SNESL formalization

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0 Level-0

Draft version 0.0.6:

- bug fix in the main correctness theroem (but not all fixed)
- changed the definition of store similarity and related lemmas
- some other trivial bug fixes, and adjustment of some subsection

0.1 Source language syntax

SNESL Expressions:

$$e ::= x \mid \textbf{let } x = e_1 \textbf{ in } e_2 \mid \phi(x_1, ..., x_k) \mid \{e : x \textbf{ in } y \textbf{ using } \cdot\}$$

$$\phi = \textbf{const}_n \mid \textbf{iota} \mid \textbf{plus}$$

Values:

$$n \in \mathbf{Z}$$

$$v ::= n \mid \{v_1, ..., v_k\}$$

0.2 Type system

$$\tau ::= \textbf{int} | \{\tau_1\}$$

Type environment $\Gamma = [x_1 \mapsto \tau_1, ..., x_i \mapsto \tau_i]$.

- Expression typing rules:

  Judgment $\boxed{\Gamma \vdash e : \tau}$

$$\frac{}{\Gamma \vdash x : \tau} (\Gamma(x) = \tau) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : \tau}$$

$$\frac{\phi : (\tau_1, ..., \tau_k) \to \tau}{\Gamma \vdash \phi(x_1, ..., x_k) : \tau} ((\Gamma(x_i) = \tau_i)_{i=1}^k) \qquad \frac{[x \mapsto \tau_1] \vdash e : \tau}{\Gamma \vdash \{e : x \textbf{ in } y \textbf{ using } \cdot\} : \{\tau\}} (\Gamma(y) = \{\tau_1\})$$

- Auxiliary Judgment $\boxed{\phi : (\tau_1, ..., \tau_k) \to \tau}$

$$\frac{}{\textbf{const}_n : () \to \textbf{int}} \qquad \frac{}{\textbf{iota} : (\textbf{int}) \to \{\textbf{int}\}} \qquad \frac{}{\textbf{plus} : (\textbf{int}, \textbf{int}) \to \textbf{int}}$$

- Value typing rules:

  Judgment $\boxed{v : \tau}$

$$\frac{}{n : \mathbf{int}} \qquad \frac{(v_i : \tau)_{i=1}^{k}}{\{v_1, ..., v_k\} : \{\tau\}}$$

0.3 Source language semantics

$\rho = [x_1 \mapsto v_1, ..., x_i \mapsto v_i]$

- Judgment $\boxed{\rho \ \vdash \ e \downarrow v}$

$$\frac{}{\rho \ \vdash \ x \downarrow v}\ (\rho(x) = v) \qquad \frac{\rho \ \vdash \ e_1 \downarrow v_1 \qquad \rho[x \mapsto v_1] \ \vdash \ e_2 \downarrow v}{\rho \ \vdash \ \mathbf{let}\ e_1 = x\ \mathbf{in}\ e_2 \downarrow v}$$

$$\frac{\phi(v_1, ..., v_k) \ \vdash \ v}{\rho \ \vdash \ \phi(x_1, ..., x_k) \downarrow v}\ ((\rho(x_i) = v_i)_{i=1}^{k})$$

$$\frac{([x \mapsto v_i] \ \vdash \ e \downarrow v_i')_{i=1}^{k}}{\rho \ \vdash \ \{e : x\ \mathbf{in}\ y\ \mathbf{using}\ \cdot\} \downarrow \{v_1', ..., v_k'\}}\ (\rho(y) = \{v_1, ..., v_k\})$$

- Auxiliary Judgment $\boxed{\phi(v_1, ..., v_k) \ \vdash \ v}$

$$\frac{}{\mathbf{const}_n() \ \vdash \ n} \qquad \frac{}{\mathbf{iota}(n) \ \vdash \ \{0, 1, ..., n-1\}}\ (n \geq 0)$$

$$\frac{}{\mathbf{plus}(n_1, n_2) \ \vdash \ n_3}\ (n_3 = n_1 + n_2)$$

0.4 SVCODE syntax

(1) Stream id:

$$s \in \mathbf{SId} = \mathbf{N} = \{0, 1, 2...\}$$

(2) Stream tree:

$$\mathbf{STree} \ni st ::= s \mid (st_1, s)$$

(3) SVCODE operations:

$$\psi ::= \mathtt{Const_a} \mid \mathtt{ToFlags} \mid \mathtt{Usum} \mid \mathtt{MapTwo}_{\oplus} \mid \mathtt{ScanPlus}_{n_0}$$

where $\oplus$ stands for some binary operation on **int**.

(4) SVCODE program:

$$\begin{aligned} p ::= \ & \epsilon \\ & \mid s := \psi(s_1, ..., s_i) \\ & \mid st := \mathtt{WithCtrl}(s, p) \\ & \mid p_1; p_2 \end{aligned}$$

(5) Target language values:

$$b \in \{\mathtt{T}, \mathtt{F}\}$$

$$a ::= n \mid b \mid ()$$

$$\vec{b} = \langle b_1, ..., b_i \rangle$$
$$\vec{a} = \langle a_1, ..., a_i \rangle$$
$$\mathbf{SVal} \ni w ::= \vec{a} \mid (w, \vec{b})$$

(6) Some notations and operations:

- For some a_0 and $\vec{a} = \langle a_1, ..., a_i \rangle$, let $\langle a_0 | \vec{a} \rangle = \langle a_0, a_1, ..., a_i \rangle$.
- $++ : \mathbf{SVal} \to \mathbf{SVal} \to \mathbf{SVal}$
  $\langle a_1, ..., a_i \rangle ++ \langle a'_1, ..., a'_j \rangle = \langle a_1, ..., a_i, a'_1, ..., a'_j \rangle$
  $(w_1, \vec{b}_1) ++ (w_2, \vec{b}_2) = (w_1 ++ w_2, \vec{b}_1 ++ \vec{b}_2)$
- `sids` is a function that converts a $st \in \mathbf{STree}$ to a set of $s \in \mathbf{SId}$:
  $\texttt{sids}(s) = \{s\}$
  $\texttt{sids}((st, s)) = \texttt{sids}(st) \cup \{s\}$
- For some set of **SId**, t, and some $s \in \mathbf{SId}$, let $t \lessdot s$ denote $\forall s' \in t. s' < s$.

0.5 SVCODE semantics

SVCODE runtime environment $\sigma = [s_1 \mapsto \vec{a}_1, ..., s_i \mapsto \vec{a}_i]$.

- Judgment $\boxed{\langle p, \sigma \rangle \downarrow^{\vec{c}} \sigma'}$

  $\vec{c}$ is the control stream.

  P-Empty : $\dfrac{}{\langle \epsilon, \sigma \rangle \downarrow^{\vec{c}} \sigma}$

  P-Xducer : $\dfrac{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{c}} \vec{a}}{\langle s := \psi(s_1, ..., s_k), \sigma \rangle \downarrow^{\vec{c}} \sigma[s \mapsto \vec{a}]} \; ((\sigma(s_i) = \vec{a}_i)_{i=1}^{k})$

  P-Wc-Emp : $\dfrac{}{\langle st := \texttt{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{c}} \sigma[s_1 \mapsto \langle\rangle, ..., s_i \mapsto \langle\rangle]} \; (\sigma(s) = \langle\rangle, \texttt{sids}(st) = \{s_1, ..., s_i\})$

  P-Wc-Nonemp : $\dfrac{\langle p, \sigma \rangle \downarrow^{\vec{a}_s} \sigma''}{\langle st := \texttt{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{c}} \sigma[s_1 \mapsto \sigma''(s_1), ..., s_i \mapsto \sigma''(s_i)]} \begin{pmatrix} \sigma(s) = \vec{a}_s = \langle a_0 | \vec{a} \rangle \\ \texttt{sids}(st) = \{s_1, ..., s_i\} \end{pmatrix}$

  P-Seq : $\dfrac{\langle p_1, \sigma \rangle \downarrow^{\vec{c}} \sigma'' \qquad \langle p_2, \sigma'' \rangle \downarrow^{\vec{c}} \sigma'}{\langle p_1; p_2, \sigma \rangle \downarrow^{\vec{c}} \sigma'}$

- *Transducer* semantics:

  Judgment $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{c}} \vec{a}}$

  P-X-Loop : $\dfrac{\psi(\vec{a}_{11}, ..., \vec{a}_{k1}) \Downarrow \vec{a}_1 \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \downarrow^{\vec{c}} \vec{a}_2}{\psi(\vec{a}_{11} ++ \vec{a}_{12}, ..., \vec{a}_{k1} ++ \vec{a}_{k2}) \downarrow^{\langle a_0 | \vec{c} \rangle} \vec{a}} \; (\vec{a} = \vec{a}_1 ++ \vec{a}_2)$

  P-X-Termi : $\dfrac{}{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\langle\rangle} \langle\rangle}$

- Transducer *block* semantics:

  Judgment $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow \vec{a}}$

  P-Const : $\dfrac{}{\texttt{Const}_{\texttt{a}}() \Downarrow \langle a \rangle}$ $\qquad$ P-ToFlags : $\dfrac{}{\texttt{ToFlags}(\langle n \rangle) \Downarrow \langle \texttt{F}_1, ..., \texttt{F}_n, \texttt{T} \rangle}$

$$\text{P-MapTwo}: \frac{}{\texttt{MapTwo}_{\oplus}(\langle n_1\rangle, \langle n_2\rangle) \Downarrow \langle n_3\rangle}\ (n_3 = n_1 \oplus n_2)$$

$$\text{P-UsumF}: \frac{\texttt{Usum}(\vec{b}) \Downarrow \vec{a}}{\texttt{Usum}(\langle \texttt{F}|\vec{b}\rangle) \Downarrow \langle ()|\vec{a}\rangle} \qquad \text{P-UsumT}: \frac{}{\texttt{Usum}(\langle \texttt{T}\rangle) \Downarrow \langle\rangle}$$

$$\text{P-ScanF}: \frac{\texttt{ScanPlus}_{n_0+n}(\vec{b}, \vec{a}) \Downarrow \vec{a}'}{\texttt{ScanPlus}_{n_0}(\langle \texttt{F}|\vec{b}\rangle, \langle n|\vec{a}\rangle) \Downarrow \langle n_0|\vec{a}'\rangle} \qquad \text{P-ScanT}: \frac{}{\texttt{ScanPlus}_{n_0}(\langle \texttt{T}\rangle, \langle\rangle) \Downarrow \langle\rangle}$$

Or if we want to use *unary* semantics maybe for later:

$$\frac{\psi(\langle \texttt{F}\rangle, ..., \vec{a}_{k1}) \downdownarrows \vec{a}_1 \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi(\langle \texttt{F}\rangle{+\!+}\vec{a}_{12}, ..., \vec{a}_{k1}{+\!+}\vec{a}_{k2}) \Downarrow \vec{a}}\ (\vec{a} = \vec{a}_1{+\!+}\vec{a}_2)$$

$$\frac{\psi(\langle \texttt{T}\rangle, ..., \vec{a}_k) \downdownarrows \vec{a}}{\psi(\langle \texttt{T}\rangle, ..., \vec{a}_k) \Downarrow \vec{a}}$$

– Transducer *unary* semantics:

Judgment $\boxed{\psi(\langle b\rangle, ..., \vec{a}_k) \downdownarrows \vec{a}}$

$$\frac{}{\texttt{Usum}(\langle \texttt{F}\rangle) \downdownarrows \langle ()\rangle} \qquad \frac{}{\texttt{Usum}(\langle \texttt{T}\rangle) \downdownarrows \langle\rangle}$$

– Transducer block with *accumulator*:

Judgment $\boxed{\psi_n(\vec{a}_1, ..., \vec{a}_k) \Downarrow \vec{a}}$

$$\frac{\psi_{n_0}(\langle \texttt{F}\rangle, ..., \vec{a}_{k1}) \downdownarrows^{n_0'} \langle n_1\rangle \qquad \psi_{n_0'}(\vec{a}_{12}, ..., \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi_{n_0}(\langle \texttt{F}\rangle{+\!+}\vec{a}_{12}, ..., \vec{a}_{k1}{+\!+}\vec{a}_{k2}) \Downarrow \langle n_1\rangle{+\!+}\vec{a}_2}$$

$$\frac{\psi_{n_0}(\langle \texttt{T}\rangle, ..., \vec{a}_k) \downdownarrows \vec{a}}{\psi_{n_0}(\langle \texttt{T}\rangle, ..., \vec{a}_k) \Downarrow \vec{a}}$$

– Transducer unary with *accumulator*:

Judgment $\boxed{\psi_n(\langle \texttt{F}\rangle, ..., \vec{a}_k) \downdownarrows^{n'} \vec{a}}$

$$\frac{}{\texttt{ScanPlus}_{n_0}(\langle \texttt{F}\rangle, \langle n\rangle) \downdownarrows^{n_0+n} \langle n_0\rangle}$$

Judgment $\boxed{\psi_n(\langle \texttt{T}\rangle, ..., \vec{a}_k) \downdownarrows \vec{a}}$

$$\frac{}{\texttt{ScanPlus}_{n_0}(\langle \texttt{T}\rangle, \langle\rangle) \downdownarrows \langle\rangle}$$

0.6 Definitions

We define some notations and operations related to σ:

(1) Let $\sigma_1 \overset{<s}{=\!=} \sigma_2$ denote $\forall s' < s. \sigma_1(s') = \sigma_2(s')$.

(2) ??? Judgment $\boxed{\sigma(st) = w}$

(3) $\mathbf{S}$ denotes a set of **SId**, can be $\emptyset$; $\mathbf{S}^+$ denotes a non-empty set of **SId**.

We define a binary relation $\overset{\mathbf{S}}{\sim}$ on stores to denote that two stores are *similar*: they have identical domains, and their bound values by $\mathbf{S}$ are the same. We call this $\mathbf{S}$ an *overlap* of these two stores.

Definition 0.1 (Stores similarity). $\sigma_1 \overset{\mathbf{S}}{\sim} \sigma_2$ *iff*
(1) $dom(\sigma_1) = dom(\sigma_2)$
(2) $\forall s \in \mathbf{S}. \sigma_1(s) = \sigma_2(s)$

According to this definition, it is only meaningful to have $\mathbf{S} \subseteq dom(\sigma_1)$ (or $dom(\sigma_2)$), and $\mathbf{S}$ can be empty. When $\mathbf{S} = dom(\sigma_1) = dom(\sigma_2)$, σ_1 and σ_2 are identical. It is easy to show that this relation $\overset{\mathbf{S}}{\sim}$ is transitive.

- If $\sigma_1 \overset{\mathbf{S}}{\sim} \sigma_2$ and $\sigma_2 \overset{\mathbf{S}}{\sim} \sigma_3$, then $\sigma_1 \overset{\mathbf{S}}{\sim} \sigma_3$.

We define another binary operation $\overset{\mathbf{S}}{\bowtie}$ on stores to denote a kind of specical concatenation of two similar stores: the *concatenation* of two similar stores is a new store, in which the bound values by $\mathbf{S}$ are from any of the parameter stores, and the others are the concatenation of the values from the two stores. In other words, a *concatenation* of two similar stores is only a concatenation of the bound values that *maybe* different in these stores.

Definition 0.2. $\sigma_1 \overset{\mathbf{S}}{\bowtie} \sigma_2 = \sigma$ *iff*
(1) $\sigma_1 \overset{\mathbf{S}}{\sim} \sigma_2$
(2) $\sigma(s) = \begin{cases} \sigma_i(s), & s \in \mathbf{S}, i \in \{1, 2\} \\ \sigma_1(s) +\!\!+ \sigma_2(s), & otherwise \end{cases}$

Lemma 0.1. *If* $\sigma_1 \overset{\mathbf{S}}{\bowtie} \sigma_2 = \sigma$*, then* $\sigma_1 \overset{\mathbf{S}}{\sim} \sigma$ *and* $\sigma_2 \overset{\mathbf{S}}{\sim} \sigma$.

This lemma says that the concatenation result of two similar stores is still similar to each of them.

Lemma 0.2. *If* $\sigma_1 \overset{\mathbf{S}_1}{\sim} \sigma_1'$, $\sigma_2 \overset{\mathbf{S}_2}{\sim} \sigma_2'$, $\sigma_1 \overset{<s}{=\!=} \sigma_2$*, and* $\sigma_1' \overset{<s}{=\!=} \sigma_2'$ *then* $\sigma_1 \overset{\mathbf{S}_1}{\bowtie} \sigma_1' \overset{<s}{=\!=} \sigma_2 \overset{\mathbf{S}_2}{\bowtie} \sigma_2'$.

Lemma 0.3. *If* $\psi(\vec{a}_{11}, ..., \vec{a}_{1k}) \downarrow^{\vec{c}_1} \vec{a}_1$*, and* $\psi(\vec{a}_{21}, ..., \vec{a}_{2k}) \downarrow^{\vec{c}_2} \vec{a}_2$*, then* $\psi(\vec{a}_{11} +\!\!+ \vec{a}_{21}, ..., \vec{a}_{1k} +\!\!+ \vec{a}_{2k}) \downarrow^{\vec{c}_1 +\!\!+ \vec{c}_2} \vec{a}_1 +\!\!+ \vec{a}_2$.

Lemma 0.4. *If* $\sigma_1 \overset{\mathbf{S}}{\sim} \sigma_2$, $\langle p, \sigma_1 \rangle \downarrow^{\vec{c}_1} \sigma_1'$ *(by some derivation* $\mathcal{P}_1$*),* $\langle p, \sigma_2 \rangle \downarrow^{\vec{c}_2} \sigma_2'$ *(by* $\mathcal{P}_2$*), and* $\mathbf{FV}(p) \cap \mathbf{S} = \emptyset$*, then* $\langle p, \sigma_1 \overset{\mathbf{S}}{\bowtie} \sigma_2 \rangle \downarrow^{\vec{c}_1 +\!\!+ \vec{c}_2} \sigma_1' \overset{\mathbf{S}}{\bowtie} \sigma_2'$ *(by* $\mathcal{P}$*).*

We need this lemma to prove that the results of single computations inside a comprehension body (i.e. p in the lemma) can be concatenated to express a parallel computation. From the other direction, we can consider this process as distributing or splitting the compuation p on even smaller degree of parallel computations, in which all the supplier streams, i.e., $\mathbf{FV}(p)$, are splitted to feed the transducers. The splitted parallel degrees are specified by the control streams, i.e., $\vec{c}_1$ and $\vec{c}_2$ in the lemma. Other untouched **SId**s in all σs (i.e., $\mathbf{S}$) have no change throughout the process.

0.7 SVCODE determinism theroem

Definition 0.3. $\vec{a}$ *is a prefix of* $\vec{a}'$ *if* $\vec{a} \sqsubseteq \vec{a}'$*:*
Judgment $\boxed{\vec{a} \sqsubseteq \vec{a}'}$

$$\frac{}{\langle\rangle \sqsubseteq \vec{a}} \qquad \frac{\vec{a} \sqsubseteq \vec{a}'}{\langle a_0 | \vec{a} \rangle \sqsubseteq \langle a_0 | \vec{a}' \rangle}$$

Lemma 0.5. *If*

(i) $(\vec{a}_i' \sqsubseteq \vec{a}_i)_{i=1}^k$ *and* $\psi(\vec{a}_1', ..., \vec{a}_k') \Downarrow \vec{a}'$*,*

(ii) $(\vec{a}_i'' \sqsubseteq \vec{a}_i)_{i=1}^k$ *and* $\psi(\vec{a}_1'', ..., \vec{a}_k'') \Downarrow \vec{a}''$

then

(i) $(\vec{a}'_i = \vec{a}''_i)_{i=1}^k$

(ii) $\vec{a}' = \vec{a}''$.

Lemma 0.6. *If* $\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{c}} \vec{a}$, *and* $\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{c}} \vec{a}'$, *then* $\vec{a} = \vec{a}'$.

Theorem 0.1 (SVCODE determinism). *If* $\langle p, \sigma \rangle \downarrow^{\vec{c}} \sigma'$ *and* $\langle p, \sigma \rangle \downarrow^{\vec{c}} \sigma''$, *then* $\sigma' = \sigma''$.

0.8 Translation

$\delta = [x_1 \mapsto st_1, ..., x_i \mapsto st_i]$

- Judgment $\boxed{\delta \ \vdash \ e \Rightarrow_{s_1}^{s_0} (p, st)}$

$$\frac{}{\delta \ \vdash \ x \Rightarrow_{s_0}^{s_0} (\epsilon, st)} \ (\delta(x) = st) \qquad \frac{\delta \ \vdash \ e_1 \Rightarrow_{s_0'}^{s_0} (p_1, st_1) \qquad \delta[x \mapsto st_1] \ \vdash \ e_2 \Rightarrow_{s_1}^{s_0'} (p_2, st)}{\delta \ \vdash \ \textbf{let } x = e_1 \textbf{ in } e_2 \Rightarrow_{s_1}^{s_0} (p_1; p_2, st)}$$

$$\frac{\phi(st_1, ..., st_k) \Rightarrow_{s_1}^{s_0} (p, st)}{\delta \ \vdash \ \phi(x_1, ..., x_k) \Rightarrow_{s_1}^{s_0} (p, st)} \ ((\delta(x_i) = st_i)_{i=1}^k)$$

$$\frac{[x \mapsto st_1] \ \vdash \ e \Rightarrow_{s_1}^{s_0+1} (p, st)}{\delta \ \vdash \ \{e : x \textbf{ in } y \textbf{ using } \cdot\} \Rightarrow_{s_1}^{s_0} (s_0 := \texttt{Usum}(s_2); st := \texttt{WithCtrl}(s_0, p), (st, s_2))} \ (\delta(y) = (st_1, s_2))$$

- Auxiliary Judgment $\boxed{\phi(st_1, ..., st_k) \Rightarrow_{s_1}^{s_0} (p, st)}$

$$\frac{}{\textbf{const}_n() \Rightarrow_{s_0+1}^{s_0} (s_0 := \texttt{Const}_\texttt{n}(), s_0)}$$

$$\frac{}{\textbf{iota}(s) \Rightarrow_{s_4}^{s_0} (p, (s_3, s_0))} \left(\begin{array}{l} s_{i+1} = s_i + 1, \forall i \in \{0, ..., 3\} \\ p = s_0 := \texttt{ToFlags}(s); \\ \quad\ s_1 := \texttt{Usum}(s_0); \\ \quad\ s_2 := \texttt{WithCtrl}(s_1, s_2 := \texttt{Const}_1()); \\ \quad\ s_3 := \texttt{ScanPlus}_0(s_0, s_2) \end{array} \right)$$

$$\frac{}{\textbf{plus}(s_1, s_2) \Rightarrow_{s_0+1}^{s_0} (s_0 := \texttt{MapTwo}_+(s_1, s_2), s_0)}$$

0.9 Value representation

- Judgment $\boxed{v \rhd_\tau w}$

$$\frac{}{n \rhd_{\textbf{int}} \langle n \rangle} \qquad \frac{(v_i \rhd_\tau w_i)_{i=1}^k}{\{v_1, ..., v_k\} \rhd_{\{\tau\}} (w, \langle \mathtt{F}_1, ..., \mathtt{F}_k, \mathtt{T} \rangle)} \ (w = w_1 {+\!\!+} ... {+\!\!+} w_k)$$

Lemma 0.7 (Value translation backwards determinism). *If* $v \rhd_\tau w$, $v' \rhd_\tau w$, *then* $v = v'$.

0.10 Correctness proof

Lemma 0.8 (???). *If* $\Gamma \ \vdash \ e : \{\tau\}$, $\rho \ \vdash \ e \downarrow \{v_1, ..., v_k\}$, *and* $\delta \ \vdash \ e \Rightarrow_{s_1}^{s_0} (p, (st, s))$, *then* $s \notin \texttt{sids}(st)$.

Lemma 0.9. *If*

(i) $\phi : (\tau_1, ..., \tau_k) \to \tau$ *(by some derivation* $\mathcal{T}$*)*

(ii) $\phi(v_1, ..., v_k) \ \vdash \ v$ *(by* $\mathcal{E}$*)*

(iii) $\phi(st_1, ..., st_k) \Rightarrow_{s_1}^{s_0} (p, st)$ *(by* $\mathcal{C}$*)*

(iv) $(v_i \rhd_{\tau_i} \sigma(st_i))_{i=1}^{k}$

(v) $\bigcup_{i=1}^{k} \mathtt{sids}(st_i) \lessdot s_0$

then

(vi) $\langle p, \sigma \rangle \downarrow^{\langle () \rangle} \sigma'$ *(by $\mathcal{P}$)*

(vii) $v \rhd_\tau \sigma'(st)$ *(by $\mathcal{R}$)*

(viii) $\sigma' \overset{\lessdot s_0}{=\!=} \sigma$

(ix) $s_0 \leq s_1$

(x) $\mathtt{sids}(st) \lessdot s_1$

Proof. By inducntion on the syntax of ϕ.

- Case $\phi = \mathbf{const}_n$
  There is only one possibility for each of $\mathcal{T}$, $\mathcal{E}$ and $\mathcal{C}$:

$$\mathcal{T} = \frac{}{\mathbf{const}_n : () \to \mathbf{int}}$$

$$\mathcal{E} = \frac{}{\vdash \mathbf{const}_n() \downarrow n}$$

$$\mathcal{C} = \frac{}{\mathbf{const}_n() \Rightarrow_{s_0+1}^{s_0} (s_0 := \mathtt{Const_n}(), s_0)}$$

  So $k = 0, \tau = \mathbf{int}, v = n, p = s_0 := \mathtt{Const_n}()$, $s_1 = s_0 + 1$, and $st = s_0$
  By P-Xducer, P-X-Loop, P-X-Termi and P-Const, we can construct $\mathcal{P}$ as follows:

$$\mathcal{P} = \frac{\dfrac{\dfrac{}{\mathtt{Const_n}() \Downarrow \langle n \rangle} \quad \dfrac{}{\mathtt{Const_n}() \downarrow^{\langle\rangle} \langle\rangle}}{\mathtt{Const_n}() \downarrow^{\langle () \rangle} \langle n \rangle}}{\langle s_0 := \mathtt{Const_n}(), \sigma \rangle \downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle n \rangle]}$$

  So $\sigma' = \sigma[s_0 \mapsto \langle n \rangle]$.
  Then we take $\mathcal{R} = \dfrac{}{n \rhd_{\mathbf{int}} \sigma'(s_0)}$.
  Also clearly, $\sigma' \overset{\lessdot s_0}{=\!=} \sigma$, $s_0 \leq s_0 + 1$, $\mathtt{sids}(s_0) \lessdot s_0 + 1$, and we are done.

- Case $\phi = \mathbf{plus}$
  We must have

$$\mathcal{T} = \frac{}{\mathbf{plus} : (\mathbf{int}, \mathbf{int}) \to \mathbf{int}}$$

$$\mathcal{E} = \frac{}{\vdash \mathbf{plus}(n_1, n_2) \downarrow n_3}$$

  where $n_3 = n_2 + n_1$, and

$$\mathcal{C} = \frac{}{\mathbf{plus}(s_1, s_2) \Rightarrow_{s_0+1}^{s_0} (s_0 := \mathtt{MapTwo}_+(s_1, s_2), s_0)}$$

  So $k = 2, \tau_1 = \tau_2 = \tau = \mathbf{int}, v_1 = n_1, v_2 = n_2, v = n_3, st_1 = s_1, st_2 = s_2, st = s_0, s_1 = s_0 + 1$ and $p = s_0 := \mathtt{MapTwo}_+(s_1, s_2)$.

  Assumption (iv) gives us $\overline{n_1 \rhd_{\mathbf{int}} \sigma(s_1)}$ and $\overline{n_2 \rhd_{\mathbf{int}} \sigma(s_2)}$, which implies $\sigma(s_1) = \langle n_1 \rangle$ and $\sigma(s_2) = \langle n_2 \rangle$ respectively.

  For (v) we have $s_1 < s_0$ and $s_2 < s_0$.

Then using P-XDUCER with $\sigma(s_1) = \langle n_1 \rangle$ and $\sigma(s_2) = \langle n_2 \rangle$, and using P-X-LOOP and P-X-TERMI, we can build $\mathcal{P}$ as follows:

$$\dfrac{\dfrac{}{\texttt{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \Downarrow \langle n_3 \rangle} \qquad \dfrac{}{\texttt{MapTwo}_+(\langle\rangle, \langle\rangle) \downarrow^{\langle\rangle} \langle\rangle}}{\dfrac{\texttt{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow^{\langle()\rangle} \langle n_3 \rangle}{\langle s_0 := \texttt{MapTwo}_+(s_1, s_2), \sigma \rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \langle n_3 \rangle]}}$$

Therefore, $\sigma' = \sigma[s_0 \mapsto \langle n_3 \rangle]$.

Now we can take $\mathcal{R} = \overline{n_3 \triangleright_{\texttt{int}} \sigma'(s_0)}$, and it is clear that $\sigma' \overset{\lessdot s_0}{=\!=} \sigma$, $s_0 \leq s_0 + 1$ and $\texttt{sids}(s_0) \lessdot s_0 + 1$ as required.

- Case $\phi = \textbf{iota}$

$\square$

Theorem 0.2. ***If***

(i) $\Gamma \vdash e : \tau$ *(by some derivation* $\mathcal{T}$*)*

(ii) $\rho \vdash e \downarrow v$ *(by some* $\mathcal{E}$*)*

(iii) $\delta \vdash e \Rightarrow^{s_0}_{s_1} (p, st)$ *(by some* $\mathcal{C}$*)*

(iv) $\forall x \in dom(\Gamma).\ \vdash \rho(x) : \Gamma(x) \wedge \texttt{sids}(\delta(x)) \lessdot s_0 \wedge \rho(x) \triangleright_{\Gamma(x)} \sigma(\delta(x))$
then

(v) $\langle p, \sigma \rangle \downarrow^{\langle()\rangle} \sigma'$ *(by some derivation* $\mathcal{P}$*)*

(vi) $v \triangleright_\tau \sigma'(st)$ *(by some* $\mathcal{R}$*)*

(vii) $\sigma' \overset{\lessdot s_0}{=\!=} \sigma$

(viii) $s_0 \leq s_1$

(ix) $\texttt{sids}(st) \lessdot s_1$

Proof. By induction on the syntax of e.

- Case $e = \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\}$.

  We must have:

  (i) $$\mathcal{T} = \dfrac{\overset{\mathcal{T}_1}{[x \mapsto \tau_1] \vdash e_1 : \tau_2}}{\Gamma \vdash \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\} : \{\tau_2\}}\ (\Gamma(y) = \{\tau_1\})$$

  (ii) $$\mathcal{E} = \dfrac{\overset{\mathcal{E}_i}{([x \mapsto v_i] \vdash e_1 \downarrow v'_i)^k_{i=1}}}{\rho \vdash \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\} \downarrow \{v'_1, ..., v'_k\}}\ (\rho(y) = \{v_1, ..., v_k\})$$

  (iii) $$\mathcal{C} = \dfrac{\overset{\mathcal{C}_1}{[x \mapsto st_1] \vdash e_1 \Rightarrow^{s_0+1}_{s_1} (p_1, st_2)}}{\delta \vdash \{e_1 : x \textbf{ in } y \textbf{ using } \cdot\} \Rightarrow^{s_0}_{s_1} (s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1), (st_2, s_2))}\ (\delta(y) = (st_1, s_2))$$

  So $p = (s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1)), \tau = \{\tau_2\}, v = \{v'_1, ..., v'_k\}, st = (st_2, s_2)$.

(iv) $\vdash\ \rho(y) : \Gamma(y)$ gives us $\vdash\ \{v_1, ..., v_k\} : \{\tau_1\}$, which must have the derivation:

$$\frac{(\vdash\ v_i : \tau_1)_{i=1}^{k}}{\vdash\ \{v_1, ..., v_k\} : \{\tau_1\}} \tag{1}$$

$\texttt{sids}(\delta(y)) \lessdot s_0$ gives us

$$\texttt{sids}(\delta(y)) = \texttt{sids}((st_1, s_2)) = \texttt{sids}(st_1) \cup \{s_2\} \lessdot s_0 \tag{2}$$

$\rho(y) \triangleright_{\Gamma(y)} \sigma(\delta(y)) = \{v_1, ..., v_k\} \triangleright_{\{\tau_1\}} \sigma((st_1, s_2))$ must have the derivation:

$$\frac{\begin{array}{c}\mathcal{R}_i \\ (v_i \triangleright_{\tau_1} w_i)_{i=1}^{k}\end{array}}{\{v_1, ..., v_k\} \triangleright_{\{\tau_1\}} (w, \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle)}\ (w = w_1 +\!\!+ ... +\!\!+ w_k) \tag{3}$$

therefore

$$\sigma(st_1) = w \tag{4}$$

and

$$\sigma(s_2) = \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle. \tag{5}$$

First we shall show:

(v) $\langle s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1), \sigma\rangle \downarrow^{\langle()\rangle} \sigma'$ by some $\mathcal{P}$

(vi) $\{v_1', ..., v_k'\} \triangleright_{\{\tau_2\}} \sigma'((st_2, s_2))$ by some $\mathcal{R}$

(vii) $\sigma' \overset{\lessdot s_0}{=\!=\!=} \sigma$

By P-Seq, we can build $\mathcal{P}$ as follow:

$$\frac{\begin{array}{c}\mathcal{P}_0 \\ \langle s_0 := \texttt{Usum}(s_2), \sigma\rangle \downarrow^{\langle()\rangle} \sigma_0\end{array} \qquad \begin{array}{c}\mathcal{P}_1 \\ \langle st_2 := \texttt{WithCtrl}(s_0, p_1), \sigma_0\rangle \downarrow^{\langle()\rangle} \sigma'\end{array}}{\langle s_0 := \texttt{Usum}(s_2); st_2 := \texttt{WithCtrl}(s_0, p_1), \sigma\rangle \downarrow^{\langle()\rangle} \sigma'}$$

By P-Xducer with $\sigma(s_2) = \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle$, we can continue to build $\mathcal{P}_0$ as follow:

$$\mathcal{P}_0 = \frac{\begin{array}{c}\mathcal{P}_0' \\ \texttt{Usum}(\langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle) \downarrow^{\langle()\rangle} \vec{a}\end{array}}{\langle s_0 := \texttt{Usum}(s_2), \sigma\rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \vec{a}]}$$

So $\sigma_0 = \sigma[s_0 \mapsto \vec{a}]$.
We split $\langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle$ into two parts: $\vec{b}_1 = \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle$ and $\vec{b}_2 = \langle\rangle$.
By P-X-loop and P-X-termi with $\vec{b}_1$ and $\vec{b}_2$, we continue building $\mathcal{P}_0'$ as follow:

$$\mathcal{P}_0' = \frac{\begin{array}{c}\mathcal{P}_0'' \\ \texttt{Usum}(\vec{b}_1) \Downarrow \vec{a}\end{array} \qquad \dfrac{}{\texttt{Usum}(\vec{b}_2) \downarrow^{\langle\rangle} \langle\rangle}}{\texttt{Usum}(\langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle) \downarrow^{\langle()\rangle} \vec{a}}$$

Then using P-UsumF k times and P-UsumT once, we obtain

$$\mathcal{P}_0'' = \frac{\dfrac{\dfrac{}{\texttt{Usum}(\langle \texttt{T}\rangle) \Downarrow \langle\rangle}}{\vdots}}{\dfrac{\texttt{Usum}(\langle \texttt{F}_2, ..., \texttt{F}_k, \texttt{T}\rangle) \Downarrow \langle()_2, ..., ()_k\rangle}{\texttt{Usum}(\langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T}\rangle) \Downarrow \langle()_1, ..., ()_k\rangle}}$$

Thus so far we have constructed $\mathcal{P}_0$ of $\langle s_0 := \texttt{Usum}(s_2), \sigma\rangle \downarrow^{\langle()\rangle} \sigma[s_0 \mapsto \langle()_1, ..., ()_k\rangle]$.

Since we have

$$\mathcal{T}_1 = \ [x \mapsto \tau_1] \ \vdash \ e_1 : \tau_2$$

$$(\mathcal{E}_i = \ [x \mapsto v_i] \ \vdash \ e_1 \Downarrow v_i' \qquad)_{i=1}^k$$

$$\mathcal{C}_1 = \ [x \mapsto st_1] \ \vdash \ e_1 \Rightarrow_{s_1}^{s_0+1} (p_1, st_2)$$

Let $\Gamma_1 = [x \mapsto \tau_1], \rho_i = [x \mapsto v_i]$ and $\delta_1 = [x \mapsto st_1]$.
From (1) and (2) it is clear that

$$\forall z \in dom(\Gamma_1). \ \vdash \ \rho_i(z) : \Gamma_1(z) \wedge \mathtt{sids}(\delta_1(z)) \lessdot s_0.$$

Let i range from 1 to k: we take $\sigma_i \overset{st_1}{\sim} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle]$ such that $\sigma_i(st_1) = w_i$.
From $\mathcal{R}_i$ in (3) we know that

$$\forall z \in dom(\Gamma_1). \rho_i(z) \rhd_{\Gamma_1(z)} \sigma_i(\delta_1(z)).$$

Then by IH (k times) on $\mathcal{T}_1$ with $\mathcal{E}_i$, $\mathcal{C}_1$ we obtain the following result:

$$(\langle p_1, \sigma_i \rangle \Downarrow^{\langle () \rangle} \sigma_i')_{i=1}^k \tag{6}$$

$$(v_i' \rhd_{\tau_2} \sigma_i'(st_2))_{i=1}^k \tag{7}$$

$$(\sigma_i' \xlongequal{< s_0+1} \sigma_i)_{i=1}^k \tag{8}$$

$$s_0 + 1 \leq s_1 \tag{9}$$

$$\mathtt{sids}(st_2) \lessdot s_1 \tag{10}$$

Assume $\mathtt{sids}(st_2) = \{s_1', ..., s_j'\}$.

There are two possibilities:

- Subcase $k = 0$, that is $\sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle](s_0) = \langle \rangle$.
  By P-Wc-Emp We build

  $$\mathcal{P}_1 = \frac{}{\langle st_2 := \mathtt{WithCtrl}(s_0, p_1), \sigma \rangle \Downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle \rangle, s_1' \mapsto \langle \rangle, ..., s_j' \mapsto \langle \rangle]}$$

  So in this subcase,

  $$\sigma' = \sigma[s_0 \mapsto \langle \rangle, s_1' \mapsto \langle \rangle, ..., s_j' \mapsto \langle \rangle].$$

  Since $k = 0$, then $v = \{\}$, $\sigma(s_2) = \langle \mathtt{T} \rangle$ (from (5)), we have
  $\sigma'(s_2) = \sigma(s_2) = \langle \mathtt{T} \rangle$ (?? not correct if $s_2 \in \mathtt{sids}(st_2)/\mathtt{sids}(st_1)$),
  and $\sigma'(st_2) = (...((\langle \rangle, \langle \rangle)_1, \langle \rangle)_2, ...)_{j-1}$.

  Therefore $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2))$, with which we construct

  $$\mathcal{R} = \frac{}{\{\} \rhd_{\{\tau_2\}} ((...(\langle \rangle, \langle \rangle)_1, ...)_{j-1}, \langle \mathtt{T} \rangle)}$$

  as required.

  Since $k = 0$, from (4) we know $\forall s' \in \mathtt{sids}(st_1). \sigma(s') = \langle \rangle$. For any $s' \in \mathtt{sids}(st_2)$ and $s' < s_0$, it must have $s' \in \mathtt{sids}(st_1)$ (because $codom(\delta_1) = \{st_1\}$), hence $\sigma(s') = \langle \rangle = \sigma'(s')$. Therefore,

  $$\sigma' \xlongequal{< s_0} \sigma.$$

- Subcase $k > 0$, that is $\sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] = \langle () | \vec{a} \rangle$ for some $\vec{a}$.
  By P-Wc-Nonemp, we take $\mathcal{P}_1 =$

  $$\frac{\dfrac{\mathcal{P}_1'}{\langle p_1, \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] \rangle \Downarrow^{\langle ()_1, ..., ()_k \rangle} \sigma''}}{\langle st_2 := \mathtt{WithCtrl}(s_0, p_1), \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] \rangle \Downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle, s_1' \mapsto \sigma''(s_1'), ..., s_j' \mapsto \sigma''(s_j')]}$$

So in this subcase

$$\sigma' = \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle, s'_1 \mapsto \sigma''(s'_1), ..., s'_j \mapsto \sigma''(s'_j)].$$

Using Lemma 0.4 (k-1) times on (6) gives us

$$\langle p_1, (\overset{st_1}{\bowtie}\ \sigma_i)_{i=1}^k \rangle \downarrow^{\langle ()_1, ..., ()_k \rangle} (\overset{st'_2}{\bowtie}\ \sigma'_i)_{i=1}^k \tag{11}$$

where $st'_2 = \texttt{sids}(st_1) \cup \texttt{sids}(st_2)$ (???) .
By Definition 0.2 we have

$$(\overset{st_1}{\bowtie}\ \sigma_i)_{i=1}^k = \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle]. \tag{12}$$

Then by Theorem 0.1 on $\mathcal{P}'_1$ with (11), we get

$$\sigma'' = (\overset{st'_2}{\bowtie}\ \sigma'_i)_{i=1}^k \tag{13}$$

Therefore, $\sigma''(st_2) = \sigma'_1(st_2) +\!\!+ ... +\!\!+ \sigma'_k(st_2)$ by Definition 0.2.
Let $\sigma'_i(st_2) = w'_i$ and $\sigma''(st_2) = w'$, then $w' = w'_1 +\!\!+ ... +\!\!+ w'_k$.

Since $\sigma'(st_2) = \sigma''(st_2) = w'$, and $\sigma'(s_2) = \sigma(s_2) = \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T} \rangle$, we now have $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2)) = (w', \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T} \rangle)$. With (7), we can construct

$$\mathcal{R} = \frac{(v'_i \rhd_{\tau_2} w'_i)_{i=1}^k}{\{v'_1, ..., v'_k\} \rhd_{\{\tau_2\}} (w', \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T} \rangle)}$$

as required.

By Lemma 0.1 on (12) we get $\sigma_i \overset{st_1}{\sim} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle]$, and similarly $\sigma'_i \overset{st'_2}{\sim} \sigma''$ from (13).

Since (8) implies

$$(\sigma'_i \overset{<s_0}{=\!=} \sigma_i)_{i=1}^k$$

using Lemma 0.2 (k-1) times, we obtain

$$\sigma'' \overset{<s_0}{=\!=} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle].$$

Therefore, $\sigma' \overset{<s_0}{=\!=} \sigma[s_0 \mapsto \langle ()_1, ..., ()_k \rangle] \overset{<s_0}{=\!=} \sigma$.

(viii) TS: $s_0 \leq s_1$
From (9) we immediately get $s_0 \leq s_1 - 1 < s_1$.

(ix) TS: $\texttt{sids}((st_2, s_2)) \lessdot s_1$
From (2) we know $s_2 < s_0$, thus $s_2 < s_0 \leq s_1$. And we already have (10). Therefore,

$$\texttt{sids}((st_2, s_2)) = \texttt{sids}(st_2) \cup \{s_2\} \lessdot s_1.$$

- Case $e = x$.
  We must have

$$\mathcal{T} = \frac{}{\Gamma \ \vdash \ x : \tau} (\Gamma(x) = \tau)$$
$$\mathcal{E} = \frac{}{\rho \ \vdash \ x \downarrow v} (\rho(x) = v)$$
$$\mathcal{C} = \frac{}{\delta \ \vdash \ x \Rightarrow_{s_0}^{s_0} (\epsilon, st)} (\delta(x) = st)$$

  So $p = \epsilon$.
  Immediately we have $\mathcal{P} = \frac{}{\langle \epsilon, \sigma \rangle \downarrow^{\langle () \rangle} \sigma}$
  So $\sigma' = \sigma$, which implies $\sigma' \overset{<s_0}{=\!=} \sigma$.
  From the assumption we already have $v \rhd_\tau \sigma(st)$, and $\texttt{sids}(st) \lessdot s_0$.
  Finally it's clear that $s_0 \leq s_0$, and we are done.

- Case $e = \textbf{let } x = e_1 \textbf{ in } e_2$.

  We must have:

$$\mathcal{T} = \cfrac{\overset{\mathcal{T}_1}{\Gamma \vdash e_1 : \tau_1} \qquad \overset{\mathcal{T}_2}{\Gamma[x \mapsto \tau_1] \vdash e_2 : \tau}}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : \tau}$$

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\rho \vdash e_1 \downarrow v_1} \qquad \overset{\mathcal{E}_2}{\rho[x \mapsto v_1] \vdash e_2 \downarrow v}}{\rho \vdash \textbf{let } x = e_1 \textbf{ in } e_2 \downarrow v}$$

$$\mathcal{C} = \cfrac{\overset{\mathcal{C}_1}{\delta \vdash e_1 \Rightarrow^{s_0}_{s_0'} (p_1, st_1)} \qquad \overset{\mathcal{C}_2}{\delta[x \mapsto st_1] \vdash e_2 \Rightarrow^{s_0'}_{s_1} (p_2, st)}}{\delta \vdash \textbf{let } x = e_1 \textbf{ in } e_2 \Rightarrow^{s_0}_{s_1} (p_1; p_2, st)}$$

  So $p = p_1; p_2$.

  By IH on $\mathcal{T}_1$ with $\mathcal{E}_1, \mathcal{C}_1$, we get

  (a) $\mathcal{P}_1$ of $\langle p_1, \sigma\rangle \downarrow^{\langle()\rangle} \sigma_1$
  (b) $\mathcal{R}_1$ of $v_1 \triangleright_{\tau_1} \sigma_1(st_1)$
  (c) $\sigma_1 \overset{\lessdot s_0}{=\!=} \sigma$
  (d) $s_0 \leq s_0'$
  (e) $\texttt{sids}(st_1) \lessdot s_0'$

  From (b), we know $\rho[x \mapsto v_1](x) : \Gamma[x \mapsto \tau_1](x)$ and $\rho[x \mapsto v_1](x) \triangleright_{\Gamma[x \mapsto \tau_1](x)} \sigma_1(\delta[x \mapsto st_1](x))$ must hold. From (e), we have $\texttt{sids}(\delta[x \mapsto st_1](x)) \lessdot s_0'$.

  Then by IH on $\mathcal{T}_2$ with $\mathcal{E}_2, \mathcal{C}_2$, we get

  (f) $\mathcal{P}_2$ of $\langle p_2, \sigma_1\rangle \downarrow^{\langle()\rangle} \sigma_2$
  (g) $\mathcal{R}_2$ of $\sigma_2 \triangleright_{\tau} \sigma_2(st)$
  (h) $\sigma_2 \overset{\lessdot s_0'}{=\!=} \sigma_1$
  (i) $s_0' \leq s_1$
  (j) $\texttt{sids}(st) \lessdot s_1$

  So we can construct:

$$\mathcal{P} = \cfrac{\overset{\mathcal{P}_1}{\langle p_1, \sigma\rangle \downarrow^{\langle()\rangle} \sigma_1} \qquad \overset{\mathcal{P}_2}{\langle p_2, \sigma_1\rangle \downarrow^{\langle()\rangle} \sigma_2}}{\langle p_1; p_2, \sigma\rangle \downarrow^{\langle()\rangle} \sigma_2}$$

  From (c), (d) and (h), it is clear that $\sigma_2 \overset{\lessdot s_0}{=\!=} \sigma_1 \overset{\lessdot s_0}{=\!=} \sigma$. From (d) and (i), $s_0 \leq s_1$.

  Take $\sigma' = \sigma_2$ (thus $\mathcal{R} = \mathcal{R}_2$) and we are done.

- Case $e = \phi(x_1, ..., x_k)$

  We must have

$$\mathcal{T} = \cfrac{\overset{\mathcal{T}_1}{\phi : (\tau_1, ..., \tau_k) \to \tau}}{\Gamma \vdash \phi(x_1, ..., x_k) : \tau} \; ((\Gamma(x_i) = \tau_i)_{i=1}^k)$$

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\vdash (v_1, ..., v_k)(\downarrow) \vdash v}}{\rho \vdash \phi(x_1, ..., x_k) \downarrow v} \; ((\rho(x_i) = v_i)_{i=1}^k)$$

$$\mathcal{C} = \cfrac{\begin{array}{c}\mathcal{C}_1\\ \phi(st_1, ..., st_k) \Rightarrow^{s_0}_{s_1} (p, st)\end{array}}{\delta \ \vdash \ \phi(x_1, ..., x_k) \Rightarrow^{s_0}_{s_1} (p, st)} \, ((\delta(x_i) = st_i)^k_{i=1})$$

From our assumption (iv), for all $i \in \{1, ..., k\}$:

(a) $\vdash \ \rho(x_i) : \Gamma(x_i)$, that is, $\vdash \ v_i : \tau_i$

(b) $\texttt{sids}(\delta(x_i)) \lessdot s_0$, that is, $\texttt{sids}(st_i) \lessdot s_0$

(c) $\rho(x_i) \rhd_{\Gamma(x_i)} \sigma(st_i)$, that is, $v_i \rhd_{\tau_i} \sigma(st_i)$

So using Lemma 0.9 on $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1, (a), (b)$ and (c) gives us exactly what we shall show.

$\square$