More Reductions for **NP** Problems

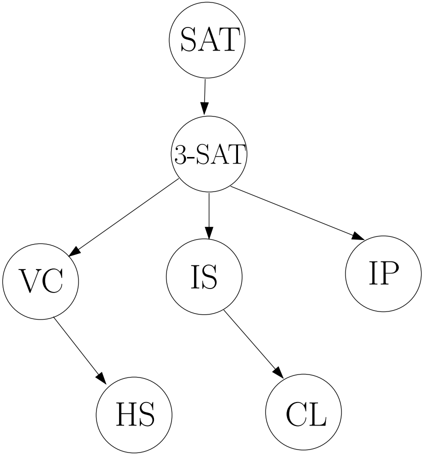
Nabil Mustafa

Computational Complexity

Reductions From Last Time

So far, we have shown the following problems **NP** complete:

- SAT , 3-CNF SAT, INDSET , CLIQUE , VERTEX-COVER , HITTING-SET , INTEGER-PROGRAMMING



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- Computationally very different from Eulerian paths
- Note that HAMILTONIAN is in **NP**
- We reduce 3-CNF SAT to HAMILTONIAN
  - For a fixed 3-CNF SAT formula, show that it can be transformed into a graph whose Hamiltonian path will give us the assignments for SAT.

Reduction Sketch

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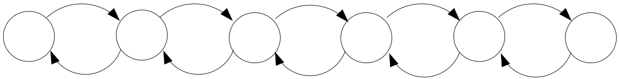

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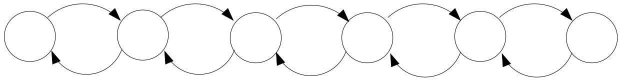
- If we are at the first (or end) vertex, only one path to follow

x_i



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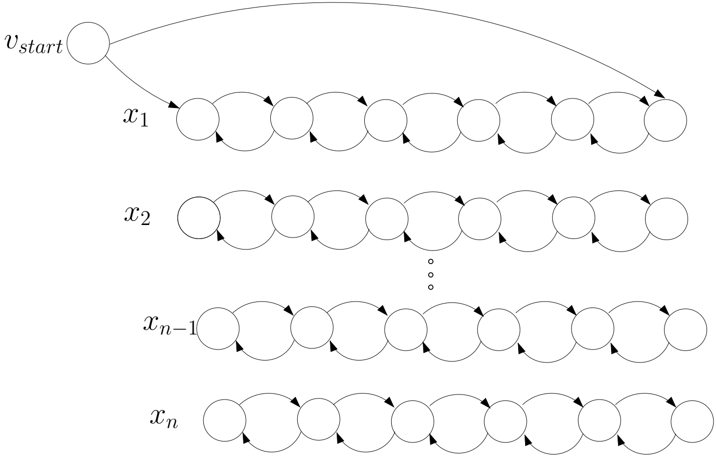

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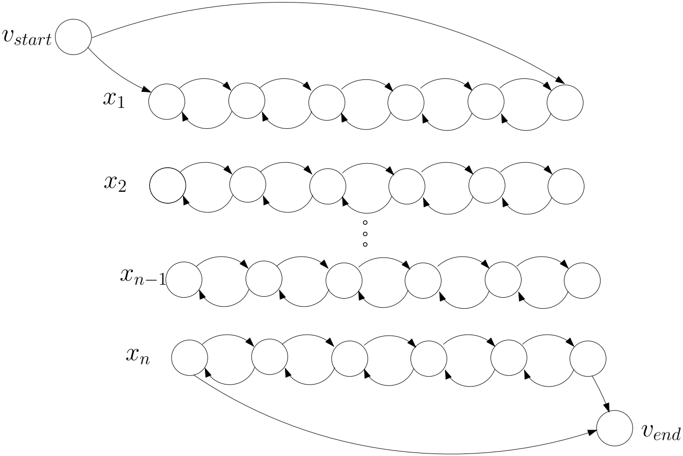

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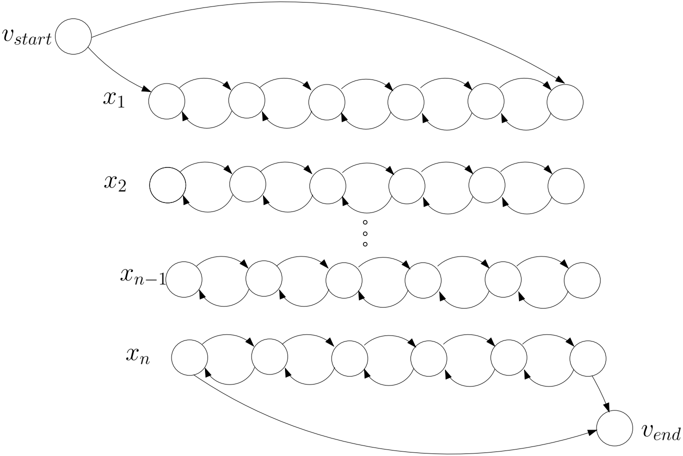

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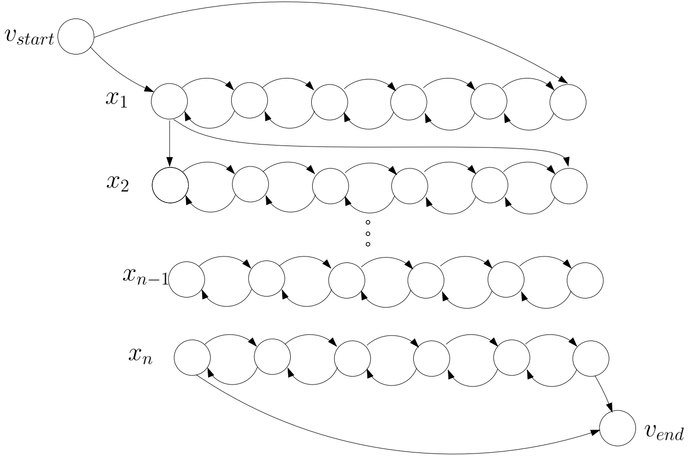

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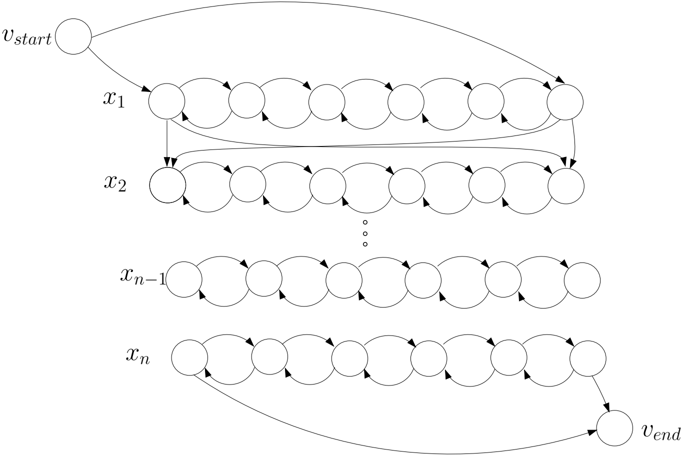

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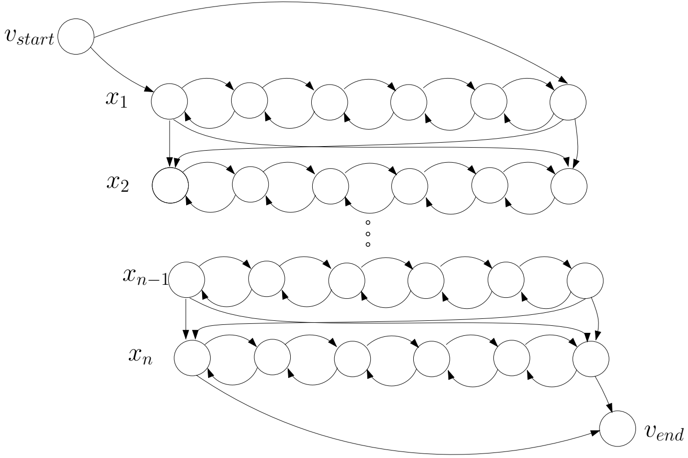
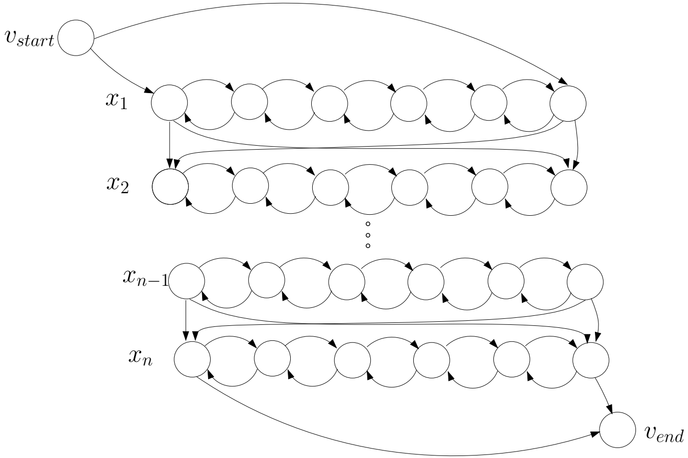


Construction Properties



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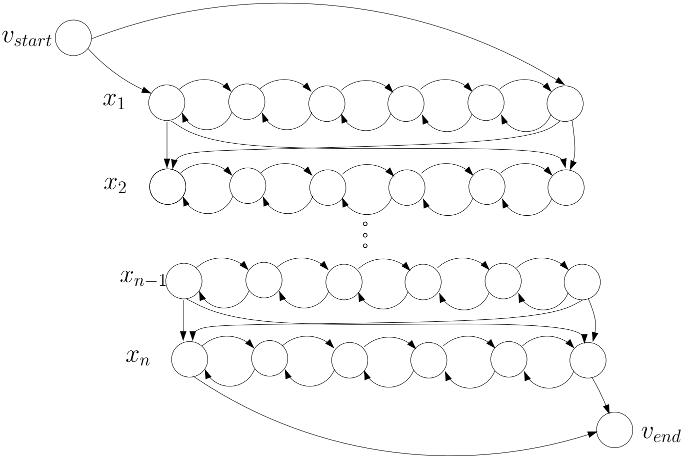

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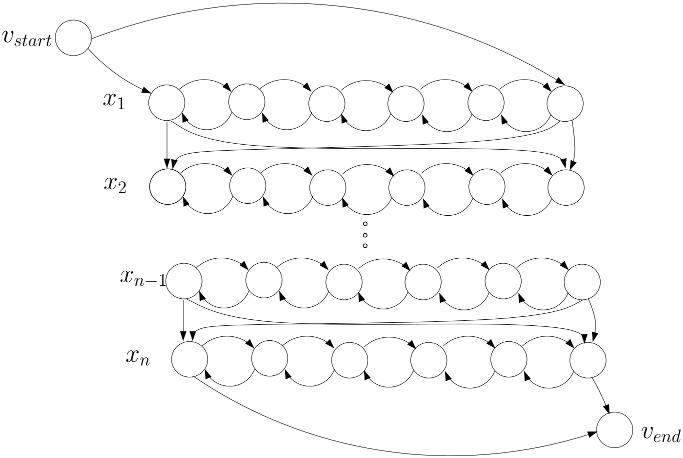
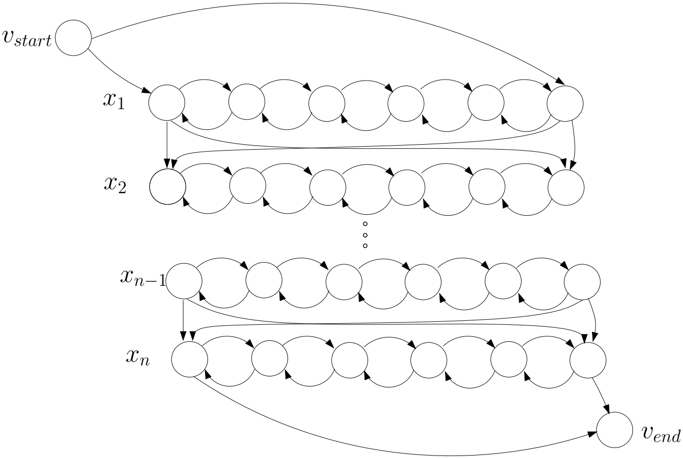


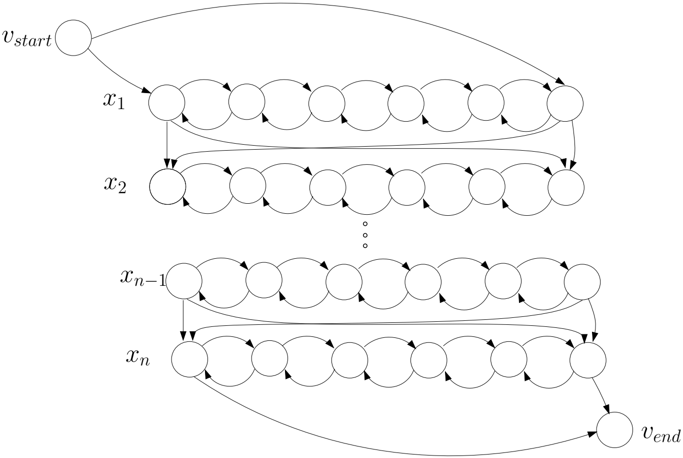
Construction Properties

- Any Hamiltonian path first traverses chain x_1, then x_2 etc.



Construction Properties

- For each chain, only two ways of traversing it.
  - Left-to-right means $x_i = 1$, right-to-left means $x_i = 0$



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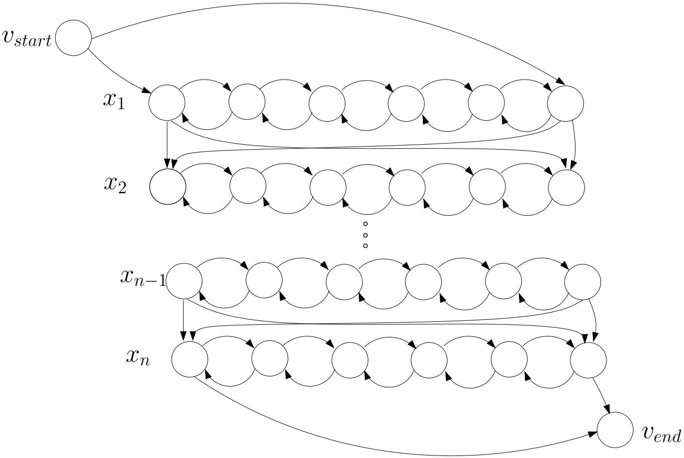

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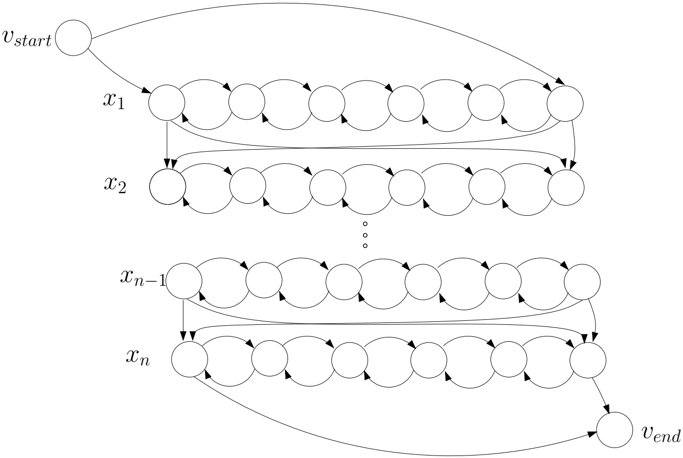
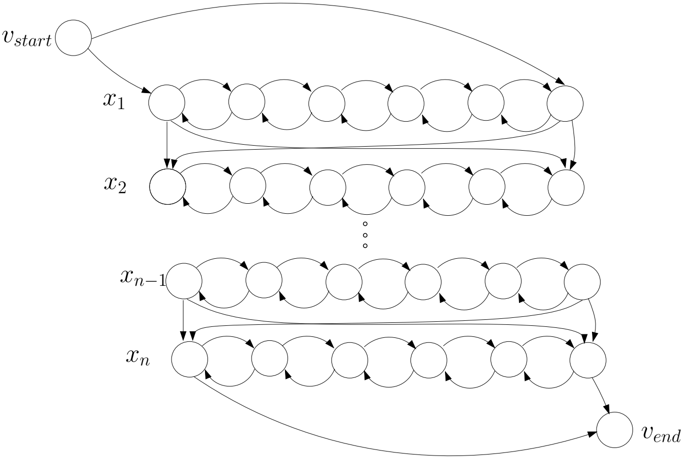


Construction Properties

- So far, no constraints – they will come from the clauses now.



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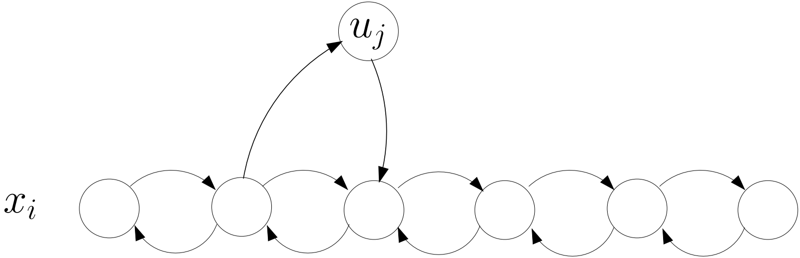

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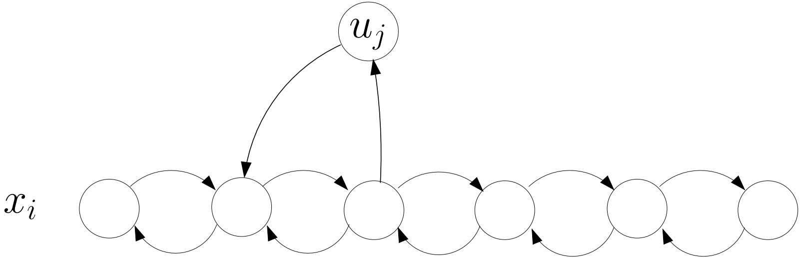


An Example

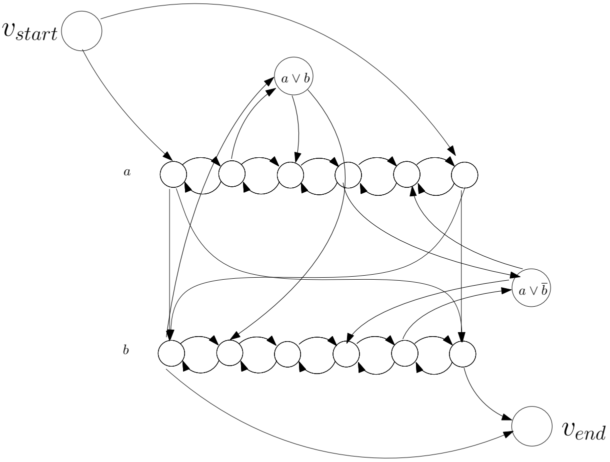
Construction of Hamiltonian path for

$$(a \vee b) \wedge (a \vee \overline{b})$$

Here:

- $\mathcal{C}_1 = (a \vee b)$
- $\mathcal{C}_2 = (a \vee \overline{b})$

The Graph Construction



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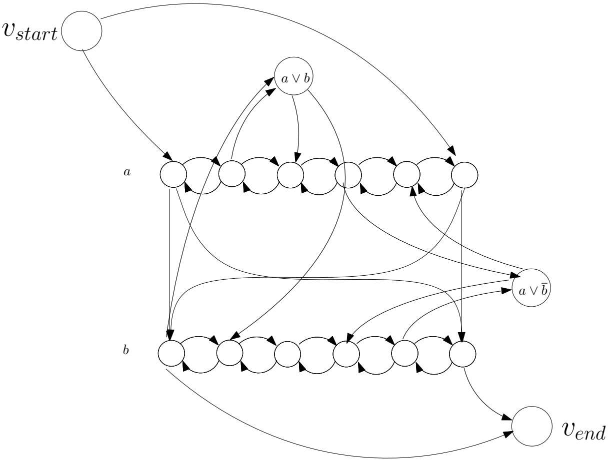
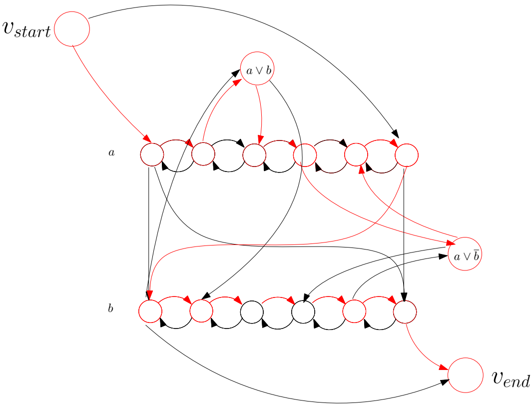


Claim

Hamiltonian path exists ONLY if you go from left to right in *a* and chose any one of the two directions for *b*.

Path Corresponding to Assignment 1

$a = 1; b = 1$



Path Corresponding to Assignment 2

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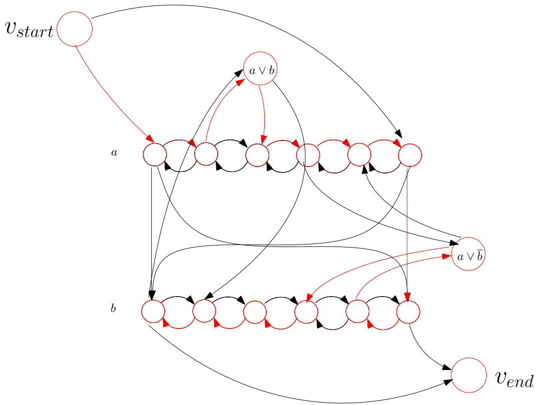

Path Corresponding to Assignment 3

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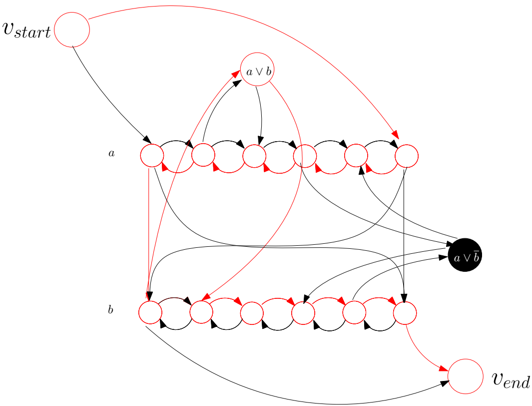

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Path Corresponding to Assignment 4

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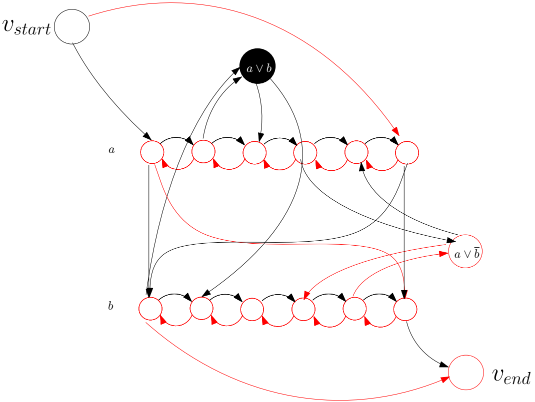

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- The above two *only* ways to visit u_j without getting stuck.

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SET-COVER : A collection $\mathcal{C} = \{S_1, \ldots, S_m\}$ of subsets of a base set X, $|X| = n$, and parameter k, find a set cover $\mathcal{C}' \subseteq \mathcal{C}$ of size k
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  - Each vertex $v_i \in V$ maps to set $C_i = \{a_j \ s.t. \ e_j$ incident to $v_i\}$

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$\square$

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EXACT-COVER : A collection $\mathcal{C} = \{S_1, \ldots, S_n\}, |S_j| = 3$ over a base set X, $|X| = 3m$, find a disjoint set cover $\mathcal{C}' \subseteq \mathcal{C}$ of size m.
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- Subset sum! Find a subset with value and weight equal to W.

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  - $v(S) = 1\ 0\ 0\ 1\ 1\ 0$
  - Easy to see that its a one-to-one mapping

An Example

$$\{1,4,6\},\{6,8,9\},\{2,4,7\},\{1,2,8\},\{2,4,6\},\{1,3,5\}$$

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$$\begin{array}{ccccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}$$

An Example

- Lets take the two sets $\{6,8,9\},\{2,4,7\}$.
- The corresponding vectors are:

$$\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

- The union of the two sets is: $\{2,4,6,7,8,9\}$
- The characteristic vector of the union is:

$$\begin{array}{ccccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}$$

- Any observations?

An Example

$$\{1,4,6\},\{6,8,9\},\{2,4,7\},\{1,2,8\},\{2,4,6\},\{1,3,5\}$$

An Example

$$\{1,4,6\}, \{6,8,9\}, \{2,4,7\}, \{1,2,8\}, \{2,4,6\}, \{1,3,5\}$$

An Example

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$$\begin{array}{ccccccccc l} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & : 2^3 + 2^1 + 2^0 = 011 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & : 2^7 + 2^5 + 2^2 = 164 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & : 2^8 + 2^6 + 2^4 = 336 \end{array}$$

An Example

$$\{1,4,6\}, \{6,8,9\}, \{2,4,7\}, \{1,2,8\}, \{2,4,6\}, \{1,3,5\}$$

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\end{array}$$

- The union covers X and the sets are pair-wise disjoint:

An Example

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- The union covers X and the sets are pair-wise disjoint:

$$\begin{array}{ccccccccc l}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & : 2^9 - 1 = 511
\end{array}$$

An Example

$$\{1,4,6\},\{6,8,9\},\{2,4,7\},\{1,2,8\},\{2,4,6\},\{1,3,5\}$$

$$\begin{array}{ccccccccc l}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & : 2^3+2^1+2^0=011 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & : 2^7+2^5+2^2=164 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & : 2^8+2^6+2^4=336
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- The union covers X and the sets are pair-wise disjoint:

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- $11+164+336=511$.

An Example

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- $11 + 164 + 336 = 511$. So set $W = 2^n - 1$, and check.

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$$
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- There exists subset equal to $W \implies$ exact-cover possible

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- Problem:

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- The union covers X and the sets are pair-wise disjoint:

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- $11 + 164 + 336 = 511$. So set $W = 2^n - 1$, and check.
- There exists subset equal to $W \implies$ exact-cover possible
- Problem: False positives.

A Problem

$$\{3,4\},\{2,4\},\{2,3,4\}$$

A Problem

$$\{3, 4\}, \{2, 4\}, \{2, 3, 4\}$$

$0 \quad 0 \quad 1 \quad 1 \quad : 3$

A Problem

$$\{3,4\}, \{2,4\}, \{2,3,4\}$$

$$\begin{array}{cccccc} 0 & 0 & 1 & 1 & : 3 \\ 0 & 1 & 0 & 1 & : 5 \end{array}$$

A Problem

$$\{3,4\},\{2,4\},\{2,3,4\}$$

$$\begin{array}{ccccl} 0 & 0 & 1 & 1 & :3 \\ 0 & 1 & 0 & 1 & :5 \\ 0 & 1 & 1 & 1 & :7 \end{array}$$

A Problem

$$\{3, 4\}, \{2, 4\}, \{2, 3, 4\}$$

$$\begin{array}{cccccl} 0 & 0 & 1 & 1 & & : 3 \\ 0 & 1 & 0 & 1 & & : 5 \\ 0 & 1 & 1 & 1 & & : 7 \end{array}$$

- The union does not cover X, and not disjoint.

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$$\{3, 4\}, \{2, 4\}, \{2, 3, 4\}$$

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$$0\,0\,1\,1 \;+\; 0\,1\,0\,1 \;+\; 0\,1\,1\,1$$

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$$\{3,4\},\{2,4\},\{2,3,4\}$$

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- The union does not cover X, and not disjoint.

$$0\,0\,1\,1 + 0\,1\,0\,1 + 0\,1\,1\,1 = 1\,1\,1\,1$$

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$$\{3,4\}, \{2,4\}, \{2,3,4\}$$

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$$0\,0\,1\,1 \ + \ 0\,1\,0\,1 \ + \ 0\,1\,1\,1 \ = \ 1\,1\,1\,1 = 15$$

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- So, where is the problem coming from?

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- Carrying! It messes up the addition.

A Solution

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$$0\ 0\ 1\ 1 \ +\ 0\ 1\ 0\ 1 \ +\ 0\ 1\ 1\ 1$$

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$$0\ 0\ 1\ 1\ +\ 0\ 1\ 0\ 1\ +\ 0\ 1\ 1\ 1\ =\ 0\ 2\ 2\ 3$$

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- Adding at most n 1's $\implies$ no carry-over at any digit

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- This solves both the problems since:

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- Adding at most n 1's $\implies$ no carry-over at any digit
- This solves both the problems since:
  - If a column has all 0's, no carry-over ensures it gets a 0.
  - If a column has more than one 1, it will get the sum, not 1.

Reductions So Far