115A, Winter 2023

Lecture 11

Fr, Feb 3

- Recall that over the last two lectures L9 & L10 we defined the notion of basis and dimension of a vector space:

- Given a vector space V, a basis for V is a subset $S \subset V$ with the properties that S is linearly independent and generates V

- We proved that if V is finitely generated (i.e., if $\exists\, S \subset V$ finite such that $\operatorname{span} S = V$) then V has a finite basis and that all basis of V have the same # of elements. We called this common number the dimension of V denoted $\boxed{\dim(V)}$.

• We now enumerate several more consequences of the replacement theorem

11.1 Corollary Let V be a finite dimensional vector space with $\dim V = n$

1° If $S \subset V$ is a set with m elements that generate V, then $m \geq n$

2° If $S \subset V$ has n elements and $\operatorname{span} S = V$, then S is a basis for V

3° If $S \subset V$ has m elements and S is lin. ind., then $m \leq n$.
If S is lin. ind. and has n elements then S is a basis

4° Any linearly independent subset S of V can be extended to a basis of V, i.e. $\exists\ S' \subset V$ subset that contains S such that S' is a basis for V.

$\boxed{\text{PR}}$. We already discussed these consequences of the Replacement Thm on Monday L9, see Remarks before 9.6 but please do the proofs as exercises!!

11.2 Theorem. Let V be a finite dimensional vector space and $W \subset V$ a subspace. Then W is finite dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$ then $W = V$.

$\boxed{\text{Proof}}$ Denote $\dim(V) = n$ and let $\{w_1, \dots, w_k\} \subset W$ be a lin. ind. subset of W. By 11.1.4°, this set can be extended to a basis of V. Thus $k \leq n$. If $\operatorname{span}(\{w_1, \dots, w_k\}) \neq W$ then let $w_{k+1} \in W \setminus \operatorname{span}(\{w_1, \dots, w_k\})$. We claim that $\{w_1, \dots, w_k, w_{k+1}\}$ follows lin. ind. Indeed, because if

(*) $\left[\sum_{i=1}^{k+1} c_i w_i = 0 \right.$ then we have

$c_{k+1} = 0$, or else we get

$$w_{k+1} = \sum_{j=1}^{k} \left(- \frac{c_j}{c_{k+1}} \right) w_j \in \operatorname{span}\{w_1, \dots, w_k\},$$

contradiction. Thus, (*) implies

$$\sum_{i=1}^{k} c_i w_i = 0,$$ which by lin. ind.

of $\{w_1, \dots, w_k\}$ implies $c_1 = c_2 = \dots = c_k = 0$

So indeed $\{w_1, \dots, w_{k+1}\}$ follows lin. ind.

Thus, since any lin. ind. subset of W has $\leq n$ many elements with n finite, we can continue the process of choosing more & more lin. ind. vectors $w_1, \dots, w_k$ in W

but since $k \leq n$ and n is finite this process must end after finitely many steps, say m steps, once we get a linearly indep. $\{w_1, \dots, w_m\}$ to span W, and from the above we have $m \leq n$. Since $\operatorname{span}\{w_1, \dots, w_m\} = W$ and $\{w_1, \dots, w_m\}$ lin. ind., $\{w_1, \dots, w_m\}$ is a basis for W, thus $\dim(W) = m \leq n = \dim(V)$.

If in addition $\dim(W) = \dim(V)$ then by 11.1.3° we must have that $\{w_1, \ldots, w_m\}$ is also a basis of V. Thus $W = V$. $\boxtimes$

11.3 Corollary. If V is fin. dim. vector space and $W \subset V$ subspace then any basis for W can be extended to a basis for V.

Pf. By Thm 11.2 we know that $m = \dim(W) \leq n$. Let $\{w_1, \ldots, w_m\} \subset W$ be a basis for W. In particular, they are lin. ind. and by 11.1.4° we can extend this set to a basis of V. $\boxtimes$

- We'll now pass to the study of

$\boxed{\text{Linear transformations between vector spaces (§2.1)}}$

11.4 Definition. Let V, W be vector spaces over the field F. A function $T : V \to W$ satisfying the properties

(a) $T(x+y) = T(x) + T(y)$, $\forall x, y \in V$

(b) $T(cx) = cT(x)$, $\forall x \in V$, $c \in F$

is called a linear transformation from V to W.

- We often just use the term linear, as an adjective/noun to call such function $T : V \to W$.

11.5. Remarks If $T : V \to W$ is linear, as above, then:

- $T(0_V) = 0_W$. Indeed, because by (b) above we have

$$\underbrace{T(\overset{\substack{\text{scalar } 0 \text{ in } F\\ \downarrow}}{0} \cdot 0_V)}_{= 0_V} = \underbrace{0\, T(0_V)}_{= 0_W}$$

2°. $T(x-y) = T(x) - T(y), \ \forall\, x, y \in V$

Indeed $T(x-y) = T(x + (-y))$

$$\overset{\text{by (a)}}{=} T(x) + T(-y) = T(x) + T((-1)\cdot y)$$

$$\overset{\text{by (b)}}{=} T(x) + (-1)\, T(y) = T(x) - T(y)$$

3°. $T\left(\sum_{i=1}^{n} c_i x_i\right) = \sum_{i=1}^{n} c_i T(x_i)$

$\forall\, x_1, \dots, x_n \in V,\ c_1, \dots, c_n \in F$

Indeed, this is clear by applying (a), (b) repeatedly.

$\boxed{\text{11.6. Example}}$ Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a, b) = (a, -b)$. Then T is linear, because on the one hand

$T((a_1, b_1) + (a_2, b_2)) =$

$= T((a_1+a_2, b_1+b_2))$

$= (a_1+a_2, -b_1-b_2)$ =

on the other hand

$T(a_1, b_1) = (a_1, -b_1)$

$T(a_2, b_2) = (a_2, -b_2)$

so $T(a_1, b_1) + T(a_2, b_2) =$

$= (a_1, -b_1) + (a_2, -b_2) = (a_1+a_2, -b_1-b_2)$

Similarly $T(c(a,b)) =$

$= T(ca, cb) = (ca, -cb)$

$= c(a, -b) = c\,T(a,b)$

$\boxed{\text{11.7 Example}}$ $T: \mathbb{R}^2 \to \mathbb{R}^2$

defined by $T(a,b) = (a, 0)$

is linear

Indeed, trivial to check

• The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(a,b) = (a, -b)$ in 12.6 is called the reflection of R^2 about the x-axis

and $T: R^2 \to R^2$, $T(a,b) = (a,0)$ in 12.7 is called the projection of R^2 on the x-axis

$\boxed{\text{11.8 Example}}$ Let V denote the set of all functions $f: \mathbb{R} \to \mathbb{R}$ that have derivative of any order. This is obviously a vector space over field $\mathbb{R}$ when endowed with usual addition of functions and multiplication of functions by scalars: if $f, g \in V$ then

$$(f+g)(t) = f(t) + g(t), \quad t \in \mathbb{R}$$

$$(cf)(t) = cf(t)$$

Define $T: V \to V$ by

$T(f) = f'$ (the derivative of f)

Then T is a linear transform. because of the known properties of the derivation of functions:

$$(f+g)' = f' + g'$$

$$(cf)' = cf' \quad , \forall f, g \in V, \; c \in \mathbb{R}$$

Obs One often denotes this vector space by $\boxed{V = C^{\infty}(\mathbb{R})}$

11.9. Example Let $V = C(\mathbb{R})$ the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Then $C(\mathbb{R})$ with the usual addition of functions and multiplication by scalars is a vector space. Let $a, b \in \mathbb{R}$, $a < b$. Then the function

$$T : C(\mathbb{R}) \to \mathbb{R}$$

defined by $T(f) = \int_a^b f(t)\,dt$

is linear because of the known properties of the integral

$$\int_a^b (f+g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$$

$$\int_a^b (cf)(t)\,dt = c\int_a^b f(t)\,dt$$

11.10 Definition Let V, W be vector spaces and $T: V \to W$ linear.
The *null space of T* (or the *kernel of T*) is the set

$$N(T) := \{x \in V : T(x) = 0_W\}$$

The *range of T* (or *image of T*) is the set $R(T) := \{T(x) : x \in V\}$

i.e. the range of the function T

11.11 Examples The projection of $\mathbb{R}^2$ onto the x axis in 12.7 defined by $T: \mathbb{R}^2 \to \mathbb{R}^2$,

$T(a,b) = (a,0)$, $\forall (a,b) \in \mathbb{R}^2$

has $N(T) = \{(0,b) : b \in \mathbb{R}\}$

i.e the y-axis

$R(T) = \{(a,0) : a \in \mathbb{R}\}$

i.e. the x axis

11.12. Theorem. Let V, W be vector space and $T: V \to W$ be linear. Then $N(T)$, $R(T)$ are subspaces of V and respectively W.

$\boxed{\text{Proof}}$. Since $T(0_V) = 0_W$

we have $0_V \in N(T)$. If

$v_1, v_2 \in N(T)$, i.e. $T(v_1) = 0_W$, $T(v_2) = 0_W$

Then $T(v_1 + v_2) \underset{\substack{\uparrow \\ T \text{ linear}}}{=} T(v_1) + T(v_2)$

$= 0_W + 0_W = 0_W$, so $x_1 + x_2 \in N(T)$

if $v \in N(T)$ then $T(v) = 0_W$

so if $c \in \mathbb{R}$ scalar then

$$T(cv) \overset{\text{because } T \text{ linear}}{=} c\underbrace{T(v)}_{=0_W} = 0_W \quad \text{so } cv \in N(T)$$

Similarity for $R(T)$

$\square$