Graph Theoretic Interpretations of the Nevanlinna Representation

Dr. Ryan Tully-Doyle, Lily Adlin, Giovani Thai, Samuel Tiscareno

Department of Mathematics, California Polytechnic State University, San Luis Obispo

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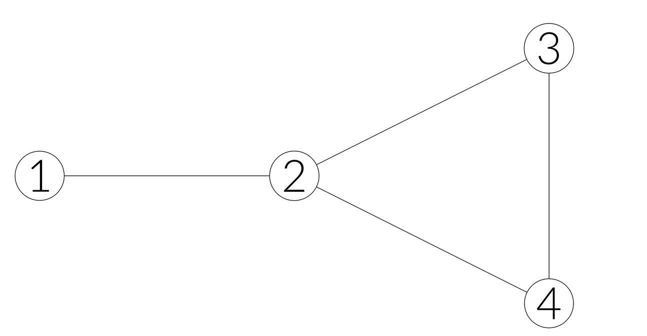
Graphs and Adjacency Matrices

A (simple, undirected) **graph** G is a collection of *vertices*, some of which are connected with *edges*.

Construct its corresponding **adjacency matrix** A that encodes which vertices are connected to each other:

$$A_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are connected} \\ 0, & \text{else.} \end{cases}$$



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Constructing the Representing Function

- Let A be the adjacency matrix of a simple, undirected graph.
- Let Y be a "vertex coloring" matrix: $Y = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$
- Let $A_C = A - zY - w(I - Y)$ denote the colored adjacency matrix.

Then

$$\boldsymbol{f(z,w) = (A_C)^{-1}_{(2,2)}}$$

studies **paths from vertex 2 to vertex 2.**

Function Properties

f is *analytic*, so we express it as a power series to investigate function behavior:

$$f(z,w) = \sum_{n=0}^{\infty} a_n z^n$$

- **Order of Vanishing**: the degree of the first non-zero coefficient.
- **Contact Order**: By [2], we can expand the *level set function* as a power series:

  $$f(z,w) = t \implies w = \Lambda(z,t)$$

  and look for the degree of the first term that depends on t.
  This measures the *similarity* of level curves as they approach the singularity at infinity.

Motivation

Coloring vertices in this way has applications in:

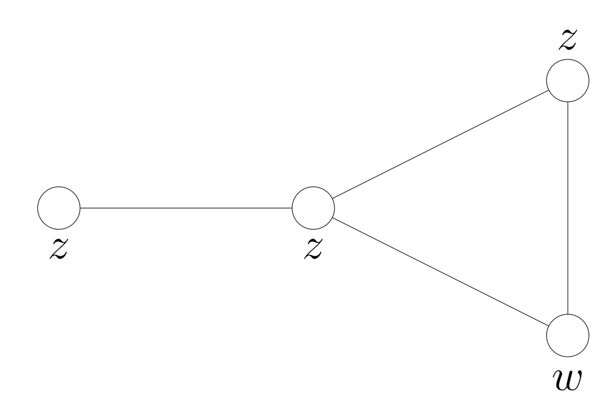
- **Network Theory**, where paths between vertices represent information of webpages.
- **Circuit Theory**, where paths between vertices represent current between resistors.

Research Question

What relationships exist between a graph's structure and the behavior of its representing function?

An Example

Color the last vertex with w, and the rest with z:



$$f(z,w) = \frac{z - wz^2}{1 - 2z - 2wz - 2z^2 + wz^3}$$

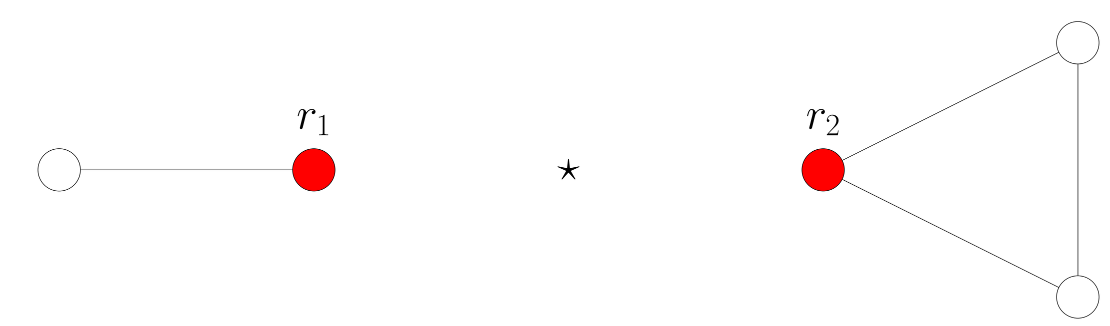
$f(z,w) = -\frac{1}{z} - \frac{1}{z^2 w} + O[\frac{1}{z}]^3 \implies$ Order of Vanishing: 1

$\Lambda(z,t) = \frac{2}{z} + \frac{-1+2t}{tz^2} + O[\frac{1}{z}]^3 \implies$ Contact Order: 2

Star Products

We can also view this graph as two smaller ones attached at a root vertex.

We call this a **Star Product.**



$$f_1(z,w) = \frac{z}{1 - z^2} \qquad f_2(z,w) = \frac{-1 + wz}{2 + w + 2z - wz^2}$$

- Order of Vanishing: 1
- Contact Order: 2

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- Contact Order: 2

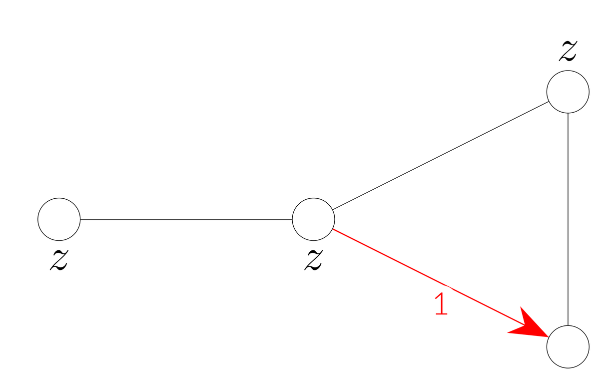
Result: Two-variable version of [1, Proposition 2.1]

If $g = 1/f$, then $\boldsymbol{g(z,w) = g_1(z,w) + g_2(z,w) - g_0(z,w)}$,

where we subtract off the representing function of the *overlap*.

Shortest Distance

Theorem: The representing function of *any* graph at a vertex i has contact order twice the shortest length between the perspective vertex and the w vertex.

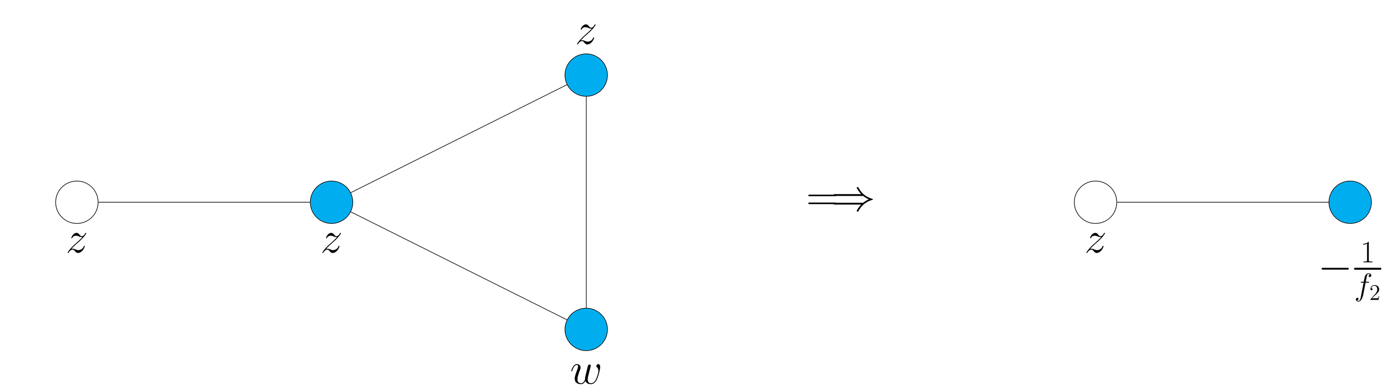


Since the distance is 1, the contact order is 2, as seen above.

Graph Reduction and Comb Products

Representing functions can be calculated by *composing* the functions of its parts.

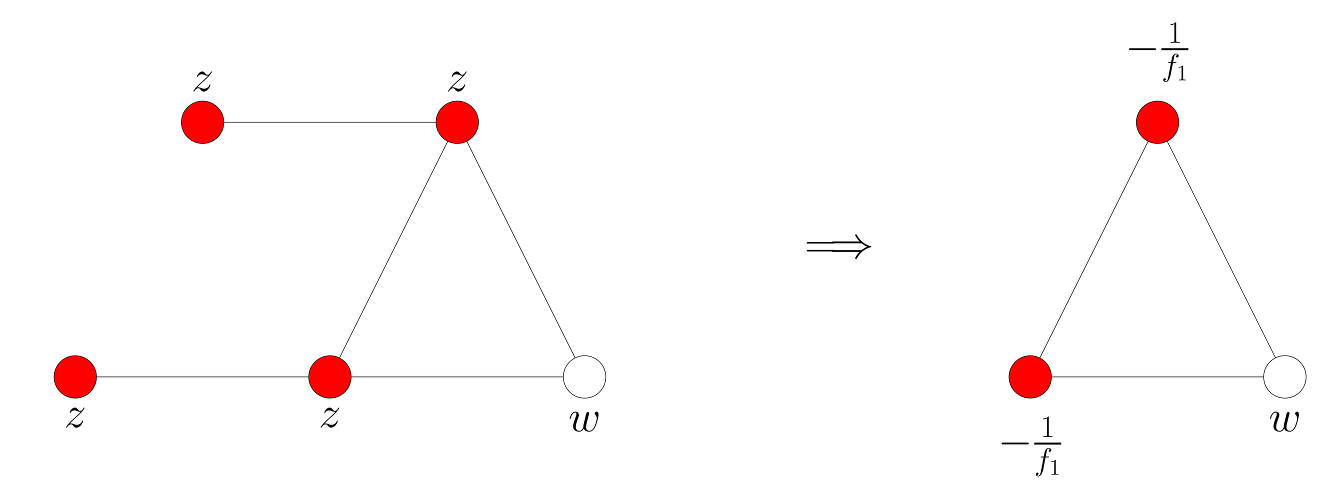
To do so, reduce the subgraph into a single vertex re-colored with its function:



and compute the representing function of the reduced graph with the reduced A_C.

Comb Products

A **z-Comb Product** is attaching a smaller graph to every z colored vertex of the larger graph.



Corollary: Because we can collapse each attached graph into a vertex labeled with its representing function, we get that function for the z-comb product, $\boldsymbol{f_{G_2 \triangleright_z G_1}(z,w) = f_{G_2}\left(-\frac{1}{f_{G_1}(z)}, w\right)}$.

Future Work and Open Questions

Application to Circuits and Networks.
How similar are other properties of representing functions to results in circuit and network theory? (e.g. total resistance of a parallel circuit)

Coloring Schemes.
We used the vertex coloring scheme consistent with [3]. What if we colored more than 1 vertex with a w? What about *linear combinations* of z's and w's on a vertex?

Eigenvalues.
What kind of information would the eigenvalues of A tell us about the asymptotic behavior of the representing function?

References and Acknowledgments

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[2] Kelly Bickel, James Eldred Pascoe, and Alan Sola. Level curve portraits of rational inner functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, pages 449–494, 2020.

[3] Yang Hong. Graphs, adjacency matrices, and corresponding functions, 2023.

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