115A/4, Winter 2023

Lecture 5

Fr, Jan 20

We'll talk today about subspaces of vector spaces

$\boxed{\text{5.1 Definition}}$ A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V

- Note that V itself is a subspace of V. Also, the set $\{0\}$, consisting of the vector $0 \in V$, is a subspace of V. It is called the zero subspace

- If W is a subset of a vector space V over a field F, and we are asked to check if W

is a subspace, then we don't need to check that all conditions $\boxed{VS1 - VS8}$ are satisfied: indeed we do know addition & scalar multiplication check the properties for all vectors in V, so all we need is in fact to check that addition of $x, y \in W$ and scalar multiplication ax for $x \in W$ keeps us within the set W.

5.2. Theorem Let V be a vector space over field F and $W \subset V$ a subset. Then W is a subspace of V if and only if the following conditions hold:

$\boxed{(a)}$ $0_V \in W$ (i.e. the zero vector of V belongs to W)

$\boxed{(b)}$ If $x, y \in W$ then $x + y \in W$
(i.e. the addition of any two vectors in W stays in W)

$\boxed{(c)}$ If $x \in W$ and $c \in F$ then $cx \in W$ (i.e. the multiplication by any scalar of a vector in W stays in W)

Proof "$\Rightarrow$" If W is a subspace of V then by Definition 5.1 it means that W with the operations of addition and scalar multiplication it "inherits" from V is a vector space. Thus $\boxed{(b) \& (c)}$ are satisfied. Also, W has by the def. of vector space a "zero element", say $0' \in W$, satisfying $x + 0' = x$, $\forall x \in W$

But we also have $x + 0 = x$ where 0 is here the zero element in V

By Thm 4.7 (a), this implies $0' = 0$, so we have $\boxed{(a)}$ as well.

"$\Leftarrow$" if $\boxed{(a), (b), (c)}$ hold

Then W with the addition and scalar multiplication of its elements viewed as elements in V satisfies $\boxed{VS1}$, $\boxed{VS2}$ and $\boxed{VS5-8}$ because we already know all these properties are satisfied in V.

All we need to show is that W has a zero vector (thus VS3) and any $x \in W$ has an additive inverse in W (or "opposite").

But condition $\boxed{(a)}$ tells us that $0 \in W$ (the zero element of V belongs to W), and this element does satisfy $x + 0 = x$ $\forall\, x \in W$ (because this holds true in fact for all elements x in V, in particular for all elements in W)

Also, by condition $\boxed{(c)}$ we have that $\forall x \in W$, the element $y = (-1)x$ belongs to W.

But by $\boxed{\text{Thm } 4.8\ (b)}$, we know that $y = (-1)x$ is the opposite of x in V. So $x \in W$ does have an additive inverse in W (what we also called an "opposite"). So $\boxed{VS\ 3\ \&\ VS\ 4}$ are satisfied as well

$\boxtimes$

5.3 Definitions Given a matrix $A = (A_{ij}) \in M_{m \times n}(F)$ its transpose is the $n \times m$ matrix $A^t \in M_{n \times m}(F)$ obtained from A by interchanging the rows with the columns, that is

$$(A^t)_{ij} = A_{ji} \quad \text{for all } 1 \le i \le n, \ 1 \le j \le m$$

For example ① if $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \\ -4 & 0 \end{pmatrix} \in M_{3\times 2}(\mathbb{R})$

then $A^t \in M_{2\times 3}(\mathbb{R})$ is

$$A^t = \begin{pmatrix} 0 & 2 & -4 \\ -1 & 3 & 0 \end{pmatrix}$$

② $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}^t = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$

③ $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -3 \end{pmatrix}^t = \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 2 & -3 \end{pmatrix}$

A symmetric matrix A is a matrix such that $A^t = A$

Note that such matrices necessarily have same # of rows & columns

i.e. $A \in M_{n \times n}(F)$.

Examples ① $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$

so it is symmetric

② Note that $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_{n \times n}(F)$ is symmetric.

5.4. Exercise. Show that the set of symmetric matrices in $M_{n \times n}(F)$ is a subspace of $M_{n \times n}(F)$.

Solution We need to check (b) & (c) of Thm 5.2 (we already noticed that the 0 matrix in $M_{n \times n}(F)$ is symmetric so we have (a) of Thm 5.2)

if $A, B \in M_{n \times n}(F)$ are symmetric then $A = (A_{ij})$ and $B = (B_{ij})$ satisfy $A = A^t$, $B = B^t$

so $A_{ij} = A_{ji}$, $B_{ij} = B_{ji}$
for all $1 \leq i, j \leq n$.

But then $(A+B)_{ij} = A_{ij} + B_{ij}$
so $((A+B)^t)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji}$
$= A_{ij} + B_{ij}$, showing that
$A+B$ is symmetric

Similarily, if $c \in F$ is a scalar
and $A = A^t$ then $(cA)_{ij} = cA_{ij}$
$((cA)^t)_{ij} = (cA)_{ji} = cA_{ij}$

so cA is symmetric $\boxtimes$

$\boxed{\text{5.5 Definitions}}$ A matrix $A \in M_{n \times n}(F)$
is called a diagonal matrix
if $A_{ij} = 0$ for all $i \neq j$, i.e. if
all entries off the diagonal of A

are equal to 0

$\boxed{\text{example}}$ $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$

is a diagonal matrix

$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in M_{4\times 4}(\mathbb{R})$

is a diagonal matrix

Note The 0 matrix in $M_{n\times n}(F)$ is a diagonal matrix, $\forall n$

$\boxed{\text{Exercise}}$ Show that the set of diagonal matrices $D_{n\times n}(F)$ in $M_{n\times n}(F)$ is a subspace.

$\boxed{\text{Sol}}$ indeed, we noticed that $0 \in D_{n\times n}(F)$

Also the sum of two diagonal matrices $A, B \in D_{n\times n}(F)$ satisfies $(A+B)_{ij} = A_{ij} + B_{ij}$ $= 0$ if $i \neq j$, because both $A_{ij} = 0$, $B_{ij} = 0$ if $i \neq j$

(since A, B are diagonal).

Similarly if $A \in D_{n\times n}(F)$ and $c \in F$ then $(cA)_{ij} = cA_{ij} = 0$ if $i \neq j$, because $A_{ij} = 0$ $\forall\, i \neq j$

$\boxtimes$

5.6 Definition Given a $n \times n$ matrix (square matrix) $A \in M_{n\times n}(F)$, the trace of A, denoted $tr(A)$ is the sum of the diagonal entries of A, i.e.

$$tr(A) = \sum_{i=1}^{n} A_{ii} \in F$$

example if $A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & -2 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

then $tr(A) = -1 + (-2) + 2 = -1$

5.7 Exercise (a) Show that the set of matrices $A \in M_{n\times n}(F)$ with $tr(A)=0$ is a subspace of $M_{n\times n}(F)$

(b) Show that the set of matrices $A \in M_{n\times n}(F)$ with $tr(A)=1$ is not a subspace of $M_{n\times n}(F)$

(c) Show that the set of matrices $A \in M_{n\times n}(F)$ with $tr(A) \leq 0$ is not a subspace

5.8 Theorem If W_1, W_2 are subspaces of a vector space V over a field F, then $W_1 \cap W_2$ is a subspace of V as well

More generally, if W_i, $i \in I$, is a collection of subspaces of V then $\bigcap_{i\in I} W_i$ is a subspace of V.

Proof. Since both W_1, W_2 are subspaces, we have

$0 \in W_1$, $0 \in W_2$, so $0 \in W_1 \cap W_2$.

Also, if $x, y \in W_1 \cap W_2$

Then: $\begin{cases} x, y \in W_1, \text{ so } x+y \in W_1 \\ x, y \in W_2, \text{ so } x+y \in W_2 \end{cases}$

Thus, $x+y \in W_1 \cap W_2$.

Similarly, if $x \in W_1 \cap W_2$ and $c \in F$

Then $\begin{cases} x \in W_1 \text{ so } cx \in W_1 \\ x \in W_2 \text{ so } cx \in W_2 \end{cases}$

Thus $cx \in W_1 \cap W_2$, and we showed That (a), (b) & (c) of Thm 5.2 are all satisfied for $W_1 \cap W_2$

General case is similar

$\blacksquare$