

SNESL formalization

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0 Level-0

Draft version 0.0.5: added the proof of the main correctness theorem and part of the proof of the built-in functions correctness theorem (page 6-12), and related definitions and lemmas.

0.1 Source language syntax

(Ignore empty sequence for now)

Expressions:

$$e ::= x \mid \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 \mid \phi(x_1, \dots, x_k) \mid \{e : x \ \mathbf{in} \ y \ \mathbf{using} \ \cdot\} \\ \phi = \mathbf{const}_n \mid \mathbf{iota} \mid \mathbf{plus}$$

Values:

$$n \in \mathbf{Z} \\ v ::= n \mid \{v_1, \dots, v_k\}$$

0.2 Type system

$$\tau ::= \mathbf{int} \mid \{\tau_1\}$$

Type environment $\Gamma = [x_1 \mapsto \tau_1, \dots, x_i \mapsto \tau_i]$.

- Expression typing rules:

Judgment $\boxed{\Gamma \vdash e : \tau}$

$$\frac{}{\Gamma \vdash x : \tau} (\Gamma(x) = \tau) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau}{\Gamma \vdash \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 : \tau} \\ \frac{\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau}{\Gamma \vdash \phi(x_1, \dots, x_k) : \tau} ((\Gamma(x_i) = \tau_i)_{i=1}^k) \qquad \frac{[x \mapsto \tau_1] \vdash e : \tau}{\Gamma \vdash \{e : x \ \mathbf{in} \ y \ \mathbf{using} \ \cdot\} : \{\tau\}} (\Gamma(y) = \{\tau_1\})$$

- Auxiliary Judgment $\boxed{\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau}$

$$\frac{}{\vdash \mathbf{const}_n : () \rightarrow \mathbf{int}} \qquad \frac{}{\vdash \mathbf{iota} : (\mathbf{int}) \rightarrow \{\mathbf{int}\}} \qquad \frac{}{\vdash \mathbf{plus} : (\mathbf{int}, \mathbf{int}) \rightarrow \mathbf{int}}$$

- Value typing rules:

Judgment $\boxed{\vdash v : \tau}$

$$\frac{}{\vdash n : \mathbf{int}} \qquad \frac{(\vdash v_i : \tau)_{i=1}^k}{\vdash \{v_1, \dots, v_k\} : \{\tau\}}$$

0.3 Source language semantics

$$\rho = [x_1 \mapsto v_1, \dots, x_i \mapsto v_i]$$

- Judgment $\boxed{\rho \vdash e \downarrow v}$

$$\frac{}{\rho \vdash x \downarrow v} (\rho(x) = v) \quad \frac{\rho \vdash e_1 \downarrow v_1 \quad \rho[x \mapsto v_1] \vdash e_2 \downarrow v}{\rho \vdash \mathbf{let} \ e_1 = x \ \mathbf{in} \ e_2 \downarrow v}$$

$$\frac{\vdash \phi(v_1, \dots, v_k) \downarrow v}{\rho \vdash \phi(x_1, \dots, x_k) \downarrow v} ((\rho(x_i) = v_i)_{i=1}^k)$$

$$\frac{([x \mapsto v_i] \vdash e \downarrow v'_i)_{i=1}^k}{\rho \vdash \{e : x \ \mathbf{in} \ y \ \mathbf{using} \ \cdot\} \downarrow \{v'_1, \dots, v'_k\}} (\rho(y) = \{v_1, \dots, v_k\})$$

- Auxiliary Judgment $\boxed{\vdash \phi(v_1, \dots, v_k) \downarrow v}$

$$\frac{}{\vdash \mathbf{const}_n() \downarrow n} \quad \frac{}{\vdash \mathbf{iota}(n) \downarrow \{0, 1, \dots, n-1\}} (n \geq 0)$$

$$\frac{}{\vdash \mathbf{plus}(n_1, n_2) \downarrow n_3} (n_3 = n_1 + n_2)$$

0.4 SVCODE syntax

- (1) Stream id:

$$s \in \mathbf{SId} = \mathbf{N} = \{0, 1, 2, \dots\}$$

- (2) Stream tree:

$$\mathbf{STree} \ni st ::= s \mid (st_1, s)$$

- (3) SVCODE operations:

$$\psi ::= \mathbf{Ctrl} \mid \mathbf{Const}_a \mid \mathbf{ToFlags} \mid \mathbf{Usum} \mid \mathbf{MapTwo}_{\oplus} \mid \mathbf{ScanPlus}_{n_0}$$

where \oplus stands for some binary operation on **int**.

- (4) SVCODE program:

$$\begin{aligned} p ::= & \epsilon \\ & \mid s := \psi(s_1, \dots, s_i) \\ & \mid st := \mathbf{WithCtrl}(s, p) \\ & \mid p_1; p_2 \end{aligned}$$

- (5) Target language values:

$$\begin{aligned} b & \in \{\mathbf{T}, \mathbf{F}\} \\ a & ::= n \mid b \mid () \\ \vec{b} & = \langle b_1, \dots, b_i \rangle \\ \vec{a} & = \langle a_1, \dots, a_i \rangle \\ \mathbf{SVal} \ni w & ::= \vec{a} \mid (w, \vec{b}) \end{aligned}$$

- (6) Some notations and operations:

- For some a_0 and $\vec{a} = \langle a_1, \dots, a_i \rangle$, let $\langle a_0 | \vec{a} \rangle = \langle a_0, a_1, \dots, a_i \rangle$.

- $++ : \mathbf{SVal} \rightarrow \mathbf{SVal} \rightarrow \mathbf{SVal}$
 $\langle a_1, \dots, a_i \rangle ++ \langle a'_1, \dots, a'_i \rangle = \langle a_1, \dots, a_i, a'_1, \dots, a'_i \rangle$
 $(w_1, \vec{b}_1) ++ (w_2, \vec{b}_2) = (w_1 ++ w_2, \vec{b}_1 ++ \vec{b}_2)$
- **sids** is a function that converts a $st \in \mathbf{STree}$ to a set of $s \in \mathbf{SId}$:
 $\mathbf{sids}(s) = \{s\}$
 $\mathbf{sids}((st, s)) = \mathbf{sids}(st) \cup \{s\}$
- For some set of **SId**, t , and some $s \in \mathbf{SId}$, let $t \leq s$ denote $\forall s' \in t. s' \leq s$.

0.5 SVCODE semantics

SVCODE runtime environment $\sigma = [s_1 \mapsto \vec{a}_1, \dots, s_i \mapsto \vec{a}_i]$.

We define some notations and operations related to σ :

(1) Let $\sigma_1 \stackrel{\leq s}{=} \sigma_2$ denote $\forall s' \leq s. \sigma_1(s') = \sigma_2(s')$.

(2) Judgment $\boxed{\sigma(st) = w}$

$$\frac{}{\sigma(s) = \vec{a}} \quad \frac{\sigma(st) = w \quad \overline{\sigma(s) = \vec{a}}}{\sigma((st, s)) = (w, \vec{a})}$$

Definition 0.1. $\sigma_1 \stackrel{st}{\sim} \sigma_2$ iff

- (1) $\text{dom}(\sigma_1) = \text{dom}(\sigma_2)$
- (2) $\forall s \in (\text{dom}(\sigma_1) - \mathbf{sids}(st)). \sigma_1(s) = \sigma_2(s)$

It is easy to show that this relation $\stackrel{st}{\sim}$ is commutative, transitive and associative.

Definition 0.2. $\sigma_1 \boxtimes^{st} \sigma_2 = \sigma$ iff

- (1) $\sigma_1 \stackrel{st}{\sim} \sigma_2$
- (2) $\sigma(s) = \begin{cases} \sigma_1(s) ++ \sigma_2(s), & s \in \mathbf{sids}(st) \\ \sigma_1(s), & \text{otherwise} \end{cases}$

Lemma 0.1. If $\sigma_1 \boxtimes^{st} \sigma_2 = \sigma$, then $\sigma_1 \stackrel{st}{\sim} \sigma$ and $\sigma_2 \stackrel{st}{\sim} \sigma$.

Lemma 0.2. If $\sigma_1 \stackrel{st_1}{\sim} \sigma'_1$, $\sigma_2 \stackrel{st_2}{\sim} \sigma'_2$, $\sigma_1 \stackrel{\leq s}{=} \sigma_2$, and $\sigma'_1 \stackrel{\leq s}{=} \sigma'_2$ then $\sigma_1 \boxtimes^{st_1} \sigma'_1 \stackrel{\leq s}{=} \sigma_2 \boxtimes^{st_2} \sigma'_2$.

SVCODE operational semantics:

- Judgment $\boxed{\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$

\vec{a}_c is the control stream.

$$\frac{}{\langle \epsilon, \sigma \rangle \downarrow^{\vec{a}_c} \sigma} \quad \frac{\psi(\vec{a}_1, \dots, \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}{\langle s := \psi(s_1, \dots, s_k), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s \mapsto \vec{a}]} ((\sigma(s_i) = \vec{a}_i)_{i=1}^k)$$

$$\frac{}{\langle st := \text{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \langle \rangle, \dots, s_i \mapsto \langle \rangle]} (\sigma(s) = \langle \rangle, \mathbf{sids}(st) = \{s_1, \dots, s_i\})$$

$$\frac{\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma''}{\langle st := \text{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \sigma''(s_1), \dots, s_i \mapsto \sigma''(s_i)]} \left(\begin{array}{l} \sigma(s) = \vec{a}_s = \langle a_0 | \vec{a} \rangle \\ \mathbf{sids}(st) = \{s_1, \dots, s_i\} \end{array} \right)$$

$$\frac{\langle p_1, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'' \quad \langle p_2, \sigma'' \rangle \downarrow^{\vec{a}_c} \sigma'}{\langle p_1; p_2, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$$

- *Transducer semantics*:

$$\text{Judgment } \boxed{\psi(\vec{a}_1, \dots, \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}$$

$$\frac{\psi(\vec{a}_{11}, \dots, \vec{a}_{k1}) \downarrow \vec{a}_1 \quad \psi(\vec{a}_{12}, \dots, \vec{a}_{k2}) \downarrow^{\vec{a}_c} \vec{a}_2}{\psi(\vec{a}_{11} ++ \vec{a}_{12}, \dots, \vec{a}_{k1} ++ \vec{a}_{k2}) \downarrow^{\langle a_0 | \vec{a}_c \rangle} \vec{a}} \quad (\vec{a} = \vec{a}_1 ++ \vec{a}_2)$$

$$\frac{}{\psi(\vec{a}_1, \dots, \vec{a}_k) \downarrow^{\langle \rangle} \langle \rangle}$$

- *Transducer block semantics*:

$$\text{Judgment } \boxed{\psi(\vec{a}_1, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

$$\frac{}{\text{Const}_a() \Downarrow \langle a \rangle} \quad \frac{}{\text{ToFlags}(\langle n \rangle) \Downarrow \langle F_1, \dots, F_n, T \rangle} \quad \frac{}{\text{MapTwo}_{\oplus}(\langle n_1 \rangle, \langle n_2 \rangle) \Downarrow \langle n_3 \rangle} \quad (n_3 = n_1 \oplus n_2)$$

$$\text{P-USUMF} : \frac{\text{Usum}(\vec{b}) \Downarrow \vec{a}}{\text{Usum}(\langle F | \vec{b} \rangle) \Downarrow \langle () | \vec{a} \rangle} \quad \text{P-USUMT} : \frac{}{\text{Usum}(\langle T \rangle) \Downarrow \langle \rangle}$$

$$\text{P-SCANF} : \frac{\text{ScanPlus}_{n_0+n}(\vec{b}, \vec{a}) \Downarrow \vec{a}'}{\text{ScanPlus}_{n_0}(\langle F | \vec{b} \rangle, \langle n | \vec{a} \rangle) \Downarrow \langle n_0 | \vec{a}' \rangle} \quad \text{P-SCANT} : \frac{}{\text{ScanPlus}_{n_0}(\langle T \rangle, \langle \rangle) \Downarrow \langle \rangle}$$

Or if we want to use *unary* semantics maybe for later:

$$\frac{\psi(\langle F \rangle, \dots, \vec{a}_{k1}) \Downarrow \vec{a}_1 \quad \psi(\vec{a}_{12}, \dots, \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi(\langle F \rangle ++ \vec{a}_{12}, \dots, \vec{a}_{k1} ++ \vec{a}_{k2}) \Downarrow \vec{a}} \quad (\vec{a} = \vec{a}_1 ++ \vec{a}_2)$$

$$\frac{\psi(\langle T \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}{\psi(\langle T \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

- Transducer *unary* semantics:

$$\text{Judgment } \boxed{\psi(\langle b \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

$$\frac{}{\text{Usum}(\langle F \rangle) \Downarrow \langle () \rangle} \quad \frac{}{\text{Usum}(\langle T \rangle) \Downarrow \langle \rangle}$$

- Transducer block with *accumulator*:

$$\text{Judgment } \boxed{\psi_n(\vec{a}_1, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

$$\frac{\psi_{n_0}(\langle F \rangle, \dots, \vec{a}_{k1}) \Downarrow^{n'_0} \langle n_1 \rangle \quad \psi_{n'_0}(\vec{a}_{12}, \dots, \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi_{n_0}(\langle F \rangle ++ \vec{a}_{12}, \dots, \vec{a}_{k1} ++ \vec{a}_{k2}) \Downarrow \langle n_1 \rangle ++ \vec{a}_2}$$

$$\frac{\psi_{n_0}(\langle T \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}{\psi_{n_0}(\langle T \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}}$$

- Transducer unary with *accumulator*:

$$\text{Judgment } \boxed{\psi_n(\langle F \rangle, \dots, \vec{a}_k) \Downarrow^{n'} \vec{a}}$$

$$\begin{array}{c}
\hline
\text{ScanPlus}_{n_0}(\langle \mathbf{F} \rangle, \langle n \rangle) \Downarrow^{n_0+n} \langle n_0 \rangle \\
\hline
\text{Judgment } \boxed{\psi_n(\langle \mathbf{T} \rangle, \dots, \vec{a}_k) \Downarrow \vec{a}} \\
\hline
\text{ScanPlus}_{n_0}(\langle \mathbf{T} \rangle, \langle \rangle) \Downarrow \langle \rangle \\
\hline
\end{array}$$

Lemma 0.3. If $\sigma_1 \stackrel{st}{\sim} \sigma_2$, $\langle p, \sigma_1 \rangle \downarrow^{\vec{a}_1} \sigma'_1$, $\langle p, \sigma_2 \rangle \downarrow^{\vec{a}_2} \sigma'_2$, then $\exists st'. \langle p, \sigma_1 \bowtie^{st} \sigma_2 \rangle \downarrow^{\vec{a}_1 + \vec{a}_2} \sigma'_1 \bowtie^{st'} \sigma'_2$

Definition 0.3. \vec{a} is a prefix of \vec{a}' if $\vec{a} \sqsubseteq \vec{a}'$:

Judgment $\boxed{\vec{a} \sqsubseteq \vec{a}'}$

$$\frac{}{\langle \rangle \sqsubseteq \vec{a}} \quad \frac{\vec{a} \sqsubseteq \vec{a}'}{\langle a_0 | \vec{a} \rangle \sqsubseteq \langle a_0 | \vec{a}' \rangle}$$

Lemma 0.4. If

- (i) $(\vec{a}'_i \sqsubseteq \vec{a}_i)_{i=1}^k$ and $\psi(\vec{a}'_1, \dots, \vec{a}'_k) \Downarrow \vec{a}'$,
- (ii) $(\vec{a}''_i \sqsubseteq \vec{a}_i)_{i=1}^k$ and $\psi(\vec{a}''_1, \dots, \vec{a}''_k) \Downarrow \vec{a}''$

then

- (i) $(\vec{a}'_i = \vec{a}''_i)_{i=1}^k$
- (ii) $\vec{a}' = \vec{a}''$.

Theorem 0.1 (SVCODE determinism). If $\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'$ and $\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma''$, then $\sigma' = \sigma''$.

0.6 Translation

$$\delta = [x_1 \mapsto st_1, \dots, x_i \mapsto st_i]$$

- Judgment $\boxed{\delta \vdash e \xrightarrow[s_1]{s_0} (p, st)}$

$$\begin{array}{c}
\frac{}{\delta \vdash x \xrightarrow[s_0]{s_0} (\epsilon, st)} (\delta(x) = st) \quad \frac{\delta \vdash e_1 \xrightarrow[s'_0]{s_0} (p_1, st_1) \quad \delta[x \mapsto st_1] \vdash e_2 \xrightarrow[s_1]{s'_0} (p_2, st)}{\delta \vdash \text{let } x = e_1 \text{ in } e_2 \xrightarrow[s_1]{s_0} (p_1; p_2, st)} \\
\\
\frac{\vdash \phi(st_1, \dots, st_k) \xrightarrow[s_1]{s_0} (p, st)}{\delta \vdash \phi(x_1, \dots, x_k) \xrightarrow[s_1]{s_0} (p, st)} ((\delta(x_i) = st_i)_{i=1}^k) \\
\\
\frac{[x \mapsto st_1] \vdash e \xrightarrow[s_1]{s_0+1} (p, st)}{\delta \vdash \{e : x \text{ in } y \text{ using } \cdot\} \xrightarrow[s_1]{s_0} (s_0 := \text{Usum}(s_2); st := \text{WithCtrl}(s_0, p), (st, s_2))} (\delta(y) = (st_1, s_2))
\end{array}$$

- Auxiliary Judgment $\boxed{\vdash \phi(st_1, \dots, st_k) \xrightarrow[s_1]{s_0} (p, st)}$

$$\begin{array}{c}
\frac{}{\vdash \text{const}_n() \xrightarrow[s_0+1]{s_0} (s_0 := \text{Const}_n(), s_0)} \\
\\
\frac{\vdash \text{iota}(s) \xrightarrow[s_4]{s_0} (p, (s_3, s_0))}{\left(\begin{array}{l} s_{i+1} = s_i + 1 \\ p = s_0 := \text{ToFlags}(s); \\ s_1 := \text{Usum}(s_0); \\ s_2 := \text{WithCtrl}(s_1, s_2 := \text{Const}_1()); \\ s_3 := \text{ScanPlus}_0(s_0, s_2) \end{array} \right)} \\
\\
\frac{}{\vdash \text{plus}(s_1, s_2) \xrightarrow[s_0+1]{s_0} (s_0 := \text{MapTwo}_+(s_1, s_2), s_0)}
\end{array}$$

0.7 Value representation

- Judgment $\boxed{v \triangleright_{\tau} w}$

$$\frac{}{n \triangleright_{\mathbf{int}} \langle n \rangle} \quad \frac{(v_i \triangleright_{\tau} w_i)_{i=1}^k}{\{v_1, \dots, v_k\} \triangleright_{\{\tau\}} (w, \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle)} (w = w_1 ++ \dots ++ w_k)$$

Lemma 0.5. *If $v \triangleright_{\tau} w$, $v' \triangleright_{\tau} w$, then $v = v'$.*

0.8 Correctness proof

Lemma 0.6 (???). *If $\Gamma \vdash e : \{\tau\}$, $\rho \vdash e \downarrow \{v_1, \dots, v_k\}$, and $\delta \vdash e \xrightarrow[s_1]{s_0} (p, (st, s))$, then $s \notin \mathbf{sids}(st)$.*

Lemma 0.7. *If*

(i) $\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau$ (by some derivation \mathcal{T})

(ii) $\vdash \phi(v_1, \dots, v_k) \downarrow v$ (by \mathcal{E})

(iii) $\vdash \phi(st_1, \dots, st_k) \xrightarrow[s_1]{s_0} (p, st)$ (by \mathcal{C})

(iv) $(v_i \triangleright_{\tau_i} \sigma(st_i))_{i=1}^k$

(v) $\bigcup_{i=1}^k \mathbf{sids}(st_i) \leq s_0$

then

(vi) $\langle p, \sigma \rangle \downarrow^{(\langle \rangle)} \sigma'$ (by \mathcal{P})

(vii) $v \triangleright_{\tau} \sigma'(st)$ (by \mathcal{V})

(viii) $\sigma' \xrightarrow[\leq s_0]{} \sigma$

(ix) $s_0 \leq s_1$

(x) $\mathbf{sids}(st) \leq s_1$

Proof. By induction on the syntax of ϕ .

- Case $\phi = \mathbf{const}_n$

There is only one possibility for each of \mathcal{T} , \mathcal{E} and \mathcal{C} :

$$\begin{aligned} \mathcal{T} &= \overline{\vdash \mathbf{const}_n : () \rightarrow \mathbf{int}} \\ \mathcal{E} &= \overline{\vdash \mathbf{const}_n() \downarrow n} \\ \mathcal{C} &= \overline{\vdash \mathbf{const}_n() \xrightarrow[s_0+1]{s_0} (s_0 := \mathbf{Const}_n(), s_0)} \end{aligned}$$

So $k = 0, \tau = \mathbf{int}, v = n, p = s_0 := \mathbf{Const}_n(), s_1 = s_0 + 1$, and $st = s_0$

Since $\mathbf{Const}_n()$ takes no arguments, there is only one possibility for \mathcal{P} :

$$\mathcal{P} = \frac{\frac{\overline{\mathbf{Const}_n() \downarrow \langle n \rangle} \quad \overline{\mathbf{Const}_n() \downarrow^{(\langle \rangle)} \langle \rangle}}{\mathbf{Const}_n() \downarrow^{(\langle \rangle)} \langle n \rangle}}{\langle s_0 := \mathbf{Const}_n(), \sigma \rangle \downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle n \rangle]}$$

So $\sigma' = \sigma[s_0 \mapsto \langle n \rangle]$.

Then we have $\mathcal{V} = \overline{n \triangleright_{\mathbf{int}} \sigma'(s_0)}$.

Also clearly, $\sigma' \xrightarrow[\leq s_0]{} \sigma$, $s_0 \leq s_0 + 1$, $\mathbf{sids}(s_0) \leq s_0 + 1$, and we are done.

- Case $\phi = \mathbf{plus}$

We must have

$$\mathcal{T} = \overline{\vdash \mathbf{plus} : (\mathbf{int}, \mathbf{int}) \rightarrow \mathbf{int}}$$

$$\mathcal{E} = \overline{\vdash \mathbf{plus}(n_1, n_2) \downarrow n_3}$$

where $n_3 = n_2 + n_1$, and

$$\mathcal{C} = \overline{\vdash \mathbf{plus}(s_1, s_2) \xrightarrow[s_0+1]{s_0} (s_0 := \mathbf{MapTwo}_+(s_1, s_2), s_0)}$$

So $k = 2, \tau_1 = \tau_2 = \tau = \mathbf{int}, v_1 = n_1, v_2 = n_2, v = n_3, st_1 = s_1, st_2 = s_2, st = s_0, s_1 = s_0 + 1$ and $p = s_0 := \mathbf{MapTwo}_+(s_1, s_2)$.

Assumption (iv) gives us $\overline{n_1 \triangleright_{\mathbf{int}} \sigma(s_1)}$ and $\overline{n_2 \triangleright_{\mathbf{int}} \sigma(s_2)}$, which implies $\sigma(s_1) = \langle n_1 \rangle$ and $\sigma(s_2) = \langle n_2 \rangle$.

For (v) we have $s_1 < s_0$ and $s_2 < s_0$.

\mathcal{P} must have the shape:

$$\frac{\frac{\mathcal{P}_1 \quad \overline{\mathbf{MapTwo}_+(\vec{a}'_1, \vec{a}'_2) \downarrow \langle \rangle \langle \rangle}}{\mathbf{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow \langle \rangle \vec{a}} \quad (\vec{a}_1 ++ \vec{a}'_1 = \langle n_1 \rangle, \vec{a}_2 ++ \vec{a}'_2 = \langle n_2 \rangle)}{\frac{\mathbf{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow \langle \rangle \vec{a}}{\langle s_0 := \mathbf{MapTwo}_+(s_1, s_2), \sigma \rangle \downarrow \langle \rangle \sigma[s_0 \mapsto \vec{a}]}} (\sigma(s_1) = \langle n_1 \rangle, \sigma(s_2) = \langle n_2 \rangle)$$

Since there is only one rule for \mathcal{P}_1 , by which \vec{a}_1 must be $\langle n_1 \rangle$ and \vec{a}_2 must be $\langle n_2 \rangle$ (, and $\vec{a}'_1 = \vec{a}'_2 = \langle \rangle$), that is,

$$\mathcal{P}_1 = \overline{\mathbf{MapTwo}_+(\langle n_1 \rangle, \langle n_2 \rangle) \downarrow \langle n_3 \rangle}$$

Therefore, $\sigma' = \sigma[s_0 \mapsto \langle n_3 \rangle]$.

Now it is clear that $\mathcal{V} = \overline{n_3 \triangleright_{\mathbf{int}} \sigma'(s_0)}$, $\sigma' \xrightarrow[s_1]{s_0} \sigma$, $s_0 \leq s_0 + 1$ and $\mathbf{sids}(s_0) < s_0 + 1$ as required.

- Case $\phi = \mathbf{iota}$

□

Theorem 0.2. *If*

- (i) $\Gamma \vdash e : \tau$ (by some derivation \mathcal{T})
- (ii) $\rho \vdash e \downarrow v$ (by \mathcal{E})
- (iii) $\delta \vdash e \xrightarrow[s_1]{s_0} (p, st)$ (by \mathcal{C})
- (iv) $\forall x \in \text{dom}(\Gamma). \vdash \rho(x) : \Gamma(x) \wedge \mathbf{sids}(\delta(x)) < s_0 \wedge \rho(x) \triangleright_{\Gamma(x)} \sigma(\delta(x))$
then
- (v) $\langle p, \sigma \rangle \downarrow \langle \rangle \sigma'$ (by \mathcal{P})
- (vi) $v \triangleright_{\tau} \sigma'(st)$ (by \mathcal{V})
- (vii) $\sigma' \xrightarrow[s_1]{s_0} \sigma$
- (viii) $s_0 \leq s_1$
- (ix) $\mathbf{sids}(st) < s_1$

Proof. By induction on the syntax of e .

- Case $e = \{e_1 : x \text{ in } y \text{ using } \cdot\}$.

We must have:

$$\begin{aligned}
\text{(i)} \quad \mathcal{T} &= \frac{\mathcal{T}_1}{\Gamma \vdash \{e_1 : x \text{ in } y \text{ using } \cdot\} : \{\tau_2\}} \quad (\Gamma(y) = \{\tau_1\}) \\
\text{(ii)} \quad \mathcal{E} &= \frac{\mathcal{E}_i}{\rho \vdash \{e_1 : x \text{ in } y \text{ using } \cdot\} \downarrow \{v'_1, \dots, v'_k\}} \quad (\rho(y) = \{v_1, \dots, v_k\}) \\
\text{(iii)} \quad \mathcal{C} &= \frac{[x \mapsto st_1] \vdash e_1 \xrightarrow[s_1]{s_0+1} (p_1, st_2)}{\delta \vdash \{e_1 : x \text{ in } y \text{ using } \cdot\} \xrightarrow[s_1]{s_0} (s_0 := \text{Usum}(s_2); st_2 := \text{WithCtrl}(s_0, p_1), (st_2, s_2))} \quad (\delta(y) = (st_1, s_2))
\end{aligned}$$

So $p = (s_0 := \text{Usum}(s_2); st_2 := \text{WithCtrl}(s_0, p_1))$, $\tau = \{\tau_2\}$, $v = \{v'_1, \dots, v'_k\}$, $st = (st_2, s_2)$.

- (iv) $\vdash \rho(y) : \Gamma(y)$ gives us $\vdash \{v_1, \dots, v_k\} : \{\tau_1\}$, which must have the derivation:

$$\frac{(\vdash v_i : \tau_1)_{i=1}^k}{\vdash \{v_1, \dots, v_k\} : \{\tau_1\}} \quad (1)$$

$\text{sids}(\delta(y)) < s_0$ gives us

$$\text{sids}(\delta(y)) = \text{sids}((st_1, s_2)) = \text{sids}(st_1) \cup \{s_2\} < s_0 \quad (2)$$

$\rho(y) \triangleright_{\Gamma(y)} \sigma(\delta(y)) = \{v_1, \dots, v_k\} \triangleright_{\{\tau_1\}} \sigma((st_1, s_2))$ must have the derivation:

$$\frac{\mathcal{V}_i}{\frac{(v_i \triangleright_{\tau_1} w_i)_{i=1}^k}{\{v_1, \dots, v_k\} \triangleright_{\{\tau_1\}} (w, \langle F_1, \dots, F_k, T \rangle)} (w = w_1 ++ \dots ++ w_k)} \quad (3)$$

therefore

$$\sigma(st_1) = w \quad (4)$$

and

$$\sigma(s_2) = \langle F_1, \dots, F_k, T \rangle. \quad (5)$$

First we shall show:

- (v) $\langle s_0 := \text{Usum}(s_2); st_2 := \text{WithCtrl}(s_0, p_1), \sigma \rangle \downarrow^{\langle () \rangle} \sigma'$
- (vi) $\{v'_1, \dots, v'_k\} \triangleright_{\{\tau_2\}} \sigma'((st_2, s_2))$ by \mathcal{V}
- (vii) $\sigma' \xrightarrow{\leq s_0} \sigma$

TS (v), we first prove $\langle s_0 := \text{Usum}(s_2), \sigma \rangle \downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$ by \mathcal{P}_0 . \mathcal{P}_0 must have the shape:

$$\frac{\mathcal{P}'_0}{\frac{\text{Usum}(\vec{b}_1) \downarrow \vec{a} \quad \text{Usum}(\vec{b}_2) \downarrow^{\langle () \rangle} \vec{a}}{\text{Usum}(\langle F_1, \dots, F_k, T \rangle) \downarrow^{\langle () \rangle} \vec{a}} (\vec{b}_1 ++ \vec{b}_2 = \langle F_1, \dots, F_k, T \rangle)} \quad (\sigma(s_2) = \langle F_1, \dots, F_k, T \rangle)$$

From the rules P-USUMF and P-USUMT we know that \vec{b}_1 must end with (and include exact) one T so that \mathcal{P}'_0 can terminate. Therefore, in our case, \vec{b}_1 can only be $\langle F_1, \dots, F_k, T \rangle$, and $\vec{b}_2 = \langle \rangle$. Then using P-USUMF k times and P-USUMT once, we obtain \mathcal{P}'_0 of

$$\text{Usum}(\vec{b}_1) \Downarrow \langle ()_1, \dots, ()_k \rangle$$

which gives us \mathcal{P}_0 of $\langle s_0 := \text{Usum}(s_2), \sigma \rangle \Downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$.

Then \mathcal{P} must have the shape:

$$\frac{\mathcal{P}_0 \quad \mathcal{P}_1}{\langle s_0 := \text{Usum}(s_2), \sigma \rangle \Downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \quad \langle st_2 := \text{WithCtrl}(s_0, p_1), \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \rangle \Downarrow^{(\langle \rangle)} \sigma'}$$

Since we have

$$\begin{aligned} \mathcal{T}_1 &= [x \mapsto \tau_1] \vdash e_1 : \tau_2 \\ (\mathcal{E}_i &= [x \mapsto v_i] \vdash e_1 \Downarrow v'_i)_{i=1}^k \\ \mathcal{C}_1 &= [x \mapsto st_1] \vdash e_1 \xrightarrow[s_1]{s_0+1} (p_1, st_2) \end{aligned}$$

Let $\Gamma_1 = [x \mapsto \tau_1]$, $\rho_i = [x \mapsto v_i]$ and $\delta_1 = [x \mapsto st_1]$.

From (1) and (2) it is clear that

$$\forall z \in \text{dom}(\Gamma_1). \vdash \rho_i(z) : \Gamma_1(z) \wedge \mathbf{sids}(\delta_1(z)) \leq s_0.$$

Let i range from 1 to k : we take $\sigma_i \stackrel{st_1}{\sim} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$ such that $\sigma_i(st_1) = w_i$.

From \mathcal{V}_i in (3) we know that

$$\forall z \in \text{dom}(\Gamma_1). \rho_i(z) \triangleright_{\Gamma_1(z)} \sigma_i(\delta_1(z)).$$

Then by IH (k times) on \mathcal{T}_1 with \mathcal{E}_i , \mathcal{C}_1 we obtain the following result:

$$\langle p_1, \sigma_i \rangle \Downarrow^{(\langle \rangle)} \sigma'_i \rangle_{i=1}^k \tag{6}$$

$$(v'_i \triangleright_{\tau_2} \sigma'_i(st_2))_{i=1}^k \tag{7}$$

$$(\sigma'_i \xrightarrow[s_1]{\leq s_0+1} \sigma_i)_{i=1}^k \tag{8}$$

$$s_0 + 1 \leq s_1 \tag{9}$$

$$\mathbf{sids}(st_2) \leq s_1 \tag{10}$$

Assume $\mathbf{sids}(st_2) = \{s'_1, \dots, s'_j\}$.

There are two possibilities:

- Subcase $k = 0$, that is $\sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle](s_0) = \langle \rangle$.

Then

$$\mathcal{P}_1 = \frac{}{\langle st_2 := \text{WithCtrl}(s_0, p_1), \sigma \rangle \Downarrow^{(\langle \rangle)} \sigma[s_0 \mapsto \langle \rangle, s'_1 \mapsto \langle \rangle, \dots, s'_j \mapsto \langle \rangle]} ,$$

thus in this subcase

$$\sigma' = \sigma[s_0 \mapsto \langle \rangle, s'_1 \mapsto \langle \rangle, \dots, s'_j \mapsto \langle \rangle].$$

Since $k = 0$, then $v = \{\}$, $\sigma(s_2) = \langle T \rangle$ (from (5)), we have

$\sigma'(s_2) = \sigma(s_2) = \langle T \rangle$ (?? not correct if $s_2 \in \mathbf{sids}(st_2)/\mathbf{sids}(st_1)$),

and $\sigma'(st_2) = (\dots((\langle \rangle, \langle \rangle)_1, \langle \rangle)_2, \dots)_{j-1}$.

Therefore $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2))$, with which we construct

$$\mathcal{V} = \frac{\{\} \triangleright_{\{\tau_2\}} ((\dots((\langle \rangle, \langle \rangle)_1, \dots)_{j-1}, \langle T \rangle))$$

as required.

Since $k = 0$, from (4) we know $\forall s' \in \mathbf{sids}(st_1). \sigma(s') = \langle \rangle$. For any $s' \in \mathbf{sids}(st_2)$ and $s' < s_0$, it must have $s' \in \mathbf{sids}(st_1)$ (because $\text{codom}(\delta_1) = \{st_1\}$), hence $\sigma(s') = \langle \rangle = \sigma'(s')$. Therefore,

$$\sigma' \stackrel{\leq s_0}{=} \sigma.$$

- Subcase $k > 0$, that is $\sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] = \langle () | \vec{a} \rangle$ for some \vec{a} .
Then $\mathcal{P}_1 =$

$$\frac{\mathcal{P}'_1 \quad \langle p_1, \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \rangle \downarrow^{\langle ()_1, \dots, ()_k \rangle} \sigma''}{\langle st_2 := \text{WithCtrl}(s_0, p_1), \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \rangle \downarrow^{\langle () \rangle} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle, s'_1 \mapsto \sigma''(s'_1), \dots, s'_j \mapsto \sigma''(s'_j)]}$$

So in this subcase

$$\sigma' = \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle, s'_1 \mapsto \sigma''(s'_1), \dots, s'_j \mapsto \sigma''(s'_j)].$$

Using Lemma 0.3 (k-1) times on (6) gives us

$$\langle p_1, (\boxtimes_{i=1}^{st_1} \sigma_i)^k \rangle \downarrow^{\langle ()_1, \dots, ()_k \rangle} (\boxtimes_{i=1}^{st'_2} \sigma'_i)^k \quad (11)$$

where $st'_2 = \mathbf{sids}(st_1) \cup \mathbf{sids}(st_2)$ (???) .

By Definition 0.2 we have

$$(\boxtimes_{i=1}^{st_1} \sigma_i)^k = \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]. \quad (12)$$

Then by Theorem 0.1 on \mathcal{P}'_1 with (11), we get

$$\sigma'' = (\boxtimes_{i=1}^{st'_2} \sigma'_i)^k \quad (13)$$

Therefore, $\sigma''(st_2) = \sigma'_1(st_2) ++ \dots ++ \sigma'_k(st_2)$ by Definition 0.2.

Let $\sigma'_i(st_2) = w'_i$ and $\sigma''(st_2) = w'$, then $w' = w'_1 ++ \dots ++ w'_k$.

Since $\sigma'(st_2) = \sigma''(st_2) = w'$, and $\sigma'(s_2) = \sigma(s_2) = \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle$, (same problem) we now have $\sigma'((st_2, s_2)) = (\sigma'(st_2), \sigma'(s_2)) = (w', \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle)$. With (7), we can construct

$$\mathcal{V} = \frac{(v'_i \triangleright_{\tau_2} w'_i)_{i=1}^k}{\{v'_1, \dots, v'_k\} \triangleright_{\{\tau_2\}} (w', \langle \mathbf{F}_1, \dots, \mathbf{F}_k, \mathbf{T} \rangle)}$$

as required.

By Lemma 0.1 on (12) we get $\sigma_i \stackrel{st_1}{\sim} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle]$, and similarly $\sigma'_i \stackrel{st'_2}{\sim} \sigma''$ from (13).

Since (8) implies

$$(\sigma'_i \stackrel{\leq s_0}{=} \sigma_i)_{i=1}^k$$

using Lemma 0.2 (k-1) times, we obtain

$$\sigma'' \stackrel{\leq s_0}{=} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle].$$

Therefore, $\sigma' \stackrel{\leq s_0}{=} \sigma[s_0 \mapsto \langle ()_1, \dots, ()_k \rangle] \stackrel{\leq s_0}{=} \sigma$.

(viii) TS: $s_0 \leq s_1$

From (9) we immediately get $s_0 \leq s_1 - 1 < s_1$.

(ix) TS: $\mathbf{sids}((st_2, s_2)) \prec s_1$

From (2) we know $s_2 < s_0$, thus $s_2 < s_0 \leq s_1$. And we already have (10). Therefore,

$$\mathbf{sids}((st_2, s_2)) = \mathbf{sids}(st_2) \cup \{s_2\} \prec s_1.$$

- Case $e = x$.

We must have

$$\begin{aligned} \mathcal{T} &= \frac{}{\Gamma \vdash x : \tau} (\Gamma(x) = \tau) \\ \mathcal{E} &= \frac{}{\rho \vdash x \downarrow v} (\rho(x) = v) \\ \mathcal{C} &= \frac{}{\delta \vdash x \xrightarrow[s_0]{s_0} (\epsilon, st)} (\delta(x) = st) \end{aligned}$$

So $p = \epsilon$.

Immediately we have $\mathcal{P} = \overline{\langle \epsilon, \sigma \rangle \downarrow^{(0)} \sigma}$

So $\sigma' = \sigma$, which implies $\sigma' \leq_{s_0} \sigma$.

From the assumption we already have $v \triangleright_{\tau} \sigma(st)$, and $\mathbf{sids}(st) \prec s_0$.

Finally it's clear that $s_0 \leq s_0$, and we are done.

- Case $e = \mathbf{let } x = e_1 \mathbf{ in } e_2$.

We must have:

$$\begin{aligned} \mathcal{T} &= \frac{\mathcal{T}_1 \quad \mathcal{T}_2}{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau} \Gamma \vdash \mathbf{let } x = e_1 \mathbf{ in } e_2 : \tau \\ \mathcal{E} &= \frac{\mathcal{E}_1 \quad \mathcal{E}_2}{\rho \vdash e_1 \downarrow v_1 \quad \rho[x \mapsto v_1] \vdash e_2 \downarrow v} \rho \vdash \mathbf{let } x = e_1 \mathbf{ in } e_2 \downarrow v \\ \mathcal{C} &= \frac{\mathcal{C}_1 \quad \mathcal{C}_2}{\delta \vdash e_1 \xrightarrow[s'_0]{s_0} (p_1, st_1) \quad \delta[x \mapsto st_1] \vdash e_2 \xrightarrow[s_1]{s'_0} (p_2, st)} \delta \vdash \mathbf{let } x = e_1 \mathbf{ in } e_2 \xrightarrow[s_1]{s_0} (p_1; p_2, st) \end{aligned}$$

So $p = p_1; p_2$.

By IH on \mathcal{T}_1 with $\mathcal{E}_1, \mathcal{C}_1$, we get

- (a) \mathcal{P}_1 of $\langle p_1, \sigma \rangle \downarrow^{(0)} \sigma_1$
- (b) \mathcal{V}_1 of $v_1 \triangleright_{\tau_1} \sigma_1(st_1)$
- (c) $\sigma_1 \leq_{s_0} \sigma$
- (d) $s_0 \leq s'_0$
- (e) $\mathbf{sids}(st_1) \prec s'_0$

From (b), we know $\rho[x \mapsto v_1](x) : \Gamma[x \mapsto \tau_1](x)$ and $\rho[x \mapsto v_1](x) \triangleright_{\Gamma[x \mapsto \tau_1](x)} \sigma_1(\delta[x \mapsto st_1](x))$ must hold. From (e), we have $\mathbf{sids}(\delta[x \mapsto st_1](x)) \prec s'_0$.

Then by IH on \mathcal{T}_2 with $\mathcal{E}_2, \mathcal{C}_2$, we get

- (f) \mathcal{P}_2 of $\langle p_2, \sigma_1 \rangle \downarrow^{(0)} \sigma_2$
- (g) \mathcal{V}_2 of $\sigma_2 \triangleright_{\tau} \sigma_2(st)$
- (h) $\sigma_2 \leq_{s'_0} \sigma_1$
- (i) $s'_0 \leq s_1$

(j) $\mathbf{sids}(st) \leq s_1$

So we can construct:

$$\mathcal{P} = \frac{\frac{\mathcal{P}_1}{\langle p_1, \sigma \rangle \downarrow^{(\langle \rangle)} \sigma_1} \quad \frac{\mathcal{P}_2}{\langle p_2, \sigma_1 \rangle \downarrow^{(\langle \rangle)} \sigma_2}}{\langle p_1; p_2, \sigma \rangle \downarrow^{(\langle \rangle)} \sigma_2}$$

From (c), (d) and (h), it is clear that $\sigma_2 \xrightarrow{\leq s_0} \sigma_1 \xrightarrow{\leq s_0} \sigma$. From (d) and (i), $s_0 \leq s_1$. Take $\sigma' = \sigma_2$ (thus $\mathcal{V} = \mathcal{V}_2$) and we are done.

- Case $e = \phi(x_1, \dots, x_k)$
We must have

$$\begin{aligned} & \mathcal{T}_1 \\ \mathcal{T} &= \frac{\vdash \phi : (\tau_1, \dots, \tau_k) \rightarrow \tau}{\Gamma \vdash \phi(x_1, \dots, x_k) : \tau} ((\Gamma(x_i) = \tau_i)_{i=1}^k) \\ & \mathcal{E}_1 \\ \mathcal{E} &= \frac{\vdash \phi(v_1, \dots, v_k) \downarrow v}{\rho \vdash \phi(x_1, \dots, x_k) \downarrow v} ((\rho(x_i) = v_i)_{i=1}^k) \\ & \mathcal{C}_1 \\ \mathcal{C} &= \frac{\vdash \phi(st_1, \dots, st_k) \xrightarrow[s_1]{s_0} (p, st)}{\delta \vdash \phi(x_1, \dots, x_k) \xrightarrow[s_1]{s_0} (p, st)} ((\delta(x_i) = st_i)_{i=1}^k) \end{aligned}$$

From our assumption (iv), for all $i \in \{1, \dots, k\}$:

- (a) $\vdash \rho(x_i) : \Gamma(x_i)$, that is, $\vdash v_i : \tau_i$
- (b) $\mathbf{sids}(\delta(x_i)) \leq s_0$, that is, $\mathbf{sids}(st_i) \leq s_0$
- (c) $\rho(x_i) \triangleright_{\Gamma(x_i)} \sigma(st_i)$, that is, $v_i \triangleright_{\tau_i} \sigma(st_i)$

So using Lemma 0.7 on $\mathcal{T}_1, \mathcal{E}_1, \mathcal{C}_1, (a), (b)$ and (c) gives us exactly what we shall show.

□