The Long Run Behavior of Markov Chains

(Chapter 10)

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Stats 102C: Introduction to Monte Carlo Methods

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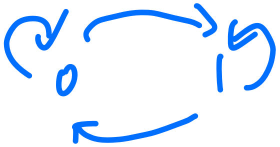
Outline

Introduction

- Consider a two-state Markov chain, with state space $\{0, 1\}$ and transition matrix

$$\mathbb{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.8 & 0.2 \\ 1 & 0.3 & 0.7 \end{array}.$$



- Suppose we are given that the chain starts at 0: $X_0 = 0$.

- We want to generate $X_1, X_2, \ldots, X_n$, for some large n (for example, $n = 10000$).

- **Goal**:

  long run

  - How often is the Markov chain in state 0?

  - How often is the Markov chain in state 1?

Introduction

R Code to generate two-state Markov chain:

```
> set.seed(9999) # for reproduceability
```

times

```
> n <- 10000 # specify length of chain

> # Specify transition matrix
> P <- rbind(c(0.8, 0.2), c(0.3, 0.7))
```

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.8 & 0.2 \\ 1 & 0.3 & 0.7 \end{array}$$

```
> # Initialize the Markov chain at state 0
> X <- 0
```

$(X_0 = 0)$

$x_1 - x_{10000}$

```
> for(i in 2:n){
+     # Specify row of P for next step probability
+     row_n <- X[i - 1] + 1
```

X[1] +1

```
+
+     # Take next step of Markov chain
+     X[i] <- sample(c(0, 1), size = 1, prob = P[row_n, ])
+ }
```

from $\{0, 1\}$

$p(0) =$ row 1

$p(1) =$ row 2

Introduction

```
> # Proportion of steps in state 0
> sum(X == 0) / n
[1] 0.6013

> # Proportion of steps in state 1
> sum(X == 1) / n
[1] 0.3987
```

What if we started the chain from a different initial state?

Introduction

```
> set.seed(999) # for reproduceability

> # Initialize the Markov chain at state 1
> X <- 1

> for(i in 2:n){
+     # Specify row of P for next step probability
+     row_n <- X[i - 1] + 1
+
+     # Take next step of Markov chain
+     X[i] <- sample(c(0, 1), size = 1, prob = P[row_n,])
+ }

> sum(X == 0) / n # Proportion of steps in state 0
[1] 0.6061

> sum(X == 1) / n # Proportion of steps in state 1
[1] 0.3939
```

Introduction

$\pi \rightarrow$ distribution

$\pi_1 \rightarrow$ stay at 1

$\pi_0 \rightarrow$ stay at 0

- Let π_0 denote the long run (relative) frequency at state 0.
  - From our simulation, we found $\pi_0 = 0.6$. (0.6061)
- Let π_1 denote the long run (relative) frequency at state 1.
  - From our simulation, we found $\pi_1 = 0.4$. (0.3939)
- Then $\pi = (\pi_0, \pi_1)$ is a distribution on the state space $\{0, 1\}$.
- How do we interpret this distribution?

π: probability distribution in space $\{0. 1\}$

$\pi = (\pi_0. \pi_1)$

Introduction

Interpretation: (pro and step)

1. The fraction of time (steps) in state j is π_j.
   - If $n = 10000$, the number of steps in state 0 is about $\pi_0 \cdot n = 0.6 \cdot 10000$. = 6000 time stay 0 → 4000 stay 1

2. The probability that long run X_n is in state j is π_j. limits
   - $\lim_{n\to\infty} P(X_n = 0|X_0 = 0) = \pi_0 = 0.6$
     (6000 time ∞)
   - $\lim_{n\to\infty} P(X_n = 0|X_0 = 1) = \pi_0 = 0.6$

     → limiting distribution
   - This distribution is independent of the initial state X_0.

$X_1 \; X_2 \; X_3 \cdots X_n$ iid

Outline

The Limiting Distribution

Definition

Let $\pi = (\overset{\text{step 0}}{\pi_0}, \overset{1}{\pi_1}, \overset{2}{\pi_2}, \overset{\cdots}{\ldots}, \overset{N}{\pi_N})$ be a probability distribution on the state space $\underline{\{0, 1, 2, \ldots, N\}}$. We say π is the **limiting distribution** of a Markov chain $\{X_0, X_1, X_2, \ldots\}$ if

$S \in (1 \cdots N)$

steps

$$\lim_{n\to\infty} P(X_n = j | X_0 = i) = \pi_j,$$

j=outcome

$p(\text{from } i \xrightarrow[n\text{ time}]{} \text{to } j) = \pi_j$

for all $i, j \in \{0, 1, 2, \ldots, N\}$.

$p(\text{from } i \xrightarrow[n\text{ time}]{} \text{to } i) = \pi_i$ $\quad i\,j \in (0\ N)$

dist exist ?

$$\begin{cases} \text{DNE } (-\infty\ \infty) & \to \text{discoverges} \\ 0 & \\ \text{other fixed number} & \end{cases}$$

} converges

The Limiting Distribution

If the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_N)$ exists, then π_j represents:

- The probability that X_n is in state j, independent of the initial state X_0.
- The long run mean (i.e., expected) fraction of time that the Markov chain $\{X_t : t = 0, 1, 2, \ldots\}$ spends in state j.

Main Questions:

- What conditions will guarantee the existence of the limiting distribution?
- If the limiting distribution exists, how do we find it?

The Limiting Distribution

if exist: as $n \longrightarrow \infty$

Question: If the limiting distribution exists, how do we find it?

Suppose we know that the limiting distribution π exists. Then, for n large enough,

$$\overset{i}{X_0} \longrightarrow \underset{\pi(k)}{\overset{k}{X_n}} \longrightarrow \underset{\pi(j)}{\overset{j}{X_{n+1}}}$$

$$P(X_n = k | X_0 = i) = \pi_k \text{ and } P(X_{n+1} = j | X_0 = i) = \pi_j,$$

for any $j, k \in \{0, 1, 2, \dots, N\}$.

$0 \longrightarrow X_n$

$0 \longrightarrow X_n \longrightarrow X_{n+1}$

How can we use these probabilities to solve for

$$\pi = (\pi_0, \pi_1, \pi_2, \dots, \pi_N)?$$

use total probability

The Limiting Distribution

Recall:

Law of Total Probability (Discrete)

Let X and Y be discrete random variables. Then

$$P(X = \overset{\overset{\text{outcome}}{\uparrow}}{x}) = \sum_y P(X = x, Y = y)$$

$$\underset{\pi(j)}{p(X_{n+1} = j)} \quad = \sum_y P(\overset{X_{n+1}}{X} = x | \overset{\overset{K}{X_n}}{Y} = y) P(\overset{X_n}{Y} = y). \quad (\text{conditional})$$

$\sum p(\text{every step}) \cdot p(\text{initial})$

$\longrightarrow \pi(x)$

The Limiting Distribution (long run)

By the Law of Total Probability, we have

$$\pi_j = P(X_{n+1} = j | X_0 = i)$$

$$= \sum_{k=0}^{N} P(X_{n+1} = j, \boxed{X_n = k} | X_0 = i)$$

$$= \sum_{k=0}^{N} P(X_{n+1} = j | X_n = k, X_0 = i) P(X_n = k | X_0 = i)$$

one step

$X_n, X_{n-1}\ X_{n-2}\ X_{n-3}\ \cdots\ X_0 \downarrow$

(Markov Property)

$$= \sum_{k=0}^{N} P_{kj} \pi_k$$

$$= p(1,2)\underset{\pi(1)}{p(1)} + p(2,3)\underset{\pi(2)}{p(2)} + \cdots p(i,j)\pi(i)$$

$$= \sum_{k=0}^{N} \pi_k P_{kj},$$

$X_n=k \mid X_{n-1}=K_1,\ X_0=i$

$X_{n-1}=K_1 \mid X_{n-2}=K_2,\ X_0=i\ \cdots$

where $P_{kj} = P(X_{n+1} = j | X_n = k)$ is the one-step transition probability from k to j. sum(p(any state $\rightarrow j$))

The Limiting Distribution

- We therefore have the relations equations

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, \quad \text{for } j = 0, 1, 2, \ldots, N.$$

(N+1 equations)

- In addition, since π is a probability distribution, then

$\pi (\pi_0, \pi_1\ \pi_2\ \cdots\ \pi_N)$

$$\sum_{i=0}^{N} \pi_i = \pi_0 + \pi_1 + \pi_2 + \cdots + \pi_N = 1.$$ first equation

another equation

- This defines a system of $\boxed{N+2}$ linear equations for $N+1$ unknowns (one equation is redundant).

- A solution π to this system of linear equations is called a **stationary distribution** of the Markov chain.

- In particular, the limiting distribution (if it exists) is a stationary distribution. $\rightarrow \pi = (\pi_0 \cdots \pi_n) \rightarrow$ stationary distribution

Example 1: Two-State Markov Chain

π exist ⟶ there exist a stationary dist

$N=1$

- Consider a two-state Markov chain, with state space $\{0, 1\}$ and transition matrix

$$\mathbb{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}.$$

2 states

- We want to find a $\pi = (\pi_0, \pi_1)$ that satisfies

$0 \to 0$

$1 \to 0$

$$① \quad \pi_0 = \pi_0 P_{00} + \pi_1 P_{10} = p(0,0) \cdot p(0) + p(1,0) \cdot p(1)$$

$1+1 = N+1$

$$② \quad \pi_1 = \pi_0 P_{01} + \pi_1 P_{11} = p(1,1) p(1) + p(0,1) p(0)$$

$1 \to 1$

$0 \to 1$

and

$$1 \quad ③ \quad \pi_0 + \pi_1 = 1.$$

solve for 3 equation → π_0 π_1

Example 1: Two-State Markov Chain

Plugging in the values from the transition matrix $\mathbb{P}$, we have

$$\begin{aligned}\pi_0 &= \pi_0 \cdot 0.8 + \pi_1 \cdot 0.3 \\ \pi_1 &= \pi_0 \cdot 0.2 + \pi_1 \cdot 0.7\end{aligned}$$

which simplify to

$$\begin{aligned}0.2\pi_0 &= 0.3\pi_1 \\ 0.3\pi_1 &= 0.2\pi_0.\end{aligned}$$

Both equations yield the relation

$$\frac{\pi_0}{\pi_1} = \frac{3}{2}.$$

Then, since $\pi_0 + \pi_1 = 1$, we have

$$\boxed{\begin{aligned}\pi_0 &= 0.6 \\ \pi_1 &= 0.4.\end{aligned}}$$

stationary distribution
↑
$\pi(0.6\ 0.4)$

This corresponds to the limiting distribution we previously found.

Outline

Stationary Distributions

Definition

Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain, with state space $\{0, 1, 2, \ldots, N\}$ and transition matrix $\boxed{\mathbb{P} = [P_{ij}]}$. The (row) vector $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_N)$ is called a **stationary distribution** of the Markov chain if it satisfies:

$$\pi = \pi p \longrightarrow \pi = [\pi_0 \cdots \pi_N)$$

$$p = \left[\begin{array}{ccc} 0 & \text{—} & N \\ | & & \\ N & & \end{array}\right]$$

- $$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, \quad \text{for } j = 0, 1, 2, \ldots, N,$$

  or, equivalently (in matrix notation), $\boxed{\pi = \pi\mathbb{P}.}$ p is transition matrix

- $$\sum_{i=0}^{N} \pi_i = 1.$$ first equation

Example 2: Identity Transition Matrix

- Consider a two-state Markov chain with transition matrix

$$\mathbb{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

$$\begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix}$$

- Notice that $\boxed{\pi I_2 = \pi}$ for every vector π.

- Therefore, there are <u>infinitely many stationary distributions for this Markov chain.</u>

- In particular, stationary distributions may not be unique.

$$\left\{ \begin{array}{l} \text{many solution} \\ \text{one solution} \\ \text{DNE} \end{array} \right.$$

Example 3: Two-State Markov Chain (Revisited)

- Consider a two-state Markov chain, with state space $\{0, 1\}$ and transition matrix
$$\mathbb{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}.$$
- Let $\pi = (\pi_0, \pi_1) = (0.6, 0.4)$ denote a stationary distribution for this Markov chain. unique
- Suppose $X_0 \sim \pi$. We want to show that $X_1 \sim \pi$.

Example 3: Two-State Markov Chain (Revisited)

if stationary

$\pi_0 = 0.6 \quad \pi_1 = 0.4)$

If $P(X_0 = 0) = \boxed{\pi_0 = 0.6}$ and $P(X_0 = 1) = \pi_1 = 0.4$, then

one step

$$\begin{aligned}
P(X_1 = 0) &= \sum_{i=0}^{1} P(X_1 = 0, X_0 = i) \\
&= P(X_1 = 0, \boxed{X_0 = 0}) + P(X_1 = 0, \boxed{X_0 = 1}) \\
&= P(X_1 = 0 | X_0 = 0) P(X_0 = 0) \\
&\quad + P(X_1 = 0 | X_0 = 1) P(X_0 = 1) \\
&= P_{00}\pi_0 + P_{10}\pi_1 \\
&= 0.8 \cdot 0.6 + 0.3 \cdot 0.4 \\
&= 0.6 \\
&= \pi_0
\end{aligned}$$

$\rightarrow 0.6$

$p(0,0)\,p(0) + p(1,0) \cdot p(1)$

$= 0.6 \times 0.8 + 0.3 \times 0.4$

$= 0.48 + 0.12$

$= 0.6 = \pi_0$

for many step

and

$$P(X_1 = 1) = 1 - 0.6 = 0.4 = \pi_1.$$

$1 - \pi_0 = \pi_1$

So $X_1 \sim \pi_1$.

one step proved

Stationary Distributions

$\pi = (\pi_0 \ \pi_1 \cdots \pi_N)$

- Let π be a stationary distribution of a Markov chain. If $X_0 \sim \underset{\pi_0}{\pi}$, then $X_1 \sim \underset{\pi_1}{\pi}$. This implies that $X_n \sim \underset{\pi_N}{\pi}$ for every n.

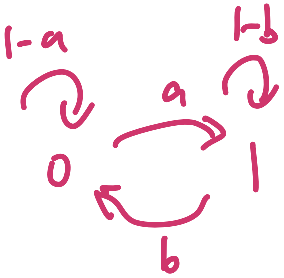
- In other words, if the initial distribution for X_0 is a stationary distribution π, then π is the common distribution for all X_n in the Markov chain (and we say the Markov chain is **stationary**).

Example 4: Two-State Markov Chain (General)

$p(0,1)=a$ $p(1,0)=b$

- Consider a two-state Markov chain, with state space $\{0, 1\}$ and transition matrix

$$\mathbb{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$



for some $a, b > 0$.

- We want to find a stationary distribution $\pi = (\pi_0, \pi_1)$.

$\pi = \pi \mathbb{P}$

Example 4: Two-State Markov Chain (General)

For $\pi = (\pi_0, \pi_1)$ to be a stationary distribution, it must satisfy

$p(0,1) = a \quad p(1,0) = b$

$$\begin{aligned} \pi &= \pi \mathbb{P} \\ \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} &= \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} \overset{\text{stay}}{\uparrow} \\ 1-a & a \\ b & 1-b \end{bmatrix} \rightarrow \text{stay} \\ &= \begin{bmatrix} (1-a)\pi_0 + b\pi_1 & a\pi_0 + (1-b)\pi_1 \end{bmatrix}, \end{aligned}$$

which equates to

$\pi_0 + \pi_1 = 1$

$$\begin{aligned} \pi_0 &= (1-a)\pi_0 + b\pi_1 \\ \pi_1 &= a\pi_0 + (1-b)\pi_1. \quad \text{solve} \end{aligned}$$

These equations give the relation $a\pi_0 = b\pi_1$, or

$$\frac{\pi_0}{\pi_1} = \frac{b}{a}. \quad \frac{0.6}{0.4} = \frac{3}{2}$$

Then, since $\pi_0 + \pi_1 = 1$, we have

$$\boxed{\pi_0 = \frac{b}{a+b} \quad \text{and} \quad \pi_1 = \frac{a}{a+b}.}$$

Outline

Conditions for the Limiting Distribution

Conditions for the Limiting Distribution

Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with state space $\{0, 1, 2, \ldots, N\}$ and transition matrix $\mathbb{P}$. The Markov chain has a limiting distribution if:

1. There is a solution to the system of equations defined by

$$\begin{cases} \pi = \pi \mathbb{P} \\ \sum_{i=0}^{N} \pi_i = 1 \end{cases}$$

   i.e., there is a stationary distribution. ① get solution

2. The Markov chain is irreducible. ② states communicate

3. The Markov chain is aperiodic. ③ period=1

Irreducible Markov Chains

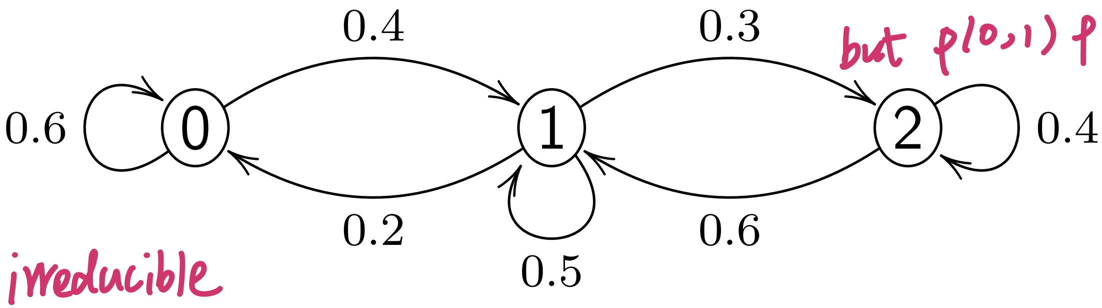
$p(i,j) \neq 0$

- Consider a three-state Markov chain, with state space $\{0, 1, 2\}$ and transition matrix

$$\mathbb{P} = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0.6 & 0.4 \end{bmatrix}.$$

- A transition state diagram is given by:

$p(0,2)=0$



but $p(0,1)\,p(1,2) \neq 0$

irreducible

- Notice that state 0 cannot jump to state 2 directly ($P_{02} = 0$), but (by following the arrows on the diagram) there is a path from state 0 to state 2: $0 \to 1 \to 2$.

Irreducible Markov Chains

Definition

State j is **accessible** from state i, denoted by $i \to j$, if there is a path from state i to state j. $P_n(i,j) \neq 0$ n path

Definition

Two states i and j **communicate**, denoted by $i \leftrightarrow j$, if each state is accessible from the other. In other words, if $i \to j$ and $j \to i$, then $i \leftrightarrow j$.

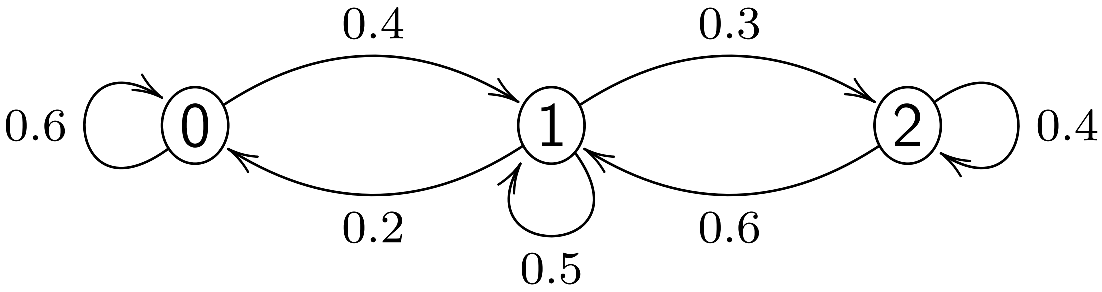
Definition

A Markov chain is **irreducible** if all states communicate with each other. A Markov chain is **reducible** if it is not irreducible.

$P_n(i,j) \neq 0$ $P_n(j,i) \neq 0$

Example 5: An Irreducible Markov Chain

- Consider a Markov chain with transition state diagram:



- There is a looped path $0 \to 1 \to 2 \to 1 \to 0$ which passes through every state in the state space and shows that every state is accessible from any other state.

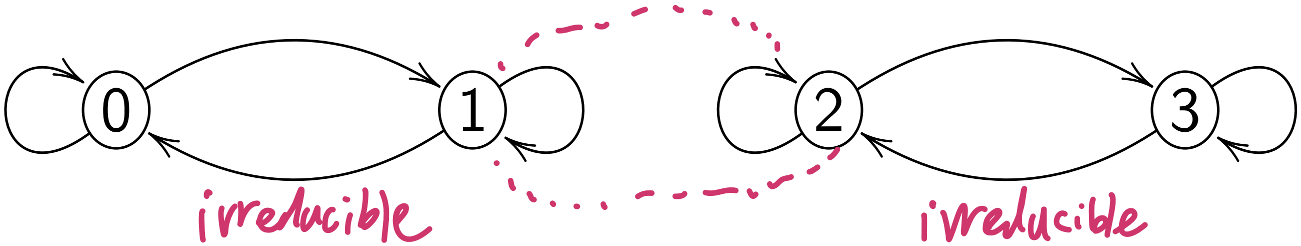
- Since all states communicate with each other, then this Markov chain is irreducible.

Example 6: A Reducible Markov Chain

$1 \xrightarrow{x} 2$ $\quad 2 \xrightarrow{x} 1$

$1 \xrightarrow{x} 3$ $\quad 2 \xrightarrow{x} 0$

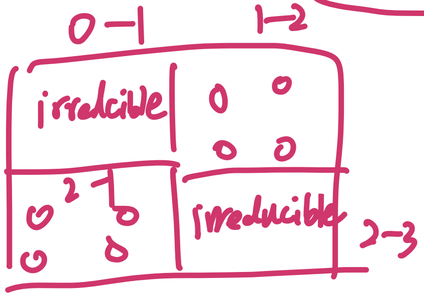
- Consider a Markov chain with transition state diagram:



- Notice that $0 \not\leftrightarrow 2$: There is no path from state 0 to state 2 (or vice versa), so 0 and 2 do not communicate.

- Not all states communicate with each other, so this Markov chain is reducible.

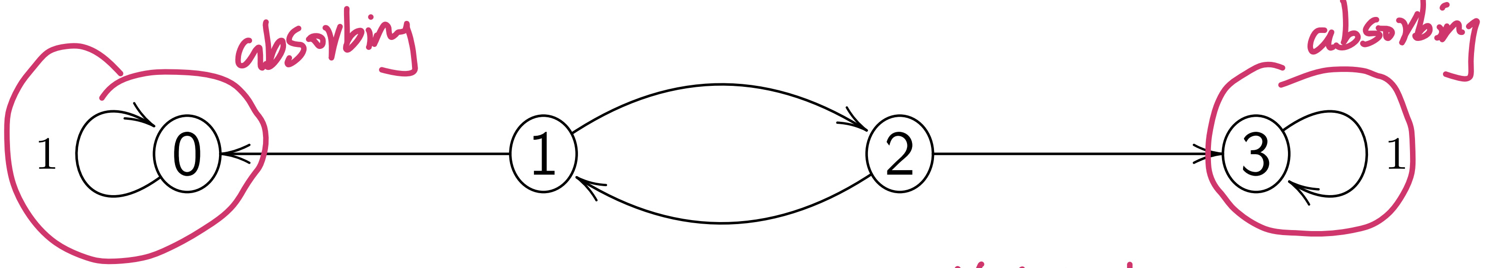
- This Markov chain can be reduced to two Markov chains, each with two states.



$\{0\ 1\}$ $\quad \{2\ 3\}$

Example 7: Absorbing States

- Consider a Markov chain with transition state diagram:



Never out

- Notice that states 0 and 3 are **absorbing states**: Once the Markov chain enters an absorbing state, it never leaves.

- Not all states communicate with each other (e.g., $2 \to 0$ but $0 \not\to 2$, so $0 \not\leftrightarrow 2$), so this Markov chain is reducible.

Example 8: Block Transition Matrix

- Consider a Markov chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix:

$$\mathbb{P} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

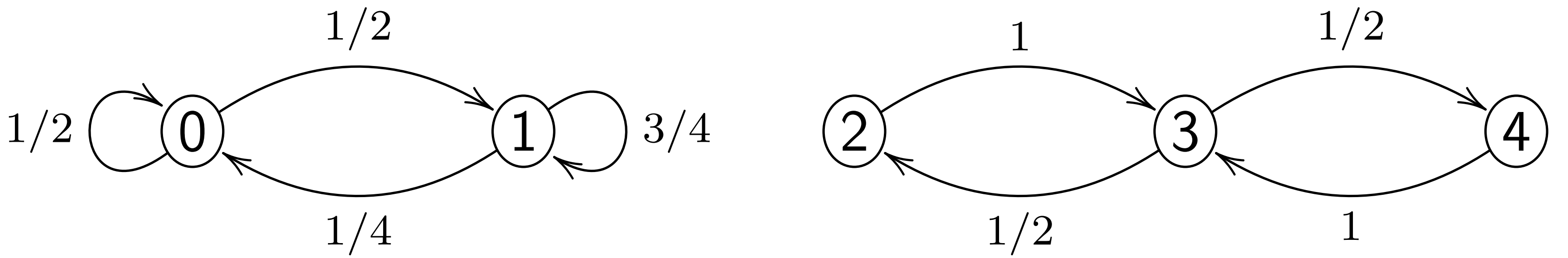
$1 \xrightarrow{\times} 2$

- Is this Markov chain irreducible or reducible?

reducible

Example 8: Block Transition Matrix

- A transition state diagram for this transition matrix is given by:



- We see that $0 \leftrightarrow 1$, $2 \leftrightarrow 3$, $3 \leftrightarrow 4$, and $2 \leftrightarrow 4$, but $\{0, 1\} \not\leftrightarrow \{2, 3, 4\}$.
- Not all states communicate with each other, so this Markov chain is reducible.

Example 8: Block Transition Matrix

- Notice that $\mathbb{P}$ is a block diagonal transition matrix:

$$\mathbb{P} = \left[\begin{array}{cc|ccc} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

- The blocks of zeroes on the block off-diagonals indicate that states within each block are not accessible from states in other blocks.

Irreducible Markov Chains

Accessibility can be defined more rigorously using n-step transition probabilities:

Definition

Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with state space $\{0, 1, 2, \ldots, N\}$. The **n-step transition probability** from state i to j is defined by

$$P_{ij}^{(n)} := P(X_n = j | X_0 = i).$$

Note that when $n = 1$, $P_{ij}^{(1)} = P_{ij}$.

Definition

State j is **accessible** from state i, denoted by $i \to j$, if there is positive probability that state j can be reached from state i in a finite number of transitions. In other words, $P_{ij}^{(n)} > 0$ for some n.

Periodicity of a Markov Chain

Definition

The **period** of a state i, denoted by $d(i)$, is the greatest common divisor (gcd) of all integers $n \geq 1$ for which $P_{ii}^{(n)} > 0$.

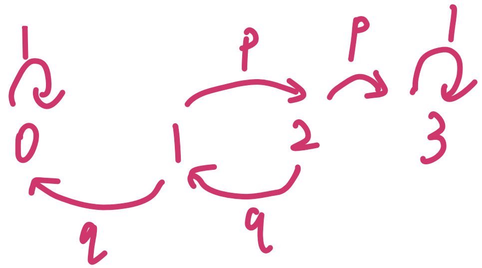
To find the period of a state i, we consider the number of steps it takes for any path starting from state i to return to state i.

Example 9: Absorbing States (Revisited)

0, 3 is absorbing

- Consider a Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix

$$\mathbb{P} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array},$$



where $0 < p, q < 1$.

move = p
back = q

reducible

- We want to find the period $d(i)$ of each state $i \in \{0, 1, 2, 3\}$.

1 $d(1) = 1 \to 2 \to 1 = 2$

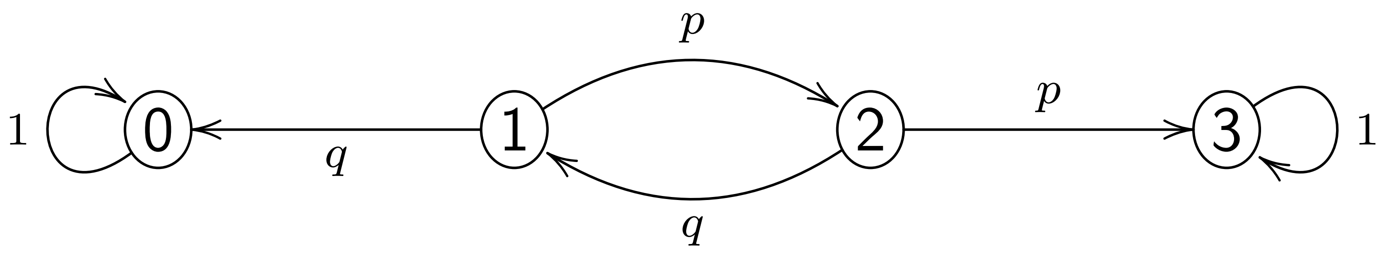
2 $d(2) : 2 \to 1 \to 2 = 2$

3 $d(3) = 1$

0 $d(0) = 1$

Example 9: Absorbing States (Revisited)

A transition state diagram for this transition matrix is given by:



$d(0)$:
- $d(0) = 1$, because $P_{00} > 0$.

$d(1)$:
- There is a path $1 \to 2 \to 1$, so $P_{11}^{(2)} > 0$.
- There is a path $1 \to 2 \to 1 \to 2 \to 1$, so $P_{11}^{(4)} > 0$.
- All number of steps n such that $P_{11}^{(n)} > 0$: $\{2, 4, 6, 8, \ldots\}$.
- $\gcd\{2, 4, 6, 8, \ldots\} = 2$, so $d(1) = 2$.

$d(2)$:
- States 1 and 2 are symmetric, so $d(2) = 2$.

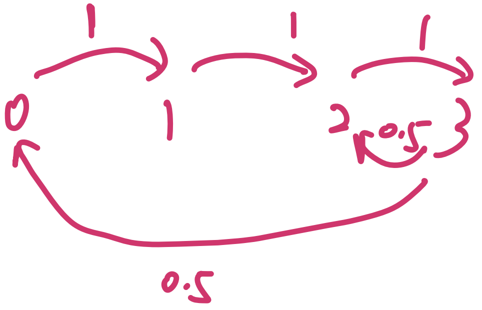
$d(3)$:
- $d(3) = 1$, because $P_{33} > 0$.

→ not aperiodic

Example 10: One Big Loop

- Consider a Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix

$$\mathbb{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & \textcircled{1} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & \textcircled{1} \\ 3 & 1/2 & 0 & 1/2 & 0 \end{array}.$$



- We want to find the period of state 0: $d(0)$.

d(0): 0 → 1 → 2 → 3 → 0 = 4

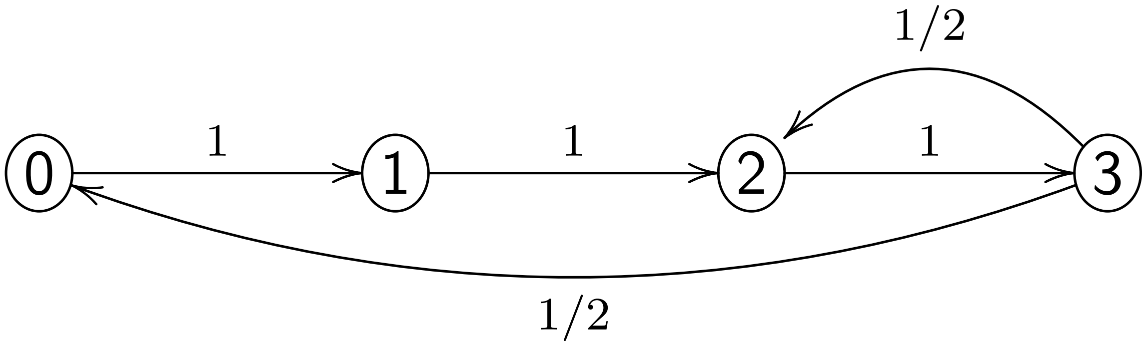
0 → 1 → 2 → 3 → 2 → 3 → 0 = 6

gcd = {4, 6, 8, 10 ···}

∴ d(0) = 2

Example 10: One Big Loop

A transition state diagram for this transition matrix is given by:



- There is a path $0 \to 1 \to 2 \to 3 \to 0$, so $P_{00}^{(4)} > 0$.
- There is a path $0 \to 1 \to 2 \to 3 \to 2 \to 3 \to 0$, so $P_{00}^{(6)} > 0$.
- All numbers of steps n such that $P_{00}^{(n)} > 0$: $\{4, 6, 8, 10, \ldots\}$.
- $\gcd\{4, 6, 8, 10, \ldots\} = 2$, so $d(0) = 2$.

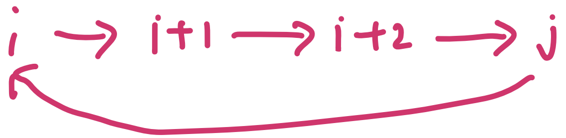
Periodicity of a Markov Chain

Definition

A Markov chain is called **aperiodic** if each state has period 1.

Some ways to check if a Markov chain is aperiodic:

- If $P_{ii} > 0$, then $d(i) = 1$.
- If $i \leftrightarrow j$ and $P_{ii} > 0$, then $d(j) = 1$. $\to P_{jj} > 0$



Periodicity of a Markov Chain

Show: If $i \leftrightarrow j$ and $P_{ii} > 0$, then $d(j) = 1$.

- Since $i \leftrightarrow j$, then there is a k-step path from $j \to i$, denoted by $j \overset{k}{\rightsquigarrow} i$, and an m-step path from $i \to j$, denoted by $i \overset{m}{\rightsquigarrow} j$, for some k, m. Then $j \overset{k}{\rightsquigarrow} i \overset{m}{\rightsquigarrow} j$ is a $(k+m)$-step path from j to j. So $P_{jj}^{(k+m)} > 0$.

- Since i has a self-loop ($P_{ii} > 0$), then $j \overset{k}{\rightsquigarrow} i \to i \overset{m}{\rightsquigarrow} j$ is a $(k+m+1)$-step path from j to j, so $P_{jj}^{(k+m+1)} > 0$.

- $\gcd(k+m, k+m+1) = 1$, so $d(j) = 1$.

The Basic Limit Theorem of Markov Chains

The Basic Limit Theorem of Markov Chains

If a Markov chain $\{X_0, X_1, X_2, \ldots\}$ is irreducible, aperiodic, and has a stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_N)$, then

$$\lim_{n\to\infty} P(X_n = j | X_0 = i) = \pi_j.$$

That is, π is the limiting distribution of the Markov chain, and π is uniquely determined by the system of equations

$$\begin{cases} \pi = \pi\mathbb{P} \\ \sum_{i=0}^{N} \pi_i = 1 \end{cases}$$

where $\mathbb{P}$ is the transition matrix of the Markov chain.