Assign7

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from section 2.3 exercises: 16,18

from section 2.4 exercises: 1 (a)-(e); 3, 6, 7, 14, 16

from section 2.5 exercises: 1 (a)-(e) (so all of them); 2 (a) and (c); 4; 5.

2.3

16 Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

a) If $rank(T) = rank(T^2)$, prove that $R(T) \cap N(T) =\{0\}$. Deduce that $V = R(T) \bigoplus N(T)$

To prove that $R(T) \cap N(T) = 0$, we will assume that $v \in R(T) \cap N(T)$ and show that $v = 0$. Since $v \in R(T)$, there exists a vector $u \in V$ such that $T(u) = v$.
Moreover, since $v \in N(T)$, we have $T(v) = T^2(u) = T(T(u)) = T(v) = 0$.

Now, since $T(u) = v$ and $T(v) = 0$, we have $T(T(u)) = 0$, which means that $u \in N(T)$ (since $N(T)$ is the nullspace of T^2). Therefore, $u \in R(T) \cap N(T)$, which implies that $u = 0$ (because v and u are linearly independent, and $\dim(R(T)) + \dim(N(T)) = \dim(V)$).

But then, $v = T(u) = 0$, which shows that $R(T) \cap N(T) = 0$.

To deduce that $V = R(T) \oplus N(T)$, we will show that $V = R(T) + N(T)$ and $R(T) \cap N(T) = 0$, which implies that $V = R(T) \oplus N(T)$ by the direct sum theorem.

We have just shown that $R(T) \cap N(T) = 0$, so it only remains to show that $V = R(T) + N(T)$.

Let $v \in V$. Then, since T is a linear transformation from V to V, we have $\dim(R(T)) + \dim(N(T)) = \dim(V)$, so there exist vectors $r \in R(T)$ and $n \in N(T)$ such that $v = r + n$.

To see this, let $u \in V$ be such that $T(u) = v$. Then, $T^2(u) \in R(T)$ (by definition of $R(T)$), and $T^2(u) = T(T(u)) = T(v)$, so $T(v) \in R(T)$. Moreover, since $T(v) \in N(T)$ (by definition of $N(T)$), we have $T(v) = 0$, which implies that $v = T(u) = T(u) + 0 \in R(T) + N(T)$.

Therefore, we have shown that $V = R(T) + N(T)$ and $R(T) \cap N(T) = 0$, which implies that $V = R(T) \oplus N(T)$.

b) Prove that $V = R(T^k) \bigoplus N(T^k)$ for some positive integer k

Since V is finite-dimensional, we know that there exists a positive integer k such that nullity(T^k) =

nullity(T^{k+1}). Moreover, we have $\text{rank}(T^k) \leq \text{rank}(T^{k-1}) \leq \cdots \leq \text{rank}(T)$.

We will now show that $V = R(T^k) + \text{null}(T^k)$, which will imply that $V = R(T^k) \oplus \text{null}(T^k)$ by the direct sum theorem.

Let $v \in V$. Then, since T is a linear transformation from V to V, we have $\text{rank}(T^k) + \text{nullity}(T^k) = \dim(V)$, so there exist vectors $r \in R(T^k)$ and $n \in \text{null}(T^k)$ such that $v = r + n$.

To see this, let $u \in V$ be such that $T^k(u) = v$. Then, $T^{2k}(u) \in R(T^k)$ (by definition of $R(T^k)$), and $T^{2k}(u) = T^k(T^k(u)) = T^k(v)$, so $T^k(v) \in R(T^k)$. Moreover, since $T^k(v) = T^{2k}(u) = 0$ (since $u \in \text{null}(T^k)$), we have $v = r + n$, where $r = T^k(u) \in R(T^k)$ and $n = u \in \text{null}(T^k)$.

Therefore, we have shown that $V = R(T^k) + \text{null}(T^k)$, which implies that $V = R(T^k) \oplus \text{null}(T^k)$. Since $\text{null}(T^k) = \text{nullity}(T^k)$ and $\text{rank}(T^k) = \text{rank}(T^{k-1}) \leq \cdots \leq \text{rank}(T)$, we have $R(T^k) \subseteq R(T^{k-1}) \subseteq \cdots \subseteq R(T)$ and $\text{null}(T^k) \supseteq \text{null}(T^{k+1}) \supseteq \cdots \supseteq \text{null}(T)$.

Therefore, we have $V = R(T^k) \oplus \text{null}(T^k) = R(T^k) \oplus L(T^k)$, where $L(T^k) = \text{null}(T^k)$. This completes the proof.

18 Let β be an ordered basis for a finite-dimensional vector space V, and let $T : V \to V$ be linear. Prove that, for any nonnegative integer k, $[T^k]\beta = ([T_\beta])^k$

We will prove the result by induction on k.

For the base case $k = 0$, we have $[T^0]\beta = [I]\beta = I$, and $([T_\beta])^0 = I^0 = I$. Therefore, the result holds for $k = 0$.

Now, assume that the result holds for some nonnegative integer k. That is, we have $[T^k]\beta = ([T]\beta)^k$. We will show that the result also holds for $k + 1$.

We have:

$[T^{k+1}]_\beta = [T(T^k)]_\beta = [T]_\beta[T^k]_\beta = [T]_\beta([T]_\beta)^k = ([T]_\beta)^{k+1}$

Therefore, we have shown that if the result holds for k, then it also holds for $k + 1$. Since the result holds for $k = 0$, it holds for all nonnegative integers k by induction. That is, for any nonnegative integer k, we have $[T^k]\beta = ([T]\beta)^k$.

2.4

1

a) FALSE

b) TRUE

c) TRUE

d) FALSE

e) TRUE

f) FALSE

g) TRUE

h) TRUE

i) TRUE

3

a) $T : R^2 \to R^3$ **defined by** $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$
NO, $dim(F^3) \neq dim(P_3(F))$

c) $T : R^3 \to R^3$ **defined by** $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$
YES, $dim(M_{2\times 2}(R) = dim(P_3(R))$

6 Prove that if A is invertible and $AB = O$, then $B = O$

Proof:
$AB = O$, Then, $A^{-1}AB = A^{-1}O = O$, $A^{-1}AB = IB = B = O$

7 Let A be an $n \times n$ matrix

a) Suppose that $A^2 = O$.Prove that A is not invertible
Proof:
By contradiction, A is invertible where $A^{-1}A = I$. Then $A^2 = AA = O$, $A^{-1}AA = IA = A^{-1}O = O$. Because I is non-zero matrix, then the only way to have this expression be satisfied is have $A = O$. Then $A^2 = AA = OO = O$.

b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.
Proof:
Assume that A is invertible, where $AA^{-1} = I$. Then $AA^{-1}B = OA^{-1} = O$, $IB = O$ Then if A is invertible, the only way to have this satisfy is to have $B = O$. Therefore, because B is a non-zero matrix, then A can't be invertible.

14 Let $V =$ { $\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F$ **} Construct an isomorphism from V to F^3**

Basis of V: $\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$
$dim(V) = 3 = dim(F^3)$

First, we will show that φ is a linear transformation. Let $A = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} d & d+e \\ 0 & f \end{pmatrix}$ be arbitrary matrices in V, and let $k \in F$ be an arbitrary scalar. Then:

You're right, there was a mistake in the algebra. Here's the corrected version:

$\varphi(kA + B) = \varphi(k\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & d+e \\ 0 & f \end{pmatrix}) = \varphi\begin{pmatrix} ka+d & ka+d+b+e \\ 0 & kf \end{pmatrix} \begin{pmatrix} ka+d \\ b+e \\ kf \end{pmatrix} = \begin{pmatrix} ka \\ 0 \\ kf \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} k \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} k\varphi \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} + \varphi \begin{pmatrix} d & d+e \\ 0 & f \end{pmatrix} k\varphi(A) + \varphi(B)$

So φ satisfies the linearity property.

Next, we will show that φ is injective, i.e., that $\ker(\varphi) = \mathbf{0}$. Let $A = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}$ be an arbitrary element of V such that $\varphi(A) = \mathbf{0}$. Then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{0}$, so $a = b = c = 0$. Therefore, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $\ker(\varphi) = \mathbf{0}$.

Finally, we will show that φ is surjective, i.e., that $\text{range}(\varphi) = F^3$. Let $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be an arbitrary element of F^3. Then $\varphi\left(\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{v}$, so $\mathbf{v}$ is in the range of φ.

Since φ is a bijective linear transformation, it is an isomorphism from V to F^3.

16 Let B be an $n \times n$ invertible matrix. Define $\Phi : M^{n\times n}(F) \to M^{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism

Proof:
To show that Φ is an isomorphism, we need to show that it is a bijective linear transformation.

First, we will show that Φ is a linear transformation. Let $A, C \in M^{n\times n}(F)$ be arbitrary matrices, and let $k \in F$ be an arbitrary scalar. Then:$\Phi(kA + C) = B^{-1}(kA + C)B = k(B^{-1}AB) + B^{-1}CB = k\Phi(A) + \Phi(C)$

So Φ satisfies the linearity property.

Next, we will show that Φ is injective, i.e., that $\ker(\Phi) = \mathbf{0}$. Let $A \in M^{n\times n}(F)$ be an arbitrary matrix such that $\Phi(A) = \mathbf{0}$. Then $B^{-1}AB = \mathbf{0}$, so $A = \mathbf{0}$ since B is invertible. Therefore, $\ker(\Phi) = \mathbf{0}$.

Finally, we will show that Φ is surjective, i.e., that $\text{range}(\Phi) = M^{n\times n}(F)$. Let $C \in M^{n\times n}(F)$ be an arbitrary matrix. Since B is invertible, B^{-1} exists, and we have $\Phi(B^{-1}CB) = B^{-1}(B^{-1}CB)B = C$, so $\text{range}(\Phi) = M^{n\times n}(F)$.

Since Φ is a bijective linear transformation, it is an isomorphism from $M^{n\times n}(F)$ to itself.

2.5

1

a)FALSE
b)TRUE
c)TRUE
d)FALSE
e)TRUE

2

a) $Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

b) $Q = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

c) $Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

d) $Q = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

4 Let T be the linear operator on R^2 defined by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$ let β be the standard ordered basis for R^2, and let $\beta =${$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$}. Use Theorem 2.23 and the fact that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ to find $[T]_{\beta'}$

Proof:

We know that $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ Then $[T]_{\beta'} = [T]_\beta Q Q^{-1} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$

5 Let T be the linear operator on $P_1(R)$ defined by $T(p(x))$, the derivatibe of $p(x)$. Let $\beta =${$1, x$} and $\beta =${$1+x, 1-x$}. Use Theorem 2.23 and the fact that $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ to find $[T]_{\beta'}$

We know that $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$[T]_{\beta'} = [T]_\beta Q Q^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$