Formalizing the implementation of Streaming NESL

Master's Thesis

Dandan Xue

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Department of Computer Science (DIKU)
University of Copenhagen

Contents

Introduction

NESL

- A functional nested data-parallel language
- Developed by Guy Blelloch in 1990s at CMU
- Highlights:
  - Highly expressive for parallel algorithms.
    - Data-parallel construct: *apply-to-each*

$$\{e_1(x) : x \textbf{ in } e_0\}$$

    - Ex: compute $\sum_{i=1}^{2} i^2$ and $\sum_{i=3}^{10} i^2$:

$$\{\textbf{sum}(\{i \times i : i \textbf{ in } s\}) : s \textbf{ in } [[1,2],[3,4,5,6,7,8,9,10]]\}$$

      ! Allocate 10 size of space for intermediate data
  - An intuitive cost model for time complexity: work–step model
    - work cost t_1: total number of operations executed
    - step cost t_∞: the longest chain of sequential dependency

Streaming NESL (SNESL)

- Experimental refinement of NESL
- Aiming at improving space-usage efficiency
- Work by Frederik Madsen and Andrzej Filinski in 2010s at DIKU
- Highlights:
  - Streaming semantics

$$
\begin{aligned}
\pi &::= \mathbf{bool} \mid \mathbf{int} \mid \mathbf{char} \mid \mathbf{real} \mid \cdots && \text{(scalar types)} \\
\tau &::= \pi \mid (\tau_1, ..., \tau_k) \mid [\tau] && \text{(concrete types)} \\
\sigma &::= \tau \mid (\sigma_1, ..., \sigma_k) \mid \{\sigma\} && \text{(streamable types)}
\end{aligned}
$$

  - A space cost model
    - sequential space s_1: the minimal space to perform the computation
    - parallel space s_∞: space needed to achieve the maximal parallel degree (NESL's case)

SNESL syntax

- Expressions

$$
\begin{aligned}
e ::= {} & a \mid x \mid (e_1, ..., e_k) \mid \textbf{let } x = e_1 \textbf{ in } e_2 \mid \phi(e_1, ..., e_k) \\
& \mid \{e_1 : x \textbf{ in } e_0\} && \text{(general comprehension)} \\
& \mid \{e_1 \mid e_0\} && \text{(restricted comprehension)}
\end{aligned}
$$

SNESL syntax

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- Primitive functions

$$
\begin{aligned}
\phi\ &::= \oplus \mid \textbf{append} \mid \textbf{concat} \mid \textbf{zip} \mid \textbf{iota} \mid \textbf{part} \mid \textbf{scan}_{\otimes} \mid \textbf{reduce}_{\otimes} \\
&\quad \mid \textbf{mkseq} \mid \textbf{the} \mid \textbf{empty} \mid \textbf{length} \mid \textbf{elt} \mid \textbf{seq} \mid \textbf{tab} \mid \cdots \\
\oplus\ &::= + \mid \times \mid / \mid == \mid \textbf{not} \mid \cdots \\
\otimes\ &::= + \mid \times \mid \textbf{max} \mid \cdots
\end{aligned}
$$

SNESL syntax

- Expressions

$$\begin{aligned} e ::= {} & a \mid x \mid (e_1, ..., e_k) \mid \textbf{let } x = e_1 \textbf{ in } e_2 \mid \phi(e_1, ..., e_k) \\ & \mid \{e_1 : x \textbf{ in } e_0\} \qquad \text{(general comprehension)} \\ & \mid \{e_1 \mid e_0\} \qquad \text{(restricted comprehension)} \end{aligned}$$

- Primitive functions

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- $\textbf{if } e_0 \textbf{ then } e_1 \textbf{ else } e_2 \equiv \textbf{let } b = e_0 \textbf{ in the}(\{e_1 \mid b\} +\!\!+ \{e_2 \mid \textbf{not}(b)\})$

SNESL primitive functions

append : $(\{\sigma\}, \{\sigma\}) \rightarrow \{\sigma\}$	syntactic sugar $++$; $\{3,1\}++\{4\} = \{3,1,4\}$
concat : $\{\{\sigma\}\} \rightarrow \{\sigma\}$	**concat**$(\{\{3,1\},\{4\}\}) = \{3,1,4\}$
zip: $(\{\sigma_1\}, ..., \{\sigma_k\}) \rightarrow \{(\sigma_1, ..., \sigma_k)\}$	**zip**$(\{1,2\},\{\mathtt{F},\mathtt{T}\}) = \{(1,\mathtt{F}),(2,\mathtt{T})\}$
iota(&) : **int** $\rightarrow$ {**int**}	&5 = {0,1,2,3,4}
part : $(\{\sigma\}, \{$**bool**$\}) \rightarrow \{\{\sigma\}\}$	**part**$(\{3,1,4\},\{\mathtt{F},\mathtt{F},\mathtt{T},\mathtt{F},\mathtt{T},\mathtt{T}\}) = \{\{3,1\},\{4\},\{\}\}$
scan$_\otimes$: {**int**} $\rightarrow$ {**int**}	**scan**$_+$(&5) $= \{0,0,1,3,6\}$
reduce$_\otimes$: {**int**} $\rightarrow$ **int**	**reduce**$_+$(&5) $= 10$
mkseq : $(\overbrace{\sigma, ..., \sigma}^{k}) \rightarrow \{\sigma\}$	**mkseq**$(1,2,3) = \{1,2,3\}$
length(#): $[\tau] \rightarrow$ **int**	#[10,20] = 2
elt(!): $([\tau],$ **int**$) \rightarrow \tau$	[3,8,2] ! 1 = 8
the : $\{\sigma\} \rightarrow \sigma$	return the element of a singleton, **the**$(\{10\}) = 10$
empty : $\{\sigma\} \rightarrow$ **bool**	**empty**$(\{1,2\}) = \mathtt{F}$, **empty**(&0) $= \mathtt{T}$
seq : $[\tau] \rightarrow \{\tau\}$	**seq**$([1,2]) = \{1,2\}$
tab : $\{\tau\} \rightarrow [\tau]$	**tab**$(\{1,2\}) = [1,2]$

SNESL example: word count

```
-- split a string into words (delimited by spaces)
function str2wds_snesl(str) =
 let flags = { x == ' ' : x in str};
     nonsps =  concat({{x | x != ' '} : x in v})
  in concat({{x|not(empty(x))}: x in part(nonsps,flags ++ {T})})
-- count the length of a stream
function slength(s) = reduce({1 : _ in s})
```

```
  $> slength(str2wds_snesl(read_file(filename)))
```

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```

```
  $> slength(str2wds_snesl(read_file(filename)))
```

- Takes constant space
- Speedups of a similar word count program from [MF16]:

Benchmark	Speedup	Chunk size	Milliseconds
wc -w	0.65	N/A	19760
snesl-1	1.00	163840	12770
snesl-2	2.06	163840	6214
snesl-4	3.54	327680	3607
snesl-6	4.73	655360	2700
snesl-8	5.84	655360	2188
snesl-10	6.74	1310720	1894

Implementation

Source language

- Simplified SNESL types

$$\pi ::= \textbf{bool} \mid \textbf{int}$$

$$\tau ::= \pi \mid (\tau_1, \tau_2) \mid \{\tau\} \qquad \text{(no vectors)}$$

$$\varphi ::= (\tau_1, ..., \tau_k) \to \tau \qquad \text{(support recursion)}$$

Source language

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- Syntax

$$e ::= \cdots \mid f(e_1, ..., e_k) \qquad \text{(user-defined function call)}$$

$$d ::= \mathbf{function}\ f(x_1 \colon \tau_1, ..., x_k \colon \tau_k) \colon \tau = e$$

Source language

- Key typing rules, $\boxed{\Gamma \vdash_\Sigma \; e : \tau}$:

$$\frac{\Gamma \vdash_\Sigma \; e_0 : \{\tau_0\} \qquad [x \mapsto \tau_0, (x_i \mapsto \tau_i)_{i=1}^{k}] \vdash_\Sigma \; e_1 : \tau}{\Gamma \vdash_\Sigma \; \{e_1 : x \textbf{ in } e_0 \textbf{ using } x_1, ..., x_k\} : \{\tau\}} \begin{pmatrix} (\Gamma(x_i) = \tau_i, \\ \tau_i \textbf{ concrete})_{i=1}^{k} \end{pmatrix}$$

- Key evaluation rules, $\boxed{\rho \vdash_\Phi e \downarrow v}$:

$$\frac{\rho \vdash_\Phi e_0 \downarrow \{v_1, ..., v_l\} \qquad ([x \mapsto v_i, (x_j \mapsto \rho(x_j))_{j=1}^{k}] \vdash_\Phi e_1 \downarrow v_i')_{i=1}^{l}}{\rho \vdash_\Phi \{e_1 : x \textbf{ in } e_0 \textbf{ using } x_1, ..., x_k\} \downarrow \{v_1', ..., v_l'\}}$$

Target language: SVCODE

- SVCODE values:
  - primitive stream: $\vec{a} ::= \langle a_1, ..., a_l \rangle$
    e.g., $\vec{a}_1 = \langle 1, 2 \rangle, \langle 0|\vec{a}_1 \rangle = \langle 0, 1, 2 \rangle$, $\vec{b} = \langle \mathtt{F}, \mathtt{T}, \mathtt{F} \rangle$
  - stream tree: $w ::= \vec{a} \mid (w_1, w_2)$

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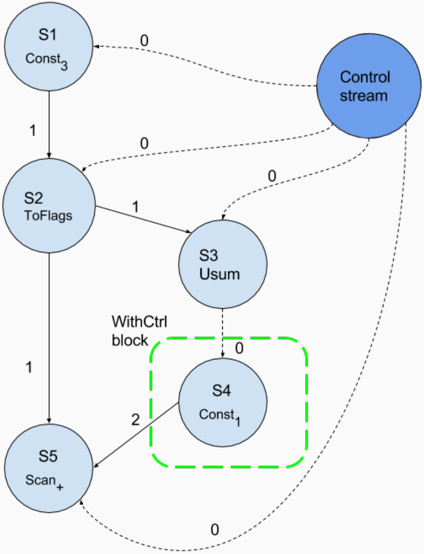
  - stream tree: $w ::= \vec{a} \mid (w_1, w_2)$
- SVCODE syntax:

$$
\begin{array}{rll}
p & ::= \epsilon \mid p_1; p_2 & \\
 & \mid s := \psi(s_1, ..., s_k) & \text{(single stream definition)} \\
 & \mid S_{out} := \mathtt{WithCtrl}(s, S_{in}, p_1) & (\mathtt{WithCtrl}\ \text{block}) \\
 & \mid (s'_1, ..., s'_{k'}) := \mathtt{SCall}\ f(s_1, ..., s_k) & \text{(function call)} \\
\psi & ::= \mathtt{Const_a} \mid \mathtt{ToFlags} \mid \mathtt{Usum} \mid \mathtt{Map}_{\oplus} \mid \mathtt{Scan}_{+} \mid \mathtt{Reduce}_{+} \mid \mathtt{Distr} & \\
 & \mid \mathtt{Pack} \mid \mathtt{UPack} \mid \mathtt{B2u} \mid \mathtt{SegConcat} \mid \mathtt{InterMerge} \mid \cdots & \text{(Xducers)} \\
s & ::= 0 \mid 1 \mid \cdots \in \mathbf{SId} = \mathbb{N} & \text{(stream ids)} \\
S & ::= \{s_1, ..., s_k\} \in \mathbb{S} & \text{(set of stream ids)}
\end{array}
$$

SVCODE dataflow DAG

```
S1  := Const_3
S2  := ToFlags S1
S3  := Usum S2
[S4]  := WithCtrl S3 []:
   S4  := Const_1
S5  := ScanPlus S2 S4
```



Value representation

- A nested sequence with a nesting depth d is represented as a flattened data stream and d segment descriptor streams.

$$\{3,1,4\} \triangleright_{\{\mathbf{int}\}} (\langle 3,1,4\rangle, \langle \mathtt{F},\mathtt{F},\mathtt{F},\mathtt{T}\rangle)$$

$$\{\{3,1\},\{4\}\} \triangleright_{\{\{\mathbf{int}\}\}} ((\langle 3,1,4\rangle, \langle \mathtt{F},\mathtt{F},\mathtt{T},\mathtt{F},\mathtt{T}\rangle), \langle \mathtt{F},\mathtt{F},\mathtt{T}\rangle)$$

Translation

- **STree** $\ni st ::= s \mid (st_1, st_2)$
- Translation symbol table $\delta ::= [x_1 \mapsto st_1, ..., x_k \mapsto st_k]$
- General comprehension translation:
  $\{i + x : i$ **in** $\&3$ **using** $x\} \Rightarrow$

```
S4 := ...   -- <10 >     x
S5 := ...   -- <F,F,F,T> descriptor of &3
S6 := ...   -- <0,1,2>  i
S7 := Usum S5;  -- 1. generate new control:   <() () ()>
S8 := Distr S4 S5; -- 2. replicate x 3 times: <10 10 10>
[S9] := WithCtrl S7 [S6,S8]: -- 3. translate  (i+x)
          S9 := Map_+ S6 S8  -- <10,11,12>
```

Translation continued

- Built-in function translation:
  - **scan**, **reduce**, **concat**, **part**, **empty**: translated to a single stream definition, e.g., $\textbf{scan}_{+}((s_d, s_b)) \Rightarrow \texttt{Scan}_{+}(s_b, s_d)$
  - **the**, **iota** translated to a few lines of code, e.g.,

$$\textbf{iota}(s) \Rightarrow \begin{pmatrix} s_0 := \texttt{ToFlags}(s); \\ s_1 := \texttt{Usum}(s_0); \\ \{s_2\} := \texttt{WithCtrl}(s_1, \emptyset, s_2 := \texttt{Const}_1()); \\ s_3 := \texttt{Scan}_{+}(s_0, s_2) \end{pmatrix}$$

  - $++_{\tau}$: translated recursively, depending on τ

Translation continued

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  - $++_\tau$: translated recursively, depending on τ
- User-defined functions: translated to SVCODE functions, unfolded at runtime when interpreting a `SCall`

SVCODE interpreters

- Eager interpreter (NESL-like)
  - assumes sufficient memory for allocating all streams at once
  - executes each instruction sequentially
- Streaming interpreter
  - limited buffer size, space-usage efficient
  - result is collected from each scheduling round
  - DAG dynamically completed
  - need effective scheduling strategy to avoid deadlock and guarantee cost preservation
- Both instrumented with cost metrics (work, step), compared with the high-level one

Recursion example

A function to compute factorial:

```
> function fact(x:int):int = if x <= 1 then 1 else x*fact(x-1)
> let s = {3,7,0,4} in {fact(n): n in s}
```

Recursion example

A function to compute factorial:

```
> function fact(x:int):int = if x <= 1 then 1 else x*fact(x-1)
> let s = {3,7,0,4} in {fact(n): n in s}
```

1st unfolding (will unfold 7 times in total):

```
--  Parameters: [S1]                    -- <3   7    0    4>
...-- compare parameters with 1,B2u:  S5 = <T   T    FT   T>
S6 := Usum S5;  -- for elements <=1     -- <         ()    >
[S7] := WithCtrl S6 []: S7 := Const_1   -- <         1     >
...
S13 := Usum S11; -- for elementes >1    -- <() ()       ()>
[S17] := WithCtrl S13 [S12]:
           S14 := Const_1               -- <1   1        1 >
           S15 := MapTwo Minus S12 S14  -- <2   6        3 >
           [S16] := SCall fact [S15]    -- <2 720        6 >
           S17 := MapTwo Times S12 S16  -- <6 5040      24>
...  -- merge results
S19 := PriSegInterS [(S7,S5),(S17,S11)];-- <6 5040 1   24>
```

Formalization

Source language: SNESL_0

- Types:

$$\tau ::= \textbf{int} \mid \{\tau_1\}$$

- Key evaluation rules with work cost, $\boxed{\rho \ \vdash \ e \downarrow v \ \$ \ W}$:
  - General comprehension:

$$\frac{([x \mapsto v_i, x_1 \mapsto \rho(x_1), \ldots, x_k \mapsto \rho(x_k)] \ \vdash \ e \downarrow v_i' \ \$ \ W_i)_{i=1}^{l}}{\rho \ \vdash \ \{e : x \textbf{ in } y \textbf{ using } x_1, \ldots, x_k\} \downarrow \{v_1', \ldots, v_l'\} \ \$ \ W}$$

where $\rho(y) = \{v_1, \ldots, v_l\}$, and $W = (k+1) \cdot (l+1) + \sum_{i=1}^{l} W_i$

  - Built-in function:

$$\frac{\phi(v_1, \ldots, v_k) \downarrow v}{\rho \ \vdash \ \phi(x_1, \ldots, x_k) \downarrow v \ \$ \ (\sum_{i=1}^{k} |v_i|) + |v|} \ ((\rho(x_i) = v_i)_{i=1}^{k})$$

Target language: SVCODE$_0$

- key semantics with work cost, $\boxed{\langle p, \sigma \rangle \Downarrow^{\vec{c}} \sigma' \ \$ \ W}$:
    - Empty new control stream ($\sigma(s_c) = \langle\rangle$):

      $$\frac{}{\langle S_{out} := \texttt{WithCtrl}(s_c, S_{in}, p_1), \sigma \rangle \Downarrow^{\vec{c}} \sigma[(s_i \mapsto \langle\rangle)_{i=1}^{k}] \ \$ \ 1}$$

      where $\forall s \in \{s_c\} \cup S_{in}.\sigma(s) = \langle\rangle$, $S_{out} = \{s_1, ..., s_k\}$

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    where $\forall s \in \{s_c\} \cup S_{in}.\sigma(s) = \langle\rangle$, $S_{out} = \{s_1, ..., s_k\}$
  - Nonempty new control stream ($\sigma(s_c) = \vec{c}_1 \neq \langle\rangle$):

$$\frac{\langle p_1, \sigma \rangle \Downarrow^{\vec{c}_1} \sigma'' \ \$ \ W_1}{\langle S_{out} := \texttt{WithCtrl}(s_c, S_{in}, p_1), \sigma \rangle \Downarrow^{\vec{c}} \sigma[(s_i \mapsto \sigma''(s_i))_{i=1}^{k}] \ \$ \ W_1 + 1}$$

Target language: SVCODE$_0$

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  - Xducers, $((\sigma(s_i) = \vec{a}_i)_{i=1}^{k})$
    $$\frac{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow^{\vec{c}} \vec{a}}{\langle s := \psi(s_1, ..., s_k), \sigma \rangle \Downarrow^{\vec{c}} \sigma[s \mapsto \vec{a}] \ \$ \ (\sum_{i=1}^{k} |\vec{a}_i|) + |\vec{a}|}$$

Xducer semantics

- General semantics, $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow^{\vec{c}} \vec{a}}$
  - $$\frac{\psi(\vec{a}_{11}, ..., \vec{a}_{k1}) \downarrow \vec{a}_{01} \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \Downarrow^{\vec{c}_0} \vec{a}_{02}}{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow^{\langle () | \vec{c}_0 \rangle} \vec{a}_0} \; ((\vec{a}_{i1} \mathbin{+\!\!+} \vec{a}_{i2} = \vec{a}_i)_{i=0}^{k})$$
  - $$\frac{}{\psi(\langle\rangle_1, ..., \langle\rangle_k) \Downarrow^{\langle\rangle} \langle\rangle}$$

Xducer semantics

- General semantics, $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow^{\vec{c}} \vec{a}}$
  - $$\frac{\psi(\vec{a}_{11}, ..., \vec{a}_{k1}) \downarrow \vec{a}_{01} \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \Downarrow^{\vec{c}_0} \vec{a}_{02}}{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow^{\langle () | \vec{c}_0 \rangle} \vec{a}_0} \; ((\vec{a}_{i1} {+\!\!+} \vec{a}_{i2} = \vec{a}_i)_{i=0}^k)$$
  - $$\frac{}{\psi(\langle\rangle_1, ..., \langle\rangle_k) \Downarrow^{\langle\rangle} \langle\rangle}$$
- Block semantics (part), $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow \vec{a}}$:

$$\frac{}{\texttt{Const}_a() \downarrow \langle a \rangle} \qquad \frac{}{\texttt{ToFlags}(\langle n \rangle) \downarrow \langle \texttt{F}_1, ..., \texttt{F}_n, \texttt{T} \rangle} \; (n \geq 0)$$

$$\frac{}{\texttt{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow \langle n_3 \rangle} \; (n_3 = n_1 + n_2)$$

$$\frac{\texttt{Usum}(\vec{b}) \downarrow \vec{a}}{\texttt{Usum}(\langle \texttt{F} | \vec{b} \rangle) \downarrow \langle () | \vec{a} \rangle} \qquad \frac{}{\texttt{Usum}(\langle \texttt{T} \rangle) \downarrow \langle\rangle}$$

Translation formalization

- General comprehension, $\boxed{\delta \ \vdash \ e \Rightarrow_{s_1}^{s_0} (p, st)}$:

$$\frac{[x \mapsto st_1, (x_i \mapsto s'_i)_{i=1}^{k}] \ \vdash \ e_1 \Rightarrow_{s''_1}^{s'_k+1} (p_1, st_2)}{\delta \ \vdash \ \{e_1 : x \ \textbf{in} \ y \ \textbf{using} \ x_1, ..., x_k\} \Rightarrow_{s''_1}^{s'_0} (p, (st_2, s_b))}$$

$$\text{with} \begin{pmatrix} \delta(y) = (st_1, s_b), (\delta(x_i) = s_i)_{i=1}^{k} \\ p = \ (\ s'_0 := \texttt{Usum}(s_b); \\ \qquad (s'_i := \texttt{Distr}(s_b, s_i);)_{i=1}^{k} \\ \qquad S_{out} := \texttt{WithCtrl}(s'_0, S_{in}, p_1)) \\ S_{in} = \overline{\overline{st_1}} \cup \{s'_1, ..., s'_k\} \\ S_{out} = \{s \mid s \in \overline{\overline{st_2}}, s \geq s'_k + 1\} \\ s'_{i+1} = \ s'_i + 1, \forall i \in \{0, ..., k-1\} \end{pmatrix}$$

Value representation formalization

- Value representation, $\boxed{v \rhd_\tau w}$:

$$\frac{}{n \rhd_{\mathbf{int}} \langle n \rangle}$$

$$\frac{v_1 \rhd_\tau w_1 \qquad \cdots \qquad v_l \rhd_\tau w_l}{\{v_1, ..., v_l\} \rhd_{\{\tau\}} (w, \langle \mathtt{F}_1, ..., \mathtt{F}_l, \mathtt{T} \rangle)} \; (w = w_1 +\!\!+_\tau \cdots +\!\!+_\tau w_l)$$

- Value recovery, $\boxed{w \lhd_\tau v, w'}$:

$$\frac{}{\langle n_0 | \vec{a} \rangle \lhd_{\mathbf{int}} n_0, \vec{a}}$$

$$\frac{w \lhd_\tau v_1, w_1 \qquad w_1 \lhd_\tau v_2, w_2 \qquad \cdots \qquad w_{l-1} \lhd_\tau v_l, w_l}{(w, \langle \mathtt{F}_1, ..., \mathtt{F}_l, \mathtt{T} | \vec{b} \rangle) \lhd_{\{\tau\}} \{v_1, ..., v_l\}, (w_l, \vec{b})}$$

- Both representation and recovery are deterministic; high-level values and low-level ones are 1-1 corresponding.

Parallelism fusion lemma

Ex. **let** $x = 10$ **in** $\{i + x : i$ **in** $\&3$ **using** $x\} \Rightarrow$:

```
S6 := ...                      -- <0, 1, 2 >  -- data stream of &3
S7 := Usum S5;                 -- <(),(),()>
S8 := Distr S4 S5;             -- <10,10,10>
[S9] := WithCtrl S7 [S6,S8]:
         S9 := Map_+ S6 S8     -- <10,11,12>  -- p
```

Parallelism fusion lemma

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Parallelism fusion lemma

Ex. **let** $x = 10$ **in** $\{i + x : i$ **in** $\&3$ **using** $x\} \Rightarrow$:

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Lemma (Parallelism fusion, simplified version)

If $\langle p, \sigma_1 \rangle \Downarrow^{\vec{c}_1} \sigma_1' \ \$ \ W_1$, *and* $\langle p, \sigma_2 \rangle \Downarrow^{\vec{c}_2} \sigma_2' \ \$ \ W_2$,
then $\langle p, \sigma_1 \bowtie \sigma_2 \rangle \Downarrow^{\vec{c}_1 ++ \vec{c}_2} \sigma_1' \bowtie \sigma_2' \ \$ \ W$, *and* $W \leq W_1 + W_2$

Correctness theorem

- If e (as well as its free variables) is well-typed, and can be evaluated to v with cost W^H, and translated to p, then executing p will generate streams that can represent v, and the cost is bounded by a constant C times W^H:

Theorem (Translation correctness, simplified version)

if *(i)* $\Gamma \vdash e : \tau$ *(ii)* $\rho \vdash e \downarrow v \ \$ \ W^H$ *(iii)* $\delta \vdash e \Rightarrow^{s_0}_{s_1} (p, st)$
(iv) $\forall x \in dom(\Gamma).\rho(x) \triangleright_{\Gamma(x)} \sigma^*(\delta(x))$
then, *(vii)* $\langle p, \sigma \rangle \Downarrow^{\langle () \rangle} \sigma' \ \$ \ W^L$ *(viii)* $v \triangleright_\tau \sigma'^*(st)$ *(ix)* $W^L \leq C \cdot W^H$

- Note: $C \geq 7$ (only for **iota**; other cases $C \sim 2$)

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- Note: $C \geq 7$ (only for **iota**; other cases $C \sim 2$)
- Corollary: Implementation correctness
  (i) if e is well-typed, closed, evaluated to v with cost W^H
  (ii) then e can be translated to p; executing p generates st that can recover v, with cost W^L bounded by $C \times W^H$
  (iii) translation, execution of p, recovery are all determinisitc

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Future work:

- Extending the proof system to support more types, primitive functions, recursion, step & space preservation, etc.
- Formalization of the streaming semantics of the target language
- Formalization of parallel Xducers
- Investigation of schedulability, deadlock, etc.
    - a characterization of streamability
    - streamable programs do not deadlock