Math115A 1/18 notes

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2023-01-19

Recall from last time:
Definition:
A vector space (or a linear space) over a field F is a set V on which two operations are defined, called addition and scalar multiplication, so that for each $x, y \in V$ we gave a unique element $x + y \in V$(x plus y) and for each $x \in V$, $a \in F$ we have a unique element $ax \in V$ (a times x) such that the following conditions are satisfied:

VS1 $x + y = y + x$, $\forall x, y \in V$ (commutative if addition)
VS2 $(x + y) + z = x + (y + z)$,$\forall x, y \in z \in V$(association of addition)
VS3 There exists an element in V denoted 0 such that $x + 0 = x, \forall x \in V$(the zero or neutral element in V)
VS4 For each $x \in V$ there exists $y \in V$ such that $x + y = 0$ (the additive inverse in V, or opposite of x)
VS5 For each $x \in V$, we have $1x = x$
VS6 For each $a, b \in F, x \in V$, we have $(ab)x = a(bx)$
VS7 For each $a \in F$ and $x, y \in V$ we have $a(x + y) = ax + ay$
VS8 For each $a, b \in F$ and $x \in V$ we have $(a + b)x = ax + by$

The most common case of field determined by us will be $F = \mathbb{R}$(the field of real numbers with its usual + and * operations)

4.1 Example

Given a field F and some $n > 1$ the set of all n-tuples $v = (v_1, ...v_n)$ with entries $v_1, ...v_n \in F$ is denoted F^n. it is a vector space over F with respect to the operations + of addition "coordinate by coordinate". if $v = (v_1, ...v_n), u = (u_1, ..., u_n)$ then $v + u = (v_1 + u_1, v_2 + u_2...., v_n + u_n)$ and scalar multiplication $a(v_1, ..., v_n) = (av_1, av_2, ..., av_n) \in F^n$
it is immediate to check that VS1-8 are all satisfied, as a consequence of the properties satisfied by + and * in the field F.
Take for instance $F = \mathbb{R}$ and $n = 3$, then $v = (-1, 2, -4), u = (\frac{1}{2}, -1, 3) \in R^3 = V$ and we have $v + u = (-1, 2, -4) + (\frac{1}{2}, -1, 3) = (-\frac{1}{2}, 1, -1)$. Also, if $a = -1$ then $av = (1, -2, 4)$

4.2 Example

if F is a field and $m, n \geq 1$ integers, then the set of all $m \times n$ matrices over F, denoted $M_{m\times n}(F)$, is a vector space over F with respect to the following operations of addition and scalar multiplication:
if $A, B \in M_{m\times n}(F)$ have entries

$$A = \begin{pmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ | & & & | \\ A_{m1} & A_{m2} & ... & A_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} B_{11} & B_{12} & ... & B_{1n} \\ B_{21} & B_{22} & ... & B_{2n} \\ | & & & | \\ B_{m1} & B_{m2} & ... & B_{mn} \end{pmatrix}$$

then

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & ... & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & ... & A_{2n} + B_{2n} \\ | & & & | \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & ... & A_{mn} + B_{mn} \end{pmatrix}$$

and if $a \in F$ then

$$aA = \begin{pmatrix} aA_{11} & aA_{12} & ... & aA_{1n} \\ aA_{21} & aA_{22} & ... & aA_{2n} \\ | & & & | \\ aA_{m1} & aA_{m2} & ... & aA_{mn} \end{pmatrix}$$

For instance, take $F = \mathbb{R}$ abd $M_{2\times 3}(\mathbb{R})$ the vector space of 2×3 matrices over $\mathbb{R}$

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & \frac{1}{2} & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 0 & 2 \\ 0 & \frac{1}{2} & -2 \end{pmatrix}$$

then $A + B = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

The vector $0 \in M_{2\times 3}(R)$ is the 2×3 matrix with all entries=0

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4.3 Example

if F is a field and S is a non-empty set, then we denote $F(S, F)$ the set of all functions $f : S \to F$ with domain S and codomain F, with addition given by:
if $f : S$ and $g : S \to F$ then $(f + g) : S \to F$ is the function on S with values in F defined by $(f + g)(s) = f(s) + g(s) \in F$ and if $a \in F$ then scalar multiply by a is $af : S \to F$ defined by $(af)(s) = af(s) \in F$
Remark if we take $S = 1, 2, 3, ..., n$ a finite set with $n \geq 1$ elements, then $F([1, 2, 3, ...n], F)$ is the same as F^n

4.4 Example

A sequence with elements in F is a function $f : [1, 2, ...] \to F$ defined an real positive integers with values(codomain) in F if $f(n) = a_n \in F, n = 1, 2, ...$ is an enumeration of the value of f then are also writes the sequence f as $(a_n)_n$

The set of sequences in F is a particular case of the Example 4.3 where $S = [1, 2, ...]$, and so it is a vector space V with respect to “entry by entry” addition of sequences and scalar multiplication given by:
$(a_n)_n + (b_n)_n = (a_n + b_n)_n$ if $c \in F$ then $c(a_n)_n = (ca_n)_n$

4.5 Example

Let $V = [(a_1, a_2) : a_1, a_2 \in \mathbb{R}]$ and for $(a_1, a_2), (b_1, b_2) \in V, c \in \mathbb{R}$ define:
$(a_1 + a_2) + (b_1 + b_2) = (a_1 + 2b_1, a_2 + 3b_2)$
$c(a_1, a_2) = (ca_1, ca_2)$

Note: V as a set is the same as $\mathbb{R}$ but we modified the definition of + (but scalar multiplication is same as in $\mathbb{R}$)
Is V a vector space with this + and ×?
No because + fails to satisfy VS1
i.e. + as defined is not commutative $(a_1 + a_2) + (b_1 + b_2) = (a_1 + 2b_1, a_2 + 3b_2)$ while $(b_1 + b_2) + (a_1 + a_2) = (b_1 + 2a_1, b_2 + 3a_2)$
For instance we take $(1, 2) + (2, 1) = (1 + 4, 2 + 3) = (5, 5)$ but $(2, 1) + (1, 2) = (2 + 2, 1 + 6) = (4, 7)$
Note Also VS8 is not satisfied because (1+1)$(a_1, a_2) = (a_1, a_2)$+$(a_1, a_2) = (3a_1, 4a_2)$ but $2(a_1, a_2) = (2a_1, 2a_2)$

4.6 Theorem

(cancellation rule for vector addition)
if V is vector space over the field F and $x, y, z \in V$ so that $x + z = y + z$ then $x = y$
Proof:
By the axioms of $(V, +)$, there exist $v \in V$ such that $z + v = 0$. Thus, we get that $x + z = y + z$ implies $(x + z) + v = (y + z) + v$ which by VS2 implies $x + (z + v) = y + (z + v)$ so $x + 0 = y + 0$ which by VS3 implies $x = y$

4.7 Theorem

Let V be vector space ever field F
(a) The vector $0 \in V$ is unique with the property VS3 i.e. if $0' \in V$ satisfies $x + 0' = x, \forall x \in V$ then $0' = 0$

(b) If $x \in V$ then there exists a unique $y \in V$ with $x + y = 0$, i.e. if $y' \in V$ satisfies $x + y' = 0$ then $y = y'$
proof
(a) if 0' satisfies $x + 0' = x, \forall x \in V$ then $0 + 0' = 0$
But by VS1 we have $0 + 0' = 0' + 0$ are by VS3 the element 0 satisfies itself the property $x + 0 = x, \forall x \in V$ which applied to $x = 0'$ gives $0' + 0 = 0'$. Thus
$0 + 0' = 0$ and $0' + 0 = 0'$ so $0 = 0'$
(b) if $x + y = 0, x + y' = 0$ then by VS1 we have $y + x = y' + x$ and by cancellation theom 4.6 thus implies $y = y'$

4.8 Theorem

If V vector space over field F then we have

(a) $0x = 0, \forall x \in V$

(b) $(-a)x = -(ax) = a(-x), \forall x \in V, a \in F$
(c) $a0 = 0, \forall a \in F$
Proof
(a) we have $0 + 0 = 0$ so $0x = (0 + 0)x = 0x + 0x$ by cancellation rule $\rightarrow 0 = 0x$
(b) $(-a)x + ax = 0, -(ax) + ax = 0, and a(-x) + ax = 0$ Therefore, ax is the inverse of $(-a)x$, $-(ax)$, and $a(-x)$, and $(-a)x = -(ax) = a(-x)$
(c) $a0 = a(0 + 0) = a0 + a0$ so by theom 4.6 (cancellation law) 0= a0