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Defuzzification: criteria and classification

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Abstract

In this paper, we contribute to the theory and development of defuzzification techniques. First we define the *core* of a fuzzy set. Then we formulate a set of criteria for defuzzification in arbitrary universes, ordered universes, and the set of the real numbers. Finally, we classify the most widely used defuzzification techniques into different groups and we examine the prototypes of each group with respect to the defuzzification criteria. We show that the maxima methods behave well with respect to the more basic defuzzification criteria, and hence are good candidates for fuzzy reasoning systems. On the other hand, the distribution methods and the area methods do not fulfill the basic criteria, but they exhibit the property of continuity that makes them suitable for fuzzy controllers. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Fuzzy systems represent and manipulate imprecise and uncertain information [1] using the theory of fuzzy sets. Many of these fuzzy systems incorporate as last step a defuzzification process that maps a fuzzy set – which in fact is the output of the core of the fuzzy system – into a crisp value. In most textbooks on fuzzy technology, the defuzzification process is treated in far lesser detail than the other processes involved. At best, they contain a chapter on defuzzification that lists a number of methods and formulas that can be used in the defuzzification process. Several factors might contribute to the shallow treatment of defuzzification: firstly, defuzzification can be seen as being not a part of the core of a fuzzy system. The system uses fuzzy sets to manipulate imprecise and/or uncertain information to overcome the shortcomings of a classical crisp system. As a consequence, it is natural that the result of the core of a fuzzy system is a fuzzy set which incorporates a

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[1] Information is considered imprecise as soon as the value of a certain variable is not precisely given, but stated in imprecise linguistic terms such as young, middle-aged for the variable age. Information is considered uncertain as soon as we are not completely certain about the (precise or imprecise) value of a given variable. Uncertain statements are of the form: it is likely that …, it is possible that … .

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representation of imprecision and/or uncertainty. Defuzzification takes all this away since it reduces the fuzzy set to a single crisp value. Secondly, defuzzification is "just the last step". The only reason for defuzzification is to interface with crisp models of the world, models that cannot handle imprecision or uncertainty. Finally, defuzzification is a process of synthesis. As such, the whole concept of defuzzification is completely opposite to the main purpose of fuzzy set theory namely the extension of crisp concepts and theories.

In this paper, we contribute to the theory and development of defuzzification techniques by formulating a set of criteria for defuzzification. We also classify the most widely used defuzzification techniques into different groups and we examine the prototypes of each group with respect to the defuzzification criteria.

1.1. Types of fuzzy systems

We can divide the fuzzy systems into two broad categories: fuzzy reasoning (or knowledge) systems, and fuzzy control systems. Both categories use fuzzy technology for their own goals and they have specific requirements on the fuzzy system as a whole, and on the defuzzification process in particular. Fuzzy knowledge systems aim at providing some kind of qualitative reasoning system for a specific domain. Fuzzy sets are used to map qualitative facts onto numerical entities that can be manipulated by the computer. The result of a "computation" in these systems is a qualitative expression based on the input to the system. In cases where these linguistic results convey sufficient information, there is no need for any defuzzification. In other cases – especially when the fuzzy system has to interfere or cooperate with classical software applications – a defuzzification step is needed to obtain a single element of the universe. Fuzzy controllers always need a crisp value as result: a result stating that a certain valve has to be opened "somewhat" is not very useful in a control system. Fuzzy sets are used as a convenient tool to define control rules and to make inferences. But the final system is a control system taking crisp inputs and computing a crisp output. We could say that a fuzzy controller always strives at *sub-term precision*: there are some linguistic terms defined on the output universe, but the result that the system is aiming at is not a term, but a numerical value. Most work on defuzzification has been done focusing on fuzzy controllers.

1.2. Definitions

We use the notations as described in [8]. Let X be an ordinary nonvoid set. A mapping A from X into the unit interval $[0,1]$ is called a fuzzy set on X. The value $A(x)$ of A in $x \in X$ is called the degree of membership of x in A. The set of all elements that have a nonzero degree of membership in A is called the *support* of A, i.e.,

$$\mathrm{supp}(A) = \{x \mid x \in X \text{ and } A(x) > 0\}.$$

The set of elements that completely belong to A is called the *kernel* of A, i.e.,

$$\ker(A) = \{x \mid x \in X \text{ and } A(x) = 1\}.$$

The set of elements having the largest degree of membership in A is called the *core* of A,[2] i.e.,

$$\mathrm{core}(A) = \{x \mid x \in X \text{ and } \neg(\exists y \in X)(A(y) > A(x))\}.$$

The weak α-*cut*, where $\alpha \in \,]0,1]$, of a fuzzy set A on X is denoted A_α and is defined as the set of all elements of X whose degree of membership in A is at least equal to α, i.e.,

$$A_\alpha = \{x \mid x \in X \text{ and } A(x) \geqslant \alpha\}.$$

[2] Remark that this definition of the core of a fuzzy set is not restricted to fuzzy sets, but also holds for L-fuzzy sets.

The class of all fuzzy sets on a given universe X will be denoted by $F(X)$. The *height* of a fuzzy set A on X is defined as

$$\mathrm{height}(A) = \sup_{x \in X} A(x).$$

When the unit interval $[0,1]$ is chosen as evaluation set – as we do in this paper – the core of A can also be written as [3]

$$\mathrm{core}(A) = \{x \mid x \in X \text{ and } (\forall y \in X)(A(y) \leqslant A(x))\}$$

or

$$\mathrm{core}(A) = \{x \mid x \in X \text{ and } A(x) = \mathrm{height}(A)\}.$$

Defuzzification is expressed by a *defuzzification operator D*. This operator maps fuzzy sets on X into elements of the universe X, i.e.,

$$D : F(X) \to X : A \to D(A).$$

1.3. Some properties of the core of a fuzzy set

Considering the importance of the core of a fuzzy set in some of the defuzzification methods, we list some properties of the core.

P.1. $\ker(A) \neq \emptyset \Rightarrow \mathrm{core}(A) = \ker(A)$,
P.2. $\mathrm{supp}(A) = \emptyset \Rightarrow \mathrm{core}(A) = X$,
P.3. $\mathrm{core}(A) = A_{\alpha_{\max}}$ where $\alpha_{\max} = \sup\{\alpha \mid \alpha \in \,]0,1] \text{ and } A_\alpha \neq \emptyset\}$,
P.4. Linguistic hedges that modify the fuzziness of a set via a power function (e.g., concentration and dilatation [8]), leave the core (and also the kernel) unchanged.

2. Criteria for defuzzification

In order to be able to evaluate existing defuzzification operators, we need a number of *criteria* which the defuzzification process can or should exhibit. We think that the main point of these criteria is not the search for the best universal defuzzification operator, but rather an evaluation of what properties are important for what types of applications. As an example, continuity and computational efficiency are of utmost importance for fuzzy controllers, but are far less critical for decision support systems.

We divide the defuzzification criteria in several groups, based on the mathematical structure that is needed in the universe X in order to be able to formulate the criteria. We distinguish between arbitrary universes, ordered universes, and universes with an algebraic structure such as the set of the real numbers $\mathbb{R}$.

Some of the criteria can be seen as a generalization of a selection from the defuzzification criteria for fuzzy numbers as proposed by Runkler [10].

2.1. Defuzzification in arbitrary universes

In the most general case, the universe just consists of a number of elements with no special inter-element relationships. A fuzzy set on a universe maps every element of the universe on an element from the unit interval $[0,1]$. Since there is no structure in the universe, the criteria in this section rely only on properties of and operations on the evaluation set $[0,1]$.

[3] This expression for the core of a fuzzy set corresponds with the one in Zadeh's seminal paper [16].

2.1.1. Core selection

The process of defuzzification of a fuzzy set A on X can be seen as the selection of a single element of X, based on the information conveyed in A. The weakest property that a fuzzy set A induces on X is an ordering of the elements of X. An element with a higher degree of membership is considered as having "more of the property that is related to the fuzzy set". For this, it seems reasonable to let the defuzzification operator make a selection amongst the elements with the largest degree of membership, i.e.,

$$\forall A \in F(X): D(A) \in \mathrm{core}(A). \tag{1}$$

We call (1) the *core selection criterion*. In [13] the selection of an element with largest degree of membership is called *semantically correct defuzzification*.

2.1.2. Scale invariance

A necessary criterion for any operation on one or more membership functions to be meaningful is that of scale invariance [9]. This means that for a defuzzification operator, the defuzzification value must not depend on a particular numerical representation within a set of permissible transforms. The defuzzification operator must be invariant under the permissible scale transformations.

Since the membership degrees can be interpreted on different scales, the exact formulation of the *scale invariance criterion* depends on the scale chosen. We will show that the stronger the scale, the weaker the criterion and vice versa.

2.1.2.1. Ordinal scale. For applications in which the only information conveyed in the fuzzy set is an ordering of the elements of X based on the degree of membership $A(x)$, the invariance is the relative order of the degrees of membership. As a consequence, the defuzzification operator should be invariant under any order preserving mapping f, i.e.,

$$D(fA) = D(A), \tag{2}$$

where

$$fA: F(X) \to [0,1]: x \to f(A(x)).$$

2.1.2.2. Interval scale. For applications in which the numerical representation of the degrees of membership has an arbitrary origin and an arbitrary unit, the degrees of membership are interpreted on an interval scale. In this case the ratio of differences of degree of membership is invariant. In the scale invariance criterion, the mapping f is restricted to a positive linear transformation, i.e.,

$$a \cdot A + b: F(X) \to [0,1]: x \to a \cdot A(x) + b \quad \forall a \in]0, +\infty[, \ \forall b \in \mathbb{R}$$

and the scale invariance criterion becomes

$$D(a \cdot A + b) = D(A) \quad \forall a \in]0, +\infty[, \ \forall b \in \mathbb{R}. \tag{3}$$

2.1.2.3. Ratio scale. For applications in which the numerical representation of the degrees of membership has a fixed origin but an arbitrary unit, the degrees of membership are interpreted on a ratio scale. In this case, the ratio of degrees of membership is invariant. The scale invariance criterion becomes

$$D(a \cdot A) = D(A) \quad \forall a \in]0, +\infty[. \tag{4}$$

This form of the scale invariance criterion corresponds to the μ-scaling constraint axiom proposed in [10].

2.1.2.4. Relative scale. When the numerical representation of the degrees of membership has a fixed unit but an arbitrary origin, the criterion becomes

$$D(A+b)=D(A), \quad \forall b \in \mathbb{R}. \tag{5}$$

This form of the scale invariance criterion relates to the μ-translation constraint axiom proposed in [10].

2.1.2.5. Absolute scale. Finally, for an absolute scale the criterion of scale invariance only has to hold for the identity transformation, i.e.,

$$D(1 \cdot A)=D(A) \tag{6}$$

which means that the criterion does not impose any restrictions.

2.2. Defuzzification in universes with an ordinal scale

In case there exists an ordinal scale in the universe, two more criteria can be defined:

2.2.1. Monotony

The *criterion of monotony* states that for two fuzzy sets A and B on X
if

$$B(D(A))=A(D(A))$$

and

$$\forall x < D(A): B(x) \leqslant A(x)$$

and

$$\forall x > D(A): B(x) \geqslant A(x)$$

then

$$D(B) \geqslant D(A) \tag{7}$$

and vice versa.

This means that when we have a defuzzification value $D(A)$ for a fuzzy set A, and for all elements on one side of $D(A)$ the degrees of membership do not increase, while for all elements on the other side of $D(A)$ the degrees of membership do not decrease, the defuzzification value should not move towards the former side. The defuzzification value can only move to that side where the elements are getting "better". This property is also mentioned in [10].

2.2.2. Triangular conorm criterion

The defuzzification value of the result of a triangular conorm operation S on two fuzzy sets A and B should reside in the interval bounded by the defuzzification values $D(A)$ and $D(B)$ [10]. Let $A, B \in F(X)$ and let $D(A) \leqslant D(B)$ then

$$D(A) \leqslant D(A \cup_{\mathrm{S}} B) \leqslant D(B). \tag{8}$$

In the conjunction of two fuzzy sets A and B where $D(A) \leqslant D(B)$, from a global point of view, the degrees of membership of the elements larger than $D(A)$ are increased more than the degrees of membership of the elements smaller than $D(A)$. As a result the defuzzification value should be larger than $D(A)$. The same arguments applied to B lead to (8).

Since the conjunction can be modeled by any triangular conorm, the *triangular conorm criterion* can be seen as a complete family of criteria. It is possible that a certain defuzzification operator fulfills the criterion only for a certain set of triangular conorms.

2.3. Defuzzification of fuzzy quantities

In this section, we recall some criteria for the defuzzification of fuzzy sets on the set of the real numbers $\mathbb{R}$.

2.3.1. x-Translation

This criterion states that the relative position of the defuzzification value should remain when the membership function is translated, i.e., let

$$B : X \to [0,1] : x \to A(x-b) \quad b \in \mathbb{R}$$

then

$$D(B) = D(A) + b. \tag{9}$$

This criterion was also proposed in [10].

2.3.2. x-Scaling

On multiplication of the x values with a constant factor, the relative position of the defuzzification value should remain, i.e., let

$$B : X \to [0,1] : x \to A(x/a), \quad a \in \mathbb{R}\backslash\{0\}$$

then

$$D(B) = a \cdot D(A). \tag{10}$$

This criterion was also proposed in [10].

2.3.3. Continuity

The *criterion of continuity* is motivated by the fact that a discontinuous defuzzification seems unnatural. A small variation in any of the degrees of membership should not result in a big change in the defuzzification value.

$$\forall x_0 \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \Delta^* > 0 \ni \forall \Delta \text{ for which } |\Delta| < \Delta^* \text{ it holds that } |D(\Delta_{x_0} A) - D(A)| < \varepsilon, \tag{11}$$

where

$$\Delta_{x_0} A(x) = \begin{cases} A(x) + \Delta & \text{for } x = x_0, \\ A(x) & \text{otherwise.} \end{cases}$$

Especially in fuzzy control systems, the criterion of continuity is of extreme importance. These systems need input–output continuity, which means that a small variation of any of the input parameters should only result in small changes of the output values. This leads to a constraint of continuity for all parts of the system: from continuous membership functions for the input terms to a continuous defuzzification operator. Note that in the framework of fuzzy control sometimes continuity is indicated as robustness.

2.4. Miscellaneous criteria

Finally, we can formulate criteria that are not directly related to any theoretical concepts or foundations, but that are of more practical importance.

2.4.1. Computational efficiency

In some applications, the computational efficiency of the defuzzification operator is extremely important. Especially in fuzzy controllers the choice of the defuzzification operator is influenced by the number of operations that is needed to calculate the defuzzification value.

2.4.2. Transparency for system design

Since the final goal is the construction of a working system, the ease of building and tuning the system for a certain choice of defuzzification operator is also an important factor. A transparent, easy to understand defuzzification operator is to be preferred over a complicated mathematical formula with no clear foundations.

3. Evaluation of defuzzification operators

Based on the defuzzification criteria presented in the previous section, we can now make an evaluation of the most widespread defuzzification operators.

It is possible to classify the existing defuzzification methods based on their technical and structural properties. We can distinguish between general defuzzification methods and specific ones, and between basic defuzzification operators and extended ones.

General defuzzification methods expect the fuzzy system to produce a single fuzzy output set, that is then defuzzified. *Specific defuzzification* methods use the typical structure of the fuzzy system to combine the defuzzification step with other processes in order to obtain a more efficient computation scheme. The specific defuzzification methods apply in general to fuzzy controllers where the aggregation and defuzzification step are combined to yield a more efficient processing.

The *extended defuzzification operators* can be differentiated from the *basic operators* by the existence of one or more parameters that have to be chosen by the designer to fully determine the defuzzification process.

We can further classify the operators based on the mathematical structure that is needed in the universe X for the operators to be meaningful very much in the way that we classified the different defuzzification criteria. Some operators can be used on arbitrary universes, others need a universe with a (relevant) ordinal scale, while some can only be used on the set of the real numbers $\mathbb{R}$.

For a rather extended classification of defuzzification operators together with an evaluation of their properties according to the defuzzification criteria, we refer to [14]. In this paper we classify the most widely used defuzzification operators into different classes based on a common basis. Generally, a class consists of a number of defuzzification methods that are built around one or more basic general defuzzification methods. These basic general defuzzification methods are extended with one or more parameters on one hand, and they are modified for speed into specific methods on the other hand. In some cases they also give rise to extended specific methods, i.e., defuzzification methods that combine different processes of the fuzzy system to gain performance and that use one or more parameters to specify their behavior.

For every class, we will evaluate the basic general defuzzification methods with the criteria proposed in the previous section.

In general all defuzzification operators can be formulated in discrete form (via Σ) as well as in continuous form (via $\int$). For simplicity, we will restrict ourselves to the discrete formulation.

3.1. Maxima methods and derivatives

The maxima methods have the common property that they select an element from the core as defuzzification value. None of the other elements is a valid candidate since they all have a smaller degree of membership and as a consequence they are less suitable. By definition, these defuzzification methods fulfill the core selection criterion.

In general this group of methods satisfies (1)–(7). The criteria that are aimed more specifically at fuzzy sets on $\mathbb{R}$ and the application in fuzzy controllers (11) are not fulfilled in general. In our opinion the primary use of the maxima defuzzification methods lies in fuzzy knowledge systems. In addition, all methods are computationally very efficient. The group includes:

3.1.1. Random choice of maxima (RCOM)

In this basic general method all elements of the core of A are considered as being equal candidates for the defuzzification value. Under the assumption that the core only contains a finite number of elements, the defuzzification value can be calculated as the outcome of a random experiment with probabilities

$$\mathrm{Prob}(D(A)=x_0)=\begin{cases}\dfrac{1}{|\mathrm{core}(A)|} & \text{if } x_0\in\mathrm{core}(A),\\ 0 & \text{otherwise.}\end{cases}$$

Because this method is nondeterministic, the only criterion it fulfills is the core selection criterion. This method can be used for arbitrary universes.

3.1.2. First of maxima and last of maxima (FOM, LOM)

The FOM method is only defined on universes with an ordinal scale. In addition it requires that the core of A has a smallest element and it selects this element as defuzzification value, i.e.,

$$\mathrm{FOM}(A)=\min\mathrm{core}(A).$$

The LOM method is also only defined on universes with an ordinal scale. It requires that the core of A has a greatest element and it selects this element as defuzzification value, i.e.,

$$\mathrm{FOM}(A)=\max\mathrm{core}(A).$$

It is readily shown that these methods fulfill the core selection criterion (1), the criterion of scale invariance (2)–(6), the criterion of monotony (7), the triangular conorm criterion for max as triangular conorm (8), and the criterion of x-translation (9) (see Appendix).

3.1.3. Middle of maxima (MOM)

Like the FOM and LOM methods, this defuzzification method requires an ordinal scale. If the core contains an odd number of elements, then the middle element of the core is selected. Otherwise, the defuzzification value depends on the implementation as follows.

For a crisp set S and $x_0\in S$ let

$$S_{<x_0}=\{x\mid x\in S \text{ and } x<x_0\}\quad\text{and}\quad S_{>x_0}=\{x\mid x\in S\text{ and } x>x_0\}.$$

If $|\mathrm{core}(A)|$ is odd then

$$|\mathrm{core}(A)_{<\mathrm{MOM}(A)}|=|\mathrm{core}(A)_{>\mathrm{MOM}(A)}|$$

else

$$|\mathrm{core}(A)_{<\mathrm{MOM}(A)}| = |\mathrm{core}(A)_{>\mathrm{MOM}(A)}| \pm 1$$

depending on the choice of implementation.

Remark that the MOM defuzzification method is deterministic. There are just two possible definitions depending on the treatment of a core with an odd number of elements. Or we can say that there are two different MOM methods: $\mathrm{MOM}_<$ and $\mathrm{MOM}_>$.

It can be shown that MOM satisfies the core selection criterion (1), the criterion of scale invariance (2)–(6), the criterion of monotony (7), and the criterion of x-translation (9) (see Appendix).

3.2. *Distribution methods and derivatives*

As was shown by Yager and Filev introducing the BADD-method [1], there exist a number of defuzzification operators that in fact first convert the membership function into a probability distribution and then compute the expected value. Considering the lack of a theoretical foundation for this conversion, the main reason for the use of these operators seems to be the property of continuity which is highly desirable in fuzzy controllers.

This group of defuzzification operators includes both general and specific methods.

3.2.1. *General distribution methods*

3.2.1.1. Center of gravity (COG). Probably the best known defuzzification operator is the *center of gravity* defuzzification method. It is a basic general defuzzification method that computes the center of gravity of the area under the membership function. As a consequence, it can only be used for fuzzy sets on $\mathbb{R}$ [4]

$$\mathrm{COG}(A) = \sum_{x_{\min}}^{x_{\max}} x \cdot A(x) \Bigg/ \sum_{x_{\min}}^{x_{\max}} A(x).$$

The formulae shows that COG calculates the expected value when A is considered to be a probability distribution.

The COG does not fulfill the core selection criterion (1), nor the criterion of scale invariance (2)–(6), in general (except for ratio scale). It satisfies the criterion of monotony (7), but not the triangular conorm criterion (8). It also satisfies the criteria of x-translation (9), x-scaling (10) and continuity (11) (see Appendix).

3.2.1.2. Mean of maxima (MeOM). The MeOM method is derived from COG. It calculates the mean of all elements of the core of a fuzzy set, which in fact equals the COG of the core.

$$\mathrm{MeOM}(A) = \frac{\sum_{x \in \mathrm{core}(A)} x}{|\mathrm{core}(A)|}.$$

MeOM does not satisfy the core selection criterion as it does not guarantee the selection of an element from the core for nonconvex fuzzy sets. For convex fuzzy sets, it satisfies the core selection criterion (1) and belongs to the group of maxima methods. Further it satisfies the criteria of scale invariance (2)–(6), monotony (7), x-translation (9) and x-scaling (10).

[4] As in most applications the universe X is a bounded subset of $\mathbb{R}$, we denote a sum over all elements of the universe as a sum from $x_{\min}$ to $x_{\max}$.

3.2.1.3. Basic defuzzification distributions (BADD). Filev and Yager [1] introduced the BADD method which is an extended version of the COG method:

$$\mathrm{BADD}(A,\gamma)=\sum_{x_{\min}}^{x_{\max}} x\cdot A(x)^{\gamma} \Bigg/ \sum_{x_{\min}}^{x_{\max}} A(x)^{\gamma} \quad \text{where } \gamma\in[0,+\infty[.$$

The parameter γ can be used to adjust the method with the special cases:

$$\mathrm{BADD}(A,0)=\mathrm{MeOS}(A) \quad \text{where MeOS} = \text{mean of support},$$

$$\mathrm{BADD}(A,1)=\mathrm{COG}(A),$$

$$\lim_{\gamma\to\infty}\mathrm{BADD}(A,\gamma)=\mathrm{MeOM}(A).$$

The BADD method was deduced based on the general principle of Yager and Filev to convert a fuzzy set into a probability distribution. This distribution is then used to calculate the expected value. The BADD method uses the distribution with probabilities

$$\mathrm{Prob}(D(A)=x_0)=\frac{A(x_0)^{\gamma}}{\sum_{x_{\min}}^{x_{\max}} A(x)^{\gamma}} \quad \text{where } \gamma\in[0,+\infty[.$$

The parameter γ can be seen as the confidence one has in the system. The higher the confidence, the more one is tempted to choose MeOM.

3.2.1.4. Generalized level set defuzzification (GLSD). Another extended method that was proposed by Filev and Yager [2] is based upon the α-cuts $\{A_{\alpha_1},\ldots,A_{\alpha_N}\}$ of the fuzzy set A:

$$\mathrm{GLSD}(A,\gamma)=\frac{\sum_{i=1}^{N} c_i m_i \gamma^i}{\sum_{i=1}^{N} c_i \gamma^i} \quad \text{with } \gamma\in\,]0,+\infty[,$$

where N equals the number of α-cuts (determined by the sampling step of the membership values), c_i the number of elements in the ith α-cut A_{α_i}, m_i the average value of the ith α-cut, and γ confidence level. The GLSD method contains as special cases:

$$\lim_{\gamma\to 0}\mathrm{GLSD}(A,\gamma)=\mathrm{MeOS}(A),$$

$$\mathrm{GLSD}(A,1)=\mathrm{COG}(A),$$

$$\lim_{\gamma\to\infty}\mathrm{GLSD}(A,\gamma)=\mathrm{MeOM}(A).$$

Again, the parameter γ can be seen as the confidence that one has in the system. For full confidence, MeOM is chosen. When there is very little confidence ($\gamma \ll 1$) the result tends to MeOS. Yager and Filev also propose an algorithm to modify the defuzzification method when the confidence in the fuzzy system increases.

3.2.1.5. Indexed center of gravity (ICOG). The ICOG method [7] computes the center of gravity of the fuzzy set that is obtained after putting all membership values below a certain threshold α equal to zero:

$$\mathrm{ICOG}(A,\alpha)=\sum_{x\in A_{\alpha}} xA(x) \Bigg/ \sum_{x\in A_{\alpha}} A(x).$$

This method contains the MeOM and COG methods as special cases:

$$\mathrm{ICOG}(A,\mathrm{height}(A))=\mathrm{MeOM}(A),$$

$$\mathrm{ICOG}(A,0)=\mathrm{COG}(A).$$

3.2.1.6. Semi-linear defuzzification (SLIDE). Another defuzzification method proposed by Yager and Filev [15]:

$$\mathrm{SLIDE}(A,\alpha,\beta)=\frac{(1-\beta)\sum_{X_L} xA(x)+\sum_{X_H} xA(x)}{(1-\beta)\sum_{X_L} A(x)+\sum_{X_H} A(x)},$$

where $\alpha\in[0,\mathrm{height}(A)]$ and $\beta\in[0,1]$ and

$$X_\mathrm{L}=\{x\mid x\in X \text{ and } A(x)<\alpha\},$$
$$X_\mathrm{H}=\{x\mid x\in X \text{ and } A(x)\geqslant\alpha\}=\mathrm{co}\,X_L.$$

This method is characterized by two parameters: a confidence level α and a rejection parameter β. For $\beta=1$ all membership degrees $A(x)<\alpha$ are rejected. When $\alpha=\mathrm{height}(A)$, the parameter β can be used to migrate from the COG-method ($\beta=0$) to the MeOM-method ($\beta=1$). The SLIDE method contains as special cases:

$$\mathrm{SLIDE}(A,0,\beta)=\mathrm{COG}(A) \qquad \forall\beta\in[0,1],$$
$$\mathrm{SLIDE}(A,\alpha,0)=\mathrm{COG}(A) \qquad \forall\alpha\in\,]0,\mathrm{height}(A)],$$
$$\mathrm{SLIDE}(A,\alpha,1)=\mathrm{ICOG}(A,\alpha) \qquad \forall\alpha\in\,]0,\mathrm{height}(A)],$$
$$\mathrm{SLIDE}(A,\mathrm{height}(A),1)=\mathrm{MeOM}(A).$$

3.2.2. Specific distribution methods

The specific distribution methods are based on the COG method, but they use the typical structure of the fuzzy system (or fuzzy controller) to obtain a more efficient computation scheme. More specific, they combine the aggregation and defuzzification steps into one computational process. For a detailed description of the building blocks of a fuzzy controller, we refer to [14]. Important for defuzzification is that the output of the controller has a number of *fuzzy output sets* A_i (typically $i=5,\dots,9$) defined on it. The fuzzy inference engine computes for each of these output sets a *degree* of applicability α_i.

In order to be able to use a general defuzzification method, the fuzzy output sets together with their degrees should be used to construct a single fuzzy set which is the output of the fuzzy system. This step is generally referred to as *aggregation*. After the aggregation, any of the general defuzzification methods can be used to compute a single numerical value.

Another possibility is to associate with every fuzzy output set A_i a characteristic *numerical value* a_i, and to calculate the defuzzification value immediately using a_i and α_i. This procedure eliminates the aggregation step, resulting in a gain in performance.

In general, the values a_i depend on the shape of the fuzzy output set A_i and on the degree α_i. However, very often the a_i are considered to depend only on the shape of A_i. In this case, they can be computed off-line, yielding another gain in performance. Usually $a_i=\mathrm{MOM}(A_i)$ is chosen for symmetrical A_i.

3.2.2.1. Fuzzy mean (FM). The FM defuzzification method is a basic specific method that can be seen as an optimization of the COG method. Like COG, it also computes a weighted sum. The difference is that FM uses pre-calculated numerical values a_i for each of the fuzzy sets of the output universe.

$$\mathrm{FM}(A)=\sum_{i=1}^{N_A}\alpha_i a_i\Bigg/\sum_{i=1}^{N_A}\alpha_i,$$

where N_A represents the number of fuzzy output sets, α_i, the degree of each output set (as a result of the inference), and a_i the numerical value of output set i.

The FM method has the advantage of being very efficient. This makes the FM method one of the most widely used defuzzification methods in fuzzy controllers.

3.2.2.2. Weighted fuzzy mean (WFM). The WFM defuzzification method is an extended specific method that is a parameterized version of the FM method [6].

$$\mathrm{WFM}(A) = \sum_{i=1}^{N_A} w_i \alpha_i a_i \Bigg/ \sum_{i=1}^{N_A} w_i \alpha_i,$$

where w_i is the weight associated with output set i. The main characteristic of this method is the possibility to introduce a degree of importance for every output set. If the weights are chosen by

$$w_i = \mathrm{area}(O_{A_i}),$$

where O_{A_i} denotes the ordinate set of A_i, the WFM method looks even more like the COG method than FM does. Every output set is interpreted as a rectangle with unit height and width w_i. Of course the fact remains that the overlap between adjacent output sets is counted more than once, which means that the implicit aggregation consists of a normal sum.

Another possibility is to replace the factor $w_i \alpha_i$ by

$$w_i(\alpha) = \mathrm{area}(O_{A_i^\alpha}),$$

where $A_i^\alpha(x) = \min(\alpha, A_i(x))\ \forall x \in \mathbb{R}$. In this case, the weights have no fixed value, but they depend on the degree to which the output term is fulfilled. Hellendoorn calls this method *center of sums* (COS) [4].

3.2.2.3. Quality method (QM). The QM method is a basic specific defuzzification method proposed in [4].

$$\mathrm{QM}(A) = \frac{\sum_{i=1}^{N_A} (\alpha_i/d_i) a_i}{\sum_{i=1}^{N_A} \alpha_i/d_i},$$

where d_i equals the width of the support of A_i. This method contains a vertical (α_i) as well as a horizontal component (d_i). The aim of the QM method is to increase the importance of the "more crisp" output sets (i.e., the sets with a more narrow support). The weight of a particular term is big if the support is small and the degree α_i big.

The QM method can be seen as a special case of the WFM method where

$$w_i = 1/d_i.$$

3.2.2.4. Extended quality method (EQM). The EQM method is an extended specific defuzzification method proposed by Hellendoorn and Thomas [4].

$$\mathrm{EQM}(A, \xi) = \frac{\sum_{i=1}^{N_A} (\alpha_i/d_i^\xi) a_i}{\sum_{i=1}^{N_A} \alpha_i/d_i^\xi} \quad \text{where } \xi \in [0, +\infty[.$$

The EQM contains as special cases many methods:

$$\mathrm{EQM}(A, 0) = \mathrm{FM}(A), \qquad \mathrm{EQM}(A, 1) = \mathrm{QM}(A).$$

For $\xi > 1$ the support is even more important than in the QM method. In another definition of the EQM method, the area under A_i is used as weight instead of the width d_i [5].

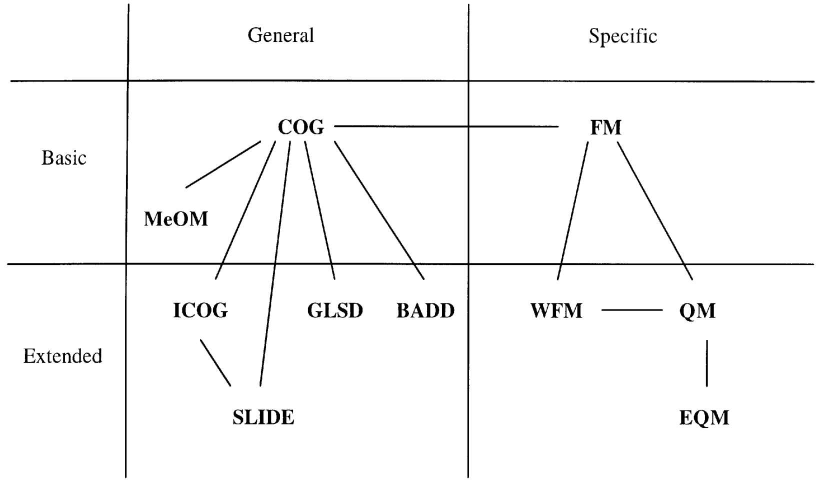

Fig. 1. Distribution methods.

3.2.3. Overview of the distribution methods

Fig. 1 gives an overview of the different distribution methods.

3.3. Area methods

The third group of defuzzification methods uses the area under the membership function to determine the defuzzification value. As for the distribution methods, this group is applicable primarily for fuzzy controllers.

3.3.1. Center of area (COA)

The COA method is a basic general defuzzification operator for ordered universes. The COA minimizes the expression

$$\left| \sum_{x=\inf(X)}^{\mathrm{COA}(A)} A(x) - \sum_{x=\mathrm{COA}(A)}^{\sup(X)} A(x) \right| .$$

The defuzzification value divides the area under the membership function in two (more or less) equal parts.

Every degree of membership is seen as a package of information, and the defuzzification value is chosen in a way that the information that indicates that the defuzzification value is smaller, is (more or less) equal to the amount of information that indicates that the defuzzification value is larger. The question that remains here is whether it makes any sense to add the membership degrees.

The COA can be calculated by using two running sums: one coming from the left and one coming from the right. Always the smallest sum is augmented by the membership value of the next element. The COA method does not satisfy the core selection criterion (1), satisfies the scale invariance criterion only for ratio scale and absolute scale (4) and (6). It satisfies the criterion of monotony, but not the triangular conorm criterion neither for max nor for the probabilistic sum. The COA method satisfies the x-translation and x-scaling (9) and (10) and the criterion of continuity (11) (see Appendix).

3.3.2. Extended center of area (ECOA)

The ECOA method [12] replaces the $A(x)$ by $A(x)^\gamma$, where $\gamma \in]0, +\infty[$ is a kind of confidence factor. The ECOA method then minimizes the expression

$$\left| \sum_{x=\inf(X)}^{\mathrm{ECOA}(A)} A(x) - \sum_{x=\mathrm{ECOA}(A)}^{\sup(X)} A(x) \right|.$$

Special cases of this are as follows:

$$\mathrm{ECOA}(A,0) = \mathrm{MeOS}(A),$$

$$\mathrm{ECOA}(A,1) = \mathrm{COA}(A),$$

$$\lim_{\gamma\to\infty} \mathrm{ECOA}(A,\gamma) = \mathrm{MeOM}(A).$$

In general, the ECOA method behaves like the basic COA method, but for $\gamma > 1$ the high membership degrees are more important.

Runkler and Glesner [12] note that the ECOA method is computational much more efficient than the BADD method.

3.4. Miscellaneous methods

Finally, there exist a number of methods that do not belong to any of the previous groups. Amongst others, this group includes.

3.4.1. Constraint decision defuzzification (CDD)

In [11] defuzzification is described as making a crisp decision under fuzzy constraints. CDD defuzzification can be seen as adding to the fuzzy system an extra fuzzy component that cooperates with a basic defuzzification operator (e.g., MOM). The fuzzy constraints are related only to the defuzzification process itself. E.g.: “the defuzzification value should be big”, “the defuzzification value should be near the COA”, etc.

The parameters of this method consist of the fuzzy constraints on one hand, and the basic defuzzification method on the other hand.

3.4.2. Fuzzy clustering defuzzification (FCD)

Genther et al. [3] proposed a defuzzification method based on fuzzy clustering. Fuzzy clustering partitions a set of input data into a number of subsets called clusters. Every element gets a membership degree in every cluster. The Fuzzy C-Means algorithm computes the center of each cluster and gives it a membership degree. The FCD method selects the center with the greatest membership degree as defuzzification value. The parameter in this method is the number of groups that the Fuzzy C-Means has to put the input data in. Genther shows that the choice of the number of groups has a big influence on the defuzzification value.

4. Conclusion

We have shown that the criteria that can be formulated for the defuzzification of a fuzzy set depend on the structure of the underlying universe and on the meaning and interpretation of the membership values. We investigated the main defuzzification methods and marked three clearly distinct classes of methods: The maxima methods, the distribution methods, and the area methods.

Table 1
Overview of defuzzification criteria

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
FOM/LOM	Yes	Yes	Yes	Yes	Yes	Yes	'max'	Yes	—	—
MOM	Yes	Yes	Yes	Yes	Yes	Yes	—	Yes	—	—
COG	—	—	—	Yes	—	Yes	—	Yes	Yes	Yes
COA	—	—	—	Yes	—	Yes	—	Yes	Yes	Yes

Note: 'max' = for *max* as triangular conorm

The maxima methods behave well with respect to the more basic defuzzification criteria such as core selection, scale invariance, monotony and the triangular conorm criterion. The two other groups have as main characteristics that they do not fulfill the basic defuzzification criteria but they provide the highly practical property of continuity. This makes them the favorite candidates to be embedded in fuzzy controllers.

Table 1 summarizes the properties of the three classes of defuzzification methods by showing the properties of the basic method of each class. The proofs of the properties can be found in the Appendix.

Appendix. Mathematical proofs and explanations

This appendix contains some proofs and explanations that were omitted from the main text in order not to lose focus on the general ideas. The proofs mainly relate to the evaluation of the defuzzification criteria for certain defuzzification operators.

A.1. First of maxima and last of maxima (FOM, LOM)

We only proof the defuzzification criteria for FOM. The proofs of LOM are completely analogous.

(1): $\mathrm{FOM}(A) = \min \mathrm{core}(A) \in \mathrm{core}(A)$
(2)–(6):

$$\begin{aligned}\mathrm{FOM}(fA) &= \min \mathrm{core}(fA)\\ &= \min \mathrm{core}(A) \quad \mathrm{core}(fA) = \mathrm{core}(A) \text{ for every order preserving mapping } f\\ &= \mathrm{FOM}(A)\end{aligned}$$

and so all of (2)–(6) are fulfilled.
(7): Let $B(D(A)) = A(D(A))$ and $\forall x < D(A): B(x) \leqslant A(x)$ and $\forall x > D(A): B(x) \geqslant A(x)$.
Further let

$$\begin{aligned}\mathrm{FOM}(B) &< \mathrm{FOM}(A)\\ &\Rightarrow \min \mathrm{core}(B) < \min \mathrm{core}(A)\\ &\Rightarrow \exists x < \min \mathrm{core}(A): B(x) = \mathrm{height}(B) \text{ and } A(x) < \mathrm{height}(A)\\ &\Rightarrow \exists x < \min \mathrm{core}(A): B(x) \geqslant \mathrm{height}(A) \text{ and } A(x) < \mathrm{height}(A)\\ &\Rightarrow \exists x < D(A): B(x) > A(x)\end{aligned}$$

this is impossible according to the definition of $B(x)$.

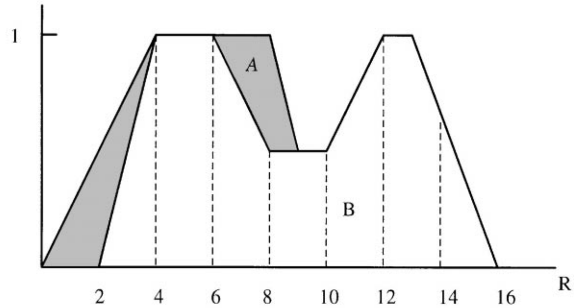

Fig. A.1.

Converse: Let $B(D(A)) = A(D(A))$ and $\forall x > D(A): B(x) \leqslant A(x)$ and $\forall x < D(A): B(x) \geqslant A(x)$. Further let

$$\mathrm{FOM}(A) < \mathrm{FOM}(B)$$

(a) height$(A) =$ height(B). Then we have

$$B(\mathrm{FOM}(A)) < \mathrm{height}\,(B)$$

$$\Rightarrow B(\mathrm{FOM}(A)) < \mathrm{height}\,(A)$$

$$\Rightarrow B(D(A)) < A(D(A)). \qquad \text{This is impossible.}$$

(b) height$(A) <$ height(B). Then we have

$$\exists x > \mathrm{FOM}(A): B(x) = \mathrm{height}(B)$$

$$\Rightarrow \exists x > \mathrm{FOM}(A): B(x) > \mathrm{height}(A)$$

$$\Rightarrow \exists x > D(A): B(x) > A(x). \qquad \text{This is impossible.}$$

(8 max):

$$\begin{aligned}\mathrm{FOM}(A \cup B) &= \min \mathrm{core}(A \cup B)\\ &= \min \mathrm{core}(A) = \mathrm{FOM}(A) \quad \text{if height}(A) > \text{height}\,(B),\\ &= \min \mathrm{core}(B) = \mathrm{FOM}(B) \quad \text{if height}(B) > \text{height}\,(A),\\ &= \min(\mathrm{FOM}(A), \mathrm{FOM}(B)) \quad \text{if height}(A) = \text{height}\,(B).\end{aligned}$$

(9) Can easily be verified.
(10) Applies only for positive values of a.
(11) Is not fulfilled as can be shown by the example in Fig. A.1: $A(x) = 1$ for $x \in [4, 8]$ and $\mathrm{FOM}(A) = 4$. Any $\Delta > 0$ added to $A(7)$ will make $\mathrm{FOM}(A)$ change from 4 to 7.

A.2. Middle of maxima (MOM)

The proofs and counter-examples for MOM are completely analogous to the ones for FOM, except for (8). (8) is not fulfilled for *max* as triangular conorm as can be seen in Fig. A.1: $\mathrm{MOM}(B) = 5.5$, $\mathrm{MOM}(A) = 6$, $\mathrm{MOM}(A \cup B) = 6.5$ and hence $\mathrm{MOM}(B) < \mathrm{MOM}(A) < \mathrm{MOM}(A \cup B)$.

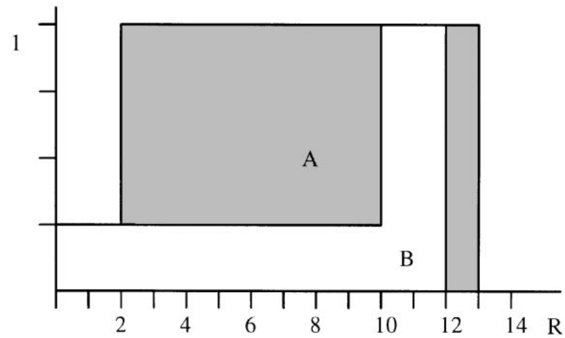

Fig. A.2.

A.3. Center of gravity (COG)

(1) Is not fulfilled as can be seen from Fig. A.2: $\text{core}(B) = [10, 12]$ and $\text{COG}(B) = 7.67$.

(2)–(6) Is only fulfilled for ratio scale (4):

$$\frac{\sum_{x_{\min}}^{x_{\max}} x(aA(x) + b)}{\sum_{x_{\min}}^{x_{\max}} aA(x) + b} = \frac{\sum_{x_{\min}}^{x_{\max}} xA(x) + \sum_{x_{\min}}^{x_{\max}} x(b/a)}{\sum_{x_{\min}}^{x_{\max}} A(x) + \sum_{x_{\min}}^{x_{\max}} (b/a)}$$

this is only equal to $\text{COG}(A)$ if $b = 0$. (7) is fulfilled because an increase in membership degrees on one side of the center of gravity, together with a decrease in membership degrees on the other side, will always make the center of gravity shift towards the former side.

(8 max) is not fulfilled, as can be seen on Fig. A.2: $\text{COG}(A) = 7.5$, $\text{COG}(B) = 7.67$, $\text{COG}(A \cup B) = 7.22$ and hence $\text{COG}(A \cup B) < \text{COG}(A) < \text{COG}(B)$.

(9) is fulfilled:

$$\text{COG}(A) = \frac{\sum_{x_{\min}}^{x_{\max}} xA(x)}{\sum_{x_{\min}}^{x_{\max}} A(x)} = \frac{\sum_{x_{\min}+b}^{x_{\max}+b} (x - b)A(x - b)}{\sum_{x_{\min}+b}^{x_{\max}+b} A(x - b)} = \frac{\sum_{x_{\min}+b}^{x_{\max}+b} xA(x - b)}{\sum_{x_{\min}+b}^{x_{\max}+b} A(x - b)} - b = \text{COG}(B) - b.$$

(10) is fulfilled:

$$\text{COG}(A) = \frac{\sum_{x_{\min}}^{x_{\max}} xA(x)}{\sum_{x_{\min}}^{x_{\max}} A(x)} = \frac{\sum_{ax_{\min}}^{ax_{\max}} (x/a)A(x/a)}{\sum_{ax_{\min}}^{ax_{\max}} A(x/a)} = \frac{1}{a}\frac{\sum_{ax_{\min}}^{ax_{\max}} xB(x)}{\sum_{ax_{\min}}^{ax_{\max}} B(x)} = \frac{1}{a}\,\text{COG}(B).$$

(11) We prove that

$$\forall x_0 \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \Delta^* > 0 \ni \forall \Delta \text{ for which } |\Delta| < \Delta^*, \text{we have } |\text{COG}(\Delta_{x_0} A) - \text{COG}(A)| < \varepsilon,$$

where

$$\Delta_{x_0} A(x) = \begin{cases} A(x) + \Delta & \text{for } x = x_0, \\ A(x) & \text{otherwise.} \end{cases}$$

We obtain successively

$$\mathrm{COG}(\Delta_{x_0}A)-\mathrm{COG}(A)=\frac{x_0\Delta+\sum_{x_{\min}}^{x_{\max}}xA(x)}{\Delta+\sum_{x_{\min}}^{x_{\max}}A(x)}-\frac{\sum_{x_{\min}}^{x_{\max}}xA(x)}{\sum_{x_{\min}}^{x_{\max}}A(x)}$$

$$=\frac{(x_0\Delta+\sum_{x_{\min}}^{x_{\max}}xA(x))\sum_{x_{\min}}^{x_{\max}}A(x)-\sum_{x_{\min}}^{x_{\max}}xA(x)(\Delta+\sum_{x_{\min}}^{x_{\max}}A(x))}{(\Delta+\sum_{x_{\min}}^{x_{\max}}A(x))\sum_{x_{\min}}^{x_{\max}}A(x)}$$

$$=\frac{-\Delta\sum_{x_{\min},x\neq x_0}^{x_{\max}}xA(x)}{(\Delta+\sum_{x_{\min}}^{x_{\max}}A(x))\sum_{x_{\min}}^{x_{\max}}A(x)}$$

$$=\frac{-\alpha\Delta}{\beta(\beta+\Delta)}\quad\text{where }\alpha=\sum_{x_{\min},x\neq x_0}^{x_{\max}}xA(x)\quad\text{and}\quad\beta=\sum_{x_{\min}}^{x_{\max}}A(x)>0.$$

So now we have to prove that $\forall\varepsilon>0,\ \exists\Delta^*>0\ni\forall\Delta$ for which $|\Delta|<\Delta^*$, we have $\left|\frac{-\alpha\Delta}{\beta(\beta+\Delta)}\right|<\varepsilon$ or, equivalently

$$\left|\frac{-\alpha\Delta}{\beta(\beta+\Delta)}\right|<\varepsilon\ \Leftrightarrow\ \left(\frac{-\alpha\Delta}{\beta(\beta+\Delta)}\right)^2<\varepsilon^2$$

$$\Leftrightarrow\ \alpha^2\Delta^2<\varepsilon^2\beta^2(\beta^2+2\Delta\beta+\Delta^2)$$

$$\Leftrightarrow\ (\alpha^2-\varepsilon^2\beta^2)\Delta^2-2\varepsilon^2\beta^3\Delta-\varepsilon^2\beta^4<0. \tag{$*$}$$

Hence, for $\Delta=0$: $-\varepsilon^2\beta^4<0$ so 0 lies within the solution interval. Solving the quadratic equation in Δ gives the solutions

$$\Delta_1=\frac{-\varepsilon\beta^2(\varepsilon\beta+\alpha)}{\alpha^2-\varepsilon^2\beta^2},\qquad \Delta_2=\frac{\varepsilon\beta^2(\alpha-\varepsilon\beta)}{\alpha^2-\varepsilon^2\beta^2}.$$

(1) Suppose $\alpha^2-\varepsilon^2\beta^2>0$.
  (1.1) If $\alpha<0$ then $\Delta_2<0$ and, because of (1), we have $\alpha<-\varepsilon\beta$. Hence $\Delta_1>0$ and condition ($*$) is fulfilled if $\Delta\in]\Delta_2,\Delta_1[$ or if $|\Delta|<\min(|\Delta_1|,|\Delta_2|)$.
  (1.2) If $\alpha>0$ then $\Delta_1<0$ and, because of (1), we have $\alpha>\varepsilon\beta$. Hence $\Delta_2>0$ and condition ($*$) is fulfilled if $\Delta\in]\Delta_1,\Delta_2[$ or if $|\Delta|<\min(|\Delta_1|,|\Delta_2|)$.
(2) Suppose $\alpha^2-\varepsilon^2\beta^2<0$.
  (2.1) If $\alpha<0$ then $\Delta_2>0$ and, because of (2), we have $\alpha>-\varepsilon\beta$. Hence $\Delta_1>0$ and condition ($*$) is fulfilled if $|\Delta|<\min(|\Delta_1|,|\Delta_2|)$.
  (2.2) If $\alpha>0$ then $\Delta_1>0$ and, because of (2), we have $\alpha<\varepsilon\beta$. Hence $\Delta_2>0$ and condition ($*$) is fulfilled if $|\Delta|<\min(|\Delta_1|,|\Delta_2|)$.
(3) Suppose $\alpha^2-\varepsilon^2\beta^2=0$.

Then condition ($*$) is fulfilled for every Δ in $\mathbb{R}^+$.

A.4. Center of area (COA)

(1) Is not fulfilled as can be seen from Fig. A.3: $\mathrm{core}(B)=[10,12]$ and $\mathrm{COA}(B)=9$.

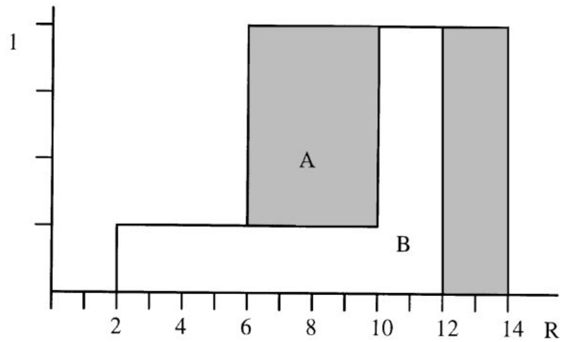

Fig. A.3.

(2) Is only fulfilled for ratio scale because

$$\sum_{x_{\min}}^{\mathrm{COA}(A)} aA(x) + b = a \sum_{x_{\min}}^{\mathrm{COA}(A)} A(x) + \sum_{x_{\min}}^{\mathrm{COA}(A)} b, \tag{A.1}$$

$$\sum_{\mathrm{COA}(A)}^{x_{\max}} aA(x) + b = a \sum_{\mathrm{COA}(A)}^{x_{\max}} A(x) + \sum_{\mathrm{COA}(A)}^{x_{\max}} b. \tag{A.2}$$

In general, (A.1) is only equal to (A.2) when $b = 0$.

(7) Is fulfilled because an increase of the degrees of membership on one side of COA(A) together with a decrease of membership degrees on the other side can result in an imbalance between the two areas. This imbalance can only be corrected by shifting the COA(A) towards the former side.

(8) Is not fulfilled as can be seen in Fig. A.3: $\mathrm{COA}(A) = 10$, $\mathrm{COA}(B) = 10$, $\mathrm{COA}(A \cup B) = 9.5$ and hence $\mathrm{COA}(A \cup B) < \mathrm{COA}(A) = \mathrm{COA}(B)$.

(9) Can easily be verified.

(10)

$$\begin{aligned}
\sum_{-\infty}^{\mathrm{COA}(A)} A(x) = \sum_{\mathrm{COA}(A)}^{+\infty} A(x) \quad &\Rightarrow \quad \sum_{-\infty}^{a\,\mathrm{COA}(A)} A\left(\frac{x}{a}\right) = \sum_{a\mathrm{COA}(A)}^{+\infty} A\left(\frac{x}{a}\right) \\
&\Rightarrow \quad \sum_{-\infty}^{a\,\mathrm{COA}(A)} B(x) = \sum_{a\,\mathrm{COA}(A)}^{+\infty} B(x) \\
&\Rightarrow \quad \mathrm{COA}(B) = a\mathrm{COA}(A)
\end{aligned}$$

(11) is fulfilled, because it can be shown easily that we can choose $\Delta^* = \varepsilon$.

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