

# SNESL formalization

Dandan Xue

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## 0 Level-0

Draft version 0.0.4: added the proof of the main correctness theorem (in process)

### 0.1 Source language syntax

(Ignore empty sequence for now)

Expressions:

$$e ::= x \mid \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 \mid \phi(x_1, \dots, x_k) \mid \{e : x \ \mathbf{in} \ y \ \mathbf{using} \ \cdot\} \\ \phi = \mathbf{const}_n \mid \mathbf{iota} \mid \mathbf{plus}$$

Values:

$$n \in \mathbf{Z} \\ v ::= n \mid \{v_1, \dots, v_k\}$$

### 0.2 Type system

$$\tau ::= \mathbf{int} \mid \{\tau_1\}$$

Type environment  $\Gamma = [x_1 \mapsto \tau_1, \dots, x_i \mapsto \tau_i]$ .

- Expression typing rules:

Judgment  $\boxed{\Gamma \vdash e : \tau}$

$$\frac{}{\Gamma \vdash x : \tau} (\Gamma(x) = \tau) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau}{\Gamma \vdash \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 : \tau} \\ \frac{\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau}{\Gamma \vdash \phi(x_1, \dots, x_k) : \tau} ((\Gamma(x_i) = \tau_i)_{i=1}^k) \qquad \frac{[x \mapsto \tau_1] \vdash e : \tau}{\Gamma \vdash \{e : x \ \mathbf{in} \ y \ \mathbf{using} \ \cdot\} : \{\tau\}} (\Gamma(y) = \{\tau_1\})$$

- Auxiliary Judgment  $\boxed{\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau}$

$$\frac{}{\vdash \mathbf{const}_n : () \rightarrow \mathbf{int}} \qquad \frac{}{\vdash \mathbf{iota} : (\mathbf{int}) \rightarrow \{\mathbf{int}\}} \qquad \frac{}{\vdash \mathbf{plus} : (\mathbf{int}, \mathbf{int}) \rightarrow \mathbf{int}}$$

- Value typing rules:

Judgment  $\boxed{\vdash v : \tau}$

$$\frac{}{\vdash n : \mathbf{int}} \qquad \frac{(\vdash v_i : \tau)_{i=1}^k}{\vdash \{v_1, \dots, v_k\} : \{\tau\}}$$

### 0.3 Source language semantics

$$\rho = [x_1 \mapsto v_1, \dots, x_i \mapsto v_i]$$

- Judgment  $\boxed{\rho \vdash e \downarrow v}$

$$\frac{}{\rho \vdash x \downarrow v} (\rho(x) = v) \quad \frac{\rho \vdash e_1 \downarrow v_1 \quad \rho[x \mapsto v_1] \vdash e_2 \downarrow v}{\rho \vdash \text{let } e_1 = x \text{ in } e_2 \downarrow v}$$

$$\frac{\vdash \phi(v_1, \dots, v_k) \downarrow v}{\rho \vdash \phi(x_1, \dots, x_k) \downarrow v} ((\rho(x_i) = v_i)_{i=1}^k)$$

$$\frac{([x \mapsto v_i] \vdash e \downarrow v'_i)_{i=1}^k}{\rho \vdash \{e : x \text{ in } y \text{ using } \cdot\} \downarrow \{v'_1, \dots, v'_k\}} (\rho(y) = \{v_1, \dots, v_k\})$$

- Auxiliary Judgment  $\boxed{\vdash \phi(v_1, \dots, v_k) \downarrow v}$

$$\frac{}{\vdash \text{const}_n() \downarrow n} \quad \frac{}{\vdash \text{iota}(n) \downarrow \{0, 1, \dots, n-1\}} (n \geq 0)$$

$$\frac{}{\vdash \text{plus}(n_1, n_2) \downarrow n_3} (n_3 = n_1 + n_2)$$

### 0.4 SVCODE syntax

- (1) Stream id:

$$s \in \mathbf{SId} = \mathbf{N} = \{0, 1, 2, \dots\}$$

- (2) Stream tree:

$$\mathbf{STree} \ni st ::= s \mid (st_1, s)$$

- (3) SVCODE operations:

$$\psi ::= \text{Ctrl} \mid \text{Const}_a \mid \text{ToFlags} \mid \text{Usum} \mid \text{MapTwo}_{\oplus} \mid \text{ScanPlus}_{n_0}$$

where  $\oplus$  stands for some binary operation on **int**.

- (4) SVCODE program:

$$\begin{aligned} p ::= & \epsilon \\ & \mid s := \psi(s_1, \dots, s_i) \\ & \mid st := \text{WithCtrl}(s, p) \\ & \mid p_1; p_2 \end{aligned}$$

- (5) Target language values:

$$\begin{aligned} b & \in \{\mathbf{T}, \mathbf{F}\} \\ a & ::= n \mid b \mid () \\ \vec{b} & = \langle b_1, \dots, b_i \rangle \\ \vec{a} & = \langle a_1, \dots, a_i \rangle \\ \mathbf{SVal} \ni w & ::= \vec{a} \mid (w, \vec{b}) \end{aligned}$$

- (6) Some notations and operations:

- For some  $a_0$  and  $\vec{a} = \langle a_1, \dots, a_i \rangle$ , let  $\langle a_0 | \vec{a} \rangle = \langle a_0, a_1, \dots, a_i \rangle$ .

- $++ : \mathbf{SVal} \rightarrow \mathbf{SVal} \rightarrow \mathbf{SVal}$   
 $\langle a_1, \dots, a_i \rangle ++ \langle a'_1, \dots, a'_i \rangle = \langle a_1, \dots, a_i, a'_1, \dots, a'_i \rangle$   
 $(w_1, \vec{b}_1) ++ (w_2, \vec{b}_2) = (w_1 ++ w_2, \vec{b}_1 ++ \vec{b}_2)$
- **sids** is a function that converts a  $st \in \mathbf{STree}$  to a set of  $s \in \mathbf{SId}$ :  
 $\mathbf{sids}(s) = \{s\}$   
 $\mathbf{sids}((st, s)) = \mathbf{sids}(st) \cup \{s\}$
- For some set of  $\mathbf{SId}$ ,  $t$ , and some  $s \in \mathbf{SId}$ , let  $t \leq s$  denote  $\forall s' \in t. s' \leq s$ .

## 0.5 SVCODE semantics

SVCODE runtime environment  $\sigma = [s_1 \mapsto \vec{a}_1, \dots, s_i \mapsto \vec{a}_i]$ .

We define some notations and operations related to  $\sigma$ :

(1) Let  $\sigma_1 \stackrel{\leq s}{=} \sigma_2$  denote  $\forall s' \leq s. \sigma_1(s') = \sigma_2(s')$ .

(2) Judgment  $\boxed{\sigma(st) = w}$

$$\frac{}{\sigma(s) = \vec{a}} \quad \frac{\sigma(st) = w \quad \overline{\sigma(s) = \vec{a}}}{\sigma((st, s)) = (w, \vec{a})}$$

**Definition 0.1.**  $\sigma_1 \stackrel{st}{\sim} \sigma_2$  iff

- (1)  $\text{dom}(\sigma_1) = \text{dom}(\sigma_2)$
- (2)  $\forall s \in (\text{dom}(\sigma_1) - \mathbf{sids}(st)). \sigma_1(s) = \sigma_2(s)$

It is easy to show that this relation  $\stackrel{st}{\sim}$  is commutative, transitive and associative.

**Definition 0.2.**  $\sigma_1 \boxtimes^{st} \sigma_2 = \sigma$  iff

- (1)  $\sigma_1 \stackrel{st}{\sim} \sigma_2$
- (2)  $\sigma(s) = \begin{cases} \sigma_1(s) ++ \sigma_2(s), & s \in \mathbf{sids}(st) \\ \sigma_1(s), & \text{otherwise} \end{cases}$

**Lemma 0.1.** If  $\sigma_1 \boxtimes^{st} \sigma_2 = \sigma$ , then  $\sigma_1 \stackrel{st}{\sim} \sigma$  and  $\sigma_2 \stackrel{st}{\sim} \sigma$ .

**Lemma 0.2.** If  $\sigma_1 \stackrel{st_1}{\sim} \sigma'_1$ ,  $\sigma_2 \stackrel{st_2}{\sim} \sigma'_2$ ,  $\sigma_1 \stackrel{\leq s}{=} \sigma_2$ , and  $\sigma'_1 \stackrel{\leq s}{=} \sigma'_2$  then  $\sigma_1 \boxtimes^{st_1} \sigma'_1 \stackrel{\leq s}{=} \sigma_2 \boxtimes^{st_2} \sigma'_2$ .

SVCODE operational semantics:

- Judgment  $\boxed{\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$

$\vec{a}_c$  is the control stream.

$$\frac{}{\langle \epsilon, \sigma \rangle \downarrow^{\vec{a}_c} \sigma} \quad \frac{\psi(\vec{a}_1, \dots, \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}{\langle s := \psi(s_1, \dots, s_k), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s \mapsto \vec{a}]} ((\sigma(s_i) = \vec{a}_i)_{i=1}^k)$$

$$\frac{}{\langle st := \text{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \langle \rangle, \dots, s_i \mapsto \langle \rangle]} (\sigma(s) = \langle \rangle, \mathbf{sids}(st) = \{s_1, \dots, s_i\})$$

$$\frac{\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma''}{\langle st := \text{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \sigma''(s_1), \dots, s_i \mapsto \sigma''(s_i)]} \left( \begin{array}{l} \sigma(s) = \vec{a}_s = \langle a_0 | \vec{a} \rangle \\ \mathbf{sids}(st) = \{s_1, \dots, s_i\} \end{array} \right)$$

$$\frac{\langle p_1, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'' \quad \langle p_2, \sigma'' \rangle \downarrow^{\vec{a}_c} \sigma'}{\langle p_1; p_2, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$$

- *Transducer semantics*:

$$\text{Judgment } \boxed{\psi(\vec{a}_1, \dots, \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}$$

$$\frac{\psi(\vec{a}_{11}, \dots, \vec{a}_{k1}) \downarrow \vec{a}_1 \quad \psi(\vec{a}_{12}, \dots, \vec{a}_{k2}) \downarrow^{\vec{a}_c} \vec{a}_2}{\psi(\vec{a}_{11} ++ \vec{a}_{12}, \dots, \vec{a}_{k1} ++ \vec{a}_{k2}) \downarrow^{\langle a_0 | \vec{a}_c \rangle} \vec{a}} \quad (\vec{a} = \vec{a}_1 ++ \vec{a}_2)$$

$$\overline{\psi(\vec{a}_1, \dots, \vec{a}_k) \downarrow^{\langle \rangle} \langle \rangle}$$

- *Transducer block semantics*:

$$\text{Judgment } \boxed{\psi(\vec{a}_1, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

$$\overline{\text{Const}_a \downarrow \langle a \rangle} \quad \overline{\text{ToFlags}(\langle n \rangle) \downarrow \langle \mathbf{F}_1, \dots, \mathbf{F}_n, \mathbf{T} \rangle} \quad \overline{\text{MapTwo}_{\oplus}(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow \langle n_3 \rangle} \quad (n_3 = n_1 \oplus n_2)$$

$$\text{P-USUMF} : \frac{\text{Usum}(\vec{b}) \downarrow \vec{a}}{\text{Usum}(\langle \mathbf{F} | \vec{b} \rangle) \downarrow \langle () | \vec{a} \rangle} \quad \text{P-USUMT} : \overline{\text{Usum}(\langle \mathbf{T} \rangle) \downarrow \langle \rangle}$$

$$\text{P-SCANF} : \frac{\text{ScanPlus}_{n_0+n}(\vec{b}, \vec{a}) \downarrow \vec{a}'}{\text{ScanPlus}_{n_0}(\langle \mathbf{F} | \vec{b} \rangle, \langle n | \vec{a} \rangle) \downarrow \langle n_0 | \vec{a}' \rangle} \quad \text{P-SCANT} : \overline{\text{ScanPlus}_{n_0}(\langle \mathbf{T} \rangle, \langle \rangle) \downarrow \langle \rangle}$$

Or if we want to use *unary* semantics maybe for later:

$$\frac{\psi(\langle \mathbf{F} \rangle, \dots, \vec{a}_{k1}) \Downarrow \vec{a}_1 \quad \psi(\vec{a}_{12}, \dots, \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi(\langle \mathbf{F} \rangle ++ \vec{a}_{12}, \dots, \vec{a}_{k1} ++ \vec{a}_{k2}) \Downarrow \vec{a}} \quad (\vec{a} = \vec{a}_1 ++ \vec{a}_2)$$

$$\frac{\psi(\langle \mathbf{T} \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}{\psi(\langle \mathbf{T} \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

- Transducer *unary* semantics:

$$\text{Judgment } \boxed{\psi(\langle b \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

$$\overline{\text{Usum}(\langle \mathbf{F} \rangle) \downarrow \langle () \rangle} \quad \overline{\text{Usum}(\langle \mathbf{T} \rangle) \downarrow \langle \rangle}$$

- Transducer block with *accumulator*:

$$\text{Judgment } \boxed{\psi_n(\vec{a}_1, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

$$\frac{\psi_{n_0}(\langle \mathbf{F} \rangle, \dots, \vec{a}_{k1}) \Downarrow^{n'_0} \langle n_1 \rangle \quad \psi_{n'_0}(\vec{a}_{12}, \dots, \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi_{n_0}(\langle \mathbf{F} \rangle ++ \vec{a}_{12}, \dots, \vec{a}_{k1} ++ \vec{a}_{k2}) \Downarrow \langle n_1 \rangle ++ \vec{a}_2}$$

$$\frac{\psi_{n_0}(\langle \mathbf{T} \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}{\psi_{n_0}(\langle \mathbf{T} \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

- Transducer unary with *accumulator*:

$$\text{Judgment } \boxed{\psi_n(\langle \mathbf{F} \rangle, \dots, \vec{a}_k) \Downarrow^{n'} \vec{a}}$$

$$\begin{array}{c}
\hline
\text{ScanPlus}_{n_0}(\langle \mathbf{F} \rangle, \langle n \rangle) \Downarrow^{n_0+n} \langle n_0 \rangle \\
\hline
\text{Judgment } \boxed{\psi_n(\langle \mathbf{T} \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}} \\
\hline
\text{ScanPlus}_{n_0}(\langle \mathbf{T} \rangle, \langle \rangle) \Downarrow \langle \rangle \\
\hline
\end{array}$$

**Theorem 0.1** (SVCODE determinism). *If  $\langle p, \sigma \rangle \Downarrow^{\vec{a}_c} \sigma'$  and  $\langle p, \sigma \rangle \Downarrow^{\vec{a}_c} \sigma''$ , then  $\sigma' = \sigma''$ .*

**Lemma 0.3.** *If  $\sigma_1 \stackrel{st}{\sim} \sigma_2$ ,  $\langle p, \sigma_1 \rangle \Downarrow^{\vec{a}_1} \sigma'_1$ ,  $\langle p, \sigma_2 \rangle \Downarrow^{\vec{a}_2} \sigma'_2$ , then  $\langle p, \sigma_1 \bowtie \sigma_2 \rangle \Downarrow^{\vec{a}_1 + \vec{a}_2} \sigma'_1 \bowtie \sigma'_2$*

**Definition 0.3.**  $\vec{a}$  is a prefix of  $\vec{a}'$  if  $\vec{a} \sqsubseteq \vec{a}'$ .

$$\begin{array}{c}
\text{Judgment } \boxed{\vec{a} \sqsubseteq \vec{a}'} \\
\hline
\langle \rangle \sqsubseteq \vec{a} \qquad \frac{\vec{a} \sqsubseteq \vec{a}'}{\langle a_0 | \vec{a} \rangle \sqsubseteq \langle a_0 | \vec{a}' \rangle}
\end{array}$$

**Lemma 0.4.** *If*

- (i)  $(\vec{a}'_i \sqsubseteq \vec{a}_i)_{i=1}^k$  and  $\psi(\vec{a}'_1, \dots, \vec{a}'_k) \Downarrow \vec{a}'$ ,
- (ii)  $(\vec{a}''_i \sqsubseteq \vec{a}_i)_{i=1}^k$  and  $\psi(\vec{a}''_1, \dots, \vec{a}''_k) \Downarrow \vec{a}''$

*then*

- (i)  $(\vec{a}'_i = \vec{a}''_i)_{i=1}^k$
- (ii)  $\vec{a}' = \vec{a}''$ .

## 0.6 Translation

$$\delta = [x_1 \mapsto st_1, \dots, x_i \mapsto st_i]$$

- Judgment  $\boxed{\delta \vdash e \xRightarrow[s_1]{s_0} (p, st)}$

$$\begin{array}{c}
\frac{}{\delta \vdash x \xRightarrow[s_0]{s_0} (\epsilon, st)} \quad (\delta(x) = st) \qquad \frac{\delta \vdash e_1 \xRightarrow[s_0]{s_0} (p_1, st_1) \quad \delta[x \mapsto st_1] \vdash e_2 \xRightarrow[s_1]{s'_0} (p_2, st)}{\delta \vdash \text{let } x = e_1 \text{ in } e_2 \xRightarrow[s_1]{s_0} (p_1; p_2, st)} \\
\\
\frac{\vdash \phi(st_1, \dots, st_k) \xRightarrow[s_1]{s_0} (p, st)}{\delta \vdash \phi(x_1, \dots, x_k) \xRightarrow[s_1]{s_0} (p, st)} \quad ((\delta(x_i) = st_i)_{i=1}^k) \\
\\
\frac{[x \mapsto st_1] \vdash e \xRightarrow[s_1]{s_0+1} (p, st)}{\delta \vdash \{e : x \text{ in } y \text{ using } \cdot\} \xRightarrow[s_1]{s_0} (s_0 := \text{Usum}(s_2); st := \text{WithCtrl}(s_0, p), (st, s_2))} \quad (\delta(y) = (st_1, s_2))
\end{array}$$

- Auxiliary Judgment  $\boxed{\vdash \phi(st_1, \dots, st_k) \xRightarrow[s_1]{s_0} (p, st)}$

$$\begin{array}{c}
\frac{}{\vdash \text{const}_n() \xRightarrow[s_0]{s_0+1} (s_0 := \text{Const}_n(), s_0)} \\
\\
\frac{\vdash \text{iota}(s) \xRightarrow[s_4]{s_0} (p, (s_3, s_0))}{\vdash \text{iota}(s) \xRightarrow[s_4]{s_0} (p, (s_3, s_0))} \left( \begin{array}{l} s_{i+1} = s_i + 1 \\ p = s_0 := \text{ToFlags}(s); \\ s_1 := \text{Usum}(s_0); \\ s_2 := \text{WithCtrl}(s_1, s_2 := \text{Const}_1()); \\ s_3 := \text{ScanPlus}_0(s_0, s_2) \end{array} \right) \\
\\
\frac{}{\vdash \text{plus}(s_1, s_2) \xRightarrow[s_0+1]{s_0} (s_0 := \text{MapTwo}_+(s_1, s_2), s_0)}
\end{array}$$

## 0.7 Value representation

- Judgment  $\boxed{v \triangleright_{\tau} w}$

$$\frac{}{n \triangleright_{\mathbf{int}} \langle n \rangle} \quad \frac{(v_i \triangleright_{\tau} w_i)_{i=1}^k}{\{v_1, \dots, v_k\} \triangleright_{\{\tau\}} (w, \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle)} (w = w_1 ++ \dots ++ w_k)$$

**Lemma 0.5.** *If  $v \triangleright_{\tau} w$ ,  $v' \triangleright_{\tau} w$ , then  $v = v'$ .*

## 0.8 Correctness proof

**Lemma 0.6** (???). *If  $\Gamma \vdash e : \{\tau\}$ ,  $\rho \vdash e \downarrow \{v_1, \dots, v_k\}$ , and  $\delta \vdash e \xrightarrow[s_1]{s_0} (p, (st, s))$ , then  $s \notin \mathbf{sids}(st)$ .*

**Lemma 0.7.** *If*

(i)  $\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau$  (by some derivation  $\mathcal{T}$ )

(ii)  $\vdash \phi(v_1, \dots, v_k) \downarrow v$  (by  $\mathcal{E}$ )

(iii)  $\vdash \phi(st_1, \dots, st_k) \xrightarrow[s_1]{s_0} (p, st)$  (by  $\mathcal{C}$ )

(iv)  $(v_i \triangleright_{\tau_i} \sigma(st_i))_{i=1}^k$

(v)  $\bigcup_{i=1}^k \mathbf{sids}(st_i) \leq s_0$

then

(vi)  $\langle p, \sigma \rangle \downarrow^{(\langle \rangle)} \sigma'$  (by  $\mathcal{P}$ )

(vii)  $v \triangleright_{\tau} \sigma'(st)$  (by  $\mathcal{V}$ )

(viii)  $\sigma' \xrightarrow[\leq s_0]{} \sigma$

(ix)  $s_0 \leq s_1$

(x)  $\mathbf{sids}(st) \leq s_1$

*Proof.* By induction on the syntax of  $\phi$ .

- Case  $\phi = \mathbf{const}_n$

There is only one possibility for each of  $\mathcal{T}$ ,  $\mathcal{E}$  and  $\mathcal{C}$ :

$$\begin{aligned} \mathcal{T} &= \frac{}{\vdash \mathbf{const}_n : () \rightarrow \mathbf{int}} \\ \mathcal{E} &= \frac{}{\vdash \mathbf{const}_n() \downarrow n} \\ \mathcal{C} &= \frac{}{\vdash \mathbf{const}_n() \xrightarrow[s_0+1]{s_0} (s_0 := \mathbf{Const}_n(), s_0)} \end{aligned}$$

So  $k = 0, \tau = \mathbf{int}, v = n, p = s_0 := \mathbf{Const}_n(), s_1 = s_0 + 1$ , and  $st = s_0$

Since  $\mathbf{Const}_n()$  takes no arguments, there is only one possibility for  $\mathcal{P}$ :

$$\mathcal{P} = \frac{\frac{\mathbf{Const}_n() \downarrow \langle n \rangle \quad \mathbf{Const}_n() \downarrow^{(\langle \rangle)} \langle \rangle}{\mathbf{Const}_n() \downarrow^{(\langle \rangle)} \langle n \rangle}}{\langle s_0 := \mathbf{Const}_n(), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle n \rangle]}$$

So  $\sigma' = \sigma[s_0 \mapsto \langle n \rangle]$ .

Then we have  $\mathcal{V} = \frac{}{n \triangleright_{\mathbf{int}} \sigma'(s_0)}$ .

Also clearly,  $\sigma' \xrightarrow[\leq s_0]{} \sigma$ ,  $s_0 \leq s_0 + 1$ ,  $\mathbf{sids}(s_0) \leq s_0 + 1$ , and we are done.

- Case  $\phi = \mathbf{plus}$   
We must have

$$\mathcal{T} = \overline{\vdash \mathbf{plus} : (\mathbf{int}, \mathbf{int}) \rightarrow \mathbf{int}}$$

$$\mathcal{E} = \overline{\vdash \mathbf{plus}(n_1, n_2) \downarrow n_3}$$

where  $n_3 = n_2 + n_1$ , and

$$\mathcal{C} = \overline{\vdash \mathbf{plus}(s_1, s_2) \xrightarrow[s_0+1]{s_0} (s_0 := \mathbf{MapTwo}_+(s_1, s_2), s_0)}$$

So  $k = 2, \tau_1 = \tau_2 = \tau = \mathbf{int}, v_1 = n_1, v_2 = n_2, v = n_3, st_1 = s_1, st_2 = s_2, st = s_0, s_1 = s_0 + 1$  and  $p = s_0 := \mathbf{MapTwo}_+(s_1, s_2)$ .

Assumption (iv) gives us  $n_1 \triangleright_{\mathbf{int}} \sigma(s_1)$  and  $n_2 \triangleright_{\mathbf{int}} \sigma(s_2)$ , which implies  $\sigma(s_1) = \langle n_1 \rangle$  and  $\sigma(s_2) = \langle n_2 \rangle$ .

For (v) we have  $s_1 < s_0$  and  $s_2 < s_0$ .

$\mathcal{P}$  must have the shape:

$$\frac{\frac{\mathcal{P}_1 \quad \overline{\mathbf{MapTwo}_+(\vec{a}_1, \vec{a}_2) \downarrow \vec{a}} \quad \overline{\mathbf{MapTwo}_+(\vec{a}'_1, \vec{a}'_2) \downarrow \langle \rangle \langle \rangle}}{\mathbf{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow \langle \rangle \vec{a}} (\vec{a}_1 ++ \vec{a}'_1 = \langle n_1 \rangle, \vec{a}_2 ++ \vec{a}'_2 = \langle n_2 \rangle)} (\sigma(s_1) = \langle n_1 \rangle, \sigma(s_2) = \langle n_2 \rangle)} (\sigma(s_0 := \mathbf{MapTwo}_+(s_1, s_2), \sigma) \downarrow \langle \rangle \sigma[s_0 \mapsto \vec{a}])$$

Since there is only one rule for  $\mathcal{P}_1$ , by which  $\vec{a}_1$  must be  $\langle n_1 \rangle$  and  $\vec{a}_2$  must be  $\langle n_2 \rangle$  (, and  $\vec{a}'_1 = \vec{a}'_2 = \langle \rangle$ ), that is,

$$\mathcal{P}_1 = \overline{\mathbf{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow \langle n_3 \rangle}$$

Therefore,  $\sigma' = \sigma[s_0 \mapsto \langle n_3 \rangle]$ .

Now it is clear that  $\mathcal{V} = n_3 \triangleright_{\mathbf{int}} \sigma'(s_0)$ ,  $\sigma' \xrightarrow{\leq s_0} \sigma$ ,  $s_0 \leq s_0 + 1$  and  $\mathbf{sids}(s_0) \leq s_0 + 1$  as required.

- Case  $\phi = \mathbf{iota}$

□

**Theorem 0.2.** *If*

- (i)  $\Gamma \vdash e : \tau$  (by some derivation  $\mathcal{T}$ )
- (ii)  $\rho \vdash e \downarrow v$  (by  $\mathcal{E}$ )
- (iii)  $\delta \vdash e \xrightarrow[s_1]{s_0} (p, st)$  (by  $\mathcal{C}$ )
- (iv)  $\forall x \in \text{dom}(\Gamma). \vdash \rho(x) : \Gamma(x) \wedge \mathbf{sids}(\delta(x)) \leq s_0 \wedge \rho(x) \triangleright_{\Gamma(x)} \sigma(\delta(x))$   
**then**
- (v)  $\langle p, \sigma \rangle \downarrow \langle \rangle \sigma'$  (by  $\mathcal{P}$ )
- (vi)  $v \triangleright_{\tau} \sigma'(st)$  (by  $\mathcal{V}$ )
- (vii)  $\sigma' \xrightarrow{\leq s_0} \sigma$
- (viii)  $s_0 \leq s_1$
- (ix)  $\mathbf{sids}(st) \leq s_1$

*Proof.* By induction on the syntax of  $e$ .

- Case  $e = \{e_1 : x \text{ in } y \text{ using } \cdot\}$ .

We must have:

$$\begin{aligned}
\text{(i)} \quad \mathcal{T} &= \frac{\mathcal{T}_1 \quad [x \mapsto \tau_1] \vdash e_1 : \tau_2}{\Gamma \vdash \{e_1 : x \text{ in } y \text{ using } \cdot\} : \{\tau_2\}} (\Gamma(y) = \{\tau_1\}) \\
\text{(ii)} \quad \mathcal{E} &= \frac{\mathcal{E}_i \quad ([x \mapsto v_i] \vdash e_1 \downarrow v'_i)_{i=1}^k}{\rho \vdash \{e_1 : x \text{ in } y \text{ using } \cdot\} \downarrow \{v'_1, \dots, v'_k\}} (\rho(y) = \{v_1, \dots, v_k\}) \\
\text{(iii)} \quad \mathcal{C} &= \frac{[x \mapsto st_1] \vdash e_1 \xrightarrow[s_1]{s_0+1} (p_1, st_2)}{\delta \vdash \{e_1 : x \text{ in } y \text{ using } \cdot\} \xrightarrow[s_1]{s_0} (s_0 := \text{Usum}(s_2); st_2 := \text{WithCtrl}(s_0, p_1), (st_2, s_2))} (\delta(y) = (st_1, s_2))
\end{aligned}$$

So  $p = (s_0 := \text{Usum}(s_2); st_2 := \text{WithCtrl}(s_0, p_1))$ ,  $\tau = \{\tau_2\}$ ,  $v = \{v'_1, \dots, v'_k\}$ ,  $st = (st_2, s_2)$ .

- (iv)  $\vdash \rho(y) : \Gamma(y)$  gives us  $\vdash \{v_1, \dots, v_k\} : \{\tau_1\}$ , which must have the derivation:

$$\frac{(\vdash v_i : \tau_1)_{i=1}^k}{\vdash \{v_1, \dots, v_k\} : \{\tau_1\}} \quad (1)$$

$\text{sids}(\delta(y)) \leq s_0$  gives us

$$\text{sids}(\delta(y)) = \text{sids}((st_1, s_2)) = \text{sids}(st_1) \cup \{s_2\} \leq s_0 \quad (2)$$

$\rho(y) \triangleright_{\Gamma(y)} \sigma(\delta(y)) = \{v_1, \dots, v_k\} \triangleright_{\{\tau_1\}} \sigma((st_1, s_2))$  must have the derivation:

$$\frac{\mathcal{V}_i \quad (v_i \triangleright_{\tau_1} w_i)_{i=1}^k}{\{v_1, \dots, v_k\} \triangleright_{\{\tau_1\}} (w, \langle F_1, \dots, F_k, T \rangle)} (w = w_1 ++ \dots ++ w_k) \quad (3)$$

therefore

$$\sigma(st_1) = w \quad (4)$$

and

$$\sigma(s_2) = \langle F_1, \dots, F_k, T \rangle. \quad (5)$$

First we shall show:

- (v)  $\langle s_0 := \text{Usum}(s_2); st_2 := \text{WithCtrl}(s_0, p_1), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma'$
- (vi)  $\{v'_1, \dots, v'_k\} \triangleright_{\{\tau_2\}} \sigma'((st_2, s_2))$  by  $\mathcal{V}$
- (vii)  $\sigma' \xrightarrow{\leq s_0} \sigma$

TS (v), we first prove  $\langle s_0 := \text{Usum}(s_2), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$  by  $\mathcal{P}_0$ .  $\mathcal{P}_0$  must have the shape:

$$\frac{\mathcal{P}'_0 \quad \frac{\text{Usum}(\vec{b}_1) \downarrow \vec{a} \quad \text{Usum}(\vec{b}_2) \downarrow^{(\langle \rangle)} \langle \rangle}{\text{Usum}(\langle F_1, \dots, F_k, T \rangle) \downarrow^{(\langle \rangle)} \vec{a}} (\vec{b}_1 ++ \vec{b}_2 = \langle F_1, \dots, F_k, T \rangle)}{\langle s_0 := \text{Usum}(s_2), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \vec{a}]} (\sigma(s_2) = \langle F_1, \dots, F_k, T \rangle)$$

From the rules P-USUMF and P-USUMT we know that  $\vec{b}_1$  must end with (and include exactly) one T so that  $\mathcal{P}'_0$  can terminate. Therefore, in our case,  $\vec{b}_1$  can only be  $\langle F_1, \dots, F_k, T \rangle$ , and  $\vec{b}_2 = \langle \rangle$ . Then using P-USUMF  $k$  times and P-USUMT once, we obtain  $\mathcal{P}'_0$  of

$$\text{Usum}(\vec{b}_1) \downarrow \langle ()_1, \dots, ()_k \rangle$$



which gives us  $\mathcal{P}_0$  of  $\langle s_0 := \mathbf{Usum}(s_2), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$ .  
Then  $\mathcal{P}$  must have the shape:

$$\frac{\mathcal{P}_0 \quad \mathcal{P}_1}{\langle s_0 := \mathbf{Usum}(s_2), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \quad \langle st_2 := \mathbf{WithCtrl}(s_0, p_1), \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \rangle \downarrow^{(\langle \rangle)} \sigma'}$$

Since we have

$$\begin{aligned} \mathcal{T}_1 &= [x \mapsto \tau_1] \vdash e_1 : \tau_2 \\ (\mathcal{E}_i &= [x \mapsto v_i] \vdash e_1 \downarrow v'_i)_{i=1}^k \\ \mathcal{C}_1 &= [x \mapsto st_1] \vdash e_1 \xrightarrow[s_1]{s_0+1} (p_1, st_2) \end{aligned}$$

Let  $\Gamma_1 = [x \mapsto \tau_1]$ ,  $\rho_i = [x \mapsto v_i]$  and  $\delta_1 = [x \mapsto st_1]$ .

From (1) and (2) it is clear that

$$\forall z \in \text{dom}(\Gamma_1). \vdash \rho_i(z) : \Gamma_1(z) \wedge \mathbf{sids}(\delta_1(z)) \leq s_0.$$

Let  $i$  range from 1 to  $k$ : we take  $\sigma_i \stackrel{st_1}{\sim} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$  such that  $\sigma_i(st_1) = w_i$ .  
From  $\mathcal{V}_i$  in (3) we know that

$$\forall z \in \text{dom}(\Gamma_1). \rho_i(z) \triangleright_{\Gamma_1(z)} \sigma_i(\delta_1(z)).$$

Then by IH ( $k$  times) on  $\mathcal{T}_1$  with  $\mathcal{E}_i, \mathcal{C}_1$  we obtain the following result:

$$(\langle p_1, \sigma_i \rangle \downarrow^{(\langle \rangle)} \sigma'_i)_{i=1}^k \tag{6}$$

$$(v'_i \triangleright_{\tau_2} \sigma'_i(st_2))_{i=1}^k \tag{7}$$

$$(\sigma'_i \xrightarrow[s_1]{\leq s_0+1} \sigma_i)_{i=1}^k \tag{8}$$

$$s_0 + 1 \leq s_1 \tag{9}$$

$$\mathbf{sids}(st_2) \leq s_1 \tag{10}$$

Assume  $\mathbf{sids}(st_2) = \{s'_1, \dots, s'_j\}$ .

There are two possibilities:

- Subcase  $k = 0$ , that is  $\sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle](s_0) = \langle \rangle$ .  
Then

$$\mathcal{P}_1 = \frac{}{\langle st_2 := \mathbf{WithCtrl}(s_0, p_1), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle \rangle, s'_1 \mapsto \langle \rangle, \dots, s'_j \mapsto \langle \rangle]} ,$$

thus in this subcase

$$\sigma' = \sigma[s_0 \mapsto \langle \rangle, s'_1 \mapsto \langle \rangle, \dots, s'_j \mapsto \langle \rangle].$$

Since  $k = 0$ , then  $v = \{\}$ ,  $\sigma(s_2) = \langle \mathbf{T} \rangle$  (from (5)), we have

$\sigma'(s_2) = \sigma(s_2) = \langle \mathbf{T} \rangle$  (?? not correct if  $s_2 \in \mathbf{sids}(st_2)/\mathbf{sids}(st_1)$ ),

and  $\sigma'(st_2) = (\dots((\langle \rangle, \langle \rangle)_1, \langle \rangle)_2, \dots)_{j-1}$ .

Therefore  $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2))$ , with which we construct

$$\mathcal{V} = \frac{\{\} \triangleright_{\{\tau_2\}} ((\dots((\langle \rangle, \langle \rangle)_1, \dots)_{j-1}, \langle \mathbf{T} \rangle))}{}$$

as required.

Since  $k = 0$ , from (4) we know  $\forall s' \in \mathbf{sids}(st_1). \sigma(s') = \langle \rangle$ . For any  $s' \in \mathbf{sids}(st_2)$  and  $s' < s_0$ , it must have  $s' \in \mathbf{sids}(st_1)$  (because  $\text{codom}(\delta_1) = \{st_1\}$ ), hence  $\sigma(s') = \langle \rangle = \sigma'(s')$ . Therefore,

$$\sigma' \xrightarrow[s_0]{\leq s_0} \sigma.$$

- Subcase  $k > 0$ , that is  $\sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] = \langle () | \vec{a} \rangle$  for some  $\vec{a}$ .  
Then  $\mathcal{P}_1 =$

$$\frac{\mathcal{P}'_1 \quad \langle p_1, \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \rangle \downarrow^{\langle ()_1, \dots, ()_k \rangle} \sigma''}{\langle st_2 := \text{WithCtrl}(s_0, p_1), \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \rangle \downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle, s'_1 \mapsto \sigma''(s'_1), \dots, s'_j \mapsto \sigma''(s'_j)]}$$

So in this subcase

$$\sigma' = \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle, s'_1 \mapsto \sigma''(s'_1), \dots, s'_j \mapsto \sigma''(s'_j)].$$

Using Lemma 0.3 (k-1) times on (6) gives us

$$\langle p_1, (\boxtimes^{st_1} \sigma_i)_{i=1}^k \rangle \downarrow^{\langle ()_1, \dots, ()_k \rangle} (\boxtimes^{st_2} \sigma'_i)_{i=1}^k \quad (11)$$

By Definition 0.2 we have

$$(\boxtimes^{st_1} \sigma_i)_{i=1}^k = \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]. \quad (12)$$

Then by Theorem 0.1 on  $\mathcal{P}'_1$  with (11), we get

$$\sigma'' = (\boxtimes^{st_2} \sigma'_i)_{i=1}^k \quad (13)$$

Therefore,  $\sigma''(st_2) = \sigma'_1(st_2) ++ \dots ++ \sigma'_k(st_2)$  by Definition 0.2.

Let  $\sigma'_i(st_2) = w'_i$  and  $\sigma''(st_2) = w'$ , then  $w' = w'_1 ++ \dots ++ w'_k$ .

Since  $\sigma'(st_2) = \sigma''(st_2) = w'$ , and  $\sigma'(s_2) = \sigma(s_2) = \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle$ , (same problem) we now have  $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2)) = (w', \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle)$ . With (7), we can construct

$$\mathcal{V} = \frac{(v'_i \triangleright_{\tau_2} w'_i)_{i=1}^k}{\{v'_1, \dots, v'_k\} \triangleright_{\{\tau_2\}} (w', \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle)}$$

as required.

By Lemma 0.1 on (12) we get  $\sigma_i \stackrel{st_1}{\sim} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$ , and similarly  $\sigma'_i \stackrel{st_2}{\sim} \sigma''$  from (13).

Since (8) implies

$$(\sigma'_i \stackrel{\leq s_0}{=} \sigma_i)_{i=1}^k$$

using Lemma 0.2 (k-1) times, we obtain

$$\sigma'' \stackrel{\leq s_0}{=} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle].$$

Therefore,  $\sigma' \stackrel{\leq s_0}{=} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \stackrel{\leq s_0}{=} \sigma$ .

(viii) TS:  $s_0 \leq s_1$

From (9) we immediately get  $s_0 \leq s_1 - 1 < s_1$ .

(ix) TS:  $\text{sids}((st_2, s_2)) < s_1$

From (2) we know  $s_2 < s_0$ , thus  $s_2 < s_0 \leq s_1$ . And we already have (10). Therefore,

$$\text{sids}((st_2, s_2)) = \text{sids}(st_2) \cup \{s_2\} < s_1.$$

- Case  $e = x$ .

We must have

$$\begin{aligned} \mathcal{T} &= \frac{}{\Gamma \vdash x : \tau} (\Gamma(x) = \tau) \\ \mathcal{E} &= \frac{}{\rho \vdash x \downarrow v} (\rho(x) = v) \end{aligned}$$

$$\mathcal{C} = \frac{}{\delta \vdash x \xrightarrow[s_0]{s_0} (\epsilon, st)} (\delta(x) = st)$$

So  $p = \epsilon$ .

Immediately we have  $\mathcal{P} = \frac{}{\langle \epsilon, \sigma \rangle \downarrow^{(\langle \rangle)} \sigma}$

So  $\sigma' = \sigma$ , which implies  $\sigma \xrightarrow[\leq s_0]{\leq s_0} \sigma$ .

From the assumption we already have  $v \triangleright_\tau \sigma(st)$ , and  $\mathbf{sids}(st) < s_0$ .

Finally it's clear that  $s_0 \leq s_0$ , and we are done.

- Case  $e = \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2$ .

We must have:

$$\begin{aligned} \mathcal{T} &= \frac{\mathcal{T}_1 \quad \mathcal{T}_2}{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau} \\ \mathcal{E} &= \frac{\mathcal{E}_1 \quad \mathcal{E}_2}{\rho \vdash e_1 \downarrow v_1 \quad \rho[x \mapsto v_1] \vdash e_2 \downarrow v} \\ \mathcal{C} &= \frac{\mathcal{C}_1 \quad \mathcal{C}_2}{\delta \vdash e_1 \xrightarrow[s'_0]{s_0} (p_1, st_1) \quad \delta[x \mapsto st_1] \vdash e_2 \xrightarrow[s_1]{s'_0} (p_2, st)} \end{aligned}$$

So  $p = p_1; p_2$ .

By IH on  $\mathcal{T}_1$  with  $\mathcal{E}_1, \mathcal{C}_1$ , we get

- (a)  $\mathcal{P}_1$  of  $\langle p_1, \sigma \rangle \downarrow^{(\langle \rangle)} \sigma_1$
- (b)  $\mathcal{V}_1$  of  $v_1 \triangleright_{\tau_1} \sigma_1(st_1)$
- (c)  $\sigma_1 \xrightarrow[\leq s_0]{\leq s_0} \sigma$
- (d)  $s_0 \leq s'_0$
- (e)  $\mathbf{sids}(st_1) < s'_0$

From (b), we know  $\rho[x \mapsto v_1](x) : \Gamma[x \mapsto \tau_1](x)$  and  $\rho[x \mapsto v_1](x) \triangleright_{\Gamma[x \mapsto \tau_1](x)} \sigma_1(\delta[x \mapsto st_1](x))$  must hold. From (e), we have  $\mathbf{sids}(\delta[x \mapsto st_1](x)) < s'_0$ .

Then by IH on  $\mathcal{T}_2$  with  $\mathcal{E}_2, \mathcal{C}_2$ , we get

- (f)  $\mathcal{P}_2$  of  $\langle p_2, \sigma_1 \rangle \downarrow^{(\langle \rangle)} \sigma_2$
- (g)  $\mathcal{V}_2$  of  $\sigma_2 \triangleright_\tau \sigma_2(st)$
- (h)  $\sigma_2 \xrightarrow[\leq s'_0]{\leq s'_0} \sigma_1$
- (i)  $s'_0 \leq s_1$
- (j)  $\mathbf{sids}(st) < s_1$

So we can construct:

$$\mathcal{P} = \frac{\mathcal{P}_1 \quad \mathcal{P}_2}{\langle p_1; p_2, \sigma \rangle \downarrow^{(\langle \rangle)} \sigma_2}$$

From (c), (d) and (h), it is clear that  $\sigma_2 \xrightarrow[\leq s_0]{\leq s_0} \sigma_1 \xrightarrow[\leq s_0]{\leq s_0} \sigma$ . From (d) and (i),  $s_0 \leq s_1$ .

Take  $\sigma' = \sigma_2$  (thus  $\mathcal{V} = \mathcal{V}_2$ ) and we are done.

- Case  $e = \phi(x_1, \dots, x_k)$   
We must have

$$\begin{aligned}
& \mathcal{T}_1 \\
\mathcal{T} &= \frac{\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau}{\Gamma \vdash \phi(x_1, \dots, x_k) : \tau} ((\Gamma(x_i) = \tau_i)_{i=1}^k) \\
& \mathcal{E}_1 \\
\mathcal{E} &= \frac{\vdash \phi(v_1, \dots, v_k) \downarrow v}{\rho \vdash \phi(x_1, \dots, x_k) \downarrow v} ((\rho(x_i) = v_i)_{i=1}^k) \\
& \mathcal{C}_1 \\
\mathcal{C} &= \frac{\vdash \phi(st_1, \dots, st_k) \xrightarrow[s_1]{s_0} (p, st)}{\delta \vdash \phi(x_1, \dots, x_k) \xrightarrow[s_1]{s_0} (p, st)} ((\delta(x_i) = st_i)_{i=1}^k)
\end{aligned}$$

From our assumption (iv), for all  $i \in \{1, \dots, k\}$ :

- (a)  $\vdash \rho(x_i) : \Gamma(x_i)$ , that is,  $\vdash v_i : \tau_i$
- (b)  $\mathbf{sids}(\delta(x_i)) \leq s_0$ , that is,  $\mathbf{sids}(st_i) \leq s_0$
- (c)  $\rho(x_i) \triangleright_{\Gamma(x_i)} \sigma(st_i)$ , that is,  $v_i \triangleright_{\tau_i} \sigma(st_i)$

So using Lemma 0.7 on  $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1, (a), (b)$  and (c) gives us exactly what we shall show.

□