

Math115A 1/20 notes

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Sub spaces of vector spaces

5.1 Definition

A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V .

Note that V itself is a subspace of V . Also, the set $\{0\}$, consisting of the vector $0 \in V$, is a subspace of V it is called the zero subspace

If W is a subset of a vector space V over a field F , and we are as lead to check if W is a subspace, then we don't want to check that all conditions VS1-VS8 are satisfied: indeed we do know addition & scalar multiplication check the multiplication check the properties for all vectors in V , so all we need is in fact to check that addition of $x, y \in W$ and scalar multiplication ax for $x \in W$ keep us within the set W .

5.2 Theorem

Let V be a vector space over field F and $W < V$ a subset. Then W is a subspace of V if and only if the following conditions field.

- (a) $0 \in W$ (i.e. the zero vector of V belong to W)
- (b) if $x, y \in W$ then $x + y \in W$ (i.e. the addition of any two vectors in W stays in W)
- (c) if $x \in W$ and $c \in F$ then $cx \in W$ (i.e. the multiplication by any scalar of a vector in x stays in W)

Proof

if W is a subspace of V then by definition 5.1 it means that W with the operations of addition and scalar multiplication it "inherits" from V is a vector space. Thus (b)&(c) are satisfied. Also, W has by the definition of vector space a "zero element", say $0' \in W$ satisfying $x + 0' = x, \forall x \in W$. But we also have $x + 0 = x$ where 0 is here the zero element in V By Theom 4.7(a) thus implies $0'0 = 0$, so we have (a) as well " \leq " if (a),(b),(c) hold.

Then W with the addition and scalar multiplication of its elements viewed as elements in V satisfies VS1, VS2, and VS5-8 because we already know all these properties are satisfied in V .

All we need to show is that W has a zero vector (thus is VS3) and any $x \in W$ has an additive inverse in W (an "opposite")

but condition (a) tells us that $0 \in W$ (no zero element of V belongs to W), and this element does satisfy $x + 0 = x, \forall x \in W$ (because this holds that in fact for all elements x in V in particular for all elements in W) Also, by condition (c) we love that $\forall x \in W$, the element $y = (-1)x$ is the opposite of x in V . So $x \in W$ does have an additive inverse in W (what we also called an "opposite"). So VS3 & VS4 are satisfied as well.

5.3 Definitions

Given a matrix $A = (A_{ij}) \in M_{m \times n}(F)$ its transpose is the $n \times m$ matrix $A^t \in M_{n \times m}(F)$ obtained from A by interchouging the row with the columns, that is $(A^t)_{ij} = A_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$

For example: /newline 1. if $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -4 & 0 \end{pmatrix} \in M_{3 \times 2}(R)$

then $A^t \in M_{2 \times 3}(R)$ is $A^t = \begin{pmatrix} 0 & 2 & -4 \\ -1 & 3 & 0 \end{pmatrix}$

2. $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}^t = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -3 \end{pmatrix}^t = \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 2 & -3 \end{pmatrix}$

A symmetric matrix A is a matrix such that $A^t = A$

Note that such matrices necessary have same # of rows and colums

i.e. $A \in M_{n \times n}(F)$

Examples:

1. $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ so it is symmetric

4. Note that $0 = \begin{pmatrix} 0 & \dots & 0 \\ | & & | \\ 2 & \dots & 0 \end{pmatrix} \in M_{m \times n}(F)$

5.4 Exercise

show that the set of symmetric matrices in $M_{m \times n}$ is a subspace of $M_{m \times n}(F)$

Solution

We need to check (b)&(c) of theom 5.2 (we already noticed that the 0 matrix in $M_{m \times n}(F)$ is symmetric as we have (a) of theom 5.2)

If $A, B \in M_{m \times n}(F)$ are symmetric then $A = (A_{ij})$ and $B = (B_{ij})$ satisfy $A = A^t$, $B = B^t$ so $A_{ij} = A_{ji}$, $B_{ij} = B_{ji}$ for all $1 \leq i, j \leq n$

But then $(A+B)_{ij} = A_{ij} + B_{ij}$ so $((A+B)^t)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A_{ij} + B_{ij}$, Showing that $A+B$ is symmetric

Similarly, if $c \in F$ is a scalar and $A = A^t$ Then $(cA)_{ij} = cA_{ij}$, $((cA)^t)_{ij} = (cA)_{ji} = cA_{ji} = cA_{ij}$ so cA is symmetric

5.5 Definitions

A matrix $A \in M_{n \times m} \in F$ is called a diagonal matrix

If $A_{ij} = 0$, $\forall i \neq j$, i.e. if all entries off the diagonal of A are equal to 0

Example: /newline $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_{3 \times 3}(R)$ is a diagonal matrix

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in M_{4 \times 4}(R) \text{ is a diagonal matrix}$$

Note the 0 matrix in $M_{m \times n}(F)$ is a diagonal matrix, $\forall n$

Exercise

Show that the set of diagonal matrices $D_{m \times n}(F)$ in $M_{m \times n}(F)$ is a subspace.

So indeed, we noticed that $0 \in D_{m \times n}(F)$

Also the sum of two diagonal matrices $A, B \in D_{m \times n}(F)$ satisfies $(A + B)_{ij} = A_{ij} + B_{ij} = 0$ if $i \neq j$, because both $A_{ij} = 0, B_{ij} = 0$ if $i \neq j$ (Since A, B are diagonal)

Similarity if $A \in D_{n \times m}(F)$ and $c \in F$ then $(cA)_{ij} = cA_{ij} = 0$ if $i \neq j$ because $A_{ij} = 0, \forall i \neq j$

5.6 Definition

Given a $n \times m$ matrix $A \in M_{m \times n}(F)$, the trace of A, denoted $\text{tr}(A)$ is the sum of the diagonal entries of A, i.e.

$$\text{tr}(A) = \sum_{i=1}^n A_{ii} \in F$$

example if $A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & -2 & 0 \\ 1 & 2 & 2 \end{pmatrix}$ then $\text{tr}(A) = -1 + (-2) + 2 = -1$

5.7 Exercise

- (a) show that the set of matrices $A \in M_{m \times n}(F)$ with $\text{tr}(A) = 0$ is a subspace of $M_{m \times n}(F)$

Solution:

Closure under addition: If $A, B \in M_{m \times n}(F)$ with $\text{tr}(A) = \text{tr}(B) = 0$, then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) = 0 + 0 = 0$. Therefore, the set is closed under matrix addition.

Closure under scalar multiplication: If $A \in M_{m \times n}(F)$ with $\text{tr}(A) = 0$ and $k \in F$, then $\text{tr}(kA) = k\text{tr}(A) = k \cdot 0 = 0$. Therefore, the set is closed under scalar multiplication.

Containing the zero vector: The zero matrix, denoted by O , is an $m \times n$ matrix whose entries are all 0. And since the trace of the zero matrix is the sum of its diagonal entries, which are all zero, we have $\text{tr}(O) = 0$. Therefore, the zero matrix is an element of the set.

- (b) Show that the set of matrices $A \in M_{m \times n}(F)$ with $\text{tr}(A) = 1$ is not a subspace of $M_{m \times n}(F)$

Solution:

Closure under addition: if $A, B \in M_{m \times n}(F)$ with $\text{tr}(A) = \text{tr}(B) = 1$, Then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) = 1 + 1 = 2$. However $1 \neq 2$. Failed by condition of closure under addition.

- (c) Show that the set of matrices $A \in M_{m \times n}(F)$ with $\text{tr}(A) \leq 0$ is not a subspace

Proof:

Closure under addition: if $A, B \in M_{m \times n}(F)$ with $\text{tr}(A) = \text{tr}(B) \leq 0$, Then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) \leq 0$. Therefore, the set is closed under matrix addition.

Closure under multiplication: if $A \in M_{m \times n}(F)$ with $\text{tr}(A) \leq 0$ and $k \in F$, Then $k\text{tr}(A)$ will be less or equal to zero when k is greater or equal than zero. Not all condition. Therefore, failed by closure under multiplication.

5.8 Theorem

If W_1, W_2 are subspace of a vector space V over a field F , then $W_1 \cap W_2$ is a subspace of V as well more generally, if $W_i, i \in I$, is a collection of subspace of V then $\cap_{i \in I} W_i$ is a subspace of V .

Proof

Since both W_1, W_2 are subspace, we have $0 \in W_1, 0 \in W_2$ so $0 \in W_1 \cap W_2$

Also, if $x, y \in W_1 \cap W_2$

Then $x, y \in W_1$, so $x + y \in W_1$ and $x, y \in W_2$, so $x + y \in W_2$

Thus, $x + y \in W_1 \cap W_2$

Similarly, if $x \in W_1 \cap W_2$ and $c \in F$

Then $x \in W_1$, so $cx \in W_1$ and $x \in W_2$, so $cx \in W_2$

Thus $cx \in W_1 \cap W_2$, and we showed that (a),(b) and (c) of theorem 5.2 are all satisfied for $W_1 \cap W_2$

General case is similar