Discrete Mathematics
CMSC 27100
Winter Quarter 2021
Final Review Answer Key

1 Basic Combinatorics

a) *Proof.* Since we only want 1 topping, one cheese, and one sauce, then we can simply multiply all possible choices together. Then our answer will be $15 \cdot 4 \cdot 3 = 15 \cdot 12 = 180$ pizza combinations. Note that what we are doing is implicitly using choice (the mathematical operation). To be more specific, since we only want 1 topping out of 15, we only have $\binom{15}{1}$ choices. Since we only want 1 cheese, we have $\binom{4}{1}$ choices, and since we only want one sauce, we have $\binom{3}{1}$ choices. Thus, we can see that $\binom{15}{1} \cdot \binom{4}{1} \cdot \binom{3}{1} = 15 \cdot 4 \cdot 3 = 180$. □

b) *Proof.* First, we know the first digit must be in the range of 1-9, which gives us 4 choices that our first digit is even. The middle two digits can be any number from 0-9, so then there are 100 choices for the middle two digits. Finally, the last digit must be odd, and there are 5 odd digits between 0-9. Thus, for our four digit number of the form $a_1a_2a_3a_4$ where a_1 has 4 choices, a_2 and a_3 have 10 choices, and a_4 has 5 choices, we get $4 \cdot 10 \cdot 10 \cdot 5 = 20 \cdot 100 = 2000$. Thus, there are 2000 4-digit numbers which begin with an even digit and end with an odd digit. □

c) *Proof.* This question may seem a bit tricky, but let us break it down step by step. First, we know the first 3 digits must be even. Second, we know the last two digits are identical. So, first consider if the choice of our first three digits will impact the choice of the last two digits. Fortunately, regardless of the first three digits we choose, our last two digits will not be impacted. This means we have 4 even-number choices for the first digit, 5 even-number choices for the second digit, and 5 even-number choices for the third digit. So, there are already $4 \cdot 5 \cdot 5 = 100$ combinations of the first three digits. Now, considering the last two digits, we know there are 10 choices for the 4-th digit, but only ONE choice for the final digit. Why is this? Well, the 4-th digit will decide what the 5-th digit is, as the two must be identical. Then there are only 10 possibilities for the last 2 digits. Thus, we have $100 \cdot 10 = 1000$ 5-digit numbers whose first three digits are even and whose last two digits are identical. □

d) *Proof.* The prime factorization of 54 is $2 \cdot 3^3$ and the prime factorization for 68 is $2^2 \cdot 17$. Then 54 has $(1+1)(3+1) = 8$ positive divisors and 68 also has $(2+1)(1+1) = 6$ positive divisors. The $gcd(54, 68) = 2$ from the prime factorization, so they share the divisors 2 and 1 in common. Thus, there are $8 + 6 - 2 = 12$ positive divisors between 1 and 100 of either 54 or 68 (remember that or is inclusive of and). □

e) *Proof.* First, we have $\frac{2000}{16} = 125$ and $\frac{2000}{28} = 71 + \frac{3}{7}$. Now, the $lcm(16, 28) = \frac{16 \cdot 28}{gcd(16,28)} = \frac{448}{4} = 112$. Then $\frac{2000}{112} = 17 + \frac{6}{7}$. Thus, there are $125 + 71 - 17 = 179$ numbers between 1 and 2000 which are either divisible by 16 or 28 (again, remember that or is inclusive). □

f) *Proof.* We choose our first digit from 9 choices, and then have 9 choices of the digits afterwards. So, we will have $9^3 = 729$ choices. □

2 Intermediate Combinatorics/Binomial Theorem

Exercise 2.1

a) *Proof.* The numberof ways to choose m out of n elements is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. $\square$

b) *Proof.* The first 4 numbers is a simple permutation problem. We can choose a number 1-20, so we start with 20 choices then 19 choices then 18 then 17. Note that we do not have to worry about duplicate tickets because the order in which you place the numbers is important. Finally, the 5-th digit of the ticket is a number 1-10 and is not impacted by the first 4 choices. Thus, the number of possible tickets is $20 \cdot 19 \cdot 18 \cdot 17 \cdot 10 = 1,162,800$. $\square$

c) *Proof.* This is nearly identical to the one on PSET 13, but let's walk through it step by step. At any given step, we can either go east or north. We must take a total of 6 steps north and a total of 5 steps east. Choosing such a path is equivalent to choosing the 6 steps out of the 11 steps where we go north, so the number of ways to do this is $\binom{11}{d} = \frac{11!}{6!5!} = 462$. $\square$

d) *Proof.* We have one group here that differs from the other two, so why don't we assign students to this group first? Call them Group A, B, and C where Group C is the one who has a different task. Then there are $\binom{9}{3}$ ways to assign people to this group. Afterwards, we have 6 students left in which they will be performing the same task (i.e. the order in which we assign Group A and B does not matter). This is similar to problem 1d) on PSET 13, and we have two indistinguishable groups to assign 6 people. Then there are $\binom{6}{3}$ ways to assign them to each group, but there are essentially two identical groups. So, assigning students to Group A and B has $\frac{\binom{6}{3}}{2}$ possibilities. Thus, the number of all possible group assignments is $\binom{9}{3} \cdot \frac{\binom{6}{3}}{2} = 84 \cdot 10 = 840$. $\square$

Exercise 2.2

Proof. By the binomial expansion formula, we have

$$(2x-3y)^4 = \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-3y) + \binom{4}{2}(2x)^2(-3y)^2 + \binom{4}{3}(2x)(-3y)^3 + \binom{4}{4}(-3y)^4 =$$
$$16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$$

$\square$

3 Advanced Combinatorics

a) *Proof.* This is equivalent to choosing 4 digits from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as once we have chosen these digits, we obtain exactly one number with these digits by putting them in decreasing order. The number of ways to do this is $\binom{10}{4} = \frac{10*9*8*7}{4*3*2*1} = 210$. $\square$

b) *Proof.* We want to take 4 unordered digits and put them into 10 ordered slots (as decreasing digits means we have the inclusion of 0 as a possibility. Then from the formula for labeled bins and unlabeled balls, we have $\binom{10+4-1}{10-1} = \binom{13}{9} = \binom{13}{4} = 715$ possible combinations for 4-digit numbers with decreasing order digits. However, we must subtract the case where we have the number 0000, and thus, our final number of combinations is $715 - 1 = 714$. $\square$

c) *Proof.* Since the cookie flavors are labeled, these will be our labeled bins. Also, because the 12-pack can contain any such flavors, these will be our unlabeled balls. Then we will have $\binom{12+8-1}{8-1} = \binom{19}{7} = 50388$ combinations of fitting 8 different types of cookies into a 12-pack. $\square$

d) The displacement of hours matters only in the total number of hours you can allocate to each project. This means we have 2 3-hour slots gone and 2 2-hour slots gone. This is a total of 10 hours which is already delegated to the projects. So, of the 21 free hours you have, there are only 21-10=11 hours left after you have put your minimum time into each one. Now we have 11 hours to place into 4 projects, which is the same as putting 11 unlabeled balls into 4 labeled bins. Then the total number of ways to allocate time to these projects is $\binom{11+4-1}{4-1} = \binom{14}{3} = 364$.

e) First, consider n to be odd. Regardless of n, we know there are 2^n total subsets of $[n]$ (think of it like a boolean: either an element of $[n]$ is in a given subset or not, which gives us 2^n total subsets possible). Now consider all possible subsets of $[n]$ which do not contain an odd number (i.e. the subsets which contain an even number or the empty set). For an odd number of $[n]$, there are $\frac{n-1}{2}$ even numbers, which means there are $2^{\frac{n-1}{2}}$ subsets which contain no odd numbers. This gives us that there are $2^n - 2^{\frac{n-1}{2}}$ subsets of an odd n which contain at least 1 odd number. For an even number of n, there are equally as many odd and even numbers, so we have $2^{\frac{n}{2}}$ subsets which contain no odd numbers. Thus, for even values of n, there are $2^n - 2^{\frac{n}{2}}$ subsets which contain at least one odd number.

f) Similar to Problem 1f) in PSET 14, we must consider how to organize these dollar amounts. Whenever we sell something worth \$1, it must be true that we sell a multiple of 5 of these items, as we need \$50 total. So, we can group \$1 with the \$5 bill when considering our combinations. First, consider all the makeups of \$50 from \$5, \$10, and \$20. Then we can have:

$$\begin{aligned}
&10 \cdot \$5 \\
&8 \cdot \$5 + \$10 \\
&6 \cdot \$5 + 2 \cdot \$10 \\
&6 \cdot \$5 + \$20 \\
&4 \cdot \$5 + 3 \cdot \$10 \\
&4 \cdot \$5 + \$10 + \$20 \\
&2 \cdot \$5 + 4 \cdot \$10 \\
&2 \cdot \$5 + 2 \cdot \$10 + \$20 \\
&2 \cdot \$5 + 2 \cdot \$20 \\
&5 \cdot \$10 \\
&3 \cdot \$10 + \$20 \\
&\$10 + 2 \cdot \$20.
\end{aligned}$$

Remember, however, that we have grouped \$1 into the \$5 to make our cases easier. Because of this, we know that the number of combinations needed to make \$50 for each case is just the number of \$5 needed plus 1. For example, consider $10 \cdot \$5$. Then we can have 10 actual

\$5 bills, or 9 actual \$5 bills and 5 \$1 bills, or 8 actual \$5 bills and 10 \$1 bills, up until we have 50 \$1 bills. Then there are 11 possible ways to arrive at \$50 using only \$5 and \$1 bills, which fits our claim of # of \$5 bills + 1. Using this technique, then we can say there are $(10+1)+(8+1)+2(6+1)+2(4+1)+3(2+1)+3(0+1) = 11+9+14+10+9+3 = 56$ total ways to raise \$50 from only \$1, \$5, \$10, and \$20 bills.

4 Advanced Combinatorics Pt. 2

Exercise 4.1

a) *Proof.* For a set of size n, there are $\binom{n}{4}$ subsets of n with a cardinality of 4. Because our set must include either 1 or n, let us try finding such possible sets that contain neither 1 nor n. Then this gives us $n-2$ selections to make and 4 possible slots to fill, which is the same as saying we have $\binom{n-2}{4}$ choices of subsets of $[n]$ with $|4|$ and that do not contain either 1 or n. Thus, we know that there must be $\binom{n}{4} - \binom{n-2}{4}$ subsets of $[n]$ that have a cardinality of 4 and contain either 1 or n. □

b) *Proof.* First, there are 2 n's, 2 u's, and 2 o's. Since the length of continuous is 10, then we have $\binom{10}{2} \cdot \binom{10-2}{2} \cdot \binom{10-2-2}{2} = \binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{2} = 45 \cdot 28 \cdot 15 = 18900$ combinations for just the n's, u's, and o's. Since the other letters of "continuous" only appear once, then we have 4! remaining possibilities for the other letters. This means we have $18900 \cdot 24 = 453600$ anagrams possible for "continuous". □

Exercise 4.2

Proof. This proof will be most easily completed via algebraic means. First, consider $\binom{2n}{2}$. Then

$$\binom{2n}{2} = \frac{(2n!}{2!(2n-2)!} = \frac{2n(2n-1)(2n-2)!}{2(2n-2)!} = \frac{2n(2n-1)}{2} = n(2n-1) = 2n^2 - n.$$

Now, considering $2\binom{n}{2}$, we have

$$2\binom{n}{2} = 2 \cdot \frac{n!}{2!(n-2)!} = 2 \cdot \frac{n(n-1)(n-2)!}{2(n-2)!} = 2 \cdot \frac{n(n-1)}{2} = n(n-1) = n^2 - n.$$

Putting these pieces together, we have $\binom{2n}{2} - 2\binom{n}{2} = 2n^2 - n - (n^2 - n) = 2n^2 - n^2 - n + n = n^2$. Thus, we have shown that $\binom{2n}{2} - 2\binom{n}{2} = n^2$. □

5 Basic Probability

Exercise 5.1

a) *Proof.* Seeing as each die roll will be an independent event, and we must have the exact order given, then the chance of this sequence is $\frac{1}{6^5} = \frac{1}{7776}$. $\square$

b) *Proof.* The chance of one of each color is a much simpler calculation. Since we do not have replacement, and the order in which we draw the colors is irrelevant, then there is first a $\frac{14}{30} = \frac{7}{15}$ chance of green, a $\frac{9}{29}$ chance of red, and a $\frac{7}{28} = \frac{1}{4}$ chance of yellow. However, we must also consider that there are $3! = 6$ different ways to draw our colors. Then the chance of one of each color is $\frac{6 \cdot 7 \cdot 9}{6 \cdot 15 \cdot 29 \cdot 4} = \frac{63}{1740} = \frac{63}{290}$. Now, the chance you get all the same color balls is the sum of the chance of 3 greens, 3 reds, and 3 yellows; more explicitly, in this respective order, we have

$$\frac{14 \cdot 13 \cdot 12}{30 \cdot 29 \cdot 28} + \frac{9 \cdot 8 \cdot 7}{30 \cdot 29 \cdot 28} + \frac{7 \cdot 6 \cdot 5}{30 \cdot 29 \cdot 28} =$$
$$\frac{2184 + 504 + 210}{30 \cdot 29 \cdot 28} =$$
$$\frac{2898}{24360} = \frac{69}{580}.$$

Thus, our chance of drawing 3 of the same colored balls is $\frac{69}{580}$. $\square$

Exercise 5.2

a) *Proof.* Note the roll of each die is independent, so then the chance of two evens is simply $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ (this is the chance of the first die being even times the chance of the second die being even). $\square$

b) *Proof.* Of the 6 rolls, we want 3 rolls to be a six, which can occur in $\binom{6}{3}$ ways. The probability of a six is $\frac{1}{6}$ and the probability of not a six is $\frac{5}{6}$. Thus, the probability of achieving three sixes would be $\binom{6}{3} \cdot (\frac{1}{6})^3 (\frac{5}{6})^3 = \frac{20 \cdot 125}{6^6} = \frac{625}{11664}$. $\square$

c) *Proof.* Given 5 positions, there are $\binom{5}{2}$ ways of getting two 3s. The chance that we have a 3 is obviously $\frac{1}{6}$ and the chance of not a 3 is $\frac{5}{6}$. Then the chance of rolling two 3s from 5 dice will be $\binom{5}{2} \cdot (\frac{1}{6})^2 \cdot (\frac{5}{6})^3 = \frac{10 \cdot 125}{6^5} = \frac{625}{3888} < \frac{1}{2}$. So, at 5 dice the odds are not in your favor. $\square$

Bonus: However, we see for a given n that the chance of two 3s is $\binom{n}{2} \cdot \frac{1}{36} \cdot (\frac{5}{6})^{n-2}$. Then

$$\binom{n}{2} \cdot \frac{1}{36} \cdot \left(\frac{5}{6}\right)^{n-2} =$$
$$\left(\frac{n(n-1)}{2}\right) \cdot \frac{5^{n-2}}{6^n} = \frac{n(n-1)5^{n-2}}{2 \cdot 6^n}.$$

Then we want such an n where $\frac{n(n-1)5^{n-2}}{2 \cdot 6^n} \geq \frac{1}{2}$, or simply $\frac{n(n-1)5^{n-2}}{6^n} \geq 1$. So, we must consider whether $n(n-1)5^{n-2} \geq 6^n$ for any $n \geq 2$. While rigorous math can (and probably should) be used to disprove this inequality, I will simply state an explanation of why it cannot be true. First, clearly 6^n will outgrow 5^{n-2}, but now we must consider the extra $n(n-1)$ term. Regardless of the value of n as n tends towards infinity, the exponential growth rate of 6^n will far outweigh any scalar affect of $n(n-1)$ has to 5^{n-2}. Intuitively, let us consider this bet: at low values of n, clearly it is hard to get exactly two 3s on such a limited number of roles. However, on large values of n, we should expect there to be many more 3s than just two. Thus, it would not be wise to accept this bet on a $1:1$ payoff.

A formal way to test this result is to find the value of n which gives us a maximum probability. Notice that we achieve a max value between $n = 11$ and $n = 12$, and afterwards we have a decreasing percentage.

6 Conditional Probabilities

a) *Proof.* Since the two cards facing up are both hearts, there are then only 11 hearts in the deck, which now only has 50 cards. Then there are 39 cards that are not hearts, so if we draw 2 cards the chance they are both not hearts is $\frac{39}{50} \cdot \frac{38}{49} = \frac{1482}{2450} = \frac{741}{1225}$. $\square$

b) *Proof.* Let A be the event that the first two coins are heads and B be the event that 3 of the 4 coin flips are heads. Then we wish to calculate $P(A \mid B)$. Using the formula for conditional probability, we have $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B|A)}{P(B)}$. The probability of B is the chance of getting 3 heads out of 4 total slots, which is simply $\frac{\binom{4}{3}}{2^4} = \frac{4}{16} = \frac{1}{4}$. The probability of A is simply $\frac{1}{2^2} = \frac{1}{4}$, as it is the chance we flip the coin twice and get two heads. Now let's calculate $P(B \mid A)$. Since we know the first two flips are heads already, and we need exactly one more head for the remaining two slots, $P(B \mid A) = \frac{\binom{2}{1}}{2^2} = \frac{1}{2}$. Thus, we arrive at $P(A \mid B) = \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{4}} = \frac{1}{2}$.

A simpler way to calculate this probability is to notice that $P(A \cap B) = \frac{1}{4} \cdot \frac{1}{2}$ and $P(B) = \frac{1}{4}$. Thus, we would have $P(A \mid B) = \frac{1}{2}$. $\square$

c) *Proof.* Let D be the dice roll and X the event of getting $D = d$ number of heads on 6 coin flips. Then we can use the law of total probability to arrive at our desired result, i.e.

$$P(X \mid D) = \frac{P(X \mid D = 1) + ... + P(X \mid D = 6)}{6}$$

Obviously, for $D = 6$, then $P(X \mid D = 6) = \frac{1}{2^6} = \frac{1}{64}$. For $D = 5$, there are 6 slots for 5 heads, meaning we will have $P(X \mid D = 5) = \frac{\binom{6}{5}}{2^6} = \frac{6}{64}$. Then notice that for $D = d$, our $P(X \mid D = d) = \frac{\binom{6}{d}}{2^6}$. This means we can say

$$P(X \mid D = 1) + ... + P(X \mid D = 6) = \sum_{i=1}^{6} \frac{\binom{6}{i}}{2^6} = \frac{\sum_{i=1}^{6} \binom{6}{i}}{2^6} = \frac{6 + 15 + 20 + 15 + 6 + 1}{64} = \frac{63}{64}.$$

Thus, $P(X \mid D) = \frac{\frac{63}{64}}{6} = \frac{63}{384} = \frac{21}{128}$. $\square$

7 Baye's Theorem and Expectation

Exercise 7.1

Proof. Call the event of having an elite player E, a good player S, and an average player A. Let G be the event that a player has won 5 out of 5 games. We wish to calculate $P(E \mid G)$, which we know is

$$P(E \mid G) = \frac{P(E)P(G \mid E)}{P(G)}.$$

Using the law of total probability, we know that $P(G) = P(G \mid E)P(E) + P(G \mid S)P(S) + P(G \mid A)P(A)$. From the given information, then we have

$$P(G \mid E)P(E) + P(G \mid S)P(S) + P(G \mid A)P(A) = (\frac{19}{20})^5(\frac{1}{100}) + (\frac{4}{5})^5(\frac{9}{50}) + (\frac{1}{2})^5(\frac{81}{100}) = \frac{19^5}{20^5 \cdot 100} + \frac{4^5 \cdot 9}{5^5 \cdot 50} + \frac{81}{32 \cdot 100}.$$

Thus, we have that $P(E \mid G) = \frac{\frac{19^5}{20^5 \cdot 100}}{\frac{19^5}{20^5 \cdot 100} + \frac{4^5 \cdot 9}{5^5 \cdot 50} + \frac{81}{32 \cdot 100}} \approx 8.5\%$. □

Exercise 7.2

a) *Proof.* Let us create two random variables, W_A and W_B, where W_A is the number of wins you have after n hands and W_B is the number of wins that your opponent has after n hands. Note here that $Range(W_A) = Range(W_B) = \{0, 1, 2, 3\}$, as after three hands won the series is over. Obviously, when $n = 1$ and $n = 2$, there is no decisive winner yet. So, consider $n = 3$ first. The chance that you have won after 3 hands is clearly $P_3(W_A = 3) = (\frac{9}{20})^3 = \frac{9^3}{20^3}$, and the chance your opponent won is $P_3(W_B = 3) = (\frac{11}{20})^3 = \frac{11^3}{20^3}$. Now let us consider the trickier cases of $n = 4$ and $n = 5$.

When $n = 4$: We have to remember that a series ends after someone has won 3 hands in a row. That means that when we get to $n = 4$, it must be true that the first 3 hands were not all won by a single person. For $n = 4$, we know that $P_4(W_A = 3) = (\binom{4}{3} - 1)(\frac{9}{20})^3(\frac{11}{20})$. Why is this the case? Well, we have to fit 3 W_A's in 4 slots, but it necessarily true that we do not have the order W_A, W_A, W_A, W_B. Then from our $\binom{4}{3}$ total arrangements, we must subtract this extra case. Using similar logic, this would mean that $P_4(W_B = 3) = (\binom{4}{3} - 1)(\frac{9}{20})(\frac{11}{20})^3$.

When $n = 5$: Considering what we had to do when $n = 4$, when we are at the 5-th hand, we must take into account this necessarily implies there were not 3 of the same person's wins in the first 4 games. So, there are $\binom{5}{3}$ arrangements of 5 wins into 3 slots, but we must make sure that the three wins do not appear in the first 4 slots. Then for $n = 5$, we will have $P_5(W_A = 3) = (\binom{5}{3} - \binom{4}{3})(\frac{9}{20})^3(\frac{11}{20})^2$ and $P_5(W_B = 3) = (\binom{5}{3} - \binom{4}{3})(\frac{9}{20})^2(\frac{11}{20})^3$.

Finding Expectation: Firstly, note that if

$$\sum_{i=3}^{5}[P_i(W_A = 3) + P_i(W_B = 3)] = 1,$$

we have done our calculations correctly. Luckily in our case this is the case. Now, to get the expectation of the series we simply modify the above sum slightly so that it becomes

$$\sum_{i=3}^{5}\left[i(P_i(W_A = 3) + P_i(W_B = 3))\right].$$

Then the expected length of this series is

$3(P_3(W_A = 3) + P_3(W_B = 3)) + 4(P_4(W_A = 3) + P_4(W_B = 3)) + 5(P_5(W_A = 3) + P_5(W_B = 3)) =$
$3\left[\frac{9^3}{20^3} + \frac{11^3}{20^3}\right] + 4\left[3\cdot(\frac{9}{20})^3(\frac{11}{20}) + 3\cdot(\frac{9}{20})(\frac{11}{20})^3)\right] + 5\left[6\cdot(\frac{9}{20})^3(\frac{11}{20})^2 + 6\cdot(\frac{9}{20})^2(\frac{11}{20})^3\right] \approx$
$4.11.$

Thus, you and your opponent should expect to play about 4.11 hands during a given series. □

b) *Proof.* We can calculate this probability using the probabilities from part a). Note that if you win after 3 hands, you gain \$30. The probability of this is $P_3(W_A = 3)$, so then you would have $30 \cdot P_3(W_A = 3) = 30 \cdot \frac{9^3}{20^3}$. We will use this technique for the remaining probabilities where you win the series. If you win after 4 hands, then you gain \$30 − \$15, so then you have $15 \cdot P_4(W_A = 3) = 15 \cdot 3 \cdot (\frac{9}{20})^3(\frac{11}{20})$. Finally, if you win after 5 hands, then you lose \$30 and gain \$30, so this won't be included in our calculations. Now we must consider all of the times you lose. If the other person wins after 3 hands, you lose \$45, so then we have $-45 \cdot P_3(W_B = 3) = -45 \cdot \frac{11^3}{20^3}$. For the sake of simplicity, I will just write out the remaining terms we need for our final sum, but the logic of how to arrive at them is similar to the previous terms. Then we will also have

$$-35 \cdot 3 \cdot (\frac{9}{20})(\frac{11}{20})^3)\text{and}$$
$$-25 \cdot 6 \cdot (\frac{9}{20})^2(\frac{11}{20})^3.$$

Adding these terms together, we will get

$$30 \cdot \frac{9^3}{20^3} + 15 \cdot 3 \cdot (\frac{9}{20})^3(\frac{11}{20}) - 45 \cdot \frac{11^3}{20^3} - 35 \cdot 3 \cdot (\frac{9}{20})(\frac{11}{20})^3) - 25 \cdot 6 \cdot (\frac{9}{20})^2(\frac{11}{20})^3 \approx$$
$$2.73 + 2.26 - 7.49 - 7.86 - 5.05 = -15.41.$$

Thus, you should expect to lose \$15.41, which means you should expect to leave the table with \$84.59. □

8 Variance and Correlation

Exercise 8.1

Proof. Firstly, in order to calculate the variances, we are going to first need the expected rating of each movie. We will denote M_1 as Movie 1 and M_2 as Movie 2. Then the mean rating for Movie 1 is $E(M_1) = \frac{8.8+9.3+7.1+3.1+9.5}{5} = \frac{37.8}{5} = 7.56$, and the mean rating for Movie 2 is $E(M_2) = \frac{7.6+9.2+8.5+7.2+6.4}{5} = \frac{38.9}{5} = 7.78$. Now, to get $Var(M_1)$, we use what we know about expected values to say

$$\begin{aligned}
&Var(M_1) = E((M_1 - E(M_1))^2) = \\
&\frac{(8.8-7.56)^2 + (9.3-7.56)^2 + (7.1-7.56)^2 + (3.1-7.56)^2 + (9.5-7.56)^2}{5} = \\
&\frac{1.5376 + 3.0276 + 0.2116 + 19.8916 + 3.7636}{5} = \frac{28.432}{5} = 5.6864,
\end{aligned}$$

and

$$\begin{aligned}
&Var(M_2) = E((M_2 - E(M_2))^2) = \\
&\frac{(7.6-7.78)^2 + (9.2-7.78)^2 + (8.5-7.78)^2 + (7.2-7.78)^2 + (6.4-7.78)^2}{5} = \\
&\frac{0.0324 + 2.0164 + 0.5184 + 0.3364 + 1.9044}{5} = \frac{4.808}{5} = 0.9616.
\end{aligned}$$

Because we want the correlation of M_1 and M_2, then we can simply find their covariance and divide it by the standard deviation of M_1 times the standard deviation of M_2. To find $Cov(M_1, M_2)$, we can say $Cov(M_1, M_2) = E((M_1 - E(M_1))(M_2 - E(M_1)))$. Then

$$\begin{aligned}
&E((M_1 - E(M_1))(M_2 - E(M_1))) = \\
&\frac{(8.8-7.56)(7.6-7.78) + (9.3-7.56)(9.2-7.78) + (7.1-7.56)(8.5-7.78) + (3.1-7.56)(7.2-7.78)}{5} + \\
&\frac{(9.5-7.56)(6.4-7.78)}{5} = \frac{(1.24)(-0.18) + (1.74)(1.42) + (-0.46)(0.72) + (-4.46)(-0.58) + (1.94)(-1.38)}{5} = \\
&\frac{-0.2232 + 2.4708 - 0.3312 + 2.5868 - 2.6772}{5} = \frac{1.826}{5} = 0.3652.
\end{aligned}$$

Thus, we will have that $Corr(M_1, M_2) = \frac{0.3652}{\sqrt{5.6864}\sqrt{0.9616}} \approx \frac{0.3652}{2.34} = 0.156$. $\square$

9 Recurrence Relations

Exercise 9.1

a) *Proof.* When $a_1 = 3$ and $a_n = 3a_{n-1} + 2$, we can first find our x value of the homogeneous part with $x^n = 3x^{n-1}$, so $x = 3$. Then to find our c value, we set $c = 3c + 2$, which means $c = -1$. Finally, to find the constant c_1 for our specific recurrence relation, we have $3 = 3c_1 - 1$ (where $3 = a_1$). This means $c_1 = \frac{4}{3}$, and thus we arrive at $a_n = \frac{4}{3}3^n - 1 = 4 \cdot 3^{n-1} - 1$. $\square$

b) *Proof.* When $b_1 = 1$ and $b_n = -b_{n-1} - 4$, we can first find our x value of the homogeneous part with $x^n = -x^{n-1}$, so $x = -1$. Then to find our c value, we set $c = -1c - 4$, which means $c = -2$. Finally, to find the constant c_1 for our specific recurrence relation, we have $1 = -1c_1 - 2$ (where $1 = b_1$). This means $c_1 = -3$, and thus we arrive at $b_n = (-3) \cdot (-1)^n - 2$. $\square$

c) *Proof.* When $c_0 = \frac{1}{3}$, $c_1 = -\frac{2}{3}$, and $c_n = c_{n-1} + 6c_{n-2} + 10$, we first consider the homogeneous part of the recurrence relation. Then we have $x^n = x^{n-1} - 6x^{n-2}$, which means $x^2 = x + 6$. Then we want to solve for x when $x^2 - x - 6 = 0$, so we have $x = 3$ and $x = -2$. Now we want our value of c, which we can easily guess to be $c = c + 6c + 10$, or that $c = \frac{-5}{3}$. To find our constants c_1' and c_2', we plug in our known information for $n = 0$ and $n = 1$, i.e. we have

$$\frac{1}{3} = c_1' + c_2' - \frac{5}{3}, c_1' = 2 - c_2'$$
$$-\frac{2}{3} = 3c_1' - 2c_2' - \frac{5}{3} = 3(2 - c_2') - 2c_2' - \frac{5}{3} = 6 - 5c_2' - \frac{5}{3}, c_2' = 1, c_1' = 1$$

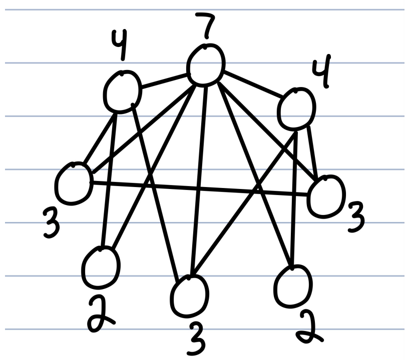
Thus, we have $c_n = 3^n + (-2)^n - \frac{5}{3}$. $\square$

10 Basic Graph Theory

Exercise 10.1

a) *Proof.* Notice that the sum of the degrees for our vertices is equal to $1+3+2+2+4+5=17$. Since it is necessary that the sum of the degrees of the vertices equals twice the number of edges in $E(G)$, then this would imply there were $\frac{17}{2}$ edges for our graph. This is clearly impossible, and thus G does not exist. □

b) *Proof.* □



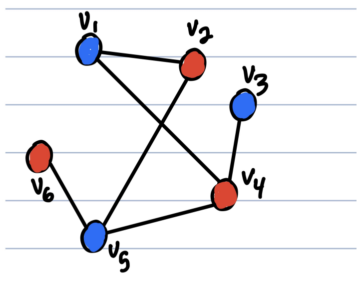
c) *Proof.* Because we have a graph with 4 vertices of which two have degree 3, we want 2 vertices to connect with every other vertex in the graph. However, this implies every vertex would have to have a degree of at least 2, which is not the case for our graph. Thus, we know such a graph is impossible to make. □

d) *Proof.* Because one of the vertices has a degree of 0, we can split this graph into two connected components, where one sub-graph has 4 vertices of degree not 0 and the other has 1 sub-graph with this vertex of degree 0. Then we see that this sub-graph with 4 vertices has a vertex with degree 4. However, since there are only 3 other vertices in this sub-graph, we cannot possibly make such a sub-graph that has only one edge between vertices. Thus, this graph G is not possible to make. □

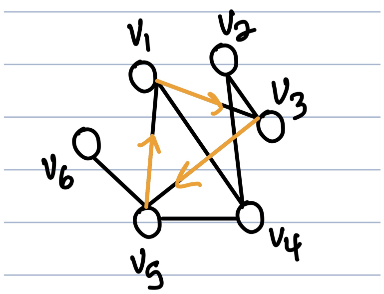
11 Bipartite Graphs and Matches

Exercise 11.1

a) *Proof.* □



b) *Proof.* The arrows give a cycle of odd length. Thus, G is not bipartite. □



Exercise 11.2

a) *Proof.* First, I will provide a match for this graph, and then give a technique on how to find matches for future graphs. First, a match of size 5 for this graph is

$$M = \{\{a_1, b_1\}, \{a_2, b_3\}, \{a_3, b_4\}, \{a_4, b_2\}, \{a_5, b_5\}\}.$$

Now, in order to find matches, a helpful tip is to start at any points in A that are only connected to one other point (the logic is that if a match existed, this edge would necessarily be included). After this, go through the remaining a_i vertices and try matching them with the vertex in B that has the least connections. For example, we see that a_1, a_3, and a_4 are each only connected to b_1, b_4, and b_2, respectively. Then $\{a_1, b_1\}$, $\{a_3, b_4\}$. and $\{a_4, b_2\}$ must be in our match of size 5. Now, we start at a_2 and see where it connects. It connects to b_3 and b_5, which both have two connections. Let us just guess that a_2 can connect to b_3. Finally, the only place left for a_5 is b_5, so everything works out. I hope you find this strategy to be useful. □

b) *Proof.* Consider U to be the set $U = \{a_1, a_2, a_3, a_5\}$. We notice that $|U| = 4$, but consider $N(U)$. We see that all the vertices in U only connect to b_2, b_3, and b_5, which means that $N(U) = \{b_2, b_3, b_5\}$. Thus, by Hall's theorem we know that there does not exist a match of size 5 for this graph. □