Discrete Mathematics
CMSC 27100
Winter Quarter 2022
Practice Midterm

1 Mathematical Sets

Exercise 1.1. Write down the following sets using mathematical notation:

a) The set of prime numbers which are greater than 10.

b) The set of integers which are either negative or prime.

c) The set of natural numbers whose roots are also natural numbers.

d) The set of natural numbers whose third root is prime and whose square is even.

Exercise 1.2. Consider the set $A = \{1, 2, 17, 31, 12\}$ and $B = \{13, 17, 16, 8, 22\}$.

a) What is $A \cap B$, $A \cup B$, and $A \setminus B$?

b) How many elements are there in $A \times B$? While you may explicitly write every such combination, doing so is not necessary.

Exercise 1.3. Consider any sets A, B, and C.

a) Prove that $A = (A \setminus B) \cup (A \cap B)$.

b) Prove that $(A \setminus B) \cap C \subset A \cap C$. Give an example of sets A, B, and C where the backwards direction is false.

2 Mathematical Sets w/ Quantifiers

Exercise 2.1. Write the following statements and their negations using mathematical notation:

a) There exists a prime number p where $p + 1$ is also prime.

b) For all natural numbers n, if n is even and has a rational root, then $\sqrt{n}$ is a natural number.

c) If z is an integer, then there exists such an integer z' where $z' > z^2$.

3 Divisibility

Exercise 3.1. Either prove the following statements or give a counterexample to show they are false.

a) For all integers z_1, z_2, z_3, p, if $z_1 z_2 | p z_3$, then $z_1 | p z_3$ and $z_2 | p z_3$.

b) Consider a to be a natural number whose root is not a natural number (e.g. 7, as $\sqrt{7} \notin N$). If $a|z$ and z is not of the form a^k with $k \in \mathbb{N}$, then the root of z must also not be a natural number.

Exercise 3.2. Find the positive divisors and sum of the positive divisors for the following numbers:

a) 120

b) 210

c) 985

Exercise 3.3. Show the following sets through mathematical notation:

a) The set of all natural numbers which have exactly 10 positive divisors.

b) The set of all natural numbers which have 6 or less positive divisors.

4 GCD and LCM w/ Euclid's Algorithm

Exercise 4.1. Find $gcd(x, y)$ and integers $a, b \in \mathbb{Z}$ such that $gcd(x, y) = ax + by$ using Euclid's algorithm.

a) $x = 34$, $y = 46$

b) $x = 87$, $y = 126$

Exercise 4.2. Find $lcm(x, y)$. For this, you may use the formula $lcm(x, y) = \frac{xy}{gcd(x,y)}$ and you may use any method you would like to find $gcd(x, y)$.

a) $x = 24$, $y = 36$

b) $x = 51$, $y = 72$

c) $x = 81$, $y = 162$

5 Proofs Via Contradiction and Induction

Exercise 5.1. Prove the following statements using contradiction:

a) If $p \in \mathbb{P}$, then $\sqrt{p} \notin \mathbb{N}$.

b) Consider $\sum_{i=1}^{n} x_i$, where each x_i is a non-negative integer. Suppose $\sum_{i=1}^{n} x_i = x_n$. Show that for each x_i with $i \in [n-1]$, then $x_i = 0$.

Exercise 5.2. Prove the following through the use of induction:

a) Consider a recursive function where $a_1 = 1$ and $a_{n+1} = 2a_n$. Show that $a_n = 2^{n-1}$.

b) Consider the sum $\sum_{i=1}^{n} i$. Show that this sum is equal to $\frac{n(n+1)}{2}$.

6 More Proofs

Exercise 6.1. Let $k \in \mathbb{N}$ and consider $a = \frac{k(k-1)}{2}$. Explain in general terms why a will always be an integer.

Exercise 6.2. Let $a, b \in \mathbb{R}$ such that $a > 0$ and $b > 0$. Prove that $a^{\log(b)} = b^{\log(a)}$ (here we take log to mean the natural log).
Hint: Consider writing a and b in terms of the exponential. Also note that $(a^x)^y = a^{xy}$.

Exercise 6.3. Let $n \in \mathbb{N}$ such that $n \geq 3$. Prove that $\binom{n}{2} + \binom{n}{3} = \binom{n+1}{3}$.

Exercise 6.4. Let $n \in \mathbb{N}$. We say that n is "3-prime" if n has exactly 3 positive divisors.

(a) Give an example of an even number with exactly three positive divisors. Afterwards, prove that all other 3-prime numbers must be odd.

(b) For a 3-prime number x, prove that the only factors are 1, x, and p for some prime number p.

(c) Prove that having a 3-prime number is equivalent to having the square of a prime. In mathematical language, this means that we must prove

$$x \text{ is 3-prime} \iff x = p^2 \text{ for a prime number } p.$$

Exercise 6.5. Bonus: Suppose you know that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Prove that for any $x \geq 0$, then $\sqrt{x\sqrt{x\sqrt{x...}}} = x$. That is, the infinite nested square root of x is equal to x.
Hint: Do not use induction. Instead, consider a way to rewrite $\sqrt{x\sqrt{x\sqrt{x...}}}$ to allow for algebraic manipulation.

7 GCD and LCM w/ Prime Factorization

Exercise 7.1. Find $gcd(x, y)$ and $lcm(x, y)$using prime factorizations.

a) $x = 12$, $y = 16$

b) $x = 49$, $y = 14$

c) $x = 72$, $y = 20$

Exercise 7.2. Prove the following or provide such a counterexample which disproves the statement:

a) If $p \in \mathbb{P}$, then for any $n \in \mathbb{N}$ with $n \neq p$, we know that $lcm(p, n) = pn$.

b) For any $x, y, z \in \mathbb{N}$, if $x|y$ and $z|y$, then $xz|y$.

c) Consider $p_1, p_2, p_3 \in \mathbb{P}$. If $x|p_1p_2p_3$ and $x < p_1p_2p_3$, then $x|p_1p_2$, $x|p_2p_3$, or $x|p_1p_3$.

d) For all $x, y \in \mathbb{Z}$, if $x|gcd(x, y)$ and $y|gcd(x, y)$ then $x = y$.

8 Modular Arithmetic

Exercise 8.1. Calculate the following:

a) $20 + 33 \mod 41$

b) $6 * 21 \mod 18$

c) $22! \mod 23$

d) $7^8 \mod 20$

Exercise 8.2. For the following, explain why or why not the inverse exists. If it exists, calculate its value.

a) 12^{-1} in $\mathbb{Z}_{21}$

b) 18^{-1} in $\mathbb{Z}_{25}$

9 Chinese Remainder Theorem and Totient Function

Exercise 9.1. Use the Chinese Remainder Theorem to find the following:

a) Find $x \in \{0, 1, ..., 265\}$ where $x \equiv 6 \mod 14$ and $x \equiv 13 \mod 19$.

b) Find $x \in \{0, 1, ..., 503\}$ where $x \equiv 4 \mod 9$, $x \equiv 7 \mod 8$, and $x \equiv 2 \mod 7$.

Exercise 9.2. Find Euler's Totient function for the following n:

a) $n = 19$

b) $n = 34$

c) $n = 105$

d) $n = 87$

e) $n = 136$

f) $n = 468$

10 Exponentiation Modulo n

Exercise 10.1. Find the following:

a) $17^{94} \mod 31$

b) $12^{62} \mod 18$

c) $9^{139} \mod 80$