Toy language formalization

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0 Level-0

0.1 Source language:

$$e ::= n \mid e_1 + e_2$$
$$(n \in \mathbf{Z}\ )$$

0.2 Source language semantics:

Judgment $\boxed{e \downarrow n}$

$$\text{E-Cons} : \frac{}{n \downarrow n} \qquad \text{E-Plus} : \frac{e_1 \downarrow n_1 \quad e_2 \downarrow n_2}{e_1 + e_2 \downarrow n_3}\ (n_1 + n_2 = n_3)$$

0.3 Target language:

$$r \in \mathbf{N}\ = \{0, 1, 2, ...\}$$
$$s ::= \mathbf{mov}\ r\ n \mid \mathbf{add}\ r_1\ r_2\ r_3$$
$$p ::= s \mid p_1; p_2$$

0.4 Target language semantics:

Environment $\sigma = [r_1 \mapsto n_1, ..., r_i \mapsto n_i]$.

Judgment $\boxed{\langle p, \sigma \rangle \downarrow \sigma'}$

.

$$\text{P-Mov} : \frac{}{\langle \mathbf{mov}\ r\ n, \sigma \rangle \downarrow \sigma[r \mapsto n]}$$
$$\text{P-Add} : \frac{}{\langle \mathbf{add}\ r_1\ r_2\ r_3, \sigma \rangle \downarrow \sigma[r_1 \mapsto n_1]}\ (\sigma(r_2) = n_2, \sigma(r_3) = n_3, n_2 + n_3 = n_1)$$
$$\text{P-Seq} : \frac{\langle p_1, \sigma \rangle \downarrow \sigma_1 \quad \langle p_2, \sigma_1 \rangle \downarrow \sigma_2}{\langle p_1; p_2, \sigma \rangle \downarrow \sigma_2}$$

0.5 Translation:

Judgment $\boxed{e \Rightarrow_{r_1}^{r_0} \texttt{let}\ p\ \texttt{in}\ r}$

Newly generated register identifiers start from r_0, end at (but not include) r_1.

$$\text{C-Cons} : \frac{}{n \Rightarrow_{r_0+1}^{r_0} \texttt{let}\ \mathbf{mov}\ r_0\ n\ \texttt{in}\ r_0}$$

$$\text{C-Plus} : \frac{e_1 \Rightarrow_{r_1'}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ r_1 \quad e_2 \Rightarrow_{r_2'}^{r_1'} \texttt{let}\ p_2\ \texttt{in}\ r_2}{e_1 + e_2 \Rightarrow_{r_2'+1}^{r_0} \texttt{let}\ p_1; (p_2; \mathbf{add}\ r_2'\ r_1\ r_2)\ \texttt{in}\ r_2' + 1}$$

0.6 Correctness theorem:

Lemma 1. *If* $e \Rightarrow^{r_0}_{r_1}$ $\texttt{let}$ p $\texttt{in}$ r*, then* $r_0 \le r_1$ *and* $r < r_1$*.*

Theorem 2. *If* $e \downarrow n$ *(by some derivation* $\mathcal{E}$*),* $e \Rightarrow^{r_0}_{r_1}$ $\texttt{let}$ p $\texttt{in}$ r *(by* $\mathcal{C}$*), then* $\langle p, \sigma \rangle \downarrow \sigma'$ *(by* $\mathcal{P}$*),* $\forall r' < r_0 . \sigma'(r') = \sigma(r')$*, and* $\sigma'(r) = n$ *.*

Proof. By induction on the syntax of e:

- Case $e = n_0$, then $n = n_0$, by E-Cons: $\mathcal{E} = \overline{n_0 \downarrow n_0}$, by C-Cons : $\mathcal{C} = \overline{n_0 \Rightarrow^{r_0}_{r_0+1} \texttt{let}\ \mathbf{mov}\ r_0\ n_0\ \texttt{in}\ r_0}$, so $p = \mathbf{mov}\ r_0\ n_0$, $r = r_0$.

  Then by P-Mov, we get $\mathcal{P} = \overline{\langle \mathbf{mov}\ r_0\ n_0, \sigma \rangle \downarrow \sigma[r_0 \mapsto n_0]}$.
  Therefore we have $\forall r' < r_0 . \sigma[r_0 \mapsto n_0](r') = \sigma(r')$, and $\sigma[r_0 \mapsto n_0](r_0) = n_0$ as required.

- Case $e = e_1 + e_2$.

  By E-Plus, $\mathcal{E}$ must have the shape: $\dfrac{\overset{\mathcal{E}_1}{e_1 \downarrow n_1} \qquad \overset{\mathcal{E}_2}{e_2 \downarrow n_2}}{e_1 + e_2 \downarrow n_1 + n_2}$, thus $n = n_1 + n_2$.

  By C-Plus, $\mathcal{C}$ must have the shape: $\dfrac{\overset{\mathcal{C}_1}{e_1 \Rightarrow^{r_0}_{r_1'} \texttt{let}\ p_1\ \texttt{in}\ r_1} \qquad \overset{\mathcal{C}_2}{e_2 \Rightarrow^{r_1'}_{r_2'} \texttt{let}\ p_2\ \texttt{in}\ r_2}}{e_1 + e_2 \Rightarrow^{r_0}_{r_2'+1} \texttt{let}\ p_1; p_2; \mathbf{add}\ r_2'\ r_1\ r_2\ \texttt{in}\ r_2'}$

  So $p = p_1; p_2; \mathbf{add}\ r_2'\ r_1\ r_2$, and $r = r_2'$.

  By IH on $\mathcal{E}_1$ with $\mathcal{C}_1$, we get $\mathcal{P}_1 = \langle p_1, \sigma \rangle \downarrow \sigma_1$ for some σ_1, $\forall r' < r_0 . \sigma_1(r') = \sigma(r')$, and $\sigma_1(r_1) = n_1$.

  Likewise, by IH on $\mathcal{E}_2$ with $\mathcal{C}_2$, we get $\mathcal{P}_2 = \langle p_2, \sigma_1 \rangle \downarrow \sigma_2$ for some σ_2, $\forall r'' < r_1' . \sigma_2(r'') = \sigma_1(r'')$, and $\sigma_2(r_2) = n_2$.

  By Lemma 1 on $\mathcal{C}_1$, $r_0 \le r_1'$, and $r_1 < r_1'$. Since $r_0 \le r_1'$, we get $\forall r''' < r_0 . \sigma_2(r''') = \sigma_1(r''') = \sigma(r''')$; since $r_1 < r_1'$, we get $\sigma_2(r_1) = \sigma_1(r_1) = n_1$.

  Use P-Seq and P-Add, we construct:

  $$\dfrac{\overset{\mathcal{P}_1}{\langle p_1, \sigma \rangle \downarrow \sigma_1} \qquad \dfrac{\overset{\mathcal{P}_2}{\langle p_2, \sigma_1 \rangle \downarrow \sigma_2} \qquad \dfrac{}{\langle \mathbf{add}\ r_2'\ r_1\ r_2, \sigma_2 \rangle \downarrow \sigma_2[r_2' \mapsto n_1 + n_2]}\ (\sigma_2(r_2) = n_2, \sigma_2(r_1) = n_1)}{\langle p_2; \mathbf{add}\ r_2'\ r_1\ r_2, \sigma_1 \rangle \downarrow \sigma_2[r_2' \mapsto n_1 + n_2]}}{\langle p_1; (p_2; \mathbf{add}\ r_2'\ r_1\ r_2), \sigma \rangle \downarrow \sigma_2[r_2' \mapsto n_1 + n_2]}$$

  Therefore, $\sigma_2[r_2' \mapsto n_1 + n_2](r_2') = n_1 + n_2 = n$. Take $\sigma' = \sigma_2$ and we are done.

$\square$

1 Level-1

1.1 Extended source language

$$e ::= ...\ |\ x\ |\ \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2$$

1.2 Extended semantics:

High-level runtime environment $\rho = [x_1 \mapsto n_1, ..., x_i \mapsto n_i]$.

Judgment $\boxed{\rho \vdash e \downarrow n}$

$$\dfrac{}{\rho \vdash x \downarrow n}\ (\rho(x) = n) \qquad \dfrac{\rho \vdash e_1 \downarrow n_1 \quad \rho[x \mapsto n_1] \vdash e_2 \downarrow n}{\rho \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 \downarrow n}$$

1.3 Target language:

(added ϵ to p)

$$r \in \mathbf{N} \;= \{0, 1, 2, ...\}$$

$$s ::= \mathbf{mov}\ r\ n \mid \mathbf{add}\ r_1\ r_2\ r_3$$

$$p ::= \epsilon \mid s \mid p_1; p_2$$

1.4 Extended target language semantics:

Judgment $\boxed{\langle p, \sigma\rangle \downarrow \sigma'}$

$$\frac{}{\langle \epsilon, \sigma\rangle \downarrow \sigma}$$

1.5 Extended translation:

Translation environment $\delta = [x_1 \mapsto r_1, ..., x_i \mapsto r_i]$.

Judgment $\boxed{\delta \ \vdash\ e \Rightarrow^{r_0}_{r_1} \texttt{let}\ p\ \texttt{in}\ r}$

$$\frac{}{\delta \ \vdash\ x \Rightarrow^{r_0}_{r_0} \texttt{let}\ \epsilon\ \texttt{in}\ r}\ (\delta(x) = r)$$

$$\frac{\delta \ \vdash\ e_1 \Rightarrow^{r_0}_{r_1'} \texttt{let}\ p_1\ \texttt{in}\ r_1 \quad \delta[x \mapsto r_1] \ \vdash\ e_2 \Rightarrow^{r_1'}_{r_2'} \texttt{let}\ p_2\ \texttt{in}\ r_2}{\delta \ \vdash\ \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 \Rightarrow^{r_0}_{r_2'} \texttt{let}\ p_1; p_2\ \texttt{in}\ r_2}$$

1.6 Correctness theorem:

Notation:
We use $\sigma_1 \overset{<r}{=\!=} \sigma_2$ to denote: $\forall r' < r.\sigma_1(r') = \sigma_2(r')$. It is easy to prove that this relation has the following properties:

- (Transitivity) if $\sigma_1 \overset{<r}{=\!=} \sigma_2$, and $\sigma_2 \overset{<r}{=\!=} \sigma_3$, then $\sigma_1 \overset{<r}{=\!=} \sigma_3$.
- if $\sigma_1 \overset{<r}{=\!=} \sigma_2$, $r' < r$, then $\sigma_1 \overset{<r'}{=\!=} \sigma_2$

Lemma 3. *If* $\delta \ \vdash\ e \Rightarrow^{r_0}_{r_1} \texttt{let}\ p\ \texttt{in}\ r$, *then* $r_0 \leq r_1$.

Theorem 4. *If* $\rho \ \vdash\ e \downarrow n$, $\delta \ \vdash\ e \Rightarrow^{r_0}_{r'} \texttt{let}\ p\ \texttt{in}\ r$, *and* $\forall x \in dom(\rho).\delta(x) < r_0 \wedge \rho(x) = \sigma(\delta(x))$, *then* $\langle p, \sigma\rangle \downarrow \sigma'$, $\sigma'(r) = n$, $\sigma' \overset{<r_0}{=\!=} \sigma$, *and* $r < r'$.

2 Level-2

2.1 Extended source language:

$$e ::= ... \mid (e_1, e_2) \mid \mathbf{fst}(e) \mid \mathbf{snd}(e)$$

2.2 Added values:

$$v ::= n \mid (v_1, v_2)$$

2.3 Added source language type system:

$$\tau ::= \textbf{Int} \mid (\tau_1, \tau_2)$$

Type environment $\Gamma = [x_1 \mapsto \tau_1, ..., x_i \mapsto \tau_i]$.

- Judgment $\boxed{\Gamma \vdash e : \tau}$

$$\frac{}{\Gamma \vdash n : \textbf{Int}} \qquad \frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 + e_2 : \tau}$$

$$\frac{}{\Gamma \vdash x : \tau}\ (\Gamma(x) = \tau) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau}{\Gamma \vdash \textbf{let}\ x = e_1\ \textbf{in}\ e_2 : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : (\tau_1, \tau_2)} \qquad \frac{\Gamma \vdash e : (\tau_1, \tau_2)}{\Gamma \vdash \textbf{fst}(e) : \tau_1} \qquad \frac{\Gamma \vdash e : (\tau_1, \tau_2)}{\Gamma \vdash \textbf{snd}(e) : \tau_2}$$

- Judgment $\boxed{\vdash v : \tau}$

$$\frac{}{\vdash n : \textbf{Int}} \qquad \frac{\vdash v_1 : \tau_1 \quad \vdash v_2 : \tau_2}{\vdash (v_1, v_2) : (\tau_1, \tau_2)}$$

- Auxiliary Judgment $\boxed{\vdash \textbf{gplus}\ (v_1, v_2) : \tau}$

  (general plus operation typing rules)

$$\frac{}{\vdash \textbf{gplus}\ (n_1, n_2) : \textbf{Int}} \qquad \frac{\vdash \textbf{gplus}\ (v_{10}, v_{20}) : \tau_1 \quad \vdash \textbf{gplus}\ (v_{11}, v_{21}) : \tau_2}{\vdash \textbf{gplus}\ ((v_{10}, v_{11}), (v_{20}, v_{21})) : (\tau_1, \tau_2)}$$

2.4 Extended semantics:

Judgment $\boxed{\rho \vdash e \downarrow v}$

(fixed runtime environment $\rho = [x_1 \mapsto v_1, ..., x_i \mapsto v_i]$)

$$\frac{}{\rho \vdash n \downarrow n} \qquad \frac{\rho \vdash e_1 \downarrow v_1 \quad \rho \vdash e_2 \downarrow v_2 \quad \textbf{gplus}(v_1, v_2) \downarrow v_3}{\rho \vdash e_1 + e_2 \downarrow v_3}$$

$$\frac{}{\rho \vdash x \downarrow v}\ (\rho(x) = v) \qquad \frac{\rho \vdash e_1 \downarrow v_1 \quad \rho[x \mapsto v_1] \vdash e_2 \downarrow v}{\rho \vdash \textbf{let}\ x = e_1\ \textbf{in}\ e_2 \downarrow v}$$

$$\frac{\rho \vdash e_1 \downarrow v_1 \quad \rho \vdash e_2 \downarrow v_2}{\rho \vdash (e_1, e_2) \downarrow (v_1, v_2)} \qquad \frac{\rho \vdash e \downarrow (v_1, v_2)}{\rho \vdash \textbf{fst}(e) \downarrow v_1} \qquad \frac{\rho \vdash e \downarrow (v_1, v_2)}{\rho \vdash \textbf{snd}(e) \downarrow v_2}$$

Auxiliary Judgment $\boxed{\textbf{gplus}(v_1, v_2) \downarrow v_3}$

$$\frac{}{\textbf{gplus}(n_1, n_2) \downarrow n_3}\ (n_1 + n_2 = n_3) \qquad \frac{\textbf{gplus}(v_{10}, v_{20}) \downarrow v_{30} \quad \textbf{gplus}(v_{11}, v_{21}) \downarrow v_{31}}{\textbf{gplus}((v_{10}, v_{11}), (v_{20}, v_{21})) \downarrow (v_{30}, v_{31})}$$

2.5 Target language:

$$rs ::= r \mid (rs_1, rs_2)$$

s, p and semantics no change.

Define a function $rset$ to convert rs to a set of r:
$rset(r) = \{r\}$
$rset((rs_1, rs_2)) = rset(rs_1) \cup rset(rs_2)$

2.6 Extended translation:

Judgment $\boxed{\delta \ \vdash \ e \Rightarrow_{r_1}^{r_0} \texttt{let}\ p\ \texttt{in}\ rs}$

(fixed $\delta = [x_1 \mapsto rs_1, ..., x_i \mapsto rs_i]$)

$$\frac{\delta \ \vdash \ e_1 \Rightarrow_{r_1}^{r} \texttt{let}\ p_1\ \texttt{in}\ rs_1 \quad \delta \ \vdash \ e_2 \Rightarrow_{r_2}^{r_1} \texttt{let}\ p_2\ \texttt{in}\ rs_2 \quad \textbf{transPlus}(rs_1, rs_2) \Rightarrow_{r_3}^{r_2} \texttt{let}\ p_3\ \texttt{in}\ rs_3}{\delta \ \vdash \ e_1 + e_2 \Rightarrow_{r_3}^{r} \texttt{let}\ p1; (p2; p3)\ \texttt{in}\ rs3}$$

$$\frac{\delta \ \vdash \ e_1 \Rightarrow_{r_1}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ rs_1 \quad \delta[x \mapsto rs_1] \ \vdash \ e_2 \Rightarrow_{r_2}^{r_1} \texttt{let}\ p_2\ \texttt{in}\ rs_2}{\delta \ \vdash \ \textbf{let}\ x = e_1\ \textbf{in}\ e_2 \Rightarrow_{r_2}^{r_0} \texttt{let}\ p_1; p_2\ \texttt{in}\ rs_2}$$

$$\frac{\delta \ \vdash \ e_1 \Rightarrow_{r_1}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ rs_1 \quad \delta \ \vdash \ e_2 \Rightarrow_{r_2}^{r_1} \texttt{let}\ p_2\ \texttt{in}\ rs_2}{\delta \ \vdash \ (e_1, e_2) \Rightarrow_{r_2}^{r_0} \texttt{let}\ p_1; p_2\ \texttt{in}\ (rs_1, rs_2)}$$

$$\frac{\delta \ \vdash \ e \Rightarrow_{r_1}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ (rs_1, rs_2)}{\delta \ \vdash \ \textbf{fst}(e) \Rightarrow_{r_1}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ rs_1} \qquad \frac{\delta \ \vdash \ e \Rightarrow_{r_1}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ (rs_1, rs_2)}{\delta \ \vdash \ \textbf{snd}(e) \Rightarrow_{r_1}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ rs_2}$$

Auxiliary Judgment $\boxed{\textbf{transPlus}(rs_1, rs_2) \Rightarrow_{r_1}^{r_0} \texttt{let}\ p\ \texttt{in}\ rs_3}$

$$\frac{}{\textbf{transPlus}(r_1, r_2) \Rightarrow_{r_3+1}^{r_3} \texttt{let}\ \textbf{add}\ r_3\ r_1\ r_2\ \texttt{in}\ r_3}$$

$$\frac{\textbf{transPlus}(rs_{10}, rs_{20}) \Rightarrow_{r_1}^{r_0} \texttt{let}\ p_1\ \texttt{in}\ rs_{30} \quad \textbf{transPlus}(rs_{11}, rs_{21}) \Rightarrow_{r_2}^{r_1} \texttt{let}\ p_2\ \texttt{in}\ rs_{31}}{\textbf{transPlus}((rs_{10}, rs_{11}), (rs_{20}, rs_{21})) \Rightarrow_{r_2}^{r_0} \texttt{let}\ p_1; p_2\ \texttt{in}\ (rs_{30}, rs_{31})}$$

2.7 Value representation:

Judgment $\boxed{\sigma \ \vdash \ v \rhd_\tau rs}$

($v : \tau$ can be represented as rs in σ)

$$\frac{}{\sigma \ \vdash \ n \rhd_{\textbf{Int}} r}\ (\sigma(r) = n) \qquad \frac{\sigma \ \vdash \ v_1 \rhd_{\tau_1} rs_1 \quad \sigma \ \vdash \ v_2 \rhd_{\tau_2} rs_2}{\sigma \ \vdash \ (v_1, v_2) \rhd_{(\tau_1, \tau_2)} (rs_1, rs_2)}$$

2.8 Correctness theorem:

Notation:
For some set s and some r, we use $s \lessdot r$ to denote : $\forall r' \in s.r' < r$. It is easy to show that this relation has the following properties:

- $s_1 \lessdot r, s_2 \lessdot r \Leftrightarrow s_1 \cup s_2 \lessdot r$.
- if $s \lessdot r$, $r < r'$, then $s \lessdot r'$

Lemma 5. *If* $\textbf{transPlus}(rs_1, rs_2) \Rightarrow_{r_1}^{r_0} \texttt{let}\ p\ \texttt{in}\ rs_3$, *then* $r_0 \leq r_1$.

Lemma 6. *If* $\delta \ \vdash \ e \Rightarrow_{r_1}^{r_0} \texttt{let}\ p\ \texttt{in}\ rs$, *then* $r_0 \leq r_1$.

Lemma 7. *If*

(i) $\sigma \vdash v_1 \rhd_\tau rs_1$, *and* $\sigma \vdash v_2 \rhd_\tau rs_2$

(ii) $\mathbf{gplus}(v_1, v_2) \downarrow v_3$

(iii) $\mathbf{transPlus}(rs_1, rs_2) \Rightarrow_{r_1}^{r_0} \texttt{let } p \texttt{ in } rs_3$

(iv) $rset((rs_1, rs_2)) \lessdot r_0$

then

(i) $\langle p, \sigma \rangle \downarrow \sigma'$

(ii) $\sigma' \vdash v_3 \rhd_\tau rs_3$

(iii) $\sigma' \xlongequal{\lessdot r_0} \sigma$

(iv) $rset(rs_3) \lessdot r_1$.

Theorem 8. *If*

(i) $\Gamma \vdash e : \tau$ *(by some derivation* $\mathcal{T}$*)*

(ii) $\rho \vdash e \downarrow v$ *(by* $\mathcal{E}$*)*

(iii) $\delta \vdash e \Rightarrow_{r}^{r_0} \texttt{let } p \texttt{ in } rs$ *(by* $\mathcal{C}$*)*

(iv) $\forall x \in dom(\Gamma).\rho(x) : \Gamma(x) \wedge rset(\delta(x)) \lessdot r_0 \wedge \sigma \vdash \rho(x) \rhd_{\Gamma(x)} \delta(x)$

then

(i) $\langle p, \sigma \rangle \downarrow \sigma'$ *(by* $\mathcal{P}$*)*

(ii) $\sigma' \vdash v \rhd_\tau rs$ *(by* $\mathcal{V}$*)*

(iii) $\sigma' \xlongequal{\lessdot r_0} \sigma$

(iv) $rset(rs) \lessdot r$

Proof. By induction on the syntax of e.

- Case $e = e_1 + e_2$.

  Then must have: $$\mathcal{T} = \dfrac{\overset{\mathcal{T}_1}{\Gamma \vdash e_1 : \tau} \qquad \overset{\mathcal{T}_2}{\Gamma \vdash e_2 : \tau}}{\Gamma \vdash e_1 + e_2 : \tau}$$

  $$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\rho \vdash e_1 \downarrow v_1} \qquad \overset{\mathcal{E}_2}{\rho \vdash e_2 \downarrow v_2} \qquad \overset{\mathcal{E}_3}{\mathbf{gplus}(v_1, v_2) \downarrow v}}{\rho \vdash e_1 + e_2 \downarrow v}$$

  $$\mathcal{C} = \dfrac{\overset{\mathcal{C}_1}{\delta \vdash e_1 \Rightarrow_{r_1}^{r_0} \texttt{let } p_1 \texttt{ in } rs_1} \qquad \overset{\mathcal{C}_2}{\delta \vdash e_2 \Rightarrow_{r_2}^{r_1} \texttt{let } p_2 \texttt{ in } rs_2} \qquad \overset{\mathcal{C}_3}{\mathbf{transPlus}(rs_1, rs_2) \Rightarrow_{r_3}^{r_2} \texttt{let } p_3 \texttt{ in } rs_3}}{\delta \vdash e_1 + e_2 \Rightarrow_{r_3}^{r_0} \texttt{let } p1; (p2; p3) \texttt{ in } rs_3}$$

  So $p = p1; p2; p3, rs = rs_3, r = r_3$.
  By IH on $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1$, we get $\mathcal{P}_1$ of $\langle p_1, \sigma \rangle \downarrow \sigma_1$, $\mathcal{V}_1$ of $\sigma_1 \vdash v_1 \rhd_\tau rs_1$, $\sigma_1 \xlongequal{\lessdot r_0} \sigma$, and $rset(rs_1) \lessdot r_1$.
  Likewise, by IH on $\mathcal{T}_2, \mathcal{E}_2, \mathcal{C}_2$, we get $\mathcal{P}_2$ of $\langle p_2, \sigma_1 \rangle \downarrow \sigma_2$, $\mathcal{V}_2$ of $\sigma_2 \vdash v_2 \rhd_\tau rs_2$, $\sigma_2 \xlongequal{\lessdot r_1} \sigma_1$, and $rset(rs_2) \lessdot r_2$.

  By lemma 6 on $\mathcal{C}_2$, we get $r_1 \leq r_2$, hence $rset(rs_1) \lessdot r_2$. Therefore, $rset(rs_1) \cup rset(rs_2) = rset((rs_1, rs_2)) = \lessdot r_2$.

Now by Lemma 7 on $\mathcal{V}_1, \mathcal{V}_2, \mathcal{E}_3, \mathcal{C}_3$, we get $\mathcal{P}_3$ of $\langle p_3, \sigma_2\rangle \downarrow \sigma_3$, $\sigma_3 \vdash v \rhd_\tau rs_3$, $\sigma_3 \xlongequal{<r_2} \sigma_2$, and $rset(rs_3) \lessdot r_3$.

Then we can construct:

$$\cfrac{\overset{\mathcal{P}_1}{\langle p_1, \sigma\rangle \downarrow \sigma_1} \qquad \cfrac{\overset{\mathcal{P}_2}{\langle p_2, \sigma_1\rangle \downarrow \sigma_2} \qquad \overset{\mathcal{P}_3}{\langle p_3, \sigma_2\rangle \downarrow \sigma_3}}{\langle p_2; p_3, \sigma_1\rangle \downarrow \sigma_3}}{\langle p_1; (p_2; p_3), \sigma\rangle \downarrow \sigma_3}$$

By lemma 6 on $\mathcal{C}_1$, we get $r_0 \le r_1$. Therefore, $r_0 \le r_1 \le r_2$. $\sigma_3 \xlongequal{<r_0} \sigma_2 \xlongequal{<r_0} \sigma_1 \xlongequal{<r_0} \sigma$.

Take $\sigma' = \sigma_3$ and we are done.

- Case $e = (e_1, e_2)$.

Must have: $\mathcal{T} = \cfrac{\overset{\mathcal{T}_1}{\Gamma \vdash e_1 : \tau_1} \qquad \overset{\mathcal{T}_2}{\Gamma \vdash e_2 : \tau_2}}{\Gamma \vdash (e_1, e_2) : (\tau_1, \tau_2)}$

$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\rho \vdash e_1 \downarrow v_1} \qquad \overset{\mathcal{E}_2}{\rho \vdash e_2 \downarrow v_2}}{\rho \vdash (e_1, e_2) \downarrow (v_1, v_2)}$

$\mathcal{C} = \cfrac{\overset{\mathcal{C}_1}{\delta \vdash e_1 \Rightarrow^{r_0}_{r_1} \texttt{let } p_1 \texttt{ in } rs_1} \qquad \overset{\mathcal{C}_2}{\delta \vdash e_2 \Rightarrow^{r_1}_{r_2} \texttt{let } p_2 \texttt{ in } rs_2}}{\delta \vdash (e_1, e_2) \Rightarrow^{r_0}_{r_2} \texttt{let } p1; p2 \texttt{ in } (rs_1, rs_2)}$

So $\tau = (\tau_1, \tau_2), v = (v_1, v_2), rs = (rs_1, rs_2), r = r_2$.
By IH on $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1$, we get $\mathcal{P}_1$ of $\langle p_1, \sigma\rangle \downarrow \sigma_1$, $\mathcal{V}_1$ of $\sigma_1 \vdash v_1 \rhd_{\tau_1} rs_1$, $\sigma_1 \xlongequal{<r_0} \sigma$, and $rset(rs_1) \lessdot r_1$.
Likewise, by IH on $\mathcal{T}_2, \mathcal{E}_2, \mathcal{C}_2$, we get $\mathcal{P}_2$ of $\langle p_2, \sigma_1\rangle \downarrow \sigma_2$, $\mathcal{V}_2$ of $\sigma_2 \vdash v_2 \rhd_{\tau_2} rs_2$, $\sigma_2 \xlongequal{<r_1} \sigma_1$, and $rset(rs_2) \lessdot r_2$.

Then we can construct:

$$\cfrac{\overset{\mathcal{P}_1}{\langle p_1, \sigma\rangle \downarrow \sigma_1} \qquad \overset{\mathcal{P}_2}{\langle p_2, \sigma_1\rangle \downarrow \sigma_2}}{\langle p_1; p_2, \sigma\rangle \downarrow \sigma_2}$$

By lemma 6 on $\mathcal{C}_1$, we get $r_0 \le r_1$, hence $\sigma_2 \xlongequal{<r_0} \sigma_1 \xlongequal{<r_0} \sigma$.

Since $rset(rs_1) \lessdot r_1$, we have $\forall r' \in rset(rs_1).\sigma_2(r') = \sigma_1(r')$ (by $\sigma_2 \xlongequal{<r_1} \sigma_1$). Therefore, there exists some $\mathcal{V}_1'$ of $\sigma_2 \vdash v_1 \rhd_{\tau_1} rs_1$.

Then we can construct: $\cfrac{\overset{\mathcal{V}_1'}{\sigma_2 \vdash v_1 \rhd_{\tau_1} rs_1} \qquad \overset{\mathcal{V}_2}{\sigma_2 \vdash v_2 \rhd_{\tau_2} rs_2}}{\sigma_2 \vdash (v_1, v_2) \rhd_{(\tau_1, \tau_2)} (rs_1, rs_2)}$

By lemma 6 on $\mathcal{C}_2$, we get $r_1 \le r_2$, hence $rset(rs_1) \lessdot r_2$. Therefore, $rset(rs) = rset((rs_1, rs_2)) = (rset(rs_1) \cup rset(rs_2)) \lessdot r_2$.

Take $\sigma' = \sigma_2$ and we are done.

- Case $e = \textbf{let } x = e_1 \textbf{ in } e_2$.

Must have: $\mathcal{T} = \cfrac{\overset{\mathcal{T}_1}{\Gamma \vdash e_1 : \tau_1} \qquad \overset{\mathcal{T}_2}{\Gamma[x \mapsto \tau_1] \vdash e_2 : \tau}}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : \tau}$

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\rho \vdash e_1 \downarrow v_1} \qquad \overset{\mathcal{E}_2}{\rho[x \mapsto v_1] \vdash e_2 \downarrow v}}{\rho \vdash \textbf{let } x = e_1 \textbf{ in } e_2 \downarrow v}$$

$$\mathcal{C} = \cfrac{\overset{\mathcal{C}_1}{\delta \vdash e_1 \Rightarrow^{r_0}_{r_1} \texttt{let } p_1 \texttt{ in } rs_1} \qquad \overset{\mathcal{C}_2}{\delta[x \mapsto rs_1] \vdash e_2 \Rightarrow^{r_1}_{r_2} \texttt{let } p_2 \texttt{ in } rs}}{\delta \vdash \textbf{let } x = e_1 \textbf{ in } e_2 \Rightarrow^{r_0}_{r_2} \texttt{let } p_1; p_2 \texttt{ in } rs}$$

So $p = p_1; p_2, r = r_2$.
By IH on $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1$, we get $\mathcal{P}_1 = \langle p_1, \sigma \rangle \downarrow \sigma_1$, $\mathcal{V}_1 = \sigma_1 \vdash v_1 \triangleright_{\tau_1} rs_1$, $\sigma_1 \xlongequal{<r_0} \sigma$ and $rset(rs_1) \lessdot r_1$.

Since from $\mathcal{V}_1$ we know $v_1 : \tau_1$, then $\rho[x \mapsto v_1](x) : \Gamma[x \mapsto \tau_1](x)$ and $\sigma_1 \vdash \rho[x \mapsto v_1](x) \triangleright_{\Gamma[x \mapsto \tau_1](x)} \delta[x \mapsto rs_1](x)$ must hold. Also, we already have $rset(\delta[x \mapsto rs_1](x)) \lessdot r_1$. Then by IH on $\mathcal{T}_2, \mathcal{E}_2, \mathcal{C}_2$, we get $\mathcal{P}_2 = \langle p_2, \sigma_1 \rangle \downarrow \sigma_2$, $\mathcal{V}_2 = \sigma_2 \vdash v \triangleright_\tau rs$, $\sigma_2 \xlongequal{<r_1} \sigma_1$, and $rset(rs) \lessdot r_2$.

So we can construct $\cfrac{\overset{\mathcal{P}_1}{\langle p_1, \sigma \rangle \downarrow \sigma_1} \qquad \overset{\mathcal{P}_2}{\langle p_2, \sigma_1 \rangle \downarrow \sigma_2}}{\langle p_1; p_2, \sigma \rangle \downarrow \sigma_2}$

By lemma 6 on $\mathcal{C}_1$: $r_0 \leq r_1$, hence $\sigma_2 \xlongequal{<r_0} \sigma_1 \xlongequal{<r_0} \sigma$.

Take $\sigma' = \sigma_2$ and we are done.

- Case $e = n$.

  Must have $\mathcal{T} = \overline{\Gamma \vdash n : \textbf{Int}}$, $\mathcal{E} = \overline{\rho \vdash n \downarrow n}$, and $\mathcal{C} = \overline{\delta \vdash n \Rightarrow^{r_0}_{r_0+1} \texttt{let } \textbf{mov } r_0\ n \texttt{ in } r_0}$.
  So $p = \textbf{mov } r_0\ n, rs = r_0, v = n, r = r_0$, and $\tau = \textbf{Int}$.
  Then immediately we get $\overline{\langle \textbf{mov } r_0\ n, \sigma \rangle \downarrow \sigma[r_0 \mapsto n]}$, $\sigma[r_0 \mapsto n] \vdash n \triangleright_{\textbf{Int}} r_0$, $\sigma[r_0 \mapsto n] \xlongequal{<r_0} \sigma(r)$, and $rset(r) = \{r_0\} \lessdot r_0 + 1$ as required.

- Case $e{=}x$.

  Must have $\mathcal{T} = \overline{\Gamma \vdash x : \tau}\ (\Gamma(x) = \tau)$, $\mathcal{E} = \overline{\rho \vdash x \downarrow v}\ (\rho(x) = v)$,
  and $\mathcal{C} = \overline{\delta \vdash x \Rightarrow^{r_0}_{r_0} \texttt{let } \epsilon \texttt{ in } rs}\ (\delta(x) = rs)$.
  So $p = \epsilon$.
  Immediately we get $\langle \epsilon, \sigma \rangle \downarrow \sigma$, $\sigma \vdash v \triangleright_\tau rs$, $\sigma \xlongequal{<r_0} \sigma$ and $rset(rs) \lessdot r_0$ (from assumption) as required.

- Case $e = \textbf{fst}(e_1)$.

  Must have: $\mathcal{T} = \cfrac{\overset{\mathcal{T}_1}{\Gamma \vdash e_1 : (\tau_1, \tau_2)}}{\Gamma \vdash \textbf{fst}(e_1) : \tau_1}$ for some τ_2,

  $$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\rho \vdash e_1 \downarrow (v_1, v_2)}}{\rho \vdash \textbf{fst}(e_1) \downarrow v_1} \text{ for some } v_2,$$

  $$\mathcal{C} = \cfrac{\overset{\mathcal{C}_1}{\delta \vdash e_1 \Rightarrow^{r_0}_{r_1} \texttt{let } p \texttt{ in } (rs_1, rs_2)}}{\delta \vdash \textbf{fst}(e_1) \Rightarrow^{r_0}_{r_1} \texttt{let } p \texttt{ in } rs_1} \text{ for some } rs_2.$$

  So $\tau = \tau_1, v = v_1, rs = rs_1, r = r_1$.
  By IH on $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1$, we get $\mathcal{P}$ of $\langle p, \sigma \rangle \downarrow \sigma_1$, $\mathcal{V}_1$ of $\sigma \vdash (v_1, v_2) \triangleright_{(\tau_1, \tau_2)} (rs_1, rs_2)$, $\sigma_1 \xlongequal{<r_0} \sigma$, and $rset((rs_1, rs_2)) \lessdot r_1$.

  Since $rset((rs_1, rs_2)) = rset(rs_1) \cup rset(rs_2)$, therefore $rset(rs_1) \lessdot r_1$ must hold.

$\mathcal{V}_1$ must have the shape: $$\dfrac{\overset{\mathcal{V}}{\sigma_1 \;\vdash\; v_1 \triangleright_{\tau_1} rs_1} \qquad \sigma_1 \;\vdash\; v_2 \triangleright_{\tau_2} rs_2}{\sigma_1 \;\vdash\; (v_1, v_2) \triangleright_{(\tau_1,\tau_2)} (rs_1, rs_2)}$$

So now we have $\mathcal{V}$. Take $\sigma' = \sigma_1$ and we are done.

- The case where $e = \mathbf{snd}(e_1)$ is analogous to the case above.

$\square$