HW4

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```
library(ggplot2)
library(truncnorm)
```

Problem 1: Suppose $\theta = \int_0^2 xe^{-3x}$. Please write corresponding algorithms and functions to answer (a) - (d)

(a) Compute the Monte Carlo estimate of θ without any means of variance reduction.

```
fx <- function(x) { x * exp(-3 * x) }

m <- 10000
u <- runif(m, 0, 2)
mc1_estimates <- fx(u)

theta_mc1 <- mean(mc1_estimates)
```

(b) Compute the Monte Carlo estimate of θ using the antithetic variate approach.

```
m <- 10000
u <- runif(m / 2, 0, 2)
v <- 2 - u  # Antithetic variates
uv <- c(u, v)
anti_mc_estimates <- fx(uv)

theta_anti_mc <- mean(anti_mc_estimates)
```

(c) Compute the Monte Carlo estimate of θ using the control variate approach.

```
f <- function(u){
  exp(u)
}
g <- function(u){
  u*exp(-3*u)
}

u <- runif(10000, 0, 2)
B <-f(u)
A <-g(u)
cor(A,B)
```

```
## [1] -0.8426329
```

```
a <- -cov(A,B)/var(B)
u <- runif(10000, 0, 2)
T1 <- g(u)
T2 <- T1 + a * (f(u) - (1 - exp(-1)) )
cbind(mean(T1), mean(T2))
```

```
##            [,1]     [,2]
## [1,] 0.05452277 0.103894
```

```
con_var <- (var(T1) - var(T2))/var(T1)
con_var
```

```
## [1] 0.7063191
```

(d) Compute the Monte Carlo estimate of θ using the stratified sampling approach.

```
m <- 10000
n_strata <- 100
stratified_estimates <- numeric(n_strata)

for (i in 1:n_strata) {
    lower <- (i - 1) * 2 / n_strata
    upper <- i * 2 / n_strata
    u <- runif(m / n_strata, min = lower, max = upper)
    stratified_estimates[i] <- mean(fx(u))
}

theta_stratified <- mean(stratified_estimates)
```

(e) Compare your four estimates with the theoretical value, and compute their variances.

```
theta_theoretical <- integrate(fx, lower = 0, upper = 2)$value
squared_deviation <- function(x) {
  (f(x) - theta_theoretical) ^ 2
}
variance_theoretical <- integrate(squared_deviation, lower = 0, upper = 2)$value


data.frame(
  method = c("theoretical" ,"simple Monte Carlo","Anti Monte Carlo", "Control Variate", "Stratified"),
  theta =c(theta_theoretical, theta_mc1, theta_anti_mc, mean(T2), theta_stratified),
  var = c(variance_theoretical, var(mc1_estimates),var(anti_mc_estimates),var(T2),var(stratified_estima
)
```

```
##               method      theta          var
## 1        theoretical 0.10918319 2.542776e+01
## 2 simple Monte Carlo 0.05416098 1.647242e-03
## 3   Anti Monte Carlo 0.05468094 1.658513e-03
## 4    Control Variate 0.10389400 4.849952e-04
## 5         Stratified 0.05457860 1.661955e-03
```

Problem 2: Consider the problem of estimating $\theta = \int_0^1 \frac{e^{-x}}{1+x^2} dx$ Given two control variates as follows, please write R code to estimate θ using the control variate approach.

$$t_1(x) = \frac{2e^{-0.5}}{\pi(1+x^2)}, \ where \ 0 < x < 1$$

```
f <- function(u){
  (2*exp(-0.5))/(pi*(1+u^2))
}
g <- function(u){
  exp(-u)/(1+u^2)
}

u <- runif(10000, 0, 1)
B <-f(u)
A <-g(u)
a <- -cov(A,B)/var(B)
u <- runif(10000, 0, 1)
T1 <- g(u)
T2 <- T1 + a * (f(u) - (1 - exp(-1)) )
var(T2)
```

```
## [1] 0.003168424
```

$$t_2(x) = \frac{e^{-x}}{1-e^{-1}}, \ where \ 0 < x < 1$$

```
f <- function(u){
  (exp(-u))/(1-exp(-1))
}
g <- function(u){
  exp(-u)/(1+u^2)
}

u <- runif(10000, 0, 1)
B <-f(u)
A <-g(u)
a <- -cov(A,B)/var(B)
u <- runif(10000, 0, 1)
T1 <- g(u)
T2 <- T1 + a * (f(u) - (1 - exp(-1)) )
var(T2)
```

```
## [1] 6.080716e-05
```

Is your estimator more efficient than the simple Monte Carlo estimator? Please provide an evidence to support your answer.

```
m <- 10000
u <- runif(m, 0, 1)
mc1_estimates <- g(u)
var(mc1_estimates)
```

```
## [1] 0.05972469
```

Since the control variate method gives me a smaller variance, I can say that my estimator are more efficient than the simple Monte Carlo estimator.

Problem 3: Suppose $X \sim f(x)$, Let $\theta = \int g(x)f(x)dx = E_f[g(X)]$. We draw m iid copies $X_1, ...X_m$ from $\phi(x)$, which is different from $f(x)$, and define $W(x) = \frac{f(x)}{\phi(x)}$

(a) Prove $E_f[g(X)] = E_\phi[g(X)W(X)]$

$$\phi(x) = \frac{f(x)}{W(x)}$$

$$E_\phi[g(X)W(X)] = \int g(x)W(x)\phi(x)dx$$

$$E_\phi[g(X)W(X)] = \int g(x)\frac{f(x)}{\phi(x)}\phi(x)dx = \int g(x)f(x)dx$$

$$E_f[g(X)] = \int g(x)f(x)dx = E_\phi[g(X)W(X)]$$

(b) Prove $E_\phi[W(X)] = 1$

$$E_\phi[W(X)] = \int W(x)\phi(x)dx$$

$$E_\phi[W(X)] = \int \frac{f(x)}{\phi(x)}\phi(x)dx = \int f(x)dx = 1$$

(c) Let $\hat{\theta} = \Sigma_{i=1}^m g(X)W(X)/m$. Find $E[\hat{\theta}]$ and $Var[\hat{\theta}]$

$$E(\hat{\theta}) = E(\Sigma_{i=1}^m g(X_i)W(X_i)/m) = \frac{1}{m}\Sigma_{i=1}^m E[g(X_i)W(X_i)]$$

Since that X_i are iid samples and $E_f[g(X)] = E_\phi[g(X)W(X)]$

$$E(\hat{\theta}) = E[g(X)W(X)] = E_f[g(X)]$$

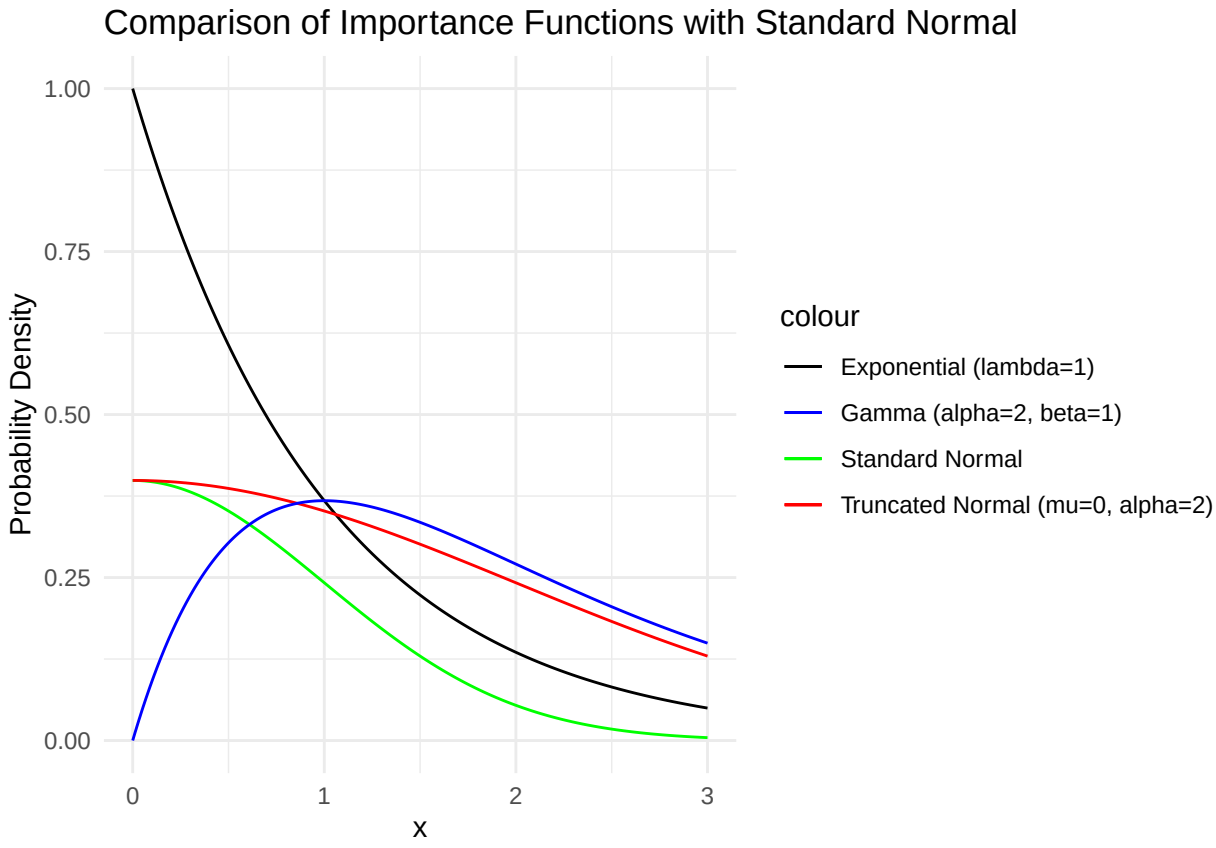
$$Var[\hat{\theta}] = Var[\Sigma_{i=1}^m g(X_i)W(X_i)/m] = \frac{1}{m^2}\Sigma_{i=1}^m Var[g(X_i)W(X_i)] = \frac{1}{m}Var[g(X)W(X)]$$

$$Var[\hat{\theta}] = \frac{1}{m}(E[g(X)^2W(X)^2] - E_f[g(X)]^2)$$

Problem 4: Given $X \sim N(0,1)$, we want to compute $\theta = P(X > C)$ where C is a positive constant.

(a) Find three importance functions that are supported on $(0, \infty)$, and explain which of your importance functions should produce the smaller $Var[\hat{\theta}]$. Please graph plots to support your answer.

```r
# Define the range for x
x <- seq(0, 3, length.out = 1000)

# Standard normal distribution
std_normal <- dnorm(x, mean = 0, sd = 1)

# Exponential distribution (lambda = 1)
exp_dist <- dexp(x, rate = 1)

# Truncated normal distribution (mu = 1, sigma = 0.5)
trunc_normal <- dtruncnorm(x, a = 0, b = Inf, mean = 0, sd = 2)

# Gamma distribution (alpha = 2, beta = 1)
gamma_dist <- dgamma(x, shape = 2, rate = 1)

df <- data.frame(x, std_normal, exp_dist, trunc_normal, gamma_dist)

ggplot(df, aes(x)) +
  geom_line(aes(y = std_normal, color = "Standard Normal")) +
  geom_line(aes(y = exp_dist, color = "Exponential (lambda=1)")) +
  geom_line(aes(y = trunc_normal, color = "Truncated Normal (mu=0, alpha=2)")) +
  geom_line(aes(y = gamma_dist, color = "Gamma (alpha=2, beta=1)")) +
  labs(title = "Comparison of Importance Functions with Standard Normal",
       x = "x", y = "Probability Density") +
  scale_color_manual(values = c("black", "blue", "green", "red")) +
  theme_minimal()
```



Among these, the Truncated Normal distribution is likely to yield the smallest $Var[\hat{\theta}]$ if its parameters are chosen correctly, as it can most closely match the tail of the standard normal distribution.

(b) Write a function to compute a Monte Carlo estimate of θ using importance functions you proposed in (a). Note: You can use the built-in functions in R to generate random variables.

```
importance_sampling <- function(importance_dist, params, C, n) {

  if (importance_dist == "exponential") {
    samples <- rexp(n, rate=params$lambda)
  } else if (importance_dist == "truncated_normal") {
    samples <- rtruncnorm(n, a=0, mean=params$mu, sd=params$sigma)
  } else if (importance_dist == "gamma") {
    samples <- rgamma(n, shape=params$alpha, rate=params$beta)
  }

  f <- dnorm(samples, mean=0, sd=1)
  g <- switch(importance_dist,
              "exponential" = dexp(samples, rate=params$lambda),
              "truncated_normal" = dtruncnorm(samples, a=0, mean=params$mu, sd=params$sigma),
              "gamma" = dgamma(samples, shape=params$alpha, rate=params$beta))

  weights <- f / g

  # Estimate theta
  theta_hat <- mean(weights * (samples > C))
  return(theta_hat)
}
```

(c) Compare your estimates with the theoretical values for C = 0.25, 0.5, 1, 2.

```
C_values <- c(0.25, 0.5, 1, 2)

n_samples <- 10000

results <- sapply(C_values, function(C) {

  theoretical <- 1 - pnorm(C)

  mc_exp <- importance_sampling("exponential", list(lambda = 1), C, n_samples)
  mc_trunc_normal <- importance_sampling("truncated_normal", list(mu = 0, sigma = 2), C, n_samples)
  mc_gamma <- importance_sampling("gamma", list(alpha = 2, beta = 1), C, n_samples)

  c(Theoretical = theoretical, Exponential = mc_exp, Truncated_Normal = mc_trunc_normal, Gamma = mc_gam
})

results_df <- data.frame(C = C_values, t(results))
results_df
```

```
##      C Theoretical Exponential Truncated_Normal      Gamma
## 1 0.25  0.40129367  0.40012204       0.39827862 0.39900577
## 2 0.50  0.30853754  0.30910281       0.30657707 0.31006330
## 3 1.00  0.15865525  0.15943734       0.15909829 0.16029531
## 4 2.00  0.02275013  0.02184473       0.02291529 0.02323517
```

Problem 5: The density function $f(x)$ is proportional to $q(x) = e^{-x^2\sqrt{x}}[sin(x)]^2$, for $x \in R_+$. Consider a trial distribution $h(x) = \frac{r(x)}{Z_r}$. We want to use importance sampling to estimate $E_f(X^2)$. Consider the following choices of un-normalized densities for trial distributions:

(i) $r_1(x) = e^{-2x}$,
(ii) $r_2(x) = x^{-1/2}e^{-x/2}$,
(iii) $r_3(x) = (2\pi)^{-1}(1 + x^2/4)^{-1}$,

for $x \in R_+$

(a) Write three functions (E1, E2, and E3) to estimate $E_f(X^2)$ using importance sampling with the above un-normalized densities.

```
qx <- function(x) {
    exp(-x^2 * sqrt(x)) * sin(x)^2
}

E1 <- function(n){

  X <- rexp(n,2)
  r1 <- function(x){exp(-2*x)}
  W <- qx(X)/r1(X)
  return(mean(W*X^2))

}

E2 <- function(n){

  X <- rexp(n,0.5)
  r2 <- function(x){x^(-1/2) * exp(-x/2)}
  W <- qx(X)/r2(X)
  return(mean(W*X^2))

}

E3 <- function(n){

  X <- abs(rcauchy(n, 2))
  r3 <- function(x){(2 * pi)^(-1) * (1 + x^2 / 4)^(-1)}
  W <- qx(X)/r3(X)
  return(mean(W*X^2))

}
```

(b) Compare your estimates with the theoretical value.

```
data.frame(
  theoretical_value = integrate(function(x) x^2 * qx(x), lower = 0, upper = Inf)$value,
  E1 = E1(10000),
  E2 = E2(10000),
  E3 = E3(10000)
  )
```

```
##   theoretical_value        E1       E2        E3
## 1         0.2313278 0.4627397 0.120512 0.4293942
```

(c) Which trial distribution above is more efficient? Explain.

E2 is the closest to the theoretical value, making it the most accurate in this case. Both E1 and E3 significantly overestimate the value.

(d) Estimate the normalizing constant of $q(x)$.

```
Q <- integrate(qx, lower = 0, upper = Inf)
Z <- 1 / Q$value
Z
```

```
## [1] 3.852986
```