The use of GARCH models to estimate the Value-at-Risk

Claudio Monaco
Riot Investment Society

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Abstract

This paper investigates the estimation of Value at Risk (VaR) using Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models, with a focus on the S&P 500 index. Given the presence of volatility clustering and heteroskedasticity in financial time series, a GARCH(2,1) model with a skewed Student's t-distribution is employed to capture the asymmetric and heavy-tailed nature of returns. The model's reliability is assessed through statistical tests. Furthermore, Conditional VaR (CVaR) is computed to provide a more comprehensive risk measure. The findings highlight the importance of modeling volatility dynamics for effective risk management in financial markets.

Contents

1 Introduction

Most data in financial econometrics occur sequentially through time, every now and then following a certain dynamic and probabilistic law, called stochastic process. However, stochastic processes do not capture all the variations occurred in a time-series. They instead represent an evolving path that occurs through time, based on certain parameters that, nevertheless, are not all the factors that have an impact on a given random variable's realization. In fact, while on one hand a stochastic process is characterized by a given collection of parameters, a time-series is a concrete realization parameters can be estimated upon. The difference between the value of a stochastic process, at a certain point in time, its empirical counterpart (i.e. the data), is due to errors that have an impact on the volatility of a given asset or portfolio. The need to model and control volatility is the reason why econometrics provides volatility models in order to take it into consideration and make forecasts about the future.

The American Heritage Dictionary, Fourth Edition, defines risk (i.e. volatility) as the "possibility of suffering harm or loss; danger". In finance, by harm or loss, a decrease in monetary values is intended. However, risk is a double-edged sword: it can manifest itself in the form of either downside or upside risk, respectively the decrease or increase of a financial value, whatever the kind of financial instrument or portfolio. However, in Value-at-Risk estimation, risk is considered as the possibility of suffering a loss. In particular, VaR represents the loss a portfolio may suffer in the most pessimistic scenario.

This paper investigates the features of Value-at-Risk and provides a real-world case for its estimation through autoregressive models, in particular through a GARCH model.

2 Literature Review

In 1982, Engle introduced the ARCH model and, in doing so, modern financial econometrics. Measuring and modeling volatility is the pillar and main goal of the field.

2.1 ARCH models

Volatility represents today a key component in financial markets and, for some market agents, an opportunity. Moreover, volatility is almost never constant in financial markets. It is very difficult to find financial instruments that lack of time-varying volatility in their returns. Volatility is a measure of how much a variable deviates from its mean or, in probabilistic terms, how much a random variable is expected to deviate from its expected value.
What most researchers are concerned about is to elaborate an estimate of where reality will go, on average and with a certain confidence interval, based on observed data. This is usually possible with linear regression models, which relate an output variable, called dependent variable, to one or more input variables, called independent variables. The most simple ones (e.g. Ordinary Least Squares) rely on different assumptions, such as homoskedasticity, that is a condition for which the variability of the studied random variable does not change and does not depend on the independent variable(s). The most common case when this condition is violated is the time-series case, which describes the evolution of one or more random variables through time and usually presents dependence not only of the variable on its past values, but also on the volatility on its past levels. In financial econometrics, AutoRegressive models (AR) are a key tool to model volatility and forecast the evolution of prices and values of portfolios. Among the AR family of models, ARCH models cover an important role, given their simplicity and capability of accurate descriptions of the dynamics of asset volatility. They allow to represent the current state of a value (e.g. an asset price) as a function of its value and volatility in the past, up to a given q time-distance. A Q^{th} order ARCH (AutoRegressive Conditional Heteroskedasticity), $ARCH\left(q\right)$,

is defined as follows:

$$\begin{aligned} r_t|\mathcal{F}_{t-1} &\sim \mathcal{N}\left(\mu_t, \sigma_t^2\right) \\ \sigma_t^2 &= \omega + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + ... + \alpha_q\varepsilon_{t-q}^2 \\ \varepsilon_t &= r_t - \mu_t \end{aligned}$$

where the variable r_t represents the returns of a given financial asset and can be modelled as a P^{th} order autoregressive process $AR\left(p\right)$.
In particular, r_t can be written as a function of past returns, until time $t-p$, and of some uncorrelated noise conditionally normally distributed, as follows:

$$r_t = a_0 + a_1r_{t-1} + a_1r_{t-1} + ... + a_pr_{t-p} + \varepsilon_t$$

where the noise as above, the error term ε_t, is defined as white noise, of which the definition is what follows:

$$\begin{aligned} &\mathbb{E}\left[\varepsilon_t\right] = 0 \\ &\mathbb{V}ar\left(\varepsilon_t\right) = \sigma^2 \\ &\mathbb{E}\left[\varepsilon_t\varepsilon_s\right] = 0, t \neq s \end{aligned}$$

Given the distribution of the financial return conditional on the past information, if

$$\mathbb{E}\left[|\varepsilon_t|\right] < \infty, \quad \forall t \tag{1}$$

then the expected value of the errors exists for every t and

$$\mathbb{E}\left[\varepsilon_t|\mathcal{F}_{t-1}\right] = 0, \quad \forall t \tag{2}$$

Conditions (1) and (2) define the error ε_t as a martingale difference sequence. Even if financial markets are a tunnel of volatility and randomness, financial returns are a function of the past information, $\mathcal{F}_{t-1}$, which represents a set of recorded realizations of other random variables. These returns, given the past information that contains itself past returns, are normally distributed with expected value μ_t and variance σ_t^2, conditionally on $\mathcal{F}_{t-1}$. Given that the past information is recorded, the variance of financial returns only depends on the errors, also called market shocks. Computing the variance of financial returns observed until time t, in fact, σ_t^2 is obtained as a result and the fact that it coincides with the variance of the error, conditional on the past information, must be taken into consideration. The proof of this is immediate and based on the fact that the expected value of the financial return of the considered asset price conditional on the past information is a constant. Mathematically, this concept can be formalized as follows:

$$\begin{aligned}\mathbb{V}ar\left(r_t\right) &= \mathbb{V}ar\left(a_0 + \sum_{i=0}^{p} a_i r_{t-i} + \varepsilon_t\right) \\ &= \mathbb{V}ar\left(\mu_t + \varepsilon_t\right) \\ &= \mathbb{V}ar\left(\varepsilon_t\right)\end{aligned}$$

Furthermore, if the residual ε_t behaves as a martingale difference sequence, then its variance coincides with its second moment:

$$\begin{aligned}\mathbb{V}ar\left(\varepsilon_t\right) &= \mathbb{E}\left[\varepsilon_t^2\right] - \mathbb{E}\left[\varepsilon_t\right]^2 \\ &= \mathbb{E}\left[\varepsilon_t^2\right]\end{aligned}$$

If tests on the estimated residuals of an autoregressive model violate at least one of these conditions, it is necessary to add a volatility model in order to capture more variations due to time-varying volatilities.

2.2 GARCH models

ARCH models usually need 5-8 lags in order to appropriately model conditional variance. GARCH (Generalized AutoRegressive Conditional Heteroskedasticity) models can capture long-memory volatility using fewer parameters. In fact, GARCH models capture volatility clustering (alternation of periods of low volatility with periods with high volatility) more effectively, thanks to the inclusion of lagged variances. The standard GARCH model, $GARCH\left(q,s\right)$ can be represented as follows:

$$\begin{aligned}&r_t = \mu_t + \varepsilon_t \\ &\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \\ &\varepsilon_t = \sigma_t e_t \\ &\varepsilon_t \sim \mathcal{N}\left(0,1\right) i.i.d.\end{aligned}$$

Also in this case, given this formulation, the conditional mean results to end up with being equal to zero. Given this, it is immediate to deduct how a

$GARCH\,(1,1)$'s volatility is defined:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

Including the lagged variance makes an $ARCH\,(\infty)$ arise. Begin with backward substituting σ_{t-1}^2:

$$\begin{aligned}
\sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \\
&= \omega + \alpha\varepsilon_{t-1}^2 + \beta\left(\omega + \alpha\varepsilon_{t-2}^2 + \beta\sigma_{t-2}^2\right) \\
&= \omega + \alpha\varepsilon_{t-1}^2 + \beta\omega + \beta\alpha\varepsilon_{t-2}^2 + \beta^2\sigma_{t-2}^2 \\
&= \omega + \alpha\varepsilon_{t-1}^2 + \beta\omega + \beta\alpha\varepsilon_{t-2}^2 + \beta^2\left(\omega + \alpha\varepsilon_{t-3}^2 + \beta\sigma_{t-3}^2\right) \\
&\ldots \\
&= \sum_{i=0}^{\infty}\omega\beta^i + \alpha\sum_{i=0}^{\infty}\beta^i\varepsilon_{t-i-1}^2 \\
&= \frac{\omega}{1-\beta} + \alpha\sum_{i=0}^{\infty}\beta^i\varepsilon_{t-i-1}^2
\end{aligned}$$

leading to the conclusion that the volatility depends on a constant $\frac{\omega}{1-\beta}$ and a weighted average of past squared shocks with weights α, $\beta_1\alpha$, $\beta_2\alpha$, ...

The $GARCH\,(1,1)$ model can be represented as a time series of the squared current shock by adding $\varepsilon_t^2 - \sigma_t^2$ to both sides:

$$\begin{aligned}
\sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \\
\sigma_t^2 + \varepsilon_t^2 - \sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \varepsilon_t^2 - \sigma_t^2 \\
\varepsilon_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \varepsilon_t^2 - \sigma_t^2 \\
&= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \beta\varepsilon_{t-1}^2 - \beta\varepsilon_{t-1}^2 + \varepsilon_t^2 - \sigma_t^2 \\
&= \omega + \alpha\varepsilon_{t-1}^2 + \beta\varepsilon_{t-1}^2 - \beta\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right) + \varepsilon_t^2 - \sigma_t^2 \\
&= \omega + \alpha\varepsilon_{t-1}^2 + \beta\varepsilon_{t-1}^2 - \beta\nu_{t-1} + \nu_t \\
&= \omega + (\alpha+\beta)\,\varepsilon_{t-1}^2 - \beta\nu_{t-1} + \nu_t
\end{aligned}$$

where $\nu_t = \varepsilon_t^2 - \sigma_t^2$ is the volatility surprise, that is the exceeding unexpected volatility.

2.3 Value-at-Risk defined

The Value-at-Risk (VaR) measures the potential loss in the value of a portfolio over a certain period of time for a given confidence level. The time span obviously depends on the application and the aim of the research. Considering a future portfolio or an asset value V_{t+h} at time $t+h$, the α VaR in a time horizon of length h, the Value-at-Risk is defined as:

$$\mathbb{P}\left(V_{t+h} < V^{\alpha}_{t+h} | \mathcal{F}_t\right) = \alpha$$

where, for instance, for $h = 1$,

$$VaR_{t+1}|\mathcal{F}_t \sim \mathcal{N}\left(\mu_{t+1}, \sigma^2_{t+1}\right)$$

and $VaR_{t+1}|\mathcal{F}_t$ follows the dynamic of an $AR(p)$ process with distribution parameters

$$\begin{aligned}
\mu_{t+1} &= a_0 + a_1 Var_t + a_2 VaR_{t-1} + ... + a_p VaR_{t-p+1} \\
\sigma^2_{t+1} &= \omega + \alpha\varepsilon^2_t + \beta\sigma^2_t
\end{aligned}$$

where the volatility can be modeled through a $GARCH(1,1)$.
The value VaR^{α}_{t+h} being the first percentile. The Value-at-Risk is a forecast based on the currently available information set, of a possible pessimistic scenario in which the value of the portfolio is smaller than this value at the time the VaR refers to. A confidence interval of 95% is usually adopted.
Standardizing the VaR, it is possible to observe that

$$\begin{aligned}
\alpha &= \mathbb{P}\left(VaR_{t+1} < VaR^{\alpha}_{t+1}|\mathcal{F}_t\right) \\
&= \mathbb{P}\left(\frac{VaR_{t+1} - \mathbb{E}\left[VaR_{t+1}|\mathcal{F}_t\right]}{\sqrt{\mathbb{V}ar\left(VaR_{t+1}|\mathcal{F}_t\right)}} < \frac{VaR^{\alpha}_{t+1} - \mathbb{E}\left[VaR_{t+1}|\mathcal{F}_t\right]}{\sqrt{\mathbb{V}ar\left(VaR_{t+1}|\mathcal{F}_t\right)}}|\mathcal{F}_t\right) \\
&= \mathbb{P}\left(\frac{\varepsilon_{t+1}}{\sqrt{\omega + \alpha\varepsilon^2_t + \beta\sigma^2_t}} < \frac{VaR^{\alpha}_{t+1} - \mathbb{E}\left[VaR_{t+1}|\mathcal{F}_t\right]}{\sqrt{\omega + \alpha\varepsilon^2_t + \beta\sigma^2_t}}|\mathcal{F}_t\right) \\
&= \mathbb{P}\left(e_{t+1} < q^{\alpha}_{t+1}|\mathcal{F}_t\right)
\end{aligned}$$

where:

- e_t represents the standardized shock;
- q^{α}_t represents the $\alpha \times 100$-th percentile in the standard normal distribution.

3 Methodology

First of all, the daily time-series of the S&P500 index's time-series has been taken into consideration and imported from Yahoo Finance. The considered time span is from the beginning of 2010 to the end of 2024. However, given that the index's price clearly does not behave as a stationary time-series (thanks to long-term economic growth), the log returns r_t have been computed as follows:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{3}$$

where P_t is the closing price of the index at the end of day t.

3.1 Preparation of the necessary data

The log returns associated to the studied index appear (Figure 1) to follow a normal distribution, but to be (almost) sure it is needed to perform a Kolmogorov-Smirnov test for normality. In fact, the Kolmogorov-Smirnov test for normality strongly rejects the null hypothesis of normality of the data at every reasonable significance level, leading to an additional attempt to test for the skewed Student's t distribution, in order to acknowledge that the analysis will be based on data characterized by fat tails.
At 5% significance level, the test fails to reject the null hypothesis that the log returns are distributed as a skewed Student's t random variable. Considered this, the analysis goes ahead testing for stationarity in the log returns through the Augmented Dickey-Fuller test. This is a crucial step and failing to reject the null hypothesis of presence of unit roots in the time-series forces the researcher to build an analytic strategy that relies on a different manipulation of the data. Figure 2 gives a hint of what the result will be, given that it is visible that the log return goes back to its mean most of the time and always evolves around it.

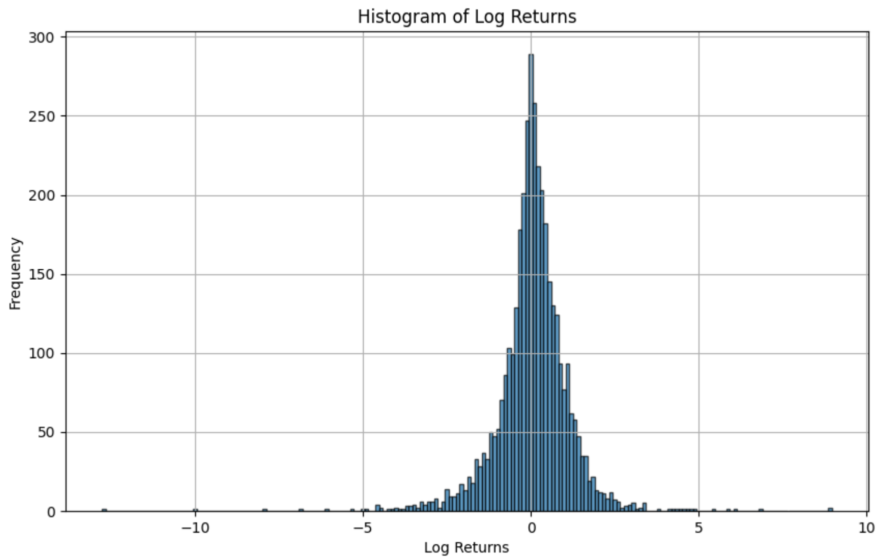

Figure 1: Histogram of the log returns.

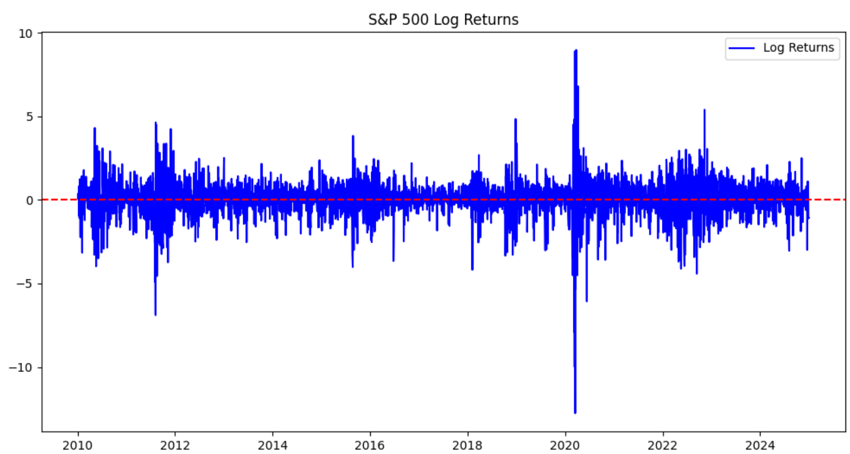

Figure 2: Plot of S&P500's log returns.

As expected, the test leads to the rejection of the null hypothesis, meaning

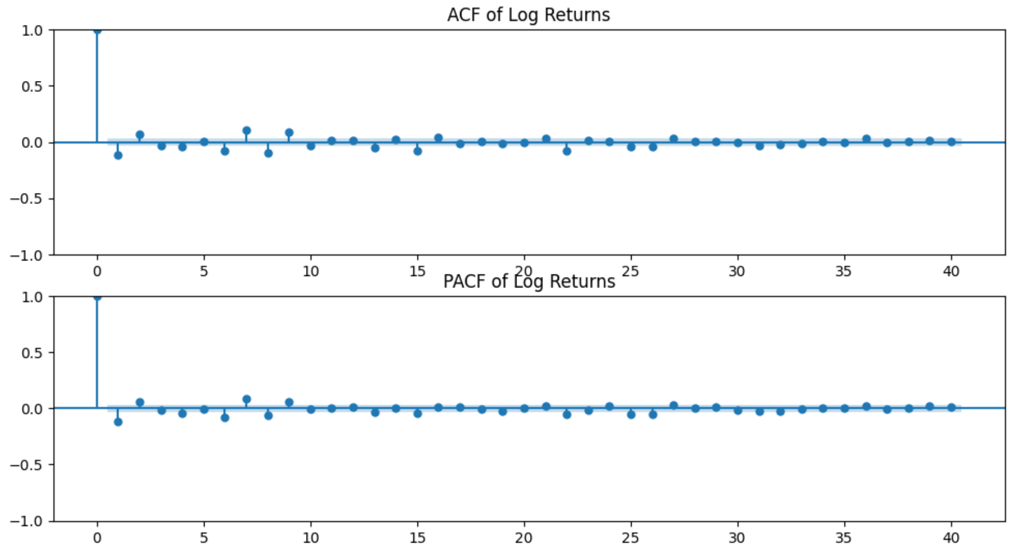

Figure 3: ACF and PACF of log returns.

that the data is stationary. This is a great achievement, since many of the statistical tests that will be used here assume stationarity and consistent properties of the variables analyzed.

3.2 Check for the need of a volatility model and its estimation

The follow-up step is to estimate the relationship between the current log return of the index and its lagged values. Before this, however, it is necessary to figure out how many lags to take into account. This is possible after plotting and studying the autocorrelation and partial autocorrelation functions of the log returns as in Figure 3. Looking at the ACF and PACF associated to the log returns, it looks like the most reasonable choice for the estimation of the autoregression equation is the following:

$$r_t = a_0 + a_1 r_{t-1} + a_2 r_{t-2} + a_6 r_{t-6} + a_7 r_{t-7} + a_8 r_{t-8} + \varepsilon_t \tag{4}$$

where ε_t is desired to behave like a white noise. To check for this, it is necessary to perform the Engle's test for heteroskedasticity, which is strongly rejected, leading to the conclusion that the residuals are heteroskedastic, violating the Gauss-Markov assumptions, and a volatility component is needed

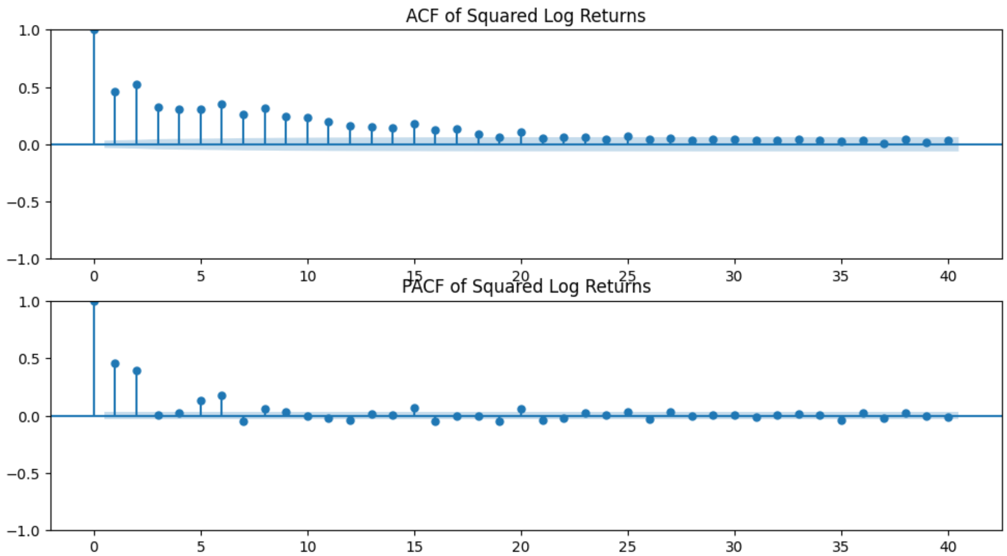

Figure 4: ACF and PACF of squared log returns.

in the model. For the volatility component, it is again essential to choose the lags for the variance component. To do this, the ACF and PACF of the squared log returns must be checked in Figure 4. The ACF shows a 20-days-long decreasing variance persistence (volatility clustering), which means that periods of high (low) volatility are followed by periods of high (low) volatility, while the PACF shows that the present volatility is affected by up to two lagged shocks. This feature suggests, together with the use of the AIC (Akaike Information Criterion), the choice of a $GARCH\,(2,1)$ model, which takes the form:

$$r_t = \mu + \varepsilon_t \quad , \qquad \varepsilon_t \sim \text{Skew-t}\,(\nu, \lambda) \tag{5}$$
$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \beta_1 \sigma_{t-1}^2 \tag{6}$$

and is chosen over a $GARCH\,(1,1)$, giving the output in Figure 5:

```
                       Constant Mean - GARCH Model Results
==============================================================================
Dep. Variable:                  Log Returns   R-squared:                       0.000
Mean Model:                   Constant Mean   Adj. R-squared:                  0.000
Vol Model:                            GARCH   Log-Likelihood:               -4716.66
Distribution:     Standardized Skew Student's t   AIC:                      9447.32
Method:                  Maximum Likelihood   BIC:                           9490.96
                                              No. Observations:                 3772
Date:                      Thu, Feb 06 2025   Df Residuals:                     3771
Time:                              16:30:35   Df Model:                            1
                                 Mean Model
=============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
-----------------------------------------------------------------------------
mu             0.0736  1.159e-02      6.348  2.187e-10 [5.086e-02,9.629e-02]
                              Volatility Model
=============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
-----------------------------------------------------------------------------
omega          0.0292  6.688e-03      4.367  1.262e-05 [1.609e-02,4.231e-02]
alpha[1]       0.1056  2.501e-02      4.224  2.400e-05   [5.662e-02,  0.155]
alpha[2]       0.0819  3.331e-02      2.457  1.399e-02   [1.657e-02,  0.147]
beta[1]        0.7960  2.432e-02     32.726 6.671e-235     [  0.748,  0.844]
                                Distribution
============================================================================
...
lambda        -0.1168  2.152e-02     -5.429  5.658e-08 [ -0.159,-7.464e-02]
============================================================================

Covariance estimator: robust
```

Figure 5: GARCH estimation: output

The model estimation relies on the probability law of the log returns. They indeed follow a skewed Student's t distribution, and so will the residuals in the analysis. In fact, the model and the estimated coefficients result jointly significant.
The output shows that the mean return is statistically different from zero and equal to 7.36%, as well as the coefficients in the volatility model. The volatility coefficients result:

$$\begin{aligned}
\omega &= 0.0292 \\
\alpha_1 &= 0.1056 \\
\alpha_2 &= 0.0819 \\
\beta_1 &= 0.7960
\end{aligned}$$

significantly different from zero and positive, avoiding the variance to end in the negative domain and supporting the theory. Also, the sum of the coefficients, α_1, α_2 and β_1, in the volatility model results close to but smaller than

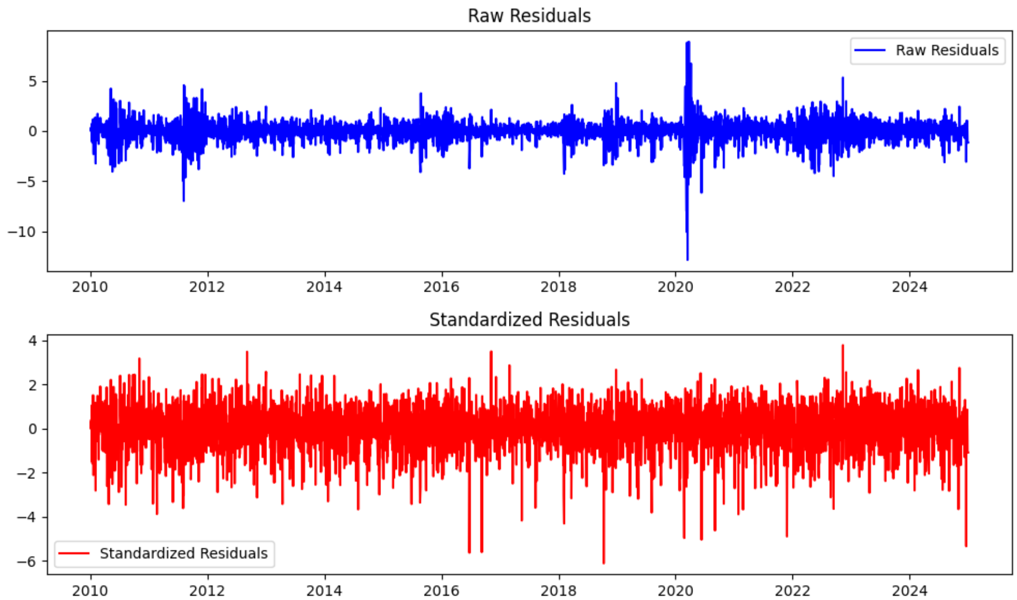

Figure 6: Raw and standardized estimated residuals.

1, meaning that the volatility is highly persistent. To conclude the output analysis, the obtained lambda (λ) is significant and negative, confirming the assumption regarding the error's probabilistic distribution in the model estimation.

Observing Figure 6, the estimated standardized residuals appear to behave as a stationary process. To confirm this, an Augmented Dickey-Fuller test is performed on both them and their standardized version, which takes into account the volatility forecast. This test confirms stationarity, in both cases. The estimated residuals, according to the ACF and PACF analysis in Figure 7, are uncorrelated, meaning that they behave as a white noise, according to the Kolmogorov-Smirnov test for the skewed Student's t distribution. Given their stationarity and their autocorrelation, the estimated residuals can be referred to as a white noise, giving more reliability to the model.

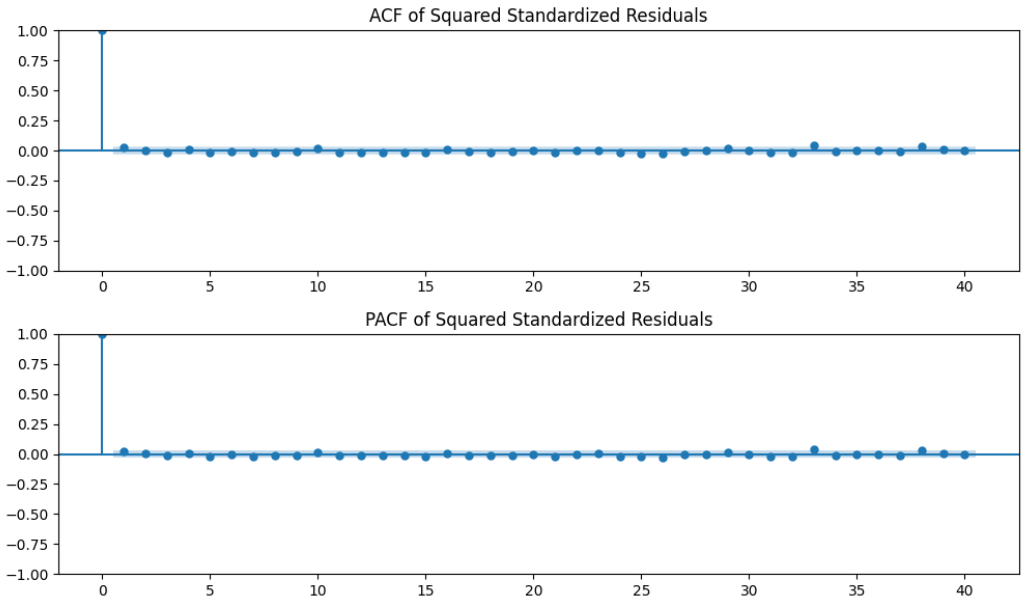

Figure 7: ACF and PACF of squared estimated standardized residuals.

Also, a Ljung-Box test performed on the data reveals that the variables of the model do not exhibit autocorrelation, validating the model once again. The model is now reliable.

4 Results

The next step of the investigation is the analysis of the Value-at-Risk estimate. Recall VaR estimates the potential loss in the value of a portfolio over a given time period with a specified confidence level, which is now 95%. Thus, it is not a level value, but is indeed a variation, i.e. a shock. In fact, the estimated GARCH model takes into account the dispersion of the residuals around the mean. Given that the periodicity of the studied returns is daily, the investigation focuses on the same time period. More specifically, the model leads to an estimate of the 1-day VaR, at a 95% confidence level, of 1.7193%. Furthermore, the comparison between the expected number of violations (188) and the actual number of violations (177) concludes that the model is reliable. This measure is unconditional, that is assumes returns to follow a constant distribution over time. The conditional VaR, usually referred to as the CVaR, is a different estimate that takes into account not only the variability in the model parameters' values, but also their nature and general behavior. Also, the CVaR is more accurate than the VaR because it captures the expected loss beyond the VaR. In mathematical terms, the Expected Shortfall is defined as

$$ES_\alpha = \mathbb{E}\left[X | X \leq VaR_\alpha\right] \tag{7}$$

where X is the loss random variable, VaR_α is the quantile loss at probability α and ES_α represents the expected loss given that losses exceed the VaR. In other terms, the Expected Shortfall considers the average loss in the worst scenarios. Also, VaR can underestimate risk when distributions have fat tails.

To conclude, the CVaR is the one-step-ahead consideration after the VaR. While the VaR represents a limit with the chosen confidence level, the CVaR provides an estimate of how big the loss will be in the case the VaR is negatively exceeded. Applying this reasoning to the studied data, while the VaR estimate collocates the maximum loss at 95% confidence level at 1.7193%, the CVaR is estimated to be 2.9882% in the case the previous measure is exceeded, as Figure 8 clearly shows. The distance between these two measures is justified by the probabilistic distribution of the residuals, which is negatively skewed.

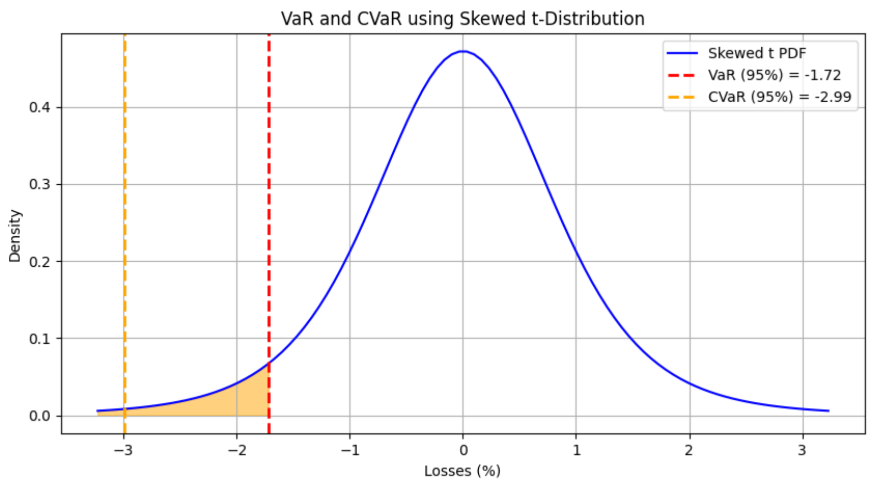

Figure 8: VaR vs CVaR graph.

Despite the results already looking satisfying and correct, there is the need for one last confirmation of the validity of the model. The tool used to do so is the Kupiec test, also called Kupiec's Proportion of Failures (POF) test. It assesses whether the actual number of VaR violations is consistent with the expected number of violations, given the model's confidence level. The Kupiec's statistic is built as follows:

- H_0: $L_0 = (1-\alpha)^{T-X}\,\alpha^X$;
- H_1: $L_1 = (1-\hat{p})^{T-X}\,\hat{p}^X$

where L represents the likelihood function; X is the number of observed exceedances and is distributed as a binomial random variable with parameters (T, α); T is the number of total observations; and α is the expected number of violations probability. The likelihood-ratio statistic is:

$$LR_{POF} = -2\ln\left(\frac{L_0}{L_1}\right) = -2\left[(T-X)\ln\left(\frac{1-\alpha}{1-\hat{p}}\right) + X\ln\left(\frac{\alpha}{\hat{p}}\right)\right]$$

where LR_{POF}, under the null hypothesis, follows a Chi-Squared distribution with one degree of freedom (χ^2_1). If the null hypothesis is rejected, then the model considered is adequate, and this is what happens performing the Kupiec test on the considered data.

5 Discussion and Conclusion

The results of this study confirm that modeling volatility using GARCH-based approaches improves Value at Risk (VaR) estimation. The application of a GARCH(2,1) model with a skewed Student's t-distribution effectively captures the asymmetric and fat-tailed nature of financial returns, addressing the limitations of standard normality assumptions. The high persistence of volatility found in the S&P500 returns further justifies the use of conditional variance models instead of static risk measures.

Through backtesting, the estimated VaR at a 95% confidence level was found to be reliable, with actual violations closely aligning with theoretical expectations. However, VaR alone provides an incomplete picture of tail risk. The estimation of Conditional VaR (CVaR) demonstrates that losses, when exceeding VaR, tend to be significantly larger than the threshold itself. This highlights the importance of Expected Shortfall measures in financial risk management.

While the model is robust for daily risk estimation, its effectiveness in extreme market conditions could be further tested using stress testing or regime-switching models. Additionally, incorporating exogenous macroeconomic factors (e.g., interest rates, inflation) into the volatility model could improve forecasting accuracy.

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