# SOSC 13310, Spring 2022
# Final Project
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Recall the two player battlefield game involving army $A$ and army $B$, where we have each individual soldier represented as $a_i/b_i$ and the army strength given by

$$A = \int_0^1 a_i di \text{ and}$$
$$B = \int_0^1 b_j dj.$$

Consider instead a game consisting of three armies lead by generals $A$, $B$, and $C$. In this game, the individual soldiers do not have choices independent of each other; instead each general can either choose to fully commit his army to war or to not engage in battle.

The setup will be as follows:
There are two sides of the war which we will call the "militants" and the "freedom fighters". Army $C$ will always choose the side of the "militants", and armies $A$ and $B$ have three choices. Either they can withhold from fighting or choose one of the two sides to fight for. Army $A$ and $B$ have a cost $c_A$ and $c_B$ for initiating battle, and there is neither a cost nor benefit to refusing to fight. Additionally, they get a reward $r_F$ for winning the fight on the freedom fighters side and a reward $r_M$ for winning on the side of the militants. We will impose the condition of $r_F > r_M > c_i$ for $i \in \{A, B\}$. Note that if there is no opposing force on the other side of the war then the army only incurs the cost of preparing for war.

We will standardize the strength of army $C$ to be 1. Each army $A$ and $B$ has a strength $\theta_i$ which is observed only by them and is drawn from the set $\{L, H\}$. Here, assume $H$ is drawn uniformly from $[\frac{1}{2}, 1]$ and $L$ is drawn uniformly from $[0, \frac{1}{2}]$. Our prior distribution is given by

$$P(\theta_i = H) = \mu.$$

This game differs from the original battlefield game in another way as well. In this version, the order of events is sequential, with army $A$ choosing their position first and army $B$ afterwards deciding their stance. Then we have a few questions we wish to answer:

1) What decision will army $A$ or $B$ make based upon their strength?

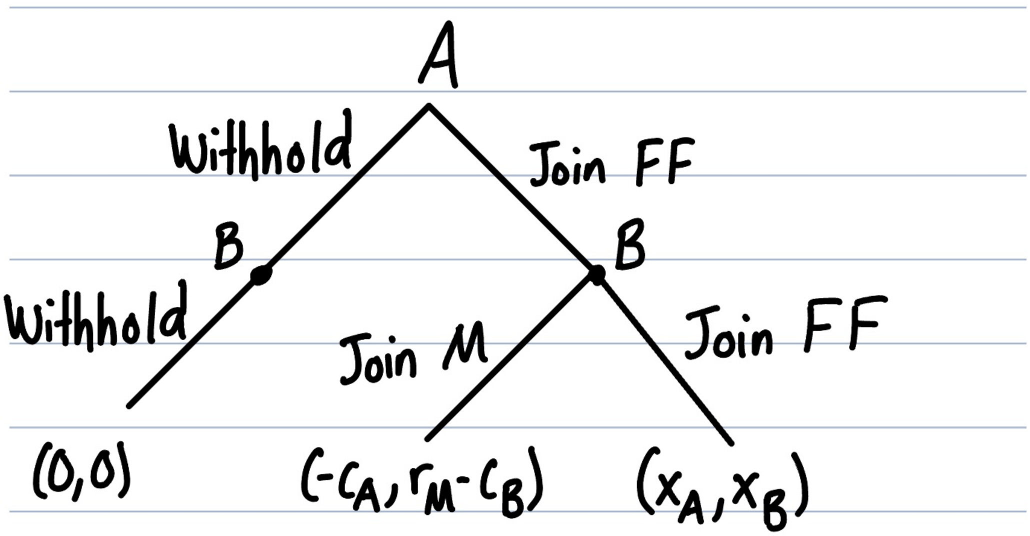
2) What is the expected payoff for army $A$ and army $B$?

Since army $A$ goes first, consider the case where they observe $\theta_A = L$. While there are technically three potential options for army $A$, they will never want to choose the side of the militants since $\theta_A + 1 > \theta_B$, i.e. they would have profited from withholding instead. So, army $A$ will either choose to join the freedom fighters or withhold from the battle. If they withhold from the battle, then they know that army $B$ will also withhold since the payoff for choosing a side would then be $-c_B$. If $A$ instead chooses to join the freedom fighters, then army $B$ is faced with three subsequent choices.

1) If they choose to join army $A$, then the combined strength of $\theta_A + \theta_B > 1$ in order to receive the payoff of $r_F - c_B$.

2) If they choose to join army $C$, then they are guaranteed a payoff of $r_M - c_B$.

Note that option 2 necessarily means that army $B$ will not withhold since $r_M - c_B > 0$. Then the sequential game can be simplified to:




Note that we currently have an unknown payoff structure for $x_A$ and $x_B$ since the payoffs are dependent on the strengths $\theta_i$ for each player $A$ and $B$. We currently assume that player $A$ has a type of $L$, so let us try to solve for this $x_A$ and $x_B$. We know that $P(\theta_B = L) = 1 - \mu$, which means that $A$ will have a payoff of $-c_A$ with probability $1 - \mu$. On the other hand, we know that $P(\theta_B = H) = \mu$. In this case, it is still possible that $\theta_A + \theta_B \leq 1$, which would result in a payoff of $-c_A$. However, note that army $A$ knows their own value of $L$, so we can replace $\theta_A$ with $L$. Since we have already accounted for the case of $\theta_B = L$, then the payoff of $x_A$ will be given by

$$\begin{aligned} x_A &= (-c_A)(1-\mu) + P(\theta_B = H)((-c_A)P(L+H \leq 1) + (r_F - c_A)P(L+H > 1)) \\ x_B &= (-c_B)(1-\mu) + P(\theta_B = H)((-c_B)P(L+H \leq 1) + (r_F - c_B)P(L+H > 1)). \end{aligned}$$

Then we must solve for $P(L+H>1)$ (which gives $P(L+H \leq 1) = 1 - P(L+H>1)$) to assess the actions the players will wish to take. We know that

$$P(L+H>1) = P(H \leq 1-L) = \frac{1-(1-L)}{\frac{1}{2}} = 2L$$

since we know $H$ has a uniform distribution in $[\frac{1}{2}, 1]$. Plugging into our previous equation, this then gives us the payoffs of

$$\begin{aligned} x_A &= -c_A(1-\mu) + \mu(-c_A(1-2L) + (r_F - c_A)2L) = -c_A + c_A\mu + \mu(2Lr_F - c_A) = 2Lr_F\mu - c_A \\ x_B &= -c_B(1-\mu) + \mu(-c_B(1-2L) + (r_F - c_A)2L) = -c_B + c_B\mu + \mu(2Lr_F - c_B) = 2Lr_F\mu - c_B. \end{aligned}$$

Now, we know that army $B$ will only join the freedom fighters alongside army $A$ if the payoff $x_B \geq r_M - c_B$, i.e. that

$$r_M - c_B \leq 2Lr_F\mu - c_B \implies \frac{r_M}{2Lr_F} \leq \mu.$$

Then given $L$ for army $A$, general $A$ knows that it must be the case that $r_M < 2Lr_F$ in order for

$$\frac{r_M}{2Lr_F} < 1.$$

If it is the case that $r_M \geq 2Lr_F$, then army $A$ should choose to withhold from the battle. Should army $A$ notice that $r_M < 2Lr_F$ and that

$$\mu \geq \frac{r_M}{2Lr_F},$$

they must then consider their own payoff. It must be the case that

$$x_A - c_A \geq 0,$$

as this would mean the expected payoff for joining the freedom fighters is higher than simply withholding. Then it must be so that

$$2Lr_F\mu - c_A - c_A \geq 0 \implies \mu \geq \frac{c_A}{Lr_F}.$$

If both

$$\mu \geq \frac{c_A}{Lr_F} \text{ and } \mu \geq \frac{r_M}{2Lr_F},$$

then army $A$ will choose to fight for the freedom fighters. With this logic, general $B$ then knows that $\theta_A$ is such that

$$\frac{r_M}{2\mu r_F} \leq \theta_A \text{ and } \frac{c_A}{\mu r_F} \leq \theta_A.$$

Therefore, if army $B$ observes that

$$\theta_B + \frac{r_M}{2\mu r_F} > 1 \text{ and } \theta_B + \frac{c_A}{\mu r_F} > 1,$$

then general $B$ knows with certainty that he will win by joining the freedom fighters. If instead he sees that either

$$\theta_B + \frac{r_M}{2\mu r_F} \leq 1 \text{ or } \theta_B + \frac{c_A}{\mu r_F} \leq 1$$

he is unsure of the outcome of joining the freedom fighters. In order to decide what he will do, we must know also consider the case of when $\theta_A = H$ to understand general $B$'s thought process. In the case of $\theta_A = H$, the payoff structure for both $x_A$ and $x_B$ change because a low type from army $B$ no longer means instant defeat. Therefore, we now have

$$x_A = \mu(r_F - c_A) + P(\theta_B = L)((r_F - c_A)P(H + L > 1) - c_A P(H + L \leq 1)) \text{ and}$$
$$x_B = \mu(r_F - c_B) + P(\theta_B = L)((r_F - c_B)P(H + L > 1) - c_B P(H + L \leq 1)).$$

Like before, we can solve for $P(H + L > 1)$ as

$$P(H + L > 1) = P(L > 1 - H) = \frac{\frac{1}{2} - (1 - H)}{\frac{1}{2}} = 2H - 1.$$

Then we can fill in for $x_A$ and $x_B$ so that

$$x_A = \mu(r_F - c_A) + (1 - \mu)((r_F - c_A)(2H - 1) - c_A(2 - 2H)) = r_F\mu - c_A\mu + (1 - \mu)(2Hr_F - c_A) = \mu(r_F - 2Hr_F) + 2Hr_F - c_A$$
$$x_B = \mu(r_F - c_B) + (1 - \mu)((r_F - c_B)(2H - 1) - c_B(2 - 2H)) = r_F\mu - c_B\mu + (1 - \mu)(2Hr_F - c_B) = \mu(r_F - 2Hr_F) + 2Hr_F - c_B.$$

Again, we know that general $B$ prefers $x_B$ when $x_B \geq r_M - c_B$, so when

$$\mu(r_F - 2Hr_F) + 2Hr_F - c_B \geq r_M - c_B \implies \mu \leq \frac{r_M - 2Hr_F}{r_F - 2Hr_F},$$

as $(r_F - 2Hr_F) < 0$. However, we know that

$$r_M - 2Hr_F < r_F - 2Hr_F \implies \frac{r_M - 2Hr_F}{r_F - 2Hr_F} > 1,$$

i.e. that this assumption is unnecessary. So army $B$ would prefer to always fight with the freedom fighters if $\theta_A = H$. Now army $A$ must consider their respective payoffs and decide the best choice, so again we solve for

$$x_A - c_A \geq 0 \implies \mu(r_F - 2Hr_F) + 2Hr_F - c_A - c_A \geq 0 \implies \mu \leq \frac{2c_A - 2Hr_F}{r_F - 2Hr_F}.$$

So when army $A$ is a high type, they will choose to fight for the freedom fighters if

$$\mu \leq \frac{2c_A - 2Hr_F}{r_F - 2Hr_F}.$$

This implies that if general $A$ makes the choice to fight, then $c_A \leq Hr_F$ or that $\frac{c_A}{r_F} \leq H$. Note from before that army $A$ will only fight as a low type if

$$\frac{c_A}{Lr_F} \leq \mu.$$

Since $L \leq \frac{1}{2}$, then a low type army $A$ would never choose to fight if $\frac{c_A}{r_F} > \frac{1}{2}$. Therefore, if general $B$ notices $\frac{c_A}{r_F} > \frac{1}{2}$ yet army $A$ still decides to fight, then he knows that $\theta_A = H$ and is such that $\theta_A \geq \frac{c_A}{r_F}$ or $\theta_A = 1$. However, we have previously shown that if army $B$ believes that army $A$ has an $H$ type, then they will fight with certainty. So, if general $A$ fights and $\frac{c_A}{r_F} > \frac{1}{2}$, army $B$ will fight for the freedom fighters with certainty. If instead $\frac{c_A}{r_F} \leq \frac{1}{2}$, then army $B$ must observe the other condition for the low type of general $A$ which is that

$$\mu \geq \frac{r_M}{2Lr_F}.$$

Here, notice that it must be the case that $\frac{r_M}{2r_F} \leq \frac{1}{2}$ as we have defined them such that $r_F > r_M$. So then we have

$$\frac{c_A}{r_F} \leq \frac{1}{2} \text{ and } \frac{r_M}{2r_F} \leq \frac{1}{2}$$

and must consider what army $A$ would do based on their type. Army $B$ knows that general $A$ only wants to fight as a high type if

$$\mu \leq \frac{2c_A - 2Hr_F}{r_F - 2Hr_F},$$

and as a low type if

$$\mu \geq \frac{c_A}{Lr_F} \text{ and } \mu \geq \frac{r_M}{2Lr_F}.$$

If army $B$ is a low type, then they would only want to join alongside $A$ only if they were a high type. As a high type, army $B$ would again want to join if $A$ was a high type, but there would also be an incentive to join even if $A$ was a low type. Therefore, if army $A$ is a high type they should attack with certainty because general $B$ would want to fight alongside army $A$ regardless of $\theta_B$. If instead general $A$ sees that $\theta_A$ is a low type, then army $A$ should withdraw with probability $1-\mu$ and join the freedom fighters with probability $\mu$ (assuming the constraints are met). With this strategy, army $B$ will at least join the freedom fighters with probability $\mu$ since

$$P(\theta_A = H) = \mu$$

and since they want to fight with certainty if $\theta_A = H$. However, in the other $1-\mu$ chance, army $B$ would still join as a high type as army $A$ has explicitly only joined the freedom fighters if

$$\mu \geq \frac{r_M}{2Lr_F},$$

which was the condition for army $B$ to join given $\theta_A = L$. If army $B$ is a low type, then they would instead join the militants alongside army $C$ to collect their rewards $r_M - c_B$. So when

$$\frac{c_A}{r_F} \leq \frac{1}{2},$$

army $A$ wants to fight as a high type and wants to fight as a low type with probability

$$\mu \cdot \mu = \mu^2,$$

which is the chance that army $A$ would even decide to fight and the chance that army $B$ would accept (i.e. the chance that $\theta_B = H$).