HW1

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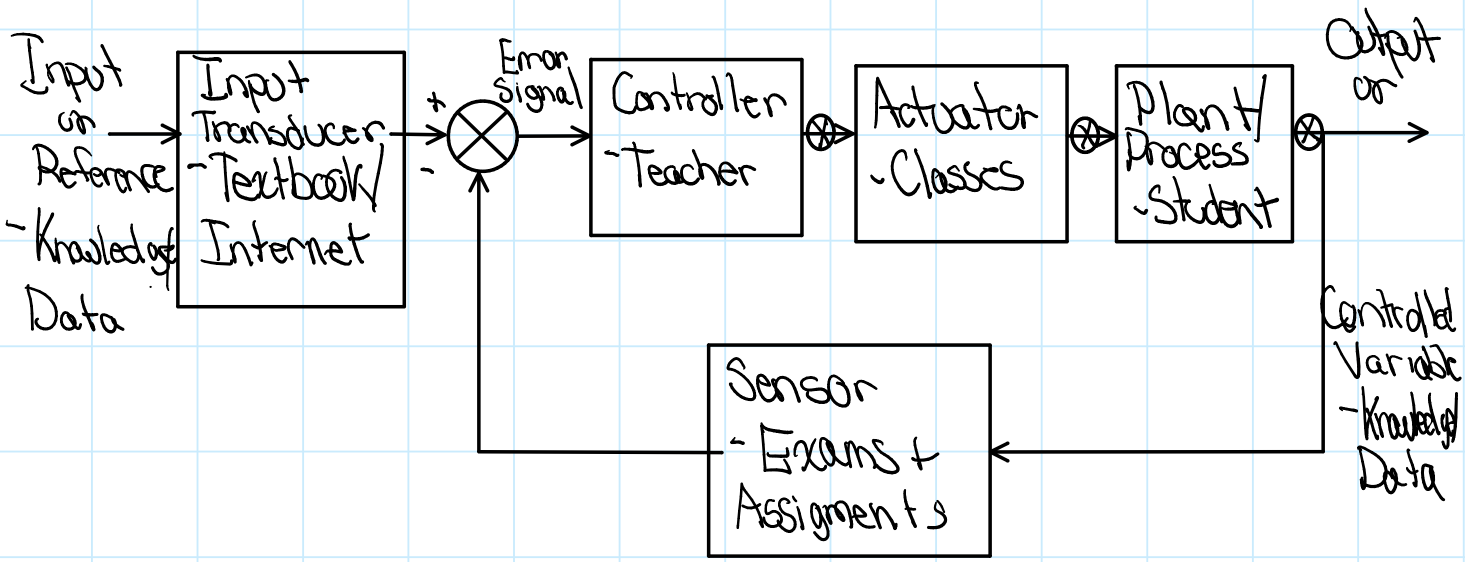
ENME462
Section 0105

1. The student-teacher learning process can be described as a closed-loop control system. Construct a closed-loop system model of this process and identify each block of the system (plant, controller, sensor etc.).



2. A system is described by the following ordinary differential equation:

$$\dddot{y} + 3\ddot{y} + 5\dot{y} + y = \dddot{x} + 4\ddot{x} + 6\dot{x} + 8x$$

Find the expression for the transfer function of the system.

x - input
y - output
Assume - ICs = 0

$$\dddot{y} + 3\ddot{y} + 5\dot{y} + y = \dddot{x} + 4\ddot{x} + 6\dot{x} + 8x$$

$$s^3 Y(s) + 3s^2 Y(s) + 5sY(s) + Y(s)$$
$$= s^3 X(s) + 4s^2 X(s) + 6s X(s) + 8X(s)$$

$$Y(s)(s^3 + 3s^2 + 5s + 1)$$
$$= X(s)(s^3 + 4s^2 + 6s + 8)$$

$$\boxed{G(s) = \frac{Y(s)}{X(s)} = \frac{s^3 + 4s^2 + 6s + 8}{s^3 + 3s^2 + 5s + 1}}$$

$$X(s) \longrightarrow \boxed{\frac{s^3+4s^2+6s+8}{s^3+3s^2+5s+1}} \longrightarrow Y(s)$$

3. Write the differential equation that is equivalent to the block diagram shown below:

$$R(s) \longrightarrow \boxed{\frac{s^4+3s^3+2s^2+s+1}{s^5+4s^4+3s^3+2s^2+3s+2}} \longrightarrow C(s)$$

Assume - ICs = 0

$r(t) = 3t^3$

Assume that $r(t) = 3t^3$.

$$G(s) = \frac{C(s)}{R(s)} = \frac{s^4+3s^3+2s^2+s+1}{s^5+4s^4+3s^3+2s^2+3s+2}$$

$$C(s)(s^5+4s^4+3s^3+2s^2+3s+2)$$

$$= R(s)(s^4+3s^3+2s^2+s+1)$$

$$\frac{d^5c}{dt^5} + 4\frac{d^4c}{dt^4} + 3\frac{d^3c}{dt^3} + 2\frac{d^2c}{dt^2} + 3\frac{dc}{dt} + 2c$$

$$= \frac{d^4r}{dt^5} + 3\frac{d^3r}{dt^3} + 2\frac{d^2r}{dt^2} + \frac{dr}{dt} + r$$

$$r = 3t^3 \qquad \frac{dr}{dt} = 9t^2 \qquad \frac{d^2r}{dt^2} = 18t$$

$$\frac{d^3r}{dt^3} = 18 \qquad \frac{d^4r}{dt^4} = 0$$

$$\frac{d^5c}{dt^5} + 4\frac{d^4c}{dt^4} + 3\frac{d^3c}{dt^3} + 2\frac{d^2c}{dt^2} + 3\frac{dc}{dt} + 2c$$

$$= 3(18) + 2(18t) + 9t^2 + 18t$$

$$\boxed{\frac{d^5c}{dt^5} + 4\frac{d^4c}{dt^4} + 3\frac{d^3c}{dt^3} + 2\frac{d^2c}{dt^2} + 3\frac{dc}{dt} + 2c = 54 + 36t + 9t^2 + 18t}$$

4. Find the inverse Laplace Transform f(t) of $F(s) = \frac{s-1}{s(s^2+s+1)}$. Then, find the final value of f(t) using the Final Value Theorem and verify your result using $\lim_{t\to\infty} f(t)$.

$$F(s) = \frac{s-1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$s-1 = (A)(s^2+s+1) + (Bs+C)(s)$$

$$s-1 = As^2 + As + A + Bs^2 + Cs$$

$$(0)s^2 = (A+B)s^2 \qquad A+B=0$$

$$(1)s = (A+C)s \qquad A+C=1$$

$$-1 = A \qquad A=-1$$

$$B=-A=1$$

$$C=1-A=2$$

$$F(s) = \frac{-1}{} + \frac{s+2}{}$$

$$F(s) = \frac{-1}{s} + \frac{s+2}{s^2+s+1}$$

$$= \frac{-1}{s} + \frac{s+2}{(s+\frac{1}{2})^2 - \frac{1}{4} + \frac{4}{4}}$$

$$= \frac{-1}{s} + \frac{s+2}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$= \frac{-1}{s} + \frac{s+2}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$= \frac{-1}{s} + \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \frac{\frac{3}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$\frac{3}{2} = \frac{\sqrt{3}}{2} C \qquad C = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$

$$= -\frac{1}{s} + \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \sqrt{3}\left(\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right)$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$= -\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right]$$

$$+\sqrt{3}\,\mathcal{L}^{-1}\left[\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right] = -\mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$+\mathcal{L}^{-1}\left[\frac{s+k}{(s+k)^2+a^2}\right] + \mathcal{L}^{-1}\left[\frac{a}{(s+k)^2+a^2}\right]$$

$$= -1 + e^{-kt}\cos at + \sqrt{3}e^{-kt}\sin at$$

$$= \boxed{-1 + e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + \sqrt{3}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$$

$$f(\infty) = \lim_{s\to 0} sF(s) =$$

$$= \lim_{s\to 0} \frac{s(s-1)}{s(s^2+s+1)}$$

$$= \lim_{s\to 0} \frac{s-1}{s^2+s+1} = \boxed{-1}$$

$$s \to 0 \quad s^2+s+1$$

$$f(\infty) = \lim_{t \to \infty} f(t)$$

$$= \lim_{t \to \infty} -1 + \cancel{e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t} + \cancel{\sqrt{3}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$$

$$= \boxed{-1}$$

5. Solve the following differential equations using Laplace transforms, assuming that the forcing functions are zero prior to t=0⁻.

a) $\dot{x} + 7x = 5\cos 2t$

b) $\ddot{x} + 6\dot{x} + 8x = 5\sin 3t$ Assume - ICs = 0

c) $\ddot{x} + 8\dot{x} + 25x = 10$

a) $\dot{x} + 7x = 5\cos 2t$

$$\mathcal{L}[\dot{x} + 7x] = \mathcal{L}[5\cos 2t]$$

$$sX(s) + 7X(s) = \frac{5s}{s^2+4}$$

$$X(s)(s+7) = \frac{5s}{s^2+4}$$

$$X(s) = \frac{5s}{(s^2+4)(s+7)}$$

$$= \frac{As+B}{} + \frac{C}{}$$

$$= \frac{As+B}{s^2+4} + \frac{C}{s+7}$$

$$5s = (As+B)(s+7) + (C)(s^2+4)$$

$$5s = As^2 + 7As + Bs + 7B + Cs^2 + 4C$$

$$(0)s^2 = (A+C)s^2 \qquad A+C=0$$

$$(5)s = (7A+B)s \qquad 7A+B=5$$

$$0 = 7B+4C \qquad 7B+4C=0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 7 & 1 & 0 & 5 \\ 0 & 7 & 4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{35}{53} \\ 0 & 1 & 0 & \frac{20}{53} \\ 0 & 0 & 1 & -\frac{35}{53} \end{array}\right]$$

$$A = \frac{35}{53} \qquad B = \frac{20}{53} \qquad C = -\frac{35}{53}$$

$$X(s) = \frac{\frac{35}{53}s + \frac{20}{53}}{s^2+4} + \frac{-\frac{35}{53}}{s+7}$$

$$= \frac{35}{53}\left(\frac{s}{s^2+4}\right) + \frac{10}{53}\left(\frac{2}{s^2+4}\right)$$

$$-\frac{35}{53}\left(\frac{1}{s+7}\right)$$

$$x(t)=\mathcal{L}^{-1}[X(s)]=\frac{35}{53}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]+\frac{16}{53}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]-\frac{35}{53}\mathcal{L}^{-1}\left[\frac{1}{s+7}\right]$$

$$=\frac{35}{53}\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]+\frac{16}{53}\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]-\frac{35}{53}\mathcal{L}^{-1}\left[\frac{1}{s+k}\right]=\frac{35}{53}\cos at+\frac{16}{53}\sin at-\frac{35}{53}e^{-kt}$$

$$=\boxed{\frac{35}{53}\cos 2t+\frac{16}{53}\sin 2t-\frac{35}{53}e^{-7t}}$$

b) $\ddot{x}+6\dot{x}+8x=5\sin 3t$

$\mathcal{L}[\ddot{x}+6\dot{x}+8x]=\mathcal{L}[5\sin 3t]$

$$\mathcal{L}[\ddot{x} + 6\dot{x} + 8x] = \mathcal{L}[5\sin 3t]$$

$$s^2X(s) + 6sX(s) + 8X(s) = \frac{5(3)}{s^2+9}$$

$$X(s)(s^2+6s+8) = \frac{15}{s^2+9}$$

$$X(s) = \frac{15}{(s^2+9)(s^2+6s+8)}$$

$$= \frac{15}{(s^2+9)(s+2)(s+4)}$$

$$= \frac{As+B}{s^2+9} + \frac{C}{s+2} + \frac{D}{s+4}$$

$$15 = (As+B)(s+2)(s+4)$$
$$+ (C)(s^2+4)(s+4)$$
$$+ (D)(s^2+9)(s+2)$$

$$15 = (As^2 + 2As + Bs + 2B)(s+4)$$
$$+ (Cs^2 + 9C)(s+4)$$
$$+ (Ds^2 + 9D)(s+2)$$

$$15 = As^3 + 4As^2 + 2As^2 + 8As$$

$$15 = As^3 + 4As^2 + 2As^2 + 8As$$
$$+ Bs^2 + 4Bs + 2Bs + 8B$$
$$+ Cs^3 + 4Cs^2 + 9Cs + 36C$$
$$+ Ds^3 + 2Ds^2 + 9Ds + 18D$$

$$(0)s^3 = (A+C+D)s^3 \quad A+C+D=0$$

$$(0)s^2 = (6A+B+4C+2D)s^2$$

$$6A+B+4C+2D=0$$

$$(0)s = (8A+6B+9C+9D)s$$

$$8A+6B+9C+9D=0$$

$$15 = 8B+36C+18D$$

$$8B+36C+18D=0$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 6 & 1 & 4 & 2 & 0 \\ 8 & 6 & 9 & 9 & 0 \\ 0 & 8 & 36 & 18 & 15 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{18}{25} \\ 0 & 1 & 0 & 0 & -\frac{3}{65} \\ 0 & 0 & 1 & 0 & \frac{15}{26} \\ 0 & 0 & 0 & 1 & -\frac{3}{10} \end{array}\right]$$

$A = -\underline{18}$ $\quad B = -\underline{3}$ $\quad C = \underline{15}$ $\quad D = -\underline{3}$

$A = -\frac{18}{65}$ $B = -\frac{3}{65}$ $C = \frac{15}{26}$ $D = -\frac{3}{10}$

$$X(s) = \frac{-\frac{18}{65}s - \frac{3}{65}}{s^2+4} + \frac{\frac{15}{26}}{s+2} + \frac{-\frac{3}{10}}{s+4}$$

$$= -\frac{18}{65}\left(\frac{s}{s^2+4}\right) - \frac{1}{65}\left(\frac{3}{s^2+4}\right)$$

$$+ \frac{15}{26}\left(\frac{1}{s+2}\right) - \frac{3}{10}\left(\frac{1}{s+4}\right)$$

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$= -\frac{18}{65}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{65}\mathcal{L}^{-1}\left[\frac{3}{s^2+4}\right]$$

$$+ \frac{15}{26}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \frac{3}{10}\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]$$

$$= -\frac{18}{65}\mathcal{L}^{-1}\left[\frac{s}{s^2+a}\right] - \frac{1}{65}\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$+ \frac{15}{26}\mathcal{L}^{-1}\left[\frac{1}{s+k}\right] - \frac{3}{10}\mathcal{L}^{-1}\left[\frac{1}{s+k}\right]$$

$$= -\frac{18}{65}\cos at - \frac{1}{65}\sin at$$

$$+ \frac{15}{26} e^{-kt} - \frac{3}{10} e^{-kt}$$

$$\boxed{= -\frac{18}{65}\cos 3t - \frac{1}{65}\sin 3t + \frac{15}{26} e^{-2t} - \frac{3}{10} e^{-4t}}$$

c) $\ddot{x} + 8\dot{x} + 25x = 10$

$$\mathcal{L}[\ddot{x} + 8\dot{x} + 25x] = \mathcal{L}[10]$$

$$s^2 X(s) + 8s X(s) + 25 X(s) = \frac{10}{s}$$

$$X(s)(s^2 + 8s + 25) = \frac{10}{s}$$

$$X(s) = \frac{10}{(s^2+8s+25)(s)}$$

$$= \frac{As+B}{s^2+8s+25} + \frac{C}{s}$$

$$10 = (As+B)(s) + (C)(s^2+8s+25)$$

$$10 = (As+B)(s) + (C)(s^2+8s+25)$$

$$10 = As^2 + Bs + Cs^2 + 8Cs + 25C$$

$$(0)s^2 = (A+C)s^2 \qquad A+C=0$$

$$(0)s = (B+8C)s \qquad B+8C=0$$

$$10 = 25C \qquad C = \frac{10}{25}$$

$$A = -C = -\frac{10}{25} \qquad B = -8C = -\frac{80}{25}$$

$$Y(s) = \frac{-\frac{10}{25}s - \frac{80}{25}}{s^2+8s+25} + \frac{\frac{10}{25}}{s}$$

$$= \frac{-\frac{10}{25}s - \frac{80}{25}}{(s+4)^2 - 16 + 25} + \frac{\frac{10}{25}}{s}$$

$$= \frac{-\frac{10}{25}s - \frac{80}{25}}{(s+4)^2 + (3)^2} + \frac{\frac{10}{25}}{s}$$

$$= -10\left(s+8 \qquad \right) + \frac{10}{25}$$

$$= -\frac{16}{25}\left(\frac{s+8}{(s+4)^2+(3)^2}\right) + \frac{\frac{16}{25}}{s}$$

$$= -\frac{16}{25}\left(\frac{s+4}{(s+4)^2+(3)^2}\right) - \frac{16}{25}\left(\frac{4}{(s+4)^2+(3)^2}\right)$$

$$+ \frac{\frac{16}{25}}{s}$$

$$4 = 3c \qquad c = \frac{4}{3}$$

$$X(s) = -\frac{16}{25}\left(\frac{s+4}{(s+4)^2+(3)^2}\right) - \frac{16}{25}\cdot\frac{4}{3}\left(\frac{3}{(s+4)^2+(3)^2}\right)$$

$$+ \frac{16}{25}\left(\frac{1}{s}\right)$$

$$= -\frac{16}{25}\left(\frac{s+4}{(s+4)^2+(3)^2}\right) - \frac{64}{75}\left(\frac{3}{(s+4)^2+(3)^2}\right)$$

$$+ \frac{16}{25}(s)$$

$$x(t) = \mathcal{L}^{-1}[X(s)] = -\frac{16}{25}\mathcal{L}^{-1}\left[\frac{s+4}{(s+4)^2+(3)^2}\right]$$

$$-\frac{64}{\cdots}\mathcal{L}^{-1}\left[\frac{3}{\cdots}\right] + \frac{16}{\cdots}\mathcal{L}^{-1}\left[\frac{1}{\cdots}\right]$$

$$-\frac{46}{75}\mathcal{L}^{-1}\left[\frac{3}{(s+4)^2+(3)^2}\right]+\frac{16}{25}\mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$= -\frac{16}{25}\mathcal{L}^{-1}\left[\frac{s+k}{(s+k)^2+a^2}\right]-\frac{46}{75}\left[\frac{a}{(s+k)^2+a^2}\right]$$

$$+\frac{16}{25}\mathcal{L}^{-1}\left[\frac{1}{s}\right] = -\frac{16}{25}e^{-kt}\cos at$$

$$-\frac{46}{75}e^{-kt}\sin at+\frac{16}{25}$$

$$\boxed{= -\frac{16}{25}e^{-4t}\cos 3t-\frac{46}{75}e^{-4t}\sin 3t+\frac{16}{25}}$$

6. The water level *h(t)* in a tank is controlled by an open-loop system as shown below. An armature-controlled DC motor turns a shaft that opens/closes a valve. The inductance of the armature circuit is negligible: L_a=0. Also, the rotational friction of the motor shaft and the valve is negligible. The height of the water in the tank is given by: $h(t)=\int[1.6\theta(t)-h(t)]dt$. The motor torque constant K_t=10 and the inertia of the motor-valve assembly is J=6×10^{-3} kg·m^2. ?

$K_t = 9.55\times10^{-3}\,K_e$

(a) Determine the differential equation for *h(t)* and *v(t)*.

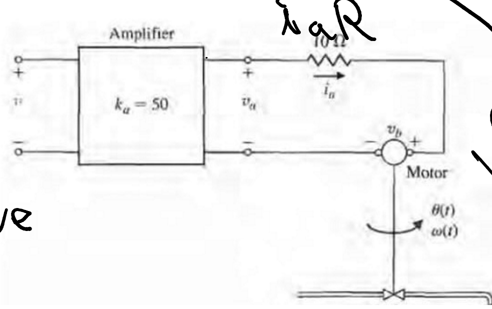
$K_e = \frac{K_t}{9.55\times10^{-3}} = \frac{10}{9.55\times10^{-3}}$

(b) Determine the transfer function *H(s)/V(s)*.

$= 1047.12$

KVL

$i_a R$



V_b

K_t = Motor Torque

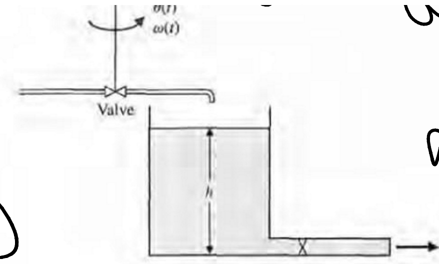
K_b = Back

$\omega(t) = \frac{d\theta(t)}{dt}$

Torque

K_e = Back EMF

K_a = Gain = $\frac{V_a(t)}{V(t)}$



$\omega(t) = \frac{d\theta(t)}{dt}$

$\alpha(t) = \frac{d^2\theta(t)}{dt^2}$

(a) $L_a = 0$ $\quad h(t) = \int [1.6\theta(t) - h(t)]dt$

$K_t = 10 \quad J = 6\times10^{-3}\ kg \cdot m^2 \quad K_a = 50$

$V(t) = \frac{1}{K_a}(V_a(t))$ $\quad R = 10\Omega$ $\quad$ Faradays Law

$\sum V_{loop} = 0$

$V_a(t) = i_a(t)R + V_b(t)$

$= i_a(t)R + K_e \frac{d\theta(t)}{dt}$

(1) $V(t) = \frac{1}{K_a}\left(i_a(t)R + K_e \frac{d\theta(t)}{dt}\right)$

$\frac{dh(t)}{dt} = \frac{d}{dt}\int 1.6\theta(t) - h(t)\,dt$

$\frac{dh(t)}{dt} = 1.6\theta(t) - h(t)$

$\frac{d^2h(t)}{dt^2} = 1.6\frac{d\theta(t)}{dt} - \frac{dh(t)}{dt}$

$$dt^2 \qquad dt \qquad dt$$

$$(2)\ \frac{d\theta(t)}{dt} = 0.625\left(\frac{d^2h(t)}{dt^2} + \frac{dh(t)}{dt}\right)$$

$$(3)\ \frac{d^2\theta(t)}{dt} = 0.625\left(\frac{d^3h(t)}{dt^3} + \frac{d^2h(t)}{dt^2}\right)$$

Fleming's Left Hand Rule

$$\Sigma M = J\alpha(t) = J\,d^2\theta(t) = K_t i_a(t)$$

$$(4)\ i_a(t) = \frac{J}{K_t}\frac{d^2\theta(t)}{dt} \quad (4)$$

$$v(t) = \frac{1}{K_a}\left(\frac{JR}{K_t}\frac{d^2\theta(t)}{dt} + K_e\frac{d\theta(t)}{dt}\right)$$

$$= \frac{1}{K_a}\left[\frac{JR}{K_t}\left(0.625\left(\frac{d^3h(t)}{dt^3} + \frac{d^2h(t)}{dt^2}\right)\right)\right.$$

$$\left. + K_e\left(0.625\left(\frac{d^2h(t)}{dt^2} + \frac{dh(t)}{dt}\right)\right)\right]$$

$$K_aV(t) = \frac{0.625JR}{K_t}\frac{d^3h(t)}{dt^3}$$

$$+ \frac{0.625JR}{}\frac{d^2h(t)}{} + 0.625K_e\frac{d^2h(t)}{}$$

$$+\frac{0.625JR}{K_t}\frac{d^2h(t)}{dt^2}+0.625K_e\frac{d^2h(t)}{dt^2}$$

$$+0.625K_e\frac{dh(t)}{dt}$$

$$\boxed{\left(\frac{K_aK_t}{0.625JR}\right)V(t)=\frac{d^3h(t)}{dt^3}+\left(1+\frac{K_eK_t}{JR}\right)\frac{d^2h(t)}{dt^2}+\left(\frac{K_eK_t}{JR}\right)\frac{dh(t)}{dt}}$$

$$\boxed{13333.33V(t)=\frac{dh^3(t)}{dt^3}+(1+166.67K_e)\frac{d^2h(t)}{dt^2}+166.67K_e\frac{dh(t)}{dt}}$$

(b) $\left(\frac{K_aK_t}{0.625JR}\right)V(s)=s^3H(s)$

$$+\left(1+\frac{K_eK_t}{JR}\right)s^2H(s)$$

$$+\left(K_eK_t\right)sH(s)$$

$$+\left(\frac{K_e K_t}{JR}\right) s H(s)$$

$$H(s)\left(s^3 + \left(1+\frac{K_e K_t}{JR}\right)s^2 + \left(\frac{K_e K_t}{JR}\right)s\right)$$

$$\boxed{\frac{H(s)}{V(s)} = \frac{\left(\frac{K_a K_t}{0.625\,JR}\right)}{\left(s^3 + \left(1+\frac{K_e K_t}{JR}\right)s^2 + \left(\frac{K_e K_t}{JR}\right)s\right)}}$$

$$\boxed{\frac{H(s)}{V(s)} = \frac{13333.33}{\left(s^3 + (1+166.67K_e)s^2 + (166.67K_e)s\right)}}$$