1. Laplace Transform Table

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

2. Laplace Transform Theorems

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) \;= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] \;= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] \;= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] \;= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] \;= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] \;= sF(s)-f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] \;= s^2F(s)-sf(0-)-f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] \;= s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] \;= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) \;= \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) \;= \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

3. Partial Fraction Expansion (PFE)

We are familiar with combining fractions over common denominators:

$$\frac{2}{x+1}+\frac{3}{x+2}=\frac{2x+4+3x+3}{(x+1)(x+2)}=\frac{5x+7}{x^2+3x+2}$$

The inverse of this procedure is PFE.

(1) Steps for PFE

1) Check that the degree of the numerator must be less than the degree of the denominator. This will be mostly the case in ENME462.
2) Factor out the denominator into 1st-order and 2nd-order rational factors.
3) Find numerator coefficients.

Example 1: Find PFE of $\frac{5x+7}{x^2+3x+2}$.

Step 1: OK.

Step 2:

$$x^2+3x+2=(x+1)(x+2)$$

Step 3:

$$\frac{A}{x+1}+\frac{B}{x+2}=\frac{5x+7}{x^2+3x+2}$$
$$A(x+2)+B(x+1)=5x+7$$
$$x=-1,\rightarrow\ A=2$$
$$x=-2,\rightarrow\ B=3$$

(2) Factors Anticipated

1) For Linear terms $ax+b$ we have:
$$\frac{A}{ax+b}$$
2) For quadratic terms (quadratics that don’t have real roots):
$$\frac{Ax+B}{ax^2+bx+c}$$
3) For repeated roots $(ax+b)^3$:
$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

Example 2:

$$\frac{2x^2+1}{x^3-x^2-8x+12}=\frac{2x^2+1}{(x-2)^2(x+3)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+3}$$

Example 3:

$$\frac{x^2+1}{(x^2+x+2)(x+7)}=\frac{Ax+b}{x^2+x+2}+\frac{C}{x+7}$$

Example 4:

$$\frac{1}{(x^2+2x+5)^2(x-1)(x+2)}=\frac{Ax+B}{x^2+2x+5}+\frac{Cx+D}{(x^2+2x+5)^2}+\frac{E}{x-1}+\frac{F}{x+2}$$

Example 5:

$$\frac{1}{(x^2-3)^2}=\frac{1}{\left(x-\sqrt{3}\right)^2\left(x+\sqrt{3}\right)^2}=\frac{A}{\left(x-\sqrt{3}\right)}+\frac{B}{\left(x-\sqrt{3}\right)^2}+\frac{C}{\left(x+\sqrt{3}\right)}+\frac{D}{\left(x+\sqrt{3}\right)^2}$$

(3) Determination of Numerator Coefficients

Method 1

Multiply both sides by the denominator and choose values for the variable to derive coefficients.

Example 6:

$$\frac{2x-1}{(x+2)^2(x-3)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-3}$$

$$2x-1=A(x+2)(x-3)+B(x-3)+C(x+2)^2$$

Now we choose values for the variable x:

$$x=3,\rightarrow C=\frac{1}{5}$$

$$x=-2,\rightarrow B=1$$

$$x=0,\rightarrow A=-\frac{1}{5}$$

Method 2

Multiply both sides by the denominator and place the coefficients of the variable equal on both sides of the equation:

Example 7:

$$2x-1=Ax^2-Ax-6A+Bx-3B+Cx^2+4Cx+4C$$

$$2x-1=(A+C)x^2+(-A+B+4C)x+(-6A-3B+4C)$$

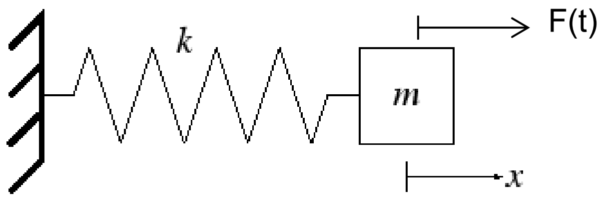
$$0=A+C$$

$$2=-A+B+4C$$

$$-1=-6A-3B+4C$$

4. Solution to ODE: A Mass-Spring System Example

The objective is to compare different ways to solve 1-DoF vibration problem: (i) hand calculation, (ii) use of ODE function in MATLAB, and (iii) use of Laplace transform and SIMULINK.



(i) Hand calculation via convolution integral

The equation of motion is given by:

$$m\ddot{x}(t) + kx(t) = F(t) \tag{1}$$

The response of the system to an external force input can be obtained using the convolution integral:

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \tag{2}$$

In case the force input F(t) is a step input, i.e. F(t)=u(t), (2) reduces to:

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau = \frac{1}{m\omega_n}\frac{1}{\omega_n}\cos\omega_n(t-\tau)|_0^t = \frac{1}{m\omega_n^2}(1-\cos\omega_n t) \\ &= \frac{1}{k}(1-\cos\omega_n t) \end{aligned} \tag{3}$$

The above response can be graphically represented using the following MATLAB commands:

```
clear all; clc

t          = 0:0.01:10; % Time in Seconds with increments of 0.01
k          = 2;         % Spring Stiffness (N/m)
m          = 1;         % mass [kg]
omega      = sqrt(k/m); % radians/sec

x = (1/k) * (1 - cos(omega*t));

plot(t,x)
axis([0,10,-0.01,1.2])
xlabel 'Time [s]'
ylabel 'Displacement [m]'
```

(ii) Use of MATLAB ODE function

To use the ODE solvers in MATLAB, the differential equation must be reformulated into a set of first-order differential equations. For this purpose, define the following variables:

$$\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= \dot{x}(t) \end{aligned} \tag{4}$$

Then (1) can be rewritten into the following:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{m}x_1(t) + \dfrac{1}{m}F(t) \end{cases} \tag{5}$$

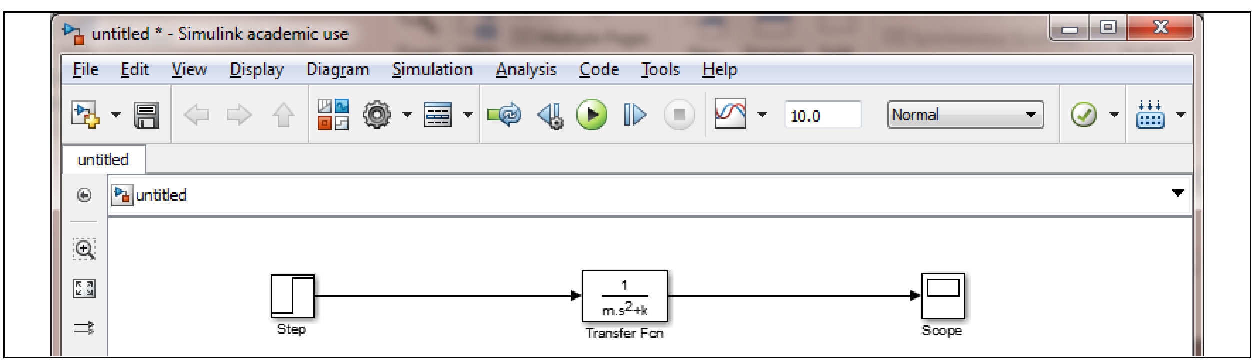
Using (5), the following MATLAB scripts can be created to solve (1) numerically:

```
clear all; clc

x_o = [0,0];
t_o = 0;
t_f = 10;

[t,x] = ode45('S_M_ODE',[t_o,t_f],x_o);

plot(t,x(:,1))
axis([0,10,-0.01,1.2])
xlabel 'Time [s]'
ylabel 'Displacement [m]'

function [ xp ] = S_M_ODE( t , x )

xp=zeros(2,1);

k         = 2;            % Spring Stiffness [N/m]
m         = 1;            % Mass [kg]
F         = 1;

xp(1) = x(2);
xp(2) = -k/m*x(1)+1/m*F;

end
```

(iii) Use of Laplace transform in SIMULINK
Using the Laplace transform, (1) becomes:

$$m\ddot{x} + kx = F(t) \;\rightarrow ms^2X(x) + kX(s) = F(s) \rightarrow X(s) = \frac{1}{ms^2 + k}F(s) \tag{3}$$

Using MATLAB's SIMULINK, the following block diagram can be constructed to numerically simulate the response of the 1-DoF system to step input.



Assignments

1. Implement the example MATLAB/SIMULINK codes for the mass-spring system and compare the plots showing the displacement responses. Show that the responses derived from different approaches are identical.

Submit the following through Gradescope:
[1] Plot comparing the responses of the mass-damper-spring system derived from the 3 approaches mentioned above
[12] MATLAB code(s) written to create the plot