Composition Methods

Chapter 3

STATS 102C: Introduction to Monte Carlo Methods

Introduction

- The inverse CDF method is a way of transforming the uniform random variable $U \sim Unif(0, 1)$ into another random variable $X \sim F^{-1}(U)$ in order to sample from X.
- In addition to the inverse CDF transform, there are other types of transformations that can be applied in order to simulate random variables.
- These transformation (or composition) methods allow us a way to leverage sampling from simpler distributions to sample from more complicated distributions.

Some Well Know Compositions

- If $Z \sim N(0,1)$, then

$$V = Z^2 \sim \chi^2(1)$$

  has a chi-square distribution with 1 degree of freedom.

- If $U \sim \chi^2(m)$ and $V \sim \chi^2(n)$ are independent, then

$$F = \frac{U/m}{V/n}$$

  has an F-distribution with (m,n) degrees of freedom.

- if $Z \sim N(0,1)$ and $V \sim \chi^2(n)$ are independent, then

$$T = \frac{Z}{\sqrt{V/n}}$$

  has a Student's t-distribution with n degrees of freedom.

- if $U, V \sim Unif(0,1)$ are independent, then

$$Z_1 = \sqrt{-2\log U}\cos 2\pi V,$$

$$Z_2 = \sqrt{-2\log U}\sin 2\pi V$$

are independent standard normal variables.

- If $U \sim Gamma(r, \lambda)$ and $V \sim Gamm(s, \lambda)$ are independent, then

$$X = \frac{U}{U+V}$$

has the $Beta(r, s)$ distribution.

Convolutions

- Let $X_1, X_2, \ldots, X_m$ be independent random variables. The **convolution** of $X_1, X_2, \ldots, X_m$ is the sum

$$S = \sum_{i=1}^{m} X_i = X_1 + X_2 + \cdots + X_m.$$

- Many common random variables can be expressed as a convolution.
- To simulate from a convolution, we can generate samples from $X_1, X_2, \ldots, X_m$ and compute the sum.

Some Common Convolutions

- **Binomial**: Let $X_1, X_2, \ldots, X_m \overset{iid}{\sim} \mathrm{Bernoulli}(p)$ be Bernoulli random variables with parameter p, i.e., for any i, $X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases}$ Then

$$S = \sum_{i=1}^{m} X_i = X_1 + X_2 + \cdots + X_m \sim Bin(m, p)$$

has a binomial distribution with parameters m and p.

- **Poisson**: If $X_i \sim \mathrm{Pois}(\lambda_i)$, $i = 1, 2, \ldots, m$, are independent Poisson random variables, for $\lambda_i > 0$ for all i, then

$$S = \sum_{i=1}^{m} X_i = X_1 + X_2 + \cdots + X_m \sim \mathrm{Pois}\left(\sum_{i=1}^{m} \lambda_i\right)$$

has a Poisson distribution with mean parameter $\sum_{i=1}^{m} \lambda_i$.

- **Negative Binomial**: If $X_1, X_2, \ldots, X_m \overset{iid}{\sim} Geom(p)$, then

$$S = \sum_{i=1}^{m} X_i = X_1 + X_2 + \cdots + X_m \sim \mathrm{NegBin}(m, p)$$

has a negative binomial distribution with parameters m and p.

- **Chi-square**: If $Z_1, Z_2, \ldots, Z_m \overset{iid}{\sim} N(0, 1)$, then

$$S = \sum_{i=1}^{m} Z_i^2 = Z_1^2 + Z_2^2 + \cdots + Z_m^2 \sim \chi^2(m)$$

has a chi-square distribution with m degrees of freedom.

- **Gamma**: If $X_1, X_2, \ldots, X_m \overset{iid}{\sim} Exp(\lambda)$, for $\lambda > 0$, then

$$S = \sum_{i=1}^{m} X_i = X_1 + X_2 + \cdots + X_m \sim \mathrm{Gamma}(m, \lambda)$$

has a gamma distribution with parameters m and λ.

Mixtures

- A random variable X is a discrete mixture if the distribution of X is a weighted sum $F(x) = \sum \theta_i F(x)$ for some sequence of random variables $X_1, X_2, \dots$ and $\theta_i > 0$ such that $\sum_i \theta_i = 1$. The constants θ_i are called the mixing weights or mixing probabilities.
- A random variable X is a continuous mixture if the distribution of X is $F_X(x) = \int_{-\infty}^{\infty} F_{X|Y=y}(x) f_Y(y) dy$ for a family $X|Y = y$ indexed by the real numbers y and weighting function f_Y such that $\int_{-\infty}^{\infty} f_Y(y) dy = 1$.

Multivariate Normal Distribution

- The multivariate normal density is obtained by replacing the univariate distance by the multivariate generalized distance in the p-dimensional normal density function.
- When the replacement is made, the univariate normalizing constant must be changed to a more general constant that makes the *volume* under the surface of the multivariate density function unity for any p.
- $f(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$ where $-\infty < x_i < \infty$, $i = 1, \dots, p$.
- The $p \times 1$ vector μ represents the expected value of the random vector x, and the $p \times p$ matrix Σ is the variance-covariance matrix of x.
- The symmetric matrix Σ is positive definite.

Cholesky decomposition

A Cholesky decomposition of Σ is a decomposition of the form

$$\Sigma = A^T A,$$

where A is a lower triangular matrix with real and positive diagonal entries. If Σ is positive-definite then it has a unique Cholesky decomposition.