1 Preface

So I had this question in my mind today, of whether or not it can be proven that if a theorem about some operation holds true for 2 operands, it also holds true for infinitely many operands.

Upon some thinking the answer I reached was hell no!

So, what else is required, then?

2 Proof

Let H be a set with an associative operation $*$ such that for all $a, b, c \in H$, we have

$$(a * b) * c = a * (b * c).$$

Suppose also that for all $a, b \in H$, the operation satisfies $a * b = d$ for some fixed element $d \in H$.

We aim to show by induction that for any N-operand expression of the form

$$a * b * c * \cdots * z,$$

where z is the N-th operand, the result of the operation is d.

2.1 Base Case

For the base case of two operands, we are given that for any $a, b \in H$,

$$a * b = d.$$

Thus, the base case holds.

2.2 Inductive Hypothesis

Assume that for any k elements $a_1, a_2, \ldots, a_k \in H$, we have

$$a_1 * a_2 * \cdots * a_k = d.$$

This is our inductive hypothesis.

2.3 Inductive Step

We must show that if the hypothesis holds for k operands, then it also holds for $k + 1$ operands. That is, we want to prove

$$a_1 * a_2 * \cdots * a_k * a_{k+1} = d.$$

Using associativity, we can group the expression as follows:

$$a_1 * a_2 * \cdots * a_k * a_{k+1} = (a_1 * a_2 * \cdots * a_k) * a_{k+1}.$$

By the inductive hypothesis, we know that $a_1 * a_2 * \cdots * a_k = d$. Substituting this into the expression, we get:

$$(a_1 * a_2 * \cdots * a_k) * a_{k+1} = d * a_{k+1}.$$

Finally, by the given assumption that $a * b = d$ for all $a, b \in H$, we know that $d * a_{k+1} = d$.

Thus, we have shown:

$$a_1 * a_2 * \cdots * a_k * a_{k+1} = d.$$

2.4 Conclusion

By induction, we have proven that for any N-operand expression $a*b*c*\cdots*z$ with $a, b, c, \ldots, z \in H$, the result of the operation is d. Therefore,

$$a * b * c * \cdots * z = d$$

for any finite number of operands in H, which completes the proof.

2.5 Multiplication of Complex Numbers in Polar Form

Let us also consider the multiplication of two complex numbers in polar form.

If we have two complex numbers given by:

$$z_1 = r_1 \text{cis}(\theta_1) \quad \text{and} \quad z_2 = r_2 \text{cis}(\theta_2),$$

where r_1 and r_2 are the magnitudes (moduli) and θ_1 and θ_2 are the angles (arguments), the multiplication is given by:

$$z_1 * z_2 = (r_1 \text{cis}(\theta_1)) * (r_2 \text{cis}(\theta_2)) = r_1 r_2 \text{cis}(\theta_1 + \theta_2).$$

Now this by itself is quite useful, but given multiplication is associative in $\mathbb{C}$ as it is a field, we can now multiply however many complex numbers we like!