A Mathematical Framework for Music

1 Foundations

It starts with a single note, which we shall call A_1, vibrating at a frequency X. Shortly after, a note A_2 follows, vibrating at a frequency $2X$. The difference between A_1 and A_2 is an **octave**.

We find notes with simple frequency ratios (small integer numerators and denominators) pleasing to the ear.

$$q = 2^{1/n} \tag{1}$$

Where n is the number of notes per octave. Any note on the instrument can be located within a geometric series, one with ratio q and one with ratio $1/q$. Each octave on the instrument corresponds to one such geometric series, and the set of all octaves forms the complete set of such series.

It will follow that q is a semi-tone, and that the first number in such a series, A_1 is the tonal center. our "base note".

2 Notes and Scales

Each note's frequency is given by:

$$f_k = A \cdot q^k \tag{2}$$

where k is the note index.

Scales are subsets of this sequence. For example, in a 12-tone equal temperament system, the major scale consists of the indices:

$$\{0, 2, 4, 5, 7, 9, 11\} \tag{3}$$

relative to the starting note.

3 Harmonic Intervals

Two notes f_k and f_m form a ratio:

$$r = \frac{f_m}{f_k} = q^{m-k} \tag{4}$$

which can be approximated by simple fractions:

- Octave: 2/1
- Perfect Fifth: 3/2
- Perfect Fourth: 4/3
- Major Third: 5/4
- Minor Third: 6/5

So, for example:
We know that $1.059 \approx 2^{1/12}$.

$$3/2 = 1.5 = (1.059)^{m-k}$$

Taking the logarithm of both sides:

$$\log(1.5) = (m-k) \cdot \log(1.059)$$

Solving for $m-k$:

$$m-k \approx \frac{\log(1.5)}{\log(1.059)} \approx 7$$

Which is perfectly consistant with reality, given that it's exactly 7 half steps away on a piano!

4 Chords as Frequency Sets

A chord is a set of notes whose frequencies maintain harmonic relationships. For example:

- **Major Chord** (C-E-G): Frequencies in $4:5:6$ ratio.
- **Minor Chord** (C-E♭-G): Frequencies in $10:12:15$.
- **Diminished Chord** (C-E♭-G♭): Frequencies in $160:192:231$.

5 Modulation as a Transform

Modulation (changing keys) can be described as shifting all notes by m semitones:

$$f'_k = A \cdot q^{k+m} \tag{5}$$

which is equivalent to multiplying all frequencies by q^m.

6 Definitions of Musical Terms

Key: A key defines the tonal center and set of notes forming a scale in a piece of music.

Consonance: The quality of notes sounding stable or pleasant together, often corresponding to simple frequency ratios.

Dissonance: The quality of notes sounding unstable or tense, often corresponding to more complex frequency ratios.

Perfect Fifth: The interval between two notes with a frequency ratio of 3/2. It is called "perfect" due to its strong harmonic stability and is one of the fundamental building blocks of Western harmony.