Mathematics for Decisions

Basics of Linear Programming

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March 2020

Overview

Linear Programming

A *Linear Programming* (LP) problem is a mathematical programming problem of the form:

$$\begin{aligned} \text{min } \textit{or} \text{ max} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

where $x \in \mathbb{R}^n$ are the decision variables, $b \in \mathbb{R}^m$ is the vector of known values, $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$ is the vector of the coefficients in the objective function.

Canonical and standard form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Canonical

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Standard

The two formulations are equivalent but the conversion from one form to the other may change the number of constraints and variables. Rules to follow:

- Conversion from "min" to "max", changing the sign of c^T
- Constraints conversion from "$\leq$" to "$=$", introducing **slack variables**: $\mathbf{a}_i^T\mathbf{x} \geq b_i \rightarrow \mathbf{a}_i^T\mathbf{x} + s_i = b_i$, with $s_i \geq 0$;
- Free variables: if x_i free, then $x_i = x_i^+ - x_i^-$, with $x_i^+, x_i^- \geq 0$;

Integer, Mixed Integer and Binary Linear Programming

- Integer LP (ILP): when all variables must assume integer values;
- Mixed Integer LP (MILP): when some variables are integer and other continuous;
- Binary LP (0-1 LP): when all variables can only assume 0 or 1 as values.

Geometry of Linear Programming

- **(Convex) Polyhedron**: intersection of a finite number of affine half-spaces and hyperplanes.
- **Feasible region**: set of feasible solutions $\mathbf{x} \in \mathbb{R}^n$ that satisfy all the linear inequalities $\rightarrow$ It's a polyhedron.
- **Polytope**: a bounded polyhedron.
- **Vertex** or **Extreme point**: a point $\mathbf{x}$ of a polyhedron P that cannot be expressed as a strict convex combination of other two points of the polyhedron, i.e., there exist no $\mathbf{y}, \mathbf{z} \in P, \mathbf{y} \neq \mathbf{z}$ and $\lambda \in (0, 1)$ such that $\mathbf{x} = \lambda \mathbf{y} + (1 - \lambda)\mathbf{z}$.
- Each polyhedron has a **finite number of vertices**.
- **Minkowski-Weyl Theorem**: every point of a polytope P can be obtained as a convex combination of its vertices $\rightarrow$ If the feasible region of a LP is a bounded polytope, then there exists at least one optimal vertex of P.

Vertices and basic solutions

- The optimal solution of an LP is a vertex: we can start from one vertex arbitrarily and iterate through the vertices, moving to an adjacent one, until the optimal is found.
- **Basis of A**: a collection of m linearly independent columns of A
- **Basic and non-basic variables**: $\mathbf{x}_B$ and $\mathbf{x}_F$

$$A\mathbf{x} = \mathbf{b} \text{ can be written as } B\mathbf{x}_B + F\mathbf{x}_F = \mathbf{b}$$

- When $\mathbf{x}_F = 0$, $\mathbf{x}_B = B^{-1}\mathbf{b}$ is the **basic solution** associated to the basis B. It is:
  - *feasible*, if $B^{-1}\mathbf{b} \geq 0$;
  - *degenerate*, if $B^{-1}\mathbf{b}$ has one or more zero components.
- A point $\mathbf{x}$ of the polyhedron $P := \{\mathbf{x} \geq 0 : A\mathbf{x} = \mathbf{b}\}$ is a **vertex** iff $\mathbf{x}$ is a basic feasible solution of $A\mathbf{x} = \mathbf{b}$.

The Simplex method

How to solve an LP?

- Enumerate all possible vertices, i.e., all the basic solutions to the problem $\rightarrow$ Number of vertices $= \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- Improving this procedure:
    - Verify the optimality of the current solution;
    - Find a way to move from a basic feasible solution to another adjacent with a better value of the objective function.
- **Tableau form**:
    - Exercise at the blackboard
    - See also `https://www.youtube.com/watch?v=XK26I9eoSl8` and `https://www.hec.ca/en/cams/help/topics/The_steps_of_the_simplex_algorithm.pdf`

Particular cases

- Loop between entering and exiting variables
- Empty feasible region
- Unlimited solution
- Multiple optimal solutions

Exam 31/07/2017 - Ex. 7

Consider the following LP problem:

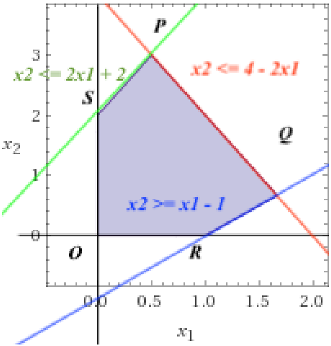
$$
\begin{aligned}
\max 3x_1 + 2x_2 & \\
2x_1 + x_2 &\leq 4 \\
-2x_1 + x_2 &\leq 2 \\
x_1 - x_2 &\leq 1 \\
x_1, x_2 \geq 0. &
\end{aligned}
$$

1. Solve it with the graphical method, specifying the objective function and the variables values at optimum.

Note: the problem is presented in the canonical form, not the standard one.

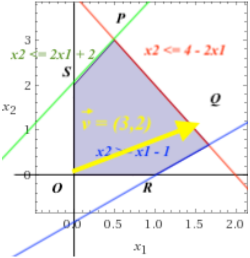
Graphical method (I)

We represent the three constraints in the plane (x_1, x_2) obtaining through their intersection the feasible region:



Graphical method (II)

- The feasible region has five vertices:
  - O $= x_1 \geq 0 \cap x_2 >= 0$;
  - P $= cons_1 \cap cons_2$;
  - Q $= cons_1 \cap cons_3$;
  - R $= cons_3 \cap x_1 \geq 0$;
  - S $= cons_2 \cap x_2 \geq 0$.
- The objective function can be seen as a family of straight lines moving towards the direction where the function is maximized: (3, 2)



Graphical method (III)

- The optimal solution is given by point $P = (\frac{1}{2}, 3)$, the last vertex reached by the family of straight lines;
- Here, the objective function is $\frac{15}{2}$.

Vertices, variables and basic solutions, non-basic variables, ...

- Given a system of linear constraints in n variables, we call **solution** a point $x \in \mathbb{R}^n$ that satisfies all the constraints;
- We write the problem in the standard form, introducing the three slack variables s_1, s_2 e s_3:

$$
\begin{array}{ll}
\max & 3x_1 + 2x_2 \\
\text{s.t.} & 2x_1 + x_2 + s_1 = 4 \\
& -2x_1 + x_2 + s2 = 2 \\
& x_1 - x_2 + s_3 = 1 \\
& x_1, x_2 \geq 0 \\
& s_1, s_2, s_3 \geq 0
\end{array}
$$

Associated bases

2. Determine the bases associated to the vertices of the feasible region;

We can rewrite $Ax = b$ as $(B|N) \cdot (x_B|x_N)^T = b$ where:

- $B =$ basic matrix ($m \times m$, composed of m columns of A)
- $N =$ non-basic matrix
- $x_B =$ basic variables
- $x_N =$ non-basic variables

Simplex

3. Specify the sequence of the bases visited by the Simplex method to reach the optimal solution (choose x_1 as the first entering variable);

Reduced costs

4. Determine the values of the reduced costs related to the basic solutions associated to the following vertices, expressed as intersections of straight lines in$\mathbb{R}^2$:
    - Eq.-Constraint 1 $\cap$ Eq.-Constraint 2
    - Eq.-Constraint 1 $\cap$ Eq.-Constraint 3

Opposite direction of the gradient vector

5. Verify that the opposite direction of the gradient vector can be expressed as a nonnegative linear combination of the gradients for **active** constraints **only** in the optimal vertex (keep in mind that, since it is a maximization problem, constraints have to be expressed with $\leq$; e.g., $x_1 \geq 0$ has to be rewritten as $-x_1 \leq 0$).

Duality in Linear Programming

Any **primal** LP in maximization form is associated to a **dual** LP in minimization form:

Primal Problem	Dual Problem
opt=max	opt=min
Constraint i :	Variable i :
<= form	y_i >= 0
= form	y_i urs
Variable j:	Constraint j:
x_j >= 0	>= form
x_j urs	= form

Duality theorems

Given the primal problem $P : \max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$ and the dual problem $D : \min \mathbf{b}^T\mathbf{u}$ s.t. $A^T\mathbf{u} \geq \mathbf{c}, \mathbf{u} \geq 0$:

- The dual of the dual problem D is the primal P.
- **Weak duality**: $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u}$.
- **Strong duality**: P has a finite optimal solution iff D has it too and the value of the two objective function is the same $\rightarrow \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{u}$.
- Economic interpretation

Relationships between Primal and Dual

		DUAL		
		FINITE OPTIMAL SOLUTION	UNBOUNDED	INFEASIBLE
PRIMAL	FINITE OPTIMAL SOLUTION	YES	NO	NO
	UNBOUNDED	NO	NO	YES
	INFEASIBLE	NO	YES	YES

Optimality conditions

Two vectors $\bar{\mathbf{x}} \in \mathbb{R}^n$ and $\bar{\mathbf{u}} \in \mathbb{R}^m$ are optimal for the primal problem P and the dual problem D iff the following optimality conditions hold:

1. $A\bar{\mathbf{x}} \geq \mathbf{b}, \bar{\mathbf{x}} \geq 0$ (primal feasibility)
2. $\mathbf{c}^T \geq \bar{\mathbf{u}}^T A, \bar{\mathbf{u}} \geq 0$ (dual feasibility)
3. $\bar{\mathbf{u}}^T(A\bar{\mathbf{x}} - b) = 0$ (complementary slackness)
4. $(\mathbf{c}^T - \bar{\mathbf{u}}^T A)\bar{\mathbf{x}} = 0$ (complementary slackness)

Sensitivity analysis

Once we get the optimal solution, are we done?

- We could investigate how much the solution is **stable**, w.r.t. changes the parameter data;
- Do not forget that we are solving a **model** of the problem, not the problem itself! Thus the less sensible is the solution, the more reliable is the model;
- **Sensitivity analysis**: study of perturbations of initial data whereby conditions:
  - $B^{-1}\mathbf{b} \geq 0$ (primal feasibility for $\bar{\mathbf{x}}$);
  - $\bar{\mathbf{c}}^T := \mathbf{c}^T - \mathbf{c}_B^T B^{-1} A \geq 0^T$ (dual feasibility for $\bar{\mathbf{u}}$, where $\bar{\mathbf{u}}^T := \mathbf{c}_B^T B^{-1}$).
- The basis B remains optimal (not the solution $\mathbf{x}$).
- We'll study three cases:
  - Changes in the right-hand sides
  - Changes in the costs of non-basic variables
  - Changes in the costs of basic variables

Changes in the right-hand sides

We consider a change of $\Delta\mathbf{b}$:

- $B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq 0$
- $\bar{\mathbf{c}}^T := \mathbf{c}^T - \mathbf{c}_B^T B^{-1} A \geq 0^T$ (unchanged).

So basis B remain feasible and optimal iff:

$$B^{-1}\mathbf{b} \geq -B^{-1}\Delta\mathbf{b}$$

The optimal value changes from $\mathbf{c}_B^T B^{-1}\mathbf{b}$ to $\mathbf{c}_B^T B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \rightarrow$ $\Delta z := (\mathbf{c}_B^T B^{-1})\Delta\mathbf{b} = \bar{\mathbf{u}}^T \Delta\mathbf{b}$

The dual variables $\bar{u}_i$, $i = 1, \ldots, m$, measure the **sensitivity** of the optimal value of the objective function w.r.t. changes Δb_i of the right-hand sides.

Changes in the costs of non-basic variables

Now we consider a change $\Delta \mathbf{c}_F^T$ and let $\mathbf{c}$ and $\widetilde{\mathbf{c}}$ be the reduced cost vectors before and after change $\Delta \mathbf{c}_F^T$.

- $B^{-1}\mathbf{b} \geq 0$ (unchanged);
- $\widetilde{\mathbf{c}}^T := [\widetilde{\mathbf{c}}_B^T, \widetilde{\mathbf{c}}_F^T] = [0^T, (\mathbf{c}_F^T + \Delta \mathbf{c}_F^T) - \mathbf{c}_B^T B^{-1} F] \geq 0^T$.

As before, we want B to remain optimal, and this happens iff:

$$\widetilde{\mathbf{c}}^T = \mathbf{c}_F^T - \mathbf{c}_B^T B^{-1} F + \Delta \mathbf{c}_F^T = \bar{\mathbf{c}}_F^T + \Delta \mathbf{c}_F^T \geq 0 \iff \Delta \mathbf{c}_F \geq -\bar{\mathbf{c}}_F.$$

So we obtain $n - m$ inequalities, independent from each other:

$$\Delta c_j \geq -\bar{c}_j, \forall x_j \text{ non-basic}$$

The reduced cost $\bar{c}_j \geq 0$ can be interpreted as the maximum **decrease** in cost c_j under which B remains optimal.

Changes in the costs of basic variables

Finally we consider a change $\Delta\mathbf{c}_B^T$ and, as before, and let $\mathbf{c}$ and $\widetilde{\mathbf{c}}$ be the reduced cost vectors before and after change $\Delta\mathbf{c}_B^T$.

- $B^{-1}\mathbf{b} \geq 0$ (unchanged);
- $\widetilde{\mathbf{c}}^T := [\widetilde{\mathbf{c}}_B^T, \widetilde{\mathbf{c}}_F^T] = [0^T, \mathbf{c}_F^T - (\mathbf{c}_B^T + \Delta\mathbf{c}_B^T)B^{-1}F] \geq 0^T$.

So B remains optimal iff:

$$\widetilde{\mathbf{c}}_F^T := \mathbf{c}_F^T - \mathbf{c}_B^T B^{-1} F - \Delta\mathbf{c}_B^T B^{-1} F \geq 0^T \iff \Delta\mathbf{c}_B^T B^{-1} F \leq \bar{\mathbf{c}}_F^T$$

We have obtained a system which defines a polyhedron in $\mathbb{R}^m$, whose points correspond to the vectors $\Delta\mathbf{c}_B$ for which B does not change.

References

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