Discrete Mathematics
CMSC 27100
Winter Quarter 2021
Final Review Questions

1 Basic Combinatrorics

a) You order from a pizza shop, where they offer 15 toppings, 4 types of cheeses, and 3 different sauces. You decide to order a pizza with one topping, one cheese, and one sauce. How many different pizza combinations are possible?

b) How many 4 digit numbers exist which begin with an even digit and end with an odd digit?

c) How many 5 digit numbers have exist where the first three digits are even and the last two digits are identical to each other?

d) How many numbers exist between 1 and 100 which are a positive divisor of 54 or 68?

e) How many numbers between 1 and 2000 are either divisible by 16 or divisible by 28?

f) How many 3 digit numbers exist for which each pair of two consecutive digits are distinct?

2 Intermediate Combinatorics/Binomial Theorem

Exercise 2.1. Deduce the total number of possibilities for the following:

a) Take $n, m \in \mathbb{N}$ where $n \geq m$. Is there a way to generalize how many subsets of $[n]$ will have exactly m elements? Either write the formula or explain why this is not possible.

b) Your friend has created his own mini-Powerball game for a statistics class, where you pick 4 distinct numbers 1-20 and a 5-th number from 1-10 which does not have to be distinct from the first 4 digits. How many possible ticket combinations are there?
Note 1: A valid ticket would be $(1, 19, 7, 8, 7)$, but an invalid ticket would be $(10, 19, 17, 10, 3)$ as 10 cannot appear twice in the first 4 numbers.
Note 2: We will consider the ticket $(1, 19, 7, 8, 7)$ to be distinct from $(1, 7, 19, 8, 7)$. This means that a ticket's order matters.

c) Downtown Chicago, unlike many East Coast cities, is almost entirely grid-like in its layout (the history of this matter is irrelevant to the question, but you can look up government policy under Thomas Jefferson if you are curious). You and your friend wish to get to a someone's apartment, which is 6 blocks north and 5 blocks east of your current location. Naturally, as a UChicago student you have a strong desire to calculate the total number of possibilities to get to their apartment. How many ways are there to arrive at the apartment?

d) Given a set of 9 students, how many ways can be break up this up into teams of 3 such that the first two teams are doing the same task but the last team is doing a different task? For example, imagine we have students 1-9 that are split into groups of 3 denoted Group A, B, and C. Group A and B are in charge of concessions while Group C is in charge of selling merchandise in the stands. Your hint is to consider which order does and does not matter.

Exercise 2.2. Expand $(2x - 3y)^4$ using Binomial Theorem. Give your answer in terms of x and y with only coefficients to accompany them (i.e. do not leave mathematical choice notation in your answer).

3 Advanced Combinatorics

a) How many four digit numbers exist such that each digit is strictly smaller than the last? For example, 4321 is such a number while 9856 and 8433 are not.

b) How many four digit numbers exist such that each digit is decreasing? (We do not say "strictly" here, which means 8433 would be an included number).

c) Insomnia Cookies is having a sale on 12-pack cookies, of which you are allowed to pick from a selection of 8 different flavors. How many arrangement of cookies are possible given the size of the pack and the variety of flavors?

d) Unfortunately, as a UChicago student you may find yourself with multiple projects due during the same week. Luckily, you are good at budgeting your time and have drawn out a schedule of 4 potential projects. For two of the projects, you are going to need at least 3 hours each and for the other two you will need at least 2 hours each. Assuming you have 21 hours to spare between these projects, how many ways can you delegate your time towards them?

e) Given a set $[n]$ for $n \in \mathbb{N}$, how can we generalize the number of subsets which have at least one odd number?

Hint: Consider the cases for both n being even and n being odd.

f) You are the treasurer of an on-campus RSO and want to raise \$50 for an organized event. You decide you are going to sell items worth either $\$1, \$5, \$10$, or \$20. How many ways are there to come up with \$50 by selling such items?

4 Advanced Combinatorics Pt. 2

Exercise 4.1. Solve the following:

a) Let $n \in \mathbb{N}$ and $n \geq 6$. How many subsets of $[n]$ exist that are of size $|4|$ and contain 1 or n.

b) How many anagrams exist of the word "continuous"?

Exercise 4.2. Prove for $n \in \mathbb{N}$ where $n \geq 2$, we have $\binom{2n}{2} - 2\binom{n}{2} = n^2$.

5 Basic Probability

Exercise 5.1. Compute the following probabilities:

a) If you roll a fair 5-sided die 5 times, what is the chance of the exact pattern 1,2,3,3,5?

b) Consider an urn with 3 colors of balls: green, red, and yellow. There are 14 green balls, 9 red balls, and 7 yellow balls. Suppose you will draw 3 balls from the urn without replacement. What is the chance you get exactly one of each color? What is the chance that you pull 3 balls of the exact same color?

Exercise 5.2. Suppose you have a n 6-sided die (assume these dice are fair, i.e. $P(1) = P(2) = ... = P(6) = \frac{1}{6}$).

a) What is the chance you roll two dice and they are both even?

b) Suppose you have 6 of these dice. What is the chance of rolling exactly three 6s?

c) Suppose you and your friend make a bet with dice. Your friend tells you that you win if exactly 2 of the dice are 3s and you lose otherwise. They are willing to accept that you roll 5 dice for this bet. Are the odds in your favor?

Bonus Question for 2c): Is there a value of n where the bet would be in your favor? A rigorous explanation is not needed: some basic calculations and a brief explanation of any complicated math will suffice.

6 Conditional Probabilities

a) You have a standard deck of playing cards, and have already drawn two hearts from the deck. You grab the next two cards from the top of the deck without replacing the two hearts from before. What is the probability that your two cards are both NOT hearts?

b) You flip a coin 4 times. What is the probability that the first two coin flips are heads given exactly 3 of the 4 coins are heads?

c) Let's say you have a fair 6-sided die and roll this die to get a number n. What is the probability you will have n number of heads after flipping a fair coin 6 times?

7 Baye's Theorem and Expectation

Exercise 7.1. Let's say you are a college coach looking for a great tennis player at nearby high schools. You know from the past that 1% of players are the elite tier you are looking for, 18% are good enough as backups, and the rest are not ready for the college level. Elite players have a 95% win rate, good players have a 80% win rate, and the rest only win half the time. If you are scouting a player and see that they have won 5 out of his 5 games so far, what is the probability that they are an elite player?

Exercise 7.2. You are playing poker at a casino against someone who is more advanced than you. You two plan to 5 hands, but will move from the table after someone has won 3 hands (you may assume there are no ties). Because the other player is better, they will win about 55% of the hands.

a) What is the probability that you will win the series after 3,4, or 5 hands? What is the expected length of the series?

b) Say you start with \$100 at the table. After winning a hand, add \$10 to your hand and after losing a hand take away \$15 from your hand. How much money should you expect to have when leaving the table?

8 Variance and Correlation

Exercise 8.1. You are excited for two upcoming movies and decide to read some of the critics reviews for each of them. Below is

Critic #	Movie 1	Movie 2
Critic 1	8.8	7.6
Critic 2	9.3	9.2
Critic 3	7.1	8.5
Critic 4	3.1	7.2
Critic 5	9.5	6.4

What is the variance of the ratings for each movie? What is the covariance and correlation between the ratings for both movies?

9 Recurrence Relations

Exercise 9.1. Solve the following (assuming $n \geq 2$):

a) $a_1 = 3$ and $a_n = 3a_{n-1} + 2$

b) $b_1 = 1$ and $b_n = -b_{n-1} - 4$

c) $c_0 = 4$, $c_1 = 8$, and $c_n = c_{n-1} + 6c_{n-2} + 10$

10 Basic Graph Theory

Exercise 10.1. For the following sets of degrees, either construct an undirected graph or explain why such a graph does not exist (these graphs also must also only have one edge between two vertices).

a) G has 6 vertices with degrees 1,3,2,2,4,5.

b) G has 8 vertices with degrees 2,2,3,4,7,4,3,3

c) G has 4 vertices with degrees 1,1,3,3

d) G has 5 vertices with degrees 0,3,3,2,4

11 Bipartite Graphs and Matchings

Exercise 11.1. For each of the following graphs, either show that the graph is bipartite or show there exists a cycle of odd length in the graph.

a) $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and

$$E(G) = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}\}.$$

b) $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and

$$E(G) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_5, v_6\}\}.$$

Exercise 11.2. For the following bipartite graphs, either give a matching of size 5 or show that there is no such matching.

a) $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$, and

$$E(G) = \{\{a_1, b_1\}, \{a_1, b_4\}, \{a_2, b_3\}, \{a_2, b_4\}, \{a_2, b_5\}, \\ \{a_3, b_4\}, \{a_4, b_2\}, \{a_4, b_5\}, \{a_5, b_3\}, \{a_5, b_5\}\}.$$

b) $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$, and

$$E(G) = \{\{a_1, b_2\}, \{a_1, b_5\}, \{a_2, b_2\}, \{a_2, b_5\}, \{a_3, b_2\}, \\ \{a_3, b_5\}, \{a_4, b_1\}, \{a_4, b_3\}, \{a_4, b_4\}, \{a_5, b_1\}, \{a_5, b_3\}\}.$$