Note 6

Haojie Liu

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Convolution Methods

$$x_i \sim Ber(p)$$
$$\Sigma x_i \sim Binomial(n,p)$$
$$x_i \sim Geo(p)$$
$$\Sigma x_i \sim NB(n,p,r)$$

$$x_i \sim Possion(\lambda_i)$$
$$\Sigma x_i \sim Possion(\Sigma \lambda_i)$$

$$x \sim X^2_{(1)}$$
$$\Sigma x_i \sim X^2_{(n)}$$

Composition Methods

$$z \sim N(0,1)$$
$$v = z^2 \sim X^2_{(1)}$$
$$V = \Sigma v_i \sim X^2_{(k)}$$
$$\frac{z}{\sqrt{\frac{V}{k}}} \sim t_k$$
$$U \sim X^2_{(n)}$$
$$\frac{\frac{V}{k}}{\frac{U}{n}} \sim F_{k,n}$$

Transformation of Random Variable

$$x \sim f_X(x)$$
$$y = h(x) \quad c.t. : h(t) \; must \; be \; one - to - one \; function$$
$$x = h^{-1}(y) = g(y)$$

1. $y \sim f_Y(y)$
2. compute $h^{-1}y = g(y)$
3. deliver $x = g(y)$

To find $f_Y(y)$:

scenario 1:

$$F_Y(y) = P(Y \leq y) = P(h(x) \leq y) = P_X(X \leq h^{-1}(y)) = P_X(X \leq g(y)) = F_X(g(y))$$
$$\frac{d}{dy}F_Y(y) = g'(y) f_X(g(y)) = f_Y(y)$$

scenario 2:

$$F_Y(y) = P_Y(Y \leq y) = P(h(x) \leq y) = P(X > h^{-1}(y)) = P_X(X > g(y)) = 1 - F_X(g(x))$$
$$\frac{d}{dy}F_Y(y) = -g'(y) f_X(g(y)) = f_Y(y)$$

universal:

$$f_Y(y) = |g'(y)| f_X(g(y))$$

Ex:

$$x \sim unif(0, 1)$$
$$1 - x \sim unif(0, 1)$$
$$y = h(x) = 1 - x \sim unif(0, 1)$$
$$h^{-1}(y) = 1 - y = g(y)$$
$$f_Y(y) = |-1| f_X(1 - y) = 1 \sim unif(0, 1)$$

Ex:

$$X \sim N(0, 1)$$
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$
$$y = h(x) = x^2$$

$$x = h^{-1}(y) = \pm\sqrt{y} = g(y)$$
$$f_Y(y) = |\frac{1}{2}y^{-\frac{1}{2}}|\frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}}$$

For $X^2_{(k)}$, $f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}x^{\frac{k}{2}-1}e^{\frac{-x}{2}}$

$$k = 1; f(x|k = 1) = \frac{1}{\sqrt{2\pi}}x^{-\frac{1}{2}}e^{\frac{-x}{2}}$$

Mixture Models

$$f_Y(y) = \frac{1}{2}f_+(y) + \frac{1}{2}f_-(y)$$
$$f(y) = \Sigma\theta_i f_i(y);\ \Sigma\theta_i = 1$$

Ex:

$$f_X(x) = \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x)$$
$$f_1(x) = N(0, 1)$$
$$f_2(x) = N(3, 1)$$

1. Verify if $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Ex:

$$f(x) \sim N(0, 1)$$
$$f(y) \sim N(0, 1)$$
$$f_{XY}(x, y) = f(x)f(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2+y^2}{2}}$$

$$r = h_1(x, y) = \sqrt{x^2 + y^2}$$
$$\theta = h_2(x, y) = tan^{-1}(\frac{y}{x})$$
$$x = h^{-1}(r, \theta) = g(r, \theta) = r\cos(\theta)$$
$$y = g_2(r, \theta) = r\sin(\theta)$$
$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\cos(\theta) = \frac{x}{r}$$
$$\sin(\theta) = \frac{y}{r}$$

$$f_{R,\theta}(r, \theta) = f_{xy}(g_1(r, \theta), g_2(r, \theta))|J|$$
$$J = \begin{pmatrix} \frac{dg_1(r,\theta)}{dr} & \frac{dg_1(r,\theta)}{d\theta} \\ \frac{dg_2(r,\theta)}{dr} & \frac{dg_2(r,\theta)}{d\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & -r\cos\theta \end{pmatrix}$$
$$|J| = r\cos^2\theta + r\sin^2\theta = r$$
$$\frac{1}{\sqrt{2\pi}}e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}}r = \frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}}r$$