1 Preface

The set of integers $\mathbb{Z}$ can be constructed from the set of natural numbers $\mathbb{N}$ by introducing the concept of ordered pairs to represent negative numbers and zero.

We let $R \subseteq N \times N$ be an equivalence relation on it. Defined as:
$(a, b)R(c, d) \iff a - b = c - d$

Which neccesitates:
$-1 := \{(2, 3), (3, 4), \ldots, (n, n + 1)\}$

Similarily:
1 := { (3,2), (4,3), . . . , (n+1, n) }

2 Operations

2.1 Addition

Therefore, It will be defined using the equivalence relationship previously established, that:

$a, b \in Z$, and $p, q, m, c \in \mathbb{N}$ s.t. $a = (p, q), b = (m, c) \subseteq R$ a+b= $(p+m, q+c)$
Since:
$(a, b) + (c, d) = (a - b) + (c - d) = a + b - (c + d)$

2.2 Multiplication

Similarily:
$(p - q) \cdot (m - c) = (pm - pc - qm + qc) = (pm + qc) - (pc + qm)$
Therefore:
$a \cdot b = (pm + qc, pc + qm)$
$\square$