surjections from X to Y.

$\boxed{\text{2.6. Example}}$ Consider the following four functions

(a) $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^2$

(b) $f_2 : \mathbb{R} \to \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $f_2(x) = x^2$

(c) $f_3 : \mathbb{R}_+ \to \mathbb{R}$, $f_3(x) = x^2$

(d) $f_4 : \mathbb{R}_+ \to \mathbb{R}_+$, $f_4(x) = x^3$

Then all f_1, f_2, f_3, f_4 are indeed well defined functions and we have:

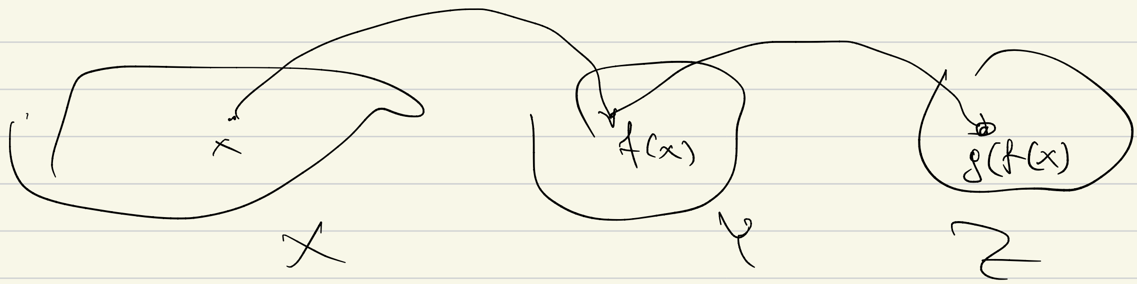
f_1 is not inj. nor surj.

f_2 is surj. but not inj.

f_3 is inj. but not surj.

f_4 is both inj & surj., so it is bijective

2.7. Definition. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. The composition of f and g is the function $g \circ f : X \to Z$ defined by $g \circ f\,(x) = g(f(x))$



2.8 Definition Let $f : X \to Y$ be a function and $X_0 \subset X$ a subset. The restriction of f to X_0, denoted $f|_{X_0} : X_0 \to Y$ is the function with domain X_0, codomain Y and assignement given by: for $x \in X_0$

$$f|_{X_0}(x) = f(x)$$

(same assignement rule as f)

2.9 **Theorem.** Let $f : X \to Y$, $g : Y \to Z$ be functions.

(a) if $g \circ f$ is injective then f is injective

(b) if $g \circ f$ is surjective then g is surjective.

(c) if f, g are injective then $g \circ f$ is injective

(d) if f, g are surjective then $g \circ f$ is surjective

Proof exercise

One important thing that you need to become comfortable with is formulating negations of mathematical statements.

Here are some samples:

2.10 Examples Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(1) Negate the following statements

(a) For all $b \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $f(x) = a$

(b) For all $x, y \in \mathbb{R}$, if $f(x) = f(y)$ then $x = y$

Non (a) reads:

There exists $b \in \mathbb{R}$ such that for any $x \in \mathbb{R}$ we have $f(x) \neq a$

Non (b) reads:

there exist $x, y \in \mathbb{R}$ such that $f(x) = f(y)$ and $x \neq y$

(2) Negate the statement below written with quantifiers:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$

Non (2) there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exist $x, y \in \mathbb{R}$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$