Mathematics for Decisions

Integer Linear Programming: Exercises about Cutting Plane methods and Gurobi

Romeo Rizzi, Alice Raffaele

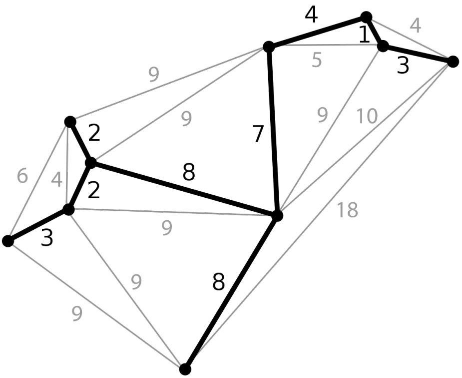
University of Verona

romeo.rizzi@univr.it, alice.raffaele@unitn.it

December 11, 2020

Overview

Minimum Spanning Tree (MST)



- Consider a graph $G = (V, E)$, with $|V| = n$ and $|E| = m$.
- Typically, MST is solved by applying a greedy algorithm (e.g., Kruskal, Prim).
- Can we model MST as an ILP?

ILP formulation (I)

- Variables: $x_{i,j}$ for each edge $(i,j) \in E$, where $x_{i,j} = 1$ if the edge is selected in the tree T, 0 otherwise.
- Constraints:
  - T must have $n-1$ edges $\rightarrow \sum_{(i,j) \in E} x_{i,j} = n-1$.
  - Being a tree, T must have no cycles $\rightarrow$ **SEC** (again!):

    $$\sum_{(i,j) \in E, i \in S, j \in S} x_{i,j} \leq |S| - 1, \forall S \subseteq V$$

    Any subset of k vertices must have at most $k-1$ edges contained in that subset.
- Objective function: minimize the total cost of selected edges $\rightarrow \min \sum_{(i,j) \in E} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the cost associated to edge (i,j).

ILP formulation (II)

$$
\begin{aligned}
& \min \sum_{(i,j)\in E} c_{i,j} x_{i,j} && (1)\\
\text{subject to} \quad & \sum_{(i,j)\in E} x_{i,j} = n-1, && (2)\\
& \sum_{(i,j)\in E, i\in S, j\in S} x_{i,j} \leq |S|-1, \forall S \subseteq V && (3)\\
& x_{i,j} \in \{0,1\}, \forall (i,j) \in E. && (4)
\end{aligned}
$$

Again, because of SEC, the problem has exponential size.
Let's consider other formulations.

Cut formulation

- For every cut, at least one edge must cross the cut. For a subset $S \subset V$, let $\delta(S)$ be the edges crossing the cut (with one endpoint in S and the other in $V \setminus S$).

$$
\begin{aligned}
\min \quad & \sum_{(i,j) \in E} c_{i,j} x_{i,j} && (1) \\
\text{subject to} \quad & \sum_{(i,j) \in E} x_{i,j} = n - 1, && (2) \\
& \sum_{(i,j) \in E, (i,j) \in \delta(S)} x_{i,j} \geq 1, \forall S \subseteq V, S \neq \emptyset, S \neq V && (3) \\
& x_{i,j} \in \{0, 1\}, \forall (i,j) \in E. && (4)
\end{aligned}
$$

Also this formulation has an exponential number of constraints.

LP relaxation

$$\min \sum_{(i,j)\in E} c_{i,j} x_{i,j} \tag{1}$$

$$\text{subject to} \quad \sum_{(i,j)\in E} x_{i,j} = n - 1, \tag{2}$$

$$\sum_{(i,j)\in E, i\in S, j\in S} x_{i,j} \leq |S| - 1, \forall S \subseteq V \tag{3}$$

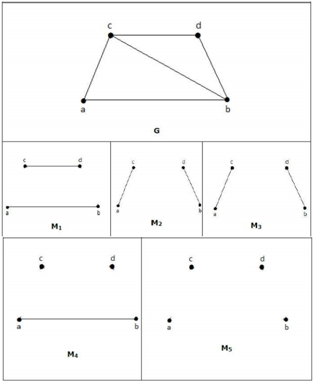
$$x_{i,j} \geq 0, \forall (i,j) \in E. \tag{4}$$

- $x_{i,j} \leq 1$ is implied by two vertex sets S.
- Unfortunately, we still have an exponential number of constraints.
- One can show that the MST is an optimal solution to the relaxation (i.e., the LP has integer extreme points).

Assignment: implement a Branch-and-Cut for MST

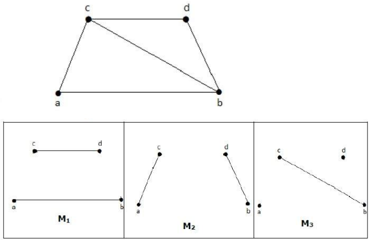
1. Choose one ILP formulation for MST.
2. Ignore the family of constraints with exponential size and implement the obtained relaxed formulation in Gurobi.
3. While the solution has a fractional coordinate, find the violated subtour elimination constraint and add it to the formulation, by exploiting an appropriate callback function (similarly to what we did together to tackle TSP); then, solve the problem again.

Matchings

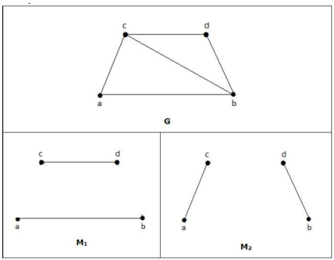


- Given a graph G, a *matching* M is a set of pairwise non adjacent edges (i.e., no two edges share a common vertex and all vertices have degree either 0 or 1).
- A matching M is *maximal* if it is not a subset of any other matching in G.
- A *maximum* matching contains the largest possible number of edges; the number of maximum matchings of G is called *matching number*.
- Note that every maximum matching is maximal, but not every maximal matching is a maximum matching.

A matching M is *perfect* if it saturates every vertex (i.e., if every vertex is incident to exactly one edge in M):



Maximal matchings of G



Maximum matchings of G

- Every perfect matching (PM) of graph is also a maximum matching of graph, because there is no chance of adding another edge in a perfect matching; thus, the matching is also maximal.
- A maximum matching of a graph is not necessarily perfect.
- If a graph G has a perfect matching, then the number of vertices $|V|$ is even. In fact, if it were odd, a vertex would remain single and not paired with any other vertex for which the degree is zero.

ILP formulation

- Consider a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$.
- Edmonds (1964) discovered the following polyhedral characterization of the matching problem, where basic solutions correspond to perfect matchings of G:

$$\min \sum_{(i,j)\in E} c_{i,j} x_{i,j} \tag{1}$$

$$\text{subject to} \quad \sum_{(i,j)\in\delta(i)} x_{i,j} = 1, \forall i \in V \tag{2}$$

$$\sum_{(i,j)\in E, i\in S, j\in V\setminus S} x_{i,j} \geq 1, \forall S \subsetneq V, |S| \text{ odd}, 3 \leq |S| \leq |V| - 3 \tag{3}$$

$$x_{i,j} \geq 0, \forall (i,j) \in E. \tag{4}$$

Blossom inequalities and bipartite relaxation

- Constraints (3) are called *blossom inequalities*.
- Even if the number of blossom inequalities is exponential in the size of the graph, for any point not in the perfect matching polytope, a violated (blossom) inequality can be found in polynomial time.
- This leads to a natural cutting plane algorithm, starting with the so called *bipartite relaxation*:

$$\min \sum_{(i,j)\in E} c_{i,j} x_{i,j} \tag{1}$$

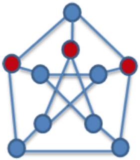
$$\text{subject to} \quad \sum_{(i,j)\in\delta(i)} x_{i,j} = 1, \forall i \in V \tag{2}$$

$$x_{i,j} \geq 0, \forall (i,j) \in E. \tag{3}$$

Assignment: implement a Branch-and-Cut for PM

1. Consider the LP formulation for PM, ignoring the family of constraints given by blossom inequalities, and implement the bipartite relaxation in Gurobi.
2. While the solution has a fractional coordinate, find the violated blossom inequality and add it to the formulation, by exploiting an appropriate callback function; then, solve the problem again.

Maximum Stable Set (MSS)



- Consider a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$.
- A *stable set* (or *independent set*) is a subset of V such that every induced subgraph is empty (i.e., no two vertices are adijacent).
- Such a set is said to be *maximal* if it is not a subset of another independent set.
- The cardinality of the largest independent set of a graph G, is known as the *independence number of G* and denoted by $\alpha(G) \rightarrow$ *Maximum Stable Set* problem

ILP formulation

$$
\begin{aligned}
& \max \sum_{i \in V} x_i && (1)\\
\text{subject to} \quad & x_i + x_j \leq 1, \forall (i,j) \in E && (2)\\
& x_i \in \{0,1\}. && (3)
\end{aligned}
$$

Even if the number of constraints is polynomial, the problem is known to be NP-hard and so does ILP as well. This suggests the use of Branch-and-Cut algorithms.

LP relaxation

$$\max \sum_{i \in V} x_i \tag{1}$$

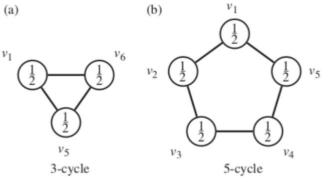
$$\text{subject to} \quad x_i + x_j \leq 1, \forall (i,j) \in E \tag{2}$$

$$x_i \geq 0. \tag{3}$$

- The basic solutions of the LP relaxation are $\left(0, \frac{1}{2}, 1\right)$-valued (Balinski, 1970; Nemhauser and Trotter, 1974).
- The two polytopes defined by the ILP and LP relaxations coincide iff the graph G is bipartite (Grotschel et al., 1988), thus the problem can be solved in polynomial time.
- For general graphs, the ILP formulation might be incomplete.

Odd-cycle inequalities

- One of the simplest non-bipartite graphs are the ones induced by odd cycles:



- We introduce a new class of inequalities to avoid *odd cycles*:

$$\sum_{v_i \in V(C)} x_i \leq \frac{|V(C)|-1}{2}, \forall \text{ odd cycle } C \in G$$

By construction, the odd-cycle inequalities are valid for the stable set polytope (the cardinality of any stable set in an odd cycle can be at most the greatest integer smaller than half the length of the cycle).

Assignment: implement a Branch-and-Cut for MSS

1. Consider the LP formulation for MSS.
2. While the solution has a fractional coordinate, find the violated odd-cycle inequality and add it to the formulation, by exploiting an appropriate callback function; then, solve the problem again.

References

- Clifford Stein, *Minimum Spanning Tree Formulation*, http://www.columbia.edu/~cs2035/courses/ieor6614.S16/mst-lp.pdf
- Jack Edmonds, *Maximum Matching and a Polyhedron With 0,1-Vertices*, https://nvlpubs.nist.gov/nistpubs/jres/69B/jresv69Bn1-2p125_A1b.pdf
- Karthekeyan Chandrasekaran, Laszlo A. Vegh and Santosh S. Vempala, *The Cutting Plane Method is Polynomial for Perfect Matchings*, http://eprints.lse.ac.uk/66123/1/Vegh_Cutting%20plane%20method.pdf
- Steffen Rebennack, Gerhard Reinelt and Panos M. Pardalos, *A tutorial on branch and cut algorithms for the maximum stable set problem*, https://onlinelibrary.wiley.com/doi/pdf/10.1111/j.1475-3995.2011.00805.x