Math 115A assign1

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Question 1

Prove in details de Morgan's law I didn't solve in class: for any sets X, A, B we have $X - (A \cap B) = (X - A) \cup (X - B)$
Proof:
1. If $x \in X - (A \cap B)$ Then, $(X - A) \in x$ and $(X - B) \in x$. Thus $x \in (X - A) \cup (X - B)$

2. Assume $x \in (X - A) \cup (X - B)$, which means that $A \notin x$ and $B \notin x$. Base on the other side of the equation, $x \in X - (A \cap B)$, we can assume by contradiction that this is not true. Thus $x \notin X - (A \cup B)$, which means $x \in A \cap B$. So we have at the same time $x \notin A \cap B$ and $x \in A \cap B$ which is a contradiction

Question 2

Let A, B be subsets of the set of real numbers $\mathbb{R}$ For each one of the statements below write its negation.

(a) For all $x \in A$ there exists $b \in B$ such that $b > x$

Negation:
There is no $b \in B$ for all $x \in A$ such that $b > x$

(b) There is an $x \in A$ such that for all $b \in B$ we have $b > x$

Negation:
There is no such $x \in A$ for all $b \in B$ where $b > x$

Question 3

Consider the set $\mathbb{Z}_3$={0,1,2} and define on it the operation $+$ and $*$ as follows:
For $+$:
$0 + 0 = 0; 0 + 1 = 1; 0 + 2 = 2$
$1 + 0 = 1; 1 + 1 = 2; 1 + 2 = 0$
$2 + 0 = 2; 2 + 1 = 0; 2 + 2 = 1$
For *:
$0 * 0 = 0; 0 * 1 = 0; 0 * 2 = 0$
$1 * 0 = 0; 1 * 1 = 1; 1 * 2 = 2$
$2 * 0 = 0; 2 * 1 = 2; 2 * 2 = 1$

Show that $(\mathbb{Z}_3, +, *)$

proof:
To be determine as a field, it has to follow the five laws of field. It does satisfied(F1), where $a + b = b + a$ and $a * b = b * a$. It does satisfied (F2), where $(a + b) + c = a + (b + c)$ and $(a * b) * c = a * (b * c)$. It does satisfied

(F3) where the set contains 0 and 1 such that $0 + a = a$ and $1 * a = a$, $\forall a \in F$. It does satisfied (F4) where all element, except 0, has a inverse element that is belong to the set. Also, it does satisfied (F5), where $a * (b + c) = a * b + a * c$

Question 4

Consider the subset of $\mathbb{C}$:
$Q(i) := x + yi : x, y \in Q$
Prove that Q(i) with addition and multiplication inherited from $\mathbb{C}$(i.e. defined the some way as for all complex numbers) is field

proof:
To be determine as a field, it has to follow the five laws of field. It does satisfied(F1), where $(a + bi) + (c + di) = (c + di) + (a + bi)$ and $(a + bi) * (c + di) = (c + di) * (a + bi)$. It does satisfied (F2), where $((a + bi) + (c + di)) + (x + yi) = (a + bi) + ((c + di) + (x + yi))$ and $((a + bi) * (c + di)) * (x + yi) = (a + bi) * ((c + di) * (x + yi))$. It does satisfied (F3) where the set contains 0 and 1 such that $0 + (a + bi) = (a + bi)$ and $1 * (a + bi) = (a + bi)$, $\forall a \in F$. It does satisfied (F4) where all element, except 0, has a inverse element that is belong to the set. Also, it does satisfied (F5), where $(a + bi) * ((c + di) + (x + yi)) = (a + bi) * (c + di) + (a + bi) * (x + yi)$

Question 5

Let $(F, +, *)$ be a field, with its operations of addition $+$ and multiplication $*$. Let $E \in F$ be a subset and assume the following fields
* **For all $a, b \in E$, $a + b \in E$ and $a * b \in E$. (E is closed to addition & multiplication)**
* $0 \in E$**,** $1 \in E$
* **If $a \in E$ then $-a \in E$**
* **If $a \in E$, $a \neq 0$ then $a^{-n} \in E$**
* **Prove that E with the solve operations $+$ and $*$ as F, is a Field. (i.e. with the operations it inherits from F)**

proof:
As known that $E \in F$ and set F is a field, so set E can satisfied (F1)(F2)and (F5). Also, we know that $0 \in E$ and $1 \in E$ where satisfied (F3). Finally, we do have that, "if $a \in E$ then $-a \in E$ and if $a \in E, a \neq 0$ then $a^{-n} \in E$". Therefore, satisfied (F4) and E is a field.

Question 6

Let $(F, +, *)$ be a field.
(a) Prove that $(-1) * a = -a$ and $(-2) * a = -(2 * a)$ for any $a \in F$
(b) Prove that for any $a \in F$, $a \neq 0$, we have $(-a)^{-1} = -a^{-1}$

proof:
a) According to definition of field, (F2), where $(a * b) * c = a * (b * c)$ then$(-1) * a = (-1)(1)(a) = -a$ and $(-2) * a = (-1) * (2) * (a) = -(2 * a)$
b) According to definition of field, (F2), where $(a * b) * c = a * (b * c)$ then$(-a)^{-1} = (-1)^{-1} * (1)^{-1} * (a)^{-1} = (-1) * (1) * (a)^{-1} = -a^{-1}$