115A, Winter 2023

Lecture 12

Mo, Feb 6th

• Recall that we defined the notion of linear transformation from one vector space V to a vector space W, $T: V \to W$, when it satisfies $T(v_1 + v_2) = T(v_1) + T(v_2)$, $T(cv) = cT(v)$ $\forall\ v_1, v_2, v \in V$, $\forall\ c \in F$.

• We also defined the null space (or kernel) of a linear $T: V \to W$ by $N(T) = \{ v \in V : T(v) = 0_W \}$ and range of T, by $R(T) = \{ T(v) : v \in V \}$

• We proved in Thm 11.12 that $N(T)$, $R(T)$ are subspaces of V respectively W.

12.1 Theorem Let V, W be vector spaces and $T: V \to W$ linear.
If $S = \{ v_1, \dots, v_n \} \subset V$ is a basis for V

Then $R(T) = \text{span}(T(S))$

$= \text{span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$

$\boxed{\text{Pf}}$ We have $T(S) \subset T(V) = R(T)$ by the definitions, so $T(v_1), \ldots, T(v_n) \in R(T)$ and since $R(T)$ is a subspace we get $\text{span}(\{T(v_1), \ldots, T(v_n)\}) \subset R(T)$.

If $w \in R(T)$ is a vector in the range of T, then there exists $v \in V$ such that $T(v) = w$. But V is spanned by $\{v_1, \ldots, v_n\}$ (because $\{v_1 \ldots v_n\}$ is a basis for V), so $\exists$ $c_1, \ldots, c_n \in F$ such that $v = \sum_{i=1}^{n} c_i v_i$.

Thus $w = T(v) = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i T(v_i)$

$\in \text{span}\{T(v_1), \ldots, T(v_n)\}$.

$\boxtimes$

12.2. Definitions Let V, W be vector spaces and $T: V \longrightarrow W$ linear if $N(T)$, $R(T)$ are finite dimensional

Then we define the nullity of T, denoted nullity(T), to be the dimension of $N(T)$

and the rank of T, denoted rank(T), to be the dimension of $R(T)$

So
$$\begin{cases} \text{nullity}(T) \overset{\text{def}}{=} \dim(N(T)) \\ \text{rank}(T) = \dim(R(T)) \end{cases}$$

12.3. Theorem (Dimension Theorem)

Let V, W be vector spaces and $T: V \longrightarrow W$ linear. If V is finite dimensional then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

$\boxed{\text{Proof}}$. Let n denote the dimension of V and k the dimension of $N(T)$, i.e. $n = \dim(V)$, $k = \dim(N(T))$.

Let $\{v_1, \dots, v_k\} \subset N(T)$ be a basis for $N(T)$. By Corollary 11.3

$\{v_1, \ldots, v_k\}$ can be extended to a basis $S = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

$S \subset V$ basis

We'll show that

$S_0 = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\} \subset R(T) \subset W$

is a basis for $R(T)$.

To see this, we need to show that the set S_0 spans $R(T)$ and is lin. ind.

Indeed, since $v_1, \ldots, v_k \in N(T)$ we have $T(v_1) = T(v_2) = \ldots = T(v_k) = 0$

thus

$$R(T) = \operatorname{span}\{\overbrace{T(v_1), T(v_2), \ldots, T(v_k)}^{=0}, T(v_{k+1}), \ldots, T(v_n)\} = \operatorname{span}\{T(v_{k+1}), \ldots, T(v_n)\} = \operatorname{span}(S_0)$$

To see that S_0 is lin. ind., assume

$$\sum_{i=k+1}^{n} c_i \, T(v_i) = 0 \text{ for some } c_i \in F$$

using linearity of T we get

$$T\left(\sum_{i=k+1}^{n} c_i v_i\right) = 0$$

implying that $\sum_{i=k+1}^{n} c_i v_i \in N(T)$

Since $\{v_1, \ldots, v_k\} \subset N(T)$ is a basis for $N(T)$ this implies

$\exists \; c_1, \ldots, c_k \in F$ such that

$$\sum_{i=1}^{k} c_i v_i = \sum_{i=k+1}^{n} c_i v_i$$

Thus $$\sum_{i=1}^{k} (-c_i) v_i + \sum_{i=k+1}^{n} c_i v_i = 0$$

Since $S = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$

is a basis for V, this implies $c_i = 0$ for all $i=1,2,\dots,n$

in particular $c_{k+1}, \dots, c_n = 0$

showing that $S_0 = \{T(v_{k+1}), \dots, T(v_n)\}$

is indeed linearly ind.

Thus, S_0 is a basis for $R(T)$ and so $\dim(R(T))$ equals the number of elements in S_0, which is $n-k$

Thus $\dim(V) = k + (n-k)$

$= \dim(N(T)) + \dim(R(T))$

$\boxtimes$

12.4 Theorem. Let V, W be vector spaces and $T: V \to W$ linear

Then T is one to one iff

$N(T) = \{0\}$

Pf. if T is one to one

and $v \in N(T)$, then $T(v) = 0_W$

implies $v = 0_V$, so we get

$N(T) = \{0\}$ (because $T(0_V) = 0_W$)

if conversely $N(T) = \{0_W\}$

and we have $T(v_1) = T(v_2)$

for some $v_1, v_2 \in V$, then

$$0_W = T(v_1) - T(v_2) \overset{\text{because } T \text{ is linear}}{=} T(v_1 - v_2)$$

so $v_1 - v_2 \in N(T) = \{0_V\}$,

Thus $v_1 - v_2 = 0_V$, implying

$v_1 = v_2$, thus showing T is

one to one

$\blacksquare$

- An easy consequence of the Dimension Thm:

12.5 Theorem. Let V, W be finite dimensional vector spaces (with $\dim(V) = \dim(W)$) and $T : V \to W$ linear. Then the following conditions are equivalent:

(a) T is one to one (T is injective)

(b) T is onto (T is surjective)

(c) $\operatorname{rank}(T) = \dim(V)$

$\boxed{\text{Pf}}$ From Dimension Thm 12.3 we have

(*) $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$

We showed in Thm 12.4 that

T is one to one iff $N(T) = \{0\}$

iff $\operatorname{nullity}(T) = 0$.

Thus, by (*) we get

T one to one iff $\operatorname{rank}(T) = \dim(V)$

so (a) is equivalent to (c).

On the other hand, we have

T is onto iff $R(T) = W$ } by definition

iff $\operatorname{rank}(T) = \dim(W)$

and from (*) we get

$\dim(W) = \dim(V)$ iff $\text{nullity}(T) = 0$

i.e. iff T is one to one

Thus shows that (b) is equivalent to (a), so altogether (a), (b), (c) are all equivalent

- One nice consequence of Thms 12.3 - 12.5 is that it allows us to show to deduce very easily that certain linear transformations are one to one / onto by looking at dimensions.

$\boxed{\text{12.6 Exercise}}$ Show that $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a,b,c) = (a-b, 2c)$ is onto.

$\boxed{\text{Solution}}$ It is easier to see what the kernel of T is:

$$N(T) = \{(a,b,c) \in \mathbb{R}^3 : T(a,b,c) = 0\}$$

which by the definition of T means

$(a,b,c) \in N(T)$ iff $a-b=0$, $2c=0$

so $c=0$ and $a=b$ so

$$N(T) = \{(a,a,0) : a \in \mathbb{R}\}$$

Thus nullity $(N(T)) = 1$ so by 12.3

$$3 = \dim \mathbb{R}^3 = \dim(N(T)) + \dim(R(T))$$
$$= 1 + \dim(R(T))$$

so $\dim(R(T)) = 3-1=2$.

12.6. Theorem Let V, W be vector spaces and $T: V \to W$ linear.
Assume $\dim(V) = n$ is finite and let $\{v_1, \dots, v_n\} \subset V$ be a basis for V.
Then for any $w_1, \dots, w_n \in W$ there exists exactly one linear transformation $T: V \to W$ such that $T(v_1) = w_1, \dots, T(v_n) = w_n$.

$\boxed{\text{Pf}}$. Since $\{v_1, \dots, v_n\} \subset V$ is a basis any $v \in V$ can be uniquely expressed as $(*)$ $\left\{ v = \sum_{i=1}^{n} c_i v_i \right.$ for some scalars $c_1, \dots, c_n \in F$

We define $T: V \to W$

by $T(x) = \sum_{i=1}^{n} c_i w_i$.

To see T is linear, let $u, v \in V$ and $u = \sum_{i=1}^{n} a_i v_i$, $v = \sum_{i=1}^{n} c_i v_i$ be their expression as linear combinations in basis $\{v_1, \dots v_n\}$.

By the definition (*) of T, we have:

$$T(v) = \sum_{i=1}^{n} c_i w_i$$

$$T(u) = \sum_{i=1}^{n} a_i w_i$$

and since

$$u + v = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (a_i + c_i) v_i$$

we also have

$$T(u+v) = \sum_{i=1}^{n} (a_i + c_i) w_i = \sum_{i=1}^{n} a_i w_i + \sum_{i=1}^{n} c_i w_i$$

$$= T(u) + T(v)$$

Thus $T(u+v) = T(u) + T(v)$.

Similarly $T(cv) = cT(v)$

So T is linear. To see uniqueness of T, assume we have another linear transformation $U : V \to W$

satisfying $U(v_i) = w_i$, $i=1,\dots,n$

Then by linearity of U we get

$$U(v) = U\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i U(v_i) = \sum_{i=1}^{n} c_i w_i$$

$$= \sum_{i=1}^{n} c_i T(v_i) = T\left(\sum_{i=1}^{n} c_i v_i\right) = T(v)$$

and since any $v \in V$ is of the form $v = \sum_{i=1}^{n} c_i v_i$ for some scalars $c_1, \dots, c_n \in F$, it follows that

$U(v) = T(v)$ for all $v \in V$.

$\boxtimes$

- Another way to formulate the uniqueness part in Thm 12.6 is this:

12.7 Corollary. Let V, W be vector spaces, with $\{v_1, \dots, v_n\} \subset V$ a basis. If $T, U : V \to W$ are linear and $T(v_i) = U(v_i)$ $\forall\, i = 1, 2, \dots, n$ Then $U = T$.

$\boxed{\text{Pf}}$ clear. $\boxtimes$