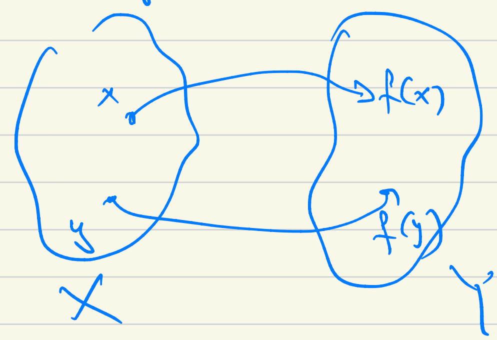
115A, Winter 2023

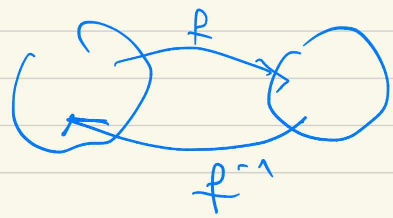
lecture 17

Fr, Feb 17

• Recall that a function $f : X \to Y$ is bijective if it is both one to one (injective) and onto (surjective).

in other words, 

a bijective function $f : X \to Y$ gives an "identification" between the sets X and Y

You may know from other classes that a bijective function $f : X \to Y$ has an inverse, 

i.e. a function $g : Y \to X$ satisfying

$$f \circ g = id_Y \quad , \quad g \circ f = id_X$$

or $f(g(y)) = y$, $g(f(x)) = x$, $\forall x \in X, y \in Y$

The function g is then called the inverse of f and is denoted f^{-1}

• When the function is a linear transformation $T: V \to W$ between vector spaces, we want to understand how its inverse as a function (in case T is bijective) fairs with vec. space structure of V & W. This is what we'll do today & Monday.

• Before we start, we recall one more time this basic fact:

if a function $f: X \to Y$ is bijective then it has an inverse, i.e. $\exists\ g: Y \to X$ such that $g \circ f = id_Y$, $f \circ g = id_X$

Conversely if f has an inverse (so $g: Y \to X$ with $g \circ f = id_Y$, $f \circ g = id_X$) then f is bijective.

• it is immediate to see that if f has an inverse, the inverse is unique. We denote it $g = f^{-1}$.

$\boxed{\text{17.1. Definition}}$ Let V, W be vector spaces $/_F$ and let $T : V \to W$ be linear. A function $U : W \to V$ is an inverse of T if $U \circ T = Id_V$, $T \circ U = id_W$. When T has an inverse, we say T is invertible

[We recalled above that the inverse of a function is unique]

We denote the (unique) function $U : W \to V$ satisfying this property by $U = T^{-1}$, call it ~~the~~ the inverse of T.

17.2 Properties (a). $V \xrightarrow{T} W \xrightarrow{U} Z$ with T, U invertible then $U \circ T : V \to Z$ invertible and one has $(UT)^{-1} = T^{-1}U^{-1}$

(b) $V \xrightarrow{T} W$ invertible, then

T^{-1} invertible and one has $(T^{-1})^{-1} = T$.

(c). If $V \xrightarrow{T} W$ with V, W fin. dim vector spaces / $\mathbb{F}$ and $\dim V = \dim W$. Then T invertible iff $R(T) = W$ iff $N(T) = \{0\}$.

Pf. (a), (b) are general properties for bijective functions between sets. (c) is just Thm 12.5 ☒

17.3 Theorem. Let $V \xrightarrow{T} W$ be vector spaces with T linear and invertible. Then T^{-1} is linear!

Pf. Let $y_1, y_2, y \in W$. We want to show

(a) $\underbrace{T^{-1}(y_1+y_2)}_{x} = \underbrace{T^{-1}(y_1) + T^{-1}(y_2)}_{x'}$

(b) $T^{-1}(cy) = c\,T^{-1}(y)$, $\forall c \in F$

Since T is one to one,
we have that $x, x' \in X$ are equal, $x = x'$
iff $T(x) = T(x')$. We apply this
fact to $x = T^{-1}(y_1 + y_2)$

$x' = T^{-1}(y_1) + T^{-1}(y_2)$

We have $T(x) = \underbrace{T(T^{-1}}_{T \circ T^{-1} = id}(y_1 + y_2)) = y_1 + y_2$

and $T(x') = T(T^{-1}(y_1) + T^{-1}(y_2))$

$\underset{T \text{ is linear}}{=} \underbrace{T(T^{-1}}_{= id}(y_1)) + \underbrace{T(T^{-1}}_{= id}(y_2))$

$= y_1 + y_2$

Thus, $T(x) = T(x')$ so we must have $x = x'$, i.e. we get (a)

(b) is similar (exercise)

$\boxtimes$

17.4 Theorem $V \xrightarrow{T} W$ with V, W vec. spaces$/_F$
& T linear.

(a). If $\dim V < \infty$ and T is onto then $\dim W < \infty$ and in fact $\dim W \le \dim V$

(b). If $\dim W < \infty$ and T is 1 to 1 then $\dim V < \infty$ and $\dim V \le \dim W$

(c). If T is invertible (equivalently T is 1 to 1 & onto) then $\dim V < \infty$ iff $\dim W < \infty$

If this is the case then $\dim V = \dim W$

Pf. (a) If $\dim V < \infty$ and $\{v_1, \dots, v_n\}$ basis for V then T onto means $R(T) = W$, which implies $T(v_1), \dots, T(v_n)$ span W so $\dim W = \dim R(T) \le n = \dim V$

(b). If $\dim W < \infty$ and T 1 to 1 then $R(T) \subset W$ so $\dim R(T) \le \dim W$. But also if $\{v_1, \dots, v_n\} \subset V$ basis then $\{T(v_1), \dots, T(v_n)\}$

span $R(T)$ and are also lin. ind, because T is one to one.
so $\{T(v_1), \dots, T(v_n)\}$ basis in $R(T)$
so $\dim V = \dim(R(T)) \leq \dim W$.

(c). if T has inverse then T is both 1 to 1 and onto

so by (a), (b) we have

$$\dim V < \infty \text{ iff } \dim W < \infty$$

with $\dim V \leq \dim W$
$\geq$
$=$

17.5. Definition Let $A \in M_{n \times n}(F)$.
We say that the matrix A is invertible if there exists $B \in M_{n \times n}(F)$ such that $AB = I_n = BA$

- Note that, like for inverses

of functions, if $A \in M_{n\times n}(A)$ is invertible and $B \in M_{n\times n}(A)$ satisfies $AB = BA = I$ then B is unique with this property.

Indeed, if $C \in M_{n\times n}(F)$ also satisfies $AC = CA = I_n$ then we have

$$C = CI_n = C(\overbrace{AB}^{I_n})$$
$$= \underbrace{(CA)}_{=I_n} B = I_n B = B$$

so $C = B$.

We denote the (unique) B satisfying $AB = BA = I_n$ by A^{-1}, call it the inverse of A

17.6 Theorem. Let V, W be fin. dim. vector spaces$/_F$ with ordered bases β for V, γ for W.

Then T is invertible transf. iff $[T]_\beta^\gamma$ is invertible matrix.

And if this is the case, then we have

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$$

$\boxed{\text{Pf}}$. "$\Longrightarrow$" if T is invertible

Then $\dim V = \dim W$ let's denote by n the common dimension of V, W.

Thus, $[T]_\beta^\gamma \in M_{n\times n}(F)$.

Since we have $T^{-1}T = I_n$

$TT^{-1} = I_n$, from Theorem 15.6 it follows that

$$[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = I_n$$

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = I_n$$

So the matrix $B = [T^{-1}]_\gamma^\beta$

$\in M_{n\times n}(F)$ satisfies

$$B\,[T]_\beta^\gamma = I_n$$
$$[T]_\beta^\gamma\, B = I_n$$

Thus, $[T]_\beta^\gamma$ is invertible matrix and its inverse matrix $([T]_\beta^\gamma)^{-1}$ is equal to $B = [T^{-1}]_\gamma^\beta$

" $\Leftarrow$ " Assume $[T]_\beta^\gamma \in M_{n\times n}(F)$ is an invertible matrix, i.e. $\exists\ B \in M_{n\times n}(F)$ with

$$B\,[T]_\beta^\gamma = I_n$$
$$[T]_\beta^\gamma\, B = I_n$$

But then by (Theorem ... Lect. 14) there exists $U \in \mathcal{L}(W, V)$ such that $[U]_\gamma^\beta = B$, i.e. such that

$$U(w_j) = \sum_{i=1}^{n} B_{ij} v_i \quad \text{for } j = 1, \ldots, n$$

(where $\beta = \{v_1, \ldots, v_n\} \subset V$, $\gamma = \{w_1, \ldots, w_n\} \subset W$)

Since we have (by Thm 15.6)

$$[UT]_\beta^\beta = [U]_\gamma^\beta [T]_\beta^\gamma$$

$$= B [T]_\beta^\gamma = I_n$$

and since UT has matrix rep equal to I_n iff $UT = id_V$ it follows that

$$UT = id_V$$

Similarly $TU = id_W$

$\boxtimes$

Do exercises & aplications at blackboard !