The Inverse CDF Method

Chapter 2

STATS 102C: Introduction to Monte Carlo Methods

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Introduction

- One of the fundamental tools required in computational statistics is the ability to simulate random variables from various probability distributions.
- Generate random numbers via True Random Number Generator (TRNG) or Pseudorandom Number Generator (PRNG).
- A pseudorandom number generator uses a hidden deterministic algorithm that starts from an initial number (called the **seed**) and generates pseudorandom numbers from it.
- Pseudorandom numbers are **statistically random**, in that they are "random enough" for statistical analysis and inference.
- The user can often specify (or **set**) the seed so that the "random" numbers that are generated from a given function are the same every time the function is run.
- More on pseudorandomness: https://en.wikipedia.org/wiki/Pseudorandomness

Uniform Assumption

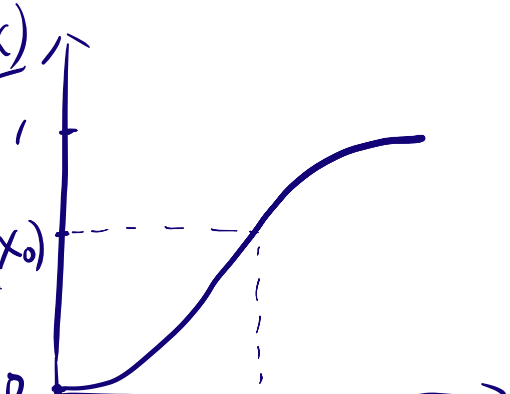
We will rely on the basic assumption that we can generate samples from the uniform distribution on the interval $(0, 1)$.

- Probability density function: $f(x) = \begin{cases} 1 & \text{for } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$
- We will not be concerned with the details of how to generate from $Unif(0, 1)$.
- In R: `runif()` can be used to generate from $Unif(0, 1)$.
- R uses the Mersenne Twister pseudorandom number generator: Mersenne Twister.

$\underline{F_X(X) \sim Unif(0,1)}$

- **Can we start from this assumption to generate samples from other distributions?**

$F_X(x)$



$F_U(u) = u$

set $Y (= F_X(Y)$

$u = F_X(x_0)$

$P(Y < y) = y$ ...

$Y \sim Unif(0,1) \longrightarrow x$

$x_0 = F^{-1}(u)$

The Inverse Transform Method

Probability Integral Transformation

- If X is a continuous random variable with CDF $F_X(x)$, then $U = F_X(x) \sim Unif(0,1)$ for every x.
- Define $F_X^{-1}(u) = inf\{x : F_X(x) = u\}$, $0 < u < 1$
- Set $Y = F_X^{-1}(u)$
- What is Y's distribution? $F_X(x) = u$

$x = F_X^{-1}(u)$

Algorithm

1. Derive the inverse function $F_X^{-1}(u)$
2. Generate $u \sim U(0,1)$ $\quad u_1 \; u_2 \; \cdots \; u_m$
3. Deliver $x = F_X^{-1}(u)$

$F_X^{-1}(u_1) = x_1$

$F_X^{-1}(u_2) = x_2$

$\vdots$

Example: Uniform Distribution

- Let $f(x)$ denote the probability density function of $Unif(a, b)$:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } x \in (a, b) \\ 0 & \text{otherwise} \end{cases}$$

- The CDF of $U(a, b)$ is then $F(x) = \int_a^x \frac{1}{b-a} dt = \frac{x-a}{b-a}$, for $a \leq x \leq b$.

- How can we sample from $U(a, b)$?

$$u = \frac{x-a}{b-a} \Rightarrow x = \underbrace{\boxed{u(b-a)+a}}_{F^{-1}(u)}$$

$u_1 \; u_2 \; \cdots \; u_m \sim Unif(0,1)$

$u_1(b-a)+a \Rightarrow x_1$

$u_2(b-a)+a \Rightarrow x_2$

$\vdots$

x_m

Example: Exponential Distribution

- Let $f(x)$ denote the PDF of the exponential distribution with rate parameter λ:
  $f(x) = \lambda e^{-\lambda x}, \quad$ for $x \geq 0$.

$1 - u$

- The CDF of $Exp(\lambda)$ is then
  $F(x) = \int_0^x \lambda e^{-\lambda t} dt = \underline{1 - e^{-\lambda x}}, \quad$ for $x \geq 0$, and $\lambda > 0$.
- How can we sample from $Exp(\lambda)$?

$$u = 1 - e^{-\lambda x}$$
$$e^{-\lambda x} = 1 - u$$
$$-\lambda x = \log(1-u)$$
$$x = -\frac{\log(1-u)}{\lambda}$$

$u_1, u_2 \cdots u_m \sim Unif(0,1)$

$$x_1 = -\frac{\log(u_1)}{\lambda}$$
$$\vdots$$
$$x_m = -\frac{\log(u_m)}{\lambda}$$

Example: Polynomial Density

$$F_X(x) = \int_0^x k t^{k-1}\, dt = x^k$$

Let $f(x)$ be the PDF defined by

$$f(x) = kx^{k-1}, \text{ for } k > 0,\ 0 < x < 1$$

$u = x^k$

$x = u^{\frac{1}{k}}$

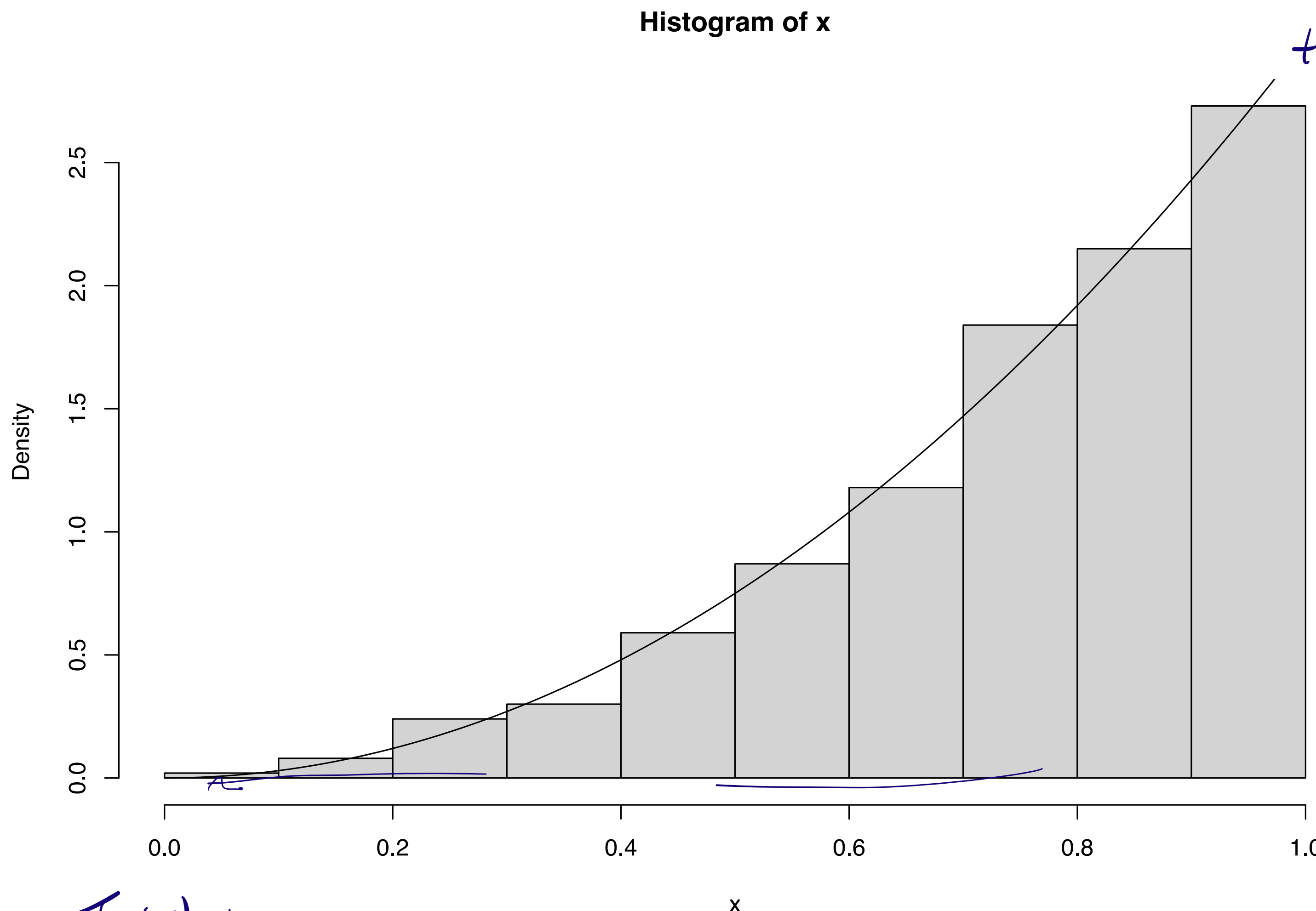
Use the inverse CDF to describe how to sample from this distribution.

The R code to generate samples from the density $f(x) = 3x^2$ (Example 3.2 from the textbook)

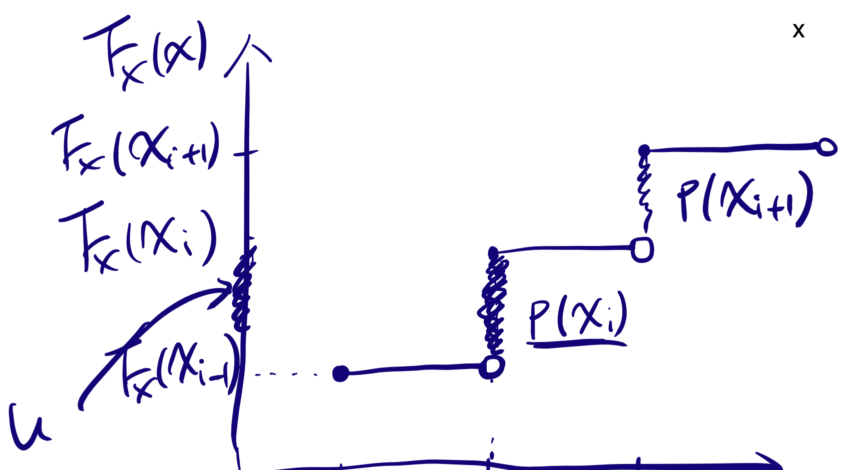
$k=3$

```
n <- 1000
u <- runif(n)
x <- u^(1/3)
hist(x, prob = TRUE) #histogrme of sample
y <- seq(0, 1, .01)
lines(y, 3 * y^2) #density curve f(x)
```

Example: Polynomial Density (Cont.)



theoratical
curve



Discrete Case

x_{i-1} x_i x_{i+1} x

The inverse transform method can also be applied to discrete distributions. If X is a discrete random variable and

$$\cdots < x_{(i-1)} < x_{(i)} < x_{(i+1)} < \cdots$$

are the points of discontinuity of $F_X(x)$, then the inverse transformation is $F_X^{-1}(u) = x_{(i)}$, where $F_X(x_{(i-1)}) < u \le F_X(x_{(i)})$.

$$F_X^{-1}(u) = \inf\{x : F(x) = u\}$$

$$F_X^{-1}(u) = \inf\{x_{(i)} : F(x_{i-1}) < u \le F(x_i)\}$$

$$\inf\{x_{(i+1)} : F(x_i) < u \le F(x_{i+1})\}$$

show $P(F_X(x_{i-1}) < U \le F_X(x_i)) = P(x_i)$

$$= F_X(x_i) - F_X(x_{i-1})$$

$$= P(X \le x_i) - P(X \le x_{i-1})$$

$$= P(X = x_i) + P(X = x_{i-1}) + P(X = x_{i-2}) + \cdots + P(X = x_1)$$

$$- [P(X = x_{i-1}) + P(X = x_{i-2}) + \cdots + P(X = x_1)]$$

$$= P(x_i)$$

Target dist.

x	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

generate $u \sim \text{Unif}(0,1)$

If $u \le F_X(0) = \frac{1}{4}$, return 0

If $F_X(0) < u \le F_X(1)$, return 1

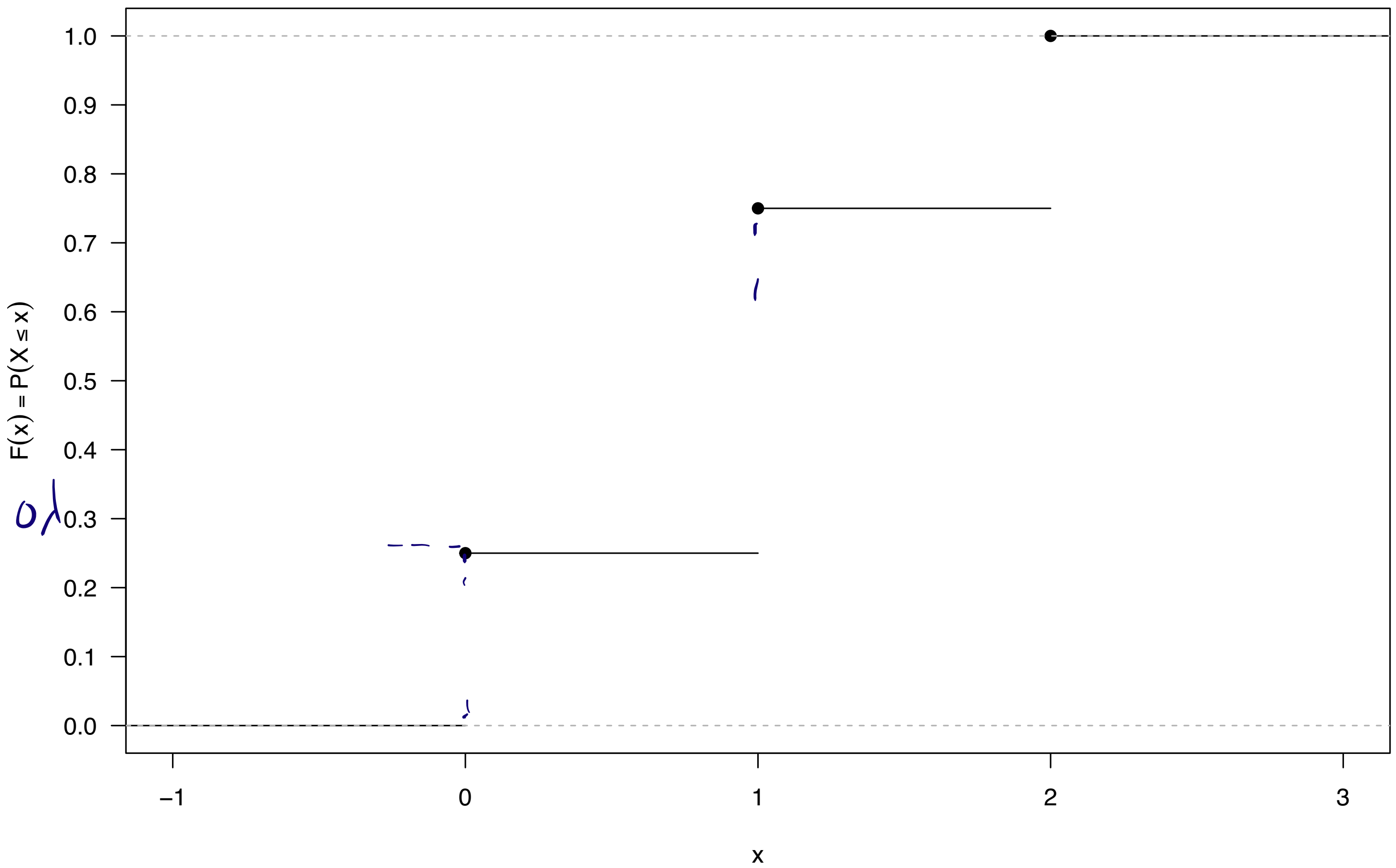
If $u > F_X(1)$, return 2

x = sample (x = c(0, 1, 2), prob = c($\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$), replace = T, size = 1000)

$x = 0, 1, 2, \ldots,$ $\qquad x = 1, 2, \ldots$

R Code for visualizing finite discrete distribution

```r
X <- rep(c(0, 1, 2), c(2, 4, 2))
plot(ecdf(X), ylab = expression(F(x) == P(X <= x)), main = "", las = 1)
axis(2, at = seq(0.1, 0.9, by = 0.2), las = 1)
```



Example: Geometric Distribution

- Let $p(x)$ denote the PDF of the geometric distribution with rate parameter p:
  $p(x) = q^x p$, where $q = 1 - p$, $x = 0, 1, 2, \ldots$.

- The CDF is then $F(x) = \sum_{t=0}^{x} q^t p = 1 - q^{x+1}$.

- For each sample element we need to generate random uniform u and solve

$$F(x-1) < u \leq F(x)$$

$$\left\lceil \frac{\log(u)}{\log q} \right\rceil - 1 = x$$

$$1 - q^x < u \leq 1 - q^{x+1}$$

$$q^x > 1 - u \geq q^{x+1}$$

$$x \log q > \log(1-u) \geq (x+1) \log q$$

$$x < \boxed{\frac{\log(1-u)}{\log q} \leq x+1}$$

Logarithmic distribution Example 3.6 Rizzo's book

$$P(x) = \frac{\theta^x}{-\log_{10}(1-\theta)x}, \quad x = 1, 2, \ldots$$

$$0 < \theta < 1$$

u

$F_x(x_i) \qquad F(x_{i+1})$

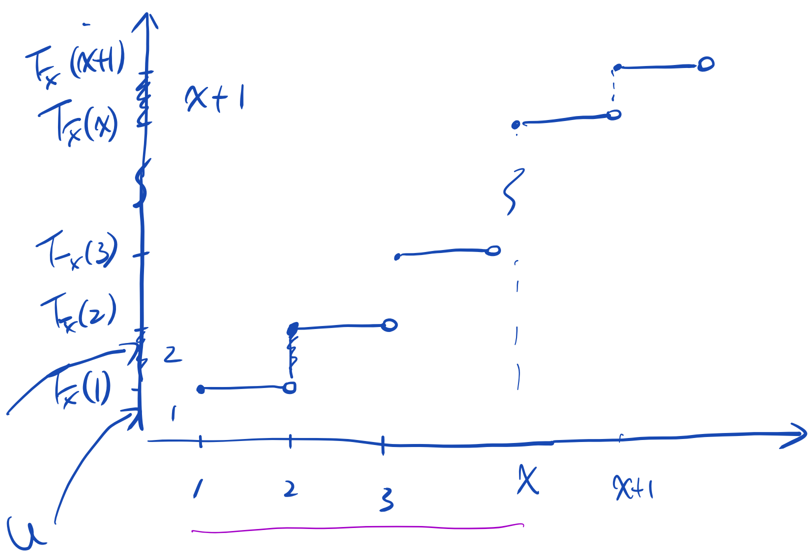
$$P(x=1) = \frac{\theta}{-\log_{10}(1-\theta)}$$

$$F(x) = \underline{P(x=1) + P(x=2) + \cdots} + P(x-1) + P(x)$$

$$= F(x-1) + P(x)$$

$$F(x+1) = F(x) + P(x+1)$$

$$P(x+1) = f(P(x))$$



$$F(x_{i-1}) < u \leq F(x_i), \quad x_i$$

$$u \sim \text{Unif}(0,1)$$

$$\underline{F_x} = \begin{pmatrix} F_x(1) \\ F_x(2) \\ F_x(3) \\ \vdots \\ F_x(x) \\ F_x(x+1) \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} u_0 \geq F_x(1) \\ u_0 \geq F_x(2) \\ \vdots \\ u_0 \geq F_x(x) \\ u_0 \geq F_x(x+1) \\ \vdots \end{pmatrix} = V$$

assume $\underline{u_0 < F_x(1)}$

$\underline{\text{sum}(u_0 \geq F_x)} = 0 + 1$

assume $u_0 < F_x(4)$

$\text{Sum}(u_0 \geq F_x) = 3 + 1$

Conclusion

- The inverse CDF method applies very generally to many distributions.
- The method relies on a closed form expression for $F^{-1}(u)$: Given $F(x) = u$, we assume we can derive $x = F^{-1}(u)$.
- However, there are random variables for which this is not possible.