SNESL formalization

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September 12, 2017

0 Level-0

Draft version 0.0.5: (almost) finished the proof of the main correctness theorem and the built-in function correctness theorem.

0.1 Source language syntax

(Ignore empty sequence for now)
Expressions:

$$e ::= x \mid \textbf{let } x = e_1 \textbf{ in } e_2 \mid \phi(x_1, ..., x_k) \mid \{e : x \textbf{ in } y \textbf{ using } \cdot\}$$

$$\phi = \textbf{const}_n \mid \textbf{iota} \mid \textbf{plus}$$

Values:

$$n \in \mathbf{Z}$$

$$v ::= n \mid \{v_1, ..., v_k\}$$

0.2 Type system

$$\tau ::= \textbf{int} | \{\tau_1\}$$

Type environment $\Gamma = [x_1 \mapsto \tau_1, ..., x_i \mapsto \tau_i]$.

- Expression typing rules:

  Judgment $\boxed{\Gamma \vdash e : \tau}$

$$\frac{}{\Gamma \vdash x : \tau}\ (\Gamma(x) = \tau) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : \tau}$$

$$\frac{\vdash \phi : (\tau_1, ..., \tau_k) \to \tau}{\Gamma \vdash \phi(x_1, ..., x_k) : \tau}\ ((\Gamma(x_i) = \tau_i)_{i=1}^k) \qquad \frac{[x \mapsto \tau_1] \vdash e : \tau}{\Gamma \vdash \{e : x \textbf{ in } y \textbf{ using } \cdot\} : \{\tau\}}\ (\Gamma(y) = \{\tau_1\})$$

- Auxiliary Judgment $\boxed{\vdash \phi : (\tau_1, ..., \tau_k) \to \tau}$

$$\frac{}{\vdash \textbf{const}_n : () \to \textbf{int}} \qquad \frac{}{\vdash \textbf{iota} : (\textbf{int}) \to \{\textbf{int}\}} \qquad \frac{}{\vdash \textbf{plus} : (\textbf{int}, \textbf{int}) \to \textbf{int}}$$

- Value typing rules:

  Judgment $\boxed{\vdash v : \tau}$

$$\frac{}{\vdash n : \textbf{int}} \qquad \frac{(\vdash v_i : \tau)_{i=1}^k}{\vdash \{v_1, ..., v_k\} : \{\tau\}}$$

0.3 Source language semantics

$\rho = [x_1 \mapsto v_1, ..., x_i \mapsto v_i]$

- Judgment $\boxed{\rho \ \vdash \ e \downarrow v}$

$$\frac{}{\rho \ \vdash \ x \downarrow v} \ (\rho(x) = v) \qquad \frac{\rho \ \vdash \ e_1 \downarrow v_1 \qquad \rho[x \mapsto v_1] \ \vdash \ e_2 \downarrow v}{\rho \ \vdash \ \mathbf{let}\ e_1 = x \ \mathbf{in}\ e_2 \downarrow v}$$

$$\frac{\vdash \ \phi(v_1, ..., v_k) \downarrow v}{\rho \ \vdash \ \phi(x_1, ..., x_k) \downarrow v} \ ((\rho(x_i) = v_i)_{i=1}^k)$$

$$\frac{([x \mapsto v_i] \ \vdash \ e \downarrow v_i')_{i=1}^k}{\rho \ \vdash \ \{e : x \ \mathbf{in}\ y \ \mathbf{using}\ \cdot\} \downarrow \{v_1', ..., v_k'\}} \ (\rho(y) = \{v_1, ..., v_k\})$$

- Auxiliary Judgment $\boxed{\vdash \ \phi(v_1, ..., v_k) \downarrow v}$

$$\frac{}{\vdash \ \mathbf{const}_n() \downarrow n} \qquad \frac{}{\vdash \ \mathbf{iota}(n) \downarrow \{0, 1, ..., n-1\}} \ (n \geq 0)$$

$$\frac{}{\vdash \ \mathbf{plus}(n_1, n_2) \downarrow n_3} \ (n_3 = n_1 + n_2)$$

0.4 SVCODE syntax

(1) Stream id:

$$s \in \mathbf{SId} = \mathbf{N} = \{0, 1, 2...\}$$

(2) Stream tree:

$$\mathbf{STree} \ni st ::= s \mid (st_1, s)$$

(3) SVCODE operations:

$$\psi ::= \mathtt{Ctrl} \mid \mathtt{Const}_{\mathtt{a}} \mid \mathtt{ToFlags} \mid \mathtt{Usum} \mid \mathtt{MapTwo}_{\oplus} \mid \mathtt{ScanPlus}_{n_0}$$

where $\oplus$ stands for some binary operation on **int**.

(4) SVCODE program:

$$\begin{aligned} p ::= \ & \epsilon \\ & \mid s := \psi(s_1, ..., s_i) \\ & \mid st := \mathtt{WithCtrl}(s, p) \\ & \mid p_1; p_2 \end{aligned}$$

(5) Target language values:

$$b \in \{\mathtt{T}, \mathtt{F}\}$$

$$a ::= n \mid b \mid ()$$

$$\vec{b} = \langle b_1, ..., b_i \rangle$$

$$\vec{a} = \langle a_1, ..., a_i \rangle$$

$$\mathbf{SVal} \ni w ::= \vec{a} \mid (w, \vec{b})$$

(6) Some notations and operations:

- For some a_0 and $\vec{a} = \langle a_1, ..., a_i \rangle$, let $\langle a_0 | \vec{a} \rangle = \langle a_0, a_1, ..., a_i \rangle$.

- $++ : \mathbf{SVal} \to \mathbf{SVal} \to \mathbf{SVal}$
  $\langle a_1, ..., a_i \rangle ++ \langle a'_1, ..., a'_i \rangle = \langle a_1, ..., a_i, a'_1, ..., a'_i \rangle$
  $(w_1, \vec{b}_1) ++ (w_2, \vec{b}_2) = (w_1 ++ w_2, \vec{b}_1 ++ \vec{b}_2)$
- $\mathtt{sids}$ is a function that converts a $st \in \mathbf{STree}$ to a set of $s \in \mathbf{SId}$:
  $\mathtt{sids}(s) = \{s\}$
  $\mathtt{sids}((st, s)) = \mathtt{sids}(st) \cup \{s\}$
- For some set of $\mathbf{SId}$, t, and some $s \in \mathbf{SId}$, let $t \lessdot s$ denote $\forall s' \in t. s' < s$.

0.5 SVCODE semantics

SVCODE runtime environment $\sigma = [s_1 \mapsto \vec{a}_1, ..., s_i \mapsto \vec{a}_i]$.
We define some notations and operations related to σ:

(1) Let $\sigma_1 \overset{<s}{=\!=} \sigma_2$ denote $\forall s' < s. \sigma_1(s') = \sigma_2(s')$.

(2) Judgment $\boxed{\sigma(st) = w}$

$$\frac{}{\sigma(s) = \vec{a}} \qquad \frac{\sigma(st) = w \qquad \overline{\sigma(s) = \vec{a}}}{\sigma((st, s)) = (w, \vec{a})}$$

Definition 0.1. $\sigma_1 \overset{st}{\sim} \sigma_2$ *iff*
(1) $dom(\sigma_1) = dom(\sigma_2)$
(2) $\forall s \in (dom(\sigma_1) - \mathtt{sids}(st)). \sigma_1(s) = \sigma_2(s)$

It is easy to show that this relation $\overset{st}{\sim}$ is commutative, transitive and associative.

Definition 0.2. $\sigma_1 \overset{st}{\bowtie} \sigma_2 = \sigma$ *iff*
(1) $\sigma_1 \overset{st}{\sim} \sigma_2$
(2) $\sigma(s) = \begin{cases} \sigma_1(s) ++ \sigma_2(s), & s \in \mathtt{sids}(st) \\ \sigma_1(s), & otherwise \end{cases}$

Lemma 0.1. *If* $\sigma_1 \overset{st}{\bowtie} \sigma_2 = \sigma$, *then* $\sigma_1 \overset{st}{\sim} \sigma$ *and* $\sigma_2 \overset{st}{\sim} \sigma$.

Lemma 0.2. *If* $\sigma_1 \overset{st_1}{\sim} \sigma'_1$, $\sigma_2 \overset{st_2}{\sim} \sigma'_2$, $\sigma_1 \overset{<s}{=\!=} \sigma_2$, *and* $\sigma'_1 \overset{<s}{=\!=} \sigma'_2$ *then* $\sigma_1 \overset{st_1}{\bowtie} \sigma'_1 \overset{<s}{=\!=} \sigma_2 \overset{st_2}{\bowtie} \sigma'_2$.

SVCODE operational semantics:

- Judgment $\boxed{\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$
  $\vec{a}_c$ is the control stream.

$$\frac{}{\langle \epsilon, \sigma \rangle \downarrow^{\vec{a}_c} \sigma} \qquad \frac{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}{\langle s := \psi(s_1, ..., s_k), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s \mapsto \vec{a}]} \; ((\sigma(s_i) = \vec{a}_i)_{i=1}^k)$$

$$\frac{}{\langle st := \mathtt{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \langle\rangle, ..., s_i \mapsto \langle\rangle]} \; (\sigma(s) = \langle\rangle, \mathtt{sids}(st) = \{s_1, ..., s_i\})$$

$$\frac{\langle p, \sigma \rangle \downarrow^{\vec{a}_s} \sigma''}{\langle st := \mathtt{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \sigma''(s_1), ..., s_i \mapsto \sigma''(s_i)]} \; \begin{pmatrix} \sigma(s) = \vec{a}_s = \langle a_0 | \vec{a} \rangle \\ \mathtt{sids}(st) = \{s_1, ..., s_i\} \end{pmatrix}$$

$$\frac{\langle p_1, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'' \qquad \langle p_2, \sigma'' \rangle \downarrow^{\vec{a}_c} \sigma'}{\langle p_1; p_2, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$$

- *Transducer* semantics:

  Judgment $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}$

  $$\frac{\psi(\vec{a}_{11}, ..., \vec{a}_{k1}) \Downarrow \vec{a}_1 \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \downarrow^{\vec{a}_c} \vec{a}_2}{\psi(\vec{a}_{11}{+\!\!+}\vec{a}_{12}, ..., \vec{a}_{k1}{+\!\!+}\vec{a}_{k2}) \downarrow^{\langle a_0|\vec{a}_c\rangle} \vec{a}} \ (\vec{a} = \vec{a}_1{+\!\!+}\vec{a}_2)$$

  $$\frac{}{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\langle\rangle} \langle\rangle}$$

- Transducer *block* semantics:

  Judgment $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow \vec{a}}$

  $$\frac{}{\texttt{Const}_\texttt{a} \Downarrow \langle a\rangle} \qquad \frac{}{\texttt{ToFlags}(\langle n\rangle) \Downarrow \langle \texttt{F}_1, ..., \texttt{F}_n, \texttt{T}\rangle} \qquad \frac{}{\texttt{MapTwo}_\oplus(\langle n_1\rangle, \langle n_2\rangle) \Downarrow \langle n_3\rangle} \ (n_3 = n_1 \oplus n_2)$$

  $$\text{P-USUMF}: \frac{\texttt{Usum}(\vec{b}) \Downarrow \vec{a}}{\texttt{Usum}(\langle \texttt{F}|\vec{b}\rangle) \Downarrow \langle ()|\vec{a}\rangle} \qquad \text{P-USUMT}: \frac{}{\texttt{Usum}(\langle \texttt{T}\rangle) \Downarrow \langle\rangle}$$

  $$\text{P-SCANF}: \frac{\texttt{ScanPlus}_{n_0+n}(\vec{b}, \vec{a}) \Downarrow \vec{a}'}{\texttt{ScanPlus}_{n_0}(\langle \texttt{F}|\vec{b}\rangle, \langle n|\vec{a}\rangle) \Downarrow \langle n_0|\vec{a}'\rangle} \qquad \text{P-SCANT}: \frac{}{\texttt{ScanPlus}_{n_0}(\langle \texttt{T}\rangle, \langle\rangle) \Downarrow \langle\rangle}$$

  Or if we want to use *unary* semantics maybe for later:

  $$\frac{\psi(\langle \texttt{F}\rangle, ..., \vec{a}_{k1}) \downdownarrows \vec{a}_1 \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi(\langle \texttt{F}\rangle{+\!\!+}\vec{a}_{12}, ..., \vec{a}_{k1}{+\!\!+}\vec{a}_{k2}) \Downarrow \vec{a}} \ (\vec{a} = \vec{a}_1{+\!\!+}\vec{a}_2)$$

  $$\frac{\psi(\langle \texttt{T}\rangle, ..., \vec{a}_k) \downdownarrows \vec{a}}{\psi(\langle \texttt{T}\rangle, ..., \vec{a}_k) \Downarrow \vec{a}}$$

  - Transducer *unary* semantics:

    Judgment $\boxed{\psi(\langle b\rangle, ..., \vec{a}_k) \downdownarrows \vec{a}}$

    $$\frac{}{\texttt{Usum}(\langle \texttt{F}\rangle) \Downarrow \langle ()\rangle} \qquad \frac{}{\texttt{Usum}(\langle \texttt{T}\rangle) \Downarrow \langle\rangle}$$

  - Transducer block with *accumulator*:

    Judgment $\boxed{\psi_n(\vec{a}_1, ..., \vec{a}_k) \Downarrow \vec{a}}$

    $$\frac{\psi_{n_0}(\langle \texttt{F}\rangle, ..., \vec{a}_{k1}) \downdownarrows^{n_0'} \langle n_1\rangle \qquad \psi_{n_0'}(\vec{a}_{12}, ..., \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi_{n_0}(\langle \texttt{F}\rangle{+\!\!+}\vec{a}_{12}, ..., \vec{a}_{k1}{+\!\!+}\vec{a}_{k2}) \Downarrow \langle n_1\rangle{+\!\!+}\vec{a}_2}$$

    $$\frac{\psi_{n_0}(\langle \texttt{T}\rangle, ..., \vec{a}_k) \downdownarrows \vec{a}}{\psi_{n_0}(\langle \texttt{T}\rangle, ..., \vec{a}_k) \Downarrow \vec{a}}$$

  - Transducer unary with *accumulator*:

    Judgment $\boxed{\psi_n(\langle \texttt{F}\rangle, ..., \vec{a}_k) \downdownarrows^{n'} \vec{a}}$

$$\frac{}{\texttt{ScanPlus}_{n_0}(\langle \texttt{F}\rangle, \langle n\rangle) \Downarrow^{n_0+n} \langle n_0\rangle}$$

Judgment $\boxed{\psi_n(\langle \texttt{T}\rangle, ..., \vec{a}_k) \Downarrow \vec{a}}$

$$\frac{}{\texttt{ScanPlus}_{n_0}(\langle \texttt{T}\rangle, \langle\rangle) \Downarrow \langle\rangle}$$

Theorem 0.1 (SVCODE determinism)**.** *If $\langle p, \sigma\rangle \downarrow^{\vec{a}_c} \sigma'$ and $\langle p, \sigma\rangle \downarrow^{\vec{a}_c} \sigma''$, then $\sigma' = \sigma''$.*

Lemma 0.3. *If $\sigma_1 \overset{st}{\sim} \sigma_2$, $\langle p, \sigma_1\rangle \downarrow^{\vec{a}_1} \sigma_1'$, $\langle p, \sigma_2\rangle \downarrow^{\vec{a}_2} \sigma_2'$, then $\langle p, \sigma_1 \overset{st}{\bowtie} \sigma_2\rangle \downarrow^{\vec{a}_1 ++ \vec{a}_2} \sigma_1' \bowtie \sigma_2'$*

Definition 0.3. *$\vec{a}$ is a prefix of $\vec{a}'$ if $\vec{a} \sqsubseteq \vec{a}'$.*

Judgment $\boxed{\vec{a} \sqsubseteq \vec{a}'}$

$$\frac{}{\langle\rangle \sqsubseteq \vec{a}} \qquad \frac{\vec{a} \sqsubseteq \vec{a}'}{\langle a_0 | \vec{a}\rangle \sqsubseteq \langle a_0 | \vec{a}'\rangle}$$

Lemma 0.4. *If*

(i) $(\vec{a}_i' \sqsubseteq \vec{a}_i)_{i=1}^k$ and $\psi(\vec{a}_1', ..., \vec{a}_k') \Downarrow \vec{a}'$,

(ii) $(\vec{a}_i'' \sqsubseteq \vec{a}_i)_{i=1}^k$ and $\psi(\vec{a}_1'', ..., \vec{a}_k'') \Downarrow \vec{a}''$

then

(i) $(\vec{a}_i' = \vec{a}_i'')_{i=1}^k$

(ii) $\vec{a}' = \vec{a}''$.

0.6 Translation

$\delta = [x_1 \mapsto st_1, ..., x_i \mapsto st_i]$

- Judgment $\boxed{\delta \vdash e \underset{s_1}{\overset{s_0}{\Rightarrow}} (p, st)}$

$$\frac{}{\delta \vdash x \underset{s_0}{\overset{s_0}{\Rightarrow}} (\epsilon, st)} (\delta(x) = st) \qquad \frac{\delta \vdash e_1 \underset{s_0'}{\overset{s_0}{\Rightarrow}} (p_1, st_1) \qquad \delta[x \mapsto st_1] \vdash e_2 \underset{s_1}{\overset{s_0'}{\Rightarrow}} (p_2, st)}{\delta \vdash \textbf{let } x = e_1 \textbf{ in } e_2 \underset{s_1}{\overset{s_0}{\Rightarrow}} (p_1; p_2, st)}$$

$$\frac{\vdash \phi(st_1, ..., st_k) \underset{s_1}{\overset{s_0}{\Rightarrow}} (p, st)}{\delta \vdash \phi(x_1, ..., x_k) \underset{s_1}{\overset{s_0}{\Rightarrow}} (p, st)} ((\delta(x_i) = st_i)_{i=1}^k)$$

$$\frac{[x \mapsto st_1] \vdash e \underset{s_1}{\overset{s_0+1}{\Longrightarrow}} (p, st)}{\delta \vdash \{e : x \textbf{ in } y \textbf{ using } \cdot\} \underset{s_1}{\overset{s_0}{\Rightarrow}} (s_0 := \texttt{Usum}(s_2); st := \texttt{WithCtrl}(s_0, p), (st, s_2))} (\delta(y) = (st_1, s_2))$$

- Auxiliary Judgment $\boxed{\vdash \phi(st_1, ..., st_k) \underset{s_1}{\overset{s_0}{\Rightarrow}} (p, st)}$

$$\frac{}{\vdash \textbf{const}_n() \underset{s_0}{\overset{s_0+1}{\Longrightarrow}} (s_0 := \texttt{Const}_\texttt{n}(), s_0)}$$

$$\frac{}{\vdash \textbf{iota}(s) \underset{s_4}{\overset{s_0}{\Rightarrow}} (p, (s_3, s_0))} \begin{pmatrix} s_{i+1} = s_i + 1 \\ p = s_0 := \texttt{ToFlags}(s); \\ \quad s_1 := \texttt{Usum}(s_0); \\ \quad s_2 := \texttt{WithCtrl}(s_1, s_2 := \texttt{Const}_1()); \\ \quad s_3 := \texttt{ScanPlus}_0(s_0, s_2) \end{pmatrix}$$

$$\frac{}{\vdash \textbf{plus}(s_1, s_2) \underset{s_0+1}{\overset{s_0}{\Longrightarrow}} (s_0 := \texttt{MapTwo}_+(s_1, s_2), s_0)}$$

0.7 Value representation

- Judgment $\boxed{v \triangleright_\tau w}$

$$\frac{}{n \triangleright_{\mathbf{int}} \langle n \rangle} \qquad \frac{(v_i \triangleright_\tau w_i)_{i=1}^k}{\{v_1, ..., v_k\} \triangleright_{\{\tau\}} (w, \langle \mathtt{F}_1, ..., \mathtt{F}_k, \mathtt{T} \rangle)} \ (w = w_1 {+\!\!+} ... {+\!\!+} w_k)$$

Lemma 0.5. *If $v \triangleright_\tau w$, $v' \triangleright_\tau w$, then $v = v'$.*

0.8 Correctness proof

Lemma 0.6 (???). *If $\Gamma \ \vdash \ e : \{\tau\}$, $\rho \ \vdash \ e \downarrow \{v_1, ..., v_k\}$, and $\delta \ \vdash \ e \overset{s_0}{\underset{s_1}{\Rightarrow}} (p, (st, s))$, then $s \notin \mathtt{sids}(st)$.*

Lemma 0.7. *If*

(i) $\vdash \ \phi : (\tau_1, ..., \tau_k) \to \tau$ *(by some derivation $\mathcal{T}$)*

(ii) $\vdash \ \phi(v_1, ..., v_k) \downarrow v$ *(by $\mathcal{E}$)*

(iii) $\vdash \ \phi(st_1, ..., st_k) \overset{s_0}{\underset{s_1}{\Rightarrow}} (p, st)$ *(by $\mathcal{C}$)*

(iv) $(v_i \triangleright_{\tau_i} \sigma(st_i))_{i=1}^k$

(v) $\bigcup_{i=1}^k \mathtt{sids}(st_i) \lessdot s_0$

then

(vi) $\langle p, \sigma \rangle \downarrow^{\langle () \rangle} \sigma'$ *(by $\mathcal{P}$)*

(vii) $v \triangleright_\tau \sigma'(st)$ *(by $\mathcal{V}$)*

(viii) $\sigma' \overset{\lessdot s_0}{=\!=\!=} \sigma$

(ix) $s_0 \leq s_1$

(x) $\mathtt{sids}(st) \lessdot s_1$

Proof. By inducntion on the syntax of ϕ.

- Case $\phi = \mathbf{const}_n$
  There is only one possibility for each of $\mathcal{T}$, $\mathcal{E}$ and $\mathcal{C}$:

$$\mathcal{T} = \frac{}{\vdash \ \mathbf{const}_n : () \to \mathbf{int}}$$

$$\mathcal{E} = \frac{}{\vdash \ \mathbf{const}_n() \downarrow n}$$

$$\mathcal{C} = \frac{}{\vdash \ \mathbf{const}_n() \xrightarrow[s_0+1]{s_0} (s_0 := \mathtt{Const_n}(), s_0)}$$

  So $k = 0, \tau = \mathbf{int}, v = n, p = s_0 := \mathtt{Const_n}()$, $s_1 = s_0 + 1$, and $st = s_0$

  Since $\mathtt{Const_n}()$ takes no arguments, there is only one possibility for $\mathcal{P}$:

$$\mathcal{P} = \frac{\dfrac{\dfrac{}{\mathtt{Const_n}() \Downarrow \langle n \rangle} \quad \dfrac{}{\mathtt{Const_n}() \downarrow^{\langle\rangle} \langle\rangle}}{\mathtt{Const_n}() \downarrow^{\langle () \rangle} \langle n \rangle}}{\langle s_0 := \mathtt{Const_n}(), \sigma \rangle \downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle n \rangle]}$$

  So $\sigma' = \sigma[s_0 \mapsto \langle n \rangle]$.

  Then we have $\mathcal{V} = \dfrac{}{n \triangleright_{\mathbf{int}} \sigma'(s_0)}$.

  Also clearly, $\sigma' \overset{\lessdot s_0}{=\!=\!=} \sigma$, $s_0 \leq s_0 + 1$, $\mathtt{sids}(s_0) \lessdot s_0 + 1$, and we are done.

- Case $\phi = \mathbf{plus}$
  We must have
$$\mathcal{T} = \frac{}{\vdash\ \mathbf{plus} : (\mathbf{int}, \mathbf{int}) \to \mathbf{int}}$$
$$\mathcal{E} = \frac{}{\vdash\ \mathbf{plus}(n_1, n_2) \downarrow n_3}$$
  where $n_3 = n_2 + n_1$, and
$$\mathcal{C} = \frac{}{\vdash\ \mathbf{plus}(s_1, s_2) \xRightarrow[s_0+1]{s_0} (s_0 := \texttt{MapTwo}_+(s_1, s_2), s_0)}$$
  So $k = 2, \tau_1 = \tau_2 = \tau = \mathbf{int}, v_1 = n_1, v_2 = n_2, v = n_3, st_1 = s_1, st_2 = s_2, st = s_0, s_1 = s_0 + 1$ and $p = s_0 := \texttt{MapTwo}_+(s_1, s_2)$.

  Assumption (iv) gives us $n_1 \triangleright_{\mathbf{int}} \sigma(s_1)$ and $n_2 \triangleright_{\mathbf{int}} \sigma(s_2)$, which implies $\sigma(s_1) = \langle n_1 \rangle$ and $\sigma(s_2) = \langle n_2 \rangle$.

  For (v) we have $s_1 < s_0$ and $s_2 < s_0$.

  $\mathcal{P}$ must have the shape:
$$\dfrac{\dfrac{\begin{array}{c}\mathcal{P}_1\\ \texttt{MapTwo}_+(\vec{a}_1, \vec{a}_2) \Downarrow \vec{a}\end{array} \qquad \dfrac{}{\texttt{MapTwo}_+(\vec{a}_1', \vec{a}_2') \downarrow^{\langle\rangle} \langle\rangle}}{\texttt{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow^{\langle()\rangle} \vec{a}}\ (\vec{a}_1 {+\!\!+} \vec{a}_1' = \langle n_1 \rangle, \vec{a}_2 {+\!\!+} \vec{a}_2' = \langle n_2 \rangle)}{\langle s_0 := \texttt{MapTwo}_+(s_1, s_2), \sigma \rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \vec{a}]}\ (\sigma(s_1) = \langle n_1 \rangle, \sigma(s_2) = \langle n_2 \rangle)$$
  Since there is only one rule for $\mathcal{P}_1$, by which $\vec{a}_1$ must be $\langle n_1 \rangle$ and $\vec{a}_2$ must be $\langle n_2 \rangle$ (, and $\vec{a}_1' = \vec{a}_2' = \langle\rangle$), that is,
$$\mathcal{P}_1 = \frac{}{\texttt{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \Downarrow \langle n_3 \rangle}$$
  Therefore, $\sigma' = \sigma[s_0 \mapsto \langle n_3 \rangle]$.
  Now it is clear that $\mathcal{V} = \overline{n_3 \triangleright_{\mathbf{int}} \sigma'(s_0)}$, $\sigma' \overset{<s_0}{=\!=\!=} \sigma$, $s_0 \leq s_0 + 1$ and $\texttt{sids}(s_0) \lessdot s_0 + 1$ as required.

- Case $\phi = \mathbf{iota}$

$\square$

Theorem 0.2. ***If***

(i) $\Gamma\ \vdash\ e : \tau$ *(by some derivation $\mathcal{T}$)*

(ii) $\rho\ \vdash\ e \downarrow v$ *(by $\mathcal{E}$)*

(iii) $\delta\ \vdash\ e \xRightarrow[s_1]{s_0} (p, st)$ *(by $\mathcal{C}$)*

(iv) $\forall x \in dom(\Gamma).\ \vdash\ \rho(x) : \Gamma(x) \wedge \texttt{sids}(\delta(x)) \lessdot s_0 \wedge \rho(x) \triangleright_{\Gamma(x)} \sigma(\delta(x))$
then

(v) $\langle p, \sigma \rangle \downarrow^{\langle()\rangle} \sigma'$ *(by $\mathcal{P}$)*

(vi) $v \triangleright_\tau \sigma'(st)$ *(by $\mathcal{V}$)*

(vii) $\sigma' \overset{<s_0}{=\!=\!=} \sigma$

(viii) $s_0 \leq s_1$

(ix) $\texttt{sids}(st) \lessdot s_1$

Proof. By induction on the syntax of e.

- Case $e = \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\}$.

  We must have:

$$\text{(i)}\quad \mathcal{T} = \cfrac{\overset{\mathcal{T}_1}{[x \mapsto \tau_1] \vdash e_1 : \tau_2}}{\Gamma \vdash \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\} : \{\tau_2\}}\ (\Gamma(y) = \{\tau_1\})$$

$$\text{(ii)}\quad \mathcal{E} = \cfrac{\overset{\mathcal{E}_i}{([x \mapsto v_i] \vdash e_1 \downarrow v_i')_{i=1}^{k}}}{\rho \vdash \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\} \downarrow \{v_1', ..., v_k'\}}\ (\rho(y) = \{v_1, ..., v_k\})$$

$$\text{(iii)}\quad \mathcal{C} = \cfrac{\overset{\mathcal{C}_1}{[x \mapsto st_1] \vdash e_1 \xrightarrow[s_1]{s_0+1} (p_1, st_2)}}{\delta \vdash \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\} \xrightarrow[s_1]{s_0} (s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1), (st_2, s_2))}\ (\delta(y) = (st_1, s_2))$$

  So $p = (s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1)), \tau = \{\tau_2\}, v = \{v_1', ..., v_k'\}, st = (st_2, s_2)$.

  (iv) $\vdash \rho(y) : \Gamma(y)$ gives us $\vdash \{v_1, ..., v_k\} : \{\tau_1\}$, which must have the derivation:

$$\frac{(\vdash v_i : \tau_1)_{i=1}^{k}}{\vdash \{v_1, ..., v_k\} : \{\tau_1\}} \tag{1}$$

  $\texttt{sids}(\delta(y)) \lessdot s_0$ gives us

$$\texttt{sids}(\delta(y)) = \texttt{sids}((st_1, s_2)) = \texttt{sids}(st_1) \cup \{s_2\} \lessdot s_0 \tag{2}$$

  $\rho(y) \rhd_{\Gamma(y)} \sigma(\delta(y)) = \{v_1, ..., v_k\} \rhd_{\{\tau_1\}} \sigma((st_1, s_2))$ must have the derivation:

$$\cfrac{\overset{\mathcal{V}_i}{(v_i \rhd_{\tau_1} w_i)_{i=1}^{k}}}{\{v_1, ..., v_k\} \rhd_{\{\tau_1\}} (w, \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle)}\ (w = w_1 +\!\!+ ... +\!\!+ w_k) \tag{3}$$

  therefore

$$\sigma(st_1) = w \tag{4}$$

  and

$$\sigma(s_2) = \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle. \tag{5}$$

  First we shall show:

  (v) $\langle s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1), \sigma\rangle \downarrow^{\langle()\rangle} \sigma'$

  (vi) $\{v_1', ..., v_k'\} \rhd_{\{\tau_2\}} \sigma'((st_2, s_2))$ by $\mathcal{V}$

  (vii) $\sigma' \overset{<s_0}{=\!=} \sigma$

  TS (v), we first prove $\langle s_0 := \texttt{Usum}(s_2), \sigma\rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \langle()_1, ..., ()_k\rangle]$ by $\mathcal{P}_0$.
  $\mathcal{P}_0$ must have the shape:

$$\cfrac{\cfrac{\overset{\mathcal{P}_0'}{\texttt{Usum}(\vec{b}_1) \Downarrow \vec{a}} \qquad \cfrac{}{\texttt{Usum}(\vec{b}_2) \downarrow^{\langle\rangle} \langle\rangle}}{\texttt{Usum}(\langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle) \downarrow^{\langle()\rangle} \vec{a}}\ (\vec{b}_1 +\!\!+ \vec{b}_2 = \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle)}{\langle s_0 := \texttt{Usum}(s_2), \sigma\rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \vec{a}]}\ (\sigma(s_2) = \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle)$$

  From the rules P-UsumF and P-UsumT we know that $\vec{b}_1$ must end with (and include exactly) one $\texttt{T}$ so that $\mathcal{P}_0'$ can terminate. Therefore, in our case, $\vec{b}_1$ can only be $\langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle$, and $\vec{b}_2 = \langle\rangle$. Then using P-UsumF k times and P-UsumT once, we obtain $\mathcal{P}_0'$ of

$$\texttt{Usum}(\vec{b}_1) \Downarrow \langle()_1, ..., ()_k\rangle$$

which gives us $\mathcal{P}_0$ of $\langle s_0 := \texttt{Usum}(s_2), \sigma\rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \langle()_1, ..., ()_k\rangle]$.
Then $\mathcal{P}$ must have the shape:

$$\frac{\dfrac{\mathcal{P}_0}{\langle s_0 := \texttt{Usum}(s_2), \sigma\rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \langle()_1, ..., ()_k\rangle]} \qquad \dfrac{\mathcal{P}_1}{\langle st_2 := \texttt{WithCtrl}(s_0, p_1), \sigma[s_0 \mapsto \langle()_1, ..., ()_k\rangle]\rangle \downarrow^{\langle()\rangle} \sigma'}}{\langle s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1), \sigma\rangle \downarrow^{\langle()\rangle} \sigma'}$$

Since we have

$$\mathcal{T}_1 = \ [x \mapsto \tau_1] \ \vdash \ e_1 : \tau_2$$
$$(\mathcal{E}_i = \ [x \mapsto v_i] \ \vdash \ e_1 \downarrow v_i' \qquad)_{i=1}^k$$
$$\mathcal{C}_1 = \ [x \mapsto st_1] \ \vdash \ e_1 \xRightarrow[s_1]{s_0+1} (p_1, st_2)$$

Let $\Gamma_1 = [x \mapsto \tau_1], \rho_i = [x \mapsto v_i]$ and $\delta_1 = [x \mapsto st_1]$.
From (1) and (2) it is clear that

$$\forall z \in dom(\Gamma_1). \ \vdash \ \rho_i(z) : \Gamma_1(z) \wedge \texttt{sids}(\delta_1(z)) \lessdot s_0.$$

Let i range from 1 to k: we take $\sigma_i \overset{st_1}{\sim} \sigma[s_0 \mapsto \langle()_1, ..., ()_k\rangle]$ such that $\sigma_i(st_1) = w_i$.
From $\mathcal{V}_i$ in (3) we know that

$$\forall z \in dom(\Gamma_1). \rho_i(z) \rhd_{\Gamma_1(z)} \sigma_i(\delta_1(z)).$$

Then by IH (k times) on $\mathcal{T}_1$ with $\mathcal{E}_i$, $\mathcal{C}_1$ we obtain the following result:

$$(\langle p_1, \sigma_i\rangle \downarrow^{\langle()\rangle} \sigma_i')_{i=1}^k \tag{6}$$
$$(v_i' \rhd_{\tau_2} \sigma_i'(st_2))_{i=1}^k \tag{7}$$
$$(\sigma_i' \overset{<s_0+1}{=\!=\!=} \sigma_i)_{i=1}^k \tag{8}$$
$$s_0 + 1 \leq s_1 \tag{9}$$
$$\texttt{sids}(st_2) \lessdot s_1 \tag{10}$$

Assume $\texttt{sids}(st_2) = \{s_1', ..., s_j'\}$.

There are two possibilities:

- Subcase $k = 0$, that is $\sigma[s_0 \mapsto \langle()_1, ..., ()_k\rangle](s_0) = \langle\rangle$.
  Then

  $$\mathcal{P}_1 = \frac{}{\langle st_2 := \texttt{WithCtrl}(s_0, p_1), \sigma\rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \langle\rangle, s_1' \mapsto \langle\rangle, ..., s_j' \mapsto \langle\rangle]},$$

  thus in this subcase

  $$\sigma' = \sigma[s_0 \mapsto \langle\rangle, s_1' \mapsto \langle\rangle, ..., s_j' \mapsto \langle\rangle].$$

  Since $k = 0$, then $v = \{\}$, $\sigma(s_2) = \langle \texttt{T}\rangle$ (from (5)), we have
  $\sigma'(s_2) = \sigma(s_2) = \langle \texttt{T}\rangle$ (?? not correct if $s_2 \in \texttt{sids}(st_2)/\texttt{sids}(st_1)$),
  and $\sigma'(st_2) = (...((\langle\rangle, \langle\rangle)_1, \langle\rangle)_2, ...)_{j-1}$.

  Therefore $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2))$, with which we construct

  $$\mathcal{V} = \frac{}{\{\} \rhd_{\{\tau_2\}} ((...(\langle\rangle, \langle\rangle)_1, ...)_{j-1}, \langle \texttt{T}\rangle)}$$

  as required.

  Since $k = 0$, from (4) we know $\forall s' \in \texttt{sids}(st_1).\sigma(s') = \langle\rangle$. For any $s' \in \texttt{sids}(st_2)$ and $s' < s_0$, it must have $s' \in \texttt{sids}(st_1)$ (because $codom(\delta_1) = \{st_1\}$), hence $\sigma(s') = \langle\rangle = \sigma'(s')$. Therefore,

  $$\sigma' \overset{<s_0}{=\!=\!=} \sigma.$$

– Subcase $k > 0$, that is $\sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] = \langle () | \vec{a} \rangle$ for some $\vec{a}$.
Then $\mathcal{P}_1 =$

$$\frac{\begin{array}{c}\mathcal{P}_1' \\ \langle p_1, \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] \rangle \downarrow^{\langle ()_1, ..., ()_k \rangle} \sigma'' \end{array}}{\langle st_2 := \mathtt{WithCtrl}(s_0, p_1), \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] \rangle \downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle, s_1' \mapsto \sigma''(s_1'), ..., s_j' \mapsto \sigma''(s_j')]}$$

So in this subcase

$$\sigma' = \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle, s_1' \mapsto \sigma''(s_1'), ..., s_j' \mapsto \sigma''(s_j')].$$

Using Lemma 0.3 (k-1) times on (6) gives us

$$\langle p_1, (\overset{st_1}{\bowtie} \sigma_i)_{i=1}^k \rangle \downarrow^{\langle ()_1, ..., ()_k \rangle} (\overset{st_2}{\bowtie} \sigma_i')_{i=1}^k \tag{11}$$

By Definition 0.2 we have

$$(\overset{st_1}{\bowtie} \sigma_i)_{i=1}^k = \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle]. \tag{12}$$

Then by Theorem 0.1 on $\mathcal{P}_1'$ with (11), we get

$$\sigma'' = (\overset{st_2}{\bowtie} \sigma_i')_{i=1}^k \tag{13}$$

Therefore, $\sigma''(st_2) = \sigma_1'(st_2) {+\!\!+} ... {+\!\!+} \sigma_k'(st_2)$ by Definition 0.2.
Let $\sigma_i'(st_2) = w_i'$ and $\sigma''(st_2) = w'$, then $w' = w_1' {+\!\!+} ... {+\!\!+} w_k'$.

Since $\sigma'(st_2) = \sigma''(st_2) = w'$, and $\sigma'(s_2) = \sigma(s_2) = \langle \mathtt{F}_1, ..., \mathtt{F}_k, \mathtt{T} \rangle$,(same problem) we now have $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2)) = (w', \langle \mathtt{F}_1, ..., \mathtt{F}_k, \mathtt{T} \rangle)$. With (7), we can construct

$$\mathcal{V} = \frac{(v_i' \triangleright_{\tau_2} w_i')_{i=1}^k}{\{v_1', ..., v_k'\} \triangleright_{\{\tau_2\}} (w', \langle \mathtt{F}_1, ..., \mathtt{F}_k, \mathtt{T} \rangle)}$$

as required.

By Lemma 0.1 on (12) we get $\sigma_i \overset{st_1}{\sim} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle]$, and similarly $\sigma_i' \overset{st_2}{\sim} \sigma''$ from (13).

Since (8) implies

$$(\sigma_i' \overset{<s_0}{=\!=} \sigma_i)_{i=1}^k$$

using Lemma 0.2 (k-1) times, we obtain

$$\sigma'' \overset{<s_0}{=\!=} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle].$$

Therefore, $\sigma' \overset{<s_0}{=\!=} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] \overset{<s_0}{=\!=} \sigma$.

(viii) TS: $s_0 \leq s_1$
From (9) we immediately get $s_0 \leq s_1 - 1 < s_1$.

(ix) TS: $\mathtt{sids}((st_2, s_2)) \lessdot s_1$
From (2) we know $s_2 < s_0$, thus $s_2 < s_0 \leq s_1$. And we already have (10). Therefore,

$$\mathtt{sids}((st_2, s_2)) = \mathtt{sids}(st_2) \cup \{s_2\} \lessdot s_1.$$

- Case $e = x$.
We must have

$$\mathcal{T} = \frac{}{\Gamma \ \vdash \ x : \tau} \ (\Gamma(x) = \tau)$$
$$\mathcal{E} = \frac{}{\rho \ \vdash \ x \downarrow v} \ (\rho(x) = v)$$

$$\mathcal{C} = \frac{}{\delta \ \vdash \ x \overset{s_0}{\underset{s_0}{\Longrightarrow}} (\epsilon, st)} \ (\delta(x) = st)$$

So $p = \epsilon$.

Immediately we have $\mathcal{P} = \frac{}{\langle \epsilon, \sigma \rangle \downarrow^{\langle () \rangle} \sigma}$

So $\sigma' = \sigma$, which implies $\sigma \overset{< s_0}{=\!=\!=} \sigma$.

From the assumption we already have $v \triangleright_\tau \sigma(st)$, and $\mathtt{sids}(st) \lessdot s_0$.

Finally it's clear that $s_0 \leq s_0$, and we are done.

- Case $e = \textbf{let } x = e_1 \textbf{ in } e_2$.

  We must have:

$$\mathcal{T} = \frac{\overset{\mathcal{T}_1}{\Gamma \ \vdash \ e_1 : \tau_1} \qquad \overset{\mathcal{T}_2}{\Gamma[x \mapsto \tau_1] \ \vdash \ e_2 : \tau}}{\Gamma \ \vdash \ \textbf{let } x = e_1 \textbf{ in } e_2 : \tau}$$

$$\mathcal{E} = \frac{\overset{\mathcal{E}_1}{\rho \ \vdash \ e_1 \downarrow v_1} \qquad \overset{\mathcal{E}_2}{\rho[x \mapsto v_1] \ \vdash \ e_2 \downarrow v}}{\rho \ \vdash \ \textbf{let } x = e_1 \textbf{ in } e_2 \downarrow v}$$

$$\mathcal{C} = \frac{\overset{\mathcal{C}_1}{\delta \ \vdash \ e_1 \overset{s_0}{\underset{s_0'}{\Longrightarrow}} (p_1, st_1)} \qquad \overset{\mathcal{C}_2}{\delta[x \mapsto st_1] \ \vdash \ e_2 \overset{s_0'}{\underset{s_1}{\Longrightarrow}} (p_2, st)}}{\delta \ \vdash \ \textbf{let } x = e_1 \textbf{ in } e_2 \overset{s_0}{\underset{s_1}{\Longrightarrow}} (p_1; p_2, st)}$$

  So $p = p_1; p_2$.

  By IH on $\mathcal{T}_1$ with $\mathcal{E}_1, \mathcal{C}_1$, we get

  (a) $\mathcal{P}_1$ of $\langle p_1, \sigma \rangle \downarrow^{\langle () \rangle} \sigma_1$

  (b) $\mathcal{V}_1$ of $v_1 \triangleright_{\tau_1} \sigma_1(st_1)$

  (c) $\sigma_1 \overset{< s_0}{=\!=\!=} \sigma$

  (d) $s_0 \leq s_0'$

  (e) $\mathtt{sids}(st_1) \lessdot s_0'$

  From (b), we know $\rho[x \mapsto v_1](x) : \Gamma[x \mapsto \tau_1](x)$ and $\rho[x \mapsto v_1](x) \triangleright_{\Gamma[x \mapsto \tau_1](x)} \sigma_1(\delta[x \mapsto st_1](x))$ must hold. From (e), we have $\mathtt{sids}(\delta[x \mapsto st_1](x)) \lessdot s_0'$.

  Then by IH on $\mathcal{T}_2$ with $\mathcal{E}_2, \mathcal{C}_2$, we get

  (f) $\mathcal{P}_2$ of $\langle p_2, \sigma_1 \rangle \downarrow^{\langle () \rangle} \sigma_2$

  (g) $\mathcal{V}_2$ of $\sigma_2 \triangleright_\tau \sigma_2(st)$

  (h) $\sigma_2 \overset{< s_0'}{=\!=\!=} \sigma_1$

  (i) $s_0' \leq s_1$

  (j) $\mathtt{sids}(st) \lessdot s_1$

  So we can construct:

$$\mathcal{P} = \frac{\overset{\mathcal{P}_1}{\langle p_1, \sigma \rangle \downarrow^{\langle () \rangle} \sigma_1} \qquad \overset{\mathcal{P}_2}{\langle p_2, \sigma_1 \rangle \downarrow^{\langle () \rangle} \sigma_2}}{\langle p_1; p_2, \sigma \rangle \downarrow^{\langle () \rangle} \sigma_2}$$

  From (c), (d) and (h), it is clear that $\sigma_2 \overset{< s_0}{=\!=\!=} \sigma_1 \overset{< s_0}{=\!=\!=} \sigma$. From (d) and (i), $s_0 \leq s_1$.

  Take $\sigma' = \sigma_2$ (thus $\mathcal{V} = \mathcal{V}_2$) and we are done.

- Case $e = \phi(x_1, ..., x_k)$
  We must have

$$\mathcal{T} = \cfrac{\begin{array}{c}\mathcal{T}_1 \\ \vdash \ \phi : (\tau_1, ..., \tau_k) \to \tau\end{array}}{\Gamma \ \vdash \ \phi(x_1, ..., x_k) : \tau} \ ((\Gamma(x_i) = \tau_i)_{i=1}^k)$$

$$\mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1 \\ \vdash \ \phi(v_1, ..., v_k) \downarrow v\end{array}}{\rho \ \vdash \ \phi(x_1, ..., x_k) \downarrow v} \ ((\rho(x_i) = v_i)_{i=1}^k)$$

$$\mathcal{C} = \cfrac{\begin{array}{c}\mathcal{C}_1 \\ \vdash \ \phi(st_1, ..., st_k) \underset{s_1}{\overset{s_0}{\Longrightarrow}} (p, st)\end{array}}{\delta \ \vdash \ \phi(x_1, ..., x_k) \underset{s_1}{\overset{s_0}{\Longrightarrow}} (p, st)} \ ((\delta(x_i) = st_i)_{i=1}^k)$$

  From our assumption (iv), for all $i \in \{1, ..., k\}$:

  (a) $\vdash \ \rho(x_i) : \Gamma(x_i)$, that is, $\vdash \ v_i : \tau_i$

  (b) $\texttt{sids}(\delta(x_i)) \lessdot s_0$, that is, $\texttt{sids}(st_i) \lessdot s_0$

  (c) $\rho(x_i) \rhd_{\Gamma(x_i)} \sigma(st_i)$, that is, $v_i \rhd_{\tau_i} \sigma(st_i)$

  So using Lemma 0.7 on $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1, (a), (b)$ and (c) gives us exactly what we shall show.

$\square$