SNESL formalization

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0 Level-0

Draft version 0.0.3: changed some notations; added some definitions and lemmas; fixed some bugs.

0.1 Source language syntax

(Ignore empty sequence for now)
Expressions:

$$e ::= x \mid \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 \mid \phi(x_1, ..., x_k) \mid \{e : x\ \mathbf{in}\ y\ \mathbf{using}\ \cdot\}$$

$$\phi = \mathbf{const}_n \mid \mathbf{iota} \mid \mathbf{plus}$$

Values:

$$n \in \mathbf{Z}$$

$$v ::= n \mid \{v_1, ..., v_k\}$$

0.2 Type system

$$\tau ::= \mathbf{int} | \{\tau_1\}$$

Type environment $\Gamma = [x_1 \mapsto \tau_1, ..., x_i \mapsto \tau_i]$.

- Judgment $\boxed{\Gamma\ \vdash\ e : \tau}$

$$\frac{}{\Gamma\ \vdash\ x : \tau}\ (\Gamma(x) = \tau) \qquad \frac{\Gamma\ \vdash\ e_1 : \tau_1 \qquad \Gamma[x \mapsto \tau_1]\ \vdash\ e_2 : \tau}{\Gamma\ \vdash\ \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 : \tau}$$

$$\frac{\vdash\ \phi : (\tau_1, ..., \tau_k) \to \tau}{\Gamma\ \vdash\ \phi(x_1, ..., x_k) : \tau}\ ((\Gamma(x_i) = \tau_i)_{i=1}^k) \qquad \frac{[x \mapsto \tau_1]\ \vdash\ e : \tau}{\Gamma\ \vdash\ \{e : x\ \mathbf{in}\ y\ \mathbf{using}\ \cdot\} : \{\tau\}}\ (\Gamma(y) = \{\tau_1\})$$

- Auxiliary Judgment $\boxed{\vdash\ \phi : (\tau_1, ..., \tau_k) \to \tau}$

$$\frac{}{\vdash\ \mathbf{const}_n : \mathbf{int}} \qquad \frac{}{\vdash\ \mathbf{iota} : \mathbf{int} \to \{\mathbf{int}\}} \qquad \frac{}{\vdash\ \mathbf{plus} : \mathbf{int} \to \mathbf{int} \to \mathbf{int}}$$

0.3 Source language semantics

$\rho = [x_1 \mapsto v_1, ..., x_i \mapsto v_i]$

- Judgment $\boxed{\rho\ \vdash\ e \downarrow v}$

$$\frac{}{\rho\ \vdash\ x \downarrow v}\ (\rho(x) = v) \qquad \frac{\rho\ \vdash\ e_1 \downarrow v_1 \qquad \rho[x \mapsto v_1]\ \vdash\ e_2 \downarrow v}{\rho\ \vdash\ \mathbf{let}\ e_1 = x\ \mathbf{in}\ e_2 \downarrow v}$$

$$\frac{\vdash\ \phi(v_1,...,v_k)\downarrow v}{\rho\ \vdash\ \phi(x_1,...,x_k)\downarrow v}\ ((\rho(x_i)=v_i)_{i=1}^k) \qquad \frac{([x\mapsto v_i]\ \vdash\ e\downarrow v'_i)_{i=1}^k}{\rho\ \vdash\ \{e : x\ \mathbf{in}\ y\ \mathbf{using}\ \cdot\}\downarrow\{v'_1,...,v'_k\}}\ (\rho(y)=\{v_1,...,v_k\})$$

- Auxiliary Judgment $\boxed{\vdash\ \phi(v_1,...,v_k)\downarrow v}$

$$\frac{}{\vdash\ \mathbf{const}_n()\downarrow n} \qquad \frac{}{\vdash\ \mathbf{iota}(n)\downarrow\{0,1,...,n-1\}}\ (n\geq 0) \qquad \frac{}{\vdash\ \mathbf{plus}(n_1,n_2)\downarrow n_3}\ (n_3=n_1+n_2)$$

0.4 SVCODE syntax

Stream id:

$$s\in\mathbf{SId}=\mathbf{N}=\{0,1,2...\}$$

Stream tree:

$$\mathbf{STree}\ni st ::= s \mid (st_1, s)$$

SVCODE operations:

$$\psi ::= \mathtt{Ctrl} \mid \mathtt{Const_a} \mid \mathtt{ToFlags} \mid \mathtt{Usum} \mid \mathtt{MapTwo} \mid \mathtt{ScanPlus}$$

SVCODE program:

$$\begin{aligned} p ::= \ & \epsilon \\ & \mid s := \psi(s_1,...,s_i) \\ & \mid st := \mathtt{WithCtrl}(s,p) \\ & \mid p_1; p_2 \end{aligned}$$

Target language values:

$$b\in\{\mathtt{T},\mathtt{F}\}$$
$$a ::= n \mid b \mid ()$$
$$\vec{b}=\langle b_1,...,b_i\rangle$$
$$\vec{a}=\langle a_1,...,a_i\rangle$$
$$\mathbf{SVal}\ni w ::= \vec{a} \mid (w,\vec{b})$$

Some notations and operations:

- For some a_0 and $\vec{a}=\langle a_1,...,a_i\rangle$, $\langle a_0|\vec{a}\rangle=\langle a_0,a_1,...,a_i\rangle$.
- $++ : \mathbf{SVal}\to\mathbf{SVal}\to\mathbf{SVal}$
  $\langle a_1,...,a_i\rangle{++}\langle a'_1,...,a'_i\rangle=\langle a_1,...,a_i,a'_1,...,a'_i\rangle$
  $(w_1,\vec{b}_1){++}(w_2,\vec{b}_2)=(w_1{++}w_2,\vec{b}_1{++}\vec{b}_2)$
- $\mathtt{sids}$ converts a $st\in\mathbf{STree}$ to a set of $s\in\mathbf{SId}$:
  $\mathtt{sids}(s)=\{s\}$
  $\mathtt{sids}((st,s))=\mathtt{sids}(st)\cup\{s\}$
- For some set set of $\mathbf{SId}$ and some $s\in\mathbf{SId}$, $set\lessdot s$ denotes $\forall s'\in set.s'<s$.

0.5 SVCODE semantics

$\sigma = [s_1 \mapsto \vec{a}_1, ..., s_i \mapsto \vec{a}_i]$.
Notation $\sigma_1 \overset{<s}{=\!=} \sigma_2$ denotes $\forall s' < s. \sigma_1(s') = \sigma_2(s')$.
$\vec{a}_c$ is the control stream.

- Judgment $\boxed{\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$

$$\frac{}{\langle \epsilon, \sigma \rangle \downarrow^{\vec{a}_c} \sigma} \qquad \frac{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}{\langle s := \psi(s_1, ..., s_k), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s \mapsto \vec{a}]} \; ((\sigma(s_i) = \vec{a}_i)_{i=1}^{k})$$

$$\frac{}{\langle st := \texttt{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \langle\rangle, ..., s_i \mapsto \langle\rangle]} \; (\sigma(s) = \langle\rangle, \texttt{sids}(st) = \{s_1, ..., s_i\})$$

$$\frac{\langle p, \sigma \rangle \downarrow^{\vec{a}_s} \sigma''}{\langle st := \texttt{WithCtrl}(s, p), \sigma \rangle \downarrow^{\vec{a}_c} \sigma[s_1 \mapsto \sigma''(s_1), ..., s_i \mapsto \sigma''(s_i)]} \begin{pmatrix} \sigma(s) = \vec{a}_s = \langle a_0 | \vec{a} \rangle \\ \texttt{sids}(st) = \{s_1, ..., s_i\} \end{pmatrix}$$

$$\frac{\langle p_1, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'' \qquad \langle p_2, \sigma'' \rangle \downarrow^{\vec{a}_c} \sigma'}{\langle p_1; p_2, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'}$$

- Auxiliary Judgment $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\vec{a}_c} \vec{a}}$

$$\frac{\psi(\vec{a}_{11}, ..., \vec{a}_{k1}) \Downarrow \vec{a}_1 \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \downarrow^{\vec{a}_c} \vec{a}_2}{\psi(\vec{a}_{11} +\!\!+ \vec{a}_{12}, ..., \vec{a}_{k1} +\!\!+ \vec{a}_{k2}) \downarrow^{\langle a_0 | \vec{a}_c \rangle} \vec{a}} \; (\vec{a} = \vec{a}_1 +\!\!+ \vec{a}_2)$$

$$\frac{}{\psi(\vec{a}_1, ..., \vec{a}_k) \downarrow^{\langle\rangle} \langle\rangle}$$

- Auxiliary Judgment $\boxed{\psi(\vec{a}_1, ..., \vec{a}_k) \Downarrow \vec{a}}$

$$\frac{}{\texttt{Const}_{\texttt{a}} \Downarrow \langle a \rangle} \qquad \frac{}{\texttt{ToFlags}(\langle n \rangle) \Downarrow \langle \texttt{F}_1, ..., \texttt{F}_n, \texttt{T} \rangle} \qquad \frac{}{\texttt{MapTwo}_{\oplus}(\langle n_1 \rangle, \langle n_2 \rangle) \Downarrow \langle n_3 \rangle} \; (n_3 = n_1 \oplus n_2)$$

$$\frac{\psi(\langle \texttt{F} \rangle, ..., \vec{a}_{k1}) \Downarrow \vec{a}_1 \qquad \psi(\vec{a}_{12}, ..., \vec{a}_{k2}) \Downarrow \vec{a}_2}{\psi(\langle \texttt{F} \rangle +\!\!+ \vec{a}_{12}, ..., \vec{a}_{k1} +\!\!+ \vec{a}_{k2}) \Downarrow \vec{a}} \; (\vec{a} = \vec{a}_1 +\!\!+ \vec{a}_2)$$

$$\frac{\psi(\langle \texttt{T} \rangle, ..., \vec{a}_k) \Downarrow \vec{a}}{\psi(\langle \texttt{T} \rangle, ..., \vec{a}_k) \Downarrow \vec{a}}$$

- Auxiliary Judgment $\boxed{\psi(\langle b \rangle, ..., \vec{a}_k) \Downarrow \vec{a}}$

$$\frac{}{\texttt{Usum}(\langle \texttt{F} \rangle) \Downarrow \langle () \rangle} \qquad \frac{}{\texttt{Usum}(\langle \texttt{T} \rangle) \Downarrow \langle\rangle}$$

Theorem 0.1 (deterministic ??). *If $\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'$ and $\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma''$, then $\sigma' = \sigma''$.*

Definition 0.1 (Stream prefix).

$\vec{a}$ is a *prefix* of $\vec{a}'$:
Judgment $\boxed{\vec{a} \sqsubseteq \vec{a}'}$

$$\frac{}{\langle\rangle \sqsubseteq \vec{a}} \qquad \frac{\vec{a} \sqsubseteq \vec{a}'}{\langle a_0 | \vec{a} \rangle \sqsubseteq \langle a_0 | \vec{a}' \rangle}$$

Lemma 0.1. *If*

(i) $(\vec{a}'_i \sqsubseteq \vec{a}_i)_{i=1}^k$ *and* $\psi(\vec{a}'_1, ..., \vec{a}'_k) \Downarrow \vec{a}'$,

(ii) $(\vec{a}''_i \sqsubseteq \vec{a}_i)_{i=1}^k$ *and* $\psi(\vec{a}''_1, ..., \vec{a}''_k) \Downarrow \vec{a}''$

then $(\vec{a}'_i = \vec{a}''_i)_{i=1}^k$ *and* $\vec{a}' = \vec{a}''$.

0.6 Translation

$\delta = [x_1 \mapsto st_1, ..., x_i \mapsto st_i]$

- Judgment $\boxed{\delta \vdash e \overset{s_0}{\underset{s_1}{\Rightarrow}} (p, st)}$

$$\frac{}{\delta \vdash x \overset{s_0}{\underset{s_0}{\Rightarrow}} (\epsilon, st)} (\delta(x) = st) \qquad \frac{\delta \vdash e_1 \overset{s_0}{\underset{s'_0}{\Rightarrow}} (p_1, st_1) \qquad \delta[x \mapsto st_1] \vdash e_2 \overset{s'_0}{\underset{s_1}{\Rightarrow}} (p_2, st)}{\delta \vdash \textbf{let } x = e_1 \textbf{ in } e_2 \overset{s_0}{\underset{s_1}{\Rightarrow}} (p_1; p_2, st)}$$

$$\frac{\vdash \phi(st_1, ..., st_k) \overset{s_0}{\underset{s_1}{\Rightarrow}} (p, st)}{\delta \vdash \phi(x_1, ..., x_k) \overset{s_0}{\underset{s_1}{\Rightarrow}} (p, st)} ((\delta(x_i) = st_i)_{i=1}^k)$$

$$\frac{[x \mapsto st_1] \vdash e \overset{s_0+1}{\underset{s_1}{\Longrightarrow}} (p, st)}{\delta \vdash \{e : x \textbf{ in } y \textbf{ using } \cdot\} \overset{s_0}{\underset{s_1}{\Rightarrow}} (s_0 := \texttt{Usum}(s_2); st := \texttt{WithCtrl}(s_0, p), (st, s_2))} (\delta(y) = (st_1, s_2))$$

- Auxiliary Judgment $\boxed{\vdash \phi(st_1, ..., st_k) \overset{s_0}{\underset{s_1}{\Rightarrow}} (p, st)}$

$$\frac{}{\textbf{const}_a() \overset{s_0+1}{\underset{s_0}{\Longrightarrow}} (s_0 := \texttt{Const}_\texttt{a}, s_0)}$$

$$\frac{}{\textbf{iota}(s) \overset{s_4}{\underset{s_0}{\Rightarrow}} (p, (s_3, s_0))} \left(\begin{array}{l} s_{i+1} = s_i + 1 \\ p = s_0 := \texttt{ToFlags}(s); \\ \quad s_1 := \texttt{Usum}(s_0); \\ \quad s_2 := \texttt{WithCtrl}(s_1, s_2 := \texttt{Const}_1); \\ \quad s_3 := \texttt{ScanPlus}(s_0, s_2) \end{array} \right)$$

$$\frac{}{\textbf{plus}(s_1, s_2) \overset{s_0+1}{\underset{s_0}{\Longrightarrow}} (s_0 := \texttt{MapTwo}_+(s_1, s_2), s_0)}$$

0.7 Value representation

- Judgment $\boxed{v \triangleright_\tau w}$

$$\frac{}{n \triangleright_{\textbf{int}} \langle n \rangle} \qquad \frac{(v_i \triangleright_\tau w_i)_{i=1}^k}{\{v_1, ..., v_k\} \triangleright_{\{\tau\}} (w, \langle \texttt{F}_1, ..., \texttt{F}_k, \texttt{T} \rangle)} (w = w_1 +\!\!+ w_2 +\!\!+ ... +\!\!+ w_k)$$

Lemma 0.2. *If* $v \triangleright_\tau w$*,* $v' \triangleright_\tau w$*, then* $v = v'$.

0.8 Correctness proof

Lemma 0.3. *If*

(i) $\vdash \phi : (\tau_1, ..., \tau_k) \to \tau$

(ii) $\vdash \phi(v_1, ..., v_k) \downarrow v$

(iii) $\vdash \ \phi(st_1, ..., st_k) \xRightarrow[s_1]{s_0} (p, st)$

(iv) $(v_i \rhd_{\tau_i} st_i)_{i=1}^{k}$

(v) $\bigcup_{i=1}^{k} \texttt{sids}(st_i) \lessdot s_0$

then

(i) $\langle p, \sigma \rangle \downarrow^{\vec{a}_c} \sigma'$ *(by $\mathcal{P}$)*

(ii) $v \rhd_\tau \sigma'(st)$ *(by $\mathcal{V}$)*

(iii) $\sigma' \xlongequal{\lessdot s_0} \sigma$

(iv) $\texttt{sids}(st) \lessdot s_1$

(v) $s_0 \leq s_1$

Theorem 0.2. *If*

(i) $\Gamma \ \vdash \ e : \tau$ *(by some derivation $\mathcal{T}$)*

(ii) $\rho \ \vdash \ e \downarrow v$ *(by $\mathcal{E}$)*

(iii) $\delta \ \vdash \ e \xRightarrow[s_1]{s_0} (p, st)$ *(by $\mathcal{C}$)*

(iv) $\forall x \in dom(\Gamma).\rho(x) : \Gamma(x) \wedge \texttt{sids}(\delta(x)) \lessdot s_0 \wedge \rho(x) \rhd_{\Gamma(x)} \sigma(\delta(x))$

then

(i) $\langle p, \sigma \rangle \downarrow^{\langle () \rangle} \sigma'$ *(by $\mathcal{P}$)*

(ii) $v \rhd_\tau \sigma'(st)$ *(by $\mathcal{V}$)*

(iii) $\sigma' \xlongequal{\lessdot s_0} \sigma$

(iv) $\texttt{sids}(st) \lessdot s_1$

(v) $s_0 \leq s_1$