Note 8

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From last time:

$$f(\theta) = \int_0^\infty f(r,\theta)dr$$
$$f_R(r) = \int_0^{2\pi} f(r,\theta)d\theta$$

Monte Carlo Integration

If we have

$$\int g(x)dx = \theta$$

Sampling approach

$$x_1, ..., x_m \sim f(x)$$

use this kind of method to estimate θ

$$\theta = E(x) = \int_{-\infty}^{\infty} xf(x)dx$$
$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$
$$\theta = E(g(x)) = \int_{-\infty}^{\infty} g(x)f(x)dx = \mu$$

By LLN

$$lim_{n\to\infty} P(|\bar{x} - \mu| < \epsilon) = 1$$
$$\bar{g}(x) = \frac{\Sigma g(x_i)}{m} = \hat{\theta}$$
$$lim_{m\to\infty} P(|\hat{\theta} - \theta| < \epsilon) = 1$$

Simple Monte Carlo Integration

$$\theta = \int_0^1 g(x)dx = \int_0^1 g(x)1dx$$

Then Generate X by Unif(0,1)

$$\hat{\theta} = \frac{\Sigma g(x_i)}{m}$$

Ex:

$$\theta = \int_0^1 e^{-x^2} dx$$
$$\hat{\theta} = \frac{\Sigma e^{-x^2}}{m}$$

$$\theta = \int_a^b g(x)dx = \int_a^b g(x)\frac{b-a}{b-a}dx = b-a\int_a^b g(x)\frac{1}{b-a}dx = (b-a)E_U[g(x)]$$
$$\hat{\theta} = (b-a)\frac{\Sigma g(x_i)}{m}$$

Is $\hat{\theta}$ a consistent estimator?

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + bias^2 = Var(\hat{\theta}) + [E(\hat{\theta}) - \theta]^2$$
$$E[\bar{g}(x)] = E[\frac{\Sigma g(x_i)}{m}] = E[g(x)] = \theta$$
$$\theta - \theta = 0 : \ Unbiased$$
$$Var(\hat{\theta}) = Var[\frac{\Sigma g(x_i)}{m}] = \frac{1}{m^2}Var[g(x)] = \frac{1}{m^2}\{E[g^2(x) - E[g(x)]^2]\}$$
$$= \frac{1}{m}\{\int g^2(x)f(x)dx - \theta^2\}$$

To make the Variance small, we have many ways:

1. $m \to \infty$
2. find $f(x)$ such that $\int g^2(x)f(x)dx \simeq \theta^2$

Ex:

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = \theta$$

If $x > 0$

$$F_X(x) = 0.5 + \int_0^x \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$
$$F_X(x) = 0.5 + x\int_0^x \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\frac{1}{x}dt$$
$$F_X(x) = 0.5 + x\frac{\Sigma \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}}{m}$$

If $x < 0$

$$F_X(x) = \int_{-\infty}^x f(t)dt$$
$$F_X(x) = 1 - [0.5 + |x|\int_0^{|x|} \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\frac{1}{x}dt]$$

Change of Variable

Let x/x, set $y = \frac{t}{x}$ and $dt = xdy$

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(xy)^2} xdy$$

$$\frac{x}{\sqrt{2\pi}} \int_0^1 e^{-\frac{1}{2}(xy)^2} 1dy$$

$$\theta = \Sigma \frac{e^{-\frac{1}{2}(xy_i)^2}}{m}$$

Hit-or-Miss Apporach

$$X_1, X_2.., X_n \sim N(0,1)$$

$$\hat{F}_X(x) = \frac{\Sigma I(X_i \leq x)}{n}$$

To conclude the both method:

$$Var(\hat{F}_X(x)) = \hat{F}_X(x)[1 - \hat{F}_X(x)]\frac{1}{n}$$

$$\sqrt{Var(\hat{F}_X(x))} = SE[\hat{F}_X(x)]$$

$$SE(\hat{\theta}) = \sqrt{Var(\hat{\theta})} = \sqrt{\frac{1}{m}\sigma^2}$$

$$\hat{\sigma}^2_{MLE} \to \sigma^2$$

$$\hat{\sigma}^2_{MLE} = \frac{\Sigma[g(x_i) - \bar{g}(x)]^2}{m}$$

$$SE(\hat{\theta}) = \sqrt{\frac{\Sigma[g(x_i) - \bar{g}(x)]^2}{m^2}} = \frac{\hat{\sigma}_{MLE}}{\sqrt{m}}$$