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Giovanni Domenico Di Salvo

Proper Holomorphic Embeddings of open Riemann Surfaces into $\mathbb{C}^2$ and holomorphic mappings between complex manifolds with dense images

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Alla prima parte della mia vita.

It's time to hear the call
of all you are
of all you are!

SMKC, Boulevard of Broken Hearts

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo, under the supervision of professor Erlend Fornæss Wold and professor Tuyen Trung Truong.

The thesis is a collection of four papers, presented in chronological order of writing. The papers are preceded by an introductory chapter that provides background information and motivation for the work.

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Thanks to all the people that contributed to help me achieving such a great goal: my friends, my music, my family.

This is not the way I would write anything personal; the way I feel to express my gratitude does not match with any Ph.D. thesis acknowledgement format; I will do it in a different context.

Giovanni Domenico Di Salvo
Oslo, April 2022

List of Papers

Paper I

G. D. Di Salvo and E. F. Wold. "Proper Holomorphic Embeddings of complements of large Cantor sets in $\mathbb{C}^2$". Submitted for publication to *Arkiv för Matematik*.

Paper II

G. D. Di Salvo. "Extended explanation of Orevkov's paper on proper holomorphic embeddings of complements of Cantor sets in $\mathbb{C}^2$ and a discussion of their measure". To be submitted.

Paper III

G. D. Di Salvo, T. Ritter and E. F. Wold. "Families of Proper Holomorphic Embeddings and Carleman–type Theorems with parameters". Submitted for publication to *The Journal of Geometric Analysis*.

Paper IV

G. D. Di Salvo. "Approximation and accumulation results of holomorphic mappings with dense image". Submitted for publication to *Mathematica Scandinavica*.

Contents

Notation

We will denote by $\mathbb{C}^n$ the usual n–dimensional complex euclidean space, endowed with the euclidean topology and distance, whose coordinates are $z = (z_1, \dots, z_n)$. We denote by $\mathbb{B}^n := \{z \in \mathbb{C}^n \ : \ |z_1|^2 + \cdots + |z_n|^2 < 1\}$ the open unit ball in $\mathbb{C}^n$; correspondingly $R\mathbb{B}^n := \{z \in \mathbb{C}^n \ : \ |z_1|^2 + \cdots + |z_n|^2 < R^2\}$ is the R–ball, for $R > 0$. When $n = 1$ we might write $\triangle$ and $\triangle_R$ instead of $\mathbb{B}$ and $R\mathbb{B}$ respectively.

Given a set A inside a complex manifold X, we will denote its topological interior, closure, boundary and the set of its accumulation points by writing A° (equiv. $\operatorname{int} A$), $\overline{A}$ (equiv. $\operatorname{cl} A$), ∂A, and $\operatorname{Acc} A$ respectively, with respect to the topology of X (or to some induced topology, it will be clear by the context). When $A = \mathbb{C}^n$, then $\overline{\mathbb{C}^n} = \mathbb{C}^n \cup \{\infty_n\}$ denotes the one–point compactification; in particular for $n = 1$ we agree to write $\infty_1 = \infty$ and we get the Riemann sphere $\overline{\mathbb{C}}$ which is equivalently denoted as the projective complex line $\mathbb{P}^1(\mathbb{C})$.

We will refer to a *domain* or to a *region* in $\mathbb{C}^n$ when dealing with an open connected set.

Given a domain $D \subset \mathbb{C}^n$ (a complex manifold X respectively) we will denote by $\mathcal{O}(D)$ (by $\mathcal{O}(X)$ respectively) the $\mathbb{C}$–algebra of all complex–valued holomorphic functions on D (on X respectively). When the target space is any Y different than $\mathbb{C}$, we will highlight it by writing $\mathcal{O}(X, Y)$. If K is compact, $\mathcal{O}(K)$ will denote the $\mathbb{C}$–algebra of all functions holomorphic on some unspecified open neighborhood of K. $\mathscr{C}^k(D)$ will denote the space of the k–times continuously differentiable functions, with $k \in \mathbb{N} \cup \{\infty\}$ and $\mathscr{C}_c(D)$ will denote the algebra of continuous functions with compact support. Denote by $\operatorname{Aut} X$ the group (with respect to the composition) of all holomorphic automorphisms $X \to X$.

Given a compact K in a complex manifold X, we define its $\mathcal{O}(X)$*–convex hull* as

$$\widehat{K}_{\mathcal{O}(X)} := \{p \in X \ : \ |f(p)| \le \|f\|_K \ \ \forall f \in \mathcal{O}(X)\} \, .$$

A compact K is said $\mathcal{O}(X)$*–convex* if $K = \widehat{K}_{\mathcal{O}(X)}$ (*polynomially convex* when $X = \mathbb{C}^n$) and a complex manifold X is said *holomorphically convex* when $\widehat{K}_{\mathcal{O}(X)}$ is relatively compact for every compact $K \subset X$.

Given a metric space (X, d), a subset $A \subset X$ and a positive number δ, we define the *open δ–neighborhood* of A as $A(\delta) := \{x \in X \ : \ d(x, A) < \delta\}$.

Chapter 1

Introduction

The main concern of the present thesis is the investigation of proper holomorphic embeddings of certain open Riemann Surfaces into $\mathbb{C}^2$. To give to the reader the big picture of the work done, we present here a general overview of Complex Analysis, the area in which the subject takes place. We will start from the historical background, making a quick journey throughout the subject, spotting the main milestones from the very beginning to the nowadays state of art in the direction of the proper holomorphic embeddings area. In particular, we will dwell on the Approximation Theory, as some of the concerning results, besides being interesting on their own, turn out to be a fundamental tool for one of the main Theorems presented in the thesis.

1.1 Historical background and some of the main results in Complex Analysis

The first jolts of Complex Analysis –historically known as *Function Theory*– can be traced back to the 18th century with L. Euler (1707–1783) who exploited complex numbers to establish, among other results, the well–known connection between circular functions and exponential function:

$$e^{it} = \cos t + i \sin t, \quad t \in \mathbb{R}.$$

C. F. Gauss (1777–1855) was aware of both the importance and beauty of the subject, as some memorable lines he wrote to Bessel on December 18th, 1811 testify (see [50]). In particular, he knew about the Cauchy integral formula back then. However, neither Euler nor Gauss participate in the development of Complex Analysis in a systematic way, which took place in the 19th century by A.L Cauchy (1789–1857), B. Riemann (1826–1866), and K. Weierstrass (1815–1897). The three of them contributed with quite different methodologies.

Cauchy followed an analytical approach as to him a holomorphic function was a complex–differentiable function with continuous derivative; his discussion is based on the residue theorem and on the integral formula named after him:

Theorem 1.1.1 (Cauchy Integral Formula)**.** Let $D \subset \mathbb{C}$ be a simply connected region and $\gamma \subset D$ a Jordan closed curve. Then any function $f \in \mathcal{O}(D)$ satisfies

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{1.1.1}$$

for any z belonging to the open bounded set of $\mathbb{C}$ identified by such a curve.

Riemann had a more geometric point of view, seeing holomorphic (=conformal) functions as mappings between domains of the complex plane, trying to understand such objects searching for some "intrinsic characteristic" properties rather with formulas (which anyhow raised as an extrinsic representation for his intuitions): he classified the non–empty simply connected domains of the complex plane, in the following, well–celebrated milestone, one of his main results in the field.

Theorem 1.1.2 (Riemann Mapping Theorem)**.** Any non–empty and simply connected region $D \subsetneq \mathbb{C}$ is biholomorphically equivalent to the unit ball $\mathbb{B}$.

The nature of Weierstrass' contribution has been more algebraic as he saw holomorphic functions as those which can be developed into convergent power series. In particular, he systematized the work by Lagrange, building a whole rigorous theoretical apparatus on it.

It soon turned out how much these three apparently different approaches to Complex Analysis are indeed intertwined: in 1831 Cauchy exploited his integral representation to express holomorphic functions as power series, used by Riemann as well. On the other side, in 1841 Weierstrass exploited the integral representation to develop holomorphic function on annular domains $\{r < |z - z_0| < R\} \subset \mathbb{C}$ (where $0 \le r < R \le +\infty$) into the so–called *Laurent series*, that is the double–sided series:

$$f(z) = \sum_{n \ge 0} c_n (z - z_0)^n + \sum_{n > 0} \frac{c_{-n}}{(z - z_0)^n} \,. \tag{1.1.2}$$

Some of the major achievements in Complex Analysis include the

Theorem 1.1.3 (Liouville, 1844)**.** Every bounded entire function is constant.

From such a result the power of the subject starts to appear as it allows to prove the Fundamental Theorem of Algebra in one line: if a polynomial p never vanishes, $1/p$ is entire and bounded, hence constant, so p is. Another fundamental result is the

Theorem 1.1.4 (Identity Principle)**.** Consider $f \in \mathcal{O}(D)$ and $g \in \mathcal{O}(D')$, where $D, D' \subset \mathbb{C}$ are two domains such that $D \cap D'$ is non–empty and connected.
If $f = g$ on some $S \subset D \cap D'$ containing an accumulation point, then f and g agree on the whole $D \cap D'$.

Hence holomorphic functions are very rigid; for example, they are uniquely identified by their behavior on the real line. In particular, this theorem allows to uniquely define a holomorphic function which holomorphically extends f (or equivalently g) to the bigger domain $D \cup D'$. The phenomenon of holomorphic extension becomes very tricky and interesting in higher dimensions; it will be discussed later. Another strong property of holomorphic functions pointing out deep differences between real and complex world is the regularity: holomorphic functions are infinitely many times complex–differentiable, as one can easily see by differentiating the Cauchy Integral Formula, against every kind of real regularity. Moreover, holomorphic functions are analytic, as one can see by expanding $\frac{1}{\zeta - z} = \sum_{n \ge 0} \frac{z^n}{\zeta^{n+1}}$ in the Cauchy Integral Formula and interchanging the series with the integral. This is not true for smooth functions in the real sense: $f(x) = e^{-1/x^2}$, $x \neq 0$ and $f(0) = 0$ is $\mathscr{C}^\infty(\mathbb{R})$ but it is not the sum of its Taylor series around 0 as $f^{(k)}(0) = 0$ for all k. Behavior in a punctured neighborhood of isolated singularities may sound weird for those who come from the real world: given z_0 a point inside a domain D, then $f \in \mathcal{O}(D \setminus \{z_0\})$ admits a Laurent representation near z_0. The second series in (1.1.2) is called the *principal part* as it describes the behavior of f near z_0. In particular we distinguish three cases:

- $c_{-n} = 0$ for all $n > 0$; z_0 is called *removable singularity*, this happens if and only if f is bounded and holomorphic near z_0 as proved by Riemann;
- $c_{-n} \neq 0$ for a finite number of n; z_0 is then a *pole* of order m (being m the biggest of such n). This is equivalent to have $(z - z_0)^m f(z)$ bounded and non zero near z_0;
- $c_{-n} \neq 0$ for infinitely many values of n. In this case z_0 is called *essential singularity*.

In the last case the following Theorem holds:

Theorem 1.1.5 (Big Picard Theorem)**.** Let $f \in \mathcal{O}(D \setminus \{z_0\})$ with z_0 essential singularity. Then in every neighborhood of z_0 the function f takes every complex value, with at most one exception, infinitely many times.

At this point, it is worth mentioning thus

Theorem 1.1.6 (Little Picard Theorem, strengthened)**.** An entire function that is not a polynomial takes every complex value, with at most one exception, infinitely many times.

A special role is played by the coefficient c_{-1}. It is called *residue* of f at z_0, denoted also as $\operatorname{Res}(f, z_0)$ and it is the main character of one of the most important theorems in the field, which allows to easily compute a large number of integrals:

Theorem 1.1.7 (Residue Theorem)**.** Let D be a simply connected region in $\mathbb{C}$, $\gamma \subset D$ be a closed Jordan curve and S a finite set contained into the bounded open region defined by γ. If $f \in \mathcal{O}(D \setminus S)$, then

$$\frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta = \sum_{s \in S} \operatorname{Res}(f, s) \ .$$

1.2 Approximation Theory

Approximation theory plays a very important role in Complex Analysis, as it allows to construct holomorphic maps with certain desired properties. The proof of the main Theorem in [15] raised the necessity of the formulation and proof of an approximation theorem; to introduce it, we collect here some of the main goals achieved in the branch, following the presentation provided in [18]. This area of investigation has its roots in two classical theorems from 1885.

The first one deals with approximating continuous functions with polynomials on compact intervals of $\mathbb{R}$, which is quite remarkable as an arbitrary continuous function could behave very wildly, whether holomorphic functions are extremely regular.

Theorem 1.2.1 (Weierstrass (1885), [57])**.** Let f be a continuous function on the compact interval $[a,b] \subset \mathbb{R}$. Then for every $\epsilon > 0$ there exists a holomorphic polynomial $p \in \mathbb{C}[z]$ such that $\|f - p\|_{[a,b]} < \epsilon$.

It is worth sketching the proof, as the technique is exploited both in [38] and in [15]: we consider the family of entire functions defined as

$$f_t(z) := \frac{1}{t\sqrt{\pi}} \int_{\mathbb{R}} f(x) e^{-(x-z)^2/t^2}\, dx, \quad z \in \mathbb{C},\ t > 0 \tag{1.2.1}$$

after having continuously extended f as a function in $\mathscr{C}_c(\mathbb{R})$. Then $f_t \to f$ uniformly on compacts of $\mathbb{R}$ as $t \to 0$. The Taylor polynomial $T_N(f_t)$ of f_t of order N does the job for t small and N large enough.

The second theorem we were referring to, concerns approximation of holomorphic functions on a neighborhood of a compact in the complex plane by either holomorphic polynomials or by rational functions, depending on whether K is polynomially convex or not:

Theorem 1.2.2 (Runge (1885), [53])**.** Let $K \subset \mathbb{C}$ be compact and $f \in \mathcal{O}(K)$. Then f can be uniformly approximated on K by rational functions without poles in K and by holomorphic polynomials if $\mathbb{C} \setminus K$ is connected.

The more delicate case of approximating functions which are continuous on K and holomorphic only on its interior K° (let us denote the set of such functions $\mathscr{C}(K) \cap \mathcal{O}(K^\circ)$ by $\mathscr{A}(K)$) has been solved by Mergelyan in the 1950s, generalizing both Weierstrass' and Runge's Theorems:

Theorem 1.2.3 (Mergelyan (1951), [39], [40], [41])**.** Let $K \subset \mathbb{C}$ be compact polynomially convex. Then any $f \in \mathscr{A}(K)$ can be uniformly approximated on K by holomorphic polynomials.

Denoting by $\overline{\mathcal{O}(K)}$ the uniform closure of $\{f_{|K} : f \in \mathcal{O}(K)\}$ in $\mathscr{C}(K)$, Mergelyan's Theorem implies the so–called *Mergelyan property*:

$$\mathscr{A}(K) = \overline{\mathcal{O}(K)}\ . \tag{1.2.2}$$

Both Runge's and Mergelyan's Theorem has been generalized a few years later, achieving the respective results on open Riemann surfaces:

Theorem 1.2.4 (Runge's Theorem on Riemann surfaces, [4], [35], [52])**.** If K is a compact set in a Riemann Surface X, then any holomorphic function on a neighborhood of K can be uniformly approximated on K by meromorphic functions on X without poles in K and by holomorphic functions on X if $X \setminus K$ has no relatively compact connected components.

Theorem 1.2.5 (Bishop (1958), [6])**.** If K is a compact set without holes in an open Riemann Surface X, any $f \in \mathscr{A}(K)$ can be uniformly approximated on K by functions in $\mathcal{O}(X)$.

Theorem 1.2.5 is actually a generalization of Theorem 1.2.3, as not having holes is equivalent to polynomial convexity for a compact $K \subset \mathbb{C}$. So far we have only dealt with approximation on compact sets; it is quite natural to investigate the approximation on unbounded sets as well. The first known result in this direction is a generalization of Weiestrass' Theorem 1.2.1:

Theorem 1.2.6 (Carleman (1927), [10])**.** Given $f\colon \mathbb{R} \to \mathbb{C}$ and $\epsilon\colon \mathbb{R} \to (0, +\infty)$ continuous functions, there exists an entire function $g \in \mathcal{O}(\mathbb{C})$ such that

$$|f(x) - g(x)| < \epsilon(x) \qquad \forall x \in \mathbb{R}\ .$$

The proof is obtained inductively applying Mergelyan's Theorem 1.2.3. Many other results have been proved in the field, for which we suggest to consult the surveys [18], [8], [25], [26], [27], [28], [30], [29], [31], [32], [37], [61].

The mentioned approximation theorem in [15] is a Carleman–type Theorem. Given a continuous and positive function $\epsilon\colon \mathbb{C} \to (0, +\infty)$, a family of continuous functions $f_r\colon \mathbb{C} \to \mathbb{C}$, holomorphic on some ball $\triangle_{\rho+a}$, along with n families of unbounded Lipschitz curves $\Gamma_{k,r}$, we inductively construct a family of entire functions $g_r\colon \mathbb{C} \to \mathbb{C}$ approximating f_r with a pointwise ϵ–precision on a smaller ball $\triangle_\rho$ and all along the n families of Lipschitz curves $\Gamma_r = \cup_k \Gamma_{k,r}$. We start approximating a family of continuous functions α_r compactly supported on a single family of curves; the approximating family of functions H_r is the convolution with the gaussian kernel (1.2.1) defined on the curve Γ_r with $n = 1$; being an integral representation, the approximation is preserved plugging in a whole family of functions, each one moving on a single Lipschitz curve; this generalizes a result in [38], holding for a single Lipschitz curve. Then with a Cousin II argument, we modify H_r into a function ξ_r which preserves the approximation already achieved and is arbitrarily small on a Pac–Man bounded region disjoint from $\operatorname{supp} \alpha_r$: this is necessary to simultaneously approximate on several families of curves. With ξ_r we gain control on the behavior of the approximation losing "entireness", which is afterwards recovered exploiting a parametric version of Runge Theorem 1.2.2 we purposely proved as a lemma, getting Q_r. The simultaneous approximation on n families of curves is obtained by constructing a $Q_{k,r}$ for each of them, then setting $Q_r := \sum_k Q_{k,r}$, which is an entire approximation of α_r on compact subsets of the curves and arbitrarily small on $\triangle_\rho$. Such a construction makes us in business to prove our Carleman–type theorem: we first consider $f_{N,r}$ the truncated Taylor polynomial of f_r on $\triangle_\rho$; then we glue f_r and $f_{N,r}$ with suitable cut–off functions obtaining α_r, which we approximate with Q_r. Then $f_{N,r} + Q_r$ is an entire approximation of f_r on $\triangle_\rho \cup (\Gamma_r \cap \overline{\triangle_U})$, for some $U > \rho + a$. Reiterating the procedure, we obtain approximations $Q_r^{(m)}$ on the curves further and further, arbitrarily small on the domain on which we have already approximated, allowing $g_r := f_{N,r} + \sum_{m\geq 1} Q_r^{(m)}$ to converge to an entire approximation of f_r on $\triangle_\rho \cup \Gamma_r$.

Finally, the last paper of this thesis [16], presents four approximation theorems. Nevertheless, being the setting of these results more general than one complex variable, we will describe them further later.

1.3 Several Complex Variables

Switching from one to several complex variables, many substantial differences arise quite soon. While certain results, like Liouville Theorem or the Identity Principle, keep holding, other important results turn out to fail. For instance: Riemann Mapping Theorem does not hold anymore, as the following theorem shows:

Theorem 1.3.1 (Poincaré (1907), [48])**.** There exists no biholomorphic map between the polydisc $P^n(0,1) = \mathbb{B} \times \cdots \times \mathbb{B}$ and the ball $\mathbb{B}^n$ if $n > 1$.

Holomorphic functions in more than one variable have no isolated zeros (see [49], corollary 3.3), or –really striking– when defined on an open neighborhood of an arbitrary domain D with connected boundary, they extend holomorphically to the whole D; for example any function holomorphic on $\{z \in \mathbb{C}^n : 1/2 < |z| < 1\}$, or on a neighborhood of $\{z \in \mathbb{C}^n : |z| = 1\}$ extends to the whole ball $\mathbb{B}^n$; this is due to geometric properties of the domain of convergence of Laurent representation in several variables (see [49], Theorem 1.6) and leads to one of the most important phenomena in several variables: domains allowing simultaneous holomorphic extension across them. The first instance in this direction was provided already in 1897 by A. Hurwitz, who showed in his lecture at the first International Congress of Mathematicians that a holomorphic function of more than one variable, has no isolated singularity; anyhow, the most famous (and historically relevant) example of such a domain was found by F. Hartogs in 1906 (see [34]):

$$H = \{(z,w) \in \mathbb{C}^2 \ : \ |z| < 1, \ \ 1/2 < |w| < 1\} \cup \{|z| < 1/2, \ |w| < 1\} \ .$$

In fact any $f \in \mathcal{O}(H)$ is holomorphically extended on the whole polydisc $P^2(0,1) = \mathbb{B} \times \mathbb{B}$ by

$$F(z,w) = \frac{1}{2\pi i} \int_{|\zeta|=3/4} \frac{f(z,\zeta)}{\zeta - w} \, d\zeta \ ,$$

as F matches f on $\{|z| < 1/2, \ |w| < 3/4\}$ and it is holomorphic on $\{|z| < 1, \ |w| < 3/4\}$. This phenomena does not occur in $\mathbb{C}$, as, given any domain $D \subset \mathbb{C}$, it is enough to consider $f(z) = (z-p)^{-1}$ for some $p \in \partial D$ to obtain a function holomorphic on D which does not extend across it. To talk precisely about this occurrence, we state the following definition.

Definition 1.3.1. A function $f \in \mathcal{O}(D)$ is called *completely singular* at $p \in \partial D$ if for every U connected neighborhood of p, there exists no $h \in \mathcal{O}(U)$ such that $f = h$ on some connected component of $U \cap D$. D is a *domain of holomorphy* when there exists $f \in \mathcal{O}(D)$ completely singular at each boundary point.

The condition to find for each $p \in \partial D$ a $f_p \in \mathcal{O}(D)$ completely singular at p, seems weaker but it is indeed an equivalent condition of being a domain of holomorphy. This is always the case in $\mathbb{C}$, but not always in $\mathbb{C}^n$ with $n > 1$ as the above examples show. To characterize these domains has been a central point in complex analysis research. Few years after Hartogs' discovery, E. E. Levi had a first major insight about domains with *differentiable boundaries*, that is domains D such that for each $p \in \partial D$ there exists U neighborhood of p and a real function $r \in \mathscr{C}^2(U)$ with $dr \neq 0$, such that $D \cap U = \{z \in U \ : \ r(z) < 0\}$. In this setting, he gave a simple differentiable condition, remarkably similar to the differential characterization of euclidean convex domains:

Theorem 1.3.2 (Levi (1910/1911))**.**

(i)$_L$ If there exists a holomorphic function on $U \cap D$ which does not extend across p (in particular D is a domain of holomorphy) then

$$L_p(r;t) := \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0$$

for all $t \in \mathbb{C}^n$ such that $\sum_{j=1}^n \frac{\partial r}{\partial z_j}(p) t_j = 0$.

(ii)$_L$ If $L_p(r;t) > 0$ for all $t \neq 0$ such that $\sum_{j=1}^n \frac{\partial r}{\partial z_j}(p) t_j = 0$, then U can be chosen so that $U \cap D$ is a domain of holomorphy.

The quadratic form $t \mapsto L_p(r;t)$ is the complex hessian of r at p, commonly called *Levi form*. When the differential condition in (i)$_L$ or (ii)$_L$ in the above theorem holds, we say that D is *Levi pseudoconvex* and *strongly pseudoconvex* respectively. In particular, it follows from (ii)$_L$, that a domain that is strongly pseudoconvex near each boundary point, is locally a domain of holomorphy. Proving that a strongly pseudoconvex domain is actually a domain of holomorphy, has been a major problem for 30 years, known as *Levi problem*. It was solved in the early '50 by K. Oka, H. Bremermann, and F. Norguet. So we have a good description of domains of holomorphy when we restrict to domains with differentiable boundaries. To deal with the general case, we need the following definition.

Definition 1.3.2. An arbitrary domain $D \subset \mathbb{C}^n$ is said to be *pseudoconvex* if the mapping $z \mapsto -\log \operatorname{dist}(z, \partial D)$ is plurisubharmonic on D, where a real valued $r \in \mathscr{C}^2(D)$ is *plurisubharmonic* if $L_p(r;t) \geq 0$ for all $t \in \mathbb{C}^n$ and all $p \in D$.

The general version of the solution of the Levi Problem reads as follows:

Theorem 1.3.3 ([43], [44], [45] [9], [46])**.** A domain D in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is pseudoconvex.

We conclude this section by spending a couple of words on $\operatorname{Aut} \mathbb{C}^n$, the group of holomorphic automorphisms of $\mathbb{C}^n$. When $n = 1$ it is well known that $\operatorname{Aut} \mathbb{C}$ is given by the affine functions $z \mapsto az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$. When $n > 1$

the situation is much more complicated. To address the problem of describing $\operatorname{Aut}\mathbb{C}^n$ we give the following

Definition 1.3.3. Let $n > 1$ and write $z = (z', z_n) \in \mathbb{C}^n$. The elements of $\operatorname{Aut}\mathbb{C}^n$ of the form

$$\Psi(z) = (z', e^{f(z')} z_n + g(z')), \quad z \in \mathbb{C}^n$$

and their $\mathrm{GL}_n(\mathbb{C}^n)$–conjugates are called *shears*, where $f, g \in \mathcal{O}(\mathbb{C}^{n-1})$. Denote by $\mathcal{S}(n)$ the group generated by the shears. If $f = 0$, Ψ and their $\mathrm{SL}_n(\mathbb{C}^n)$–conjugates are called *additive shears*, while if $g = 0$, Ψ and their $\mathrm{GL}_n(\mathbb{C}^n)$–conjugates are called *over–shears*.

Even if an explicit description of $\operatorname{Aut}\mathbb{C}^n$ is not known, in 1992 E. Andersén and L. Lempert made a fundamental discovery that started a new theory.

Theorem 1.3.4 (Andersén and Lempert (1992), [3])**.** $\mathcal{S}(n)$ is dense in $\operatorname{Aut}\mathbb{C}^n$ and not equal to it.

1.4 Stein Manifolds

An analytic theory developed on euclidean spaces is naturally generalized to the setting of manifolds. Complex manifolds, in this case. As the concept of domain of holomorphy turned out to be central in the "standard theory", we wish to carry this notion to the manifold setting, together with other reasonable requests, which are embodied in the following definition provided by K. Stein:

Definition 1.4.1 (K. Stein (1951), [56])**.** A complex manifold X is said to be a *Stein Manifold* (or a *holomorphically complete manifold*) if the following hold:

$(\mathrm{i})_{SM}$ For every pair of distinct points $x, y \in X$, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.

$(\mathrm{ii})_{SM}$ For every $p \in X$ there are n holomorphic functions (where $n = \dim X$) $f_1, \ldots, f_n \in \mathcal{O}(X)$ whose differentials df_j are $\mathbb{C}$–linearly independent at p (so global functions provide local charts).

$(\mathrm{iii})_{SM}$ X is holomorphically convex.

If $X = \mathbb{C}^n$, properties $(\mathrm{i})_{SM}$ and $(\mathrm{ii})_{SM}$ are automatically fulfilled, while $(\mathrm{iii})_{SM}$ is actually equivalent to being a domain of holomorphy:

Theorem 1.4.1 (Cartan and Thullen (1932), [11])**.** A Riemann domain X over $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex.

A Riemann domain X is just a complex manifold endowed with a locally biholomorphic mapping $\pi\colon X \to \mathbb{C}^n$: as analytic continuation leads to multivalued functions, the choice of such a π avoids any kind of ambiguity. Let us highlight that this theorem generalizes the notion of domain of holomorphy to an abstract setting in which we cannot refer (a priori) to a surrounding space, so it identifies the property of being a domain of holomorphy as an intrinsic characteristic of a space.

As mentioned in Section 1.2, the last paper of this thesis [16] is about approximation theory dealing with holomorphic mappings with dense images; some results covering this topic can be found in [1] and [19]. In particular, the motivation for [16] was to generalize the main result of [22], in which F. Forstnerič and J. Winkelmann proved that the set of all holomorphic mappings $f\colon \triangle \to Y$ with dense image, is dense in $\mathcal{O}(\triangle, Y)$, where Y is a connected complex manifold. In fact, the first result collected in [16] is about approximating holomorphic embeddings $g\colon \Omega \to Y$ with the same kind of mappings with dense images, where Y is a connected complex manifold and $\Omega \subset \mathbb{C}^n$ is open, bounded star–shaped and pseudoconvex. The method exploited to prove this result relies on the technique of exposing boundary points introduced in [12] and it turned out to be valid to prove three furthermore general approximation theorems, likewise presented in [16]. The second theorem, like the first one, claims that the set of all non–constant holomorphic mappings $g\colon X \to Y$ with bounded image can be approximated by holomorphic mappings with dense image, where X is a complex manifold and Y is a Stein manifold. The remaining two theorems work the other way around, proving that mappings defined on a given compact $K \subset Y$ minus certain points (hence not a priori holomorphic) with dense image in Y, are approximated by holomorphic mappings defined on neighborhoods of K, which is a Stein compact inside a complex manifold Y in the third theorem and it is a compact inside a Stein manifold Y in the fourth one. The methodology exploited to prove these results is as follows: we start from a compact K, which in the last two theorems is given, whether it is obtained as the closure of the image $\overline{g(X)}$ of the mapping $g\colon X \to Y$ we wish to approximate in the first two; consider moreover a countable dense set $Q \subset Y$. K contains *locally exposable points* (a generalization of the notion of *peak point*), which actually belong to the boundary; pick one of them ζ and connect it to a suitable $q \in Q \setminus K$ with a smoothly embedded path $\gamma \subset Y \setminus K \cup \{q\}$. Then Theorem 1.1 in [12] (recalled in the paper in a convenient version) guarantees the existence of a holomorphic mapping f on a neighborhood of K, arbitrarily close to the identity on K minus a small ball B around ζ, and deforming such a ball B into a tube arbitrarily close to γ and touching q (namely $f(\zeta) = q$). We repeat the procedure on the compact $f(K)$ and iterate it; in the limit, we get a mapping H with dense image, as we constructed it touching new points of Q at each step and moving arbitrarily little what we have already hit, that guarantees both convergence and approximation. In the first two theorems, $H \circ g\colon X \to Y$ is the approximating holomorphic mapping with dense image. In the remaining two, H is a mapping with dense image which is approximated by partial compositions of the holomorphic mappings obtained at each step.

1.5 Proper Holomorphic Embeddings

From what we have seen so far, Stein manifolds are the natural source of holomorphic functions in a reasonably general setting. We are therefore interested in studying these objects. To represent them more concretely, we would like to identify them as subsets of some Euclidean space, in a faithful way (one–to–one and avoiding creating singularities), respecting the boundaries (which is crucial as the definition of pseudoconvexity actually relies on the boundaries) and the complex structure of a manifold. Hence, given a Stein manifold X we are looking for *proper holomorphic embeddings*

$$f\colon X \hookrightarrow \mathbb{C}^N$$

for some N. Proper mappings between topological spaces are the ones preserving compactness under inverse image. An equivalent characterization (that takes into account the boundary if defined) is that such mappings, map sequences leaving every compact into sequences leaving every compact, thus if a Stein manifold is a bounded domain inside some bigger space (e.g., any domain of holomorphy $X \subset \mathbb{C}^n$ relatively compact), every sequence approaching the boundary of X is sent to an unbounded sequence. The first question is then whether such a mapping always exists. An affirmative answer has been provided by R. Remmert in 1956:

Theorem 1.5.1 (Remmert (1956), [51])**.** Given any Stein manifold X, there exists a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^N$ for N sufficiently large.

The attention is thus towards N in function of $n = \dim X$. In particular, interest is for spotting a minimal N: the smaller N, the tighter the euclidian space, the more faithful the representation. A first result in this direction has been given in 1960/1961 independently by E. Bishop and R. Narasimhan, as they proved that N can be taken to be $2n+1$ (see [7], [42]). In 1970, O. Forster conjectured that, if $n > 1$, N could be taken to be $\lfloor \frac{3n}{2} \rfloor + 1$, proving moreover that this is minimal (see [20]): given an integer $n \geq 2$, he defined the n–dimensional Stein mainfold

$$X := \begin{cases} Y^m & n = 2m \\ Y^m \times \mathbb{C} & n = 2m+1\,, \end{cases}$$

where

$$Y := \{[x : y : z] \in \mathbb{P}^2 \;:\; x^2 + y^2 + z^2 \neq 0\}$$

is a Stein surface. Eliashberg and Gromov first, Schürmann later, turned this conjecture into the following

Theorem 1.5.2 (Eliashberg, Gromov (1992), [17] and Schürmann (1997), [54])**.** Every Stein manifold with $n = \dim X > 1$ admits a proper holomorphic embedding into $\mathbb{C}^N$, for $N = \lfloor \frac{3n}{2} \rfloor + 1$.

The proof breaks down in the case $n = 1$. Since Behnke and Stein proved that one–dimensional Stein manifolds are precisely open connected Riemann

Surfaces (see [4] and [5]), we are led to the following long–standing open problem:

Conjecture 1.5.1. Every open connected Riemann Surface admits a proper holomorphic embedding into $\mathbb{C}^2$.

The first example validating this conjecture dates back to 1970 when Kasahara and Nishino proved it in the case of the unit disk $\mathbb{B}$ ([55]). The annulus $\{1 < |z| < r\}$ and the punctured disk $\mathbb{B} \setminus \{0\}$ have been proved to verify the conjecture by Laufer (1973, see [36]) and Alexander (1977, [2]) respectively. The first quite general result has been given by J. Globevnik and B. Stensønes in 1995: they proved in [33] that any finitely connected domain in $\mathbb{C}$ without isolated boundary points verifies the conjecture. A more complete account concerning the topic of proper holomorphic embeddings can be found in [13]. To present more general results, we need to gather some definitions.

Definition 1.5.1.

- A *bordered Riemann Surface* is the interior D of a 1–dimensional compact complex manifold $\overline{D}$, not necessarily connected, with smooth boundary ∂D consisting of finitely many closed Jordan curves.
- Given a closed complex curve $\Sigma \subset \mathbb{C}^2$ (that is the closure of the holomorphic image of some open subset of $\mathbb{C}$, namely $\Sigma = \overline{F(D)}$, where $F\colon D \to \mathbb{C}^2$ is holomorphic), possibly with boundary, and a linear projection $\pi\colon \mathbb{C}^2 \to \mathbb{C}$, a point $p \in \Sigma$ is an *exposed point* (with respect to π) if the affine complex plane thru p defined by π, that is $\Lambda_p := \pi^{-1}\pi(p)$, meets Σ exactly in p ($\Lambda_p \cap \Sigma = \{p\}$) and the intersection is transverse: $T_p\Lambda_p \cap T_p\Sigma = \{0\}$.
- A family $\Gamma = \{\gamma_j(t) \ : \ j = 1, \ldots, m, \ t \in (-\infty, +\infty)\}$ of pairwise disjoint, smooth curves in $\mathbb{C}^2$ wihout self intersections has the *immediate projection property* with respect to some linear projection $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ if:
  - $|\pi(\gamma_j(t))| \to +\infty$ as $|t| \to +\infty$ for every j,
  - there exists $M > 0$ such that $\mathbb{C} \setminus (\pi(\Gamma) \cup R\overline{\mathbb{B}})$ has no relatively components for every $R \geq M$.
- A domain $\Omega \subset \mathbb{C}^n$, $n > 1$, is a *Fatou–Bieberbach domain* if it is biholomorphic to $\mathbb{C}^n$. A biholomorphism $\Phi\colon \Omega \to \mathbb{C}^n$ is called a *Fatou–Bieberbach map*.

The problem of finding proper holomorphic embeddings of such D into $\mathbb{C}^2$ can be split into the following two problems:

A) find an embedding $f\colon \overline{D} \hookrightarrow \mathbb{C}^2$ onto a smooth compact complex curve with boundary $\overline{\Sigma} := f(\overline{D}) \subset \mathbb{C}^2$, holomorphic on D.

B) obtaining properness by pushing the boundary $\partial\Sigma = f(\partial D)$ to infinity without introducing any interior double points.

E. F. Wold developed a remarkably general method in [58], [59], that allowed him to generalize Globevnik and Stensønes' result to all finitely connected and some infinitely connected planar domains [58], to every domain in the Riemann sphere $\mathbb{P}^1$ with at most countably many boundary components, none of which are points [24] (with Forstnerič), and to every subset of the torus $\mathbb{T}$ with finitely many boundary components, none of which being a point [60]. But most importantly, the works [58] and [59] have been the starting point leading Wold and Forstnerič to formulate and prove the following:

Theorem 1.5.3 (Theorem 4.2, [23])**.** A $\mathscr{C}^1$–smooth embedding $f\colon \overline{D} \hookrightarrow \mathbb{C}^2$, holomorphic on D, can be approximated in the $\mathscr{C}^1$–topology on $\overline{D} \setminus \cup_j U_j$, where U_j is a small neighborhood of a point a_j in each connected component of the boundary ∂D, by embeddings $F\colon \overline{D} \hookrightarrow \mathbb{C}^2$, holomorphic on D, such that every boundary component of $F(\overline{D})$ has an exposed point.

This result is needed to give a complete solution to problem B) (Corollary 1.2 in [23]). Another fundamental tool is the following

Lemma 1.5.1 (Lemma 1, [58])**.** Given $K \subset \mathbb{C}^2$ compact polynomially convex and $\Gamma \subset \mathbb{C}^2 \setminus K$ a set of m real pairwise disjoint smooth curves without self–intersection enjoying the immediate projection property, we can build automorphisms $\phi \in \operatorname{Aut}\mathbb{C}^2$ arbitrarily close to the identity on K and pushing the curves Γ outside a ball of arbitrarily big radius.

This Lemma turns out to be a key step of the main result in [59], in which a description of the construction of proper holomorphic embeddings into $\mathbb{C}^2$ of bordered Riemann Surfaces D whose boundary ∂D enjoys the nice projection property is provided. We report it here as resumed in [21]:

Theorem 1.5.4 (Theorem 4.14.6, [21])**.** Let $X \subset \mathbb{C}^n$ be a complex curve ($n > 1$), possibly with singularities, whose boundary $\Gamma = \partial X$ is as in definition 1.5.1. Then there exists a Fatou–Bieberbach domain $\Omega \subset \mathbb{C}^n$ such that $X \subset \Omega$ and $\partial X \subset \partial\Omega$. Moreover, a Fatou–Bieberbach map $\Phi\colon \Omega \to \mathbb{C}^n$ can be chosen to be arbitrarily close to the identity on a fixed compact inside X. In particular $\Phi_{|X}\colon X \hookrightarrow \mathbb{C}^n$ is a proper holomorphic embedding.

The last–mentioned results are the state of art regarding Conjecture 1.5.1, which remains unsolved as not much is known about A). It, therefore, makes sense to search for a counterexample: there have been discussions suggesting that 1.5.1 could have failed for complements of large Cantor sets inside the Riemann sphere. S. Y. Orevkov, wrote the paper [47] to prove the existence of a proper holomorphic embedding $\mathbb{P}^1 \setminus C \hookrightarrow \mathbb{C}^2$, without any discussion on the size of the Cantor set C. Therefore the second paper included in this thesis [14] is focused on the study of the measure of C. It turns out natural to take it very small: it is there proved that the construction of C can be performed so that C has zero Hausdorff dimension. Such a paper is also justified by the extreme synthesis and lack of details in [47], which I have provided. The construction is as follows: a sequence of continuous mappings $\gamma_n\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}^2}$ is built, such that $\gamma_n\colon \overline{\mathbb{C}} \setminus \gamma_n^{-1}(\infty_2) \to \mathbb{C}^2$ is a holomorphic embedding and $A_n = \gamma_n(\overline{\mathbb{C}})$ is compact complex curve with an increasing (with n) number of unbounded components in both coordinate directions. Call Δ_n the preimage through γ_n of the curves A_n in smaller and smaller (as n increases) neighborhoods of ∞_2, whose size depends on a divergent sequence of parameters $\{R_n\}_n$ on which the whole construction relies on; Δ_n is the disjoint union of open discs in $\mathbb{C}$, which are as many as the branches of A_n touching ∞_2. It turns out that Δ_n is a decreasing sequence and every disc tends to shrink to a point, none of which remains isolated. Therefore $C = \cap_n \Delta_n$ is a Cantor set and the limit mapping $\gamma = \lim_n \gamma_n \colon \overline{\mathbb{C}} \setminus C \to \mathbb{C}^2$ is a holomorphic embedding which turns to be proper. The diameter of the discs can be taken arbitrarily little provided $R_n \to +\infty$ fast enough, thus C can be realized to have zero Hausdorff dimension.

Since this last construction provides a proper holomorphic embedding of the complement of a Cantor set which is actually thin, here is where the first paper of this thesis [13] comes into play. We presented in it a construction of the same topological object using the techniques developed by Wold, which allow a construction leading to properly holomorphically embed a considerably large Cantor set, as follows: let Q_1 be a square in the Riemann Sphere $\mathbb{P}^1$ of side 2. At each step we remove smaller and smaller strips, alternatively vertical and horizontal, getting Q_n; set then $C_n := \overline{\mathbb{P}^1 \setminus Q_n}$. We inductively construct the holomorphic embeddings $f_n \colon C_n \hookrightarrow \mathbb{C}^2$ as a composition $\phi_n \circ g_n \circ F_n$, where $F_n \colon C_n \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding having exposed points on each of the 2^{n-1} boundary components of the image, g_n is a rational shear sending these points to infinity in $\mathbb{C}^2$, so that $X_n := g_n \circ F_n(C_n) \subset \mathbb{C}^2$ is an unbounded complex curve enjoying the immediate projection property 1.5.1. Then setting $r_n := n-2$ we have that $L_n := r_n\overline{\mathbb{B}^2} \cup K_n$ is polynomially convex for a sufficiently big polynomially convex $K_n \subset X_n$ (see the proof of Theorem 4.14.6 in [21]), so we can apply Lemma 1.5.1 and get $\phi_n \in \operatorname{Aut} \mathbb{C}^2$

1. suitably close to the identity on L_n, and

2. pushing the boundary away from a bigger ball, say $\phi_n(\partial X_n) \subset \mathbb{C}^2 \setminus r_{n+1}\overline{\mathbb{B}^2}$.

Extend f_n to a smooth embedding $\widetilde{f}_n \colon C_n \cup l_n \hookrightarrow \mathbb{C}^2$, where l_n is the set of vertical or horizontal lines dividing the rectangular components of Q_n into two equal pieces and use Mergelyan Theorem 1.2.3 to approximate $\widetilde{f}_n$ on $C_n \cup l_n$ with a holomorphic embedding $\widehat{f}_{n+1} \colon C_{n+1} \hookrightarrow \mathbb{C}^2$. Theorem 1.5.3 allows now to approximate $\widehat{f}_{n+1}$ with a holomorphic embedding $F_{n+1} \colon C_{n+1} \hookrightarrow \mathbb{C}^2$ with an exposed point on every boundary component of the image and the induction may proceed, creating a sequence of holomorphic embeddings $\{f_n\}_n$ convergent by 1 to a holomorphic embedding f on $\left(\bigcup_{n\geq 1} C_n\right)^\circ$, which equals $\mathbb{P}^1 \setminus C$, where C is a Cantor set of Lebesgue measure arbitrarily close to 4 (by choosing the strips to remove to be suitably small at each step). Properness of the limit function $f \colon \mathbb{P}^1 \setminus C \hookrightarrow \mathbb{C}^2$ follows from 2.

The last result to mention in this thesis, is the main Theorem of the third paper [15]. The concern deals with simultaneously properly holomorphically embedding a whole family of complex curves into $\mathbb{C}^2$. Namely: given a continuous family of complex curves $\{\Omega_r\}_r$ inside the Riemann sphere, the goal is to find a continuous mapping $\Xi\colon \bigcup_r \{r\} \times \Omega_r \to \mathbb{C}^2$ which properly holomorphically embeds each of them, that is $\Xi(r,\cdot)\colon \Omega_r \hookrightarrow \mathbb{C}^2$ is a proper holomorphic embedding for every r. The proof structure is similar to the one of Theorem 1.5.4; the main difference lies in the use of Lemma 1.5.1, in fact, we now need a parametric version of it. This issue is solved with the Andersén–Lempert procedure, for which the parametric version of the Carleman Theorem previously presented is fundamental.

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1.6 Summary of Papers

Paper I presents a construction of a proper holomorphic embedding $f\colon \mathbb{P}^1 \setminus C \hookrightarrow \mathbb{C}^2$, where C is a Cantor set obtained by removing smaller and smaller vertical and horizontal strips from a square of side 2, allowing to realize it to have Lebesgue measure arbitrarily close to 4.

Paper II clarifies the details of a cryptical paper by Orevkov in which a construction of a proper holomorphic embedding $\gamma\colon \mathbb{P}^1 \setminus C \hookrightarrow \mathbb{C}^2$ is performed with techniques completely different from the ones exploited in Paper I; in particular, it is proved that such a construction can be done to get the Cantor set C to have zero Hausdorff dimension.

Paper III solves the problem of simultaneously embedding properly holomorphically into $\mathbb{C}^2$ a whole family of n–connected domains $\Omega_r \subset \mathbb{P}^1$ such that none of the components of $\mathbb{P}^1 \setminus \Omega_r$ reduces to a point, by constructing a continuous mapping $\Xi\colon \bigcup_r \{r\} \times \Omega_r \to \mathbb{C}^2$ such that $\Xi(r,\cdot)\colon \Omega_r \hookrightarrow \mathbb{C}^2$ is a proper holomorphic embedding for every r. To this aim, a parametric version of both the Andersén–Lempert procedure and Carleman's Theorem is formulated and proved.

Paper IV displays four approximation theorems for manifold–valued mappings. The first one approximates holomorphic embeddings on pseudoconvex domains in $\mathbb{C}^n$ with holomorphic embeddings with dense images. The second theorem approximates holomorphic mappings on complex manifolds with bounded images with holomorphic mappings with dense images. The last two theorems work the other way around, constructing (in different settings) sequences of holomorphic mappings (embeddings in the first one) converging to a mapping with dense image defined on a given compact minus certain points (thus in general not holomorphic).

Papers

Paper I

Proper Holomorphic Embeddings of complements of large Cantor sets in $\mathbb{C}^2$

G. D. Di Salvo and E. F. Wold

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Abstract

We present a construction of a proper holomorphic embedding $f\colon \mathbb{P}^1 \setminus C \hookrightarrow \mathbb{C}^2$, where C is a Cantor set obtained by removing smaller and smaller vertical and horizontal strips from a square of side 2, allowing to realize it to have Lebesgue measure arbitrarily close to 4.

Contents

I.1 Introduction

I.1.1 The main result

A major unresolved issue, known as Forster's Conjecture, is whether or not every open Riemann surface X admits a holomorphic embedding into $\mathbb{C}^2$, and, if it does, whether it admits a *proper* holomorphic embedding. For instance, if Y is a compact Riemann surface, and if $X = Y \setminus C$ where C is a closed set whose connected components are all points, it is unknown whether X embeds (properly or not) into $\mathbb{C}^2$. We may consider two extremal cases: (i) the case where C is a finite set, and (ii) the case where C is a Cantor set, and we may further consider the simplest compact Riemann surface in this context, namely $Y = \mathbb{P}^1$. Then in case (i), it is clear that X admits a proper holomorphic embedding into $\mathbb{C}^2$, so we will consider the case (ii).

Let Q denote the square $Q = [-1, 1] \times [-1, 1] \subset \mathbb{C}$, and let μ denote the 2–dimensional Lebesgue measure on $\mathbb{C}$. A procedure for constructing a (large) Cantor set $C \subset Q$ is as follows (see Section I.3 for a more detailed description). Let l_1 denote the vertical line dividing Q into two equal pieces, choose $\delta_1 > 0$ small, and remove an open δ_1–neighbourhood of l_1 to obtain a union Q_2 of two disjoint rectangles. Next, let $l_2^j, j = 1, 2$, be horizontal lines dividing each rectangle in Q_2 into equal pieces, choose δ_2 small, and remove an open δ_2–neighbourhood of $l_2^1 \cup l_2^2$ to obtain a disjoint union Q_3 of four rectangles. Next, switch back to vertical lines, chose δ_3 small to obtain Q_4 and so forth, to obtain a sequence $\delta_j \to 0$ and nested sequence Q_j of rectangles such that $C = \cap_j Q_j$ is a Cantor set contained in Q. Our main result is the following.

Theorem I.1.1. There are sequences $\{\delta_j\}_j$ converging to zero arbitrarily fast such that the complement $\mathbb{P}^1 \setminus C$ of the resulting Cantor set admits a proper holomorphic embedding into $\mathbb{C}^2$. In particular, for any $\epsilon > 0$ we may achieve that $\mu(C) > 4 - \epsilon$.

The motivation for proving this result is that there have been speculations that considering complements of “fat” Cantor sets could lead to counterexamples to Forster’s Conjecture.

We note that Orevkov [18] showed the existence of a Cantor set $C \subset \mathbb{P}^1$ such that $\mathbb{P}^1 \setminus C$ admits a proper holomorphic embedding into $\mathbb{C}^2$. His construction is quite cryptical and it is explained in detail in [4], where it is also proved that C can be obtained to have zero Hausdorff dimension. From such a construction it seems difficult, or perhaps impossible, to achieve that C is large.

I.1.2 History

Dealing with Stein manifolds, one of the most important goals to achieve is to embed them properly holomorphically into $\mathbb{C}^N$ for some N. A first important result comes from Remmert [19], who proved in 1956 that every n–dimensional Stein manifold admits a proper holomorphic embedding into $\mathbb{C}^N$ for N big enough. Such a result was made precise by Bishop and Narasimhan who independently proved in 1960–61 that N can be taken to be $2n + 1$ (see [3] and [17]). In 1970 Forster [7] improved Bishop–Narasimhan’s result, decreasing N to $\lfloor \frac{5n}{3} \rfloor + 2$ and proving that it is not possible for N to go below $\lfloor \frac{3n}{2} \rfloor + 1$ and conjecturing that the euclidean dimension could have been improved exactly to $\lfloor \frac{3n}{2} \rfloor + 1$. Eliashberg, Gromov (1992) and Shürmann (1997) proved the following

Theorem I.1.2. [Eliashberg–Gromov [5] (1992) and Shürmann [21] (1997)] Every n–dimensional Stein manifold X, with $n \geq 2$ embeds properly holomorphically into $\mathbb{C}^N$ with $N = \lfloor \frac{3n}{2} \rfloor + 1$.

The proof of the theorem breaks down when $n = 1$; since 1–dimensional Stein manifolds are precisely open connected Riemann surfaces, Forster’s conjecture reduces to the following

Conjecture I.1.1. Every open connected Riemann surface embeds properly holomorphically into $\mathbb{C}^2$.

So far, only a few open Riemann surfaces are known to admit a proper holomorphic embedding into $\mathbb{C}^2$. The first known examples are the open unit disk in $\mathbb{C}$ (Kasahara–Nishino, 1970, [22]), open annuli in $\mathbb{C}$ (Laufer, 1973, [14]) and the punctured disk in $\mathbb{C}$ (Alexander, 1977, [1]). Later (1995) Stensønes and Globevnik proved in [12] that every finitely connected planar domain without isolated boundary points verifies the conjecture. In 2009, Wold and Forstnerič proved the best result known so far: if $\overline{D}$ is a Riemann surface with smooth enough boundary which admits a smooth embedding into $\mathbb{C}^2$, holomorphic on the interior D, then D admits a proper holomorphic embedding into $\mathbb{C}^2$ (see [10] and next section). Other remarkable results include proper holomorphic embeddings of certain Riemann surfaces into $\mathbb{C}^2$ with interpolation (see [13]), deformation of continuous mappings $f\colon S \to X$ between Stein manifolds into proper holomorphic embeddings under certain hypothesis on the dimension of the spaces (see [2]), embeddings of infinitely connected planar domains into $\mathbb{C}^2$ (see [11]), the existence of a homotopy of continuous mappings $f\colon D \to \mathbb{C} \times \mathbb{C}^*$ into proper holomorphic embedding whenever D is a finitely connected planar domain without punctures (see [20]), existence of proper holomorphic embeddings of the unit disc $\mathbb{B}$ into connected pseudoconvex Runge domains $\Omega \subset \mathbb{C}^n$ (when $n \geq 2$) whose image contains arbitrarily fixed discrete subsets of Ω (see [9]), approximation of proper embeddings on smooth curves contained in a finitely connected planar domain D into $\mathbb{C}^n$ (with $n \geq 2$) by proper holomorphic embeddings $f\colon D \hookrightarrow \mathbb{C}^n$ (see [15]), and the existence of proper holomorphic embeddings into $\mathbb{C}^2$ of certain infinitely connected domains Ω lying inside a bordered Riemann surface $\overline{D}$ knowing to admit a proper holomorphic embedding into $\mathbb{C}^2$ [16].

I.2 Preliminaries

I.2.1 Notation

We will use the following notation.

- Given $K \subset \mathbb{C}$ and a positive real number δ, we define the open subset

$$K(\delta) := \{z \in \mathbb{C} \ : \ \operatorname{dist}(z, K) < \delta\}.$$

- For a closed subset $K \subset \mathbb{P}^1$ we denote by $\mathcal{O}(K)$ the algebra of continuous functions $f \in \mathcal{C}(K)$ such that there exists an open set $U \subset \mathbb{P}^1$ containing K, and $F \in \mathcal{O}(U)$ with $F|_K = f$.

- We let $\pi_j \colon \mathbb{C}^2 \to \mathbb{C}$ denote the projection onto the j–th coordinate line, and given a point $p \in \mathbb{C}^2$ we denote the vertical complex line through p by

$$\Lambda_p := \pi_1^{-1}(\pi_1(p)) = \{(\pi_1(p), \zeta) \ : \ \zeta \in \mathbb{C}\} .$$

- If X is a domain with piecewise smooth boundary in a Riemann surface Y, $f\colon \overline{X} \to \mathbb{C}^2$ is a holomorphic map, and if $a \in \partial X$ is a smooth boundary point, we say that $f(a)$ is π_1–exposed for $f(\overline{X})$ if $f(\overline{X}) \cap \Lambda_{f(a)} = \{f(a)\}$, and $\pi_1 \circ f$ is an embedding sufficiently close to a. Similarly, for a smooth map $\gamma\colon [0,1] \to \mathbb{C}^2$, we say that $\gamma([0,1])$ is exposed at $\gamma(1)$ if $\gamma([0,1]) \cap \Lambda_{\gamma(1)} = \{\gamma(1)\}$, and $\pi_1 \circ \gamma$ is an embedding sufficiently close to 1.

I.2.2 Results

In this section, we collect the technical tools needed to prove Theorem I.1.1.

The following result is essentially Theorem 4.2 in [10]. Although (1) and (2) were not stated explicitly in [10] they are evident from the proof therein and were added to the corresponding Theorem 2.8 in [11].

Theorem I.2.1. Let X be a smoothly bounded domain in a Riemann surface Y, $f\colon \overline{X} \hookrightarrow \mathbb{C}^2$ a holomorphic embedding, and $a_1, \dots, a_m \in \partial X$. Let $\gamma_j\colon [0,1] \to \mathbb{C}^2$ ($j = 1, \dots, m$) be smooth embedded arcs with pairwise disjoint images satisfying the following properties:

- $\gamma_j([0,1]) \cap f(\overline{X}) = \gamma_j(0) = f(a_j)$ for $j = 1, \dots, m$, and
- the image $E := f(\overline{X}) \cup \bigcup_{j=1}^m \gamma_j([0,1])$ is π_1–exposed at $\gamma_j(1)$ for $j = 1, \dots, m$.

Then given an open set $V \subset \mathbb{C}^2$ containing $\bigcup_{j=1}^m \gamma_j([0,1])$, an open set $U \subset Y$ containing the points a_j that satisfies $f(\overline{U \cap X}) \subset V$, and any $\epsilon > 0$, there exists a holomorphic embedding $F\colon \overline{X} \hookrightarrow \mathbb{C}^2$ with the following properties:

(1) $\|F - f\|_{\overline{X} \setminus U} < \epsilon$,

(2) $F(\overline{U \cap X}) \subset V$, and

(3) $F(a_j) = \gamma_j(1)$, and $F(\overline{X})$ is π_1–exposed at $F(a_j)$ for $j = 1, \dots, m$.

The following is essentially Lemma 1 in [23]. The difference is that Lemma 1 was stated for π_1 instead of π_2, and for curves $\lambda\colon [0, +\infty) \to \mathbb{C}^2$ instead of $\lambda\colon (-\infty, +\infty) \to \mathbb{C}^2$ – neither make a difference for the proof.

Lemma I.2.1. Let $K \subset \mathbb{C}^2$ be a polynomially convex compact set, and let

$$\Lambda = \{\lambda_j(t) : j = 1, \dots, m, \;\; t \in (-\infty, +\infty)\}$$

be a collection of pairwise disjoint smooth curves in $\mathbb{C}^2 \setminus K$ without self intersection, enjoying the *immediate projection property* (with respect to π_2):

- $\lim_{|t| \to \infty} |\pi_2(\lambda_j(t))| = \infty$ for all j, and
- there exists an $M > 0$ such that $\mathbb{C} \setminus (R\overline{\mathbb{B}} \cup \pi_2(\Lambda))$ does not contain any relatively compact components for $R \geq M$.

Then for any $r > 0$ and $\epsilon > 0$ there exists $\phi \in \operatorname{Aut}\mathbb{C}^2$ such that the following are satisfied:

(i) $\|\phi - \operatorname{Id}\|_K < \epsilon$, and

(ii) $\phi(\Lambda) \subset \mathbb{C}^2 \setminus r\overline{\mathbb{B}^2}$.

I.3 The Induction Step

We will now describe an inductive procedure to construct a nested sequence of closed rectangles $Q_n \subset Q$, along with holomorphic embeddings $f_n \colon \overline{\mathbb{P}^1 \setminus Q_n} \to \mathbb{C}^2$ that will be used to construct a proper holomorphic embedding

$$f \colon \mathbb{P}^1 \setminus \bigcap_n Q_n \hookrightarrow \mathbb{C}^2,$$

where $C = \bigcap_n Q_n$ will be a Cantor set, where the construction will enable us to ensure that its Lebesgue measure $\mu(C)$ is arbitrarily close to 4.

Set $Q_1 := Q$ and set $C_1 := \overline{\mathbb{P}^1 \setminus Q_1}$. To construct Q_2 from Q_1 we let l_1 be the vertical line segment dividing Q_1 into two equal pieces. Then, for $0 < \delta_2 << 1$, we set

$$Q_2 := Q_1 \setminus l_1(\delta_2),$$

and we set $C_2 := \overline{\mathbb{P}^1 \setminus Q_2}$. Then C_2 is the complement of the disjoint union of 2 open rectangles $(Q_2^j)^\circ,\ j = 1, 2$, contained in Q_1.

Assume now that we have constructed a nested sequence $\{Q_j\}_{j=1}^n$, $n \geq 2$, where

$$Q_n = \bigsqcup_{j=1}^{2^{n-1}} Q_n^j$$

is the disjoint union of 2^{n-1} closed rectangles contained in Q_{n-1}, along with an increasing sequence of closed subsets $C_n := \overline{\mathbb{P}^1 \setminus Q_n}$ in $\mathbb{P}^1$. We let l_n^j be the line segment – vertical for n odd, horizontal for n even – dividing Q_n^j into two equal pieces, we set $l_n := \bigsqcup_{j=1}^{2^{n-1}} l_n^j$, and for $\delta_{n+1} > 0$ small enough we define

$$Q_{n+1} := Q_n \setminus l_n(\delta_{n+1}) =: \bigsqcup_{j=1}^{2^n} Q_{n+1}^j,$$

and

$$C_{n+1} := \overline{\mathbb{P}^1 \setminus Q_{n+1}}\ .$$

Proposition I.3.1. With the procedure above assume that we have constructed Q_n and C_n for $n \geq 1$. Let $K_n \subset C_n^\circ$ be a compact set, let $r_n > 0$, and assume that $f_n \colon C_n \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding such that

$$f_n(\overline{C_n \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}^2}\ . \tag{I.3.1}$$

Then for any $\varepsilon_n > 0$ and any $r_{n+1} > r_n$, there exist $\delta_{n+1} > 0$ arbitrarily close to zero and a holomorphic embedding $f_{n+1} \colon C_{n+1} \hookrightarrow \mathbb{C}^2$ such that

(a) $\|f_{n+1} - f_n\|_{K_n} < \varepsilon_n$,

(b) $f_{n+1}(\overline{C_{n+1} \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}^2}$,

(c) $f_{n+1}(\partial C_{n+1}) \subset \mathbb{C}^2 \setminus r_{n+1}\overline{\mathbb{B}^2}$.

Proof. We extend f_n to a smooth embedding $\widetilde{f}_n\colon C_n \cup l_n \hookrightarrow \mathbb{C}^2$ with $\widetilde{f}_n(l_n)$ lying close enough to $f_n(\partial C_n)$ so that by (I.3.1) we get

$$\widetilde{f}_n(l_n) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}^2} \,. \tag{I.3.2}$$

Now Mergelyan's theorem (see e.g., [6]) ensures the existence of a holomorphic embedding $\widehat{f}_{n+1}\colon C_n \cup l_n \hookrightarrow \mathbb{C}^2$ such that

$$\|\widehat{f}_{n+1} - \widetilde{f}_n\|_{C_n \cup l_n} < \frac{\epsilon_n}{4},$$

and

$$\widehat{f}_{n+1}((\overline{C_n \cup l_n) \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}^2} \,. \tag{I.3.3}$$

Then by choosing a preliminary $\tilde{\delta}_{n+1} > 0$ sufficiently small (to be shrunk further later), and letting the set corresponding to C_{n+1} be denoted by $\tilde{C}_{n+1}$ (and similarly for Q_{n+1}), we have that $\widehat{f}_{n+1} \in \mathcal{O}(\tilde{C}_{n+1})$, and

$$\widehat{f}_{n+1}(\overline{\tilde{C}_{n+1} \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}^2} \,. \tag{I.3.4}$$

Next, recall that $\tilde{Q}_{n+1}$ is constructed from Q_n by splitting each Q_n^j into two smaller rectangles $\tilde{Q}_n^{j,1}$ and $\tilde{Q}_n^{j,2}$, by removing the strip $l_n^j(\tilde{\delta}_{n+1})$. Choose smooth boundary points

$$\tilde{a}_{ji} \in \partial\tilde{Q}_n^{j,i} \cap \overline{l_n^j(\tilde{\delta}_{n+1})}, \tag{I.3.5}$$

for $i = 1, 2$, and $j = 1, \ldots, 2^{n-1}$, and relabel these to get 2^n boundary points a_j, one in each $\partial\tilde{Q}^j_{n+1}$.

Now choose 2^n pairwise disjoint smoothly embedded arcs $\gamma_j\colon [0,1] \hookrightarrow \mathbb{C}^2$ disjoint from $r_n\overline{\mathbb{B}^2}$, such that

$$\gamma_j([0,1]) \cap \widehat{f}_{n+1}(\tilde{C}_{n+1}) = \widehat{f}_{n+1}(a_j) = \gamma_j(0),$$

and such that each point $\gamma_j(1)$ is π_1–exposed for the surface

$$\widehat{f}_{n+1}(\tilde{C}_{n+1}) \cup \bigcup_{j=1}^{2^n} \gamma_j([0,1]).$$

Choose an open set $V \subset \mathbb{C}^2$ containing the arcs $\gamma_j([0,1])$ with $\overline{V} \cap r_n\overline{\mathbb{B}^2} = \emptyset$ and take $U \subset \mathbb{P}^1$ to be the union of sufficiently small open balls centered at the points a_j, so that $U \cap K_n = \emptyset$ and $\widehat{f}_{n+1}(\overline{U \cap \tilde{C}_{n+1}}) \subset V$. Then Theorem I.2.1 furnishes a holomorphic embedding $F_{n+1}\colon \tilde{C}_{n+1} \hookrightarrow \mathbb{C}^2$ such that $p_j := F_{n+1}(a_j) = \gamma_j(1)$ is an exposed point for $F_{n+1}(\tilde{C}_{n+1})$ for each j, and

$$\|F_{n+1} - \widehat{f}_{n+1}\|_{K_n} < \frac{\varepsilon_n}{4}, \tag{I.3.6}$$

and also

$$F_{n+1}(\overline{\tilde{C}_{n+1} \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}^2}\ . \tag{I.3.7}$$

Now choose $\alpha_j \in \mathbb{C}$, $j = 1, \dots, 2^n$, such that setting

$$g_{n+1}(z, w) := \left(z, w + \sum_{j=1}^{2^n} \frac{\alpha_j}{\pi_1(p_j) - z} \right)\ ,$$

we have that

$$\|g_{n+1} \circ F_{n+1} - F_{n+1}\|_{K_n} < \frac{\epsilon_n}{4},$$

and such that the conditions in Lemma I.2.1 are satisfied for the collection Λ of curves

$$\lambda_{ji} := g_{n+1} \circ F_{n+1}(\partial\tilde{Q}_n^{j,i}), \quad i = 1, 2,\ j = 1, \dots, 2^{n-1}\ ,$$

that are the boundary of the unbounded complex curve

$$X_{n+1} := g_{n+1} \circ F_{n+1}(\tilde{C}_{n+1})\ .$$

Note that we still have

$$g_{n+1} \circ F_{n+1}(\overline{\tilde{C}_{n+1} \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}^2}\ . \tag{I.3.8}$$

Choose $0 < \eta << 1$ such that $(r_n + \eta)\overline{\mathbb{B}^2} \cap \Lambda = \emptyset$. We may choose a compact polynomially convex set $K' \subset X_{n+1}^\circ$ with $g_{n+1} \circ F_{n+1}(K_n) \subset K'$ such that $L = (r_n + \eta)\overline{\mathbb{B}^2} \cup K'$ is polynomially convex (see e.g., Theorem 4.14.6 in [8]). Then by Lemma I.2.1 there exists $\phi_{n+1} \in \operatorname{Aut}\mathbb{C}^2$ such that

$$\phi_{n+1}(\Lambda) \subset \mathbb{C}^2 \setminus r_{n+1}\overline{\mathbb{B}^2}\ , \tag{I.3.9}$$

and

$$\|\phi_{n+1} - \operatorname{Id}\|_L < \frac{\min\{\eta, \varepsilon_n\}}{4}\ . \tag{I.3.10}$$

We consider the map $f_{n+1}\colon \tilde{C}_{n+1} \to \mathbb{C}^2$ defined by

$$f_{n+1} := \phi_{n+1} \circ g_{n+1} \circ F_{n+1}\ .$$

We have that (a) and (b) (with $\tilde{C}_{n+1}$ instead of C_{n+1}) clearly hold, but now f_{n+1} has singularites on $\partial\tilde{C}_{n+1}$. However, we now consider $0 < \delta_{n+1} < \tilde{\delta}_{n+1}$ to see what happens on C_{n+1}. As the points to expose are taken on the boundary components (see (I.3.5)), the singularities of f_{n+1} are not contained in C_{n+1} for any such δ_{n+1}, and so $f_n\colon C_{n+1} \to \mathbb{C}^2$ is holomorphic. Finally, since ∂C_{n+1} will converge to $\partial\tilde{C}_{n+1}$ as $\delta_{n+1} \to \tilde{\delta}_{n+1}$ we have (c) for δ_{n+1} sufficiently close to $\tilde{\delta}_{n+1}$.

■

I.4 Proof of Theorem I.1.1

We will prove Theorem I.1.1 via an inductive construction, where Proposition I.3.1 provides us with the inductive step. Without loss of generality, we may assume that $\epsilon < 1$.

I.4.1 The Induction Scheme

To start the induction, with the notation as in Section I.3, we define $f_1 \colon C_1 \hookrightarrow \mathbb{C}^2$ by $f_1(\zeta) := (2/\zeta, 0)$ for $\zeta \in \mathbb{C}$, and $f(\infty) := (0, 0)$. Setting $r_1 = 1$ we note that $f_1(\partial C_1) \subset \mathbb{C}^2 \setminus r_1\overline{\mathbb{B}^2}$, so if we choose $0 < \delta_1' << 1$ sufficiently close to zero, and set

$$K_1 := \mathbb{P}^1 \setminus Q_1(\delta_1'),$$

we have that $K_1 \subset C_1^\circ$ and $f_1(\overline{C_1 \setminus K_1}) \subset \mathbb{C}^2 \setminus r_1\overline{\mathbb{B}^2}$. Then the conditions in Proposition I.3.1 are satisfied with $n = 1$, and setting $\delta_2 \leq \epsilon \cdot 2^{-4}$, $r_2 = 2$, we let f_2 be the map furnished by the proposition, with ϵ_1 explained in the induction scheme below. We then choose $\delta_2' < \delta_2/2$ sufficiently close to zero such that if we set

$$K_2 := \mathbb{P}^1 \setminus Q_2(\delta_2'),$$

we have $K_2 \subset C_2^\circ$ and $f_2(\overline{C_2 \setminus K_2}) \subset \mathbb{C}^2 \setminus r_2\overline{\mathbb{B}^2}$.

Let us now state our induction hypothesis I_n for some $n \geq 2$. We assume that we have found and constructed the following.

(i)$_n$ A decreasing sequence $\delta_2 > \delta_3 > \cdots > \delta_n$ of numbers with $\delta_k \leq \epsilon \cdot 2^{-2k}$ such that $\{Q_k\}_{k=1}^n$ is a nested sequence of rectangles.

(ii)$_n$ A decreasing sequence $\delta_1' > \delta_2' > \cdots > \delta_n'$ of numbers with δ_1', δ_2' as above, and $\delta_k' < \delta_k/2$ for $k = 1, \ldots, n$, and holomorphic embeddings $f_k \colon C_k \hookrightarrow \mathbb{C}^2$ such that, setting $K_k := \mathbb{P}^1 \setminus Q_k(\delta_k')$, we have that $f_m(\overline{C_m \setminus K_k}) \subset \mathbb{C}^2 \setminus r_k\overline{\mathbb{B}^2}$ for $1 \leq k \leq m \leq n$, where $r_k \geq k$.

(iii)$_n$ A sequence of positive numbers $\{\eta_k\}_{k=2}^n$ such that if $f \colon K_k \to \mathbb{C}^2$ is a holomorphic map with $\|f - f_k\|_{K_k} < \eta_k$, then $f \colon K_{k-1} \hookrightarrow \mathbb{C}^2$ is an embedding.

(iv)$_n$ A sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{n-1}$ such that $\varepsilon_{k+j} < \eta_k \cdot 2^{-j-1}$, $j \leq n - k - 1$, with $\|f_k - f_{k-1}\|_{K_{k-1}} < \varepsilon_{k-1}$ for $k = 2, \ldots, n$.

Our constructions above gives (i)$_n$, (ii)$_n$ and (iv)$_n$ in the case $n = 2$ (possibly shrinking ϵ_1). Then, choosing η_2 small enough, gives f and f' close to f_2 and f_2' respectively (the latter by Cauchy estimates) on K_2 such that f is injective and f' never vanishes on K_1. Being K_1 compact, this is enough to achieve (iii)$_n$ when $n = 2$.

I.4.2 Passing from I_n to I_{n+1}

Let us assume that I_n is true and prove I_{n+1}. First of all we have that $(\mathrm{i})_{n+1}$, $(\mathrm{iii})_{n+1}$ and the first part of $(\mathrm{iv})_{n+1}$ are just a matter of choosing respectively δ_{n+1}, η_{n+1} and ϵ_n sufficiently small. By $(\mathrm{ii})_n$ with $k = n$, and with ϵ_n above fixed, we may apply Proposition I.3.1 to get a holomorphic embedding $f_{n+1}\colon C_{n+1} \hookrightarrow \mathbb{C}^2$ to obtain the second part of $(\mathrm{iv})_{n+1}$ and $(\mathrm{ii})_{n+1}$ with $m = n+1$ and $k = n$. Next, by choosing δ'_{n+1} sufficiently small we get $(\mathrm{ii})_{n+1}$ for $k = m = n+1$. It remains to explain how to achieve $(\mathrm{ii})_{n+1}$ for $m = n+1$ and $k = 1, \ldots, n-1$. Since

$$\overline{C_{n+1} \setminus K_k} = \overline{C_{n+1} \setminus K_n} \cup \overline{K_n \setminus K_k}$$

what is needed is $f_{n+1}(\overline{K_n \setminus K_k}) \subset \mathbb{C}^2 \setminus r_k\overline{\mathbb{B}^2}$. This follows from $(\mathrm{ii})_n$, possibly after having decreased ϵ_n.

I.4.3 Proof of Theorem I.1.1

Consider the objects constructed in the inductive scheme above. Then by $(\mathrm{iv})_n$ we have that $\lim_{j\to\infty} f_j = f$ exists on K_k for any k. We have that

$$\left(\bigcup_k C_k\right)^{\circ} = \mathbb{P}^1 \setminus \bigcap_k Q_k =: \mathbb{P}^1 \setminus C$$

and so $\lim_{j\to\infty} f_j = f$ exists on $\mathbb{P}^1 \setminus C$. Now for any fixed k we get by $(\mathrm{ii})_n$ that $f_n^{-1}(r_k\overline{\mathbb{B}^2}) \subset K_k$ for all $n > k$ and therefore $f^{-1}(r_k\overline{\mathbb{B}^2}) \subset K_k$, so f is proper. By $(\mathrm{iv})_n$ we get that $\|f - f_k\|_{K_k} < \eta_k$, hence by $(\mathrm{iii})_n$ we have that $f\colon K_{k-1} \hookrightarrow \mathbb{C}^n$ is an embedding for all k, so f is an embedding. Finally, note that when constructing a rectangle Q_{n+1} from Q_n, a crude estimate gives that one obtains Q_{n+1} by removing strips of total area bounded by $2^n \cdot \delta_n$. It follows that

$$\mu(C) = \mu\left(\bigcap_n Q_n\right) \geq 4 - \sum_{n=1}^{\infty} 2^n \cdot \delta_n \geq 4 - \epsilon \cdot \sum_{n=1}^{\infty} 2^{-n} > 4 - \epsilon.$$

$\blacksquare$

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Authors' addresses

G. D. Di Salvo University of Oslo, Postboks 1337 Blindern, 0316 Oslo, Norway, giovandd@math.uio.no

E. F. Wold University of Oslo, Postboks 1337 Blindern, 0316 Oslo, Norway, erlendfw@math.uio.no

Paper II

Extended explanation of Orevkov's paper on proper holomorphic embeddings of complements of Cantor sets in $\mathbb{C}^2$ and a discussion of their measure

G. D. Di Salvo

To be submitted.

II

Abstract

We clarify the details of a cryptical paper by Orevkov in which a construction of a proper holomorphic embedding $\gamma\colon \mathbb{P}^1 \setminus C \hookrightarrow \mathbb{C}^2$ is performed with techniques completely different from the ones exploited in Paper I; in particular, it is proved that such a construction can be done to get the Cantor set C to have zero Hausdorff dimension.

Contents

II.1 Introduction

II.1.1 The main result and motivation

A very relevant open problem in complex geometry is to investigate whether every open Riemann surface $\mathcal{R}$ admits a proper holomorphic embedding into $\mathbb{C}^2$. It has been suggested to search for a counterexample as the complement of a "thick" Cantor set C inside the Riemann Sphere, that is $\mathcal{R} = \overline{\mathbb{C}} \setminus C$. In paper [5], the author proved the existence of a proper holomorphic embedding from the complement of a Cantor set $C \subset \overline{\mathbb{C}}$ into $\mathbb{C}^2$. Nevertheless, no information dealing with the size of C is given; moreover, such a paper is quite cryptical. The motivations for the present paper are therefore two: to write in detail the construction of both the Cantor set $C \subset \overline{\mathbb{C}}$ and the proper holomorphic embedding $\gamma\colon \overline{\mathbb{C}} \setminus C \hookrightarrow \mathbb{C}^2$ originally presented in [5] and discuss the size of C, proving in particular that it can be constructed to have zero Hausdorff dimension. From the construction, it will be rather clear that it is really hard to prove that this C could have any kind of positive measure.

The main result of the present paper, originally presented by S. Orevkov in [5] without the estimate on the size of C, is the following

Theorem II.1.1. There exists a Cantor set $C \subset \overline{\mathbb{C}}$ with $\dim_H(C) = 0$ and a proper holomorphic embedding $\gamma\colon \overline{\mathbb{C}} \setminus C \hookrightarrow \mathbb{C}^2$.

It is worth mentioning that in [3] has been proved that the complement of a thick Cantor set inside $\overline{\mathbb{C}}$ does not constitute a counterexample, as it is therein provided the construction of a proper holomorphic embedding $\overline{\mathbb{C}} \setminus C \hookrightarrow \mathbb{C}^2$ where $C \subset \overline{\mathbb{C}}$ is a Cantor set of arbitrarily large measure. Namely: such C is realized as a subset of a square whose side has length 2 and for every $\epsilon > 0$ it can be built of Lebesgue measure greater than $4 - \epsilon$.

II.2 Definitions and preliminary results

II.2.1 Cantor sets

The Cantor set is defined as the intersection of the decreasing family of subsets of the unit interval of the real line $E_0 = [0, 1] \subset \mathbb{R}$ defined inductively by removing at each step the open middle third from each interval of the previous set. Thus $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and so on. Finally one defines

$$E := \bigcap_{n \geq 0} E_n \subset [0, 1].$$

Given a topological space X, we define *Cantor set* any subset $C \subset X$ homeomorphic to E. Hence a Cantor set is characterized by the topological properties of E; thus, according to Brouwer's theorem in [2], $C \subset X$ is a Cantor set if and only if it is

$(\mathrm{i})_C$ not–empty ;

$(\mathrm{ii})_C$ perfect (i.e. closed and with no isolated points) ;

$(\mathrm{iii})_C$ compact ;

$(\mathrm{iv})_C$ totally disconnected (i.e. every its connected component is a one–point set) ;

$(\mathrm{v})_C$ metrizable .

II.2.2 Sperichal Distance

The one point compactification of $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ x_1^2 + \cdots + x_{n+1}^2 = 1\}$ via stereographic projection:

$$\begin{aligned}
\varphi_n &\colon \overline{\mathbb{R}^n} \to \mathbb{S}^n \\
\varphi_n(x) &:= \frac{2}{\|x\|^2+1}\left(x, \frac{\|x\|^2-1}{2}\right), \quad x \neq \infty \ ; \\
\varphi_n(\infty) &:= (0, \ldots, 0, 1) \ .
\end{aligned}$$

This allows us to define a notion of distance on any such $\overline{\mathbb{R}^n}$ as

$$d_n(x,y) := \arccos(\varphi_n(x) \cdot \varphi_n(y)), \quad x, y \in \overline{\mathbb{R}^n} \ ,$$

(see e.g., [1], Theorem 5 pag. 444) where $\cdot$ denotes the scalar product in $\mathbb{R}^{n+1}$. The spaces $\overline{\mathbb{C}}$ and $\overline{\mathbb{C}^2}$ are just the case $n = 2$ and $n = 4$ respectively, so we have a notion of distance on them, allowing us to talk about the diameter of their subsets. Namely, for a connected subset $\Omega \subseteq \overline{\mathbb{R}^n}$ we denote its diameter as $|\Omega| := \sup\{d_n(x,y) \ : \ x, y \in \Omega\}$ and if $\Omega = \bigcup_j \omega_j$, where ω_j are pairwise disjoint connected subsets of $\overline{\mathbb{R}^n}$ then we define $|\Omega| := \sup_j |\omega_j|$.

II.2.3 Definitions and technical results

We will consider the Riemann Sphere $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and the one–point compactification of $\mathbb{C}^2$ as $\overline{\mathbb{C}^2} := \mathbb{C}^2 \cup \{\infty_2\}$. Given a subset $B \subset \mathbb{C}^2$ and a positive number $\alpha > 0$, we define the open subset

$$B(\alpha) := \{z \in \mathbb{C}^2 \ : \ \mathrm{dist}(z, B) < \alpha\} \ .$$

Given a subset Ω of $\overline{\mathbb{C}}$ (resp. $\overline{\mathbb{C}^2}$), we say that Ω is *bounded* if $\Omega \cap U = \emptyset$ for some U open neighborhood of ∞ (resp. of ∞_2), *unbounded* otherwise. We consider $\{\epsilon_n\}_{n\geq 1}, \{R_n\}_{n\geq 0}$ strictly monotone sequences of positive real numbers, with $\epsilon \to 0$ and $R_n \to +\infty$. Consider moreover $\{a_n\}_{n\geq 1} \subset \mathbb{C}$ such that $R_{n-1} < |a_n| < R_n$. We will refer to these sequences as the *parameters* of the construction. Fixing these sequences, the sets

$$\begin{aligned}
D_n :=& \{|z| < R_n\} \subset \mathbb{C} \\
C_n^v :=& D_n \times \mathbb{C} \\
C_n^h :=& \mathbb{C} \times D_n \\
C_n :=& C_n^v \cup C_n^h \\
B_n :=& \overline{C_n^v \cap C_n^h} = \overline{D}_n \times \overline{D}_n
\end{aligned}$$

and the rational shears $f_n \colon \mathbb{C}^2 \to \overline{\mathbb{C}^2}$,

$$f_n(x,y) := \begin{cases} (x, y + g_n(x)) & \text{for odd } \ n \\ (x + g_n(y), y) & \text{for even } \ n \end{cases} ,$$

where
$$g_n(z) := \frac{\epsilon_n}{z - a_n} \,,$$
are defined. Finally define the sequence of continuous mappings $\gamma_n \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}^2}$ as
$$\begin{aligned} \gamma_n(z) &:= f_n \circ \cdots \circ f_1(z, 0), \quad z \neq \infty \\ \gamma_n(\infty) &:= \infty_2 \end{aligned}$$
and consequently the sets
$$A_n := \gamma_n(\overline{\mathbb{C}}) = f_n \circ \cdots \circ f_1(\mathbb{C} \times \{0\}) \cup \{\infty_2\} \,.$$

The construction of the proper holomorphic embedding γ and of the Cantor set C in Theorem II.1.1 relies on the sets and functions just defined: we will see that a suitable choice of the sequences $\{\epsilon_n\}_n$, $\{R_n\}_n$ and $\{a_n\}_n$ allows to obtain C as the closure of the set $\overline{\bigcup_n \gamma_n^{-1}(\infty_2)}$ and the limit $\lim_n \gamma_n$ to be a proper holomorphic embedding γ on $\overline{\mathbb{C}} \setminus C$ into $\mathbb{C}^2$; in particular the faster $\{R_n\}_n$ diverges, the smaller the size of C so that it can be constructed to have zero Hausdorff dimension.

In the rest of this paper, $\{k_n\}_{n \geq 0}$ will denote the Fibonacci sequence, that is the sequence of natural numbers defined by $k_0 = k_1 = 1$ and
$$k_{n+2} = k_{n+1} + k_n, \quad \forall n \geq 0 \,.$$

The following remark gathers some known facts about the number of solutions of rational equations (see 4.23–4.25, [4]), which will be fundamental in the proof of the subsequent Lemma.

Remark II.2.1. Let $r \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function. There exists a natural number k, called *degree* of r and denoted as $\deg r$, such that for every $a \in \overline{\mathbb{C}}$, the equation $r(z) = a$ has k solutions counted with their multiplicity, which are multiple only for a finite number of $a \in \overline{\mathbb{C}}$.

(1) In particular, up to a slightly small perturbation of a, the zeros of $r(z) - a$ are simple.

(2) The zeros of r are as many as the poles, counted with their multiplicity, therefore the poles of r are all simple if and only if they are k distinct.

(3) Let t be another rational function and $w \in \overline{\mathbb{C}}$ fixed. Up to perturb $a \in \mathbb{C}$ arbitrarily little, the equations $t = w$ and $r - a = 0$ have no common solutions, as it is easily seen observing that $r^{-1}(a) \cap r^{-1}(a') = \emptyset$ for any $a' \neq a$.

Lemma II.2.1. Define a sequence of rational functions recursively:
$$\begin{aligned} r_0(z) &:= z \\ r_1(z) &:= g_1(z) \end{aligned}$$

$$r_{n+2}(z) := r_n(z) + g_{n+2}(r_{n+1}(z)) = r_n(z) + \frac{\epsilon_{n+2}}{r_{n+1}(z) - a_{n+2}}, \quad n \geq 0. \quad \text{(II.2.1)}$$

Then, up to perturbing the elements of the sequence $\{a_n\}_n \subset \mathbb{C}$ arbitrarily little, the following hold:

(i) for every $n \geq 0$, r_n and r_{n+1} do not share any pole and $r_n^{-1}(\infty) \subsetneq r_{n+2}^{-1}(\infty)$; in particular ∞ is a pole only for r_n with n even;

(ii) the poles of r_n are k_n and all simple; therefore the equation

$$r_n(z) = a_{n+1}$$

has exactly k_n solutions, all distinct.

Proof. We see that $r_n^{-1}(\infty) \cap r_{n+1}^{-1}(\infty) = \emptyset$ for all $n \geq 0$ by induction on n. The base case follows immediately. Assume the claim true for n; if by contradiction $\exists w \in r_{n+1}^{-1}(\infty) \cap r_{n+2}^{-1}(\infty)$, then by (II.2.1), it must be $r_n(w) = \infty$, so $r_n^{-1}(\infty) \cap r_{n+1}^{-1}(\infty) \neq \emptyset$, against inductive hypothesis.

The inclusion $r_n^{-1}(\infty) \subsetneq r_{n+2}^{-1}(\infty)$ for all $n \geq 0$ follows by looking at (II.2.1) as from (3) in Remark II.2.1, up to slightly perturbing a_{n+2}, one gets $\{r_n = \infty\} \cap \{r_{n+1} - a_{n+2} = 0\} = \emptyset$.

Clearly ∞ is a pole for r_0, thus what said so far ensures that ∞ is a pole for r_{2k} and not a pole for r_{2k+1}, for all $k \geq 0$. So (i) is proved.

Let us prove (ii) by induction on n. This is trivial for r_0 and r_1. Assume the claim true up to $n+1$. As observed in (3) in Remark II.2.1, up to a slightly small perturbation of a_{n+2}, the sets $\{r_n = \infty\}$ and $\{r_{n+1} - a_{n+2} = 0\}$ have empty intersection. Hence, by (II.2.1), the poles of r_{n+2} are precisely the poles of r_n (which are k_n and simple by inductive hypothesis) and the poles of $\frac{\epsilon_{n+2}}{r_{n+1} - a_{n+2}}$ (which are k_{n+1} and simple as the zeros of the denominator are, up to a slightly small perturbation of a_{n+2} as pointed out in (1) in Remark II.2.1, in fact $\deg r_{n+1} = k_{n+1}$ by inductive hypothesis). Therefore the poles of r_{n+2} are $k_n + k_{n+1} = k_{n+2}$ and all simple, hence by (2) in Remark II.2.1 one has $\deg r_{n+2} = k_{n+2}$. So up to a slightly small perturbation of a_{n+3}, (1) guarantees that the equation $r_{n+2}(z) = a_{n+3}$ has k_{n+2} solutions, all distinct. So (ii) is proved. ■

We will assume from now on to choose R_n large enough, so that $r_n(z) = a$ has k_n distinct solutions for every $|a| \geq R_n$. The following Lemma is used in Section II.3.3:

Lemma II.2.2. Let r be a rational function with a finite number of poles $r^{-1}(\infty) = \{w_1, \ldots, w_N\} \subset \overline{\mathbb{C}}$, all simple. Assume that $r(z) = a$ has k distinct solutions for every $|a| \geq T_0$, for some T_0. Then for any $T \geq T_0$ large enough, the set

$$\Omega(r, T) := \{z \in \overline{\mathbb{C}} \setminus r^{-1}(\infty) \ : \ |r(z)| > T\}$$

is the disjoint union of k connected sets, each of which homeomorphic to the punctured open disk $\{|z| < 1\} \setminus \{0\}$ and containing exactly one solution of

$r(z) = a$, for every $|a| > T$. Moreover, such k sets can be taken to have arbitrarily small diameters, provided that T is large enough.

Proof. Assume the poles of r are all distinct; being them simple one has that $N = k$. We express r on a neighborhood $V_j \subset \overline{\mathbb{C}}$ of a pole w_j as $r(z) = h_j(z)(z - w_j)^{-1}$ or $r(z) = h_j(z)z$, accordingly with w_j to be different or equal to ∞, where $h_j \in \mathcal{O}(V_j)$ is bounded and never vanishing. Let $a \in \mathbb{C}$ with $|a| > T$. The larger T, the closer z has to be to the poles for $r(z) = a$ to be solved. In particular, up to enlarging T we may assume that the V_j are pairwise disjoint, therefore $X_j := \{z \in V_j \setminus \{w_j\} \ : \ |r(z)| > T\}$ are pairwise disjoint and connected. In particular for each $\epsilon > 0$, there exists $T > 0$ sufficiently large such that the diameter of X_j is less than ϵ. Moreover the union of the X_j is $\Omega(r, T)$ and each X_j contains precisely one of the k solutions of the equation $r(z) = a$ for any $|a| > T$, for T sufficiently large: if not, there must be one of them, say X_1, such that for any T sufficiently large it contains at least two solutions of $r(z) = a$ for some $|a| > T$, leading w_1 to be a multiple pole for r. ∎

Finally, it is easily seen by induction that

$$\gamma_n = \begin{cases} (r_{n-1}, r_n) & \text{for odd } n \\ (r_n, r_{n-1}) & \text{for even } n \end{cases}, \tag{II.2.2}$$

in fact the case $n = 1$ is immediate and for n even, for example

$$\gamma_{n+1} = f_{n+1}(\gamma_n) = f_{n+1}(r_n, r_{n-1}) = (r_n, r_{n-1} + g_{n+1}(r_n)) = (r_n, r_{n+1}) .$$

II.3 General Properties

This section is devoted to the description of the objects introduced in Section II.2.3, which are the ingredients of the construction of the proper holomorphic embedding $\gamma\colon \overline{\mathbb{C}} \setminus C \hookrightarrow \mathbb{C}^2$ presented in [5]. In particular we will focus on the choice of the three sequences $\{\epsilon_n\}_{n\geq 1}, \{R_n\}_{n\geq 0}, \{a_n\}_{n\geq 1}$ upon which the whole construction is based.

II.3.1 On the set $\gamma_n^{-1}(\infty_2)$ and the sequence $\{R_n\}_{n\geq 0}$. Definition and properties of vertical and horizontal components of A_n

It follows from (II.2.2) and (i) in Lemma II.2.1 that

$$\gamma_n^{-1}(\infty_2) \subsetneq \gamma_{n+1}^{-1}(\infty_2) \tag{II.3.1}$$

holds true for all $n \geq 0$; moreover $\infty \in \gamma_n^{-1}(\infty_2)$ for every $n \geq 0$, so we can write

$$\gamma_n^{-1}(\infty_2) = \{\infty\} \cup \{t_j^{(n)}\}_{j=1}^{b_n - 1} \tag{II.3.2}$$

for some $t_j^{(n)} \in \mathbb{C}$, where $b_n = k_{n+1}$. Observe that $A_n = \gamma_n(\overline{\mathbb{C}}) \subset \overline{\mathbb{C}^2}$ is compact with $\infty_2 \in A_n$; moreover $A_n \setminus \{\infty_2\} = \gamma_n(\mathbb{C})$ is a connected complex curve in $\mathbb{C}^2$.

Proposition II.3.1. Assume $\epsilon_1, \dots, \epsilon_n$, $a_1, \dots, a_n$ and $R_1, \dots, R_{n-1}$ fixed, $|a_{j+1}| > R_j > |a_j|$ for $j = 1, \dots, n-1$. Then there exist $R_n > |a_n|$ such that what follows is true:

(a) $A_n \setminus (B_n \cup \{\infty_2\})$ has b_n disjoint components, and

(b) $A_n \subset C_n$.

We refer to the components of $A_n \setminus (B_n \cup \{\infty_2\})$ contained in C_n^h (resp. in C_n^v) as the *horizontal* (resp. *vertical*) components of A_n. Correspondingly we split b_n as $h_n + v_n$. It turns out that

$$\begin{cases} v_n = k_{n-1} \\ h_n = k_n \end{cases} (n \ \text{ even}) , \qquad \begin{cases} v_n = k_n \\ h_n = k_{n-1} \end{cases} (n \ \text{ odd}) . \tag{II.3.3}$$

Proof. The components of $A_n \setminus (B_n \cup \{\infty_2\})$ are the image, via γ_n of punctured neighborhoods in $\overline{\mathbb{C}}$ of the points (II.3.2) (which are defined once $a_n, \epsilon_n, a_j, \epsilon_j, R_j, \quad j \leq n-1$ are fixed). So, by Lemma II.2.2, the bigger R_n, the smaller these neighborhoods; hence these b_n neighborhoods are disjoint for R_n large enough, achieving (a), as γ_n is injective on $\overline{\mathbb{C}} \setminus \gamma_n^{-1}(\infty_2)$ (see Section II.3.2). Looking at (II.2.2) it is clear that unbounded components of $A_n = \gamma_n(\overline{\mathbb{C}})$ are due to the poles of r_n and r_{n-1}; being $r_n^{-1}(\infty) \cap r_{n-1}^{-1}(\infty) = \emptyset$ by (i) in Lemma II.2.1, it follows that when one component of A_n becomes unbounded, the other remains bounded. Hence, for R_n large enough, we get (b). Finally, (II.3.3) follows directly from Lemma II.2.1, once we express γ_n as in (II.2.2). ∎

II.3.2 Definition of certain sets Δ_n, K_n and some properties of the mappings γ_n

Let us then define

$$\Delta_n := \gamma_n^{-1}(A_n \setminus B_n) .$$

Given $U \subset \overline{\mathbb{C}^2}$ open neighborhood of ∞_2, $\gamma_n^{-1}(U)$ is a neighborhood of $\gamma_n^{-1}(\infty_2)$; this last set has b_n points, therefore Lemma II.2.2 guarantees that for U sufficiently small (meaning the points of U are sufficiently close to ∞_2 with respect to the spherical measure introduced in Section II.2.2; equivalently U is contained in the complement of a closed ball in $\overline{\mathbb{C}^2}$ of sufficiently large radius), $\gamma_n^{-1}(U)$ is the disjoint union of b_n open neighborhoods $U_j^{(n)}$ of the points (II.3.2). In particular $\forall \epsilon > 0 \ \exists U \subset \overline{\mathbb{C}^2}$ open neighborhood of ∞_2 such that $|U_j^{(n)}| < \epsilon$. Since

$$\Delta_n = \gamma_n^{-1}(\overline{\mathbb{C}^2} \setminus B_n) \tag{II.3.4}$$

(straightforward with double inclusion), we can always suppose to have R_n large enough such that

$$\Delta_n = \bigcup_{j=1}^{b_n} U_j^{(n)}$$

is a disjoint union. In particular for every n, there exists R_n so large that

$$|U_j^{(n)}| < \frac{1}{b_n^n} \ , \qquad \forall j = 1, \dots, b_n \ . \tag{II.3.5}$$

Let us now check that

$$\gamma_n \colon \overline{\mathbb{C}} \setminus \gamma_n^{-1}(\infty_2) \to \mathbb{C}^2$$

is a holomorphic embedding by induction on n. As $\gamma_1 \colon \overline{\mathbb{C}} \setminus \{a_1, \infty\} \to \mathbb{C}^2$, $\gamma_1(z) = (z, \frac{\epsilon_1}{z-a_1})$, the base case trivially follows. Assume the claim true for n. Then $\gamma_{n+1} = f_{n+1} \circ \gamma_n$ is a holomorphic embedding in the intersection of the subset where γ_n is such (that is $\overline{\mathbb{C}} \setminus \gamma_n^{-1}(\infty_2)$) and $\{z \in \overline{\mathbb{C}} \ : \ \gamma_n(z) \notin f_{n+1}^{-1}(\infty_2)\} = \overline{\mathbb{C}} \setminus \gamma_{n+1}^{-1}(\infty_2)$. Such an intersection is

$$\overline{\mathbb{C}} \setminus \gamma_n^{-1}(\infty_2) \cap \overline{\mathbb{C}} \setminus \gamma_{n+1}^{-1}(\infty_2) = \overline{\mathbb{C}} \setminus (\gamma_n^{-1}(\infty_2) \cup \gamma_{n+1}^{-1}(\infty_2)) = \overline{\mathbb{C}} \setminus \gamma_{n+1}^{-1}(\infty_2)$$

and we are done (the last equality follows from (II.3.1)).

Define then the following compacts in $\overline{\mathbb{C}}$:

$$\begin{aligned} &K_n := \overline{\mathbb{C}} \setminus \Delta_n = \gamma_n^{-1}(A_n \cap B_n) = \gamma_n^{-1}(B_n), \quad n \geq 1 \\ &K_0 := \emptyset \ . \end{aligned}$$

It is straighforward to see that $K_n \subset \overline{\mathbb{C}} \setminus \gamma_n^{-1}(\infty_2)$. It will follow from Proposition II.3.2 that $K_n \subsetneq K_{n+1}^{\circ}$; therefore there exists $\delta_n > 0$ such that if $h \colon \overline{\mathbb{C}} \setminus \gamma_n^{-1}(\infty_2) \to \mathbb{C}^2$ is holomorphic with

$$\|\gamma_n - h\|_{K_n} < \delta_n \ ,$$

then h is an embedding on K_{n-1}. Set $\delta_0 := 1/4$ and assume without loss of generality $\{\delta_n\}_{n \geq 0}$ to be strictly decreasing.

II.3.3 From Δ_n to Δ_{n+1}: definition of pair of pants

We will prove that for a suitable choice of the parameters, the sequences $\{\Delta_n\}_n$ and $\{K_n\}_n$ are strictly decreasing and increasing respectively; in particular, we will focus on the topological behavior of the sequence $\{\Delta_n\}_n$ when passing from step n to step $n+1$.

Proposition II.3.2. Assume $\epsilon_1, \dots, \epsilon_{n+1}$, $a_1, \dots, a_{n+1}$ and $R_1, \dots, R_n$ fixed, with $|a_{j+1}| > R_j > |a_j|$ for $j = 1, \dots, n$ and rename the $U_j^{(n)}$ to highlight which one comes from horizontal or vertical components of $A_n \setminus B_n$ as

$$\Delta_n = \bigcup_{j=1}^{h_n} H_j^{(n)} \quad \cup \quad \bigcup_{j=1}^{v_n} V_j^{(n)} = H_n \cup V_n \ .$$

Then, for any $R_{n+1} > |a_{n+1}|$ sufficiently large, we have that

$$\Delta_{n+1} \subsetneq \Delta_n \ , \tag{II.3.6}$$
$$K_n \subsetneq K^\circ_{n+1} \ , \tag{II.3.7}$$

and passing from Δ_n to Δ_{n+1}, for n even, $V_j^{(n)}$ shrinks to $V_j^{(n+1)}$, $j = 1, \ldots, v_n$, while $H_j^{(n)}$ splits itself into $H_j^{(n+1)}$ and $V_{v_n+j}^{(n+1)}$, $j = 1, \ldots, h_n$, creating the so called *pair of pants*. For n odd, a similar conclusion holds, swapping vertical and horizontal components.

Proof. Let n be even. From (II.2.1) and (II.2.2) we have

$$\gamma_n = (r_n, r_{n-1}) \ , \text{ and}$$
$$\gamma_{n+1} = (r_n, r_{n+1}) = (r_n, r_{n-1}) + \left(0, \frac{\epsilon_{n+1}}{r_n - a_{n+1}}\right) \ ,$$

from which

$$H_n = \{|r_n| > R_n\},$$
$$V_n = \{|r_{n-1}| > R_n\}, \text{ and}$$
$$H_{n+1} = \{|r_n| > R_{n+1}\}.$$

Moreover, up to slightly move a_{n+1}, (3) in Remark II.2.1 guarantees that $r_{n-1} = \infty$ and $r_n = a_{n+1}$ have disjoint solutions, so for R_{n+1} large enough we have

$$V_{n+1} = \{|r_{n-1}| > R_{n+1}\} \cup \{|r_n - a_{n+1}| < \epsilon_{n+1}/R_{n+1}\} =: V'_{n+1} \cup V''_{n+1} \ ,$$

with $V'_{n+1} \cap V''_{n+1} = \emptyset$. Since $|a_{n+1}| > R_n$, it follows that R_{n+1} sufficiently large implies $V''_{n+1} \subseteq H_n$; the inclusions $V'_{n+1} \subseteq V_n$ and $H_{n+1} \subseteq H_n$ are trivial, so we get (II.3.6) and (II.3.7), where the strict inclusions follow from (II.3.4) and Lemma II.2.2, which guarantees the $U_j^{(n+1)}$ shrunk arbitrarily little around the point they are neighborhood of, provided R_{n+1} to be large enough. Taking R_{n+1} large enough guarantees moreover that V''_{n+1} has k_n components (as the equation $r_n = a_{n+1}$ has k_n simple solutions by (ii) in Lemma II.2.1) and $H_{n+1} \cap V''_{n+1} = \emptyset$ (as $R_{n+1} > |a_{n+1}|$). Now H_n has k_n components, each of which splits into a component of H_{n+1} and a component of V''_{n+1}. So the statement on the pair of pants follows.

For n odd the argument is the same, just switching the role of vertical and horizontal objects. ∎

II.3.4 The sequence $\{\epsilon_n\}_{n\geq 1}$

Describing constraints for R_n in Sections II.3.1, II.3.2 and II.3.3, we assumed to have already fixed $a_1, \ldots, a_n$, $R_1, \ldots, R_{n-1}$, and in particular $\epsilon_1, \ldots, \epsilon_n$. Let us now see how to choose the ϵ_n.

Proposition II.3.3. There are sequences $\{a_n\}_n$, $\{\epsilon_n\}_n$ and $\{R_n\}_n$ such that

$$\| f_n - \mathrm{Id} \|_{\overline{B_j(\frac{1}{2})}} < \delta_j \cdot \frac{1}{2^{j+n}} \ , \qquad 0 \leq j < n \tag{II.3.8}$$

Proof. Let us construct the three sequences inductively on n. Case $n = 1$; take any $a_1 \in \mathbb{C}$ such that $|a_1| > 1/2$, define B_0 as the origin $(0, 0) \in \mathbb{C}^2$. Then there exists $\epsilon_1 > 0$ such that $\| f_1 - \mathrm{Id} \|_{\overline{B_0(\frac{1}{2})}} < \delta_0 \cdot \frac{1}{2^{1+0}}$ (note that $B_0(\frac{1}{2})$ is the 1/2–ball in $\mathbb{C}^2$). Assume (II.3.8) true for n. In particular $a_i, \epsilon_i, R_i,\ i \leq n-1$ and a_n, ϵ_n are defined, so we can take R_n to satisfy all the constraints discussed in the previous sections; in particular in Lemma II.2.1, Proposition II.3.1 and II.3.2, condition (II.3.5). So we can fix a_{n+1} such that $|a_{n+1}| > \sqrt{2}R_n + 1/2$. Define then $\epsilon_{n+1} > 0$ such that $\| f_{n+1} - \mathrm{Id} \|_{\overline{B_n(\frac{1}{2})}} < \delta_n \cdot \frac{1}{2^{2n+1}}$. Then (II.3.8) follows in the case $n + 1$. ∎

With such a choice of parameters, we achieve

$$\| f_n \circ \cdots \circ f_j - \mathrm{Id} \|_{B_{j-1}} < \frac{1}{2} \ , \qquad 1 \leq j \leq n \ . \tag{II.3.9}$$

The case $n = 1$ follows from the base case of (II.3.8). Assume (II.3.9) true for n and let us see it holds for $n + 1$:

$$\begin{aligned}
\| f_{n+1} \circ \cdots \circ f_j - \mathrm{Id} \|_{B_{j-1}} \leq & \| f_{n+1} - \mathrm{Id} \|_{f_n \circ \cdots \circ\, f_j(B_{j-1})} \\
& + \| f_n - \mathrm{Id} \|_{f_{n-1} \circ \cdots \circ\, f_j(B_{j-1})} + \cdots + \| f_j - \mathrm{Id} \|_{B_{j-1}} \\
\leq & \| f_{n+1} - \mathrm{Id} \|_{\overline{B_{j-1}(\frac{1}{2})}} \\
& + \| f_n - \mathrm{Id} \|_{\overline{B_{j-1}(\frac{1}{2})}} + \cdots + \| f_j - \mathrm{Id} \|_{\overline{B_{j-1}(\frac{1}{2})}} \qquad \text{(II.3.10)} \\
\leq & \frac{\delta_{j-1}}{2^{n+1+j-1}} + \cdots + \frac{\delta_{j-1}}{2^{j+j-1}} \qquad \text{(II.3.11)} \\
= & \delta_{j-1} \cdot \sum_{i=2j-1}^{n+j} \frac{1}{2^i} < \frac{1}{2} \ .
\end{aligned}$$

Hence (II.3.9) holds for any $1 \leq j \leq n + 1$ as promised. Just observe that (II.3.10) follows since

$$f_n \circ \cdots \circ\ f_j(B_{j-1}) \subset B_{j-1}(1/2) \subset \overline{B_{j-1}(1/2)}$$

holds true by inductive hypothesis for any $1 \leq j \leq n$ and (II.3.11) follows from Proposition II.3.3.

Remark II.3.1. Since the three sequences underlying the whole construction depend one on the other, it is important to highlight the order with which their elements have to be taken. We have seen in Sections II.3.1, II.3.2 and II.3.3 that in order to define R_n, the parameters

$$a_j, \epsilon_j, R_j,\ j \leq n-1, \ \text{ and } \ a_n, \epsilon_n$$

need to be already fixed; then we immediately fix a_{n+1} such that $|a_{n+1}| > \sqrt{2}R_n + 1/2$. Similarly, in Section II.3.4 it turned out that in order to define ϵ_n,

$$a_j, \epsilon_j, R_j,\ j \leq n-1, \text{ and } a_n$$

need to be previously fixed; in particular, to choose ϵ_1, some $a_1 \in \mathbb{C}$, $|a_1| > 1/2$, has to be fixed. Namely, the order to follow for choosing the sequences of parameters to perform our construction is

$$a_1, \epsilon_1, R_1, a_2, \epsilon_2, R_2, \ldots\ .$$

II.4 The Cantor set C and the proper holomorphic embedding γ

II.4.1 Definition of the Cantor set C

We are ready to define the object of our interest:

$$C := \bigcap_n \Delta_n \subset \overline{\mathbb{C}}. \tag{II.4.1}$$

Let us see C is indeed a Cantor set by proving that it fulfills all the properties (i)$_C$–(v)$_C$ stated in Section II.2. Being $\overline{\mathbb{C}}$ metrizable, we get (v)$_C$. Since for every $n \in \mathbb{N}$ one has

$$\gamma_n^{-1}(\infty_2) \subset \gamma_{n+j}^{-1}(\infty_2) \subset \Delta_{n+j} \quad \forall j \in \mathbb{N}\ ,$$

it follows that

$$\bigcup_{n \geq 1} \gamma_n^{-1}(\infty_2) \subseteq C\ , \tag{II.4.2}$$

from which $C \neq \emptyset$ and thus we get (i)$_C$. From the above construction, every component of Δ_n eventually shrinks to a point (see e.g., (II.3.5)) and since the role of vertical and horizontal components switches at each step, it follows that $\forall n \in \mathbb{N}$, $\forall j \in \{1, \ldots, b_n\}$ there will be infinitely many pair of pants inside each $U_j^{(n)}$, so there will be no isolated point. Moreover exploiting (II.3.7) one gets

$$\left(\overline{\mathbb{C}} \setminus C\right)^\circ = \left(\bigcup_n K_n\right)^\circ = \bigcup_n K_n^\circ = \bigcup_n K_n = \overline{\mathbb{C}} \setminus C \tag{II.4.3}$$

from which C is closed, and thus (ii)$_C$ is verified. Being C closed and $\overline{\mathbb{C}}$ compact, it follows that C is compact itself, so we have (iii)$_C$. Finally, if any connected component of C had more than one point, then (II.3.5) would fail for n big enough. So C is totally disconnected, that is we have (iv)$_C$ and so we have proved C is indeed a Cantor set.

II.4.2 Definition of the proper holomorphic embedding $\gamma\colon \overline{\mathbb{C}} \setminus C \hookrightarrow \mathbb{C}^2$

From time to time we will use the following fact.

Remark II.4.1. If $\{K_n\}_n$ is a normal exhaustion for a domain $\Omega \subseteq \overline{\mathbb{C}}$ and $\{z_n\}_n \subset \Omega$ converges to $z_0 \in \partial\Omega$, then we can suppose without loss of generality that $z_n \in K_n \setminus K_{n-1}$.

Consider γ_n as a mapping $K^\circ_{n+1} \to \mathbb{C}^2$; thus the natural domain to define a limit mapping is $\bigcup_{n\geq 1} K^\circ_n$ which equals $\overline{\mathbb{C}} \setminus C$ from (II.4.3). For every $n \geq k$ one has

$$\begin{aligned}
\|\gamma_{n+1} - \gamma_n\|_{K_k} &= \|(f_{n+1} - \mathrm{Id}) \circ \gamma_n\|_{\gamma_k^{-1}(B_k)} \\
&= \|(f_{n+1} - \mathrm{Id}) \circ (f_n \circ \cdots \circ f_{k+1} \circ \gamma_k)\|_{\gamma_k^{-1}(B_k)} \\
&\leq \|f_{n+1} - \mathrm{Id}\,\|_{f_n \circ\cdots\circ f_{k+1}(B_k)} \\
&\leq \|f_{n+1} - \mathrm{Id}\,\|_{\overline{B_k(\frac{1}{2})}} && \text{by (II.3.9)} \\
&< \delta_k \cdot \frac{1}{2^{n+1+k}} && \text{by (II.3.8)} \ ,
\end{aligned}$$

from which it follows that

$$\sum_{n\geq k} \|\gamma_{n+1} - \gamma_n\|_{K_k} < \sum_{n\geq k} \delta_k \cdot \frac{1}{2^{n+1+k}} < +\infty$$

for every fixed $k \geq 1$; hence $\{\gamma_n\}_n$ compactly converges to a holomorphic mapping

$$\gamma\colon \overline{\mathbb{C}} \setminus C \to \mathbb{C}^2 \ ,$$

since $\{K_n\}_n$ is a normal exhaustion for the domain. It is now easily seen that γ is a local embedding (namely: an embedding on every K_{k-1}) since for every $k \geq 1$ one has

$$\|\gamma - \gamma_k\|_{K_k} \leq \sum_{n\geq k} \|\gamma_{n+1} - \gamma_n\|_{K_k} < \sum_{n\geq k} \delta_k \cdot \frac{1}{2^{n+1+k}} < \delta_k \ .$$

Finally, global embedding property of γ will follow automatically once properness is proved, achieving thus that γ is a proper holomorphic embedding from the complement of a Cantor set into $\mathbb{C}^2$, as wanted. Consider then $\{z_n\}_n \subset \overline{\mathbb{C}} \setminus C = \bigcup_n K_n$ such that $z_n \to \partial(\overline{\mathbb{C}} \setminus C) = \partial C$. By Remark II.4.1 we may assume $z_n \in K_n \setminus K_{n-1} \subseteq \Delta_{n-1}$, from which $|\gamma_{n-1}(z_n)| \to +\infty$ as $n \to +\infty$. Nevertheless this does not imply that $|\gamma(z_n)| \to +\infty$, as it just claims the divergence of $\{\gamma_{n-1}(z_n)\}_n$ and no information on how the sequence $\{\gamma_n\}_n$ acts on the elements of $\{z_n\}_n$ is given. In particular, we need to achieve z_n to be set outside some R_n–ball (say B_{n-3}) by a whole tail of $\{\gamma_n\}_n$ (say γ_{n+j} for every $j \in \mathbb{N}$). In this way the possibility for z_n to come back inside a ball is prevented and we obtain properness of the limit function γ. Hence properness will be achieved if, for instance, the following condition is satisfied:

$$\gamma_{n+j}(K_n \setminus K_{n-1}) \subset \overline{\mathbb{C}}^2 \setminus B_{n-3} \qquad \forall n \geq 3,\ j \in \mathbb{N}\ . \tag{II.4.4}$$

As we assumed $z_n \in K_n \setminus K_{n-1}\ \forall n$, one has

$$\gamma(z_n) = \lim_j \gamma_{j+n}(z_n) \overset{(II.4.4)}{\in} \overline{\mathbb{C}}^2 \setminus B_{n-3}\ ,$$

which implies $|\gamma(z_n)| \geq R_{n-3}$ and thus $\lim_n |\gamma(z_n)| = +\infty$, as wanted. Notice that

$$\gamma_n(\gamma_n^{-1}(B_n)) = A_n \cap B_n \tag{II.4.5}$$

holds true for every $n \in \mathbb{N}$ and this allows to prove that (II.4.4) holds, in fact

$$\begin{aligned} \gamma_{n+j}(K_n \setminus K_{n-1}) &= f_{n+j} \circ \cdots \circ f_{n+1} \circ \gamma_n \left(\gamma_n^{-1}(B_n) \setminus \gamma_{n-1}^{-1}(B_{n-1})\right) && \text{by (II.4.5)} \\ &= f_{n+j} \circ \cdots \circ f_{n+1} \left((A_n \cap B_n) \setminus (f_n(B_{n-1}) \cap A_n)\right) \\ &\subseteq f_{n+j} \circ \cdots \circ f_{n+1} \left(B_n \setminus f_n(B_{n-1})\right) \\ &\subseteq f_{n+j} \circ \cdots \circ f_{n+1} \left(B_n \setminus B_{n-2}\right) && \text{by (II.3.9)} \\ &\subseteq B_{n+1} \setminus B_{n-3} && \text{by (II.3.9)} \\ &\subseteq \overline{\mathbb{C}}^2 \setminus B_{n-3}\ , \end{aligned}$$

as promised.

II.4.3 On the Hausdorff dimension of C and proof of Theorem II.1.1

Fix $\epsilon > 0$; the *Hausdorff ϵ–measure* of C is, by definition

$$\mathcal{H}^\varepsilon(C) := \liminf_{r \to 0} \left\{ \inf \left\{ \sum_i |T_i|^\varepsilon \ :\ C \subseteq \bigcup_i T_i,\ \ |T_i| < r \right\} \right\}\ .$$

Since $C \subset \Delta_n = \bigcup_{j=1}^{b_n} U_j^{(n)}$ holds true for any n and since for every $r > 0$ there exists N_r such that $1/b_n^n < r$ for all $n \geq N_r$, exploiting (II.3.5), it trivially follows that

$$\begin{aligned} \inf \left\{ \sum_i |T_i|^\varepsilon \ :\ C \subseteq \bigcup_i T_i,\ \ |T_i| < r \right\} &\leq \inf \left\{ \sum_{j=1}^{b_n} |U_j^{(n)}|^\varepsilon \ :\ \ n \geq N_r \right\} \\ &\leq \inf \left\{ \frac{1}{b_n^{\epsilon n - 1}} \ :\ \ n \geq N_r \right\} = 0\ , \end{aligned}$$

hence $\mathcal{H}^\varepsilon(C) = 0$ for any $\epsilon > 0$ fixed. Therefore, recalling that the *Hausdorff dimension* of C is defined as

$$\dim_H(C) := \inf\{d \geq 0\ :\ \mathcal{H}^d(C) = 0\}\ ,$$

we conclude that

$$\dim_H(C) = 0 \ .$$

This last observation, together with the construction of the proper holomorphic embedding $\gamma\colon \overline{\mathbb{C}}\setminus C \hookrightarrow \mathbb{C}^2$ made in Section II.4.2, completes the proof of Theorem II.1.1.

II.4.4 Two further characterizations of C

The original definition of the Cantor set C given in [5] is

$$C = \gamma^{-1}(\infty_2) \ . \tag{II.4.6}$$

Actually, γ is not defined on the Cantor set C, therefore, for this last relation to make sense, we need to extend γ on C. Being C a Cantor set in $\overline{\mathbb{C}}$, it is the boundary of $\overline{\mathbb{C}} \setminus C$, domain on which γ is proper, therefore

$$\lim_{w\to C, w\in\overline{\mathbb{C}}\setminus C} |\gamma(w)| = \infty \ ,$$

in fact if $z \in C$, being C a Cantor set, z is an accumulation point of the complement of C, which is $\bigcup_n K_n$, thus z can be hit by a sequence $\{z_n\}_n \subset \bigcup_n K_n$; by Remark II.4.1 let $z_n \in K_n \setminus K_{n-1}$. On one hand it is clear that $\lim_j \gamma_{n+j}(z_n) = \gamma(z_n)$ for every n, on the other hand (II.4.4) implies $\lim_n \lim_j \gamma_{n+j}(z_n) = \infty_2$. So we can extend γ on C by continuity defining $\gamma \equiv \infty_2$ on C. Since trivially $\gamma \neq \infty_2$ on $\overline{\mathbb{C}} \setminus C$, we have that defining C as in (II.4.6) matches with the definition (II.4.1). We conclude by characterizing C in one last way:

$$C = \mathrm{cl}\left(\bigcup_{n\geq 1} \gamma_n^{-1}(\infty_2)\right) \ . \tag{II.4.7}$$

Being C closed, (II.4.2) implies that

$$\mathrm{cl}\left(\bigcup_{n\geq 1} \gamma_n^{-1}(\infty_2)\right) \subseteq C \ .$$

Consider now the following inductive procedure: at step n we have b_n points, the elements of $\gamma_n^{-1}(\infty_2)$, and b_n neighborhoods $U_j^{(n)}$ of them. At step $n+1$ we have the old b_n points and k_n new points (that is, the elements of $\gamma_{n+1}^{-1}(\infty_2)$) and $b_{n+1} = b_n + k_n$ neighborhoods $U_j^{(n+1)}$ of them, b_n of which are just the $U_j^{(n)}$ shrunk around the points of $\gamma_n^{-1}(\infty_2)$ and the k_n remaining ones are neighborhoods of the new points $\gamma_{n+1}^{-1}(\infty_2) \setminus \gamma_n^{-1}(\infty_2)$. Now $\cup_{j=1}^{b_n} U_j^{(n)} = \Delta_n$ forms a decreasing sequence of open sets, whose intersection defines C and all the $U_j^{(n)}$ eventually shrink to the point they are neighborhood of, hence C is the limit of the above inductive procedure. Therefore, given $w \in C$, either it appears at some finite step of the procedure (hence $w \in \gamma_n^{-1}(\infty_2)$ for some n), or it appears in the limit: by construction, the points tend to accumulate (in fact there is no isolated point, see (ii)$_C$), that is $w \in \operatorname{Acc}\left(\bigcup_{n\geq 1} \gamma_n^{-1}(\infty_2)\right) \setminus \left(\bigcup_{n\geq 1} \gamma_n^{-1}(\infty_2)\right)$. So $w \in \operatorname{cl}\left(\bigcup_{n\geq 1} \gamma_n^{-1}(\infty_2)\right)$ and we are done.

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Author address

G. D. Di Salvo University of Oslo, Postboks 1337 Blindern, 0316 Oslo, Norway, giovandd@math.uio.no

Paper III

Families of Proper Holomorphic Embeddings and Carleman–type Theorems with parameters

G. D. Di Salvo, T. Ritter, and E. F. Wold

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Abstract

We solve the problem of simultaneously embedding properly holomorphically into $\mathbb{C}^2$ a whole family of n–connected domains $\Omega_r \subset \mathbb{P}^1$ such that none of the components of $\mathbb{P}^1 \setminus \Omega_r$ reduces to a point, by constructing a continuous mapping $\Xi\colon \bigcup_r \{r\} \times \Omega_r \to \mathbb{C}^2$ such that $\Xi(r,\cdot)\colon \Omega_r \hookrightarrow \mathbb{C}^2$ is a proper holomorphic embedding for every r. To this aim, a parametric version of both the Andersén–Lempert procedure and Carleman's Theorem is formulated and proved.

Contents

III

III.1 Introduction

Existence of proper holomorphic embeddings of Riemann surfaces $\mathcal{R}$ into 2–dimensional complex manifolds X, e.g., $X = \mathbb{C}^2$, with prescribed geometrical properties, e.g., being complete, has been an active area of research over the recent years. Various techniques have been developed, but in several cases, positive results have been obtained only at the cost of perturbing the complex structure of $\mathcal{R}$ (see Černe–Forstnerič [4], Alarcón [1] and Alarcón–López [2]). It can be hoped, however, that if you let r be a local parameter on the moduli space of Riemann surfaces of a given type, and you perform various constructions continuously with the parameter r near a given point r_0, then you will get a perturbation of the complex structure for each given r, but at least one perturbation will correspond to your initial r_0. Indeed this is the philosophy behind the embedding results of Globevnik–Stensønes [7]. The purpose of this article is to take a first step towards results of this type that may be generalized to larger classes of Riemann surfaces.

We will consider the following. It is known that any n–connected domain Ω in the Riemann sphere may be mapped univalently onto a domain in the Riemann sphere whose complement consists of n parallel disjoint slits with a given inclination Θ to the real axis. The univalent map achieving this is uniquely determined by Θ and the choice of a certain normalization of the Laurent series expansion at a chosen point $\zeta \in \Omega$ being sent to ∞ (see Goluzin, [8], page 213). Considering a continuous family of n–connected domains, we obtain a continuously varying family of uniformizing slit–maps.

Let $C_j \subset \mathbb{C}$ be compact disks and $I_j \subset \mathbb{R}_{>0}$ be compact intervals, $j = 1, \ldots, n$. Set $B_j := C_j \times I_j$ and $B := B_1 \times \cdots \times B_n$. Let $r = ((a_1, b_1), \ldots, (a_n, b_n))$ denote the coordinates on B, and letting $l_{r,j}$ denote the closed straight line segment which is parallel to the real axis with right end–point $a_j(r)$ and of length $b_j(r)$, we assume that $L_r := \{l_{r,1}, \ldots, l_{r,n}\}$ is a set of pairwise disjoint slits, and thus $\mathbb{P}^1 \setminus L_r$ is an n–connected domain, none of whose boundary components are isolated points. After possibly having to apply the map $z \mapsto (z - a_1(r))/b_1(r)$ we may assume that for all r we have that $l_{r,1} = [-1, 0] \subset \mathbb{C}$.

The goal is to prove the following.

Theorem III.1.1. In $B \times \mathbb{P}^1$ set

$$\Omega = (B \times \mathbb{P}^1) \setminus (\bigcup_{r \in B} \{r\} \times L_r).$$

Then there exists a continuous map $\Xi \colon \Omega \to \mathbb{C}^2$ such that for each $r \in B$ we have that $\Xi(r, \cdot) \colon \Omega_r \to \mathbb{C}^2$ is a proper holomorphic embedding.

III.2 The Setup

We will now introduce a setup to prove Theorem III.1.1. First, we need the notion of a certain directed family of curves.

Let $C > 0$ and $R > 1$. Let Γ denote the half line $\Gamma = \{x \in \mathbb{R} \subset \mathbb{C} : x \geq R - 1\}$, let $B \subset \mathbb{R}^m$ be a compact set, and denote by (r, x) the coordinates on $B \times \Gamma$. Let $h, h' = \frac{\partial h}{\partial x} \in \mathscr{C}(B \times \Gamma)$, and assume that

$$|h(r, x)| < \frac{C}{2} \quad , \quad |h'(r, x)| < \frac{1}{2}.$$

Definition III.2.1. Let $\theta \in [0, 2\pi)$. Then the set of curves

$$e^{i\theta} \cdot \{x + ih(r, x) \ : \ r \in B, \ x \in \Gamma\}$$

is referred to as being θ*–directed*, and subordinate to R, C. A family of curves is said to be θ*–directed* if it is θ–directed subordinate to R, C for sufficiently large R, C.

With the notation in the previous section, set $\psi(z) := \frac{1}{z} + 1$, $\lambda_{r,j} := \psi(l_{r,j})$, $c_j(r) := \psi(a_j(r))$. Then $\Lambda_r := \{\lambda_{r,1}, \ldots, \lambda_{r,n}\}$ is a set of disjoint slits in $\mathbb{P}^1$, where $\lambda_{r,1}$ is the negative real axis and $\lambda_{r,j}$ are circular slits (or possibly straight line segments along the real axis) for $j = 2, \ldots, n$. We set $e^{i\theta_{r,j}} := \psi'(a_j(r))/|\psi'(a_j(r))|$, i.e., we have that $e^{i\theta_{r,j}}$ is a unit tangent to the circle $\Lambda_{r,j}$ on which $\lambda_{r,j}$ lies at the point $c_j(r)$. Setting $\alpha_{r,j}(z) := e^{-i\theta_{r,j}}(z - c_j(r))$ we have that $\alpha_{r,j}(\Lambda_{r,j})$ is a circle which is tangent to the real axis at the origin, and we let $\kappa_{r,j}$ denote the signed curvature of this circle; positive if the circle is in the upper half plane, negative if the circle is in the lower half plane, and zero if the circle is the real axis.

Proposition III.2.1. Fix $j \in \{2, \ldots, n\}$ and suppose that $g_{r,j} \in \mathcal{O}(\triangle_\delta(c_j(r)))$ is a continuous family of functions, for $r \in B$. Let $\theta \in [0, 2\pi)$, and set

$$\varphi_j(r, z) := \frac{e^{i\theta}}{\alpha_{r,j}(z)} + g_{r,j}(z).$$

Then the family Γ_j of curves $\varphi(r, \lambda_{r,j})$ is $(\theta - \pi)$–directed.

Proof. It suffices to prove this for $\theta = 0$. Then $\alpha_{r,j}(\Lambda_{r,j})$ is parametrized near the origin by

$$\eta_{r,j}(x) = x + i\frac{\kappa_{r,j}}{2}x^2 + O(x^4).$$

Set $\tilde{g}_{r,j}(z) = g_{r,j}(\alpha_{r,j}^{-1}(z))$ We have that

$$\begin{aligned}
\varphi_j(r, x) &= \frac{1}{x + i\frac{\kappa_{r,j}}{2}x^2 + O(x^4)} + \tilde{g}_{r,j}(\eta_{r,j}(x)) \\
&= \frac{x - i\frac{\kappa_{r,j}}{2}x^2 + O(x^4)}{x^2 + O(x^4)} + \tilde{g}_{r,j}(\eta_{r,j}(x)) \\
&= \left(\frac{1}{x} - i\frac{\kappa_{r,j}}{2} + O(x^2)\right)(1 + O(x^2)) + \tilde{g}_{r,j}(\eta_{r,j}(x)) \\
&= \frac{1}{x} - i\frac{\kappa_{r,j}}{2} + O(x) + \tilde{g}_{r,j}(\eta_{r,j}(x)).
\end{aligned}$$

Since $g_{r,j}(z)$ is close to a constant when z is close to $c_j(r)$, the uniform bound in the definition of $(-\pi)$–directed holds. Now

$$\varphi_j'(r, x) = \frac{-1}{x^2} + v_{r,j}(x),$$

where $v_{r,j}(x)$ is bounded and scaling it to have almost unit length we see

$$x^2\varphi_j'(r, x) = -1 + x^2 v_{r,j}(x).$$

∎

Proposition III.2.2. Fix $\theta_2, \dots, \theta_n \in (0, 2\pi)$. Define $\phi_r \colon \mathbb{C} \setminus \{c_2(r), \dots, c_n(r)\} \to \mathbb{C}^2$ by

$$\phi_r(z) := \left(z, \sum_{j=2}^{n} \frac{e^{i\theta_j}}{\alpha_{r,j}(z)} \right).$$

Choose $\delta > 0$ small, and let $a, b \in \triangle_\delta(1/\sqrt{2})$, and set $A_{a,b}(z,w) := (az + bw, -bz + aw)$. Write $a = r_a e^{i\vartheta_a}, b = r_b e^{i\vartheta_b}$. Then the family Γ_1 defined by $\Gamma_1 = \{\pi_1 \circ A_{a,b} \circ \phi_r(\lambda_{r,1}) : r \in B\}$ is $(\vartheta_a - \pi)$–directed, and each family $\Gamma_j, j = 2, \dots, n$, defined by $\Gamma_j = \{\pi_1 \circ A_{a,b} \circ \phi_r(\lambda_{r,j}) : r \in B\}$ is $(\vartheta_b + \theta_j - \pi)$–directed.

Proof. For $j = 2, \dots, n$ this is just Proposition III.2.1 since for any fixed j we have that $\pi_1 \circ A_{a,b} \circ \phi_r(\lambda_{r,j})$ is parametrized by

$$\frac{r_b e^{i(\vartheta_b + \theta_j)}}{\alpha_{r,j}(z)} + \sum_{k \neq j} \left(\frac{r_b e^{i(\vartheta_b + \theta_k)}}{\alpha_{r,k}(z)} \right) + r_a e^{i\vartheta_a} z.$$

For $j = 1$ this is because $\pi_1 \circ A_{a,b} \circ \phi_r(\lambda_{r,j})$ is parametrized by $r_a e^{i\vartheta_a} z + g_r(z)$ where $g_r(z)$ is uniformly comparable to $\frac{1}{z}$. ■

III.3 Carleman approximation with parameters

We will start by introducing some notation. Afterwards, we present Theorem III.3.1, a Carleman–type theorem (see e.g., [5]), which is the main result of the present section: families of smooth functions holomorphic on a disc can be approximated by entire functions on a smaller disc and on the union of several Lipschitz curves. The proof is obtained applying inductively Corollary III.3.1, which in turn easily follows from Proposition III.3.1, a tool that allows to approximate smooth functions on compact pieces of a Lipschitz curve; Corollary III.3.1 extends the result to several curves. Proposition III.3.1 relies on three technical lemmata that will be presented in Section III.3.3.

III.3.1 The setup

Recall that $R > 1$, Γ is the half line $\Gamma := \{x \in \mathbb{R} \subset \mathbb{C} \ : \ x \geq R - 1\}$, $B \subset \mathbb{R}^m$ is a compact and (r, x) are the coordinates on $B \times \Gamma$. For $k = 1, \dots, n$ let $h_k, h_k' = \frac{\partial h_k}{\partial x} \in \mathscr{C}(B \times \Gamma)$ be such that

$$|h_k(r,x)| < \frac{C}{2} \ \ , \ \ |h_k'(r,x)| < \frac{1}{2} \tag{III.3.1}$$

for some $C > 0$, for every $(r, x) \in B \times \Gamma$, and every $k = 1, \dots, n$. Then, setting $l = 1/2$, we have that

$$|h_k(r, x_1) - h_k(r, x_2)| \leq l|x_1 - x_2|, \ \ \forall x_1, x_2 \in \Gamma, \ r \in B \ , \tag{III.3.2}$$

so h_k is l–Lipschitz and in this way we also call its graph. Let $0 = \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ and define the Lipschitz curves

$$\Gamma_{k,r} := e^{i\theta_k} \cdot \{x + ih_k(r, x) \ : \ x \in \Gamma\}$$

and their union

$$\Gamma_r := \bigcup_{k=1}^{n} \Gamma_{k,r} \ .$$

If $D \subseteq \Omega \subseteq \mathbb{C}$ are domains, a useful notation is given by setting

$$\mathcal{P}(B, \Omega, D) := \{f \in \mathscr{C}(B \times \Omega) \ : \ f(r, \cdot) \in \mathcal{O}(D) \ \ \forall r \in B\}$$

and

$$\mathcal{P}(B, \Omega) := \mathcal{P}(B, \Omega, \Omega) \ .$$

Theorem III.3.1 (Carleman–type Theorem with parameters)**.** Assume that $f \in \mathcal{P}(B, \mathbb{C}, \overline{\triangle}_{\rho+3+\frac{3C}{2}})$ for some $\rho > R$. Then for any $\epsilon \in \mathscr{C}(\mathbb{C}), \epsilon > 0$, there exists $g \in \mathcal{P}(B, \mathbb{C})$ such that

$$|g(r, z) - f(r, z)| < \epsilon(z)$$

for all $z \in \overline{\triangle}_\rho \cup \Gamma_r, \ \ r \in B$.

III.3.2 Proof of Theorem III.3.1

Fix $j \in \mathbb{N}$, $j \geq R$ and let b be some real number such that

$$j + 3 + \frac{3C}{2} < b \ .$$

For $\rho \geq C$ set

$$\psi(\rho) := \arcsin \frac{C}{\rho}$$

and define

$$S_\rho := \{se^{i\theta} : 0 < s < \infty, |\theta| < \psi(\rho)\} \quad \text{and} \quad A_{\rho,b} := \triangle_b \setminus \overline{S_\rho}.$$

Then S_ρ is the wedge in the right half–plane bounded by the straight lines passing through the origin and the intersection between $\partial\triangle_\rho$ and the lines $y = \pm C$. Up to consider a larger R, we assume $e^{i\theta_j} S_\rho \cap e^{i\theta_k} S_\rho = \emptyset$ for all $j \neq k$ for $\rho \geq R$. We define further the following sets

$$\begin{aligned}
\omega_1 &:= \{z = x + iy : j + 1 < x, |z| < b, |y| < C\} \\
\omega_2 &:= \{z = x + iy : 0 < x < j + 2, |y| < C\} \cup A_{j,b} \\
\Omega &:= \omega_1 \cup \omega_2
\end{aligned}$$

Given $\delta > 0$, we will denote the open δ–neighborhood of D as

$$D(\delta) := \{z \in \mathbb{C} \ : \ d(z, D) < \delta\} .$$

The following proposition, or rather its corollary below, is the main technical ingredient in the proof of the Carleman Theorem III.3.1. The proposition follows from Lemma III.3.1, Lemma III.3.2, and finally Lemma III.3.3 below.

Proposition III.3.1. Assume that $n = 1$. Let $\alpha\colon \bigcup_{r\in B}\{r\} \times \Gamma_r \to \mathbb{C}$ be continuous such that $\alpha(r,\cdot) \in \mathscr{C}_c(\Gamma_r)$ for every $r \in B$, with

$$\operatorname{supp}\alpha(r,\cdot) \subset \{z = x+iy \in \Gamma_r \ : \ j+3+\frac{3C}{2} < x, |z| < b, \ |y| < C/2\} \quad \forall r \in B.$$

Then for every $\epsilon > 0$ there exists $\{Q_t\}_{t>0} \subset \mathcal{P}(B, \mathbb{C})$ such that

$$\|\alpha(r,\cdot) - Q_t(r,\cdot)\|_{\Gamma_r \cap \overline{\triangle}_b} < \epsilon \tag{III.3.3}$$

for every $r \in B$, $0 < t < t_0$, and

$$Q_t \to 0 \ \text{ as } \ t \to 0 \tag{III.3.4}$$

uniformly on $B \times \omega_2(\delta)$, for some $\delta > 0$.

Corollary III.3.1. Let $\alpha\colon \bigcup_{r\in B}\{r\} \times \Gamma_r \to \mathbb{C}$ be continuous such that $\alpha(r,\cdot) \in \mathscr{C}_c(\Gamma_r)$ for every $r \in B$, with

$$\operatorname{supp}\alpha(r,\cdot) \subset \{z \in \Gamma_r \ : \ j+3+\frac{3C}{2} < |z| < b\} , \ \ \forall r \in B .$$

Then for every $\epsilon > 0$ there exists $\{Q_t\}_{t>0} \subset \mathcal{P}(B, \mathbb{C})$ such that

$$\|\alpha(r,\cdot) - Q_t(r,\cdot)\|_{\Gamma_r \cap \overline{\triangle}_b} < \epsilon \tag{III.3.5}$$

for every $r \in B$, $0 < t < t_0$, and

$$Q_t \to 0 \ \text{ as } \ t \to 0 \tag{III.3.6}$$

uniformly on $B \times \overline{\triangle}_j(\delta)$, for some $\delta > 0$.

Proof. On $e^{-i\theta_k}\Gamma_{k,r}$ define $\alpha_k(r,z) := \alpha(r, e^{i\theta_k}z)$. Using the proposition we obtain approximations $Q_{t,k}(r,z)$. Then setting

$$Q_t(r,z) := \sum_{k=1}^{n} Q_{t,k}(r, e^{-i\theta_k}z)$$

will yield the result for sufficiently small t. ∎

Proof of Theorem III.3.1: The proof is by induction on $k \geq 0$, and the induction hypothesis is the following. For every $j = 0, \ldots, k$ there exist:

(i) $g_j \in \mathcal{P}(B, \mathbb{C}, \overline{\triangle}_{\rho+j+3+\frac{3C}{2}})$,

(ii) $|g_j(r,z) - f(r,z)| < \epsilon(z)/2$ for all $z \in \overline{\triangle}_\rho \cup \Gamma_r$, $r \in B$, and

(iii) $\|g_j - g_{j-1}\|_{B \times \overline{\triangle}_{\rho+j-1}} < 2^{-j}$ for $j \geq 1$.

We start by setting $g_0 := f$; then in the case $k = 0$ we see that (i), (ii) hold, and (iii) is void. Assume now that the induction hypothesis holds for some $k \geq 0$. Fix $\eta > 0$ such that

$$g_k(r, \cdot) \in \mathcal{O}(\overline{\triangle}_{\eta+\rho+k+3+\frac{3C}{2}}),$$

and choose a cutoff function $\chi \in \mathscr{C}^\infty(\mathbb{C})$ such that $0 \leq \chi \leq 1$, such that $\chi = 0$ near $\overline{\triangle}_{\rho+k+3+\frac{3C}{2}}$, and $\chi = 1$ outside $\overline{\triangle}_{\eta+\rho+k+3+\frac{3C}{2}}$. Now g_k may be approximated on $\overline{\triangle}_{\eta+\rho+k+3+\frac{3C}{2}}$ to arbitrary precision by $h_k \in \mathscr{C}(B)[z]$ using Taylor series expansion, and so

$$h_k + \chi \cdot (g_k - h_k) =: h_k + \alpha_k$$

approximates g_k to arbitrary precision. Hence it suffices to approximate α_k to arbitrary precision by a suitable function. Multiplying α_k by a suitable cutoff function so that Corollary III.3.1 applies, we have that α_k may be approximated to arbitrary precision on

$$\bigcup_r \Gamma_r \cap \overline{\triangle}_{\rho+k+2+3+\frac{3C}{2}}$$

by a function $Q_k \in \mathcal{P}(B, \mathbb{C})$ which is arbitrarily small on $\overline{\triangle}_{\rho+k}$. Setting then $g_{k+1} := h_k + Q_k + \tilde{\chi} \cdot (\alpha_k - Q_k)$ where $\tilde{\chi}$ is a third cutoff function such that $\tilde{\chi} = 0$ near $\overline{\triangle}_{\rho+k+1+3+\frac{3C}{2}}$ and $\tilde{\chi} = 1$ near $\mathbb{C} \setminus \triangle_{\rho+k+2+3+\frac{3C}{2}}$, completes the induction step. We may finish the proof of Theorem III.3.1 by setting $g := \lim_{j\to\infty} g_j$, which exists by (iii), and the approximation holds by (ii). ■

III.3.3 Lemmata: Mergelyan–type and Runge's Theorems with parameters

The three lemmata we present and prove in this section are fundamental ingredients to formulate a Mergelyan–type Theorem (see e.g., [5]). The first one of them generalizes a theorem claimed and proved by P. Manne in his Ph.D. thesis [10] and is about the holomorphic (entire) approximation of a family of smooth functions, each of which is defined on a Lipschitz curve in the complex plane.

Lemma III.3.1. Assume that $n = 1$, and let α be as in Proposition III.3.1. Then for every $\epsilon > 0$ there exists $\{H_t\}_{t>0} \subset \mathcal{P}(B, \mathbb{C})$ such that

$$\|\alpha(r, \cdot) - H_t(r, \cdot)\|_{\Gamma_r} < \epsilon \tag{III.3.7}$$

for every $r \in B$, $0 < t < t_0$, for some $t_0 > 0$, and

$$H_t \to 0 \text{ as } t \to 0 \tag{III.3.8}$$

uniformly on $B \times (\omega_1 \cap \omega_2)(\delta)$, for some $\delta > 0$.

Proof. Extend h_1 to a function h on the whole real line by setting $h(r, x) := h_1(r, x)$ for $x \geq R - 1$ and $h(r, x) := h_1(r, 2R - 2 - x)$ for $x < R - 1$. Define $S_r := \{s + ih(r, s) \ : \ s \in \mathbb{R}\}$. Denote by $z = x + ih(r, x)$ a point in Γ_r and let $\zeta = \zeta(r, s) = s + ih(r, s)$ be a parametrization of S_r. Further, $\zeta'(r, s) = \frac{\partial \zeta}{\partial s}(r, s)$, extend $\alpha(r, \cdot)$ to $S_r \setminus \Gamma_r$ to be 0 for all $r \in B$ and define

$$\begin{aligned} H_t(r, z) &:= \int_{S_r} \alpha(r, \zeta) K_t(\zeta, z)\, d\zeta \\ &= \int_{\mathbb{R}} \alpha(r, \zeta(r, s)) K_t(\zeta(r, s), z) \zeta'(r, s)\, ds \end{aligned}$$

for $t > 0, r \in B, z \in \mathbb{C}$, where

$$K_t(\zeta, z) := \frac{1}{t\sqrt{\pi}} e^{-\frac{(\zeta - z)^2}{t^2}}$$

is the Gaussian kernel.

We start by proving (III.3.8). Let $z = x + iy \in (\omega_1 \cap \omega_2)(\delta)$. We have that

$$\begin{aligned} |H_t(r, z)| &\leq \frac{1}{t\sqrt{\pi}} \int_{\mathbb{R}} |\alpha(r, \zeta(r, s))| e^{-\frac{(s-x)^2 - (h(r,s)-y)^2}{t^2}} |\zeta'(r, s)|\ ds \\ &= \frac{1}{t\sqrt{\pi}} \int_{j+3+\frac{3}{2}C<s<b} |\alpha(r, \zeta(r, s))| e^{-\frac{(s-x)^2 - (h(r,s)-y)^2}{t^2}} |\zeta'(r, s)|\ ds\ , \end{aligned}$$

and $(s - x)^2 - (h(r, s) - y)^2 \geq (1 + \frac{3C}{2} - \delta)^2 - (\frac{3C}{2})^2$, therefore (III.3.8) follows.

For any fixed $\eta > 0$ we split S_r as

$$\begin{aligned} S_r^{(1)} &:= \{\zeta \in S_r \ : \ |\Re(\zeta - z)| \leq \eta\} = \{\zeta(r, s) \ : \ |s - x| \leq \eta\} \\ S_r^{(2)} &:= \{\zeta \in S_r \ : \ |\Re(\zeta - z)| > \eta\} = \{\zeta(r, s) \ : \ |s - x| > \eta\}\ . \end{aligned}$$

Since by (III.3.2) we have

$$|K_t(\zeta, z)| \leq \frac{1}{t\sqrt{\pi}} e^{-\frac{(s-x)^2(1-l^2)}{t^2}}\ ,$$

we immediately get the following upper bound:

$$
\begin{aligned}
\int_{S_r^{(1)}} |K_t(\zeta,z)|\, d|\zeta| \leq & \frac{1}{t\sqrt{\pi}} \int_{x-\eta}^{x+\eta} e^{-\frac{(s-x)^2(1-l^2)}{t^2}}\, ds \\
= & \frac{1}{\sqrt{\pi(1-l^2)}} \int_{|u|\leq \frac{\sqrt{1-l^2}}{t}\eta} e^{-u^2}\, du \\
\leq & \frac{1}{\sqrt{1-l^2}}\ , \qquad \text{(III.3.9)}
\end{aligned}
$$

which holds for every $z \in \Gamma_r$, $r \in B$ and $t > 0$. Similarly, one sees that for all $\epsilon > 0, \eta > 0$ there exists $t_0 > 0$ such that

$$
\int_{S_r^{(2)}} |K_t(\zeta,z)|\, d|\zeta| \leq \frac{1}{\sqrt{\pi(1-l^2)}} \int_{|u|> \frac{\sqrt{1-l^2}}{t}\eta} e^{-u^2}\, du < \epsilon \qquad \text{(III.3.10)}
$$

for every $z \in \Gamma_r,\ r \in B,\ 0 < t < t_0$. We need one last property of the kernel, that is

$$
\int_{S_r} K_t(\zeta,z)\, d\zeta = 1 \qquad \text{(III.3.11)}
$$

for all $z \in \Gamma_r$, $r \in B$ and $t > 0$. Let us consider the function

$$
F(z) := \int_{S_r} K_t(\zeta,z)\, d\zeta = \frac{1}{t\sqrt{\pi}} \int_{S_r} e^{-\frac{(\zeta-z)^2}{t^2}}\, d\zeta
$$

which is holomorphic entire. Let $z = x \in \mathbb{R}$ and define for $T > 0$

$$
\begin{aligned}
A(T) :=& \{u + i0\ :\ -T \leq u \leq T\}\ , \\
S_r(T) :=& \{\zeta \in S_r\ :\ -T \leq s \leq T\}\ ,
\end{aligned}
$$

and let $\rho_r^{\pm}(T)$ be the straight line segment between $\pm T$ and $\pm T + ih(r, \pm T)$. Set

$$
\gamma_r(T) := A(T) + \rho_r^{+}(T) - S_r(T) - \rho_r^{-}(T)
$$

which is a piecewise $\mathscr{C}^1$–smooth closed curve which is nullhomotopic, hence we get

$$
\frac{1}{t\sqrt{\pi}} \int_{\gamma_r(T)} e^{-(\frac{\zeta-x}{t})^2}\, d\zeta = 0
$$

for every $t > 0,\ r \in B,\ x \in \mathbb{R}$ and $T > 0$. On the other hand

$$
\begin{aligned}
\frac{1}{t\sqrt{\pi}} \int_{\gamma_r(T)} e^{-(\frac{\zeta-x}{t})^2}\, d\zeta = \frac{1}{t\sqrt{\pi}} \Bigg(& \int_{A(T)} e^{-(\frac{u-x}{t})^2}\, du + \int_{\rho_r^{+}(T)} e^{-(\frac{\zeta-x}{t})^2}\, d\zeta \\
& - \int_{S_r(T)} e^{-(\frac{\zeta-x}{t})^2}\, d\zeta - \int_{\rho_r^{-}(T)} e^{-(\frac{\zeta-x}{t})^2}\, d\zeta \Bigg)\ .
\end{aligned}
$$

Passing to the limit as $T \to +\infty$, the vertical contributions vanish (as h is bounded), while

$$\frac{1}{t\sqrt{\pi}} \int_{A(T)} e^{-(\frac{u-x}{t})^2}\, du \longrightarrow \frac{1}{t\sqrt{\pi}} \int_{\mathbb{R}} e^{-(\frac{u-x}{t})^2}\, du = 1$$

and

$$\frac{1}{t\sqrt{\pi}} \int_{S_r(T)} e^{-(\frac{\zeta-x}{t})^2}\, d\zeta \longrightarrow \frac{1}{t\sqrt{\pi}} \int_{S_r} e^{-(\frac{\zeta-x}{t})^2}\, d\zeta = F(x)$$

for every $t > 0$, $r \in B$ and $x \in \mathbb{R}$. This implies that the entire function F is identically 1 on the real line for every $t > 0$ and $r \in B$, so by the identity principle it is constantly 1 on the whole $\mathbb{C}$; in particular (III.3.11) holds true.

We gathered all the ingredients to prove (III.3.7). Let $\epsilon > 0$, let $\eta > 0$ such that $|\alpha(r,\zeta) - \alpha(r,z)| < \epsilon$ for all $z \in \Gamma_r$, $\zeta \in S_r^{(1)}$, for all $r \in B$. Then

$$\begin{aligned}
|H_t(r,z) - \alpha(r,z)| =& \left| \int_{S_r^{(1)}} \alpha(r,\zeta) K_t(\zeta,z)\, d\zeta + \int_{S_r^{(2)}} \alpha(r,\zeta) K_t(\zeta,z)\, d\zeta - \alpha(r,z) \right| \\
=& \left| \int_{S_r^{(1)}} \alpha(r,\zeta) K_t(\zeta,z)\, d\zeta + \right. \\
& \left. \qquad \int_{S_r^{(2)}} \alpha(r,\zeta) K_t(\zeta,z)\, d\zeta - \alpha(r,z) \int_{S_r} K_t(\zeta,z)\, d\zeta \right| \\
\leq& \left| \int_{S_r^{(1)}} (\alpha(r,\zeta) - \alpha(r,z)) K_t(\zeta,z)\, d\zeta \right| + \\
& \qquad \left| \int_{S_r^{(2)}} (\alpha(r,\zeta) - \alpha(r,z)) K_t(\zeta,z)\, d\zeta \right| \\
\leq& \epsilon \int_{S_r^{(1)}} |K_t(\zeta,z)|\, d|\zeta| + \\
& \qquad \Big(\|\alpha(r,\cdot)\|_{S_r^{(2)}} + |\alpha(r,z)| \Big) \int_{S_r^{(2)}} |K_t(\zeta,z)|\, d|\zeta| \\
\leq& \frac{\epsilon}{\sqrt{1-l^2}} + 2\epsilon \|\alpha(r,\cdot)\|_{S_r}\ ,
\end{aligned}$$

where the second equality follows from (III.3.11) and the last inequality follows from (III.3.9) and (III.3.10). So we can conclude, since this last quantity can be taken arbitrarily small for ϵ small, uniformly in $z \in \Gamma_r$ and $r \in B$, for all $0 < t < t_0$, where t_0 comes from (III.3.10). ■

The following Lemma shows how to modify the approximation constructed in Lemma III.3.1, so that, besides approximating the given smooth function, it becomes arbitrarily small on a suitable region. The price to pay is that the approximation obtained this way is no more entire; we will get "entireness" back with Lemma III.3.3, that is a parametric version of Runge's Theorem (see e.g., [5]).

Lemma III.3.2. Assume $n = 1$, and let α be as in Proposition III.3.1. Then for every $\epsilon > 0$ there exists $\{\xi_t\}_{t>0} \subset \mathcal{P}(B, \Omega(\delta))$ such that

$$\|\alpha(r,\cdot) - \xi_t(r,\cdot)\|_{\Gamma_r \cap \overline{\triangle}_b} < \epsilon \tag{III.3.12}$$

for every $r \in B$, $0 < t < t_0$, and

$$\xi_t \to 0 \ \text{ as } \ t \to 0 \tag{III.3.13}$$

uniformly on $B \times \omega_2(\delta)$, for some $\delta > 0$.

Proof. Let H_t be the map defined in Lemma III.3.1 and $\phi_i\colon \Omega(\delta) \to [0,1]$ be smooth, such that

(i) $\operatorname{supp} \phi_i \subset \omega_i(\delta)$, and

(ii) $\phi_1 + \phi_2 \equiv 1$ on $\Omega(\delta)$

for some $\delta > 0$. Define

$$g_{t,1}(r,z) := -H_t(r,z)\phi_2(z), \ \ g_{t,2}(r,z) := H_t(r,z)\phi_1(z)$$

on $B \times \Omega(\delta)$; then $g_{t,i}(r,\cdot)$ is a smooth function on $\Omega(\delta)$ and

$$g_{t,2} - g_{t,1} = H_t \tag{III.3.14}$$

holds true on $B \times \Omega(\delta)$, therefore $\frac{\partial g_{t,1}}{\partial \bar z}(r,z)$ and $\frac{\partial g_{t,2}}{\partial \bar z}(r,z)$ are the same function; call it v_t and consider $v_t(r,\cdot)$ smoothly extended on $\overline{\Omega(\delta)}$. Hence, defining

$$u_t(r,z) := \frac{1}{2\pi i} \iint_{\Omega(\delta)} \frac{v_t(r,\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta \ ,$$

we can assume without loss of generality to have smoothed the corners of ω_i so that $\Omega(\delta)$ is smoothly bounded, hence we are allowed to apply Theorem 2.2 in [3], which ensures that $u_t(r,\cdot)$ is smooth on $\overline{\Omega(\delta)}$ for every $r \in B$ and solves $\frac{\partial u_t}{\partial \bar z} = v_t$ on $B \times \overline{\Omega(\delta)}$, hence

$$\Theta_{t,i} := g_{t,i} - u_t \in \mathcal{P}(B, \Omega(\delta)) \ , \ \ i = 1,2 \ . \tag{III.3.15}$$

Then (III.3.8) and (i) imply

- $g_{t,i} \to 0$ uniformly on $P \times \omega_i(\delta)$ as $t \to 0$,
- $\frac{\partial g_{t,1}}{\partial \bar z}(r,z) = -H_t(r,z)\frac{\phi_2}{\partial \bar z}(z) \to 0$ uniformly on $B \times \omega_1(\delta)$ as $t \to 0$, and
- $\frac{\partial g_{t,2}}{\partial \bar z}(r,z) = H_t(r,z)\frac{\phi_1}{\partial \bar z}(z) \to 0$ uniformly on $B \times \omega_2(\delta)$ as $t \to 0$.

The last two imply $u_t \to 0$ uniformly on $B \times \Omega(\delta)$, hence

$$\Theta_{t,i} \to 0 \tag{III.3.16}$$

uniformly on $P \times \omega_i(\delta)$ as $t \to 0$. Since $\Theta_{t,2} - \Theta_{t,1} = H_t$ on $B \times \Omega(\delta)$, it follows from (III.3.7), (III.3.15) and (III.3.16) that

$$\xi_t := \begin{cases} H_t + \Theta_{t,1} & B \times \omega_1(\delta) \\ \Theta_{t,2} & B \times \omega_2(\delta) \end{cases}$$

satisfies the stated properties. ■

Lemma III.3.3 (Runge–type Theorem with parameters)**.** With the notation of the previous lemma, for all $\epsilon > 0$, there is $Q_t \in \mathscr{C}(B)[z]$ (polynomial with coefficients in $\mathscr{C}(B)$; in particular $\{Q_t\}_{t \geq 0} \subset \mathcal{P}(B, \mathbb{C})$) such that

$$\|\xi_t - Q_t\|_{B \times \overline{\Omega}} < \epsilon$$

for all $t > 0, r \in B$.

Proof. Observe that $\overline{\Omega}$ is compact polynomially convex. Let $\gamma = \partial\Omega(\frac{\delta}{2})$. We have that

$$(r, \zeta, z) \mapsto \frac{\xi_t(r, \zeta)}{\zeta - z}$$

is uniformly continuous on $B \times \gamma \times \overline{\Omega}$, hence for every $\epsilon > 0$ there exists $\eta > 0$, such that, dividing γ into N pieces $\gamma_1, \ldots, \gamma_N$ whose length $L(\gamma_j)$ is less than η and fixing a point $\zeta_j \in \gamma_j$ for every j,

$$\left| \frac{\xi_t(r, \zeta)}{\zeta - z} - \frac{\xi_t(r, \zeta_j)}{\zeta_j - z} \right| < \frac{1}{N} \frac{2\pi}{L(\gamma_j)} \epsilon$$

holds $\forall (r, \zeta, z) \in B \times \gamma_j \times \overline{\Omega}$. Calling $\gamma_j(1)$, $\gamma_j(0)$ the final and initial points of γ_j, for every $(r, z) \in B \times \overline{\Omega}$ one has

$$\begin{aligned} \xi_t(r, z) - \overbrace{\sum_{j=1}^{N} \frac{\gamma_j(1) - \gamma_j(0)}{2\pi i} \frac{\xi_t(r, \zeta_j)}{\zeta_j - z}}^{=:\beta_t(r,z)} &= \frac{1}{2\pi i} \int_\gamma \frac{\xi_t(r, \zeta)}{\zeta - z} \, d\zeta - \sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\gamma_j} \frac{\xi_t(r, \zeta_j)}{\zeta_j - z} \, d\zeta \\ &= \frac{1}{2\pi i} \sum_{j=1}^{N} \int_{\gamma_j} \left(\frac{\xi_t(r, \zeta)}{\zeta - z} - \frac{\xi_t(r, \zeta_j)}{\zeta_j - z} \right) d\zeta \end{aligned}$$

thus

$$\|\xi_t - \beta_t\|_{B \times \overline{\Omega}} < \epsilon \tag{III.3.17}$$

for every $t > 0$. The result now follows since each rational function $z \mapsto \frac{1}{\zeta_j - z}$ may be approximated arbitrarily well on $\overline{\Omega}$ by polynomials. ■

III.4 Andersén–Lempert Theory

We will now apply Andersén–Lempert Theory in $B\times\mathbb{C}^2$. We have that $B\subset\mathbb{R}^N$, and when talking about analytic properties of sets and functions on $B\times\mathbb{C}^2$ we will think of $B\times\mathbb{C}^2\subset\mathbb{C}^N\times\mathbb{C}^2$, $\mathbb{C}^N=\mathbb{R}^N+i\mathbb{R}^N$. For instance, by saying that $K\subset B\times\mathbb{C}^2$ is polynomially convex compact we mean polynomially convex in $\mathbb{C}^N\times\mathbb{C}^2$; this is, in fact, equivalent to K_r being polynomially convex in $\{r\}\times\mathbb{C}^2$ for each $r\in B$.

With the setup introduced in Section III.2 we now set $s_{r,j}:=\phi_r(\lambda_{r,j})$ and we set

$$S_r:=\bigcup_{j=1}^{n} s_{r,j}\ .$$

In the product space $B\times\mathbb{C}^2$ we define $S:=\{(r,(z,w)):(z,w)\in S_r\ ,\ r\in B\}$.

Proposition III.4.1. Let $K\subset(B\times\mathbb{C}^2)\setminus S$ be a compact set such that K is polynomially convex. Let $T>0$, and let $\epsilon>0$. Then there exists a continuous map $g\colon B\times\mathbb{C}^2\to\mathbb{C}^2$ such that the following hold for all $r\in B$.

(i) $g(r,\cdot)\in\operatorname{Aut}\mathbb{C}^2$,

(ii) $\|g(r,\cdot)-\operatorname{Id}\|_{K_r}<\epsilon$, and

(iii) $g(r,S_r)\subset\mathbb{C}^2\setminus T\mathbb{B}^2$.

For the following lemma we extend the map ψ defined in Section III.2 to a map $\psi\colon\mathbb{P}^1\times\mathbb{C}\to\mathbb{P}^1\times\mathbb{C}$ by setting $\psi(z,w):=(1/z+1,w)$, and we extend the map $\phi_r(z)$ to a rational map on $\mathbb{C}^2$ by setting

$$\phi_r(z,w):=\left(z,w+\sum_{j=2}^{n}\frac{e^{i\theta_j}}{\alpha_{r,j}(z)}\right). \tag{III.4.1}$$

Moreover for $T'<T''$ we set $S_r(T',T''):=\{(z,w)\in S_r:T'\leq|(z,w)|\leq T''\}$, and we let $S(T',T''):=\bigcup_r\{r\}\times S_r(T',T'')$ which is the union over r in the product space $B\times\mathbb{C}^2$. Finally define $S(T',T'')(\delta):=\bigcup_r\{r\}\times S_r(T',T'')(\delta)$ for $\delta>0$.

Lemma III.4.1. There exist $T''>T'>>T$ arbitrarily large, $\delta>0$, such that for any $\epsilon>0$ there exists an open set $U\subset B\times\mathbb{C}^2$ containing $S\cup\overline{S(T',T'')(\delta)}$ and a smooth fiber preserving map $\psi\colon[0,1]\times U\to B\times\mathbb{C}^2$ such that, for each $r\in B$ the following hold:

(i) $\psi_{r,t}(\cdot)$ is an isotopy of holomorphic embeddings, and $\psi_{r,0}(\cdot)=\operatorname{Id}$,

(ii) $\psi_{r,t}(S_r)\subset S_r$ for every $t\in[0,1]$,

(iii) $\|\psi_{r,t}-\operatorname{Id}\|_{\mathscr{C}^2(S_r(T',T'')(\delta))}<\epsilon$ for every $t\in[0,1]$, and

(iv) $\psi_{r,1}(S_r)\subset\mathbb{C}^2\setminus T\mathbb{B}^2$.

Proof. Set $\gamma_{r,j}(z,w) := (b_j(r) \cdot z + a_j(r), w)$ such that $\gamma_{r,j}[-1,0]$ parametrizes $l_{r,j}$. Setting $F_{r,j} := \phi_r \circ \psi \circ \gamma_{r,j}$ we have that $F_{r,j}[-1,0]$ parametrizes $s_{r,j}$. Fix $T > 0$ and choose $-1 < s < 0$ such that

$$\bigcup_{r \in B} \bigcup_{j=1}^{m} F_{r,j}^{-1}((T+1)\overline{\mathbb{B}^2} \cap s_{r,j}) \subset [-1, s].$$

Choose any pair T', T'' such that $F_{r,j}^{-1}(s_{r,j}(T', T'')) \subset (s, 0)$ for all r, j.

For $N \in \mathbb{N}$ define

$$\eta_{N,t}(z,w) := \left(\frac{z - t(1+s)e^{-N(z-s)} + t(1+s)e^{-N(-s)}}{1 - t(1+s)e^{-N(-s)}}, w \right).$$

Then $\eta_{N,t}$ is an isotopy of injective holomorphic maps near the real line in the z–plane, and leaves the real line invariant, fixing 0. We see that

$$\eta_{N,1}(x,0) = \left(\frac{x - (1+s)e^{-N(x-s)} + (1+s)e^{-N(-s)}}{1 - (1+s)e^{-N(-s)}}, 0 \right)$$

from which $\eta_{N,1}(s,0) = (-1,0)$ and $\lim_{x \to +\infty} \eta_{N,1}(x,0) = (+\infty, 0)$, so the interval $[s, \infty)$ is stretched to the interval $[-1, \infty)$ when $t = 1$. Note that for any $s' > s$ we have that $\lim_{N \to \infty} \eta_{N,t} = \mathrm{Id}$ uniformly on $\{\mathfrak{Re}(z) \geq s'\}$.

Now let $\sigma_{N,t}$ be the inverse isotopy to $\eta_{N,t}$, i.e., $\sigma_{N,t} = \eta_{N,1-t} \circ \eta_{N,1}^{-1}$; it is injective holomorphic near the real line in the z–plane, and by choosing N large, may be extended, arbitrarily close to the identity, to any set $\{\mathfrak{Re}(z) \geq s'\}$ for $s' > s$.

We may now define $\psi_{r,t}(\cdot)$ on $s_{r,j}$ by

$$\psi_{r,t} := F_{r,j} \circ \sigma_{N,t} \circ F_{r,j}^{-1}.$$

The claims of the lemma are satisfied by choosing N large, and δ sufficiently small. ■

Remark III.4.1. If δ is sufficiently small and ϵ further sufficiently small we get that

$$K \cup \psi_t(S \cup \overline{S(T', T'')(\delta)})$$

is polynomially convex. Observe first that $K_r \cup S_r(T', T'')$ is polynomially convex, since K_r is, and $S_r(T', T'')$ is a collection of disjoint arcs. For a sufficiently small δ' it is known that the tube $\overline{S_r(T', T'')(\delta')}$ is polynomially convex, and for sufficiently small δ' we have that $K_r \cup \overline{S_r(T', T'')(\delta')}$ is polynomially convex. Then if $\delta < \delta'$ and we consider $\psi_{t,r}$ as in the lemma with δ' instead of δ, if ϵ is small enough we get our claim, since the δ, δ' may be chosen independently of r.

Proof of Proposition III.4.1: Fix $0 < \delta << 1$. For each $\eta \in \delta\mathbb{B}^2$ we set $v_\eta = (0,1) + \eta$, and we let π_η denote the orthogonal projection onto the orthogonal complement of v_η. After applying the linear transformation

$$A(z,w) = ((1/\sqrt{2})z + (1/\sqrt{2})w, -(1/\sqrt{2})z + (1/\sqrt{2})w)$$

it follows from Proposition III.2.2 that the family $\pi_\eta(s_{r,1})$ is $(\vartheta_{1,\eta} - \pi)$–directed and that the families $\pi_\eta(s_{r,j})$ are $(\vartheta_{2,\eta} + \theta_j - \pi)$–directed, where the $\vartheta_{j,\eta}$'s vary continuously with η. From now on we will assume that have applied the transformation A without changing the notation for all sets considered above.

By increasing $T > 0$ we may assume that $K_r \subset T\mathbb{B}^2$ for all r, and we fix R as in Theorem III.3.1 such that $\pi_\eta(T\mathbb{B}^2) \subset \triangle_R$, and choose $T' < T''$ such that $\pi_\eta(S_r(T', T'')) \subset \mathbb{C} \setminus \triangle_{R+3+\frac{3}{2}C}$ for all r and all η.

Let ψ_t be the isotopy from Lemma III.4.1, extended to be the identity on some neighborhood of K which we regard as being included in U. On $\psi_{t_0}(U)$ we define the vector field $X_{t_0}(\zeta) = \frac{d}{dt}_{t=t_0} \psi_t(\psi_{t_0}^{-1}(\zeta))$ (here $\zeta = (r, x) = (r, z, w)$). The goal is to follow the standard Andersén–Lempert procedure parametrically for approximating the flow of the time dependent vector field X_t by compositions of flows of complete fields, but to modify these so that they do not move $S \setminus S(0, T'')$. The proof is the same as the corresponding proof in [9] where this was done without parameters, but we include here a sketch and some additional details. The reader is assumed to be familiar with the Andersén–Lempert–Forstnerič–Rosay construction.

Step 1: We will find flows $\sigma_{r,j}(t, x), j = 1, \ldots, m$, such that the composition

$$\sigma_{r,m} \circ \cdots \circ \sigma_{r,1}(t, x)$$

approximates ψ_t. The flows are of two forms:

$$\sigma_{r,j}(t, x) = x + ta_{r,j}(\pi_j(x))v_j \tag{III.4.2}$$

or

$$\sigma_{r,j}(t, x) = x + (e^{ta_{r,j}(\pi_j(x))} - 1)\langle x, v_j\rangle v_j. \tag{III.4.3}$$

We write $\sigma_{r,j}(t, x) = x + b_{r,j}(t, x)v_j$.

Step 2: The plan is then roughly to find a family of cutoff functions $\chi_j \in \mathscr{C}^\infty(\mathbb{C}^2)$, $0 \leq \chi_j \leq 1$, such that $\chi_j \equiv 1$ near $T'\overline{\mathbb{B}^2}$ and $\chi_j \equiv 0$ near $\mathbb{C}^2 \setminus T''\mathbb{B}^2$, and define

$$\tilde{\sigma}_{r,j}(t, x) := x + \chi_j(x)b_{r,j}(t, x)v_j,$$

in such a way that all compositions

$$\tilde{\sigma}(j)_r := \tilde{\sigma}_{r,j} \circ \cdots \circ \tilde{\sigma}_{r,1}$$

are as close to the identity as we like in $\mathscr{C}^1$–norm on $S_r(T', T'')(\delta/2)$. Note that $\tilde{\sigma}_{r,j} = \sigma_{r,j}$ on $T'\overline{\mathbb{B}^2}$ and $\tilde{\sigma}_{r,j} = \mathrm{Id}$ outside $T''\mathbb{B}^2$. In particular the families $\pi_j(\tilde{\sigma}(j)_r(S_r))$ are as close as we like to the original families $\pi_j(S_r)$ and identical outside some compact set.

Step 3: For each j we may rewrite $\tilde{\sigma}_{r,j}$ on $\tilde{\sigma}(j-1)_r(S_r)$ as

$$\tilde{\sigma}_{r,j}(t, x) = x + c_{r,j}(t, \pi_j(x))v_j \text{ or } \tilde{\sigma}_{r,j}(t, x) = x + (e^{c_{r,j}(t,\pi_j(x))} - 1)\langle x, v_j\rangle v_j.$$

where the $c_{r,j}$'s extend to be holomorphic near $\overline{\triangle}_{R+3+\frac{3}{2}C}$ and zero on $\pi_j(\tilde{\sigma}(j-1)_r(S_r(T'',\infty)))$.

Step 4: Approximate the coefficients $c_{r,j}$ in the sense of Carleman using Theorem III.3.1.

We now include some estimates explaining why the above scheme works (see also [9] where the construction is done without dependence of parameters).

Choose $\chi \in \mathscr{C}^\infty(\mathbb{C}^2)$ nonnegative such that $\chi \equiv 1$ near $T'\overline{\mathbb{B}^2}$ and $\chi \equiv 0$ near $\mathbb{C}^2 \setminus T''\mathbb{B}^2$.

We may assume that the vector fields $X_{r,t}$ satisfy $\|X_{r,t}\| < \alpha$ on $S(T',T'')(\delta)$ for any small $\alpha > 0$. Thus, freezing the vector field at time i/N to obtain a vector field X^i with a flow $\gamma^i_{r,t}$, we have that $\|\gamma^i_r(t/N,x) - x\| \leq (t/N)\alpha$ on $S_r(T',T'')(2\delta/3)$. Thus, we may assume that the compositions

$$\gamma^i_{r,t/N} \circ \cdots \circ \gamma^1_{r,t/N}$$

exist and remain arbitrarily close to the identity on $S_r(T',T'')(\delta/2)$ for $i < N$.

By Remark III.4.1 we may approximate each vector field X^i_r to arbitrary precision by a polynomial vector field, which we will still denote by X^i_r. By the parametric Andersén–Lempert observation, see Lemma 4.9.9 and the proof of Theorem 4.9.10 in [6], the family of vector fields X^i_r may be written as a sum of shear and over–shear vector fields $X^i_r = \sum_{j=1}^m Y^i_{r,j}$ with $Y^i_{r,j}(x) = g^i_{r,j}(x)v_j$ with $v_j = v_{\eta_j}$, with flows

$$\sigma^i_{r,j}(t,x) = x + b^i_{r,j}(t,x)v_j.$$

Write

$$\Theta^i_r(t,x) := \sigma^i_{r,m}(t,x) \circ \cdots \circ \sigma^i_{r,1}(t,x) =: x + f^i_r(t,x).$$

It is known that the composition

$$(\Theta^N_{r,t/nN})^n \circ \cdots \circ (\Theta^1_{r,t/nN})^n \tag{III.4.4}$$

converges to the flow of $X_{r,t}$ on $K_r \cup S_r$ as N and then n tends to infinity. Our first goal is to replace the maps $\sigma^i_{r,j}(t,x)$ by maps $x + \chi_j(x)b^i_{r,j}(t,x)v_j$ with $\chi_j(x) = 1$ for $|x| \leq T'$ and $\chi_j(x) = 0$ for $|x| \geq T''$ in the composition (III.4.4) or partial compositions of it, and show that we still get maps that are close to the identity on $S_r(T',T'')(\delta/2)$. The compositions thus obtained will remain the same on $\{|x| < T'\}$ and be the identity map outside $\{|x| \leq T''\}$.

We will now modify the flows on $S_r(T',T'')(\delta)$. We have that $\|f^i_r(t,x)\| \leq 2\alpha t$ for t sufficiently small. If we set

$$\tilde{\Theta}^i_r(t,x) = x + \chi(x)f^i_r(t,x)$$

we see that

$$\|(\tilde{\Theta}^i_{r,(t/n)})^m - \mathrm{Id}\,\| \leq 2\alpha(m/n)t$$

for n large.

We decompose in a natural way

$$\sigma(j)_r(t,x) := \sigma_{r,j} \circ \cdots \circ \sigma_{r,1}(t,x) = x + h_{r,1}(t,x)v_1 + \cdots + h_{r,j}(t,x)v_j;$$

then the $h_{r,j}(t,\cdot)$ go to zero as $t \to 0$. Now for large n define

$$\tilde{\sigma}_{r,1}(t/n,x) = x + \chi_1(x)h_{r,1}(t/n,x)v_1 \ ,$$

and by induction

$$\tilde{\sigma}_{r,j+1}(t/n,x) = x + \chi_{j+1}(\tilde{\sigma}(j)_r^{-1}(t/n,x))h_{r,j+1}(\tilde{\sigma}(j)_r^{-1}(t/n,x))v_{j+1}.$$

Then

$$\tilde{\sigma}^i_{r,m,t/n} \circ \cdots \circ \tilde{\sigma}^i_{r,1,t/n}(x) = \tilde{\Theta}^i_r(t/n,x),$$

and we get that

$$\|(\tilde{\Theta}^N_{r,t/Nn})^n \circ \cdots \circ (\tilde{\Theta}^1_{r,t/Nn})^n - \mathrm{Id}\,\| \leq 2\alpha t,$$

and corresponding estimates hold for partial compositions. Note that this shows that $S_r(T',T'')(\delta/2)$ remains in $S_r(T',T'')(2\delta/3)$ where we may assume that the $\mathscr{C}^1$–norms of the f^i_r's are arbitrarily small, and so by arguments similar to those above we get that all partial compositions are close to the identity in $\mathscr{C}^1$–norm, which will allow us to use the implicit function theorem to rewrite as in Step 3 above.

Finally, Step 4 is carried out exactly as in [9].

■

III.5 Proof of Theorem III.1.1

Proof of Theorem III.1.1. Recall that $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$ was originally defined by $\psi(z) = \frac{1}{z} + 1$ and $\psi(\mathbb{P}^1 \setminus L_r) = \mathbb{P}^1 \setminus \Lambda_r$. Then, if $0 < \theta_2 < \theta_2 < \cdots < \theta_n$, $\phi_r\colon \mathbb{C} \setminus \{c_2(r),\ldots,c_n(r)\} \to \mathbb{C}^2$ was defined by

$$\phi_r(z) = \left(z, \sum_{j=2}^{n} \frac{e^{i\theta_j}}{\alpha_{r,j}(z)} \right).$$

Let S_r and S be the sets defined in Section III.4 and set $X_r := \phi_r \circ \psi(\mathbb{P}^1 \setminus L_r) = \phi_r(\mathbb{P}^1) \setminus S_r$ which is a 1–dimensional complex manifold with boundary $\partial X_r = S_r$. Define

$$C^r_j := \mathbb{P}^1 \setminus L_r(1/j),$$

such that $\{C^r_j\}_{j=1}^{\infty}$ is a normal exhaustion of $\mathbb{P}^1 \setminus L_r$ by $\mathcal{O}(\mathbb{P}^1 \setminus L_r)$–convex compact sets. It follows that $K^r_j := \phi_r \circ \psi(C^r_j)$, $j \geq 1$ is a normal exhaustion of X_r by $\mathcal{O}(X_r)$–convex compact sets. The proof of Proposition 1 in [11] ensures the following two crucial facts:

(i) K^r_j are polynomially convex, and

(ii) given any $K \subset \mathbb{C}^2 \setminus S_r$ compact polynomially convex, the set $K \cup K_j^r$ is polynomially convex for any j large enough .

We construct now inductively a sequence of continuous mappings, whose continuous limit

$$h\colon \bigcup_{r\in B} (\{r\} \times X_r) \to \mathbb{C}^2$$

will be a fiberwise proper holomorphic embedding that we will compose with suitable mappings to prove the statement.

Proposition III.4.1 provides a continuous $g_1\colon B \times \mathbb{C}^2 \to \mathbb{C}^2$ such that, for every $r \in B$

- $g_1(r, \cdot) \in \operatorname{Aut} \mathbb{C}^2$, and
- $g_1(r, S_r) \subset \mathbb{C}^2 \setminus 1\overline{\mathbb{B}^2}$.

Assume that we have constructed $H_j\colon B\times\mathbb{C}^2 \to B\times\mathbb{C}^2$, $H_j(r,\cdot) = (r, h_j(r,\cdot))$, continuous such that for every $r \in B$ we have that

- $h_j(r, \cdot) \in \operatorname{Aut} \mathbb{C}^2$, and
- $h_j(r, S_r) \subset \mathbb{C}^2 \setminus j\overline{\mathbb{B}^2}$.

It follows from (ii) that

$$L_j^r := h_j(r, K_{m_j}^r) \cup j\overline{\mathbb{B}^2} \subset \mathbb{C}^2 \setminus h_j(r, S_r) \ .$$

is polynomially convex for sufficiently large m_j. Then for every $\epsilon_j > 0$, Proposition III.4.1 gives us $g_{j+1}\colon B \times \mathbb{C}^2 \to \mathbb{C}^2$ continuous, such that for every $r \in B$ the following hold:

- $g_{j+1}(r, \cdot) \in \operatorname{Aut} \mathbb{C}^2$,
- $\|g_{j+1}(r, \cdot) - \operatorname{Id}\|_{L_j^r} < \epsilon_j$, and
- $g_{j+1}(r, h_j(r, S_r)) \subset \mathbb{C}^2 \setminus (j+1)\overline{\mathbb{B}^2}$.

Define $G_{j+1}(r,\cdot) := (r, g_{j+1}(r,\cdot))$ and consequently $H_{j+1} := G_{j+1} \circ H_j$. Then letting $\epsilon_j \to 0$ and $m_j \to +\infty$ fast enough, the push–out method (see [6]) allows to conclude that, for every $r \in B$, the sequence $\{h_j(r,\cdot)\}_j$ converges uniformly on compact subsets of $D_r := \bigcup_{j\geq 1} h_j^{-1}(r, L_j^r)$ to a biholomorphism $h_r\colon D_r \to \mathbb{C}^2$. It is straightforward to check that $X_r \subset D_r \subset \mathbb{C}^2 \setminus S_r$, from which it follows that $S_r = \partial X_r \subseteq \partial D_r$, thus $h_r\colon X_r \to \mathbb{C}^2$ is a proper holomorphic embedding for every $r \in B$. So setting $H(r,\cdot) := h_r(\cdot)$, $\Psi(r,\cdot) := (r, \psi(\cdot))$, and $\Phi(r,\cdot) := (r, \phi_r(\cdot))$, the mapping $\Xi := H \circ \Phi \circ \Psi\colon \Omega \to \mathbb{C}^2$ proves the claim of the Theorem.

■

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Authors' addresses

G. D. Di Salvo University of Oslo, Postboks 1337 Blindern, 0316 Oslo, Norway, giovandd@math.uio.no

T. Ritter University of Stavanger, Postboks 8600, 4036 Stavanger, tyson.ritter@uis.no

E. F. Wold University of Oslo, Postboks 1337 Blindern, 0316 Oslo, Norway, erlendfw@math.uio.no

Paper IV

Approximation and accumulation results of holomorphic mappings with dense image

G. D. Di Salvo

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Abstract

We display four approximation theorems for manifold–valued mappings. The first one approximates holomorphic embeddings on pseudoconvex domains in $\mathbb{C}^n$ with holomorphic embeddings with dense images. The second theorem approximates holomorphic mappings on complex manifolds with bounded images with holomorphic mappings with dense images. The last two theorems work the other way around, constructing (in different settings) sequences of holomorphic mappings (embeddings in the first one) converging to a mapping with dense image defined on a given compact minus certain points (thus in general not holomorphic).

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IV.1 Introduction

Calling Y a connected complex manifold and $\triangle$ the unit open disc in $\mathbb{C}$, F. Forstnerič and J. Winkelmann proved in [5] that the set $\mathcal{D}$ of all holomorphic maps $f\colon \triangle \to Y$ with dense image, is dense in $\mathcal{O}(\triangle, Y)$.

We are going to present four results which generalize in different directions the above–mentioned one. The first one is the most straightforward generalization in higher dimension, which is stated for holomorphic embeddings. The second one still provides approximating holomorphic mappings with dense images but in a more general setting. Moreover, it is no more a "density result" as the approximating mappings do not belong to the set of mappings we approximate. The last two theorems work the other way around, describing mappings with dense images approximated by holomorphic mappings.

Another result accounting this topic can be found in [1]: given a closed complex submanifold $X \subsetneq \mathbb{C}^n$, for $n > 1$, there exists a *complete* (the image of every divergent path in X has infinite length in $\mathbb{C}^n$) holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^n$ with everywhere dense image; for $n = 1$ the same result holds for complete holomorphic embeddings $f\colon \mathbb{C} \hookrightarrow \mathbb{C}^2$. If moreover $X \cap \mathbb{B}^n \neq 0$ and $K \subset X \cap \mathbb{B}^n$ is compact, there exists a Runge domain $\Omega \subset X$ containing K which admits a complete holomorphic embedding $f\colon \Omega \hookrightarrow \mathbb{B}^n$ with dense image.

Finally in [3] it is proved the existence of a holomorphic injective mapping with dense image from the open unit polydisc in $\mathbb{C}^m$ to an m–dimensional paracompact connected complex manifold M.

The paper is organized as follows: we start by presenting the main results, then a technical section follows. The subsequent section shows the inductive procedure on which the proofs of the main results rely, which in turn are provided in the last section.

I would like to warmly thank E. F. Wold for directing me to write this paper.

IV.2 Main Results

All the manifolds in this paper are connected. We will denote with $\mathcal{O}(X, Y)$ the set of all holomorphic mappings $h\colon X \to Y$ and with $\mathcal{O}_{\mathrm{emb}}(X, Y)$ the set of all holomorphic embeddings.

Theorem IV.2.1. Let $\Omega \subset \mathbb{C}^n$ open, bounded, star–shaped (with respect to 0) and pseudoconvex. Let Y be a complex manifold. Then

$$\{h \in \mathcal{O}_{\mathrm{emb}}(\Omega, Y) \ : \ \overline{h(\Omega)} = Y\}$$

is dense inside $\mathcal{O}_{\mathrm{emb}}(\Omega, Y)$ (with respect the compact convergence topology).

Observe that the density is considered with respect to the topology of Y. The distance d on Y exploited throughout the paper to make computations (namely: it allows to define the distance between two mappings $f, g\colon X \to Y$ on a given $M \subset X$ by $\|f - g\|_M := \sup\{d(f(x), g(x)) \ : \ x \in M\}$) is induced by a Riemannian metric. Such d induces the same topology Y is endowed with.

Remark IV.2.1. We will implicitly assume $n \leq \dim Y$, otherwise $\mathcal{O}_{\mathrm{emb}}(\Omega, Y) = \emptyset$.

Theorem IV.2.2. Let X be a complex manifold, Y Stein manifold. Then

$$\mathscr{H}(X, Y) := \{h \in \mathcal{O}(X, Y) \ : \ \overline{h(X)} = Y\}$$

compactly approximates

$$\mathscr{G}(X, Y) := \{g \in \mathcal{O}(X, Y) \ : \ g \text{ non constant}, \ g(X) \subset\subset Y\}$$

meaning that, given $g \in \mathscr{G}(X,Y)$, for every $M \subset X$ compact and for every $\epsilon > 0$ there exists $h \in \mathscr{H}(X,Y)$ such that $\|g-h\|_M < \epsilon$.

Observe that $\mathscr{G}(X,Y) = \emptyset$ whenever X is either compact or euclidean: in the former case the holomorphic mappings are constant, in the latter Liouville holds, obtaining again g constant, as its image is bounded. An example of a domain X such that $\mathscr{G}(X,Y) \neq \emptyset$ is given by any open relatively compact domain $X \subset\subset Y$ in a Stein manifold; just consider any non–constant holomorphic bounded mapping $g\colon X \to Y$, e.g., the inclusion $\iota\colon X \hookrightarrow Y$.

To present the last two theorems, we need the following definition:

Definition IV.2.1. Let K be a compact in Y complex manifold. A point $\zeta \in K$ is *locally exposable* if there exists a $\mathscr{C}^2$–smooth strictly plurisubharmonic function ρ on some open neighborhood U of ζ such that

1. $\rho(\zeta) = 0$ and $d\rho(\zeta) \neq 0$, and
2. $\rho < 0$ on $(K \cap U) \setminus \{\zeta\}$.

Definition IV.2.1 was originally presented in [2]. It is a generalization of the more standard definition of *local peak point* (see e.g., [7] pag. 354).

Theorem IV.2.3. Let Y be a complex manifold, $K \subset Y$ connected not finite Stein compact. Then the set of locally exposable points is non–empty and for any such point $x_0 \in K$, there exist $U_k \subset Y$ open neighborhood of K and $F_k \in \mathcal{O}_{\mathrm{inj}}(U_k, Y)$ compactly convergent to $F\colon K \setminus \{x_0\} \to Y$ such that $\overline{F(K \setminus \{x_0\})} = Y$.

Theorem IV.2.4. Let Y be a Stein manifold, $K \subset Y$ connected not finite compact. Denote with Γ the closure in Y of the set of locally exposable points for K, which is non–empty. Then there exists $U_k \subset Y$ open neighborhood of K and $F_k \in \mathcal{O}(U_k, Y)$ compactly convergent to $F\colon K \setminus \Gamma \to Y$ such that $\overline{F(K \setminus \Gamma)} = Y$.

In Theorems IV.2.3 and IV.2.4 the limit mapping F is not in general holomorphic, as the domain is not in general an open set. But being F obtained as uniform limit of holomorphic mappings, the realization of F could be a consistent definition for a holomorphic map on $K \setminus \{x_0\}$ or $K \setminus \Gamma$.

IV.3 Technical Tools

The proofs will extensively exploit Theorem 1.1 in [2] and a slightly different version of it (whose proof follows automatically from the original one) which is as follows:

Theorem IV.3.1. Let Y be a complex manifold and $Y_0 \subset Y$ Stein compact. Let $\zeta \in Y_0$ be locally exposable and $\gamma\colon [0,1] \to Y$ smoothly embedded curve such that $\gamma(0) = \zeta$ and $\gamma((0,\delta]) \subset Y \setminus Y_0$ for some $\delta > 0$. Then, for every V open neighborhood of γ and for every $\epsilon > 0$, there exist the following:

1. $U \subset Y$ neighborhood of Y_0 ,

2. an arbitrarily small $V' \subset V$ neighborhood of ζ , and

3. $f\colon U \to Y$ holomorphic mapping such that

   - $f(\zeta) = \gamma(1)$;
   - $f(V') \subset V$;
   - $\| f - \mathrm{Id} \|_{Y_0 \setminus V'} < \epsilon$.

If the whole curve $\gamma((0,1])$ lies in $Y \setminus Y_0$ then f can be chosen to be injective on U.

In [2] the theorem is formulated in a slightly more general setting and considering only the case $\gamma((0,1]) \subset Y \setminus Y_0$.

Lemma IV.3.1. Let X be a metrizable topological space, $W \subset\subset X$ connected not finite. Let $f_k\colon U_k \to X$ be a sequence of continuous mappings, where $U_k \subset X$ is some open neighborhood of K_k and $K_1 := \overline{W}$, $K_{k+1} := f_k(K_k)$. Define $F_k := f_k \circ \cdots \circ f_1\colon \overline{W} \to X$ and consider

- $\{\epsilon_k\}_k$ positive real numbers ,
- $\{C_k\}_k$ *compact exhaustion* of W, that is $C_k \subset X$ compact, $C_k \subsetneq C_{k+1}$ and $\bigcup_k C_k = W$, and
- $\{V'_k\}_k$ open sets in X ,

such that, setting $r_k := \max\{d(x, F_k(C_k)) \ : \ x \in K_{k+1}\}$, the following hold:

(i)$_k$ $V'_k \subset U_k$ and $V'_k \cap K_k \neq \emptyset$,

(ii)$_k$ $\max\{r_k,\ \| f_k - \mathrm{Id} \|_{K_k \setminus V'_k}\} \leq \epsilon_k$, and

(iii)$_k$ $V'_{k+1} \cap F_k(C_k) = \emptyset$.

If F_k converges uniformly on compacts of W to $F\colon W \to X$, then for every $x \in K_{k+1}$

$$d(x, F(W)) \leq \sum_{n \geq k} \epsilon_n \ . \qquad \text{(IV.3.1)}$$

Proof. Let $x \in K_{k+1}$. Then by definition and from (ii)$_k$ one gets

$$d(x, F_k(C_k)) \leq r_k \leq \epsilon_k \ .$$

Then (iii)$_k$ implies $F_k(C_k) \subset K_{k+1} \setminus V'_{k+1}$ hence (ii)$_{k+1}$ says that f_{k+1} moves $F_k(C_k)$ less than ϵ_{k+1}, therefore

$$d(x, f_{k+1}(F_k(C_k))) \leq \epsilon_k + \epsilon_{k+1} \ .$$

Since $f_{k+1}(F_k(C_k)) = F_{k+1}(C_k) \subset F_{k+1}(C_{k+1}) \subset K_{k+2} \setminus V'_{k+2}$ we can repeat the argument getting

$$d(x, \underbrace{f_{k+2} f_{k+1}(F_k(C_k))}_{F_{k+2}(C_k)}) \leq \epsilon_k + \epsilon_{k+1} + \epsilon_{k+2} \ .$$

Inductively, for every $m \geq 0$ we get

$$d(x, F_{k+m}(C_k)) \leq \sum_{j=0}^{m} \epsilon_{k+j}$$

and passing to $\lim_{m\to\infty}$ (which is well defined in left hand side, since the distance is continuous and $\{F_n\}_n$ uniformly converges to F on C_k) we get $d(x, F(C_k)) \leq \sum_{j\geq 0} \epsilon_{k+j}$. Since $F(C_k) \subset F(W)$, we have $d(x, F(W)) \leq d(x, F(C_k))$ and we conclude. ■

Recall now a useful property of Stein manifolds ([7], §2, Proposition 2.21 and Theorem 3.24):

Theorem IV.3.2. Let Y be a Stein manifold. Then there exists ρ, a $\mathscr{C}^2$–smooth strictly plurisubharmonic exhausting function for Y, such that the set of critical points $C := \{z \in Y \ : \ d\rho(z) = 0\}$ is discrete in Y. In particular, for every $c \in \mathbb{R}$, $\{\rho < c\} \subset\subset Y$ and $Y_c := \{\rho \leq c\}$ is a Stein compact.

Remark IV.3.1. With the notation of Theorem IV.3.2, we see that every regular boundary point of a strictly pseudoconvex domain $\{\rho < c\}$ is locally exposable: take $\zeta \in \{\rho = c\} \setminus C$; we assume it is the origin in suitable local coordinates. Then considering $\tilde{\rho}(z) := \rho(z) - c - \epsilon|z|^2$ for ϵ small enough, we conclude.

IV.4 Inductive procedure

Consider $Q = \{q_n\}_n \subset Y$ such that $\overline{Q} = Y$. Fix $\epsilon > 0$ and define $\epsilon_k := \frac{\epsilon}{2^{k+1}}$. For Theorems IV.2.1 and IV.2.2 we fix a compact subset M of the domain, Ω and X respectively. In what follows, we exploit Theorem IV.3.1 to get a suitable sequence of holomorphic functions f_k allowing to reach all the points of Q which have not been already reached, so that the image of the composition tends to invade the whole codomain. Along with the f_k, we construct both a compact exhaustion $\{C_k\}_k$ of the case domain Ω, X, $K \setminus \{x_0\}$ or $K \setminus \Gamma$ so that it fulfills the hypothesis of Lemma IV.3.1, and all the other sequences needed. For the construction of the f_k, we will focus on Theorems IV.2.2 and IV.2.1; the procedure for the remaining two theorems is alike and it is afterwards explained. Similarly for the proofs: we worked out the details of the proof of Theorem IV.2.2, which is displayed by proving convergence of the composition of the f_k, approximation of the given function, and density of the image. The argument for the remaining three results is analogous and it is subsequently illustrated.

IV.4.1 Construction of f_k for Theorem IV.2.2

IV.4.1.1 Existence of locally exposable points and base of the induction

Consider $g\colon X \to Y$ holomorphic non–constant such that $g(X) \subset\subset Y$. Set $K_1 := \overline{g(X)}$ and take $c_1 \in \mathbb{R}$ such that $K_1 \subseteq Y_{c_1}$ and $K_1 \cap \partial Y_{c_1} \neq \emptyset$. Consider $\zeta_1 \in K_1 \cap \partial Y_{c_1}$; if $\zeta_1 \notin C$ then it is locally exposable, otherwise we slightly move K_1 by composing g with a suitable holomorphic small perturbation defined as follows. Assume ζ_1 is the only point in $K_1 \cap \partial Y_{c_1}$. Observe that in suitable local coordinates on $\mathbb{C}^n$, which we split as $z = (z', z'') = x + iy = (x' + iy', x'' + iy'') \in \mathbb{C}^k \oplus \mathbb{C}^{n-k}$, we can express ρ near the origin ([5], Lemma 3.10.1) as

$$\rho(z) = \rho(0) + E(x', x'', y', y'') + o(|z|^2)$$

where

$$E(x', x'', y', y'') := \sum_{j=1}^{k} (\lambda_j y_j^2 - x_j^2) + \sum_{j=k+1}^{n} (\lambda_j y_j^2 + x_j^2)$$

with $\lambda_j > 1$ for $j = 1, \ldots, k$ and $\lambda_j \geq 1$ for $j = k+1, \ldots, n$, for some $k \in \{0, 1, \ldots, n\}$. Assume ζ_1 to be the origin $0 \in \mathbb{C}^n$ in these coordinates, so $\rho(0) = \rho(\zeta_1) = c_1$. The boundary of any Y_c is defined by ρ, so we want to move a small neighborhood of ζ_1 in a suitable direction allowing it to go across the level set ∂Y_{c_1}. Take then any nonzero vector $v = (\xi' + i\eta', \xi'' + i\eta'')$ with $\xi' = 0 \in \mathbb{R}^k$. By standard results on Stein Manifolds ([6], Corollary 5.6.3), given $w \in T_{\zeta_1} Y$ there exists a holomorphic vector field $V\colon Y \to TY$ such that $V(\zeta_1) = w$. In our case we take w to correspond to v. The flow of V on some neighborhood W of K_1 is a holomorphic mapping $\phi_t\colon W \to Y$ defined for complex times sufficiently small in modulus, say $|t| < T$, $t \in \mathbb{C}$. In local coordinates around ζ_1 it is $\phi_t(0) = tv + o(|t|^2)$ and up to shrinking T, we have that

$$\rho(\phi_t(0)) = \rho(0) + t^2 E(0, \xi'', \eta', \eta'') + o(|t|^3) > \rho(0) = c_1$$

for any $|t| < T$, considering now $t \in \mathbb{R}$. By continuity of $(z, t) \mapsto \phi_t(z)$ and of ρ, the flow will drag a whole small neighborhood of ζ_1 beyond ∂Y_{c_1} and for sufficiently small times it will be the only piece of K_1 crossing the boundary (as we are assuming that $K_1 \cap \partial Y_{c_1}$ is just one point), that is: there exist $T' < T$ and U neighborhood of ζ_1 sufficiently small, so that

- $\rho(\phi_t(z)) > \rho(0), \;\; \forall z \in U \cap K_1, \;\; T'/2 < t < T'$;
- $\rho(\phi_t(z)) \leq \rho(0), \;\; \forall z \in K_1 \setminus U, \quad 0 \leq t < T'$;
- $\phi_t(U \cap K_1) \cap C = \emptyset, \;\; \forall\, T'/2 < t < T'$.

So, considering now $\widetilde{K}_1 := \phi_t(K_1)$ for any $T'/2 < t < T'$, there exists some $c > c_1$ such that $\widetilde{K}_1 \subseteq Y_c$ and $\widetilde{K}_1 \cap \partial Y_c \setminus C \neq \emptyset$.

If $K_1 \cap \partial Y_{c_1}$ contains more than one point, either it is not discrete or it is a finite set. In the former case $K_1 \cap \partial Y_{c_1} \setminus C \neq \emptyset$, in the latter we may assume $K_1 \cap \partial Y_{c_1} = \{\zeta_1, \zeta_1'\}$; then, applying the previous argument to one of these points, the piece of K_1 that is dragged across ∂Y_{c_1}, could be not only $U \cap K_1$, but also $U' \cap K_1$, for any small time (where $U, U' \subset Y$ are suitably small neighborhoods of ζ_1 and ζ_1' respectively), leading to a similar $\widetilde{K}_1$ and achieving the same conclusion. Therefore, up to consider $\phi_t \circ g$ instead of g, we may assume that $\exists \zeta_1 \in K_1 \cap \partial Y_{c_1} \setminus C$, which is then locally exposable by Remark IV.3.1 and could be sent to any point of $Y \setminus K_1$ by the holomorphic mapping f_1 provided by Theorem IV.3.1 (see next section).

Finally define $f_0 := \mathrm{Id}_Y$, $C_0 := \emptyset$, $F_0 := f_0$ and $n_0 := 0$.

IV.4.1.2 Inductive step

Assume we have the following: $K_k \subset Y$ compact, $c_k \in \mathbb{R}$ such that $K_k \subseteq Y_{c_k}$ and $\exists \zeta_k \in K_k \cap \partial Y_{c_k} \setminus C$, F_{k-1} holomorphic on some neighborhood of K_1, with $F_{k-1}(K_1) = K_k$, $C_{k-1} \subset X$ compact and $n_{k-1} \in \mathbb{N}$. Consider then a smoothly embedded curve $\gamma_k \colon [0,1] \to Y$ such that

(i)$_k$ $\gamma_k(0) = \zeta_k$;

(ii)$_k$ $\gamma_k(1) = q_{n_k}$ where $n_k := \min\{n > n_{k-1} \;:\; q_n \notin K_k\}$;

(iii)$_k$ $\gamma_k((0, \delta_k]) \subset Y \setminus Y_{c_k}$ for some $\delta_k > 0$.

Then for every V_k open neighborhood of γ_k, Theorem IV.3.1 guarantees there exist

(1)$_k$ $U_k \subset Y$ neighborhood of Y_{c_k} ;

(2)$_k$ $V_k' \subset (V_k \cap B(\zeta_k, \epsilon_k)) \setminus F_{k-1}(g(M \cup C_{k-1}))$;

(3)$_k$ $f_k \colon U_k \to Y$ holomorphic such that the following hold:

- (a)$_k$ $f_k(\zeta_k) = q_{n_k}$;
- (b)$_k$ $f_k(V_k') \subset V_k$;
- (c)$_k$ $\|f_k - \mathrm{Id}\|_{Y_{c_k} \setminus V_k'} < \epsilon_k$.

Observe that to apply Theorem IV.3.1, V_k' needs to be a neighborhood of the locally exposable point ζ_k; moreover to have convergence (see next section) it has to avoid the image of (the fixed compact M and) the compact C_{k-1}. This cannot happen if g is constant, as $F_{k-1}(g(M \cup C_{k-1})) = K_k = \{\zeta_k\}$ so V_k' would be a punctured neighborhood of ζ_k; moreover $\zeta_k \in Y_{c_k} \setminus V_k'$ and (c)$_k$ would fail for ϵ_k small enough. Up to perturbing f_k as we did with g, there exists $c_{k+1} \in \mathbb{R}$ such that $K_{k+1} \subseteq Y_{c_{k+1}}$ and $\exists \zeta_{k+1} \in K_{k+1} \cap \partial Y_{c_{k+1}} \setminus C$; then $K_{k+1} := f_k(K_k)$ is a compact in Y, $F_k := f_k \circ F_{k-1}$ is holomorphic on some neighborhood of K_1 and $F_k(K_1) = K_{k+1}$; we finally choose a compact $C_k \subset X$ large so that $\max\{d(x, F_k(g(C_k))) : x \in K_{k+1}\} \leq \epsilon_k$ and $C_{k-1} \subsetneq C_k$. The induction may proceed.

IV.4.2 Construction of f_k for Theorem IV.2.1

IV.4.2.1 Existence of a locally exposable point

Let $g \in \mathcal{O}_{\mathrm{emb}}(\Omega, Y)$. Exploiting sharshapedness and up to considering $g_\delta(z) = g((1-\delta)z)$ for $0 < \delta < 1$, we can suppose without loss of generality g to be holomorphic and injective on U a Stein neighborhood of $\overline{\Omega}$. Call $R = \max_{z \in \overline{\Omega}} |z|$, let $\zeta_0 \in \partial\Omega$ be such that $|\zeta_0| = R$ and define $\rho(z) := |z|^2 - R^2$ which is strictly plurisubharmonic, hence ζ_0 a locally exposable point by Remark IV.3.1. Define $\zeta_1 := g(\zeta_0)$ and $K_1 := g(\overline{\Omega})$.

If $\dim Y = n$, then ζ_1 is locally exposable with respect to $\rho_1 := \rho \circ g^{-1}$ and K_1 is a Stein compact, in fact $W_\alpha = g_\alpha(\Omega),\ \ 0 < \alpha < \delta$ is a neighbourhood basis of Stein domains since each W_α is biholomorphic to Ω which is holomorphically convex. In this case, ζ_1 is locally exposable and K_1 is a Stein compact asking g for mere injectivity.

Let us now prove that ζ_1 and K_1 are still a locally exposable point and a Stein compact respectively even in the case $\dim Y = m > n$. Working with local charts we can assume to work on open subsets of $\mathbb{C}^m$. Since $dg(\zeta)$ has full rank n at each point $\zeta \in U$, it is $\operatorname{Im} dg(\zeta) \simeq \mathbb{C}^n$ and up to a linear change of coordinates, we can assume $\operatorname{Im} dg(\zeta_0) = \mathbb{C}^n \times \{0\}^{m-n}$ and obviously $\operatorname{Im} dg(\zeta_0) \bigoplus \operatorname{Span}_{\mathbb{C}}(e_{n+1}, \dots, e_m) = \mathbb{C}^m$. Define then, for $(z_1, \dots, z_m) \in U \times \mathbb{C}^{m-n}$

$$\tilde{g}(z_1, \dots, z_m) := g(z_1, \dots, z_n) + z_{n+1}e_{n+1} + \cdots + z_m e_m \ .$$

Clearly $\tilde{g}(\zeta_0, 0) = \zeta_1$ and it is locally invertible near $(\zeta_0, 0)$. Call $h\colon A \to B$ the inverse, where $A, B \subset \mathbb{C}^m$ are open neighborhoods of ζ_1 and $(\zeta_0, 0)$ respectively and since

$$\pi_j \circ h(z) = z_j \ \text{ for } \ j = n+1, \dots, m \ ,$$

we have that

$$g(U) \cap A = \{z \in A \ : \ z_{n+1} = \cdots = z_m = 0\} \ ;$$

we worked around ζ_1, but the same argument holds for any other point (regardless of whether it is regular or singular) so $g(U)$ is a complex subvariety; in particular, it is *locally closed* (every point in $g(U)$ has an open neighborhood W such that $g(U) \cap W$ is closed in W), thus it admits a Stein neighborhood basis ([5], Theorem 3.1.1). The same holds for any $g(U')$, with $\overline{\Omega} \subset U' \subset U$, therefore K_1 is a Stein compact. Define now

$$\rho_1(z) := \rho \circ \alpha(z) + \sum_{j=n+1}^{m} |z_j|^2 \ ,$$

where $\alpha := \pi_{|\mathbb{C}^n} \circ h\colon A \to U$ and $\pi_{|\mathbb{C}^n}\colon \mathbb{C}^m \to \mathbb{C}^n$ is the projection on the first n coordinates. The term $\rho \circ \alpha$ is plurisubharmonic as ρ is such and α is holomorphic; moreover

$$L_w(\rho \circ \alpha; t) = L_{\alpha(w)}(\rho; \alpha'(w)t) > 0$$

for every $t \in \mathbb{C}^m$, $\pi_{|\mathbb{C}^n}(t) \neq 0$, $w \in A$, in fact $\ker \alpha'(w) = \{0\}^n \times \mathbb{C}^{m-n}$ for any $w \in A$ and ρ is strictly plurisubharmonic. Therefore we add the plurisubharmonic term $\beta(z) := \sum_{j=n+1}^{m} |z_j|^2$. Clearly

$$L_w(\beta; t) = \beta(t) > 0$$

for every $t \in \mathbb{C}^m$, $\pi_{|\mathbb{C}^{m-n}}(t) \neq 0$, $w \in A$. Therefore ρ_1 is strictly plurisubharmonic on ζ_1 (actually on the whole A). It is clear that $\rho_1(\zeta_1) = 0$ and $\rho_1 < 0$ on $A \cap K_1 \setminus \{\zeta_1\}$; then a computation shows that $d\rho_1(\zeta_1) \neq 0$, hence ζ_1 is locally exposable in K_1 with respect to ρ_1.

IV.4.2.2 Inductive procedure

Let us observe that $Y \setminus K_1$ is connected, in fact if $n = 1$, Ω is simply connected (being starshaped) thus its boundary is connected and so is its image.

Let $n \geq 2$ and $\dim Y = n$; then $g(\Omega)$ is a Stein domain, which has connected boundary for $\dim Y \geq 2$ (see [8], pag. 22).

The remaining case, by Remark IV.2.1, is $n \geq 2$ and $m = \dim Y > n$. In this case just observe that $g(\Omega)$ is a complex submanifold of complex codimension $m - n \geq 1$, so its complement is connected.

Assume we have $K_k \subset Y$ Stein compact, $\zeta_k \in K_k$ locally exposable with respect to some strictly plurisubharmonic ρ_k, $C_{k-1} \subset \Omega$ compact, F_{k-1} holomorphic injective on some neighborhood of K_1 such that $F_{k-1}(K_1) = K_k$. The construction of $f_k \colon U_k \to Y$ is as in Section IV.4.1.2, with $Y_{c_k} = K_k$ (Stein compact) and $\gamma_k((0,1]) \subset Y \backslash Y_{c_k}$ (which can be achieved since $Y \backslash Y_{c_k}$ is connected, as above), allowing f_k to be injective on U_k and setting $\zeta_{k+1} := f_k(\zeta_k) = q_{n_k}$ (which is locally exposable with respect to $\rho_{k+1} := \rho_k \circ f_k^{-1}$). Finally set $K_{k+1} := f_k(K_k)$ Stein compact, $F_k := f_k \circ F_{k-1}$ holomorphic injective on some open neighborhood of K_1 and $C_k \subset \Omega$ as in Section IV.4.1.2. In particular $F_k(K_1) = K_{k+1}$ and $\{C_k\}_k$ is a compact exhaustion of Ω.

IV.4.3 Construction of f_k for Theorem IV.2.3

Since K is a Stein compact, there exists $U \subset Y$ Stein neighborhood of K and consequently a plurisubharmonic exhausting function $\rho \colon U \to \mathbb{R}$ as recalled in Theorem IV.3.2. Then, the existence of at least one locally exposable point $x_0 \in K$ is guaranteed by the argument of Section IV.4.1.1, with x_0, K, U playing the role of ζ_1, K_1 and Y respectively and being $\{C_k\}_k$ compact exhaustion for $K \setminus \{x_0\}$. So, given any $x_0 \in K$ locally exposable, the rest of the inductive procedure is as in Section IV.4.2.2, except for M and g which here just do not play any role, with x_0, K, ρ playing the role of ζ_1, K_1 and ρ_1 respectively.

IV.4.4 Construction of f_k for Theorem IV.2.4

The construction is as in Section IV.4.1, with no M, no g, with K playing the role of K_1 and $\{C_k\}_k$ compact exhaustion for $K \setminus \Gamma$. At each step we get a locally exposable point $\zeta_k \in K_k \cap \partial Y_{c_k} \setminus C$, sent to q_{n_k} by f_k and corresponding to some $x_k \in K = K_1$, which is locally exposable as well (thus $\{x_k\}_k \subset \Gamma$).

IV.5 Proofs

IV.5.1 Proof of Theorem IV.2.2

Proof. $F_k\colon K_1 = \overline{g(X)} \to Y$ is holomorphic. Then from $(2)_{k+1}$ it follows that for every fixed j

$$F_k(g(C_j)) \subset K_{k+1} \setminus V'_{k+1} \tag{IV.5.1}$$

holds true for every $k \geq j$. Hence we get that, for every fixed j

$$\|F_{k+1} - F_k\|_{g(C_j)} = \|f_{k+1} - \operatorname{Id}\|_{F_k(g(C_j))} \leq \|f_{k+1} - \operatorname{Id}\|_{K_{k+1}\setminus V'_{k+1}} < \epsilon_k \tag{IV.5.2}$$

is true for every $k \geq j$, so $\{F_k\}_k$ converges on compact subsets of $g(X)$ to $F\colon g(X) \to Y$ holomorphic. As above, $(2)_{k+1}$ implies $F_k(g(M)) \subset K_{k+1} \setminus V'_{k+1}$ for every k, hence inequality (IV.5.2) holds true for all k, thus

$$\|F - \operatorname{Id}\|_{g(M)} \leq \sum_{k\geq 0} \|F_{k+1} - F_k\|_{g(M)} < \epsilon \ ,$$

allowing us to conclude that $\|h - g\|_M < \epsilon$, where $h = F \circ g\colon X \to Y$ is the approximating mapping. We now check it actually has dense image. If $h(X)$ is not dense in Y, then there exists an open ball $B \subset Y$ such that

$$\beta := d(B, h(X)) > 0 \ .$$

The construction of the sets K_k and the sequence $\{n_k\}_{k\geq 1}$ allows to consider a partition of Q as

$$q_{n_{k-1}}, \ldots, q_{n_k - 1} \in K_k, \quad k \geq 1 \ .$$

In this way we can define the sequence

$$k(n) := j \ \text{ for } \ n = n_{j-1}, \ldots, n_j - 1, \ \text{ for } \ j \geq 1$$

and we have $q_n \in K_{k(n)}$ holds true for all n; the sequence $n \mapsto k(n)$ is increasing and such that $k(n) \to +\infty$ as $n \to +\infty$ (otherwise $\exists \widetilde{k}, N$ such that $q_n \in K_{\widetilde{k}}$ for all $n \geq N$, so Q would not be dense). Since $g(X) \subset\subset Y$, it follows by Lemma IV.3.1 that

$$d(q_n, h(X)) = d(q_n, F(g(X))) \leq \sum_{j\geq k(n)-1} \epsilon_j \ .$$

This last sum is less than β for any $n \geq n_\beta$, for a suitably large n_β. Therefore $\{q_n\}_{n>n_\beta}$, which is still dense in Y, does not meet an open ball, contradiction. ∎

IV.5.2 Proof of Theorem IV.2.1

Proof. It is the same as the previous proof. Just observe that now F_k is defined on $K_1 = g(\overline{\Omega})$, it is holomorphic injective and so is F. Since g is injective by assumption, then the approximating mapping $h = F \circ g$ is holomorphic injective as well. ∎

IV.5.3 Proof of Theorem IV.2.3

Proof. The mappings F_k are defined, holomorphic and injective on some open neighborhood of K and converge to $F\colon K \setminus \{x_0\} \to Y$ uniformly on compacts of $K \setminus \{x_0\}$. The construction of mappings f_k ensures, as for Theorem IV.2.2, to achieve $\overline{F(K \setminus \{x_0\})} = Y$. ■

IV.5.4 Proof of Theorem IV.2.4

Proof. As for Theorem IV.2.3 (except for injectivity of F_k), with $K \setminus \Gamma$ instead of $K \setminus \{x_0\}$. ■

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Author address

G. D. Di Salvo University of Oslo, Postboks 1337 Blindern, 0316 Oslo, Norway, giovandd@math.uio.no