Graph Theoretic Interpretations of the Nevanlinna Representation

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Overview of Presentation

1. Pick Functions and the Nevanlinna Representation
2. Graph Motivation and Research Question
3. Results: Graph Products, Reduction, Distance
4. Future Work

The Nevanlinna Representation

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We can generate Pick functions using the following map:

Nevanlinna Representation

For a self-adjoint matrix A, positive contraction Y, and real vector α, the **Nevanlinna Representation** is:

$$f(z,w) = \langle (A - z_Y)^{-1}\alpha, \alpha \rangle = \langle (A - zI - w(I - Y))^{-1}\alpha, \alpha \rangle$$

which is a rational self-map on the complex upper-half plane $\mathbb{H}$.

Graph Theoretic Inputs Yield Graph Theoretic Outputs

If we input...

- Adjacency matrix of a simple, undirected graph for A
- Vertex coloring matrix: $Y = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$
- $\alpha = e_i$, the i^{th} basis vector

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Nevanlinna Representation. (with Graph Inputs)

The rational self-map on $\mathbb{H}$,

$$f(z, w) = (A_C)^{-1}_{(i,i)} = [I - (A + zI + w(I - Y))]^{-1}_{(i,i)},$$

studies paths from vertex i to vertex i.

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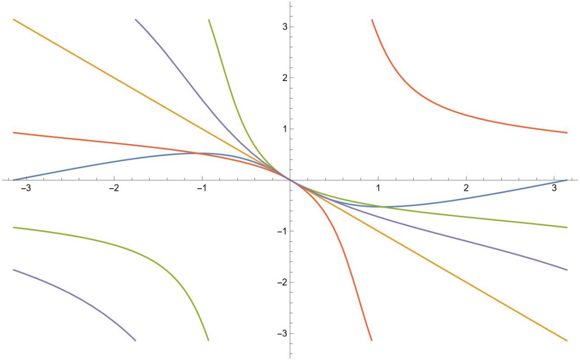
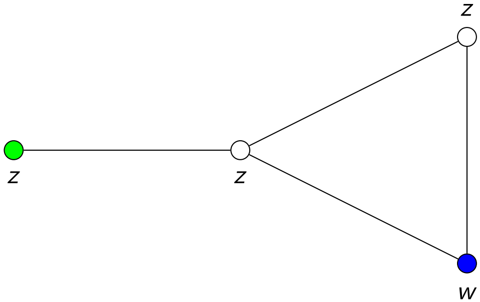


Figure: Level Curves of a rational inner function ϕ, obtained via a conformal map on the Nevanlinna output.

Example



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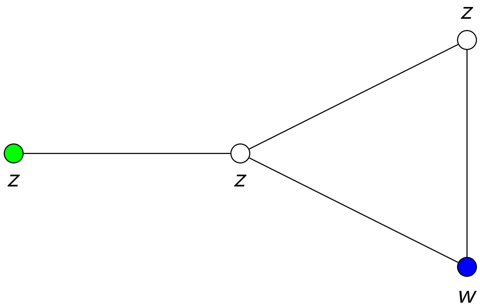

$$f(z,w) = (A_C)^{-1}_{(1,1)} = \frac{z - wz^2}{1 - 2z - 2wz - 2z^2 + wz^3}$$

$$f\left(\frac{1}{\bar{z}}, w\right) = -z - \frac{z^2}{w} + O[z]^3$$

$$\Lambda_f\left(\frac{1}{\bar{z}}, t\right) = 2z + 2z^2 + 3z^3 + \frac{(-1+4t)}{t} z^4 + O[z]^5$$

Motivating Question

What are the relationships between a graph's structure and the asymptotic behavior of its representing function?

Graph Products

Definition: Star Product

Given simple, undirected, rooted graphs $(G, r), (G', r')$, their star product, denoted $G \star G'$, attaches G and G' together by merging vertices r and r'.

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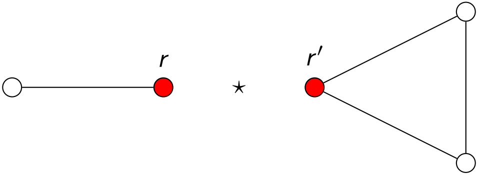

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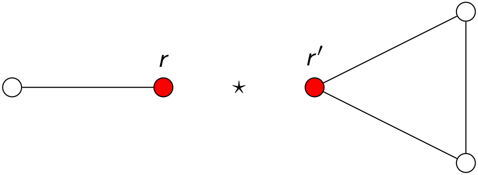
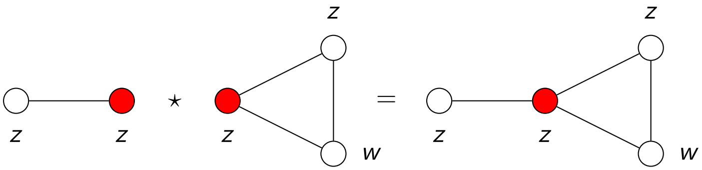


Proposition 2.1 (Arizmendi, Hasebe, Lehner, 2022)

For rooted graphs G and G', denote f as representing functions and $g = \frac{1}{f}$. Then the following formula holds:

$$g_{G \star G'}(z) = g_G(z) + g_{G'}(z) - g_0(z).$$

Star Product Representing Function



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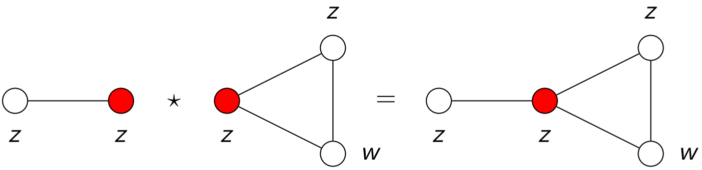

$\Downarrow f_1$ $\qquad$ $\Downarrow f_2$ $\qquad$ $\Downarrow f = (A_C)^{-1}_{(2,2)}$

$$\frac{z}{1-z^2} \qquad \frac{wz-1}{2+w+2z-wz^2} \qquad \frac{z-wz^2}{1-2z-2wz-2z^2+wz^3}$$

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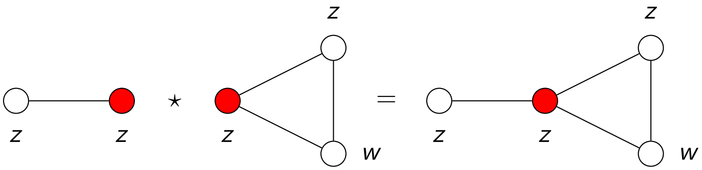

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$\Downarrow$	$\Downarrow$	$\Downarrow$
$g_1 = \frac{1}{f_1}$	$g_2 = \frac{1}{f_2}$	$g = \frac{1}{f}$

Star Product Representing Function

$$g_1 + g_2 = \frac{1 - 2z - 2wz - 3z^2 + 2wz^3}{-(z - wz^2)}$$

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&= g_1(z, w) + g_2(z, w) - g_0(z, w)
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Two-variable Star Product.

Let G and G' be rooted graphs with a shared color of z on their root vertices. Define the zero graph to be a single vertex with color z. Then:

$$g_{G \star G'}(z, w) = g_G(z, w) + g_{G'}(z, w) - g_0(z, w).$$

Connection to Circuits

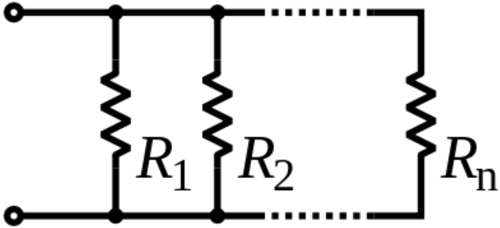

Figure: Parallel circuit. The total resistance of all components is:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$

Graph Reduction and Composition

Theorem. (Retraction)

Let G be a graph that can be written as $G = H \star K$ connected only at vertex i. Then $f_G(z, w) = f_{H'}(z, w)$, where H' is graph H with vertex i colored with $-g_K(z, w)$.

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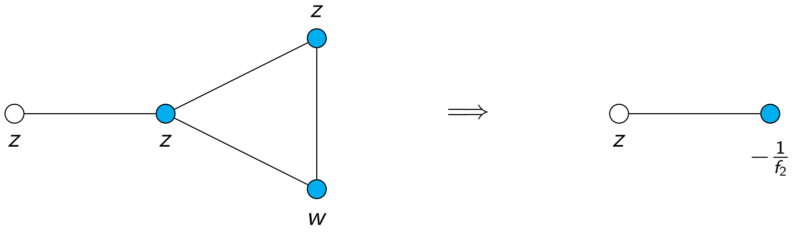

Function Asymptotics & Graph Distance

Lemma. (Order of Vanishing & Distance)

Let G be a z-colored connected graph with adjacency matrix A. Then $f_{i,j}(z)$ has *order of vanishing* $n+1$ when expanded at ∞, where n is the length of the shortest path from i to j.

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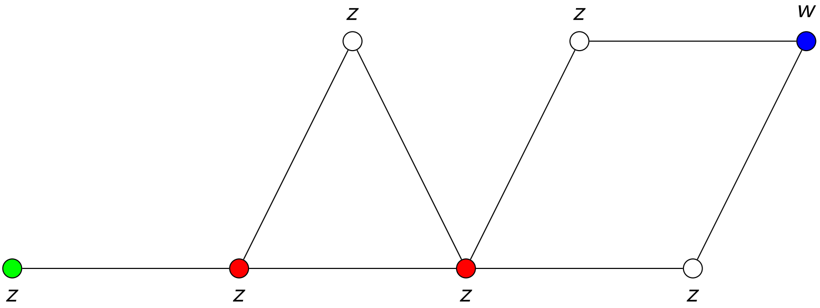
Theorem. (Contact Order & Distance)

Let G be a graph with vertex j colored w and the rest colored z. Then $f_{i,i}(z,w)$ has *contact order* $2n$, where the path length from i to j is n.

Putting it all Together



Future Work

Application to Circuits and Networks.
How similar are other properties of representing functions to results in circuit and network theory? (e.g. total resistance of a parallel circuit)

Coloring Schemes.
What if we colored more than 1 vertex with a w? *Linear combinations* of z's and w's on a vertex?

Eigenvalues.
What kind of information would the eigenvalues of A tell us about the asymptotic behavior of the representing function?

Thank you!

- Team Members: Lily Adlin, Samuel Tiscareno, Dr. Ryan Tully-Doyle
- William and Linda Frost Fund, Cal Poly Bailey College of Science and Mathematics
- CSU Bakersfield