# Math115 assign4

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2023-02-08

From Section 1.5: exercise 1 parts (b), (d), (e); exercise 2 parts (a), (b), (c); exercises 7, 16, 18, 20.

From Section 1.6: exercise 1 parts (a) through (g); exercise 2 part (a); exercise 3 part (a); exercises 4, 5, 11, 12, 15, 31.

## 1.5

**1. Label the following statements as true or false.**

b) Any set containing the zero vector is linearly dependent. **False**
d) Subsets of linearly dependent sets are linearly dependent. **False**
e) Subsets of linearly independent sets are linearly independent. **True**

**2. Determine whether the following sets are linearly dependent or linearly independent.**

a) Linearly dependent, $c_1 = 1, c_2 = -2$
b) Linearly independent
c) Linearly independent

**7. Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2\times2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.**

As long as the diagonal matrices is a $m \times n$ zero matrices the set can generate this subspace is $S = \{0\}$

**16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.**

**Proof:** Recall that the subset of a linear independent set will be independent. So we can say that any random subset $S' \in S$ is linear independent iff S is linear independent. Therefore, when all of the S' in S are linear independent we will have S, the original set, is linear independent.

**18. Let S be a set of nonzero polynomials in P(F) such that no two have the same degree. Prove that S is linearly independent.**

*Proof:* $S \in P(F)$, $S = \{1, X, X^2, X^3 ....X^n\}$ where S contains n elements. We want say that $cS = \{c_0 + c_1X + c_2X^2 + ...c_nX_n\} = 0$. There are no two element have the same degree, then the only way is set $c_0 = c_1 = c_2 = ... = c_n = 0$. Therefore, S is linear independent.

**20. Let $f, g \in F(R, R)$ be the function defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$ where $r \neq s$. Prove that f and g are linearly independent in $F(R, R)$**

*Proof:* Base on the implementation of f and g, we know that f and g will never have the same degree unless t equal to zero. Then if we want compute $f + g = 0$ we must have that $t = 0$. Therefore, f and g are linearly independent in $F(R, R)$

## 1.6

**1. Label the following statements as true or false.**

**(a) The zero vector space has no basis.**
False
**(b) Every vector space that is generated by a finite set has a basis.**
True
**(c) Every vector space has a finite basis.**
False
**(d) A vector space cannot have more than one basis.**
False
**(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.**
True
**(f) The dimension of $P_n(F)$ is n**
False
**(g) The dimension of $M_{m\times n}(F)$ is m+n**
False

**2. Determine which of the following sets are bases for $R^3$**

a) $c_1(1, 0, -1) + c_2(2, 5, 1) + c_3(0, -4, 3) = (0, 0, 0)$ $c_1 + 2c_2 = 0, 5c_2 - 4c_3 = 0, -c_1 + c_2 + 3c_3 = 0$, Then the only way to have all three statement equal is to have $c_1 = c_2 = c_3 = 0$

**3. Determine which of the following sets are bases for $P_2(R)$**

a) $c_1(-1, -1, 2) + c_2(2, 1, -2) + c_3(1, -2, 4) = 0$ and $-c_1 + 2c_2 + c_3 = 0, -c_1 + c_2 - 2c_3 = 0, 2c_1 - 2c_2 + 4c_3 = 0$, we can have that $c_1 = 7, c_2 = -3, c_3 = 1$ Then it is not a basis of $P_2(R)$

**4. Do the polynomials $x^3 - 2x^2 + 1, 4x^2 - x + 3$ and $3x - 2$ generate $P_3(R)$? Justify your answer**

$dim(P_3(R)) = 4$, then the number of element in the basis need to be 4. Therefore, its not.

**5. Is $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ a linear independent subset of $R^3$? Justify your answer**

$dim(R^3) = 3$, However the number of element in the subset is 4. Therefore, its not.

**11. Let u and v be distinct vectors of a vector space V. Show that if {u, v} is a basis for V and a and b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, bv\}$ are also bases for V.**

1) $c_1(u+v) + c_2(au) = 0$ we can have that $c_1u + c_1v + c_2au = (c_1 + ac_2)u + c_1v = 0$ Then we can say that $c_1 + ac_2 = 0, c_1 = 0$ or $c_1 = c_2 = 0$ Therefore, $\{u+v, au\}$ is linear independent

2) $c_1(au) + c_2(bv) = ac_1u + bc_2v = 0$ We can have that $ac_1 = 0$ and $bc_2 = 0$. Therefore, linear independent for $\{au, bv\}$
   and if {u, v} is a basis for V and a and b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, bv\}$ are also bases for V.

**12. Let u, v, and w be distinct vectors of a vector space V. Show that if {u, v, w} is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V**

**Proof:** $a(u+v+w) + b(v+w) + c(w) = au + (a+b)v + (a+b+c)w = 0$ and we can conclude that $a = b = c = 0$ Therefore, if {u, v, w} is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

**15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(F)$ (see Example 4 of Section 1.3). Find a basis for W. What is the dimension of W? Proof:** A basis for the set of all $n \times n$ matrices with trace equal to zero is given by the matrices $\{E_ij\}$, where $E_ij$ is a matrix with 1 in the (i, j)th position and 0 elsewhere, for $i \neq j$. This set has $n(n-1)$ matrices, so it has dimension n(n-1), i.e. The dimension of W is n(n-1).

**31. Let $W_1$ and $W_2$ be subspace of a vector space V having dimensions m and n, respectively, where $m \geq n$**

**a) Prove that** $dim(W_1 \cap W_2) \geq n$

**Proof:** To prove that the dimension of the intersection of two subspaces $W_1$ and $W_2$ is greater than or equal to n, we use the fact that the dimension of a subspace is equal to the number of linearly independent vectors in a basis for that subspace.
Since $W_1$ and $W_2$ are subspaces of V, they each have a basis with m and n linearly independent vectors, respectively. When we take the intersection of $W_1$ and $W_2$, we are left with a set of vectors that are in both $W_1$ and $W_2$. This set also forms a basis for $W_1 \cap W_2$, and since it is a subset of the basis for $W_1$ and $W_2$, it must contain at least n linearly independent vectors.
Therefore, $dim(W1 \cap W2) \geq n$.

**b) Prove that** $dim(W_1 + W_2) \geq m + n$

**Proof:** Since $dim(W_1) = m$ and $dim(W_2) = n$, there exist bases $\{v1, v2, ..., vm\}$ for $W_1$ and $\{u_1, u_2, ..., u_n\}$ for $W_2$.
Then we can have $dim(W_1+W_2) = dim(W_1)+dim(W_2)-dim(W_1 \cap W_2) = m+n-dim(W_1 \cap W_2), dim(W_1 \cap W_2) > 0$ Therefore, $dim(W_1 + W_2) \geq m + n$