Characterizing Non-Singular Graphs with Games

Giovani Thai

Cal Poly, San Luis Obispo
Frost Summer Research Program
Joint work with Dr. Jeffrey Liese; Cameron Klig, Riley Lane, Matthew Moscot

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Matrix Games

- **Two-Player:** One for row, one for column.
- **Zero-Sum:** what's gained in one player is lost in another
- **The Game:** call out row and column, resulting matrix entry is payoff to row player.

$$R = \begin{bmatrix} +1 & 0 & +2 \\ +2 & +1 & 0 \\ 0 & +2 & +1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 & -2 \\ -2 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

Matrix Games

- Players develop **optimal strategies** $\vec{x}$ and $\vec{y}$ to maximize row gain, minimize column loss
- **Optimal Value:** Expected payoff under optimal play.

Theorem. (Shapley-Snow, 1950).

Extreme optimal solutions come from invertible submatrices. If an extreme optimal solution uses every row, the matrix must be invertible.

- **Broad Plan:** Remove an arbitrary row from the matrix, show that the value goes down. Implies that row must be played to achieve the optimal value.

Rock Paper Scissors

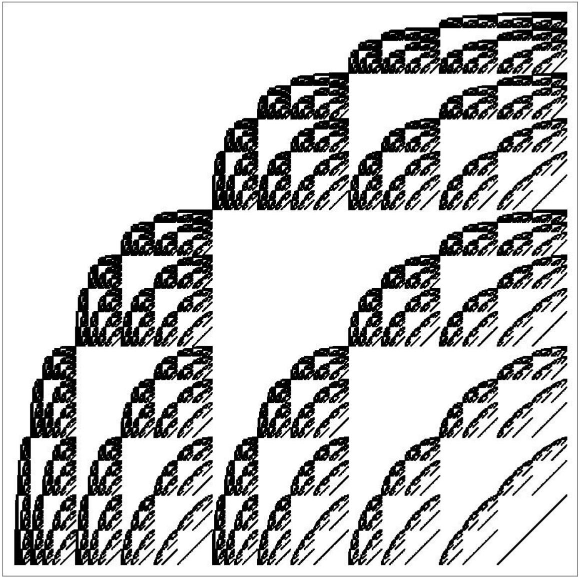
$$M = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

- $Det(M) = 9$ (M is invertible)
- Optimal Strategies:
  - Row Strategy: $\vec{x} = \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle$
  - Column Strategy: $\vec{y} = \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle$
  - $v = 1$
- If Row Player can't play rock:
  - $\vec{y} = \langle \frac{1}{3}, 0, \frac{2}{3} \rangle$
  - $v' = \frac{2}{3}$

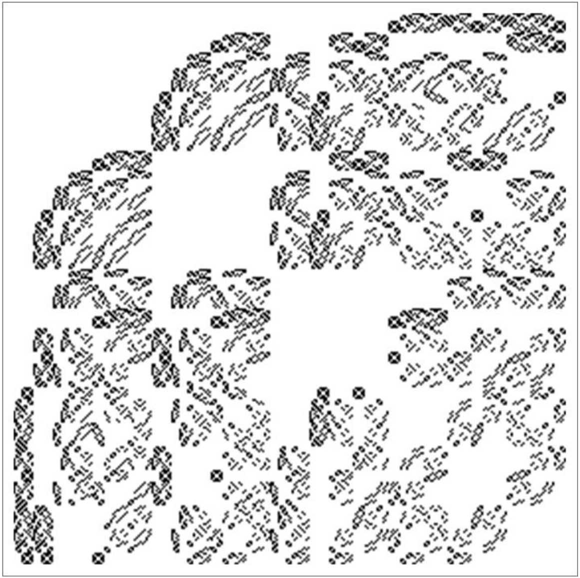
Connections to Graphs

- **Graph:** A graph G is composed of a collection of vertices (V), and an edge set (E) defining which vertices are adjacent (connected).
- **Adjacency Matrix, A(G):** $A_{i,j=} \begin{cases} 1, & \{v_i, v_j\} \in E \\ 0, & \{v_i, v_j\} \notin E. \end{cases}$
- Investigating **Vertex-Transitive** Graphs
  - No advantage in choosing/playing one vertex over another
  - Uniform probability is an optimal strategy (true for any regular graph)

Example: Kneser Graph Adjacency Matrix



Motivation: Transversal Graph Adjacency Matrix



Method

1. Remove the first row from M, denote it M'.
   - Corresponds to removing vertex 1 from play for the row player.
2. Find a **regular partition** $P = P_1, \cdots, P_k$ of the vertices.
   - Goal: group vertices based on their relationship to the removed v_1.
   - Regular: $|N(x) \cap P_j|$ is constant on P_i for all i.
   - We want $P_1 = \{v_1\}$.
   - Preserves the density of connections within and across groups.

Method

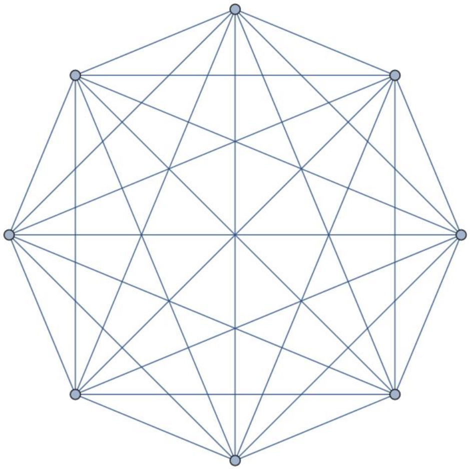
3. Based on the regular partition, create the **intersection-probability matrix**.
   - i, j-entry represents the probability that two vertices from P_i and P_j chosen uniformly at random are adjacent.

4. Remove the first row from I-P matrix (M''); find a column strategy $\vec{y}$ for M'' that reduces the value of the original game.

Complete Graph, K_n

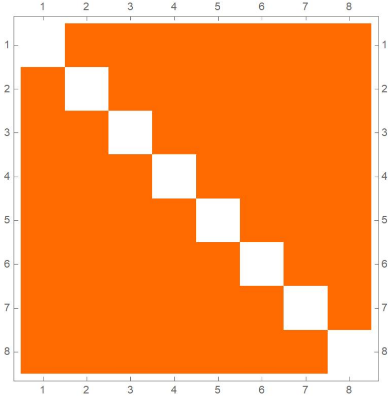
- Each vertex connected to every other vertex; $deg(v_j) = n - 1 \ \forall \ j \in [n] = \{1, \cdots, n\}$.

Ex. Complete Graph on 8 vertices:



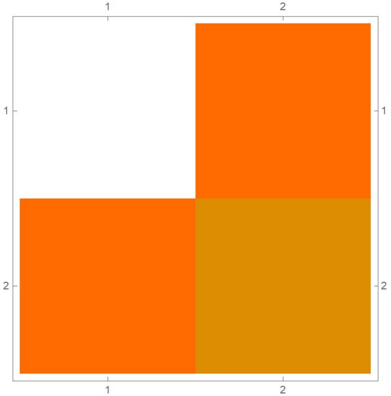
Complete Graph, K_n

Adjacency Matrix of K_8 :



Dimension of $M = A(K_8)$: 8×8

Intersection − Probability Matrix :



Dimension of I-P Matrix: 2×2
(General Dimension: 2×2*)*

Complete Graph, K_n

For K_8 :

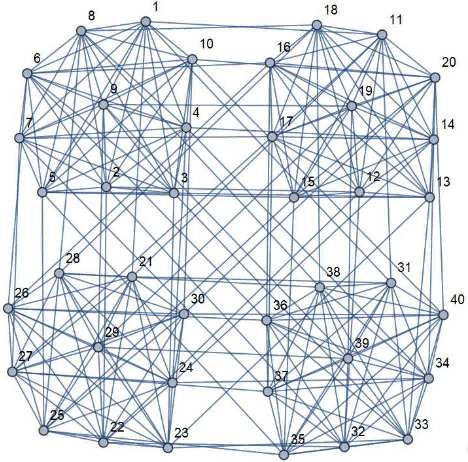
- $v = \frac{7}{8}$
- $\vec{y} = \langle 0, 1 \rangle \implies v' = \frac{6}{7}$

For K_n :

- Game Value: $v = \frac{n-1}{n}$
- Optimal Column Strategy on M'': $\vec{y} = \langle 0, 1 \rangle$
  - $M''\vec{y} = \frac{n-2}{n-1}\mathbf{1}$
  - Thus, $v' = \frac{n-2}{n-1} < \frac{n-1}{n} = v$

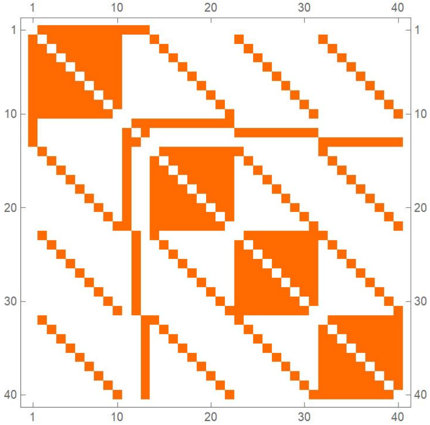
Rook's Graph, R(m, n)

- Vertices represent tiles on $m \times n$ chessboard; ordered pairs (R, C).
- A vertex is adjacent to all possible moves for a Rook in that position.
- Regular Partition:
  $\{(1,1)\}, \{(i,1) : 2 \leq i \leq m\}, \{(1,j) : 2 \leq j \leq n\}, \{(i,j) : i,j \neq 1\}$



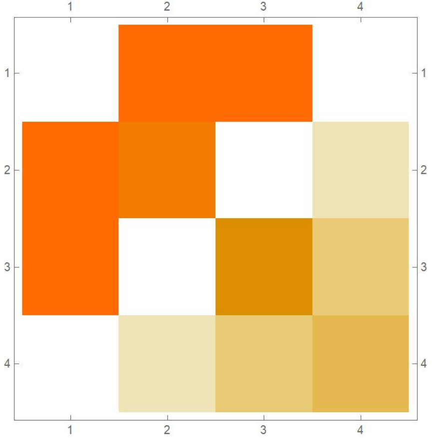
Rook's Graph, R(m, n)

Adjacency Matrix of R(4, 10):



Dimension of M : 40×40

Intersection-Probability Matrix:



Dimension of I-P Matrix: 4×4
(General Dimension: 4×4)

Rook's Graph, R(m, n)

For R(4, 10):

- $v = \frac{12}{40} = \frac{3}{10} = 0.3$
- $\vec{y} = \langle \frac{1}{20}, \frac{7}{20}, \frac{3}{5}, 0 \rangle \implies v' = \frac{41}{138} \approx 0.297$

General R(m, n):

- Game Value: $v = \frac{m+n-2}{mn}$
- Optimal Strategy on M'' :

$$\vec{y} = \begin{bmatrix} 0 & \dfrac{(m-2)(m-1)(n-3)}{mn(m-3)(n-3)-4} & \dfrac{(m-3)(n-2)(n-1)}{mn(m-3)(n-3)-4} & \dfrac{(m-1)(n-1)((m-3)(n-3)-1)}{(m-3)m(n-3)n-4} \end{bmatrix}^T$$

$$\implies M''\vec{y} = \frac{(m-3)(n-3)(m+n-2)-2}{mn(m-3)(n-3)-4} \cdot \mathbf{1}$$

$$\implies v - v' = \frac{2(m-2)(n-2)}{mn(mn(m-3)(n-3)-4)} > 0 \ \forall \ m, n \geq 4.$$

Kneser Graph, K(n, k)

- Vertices: k-element subsets of $[n] = \{1, \cdots, n\}$.
- $\{v_i, v_j\} \in E \iff$ corresponding subsets are disjoint.
- Regular Partition: based on number of elements in common with $[k] = \{1, \cdots, k\}$.

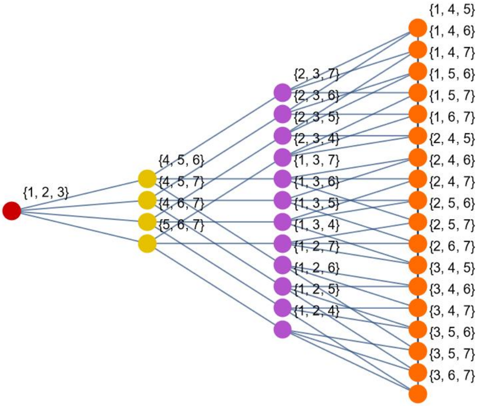
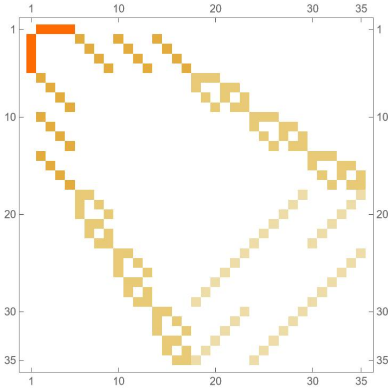


Figure: Kneser(7, 3)

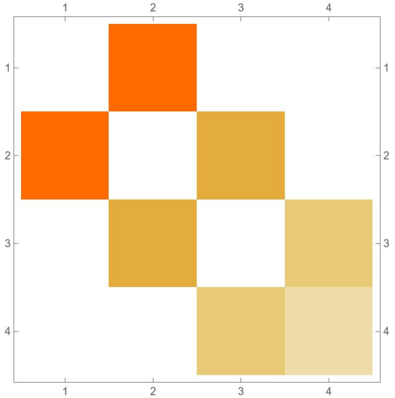
Adjacency Matrix of K(7, 3):



Dimension of M : 35×35

Game Value: $v = \frac{4}{35}$

Intersection-Probability Matrix:



Dimension of I-P Matrix: 4×4

General Dimension:$(k+1) \times (k+1)$

$\vec{y} = \langle 0, \frac{2}{9}, \frac{4}{9}, \frac{1}{3} \rangle \Rightarrow v' = \frac{4}{36} < \frac{4}{35} = v$

Kneser Graph, K(n, k)

For General Kneser Graphs:

- $v = \dfrac{\binom{n-k}{k}}{\binom{n}{k}}$
- A Column Strategy on M'' :

$$y_j = \frac{\binom{n-k}{k}\binom{n-k-1}{k}}{\binom{n}{k}\binom{n-k-1}{k}+1}\left(\frac{k\binom{k}{j}\left((-1)^{j+k}+\binom{n-k-1}{j}\right)}{\binom{n-k-1}{k-1}(n-j-k)}\right)$$

$$\forall\, j = 0, \cdots, k.$$

$$\implies M''\vec{y} = \frac{\binom{n-k}{k}\binom{n-k-1}{k}}{\binom{n}{k}\binom{n-k-1}{k}+1}\mathbf{1}$$

$$\implies v' = \frac{\binom{n-k}{k}\binom{n-k-1}{k}}{\binom{n}{k}\binom{n-k-1}{k}+1} < \frac{\binom{n-k}{k}\binom{n-k-1}{k}}{\binom{n}{k}\binom{n-k-1}{k}} = \frac{\binom{n-k}{k}}{\binom{n}{k}} = v$$

Next Steps and Further Questions

- Which graph families correspond to matrix games whose optimal row strategies must use every row? We conjecture that all non-singular vertex-transitive graphs have this property.
- What information about the spectrum of a graph can be obtained from regular partitions?
- Can we classify vertices by the effect of the value of the game if removed from the graph?

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