1 Contents

Here I will prove and dicuss some proprities of polynomials, for fun and learning.

2 Polynomials

We define a polynomial as an algebraic expression consisting of one or more terms, where each term has a constant coefficient, and an unknown raised to some natural number n.

Ex.1: $P(x) = -4x^3 + 2x^2 + 4x + 3$ Where $x \in \mathbb{R}$

Ex.2: $P(z) = -3iz^3 + 2z^2 + 4z + (i + 1)$ where $z \in \mathbb{C}$.

Generally, a polynomial P is defined as:

$P(x) := a_n \cdot x^n + a_{n-1} \cdot x^{n-1} \ldots a_2 \cdot x^2 + a_1 \cdot x + a_0$ $(n \in \mathbb{N})$

2.1 Degree, leading coefficient

Where n is said to be the degree of the polynomial, and a_n is called the leading coefficient.

The leading coefficient cannot be 0, since that cleary defines a lower degree polynomial. This is true in all cases but in the case: $P(x) = 0$.
In this case we say the degree of the polynomial is $-\infty$ or we say it has no degree. In any case, we permit this case to still be considered a polynomial

3 Addition

3.1 Theorem 1

Theorem: Addition of a polynomial P_1 to a different polynomial P_2 will always result in a polynomial whose degree is smaller or equal to that of the higher degreed polynomial.

Proof: Let P_1 be a general polynomial of degree n. Let P_2 be a general polynomial of degree k. Then:

$P_3 = P_1 + P_2 = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} \cdots + a_k \cdot x^k + a_{k-1} \cdot x^{k-1} \ldots$

Then it will be seen clearly that if $k > n$ then k is the degree and vise versa, but if k=n then it might be that the terms cancel out.

3.2 Theorem 2

Theorem: Polynomials are closed under addition

To prove that polynomials are closed under addition, we need to show that the sum of any two polynomials is also a polynomial.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where a_i are constants (coefficients) and n is a non-negative integer (degree of the polynomial). Similarly, let $g(x)$ be another polynomial:

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$

where b_i are constants and m is a non-negative integer.**Adding the Polynomials**: The sum of $f(x)$ and $g(x)$ is:

$$f(x)+g(x) = \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\right)+\left(b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0\right).$$

Combine Like Terms: When we add these two polynomials, we add the coefficients of terms with the same powers of x. This gives us a new polynomial:

$$h(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0,$$

where each c_i is the sum of the corresponding coefficients from $f(x)$ and $g(x)$:

$$c_i = a_i + b_i.$$

Conclusion: Since each c_i is a sum of constants (which is also a constant), $h(x)$ is also a polynomial. The degree of $h(x)$ will be at most $\max(n, m)$, which is a non-negative integer.

Therefore, the sum $f(x) + g(x)$ is a polynomial, proving that polynomials are **closed under addition**.

3.3 Multiplicative commutativity

$\frac{A}{B}$