Characterizing Non-Singular Graphs through Games

Dr. Jeffrey Liese, Cameron Klig, Riley Lane, Matthew Moscot, Giovani Thai
Department of Mathematics, California Polytechnic State University, San Luis Obispo

Introduction

The goal of our research was to prove that certain graph families are non-singular using a novel approach combining principles from game theory, graph theory, and linear algebra. Fundamental to our work was utilizing *Mathematica* to produce large numbers of graph examples which then allowed us to make useful conjectures.

Definitions

- A ***graph*** is a pair $G = (V, E)$ where V is a finite set of ***vertices*** and E is a set of paired elements from V whose elements are called ***edges***.
- Let G have vertices $v_1, \ldots, v_n$. The ***adjacency matrix*** $A(G)$ is an $n \times n$ matrix with (i, j) entry equal to 1 if there is an edge between v_i and v_j and 0 otherwise. This matrix is symmetric for an *undirected graph*.
- A ***graph automorphism*** is a bijective function $f : V \to V$ that preserves edges, and a graph is called ***vertex-transitive*** if, for any vertices $v_1, v_2 \in V$, there exists an automorphism f such that $f(v_1) = v_2$.
- In a ***two-player, zero-sum matrix game***, two players each independently choose a row i and a column j, respectively of a matrix A. The column player then pays the the row player an amount equal to A_{ij}.
- An ***optimal solution*** consists of a row strategy that maximizes that player's earnings, a column strategy that minimizes that player's losses, and the value of the game (which is the expected payout under optimal play.)

Our Method

Theorem 1 (**Shapley-Snow**). *Extreme optimal solutions of matrix games come from invertible submatrices; if an extreme optimal solution uses every row (i.e. plays every row with positive probability), the matrix must be invertible.*

We use the above theorem and show that for certain matrices, every optimal solution uses every row by witnessing a decrease in the value of the game when removing an arbitrary row. Outline of method:

1. Let M be the adjacency matrix of a graph G and play a matrix game on M. If G is vertex transitive, we only need to consider the removal of any one specific row, so let M' be the matrix obtained by removing row v from M.
2. Produce a regular partition P of the vertices where v is a singleton.
3. Construct the intersection-probability matrix, a matrix whose rows and columns are indexed by the components of P and the associated matrix entry is the density of edges between the components. Remove the row corresponding to the singleton v to obtain M''.
4. Find a column strategy on M'' that shows that the game value on M'' is strictly less than the game value on M. (This is usually the hard part!)

Example: The Kneser (7, 3) Graph

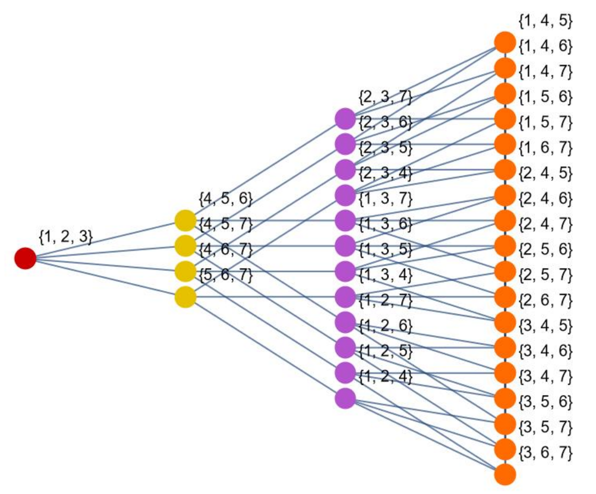


Fig. 1: The Kneser Graph (7,3). Vertices are the 3-element subsets of $\{1, \ldots, 7\}$; two vertices are adjacent iff they are disjoint. Vertex color shows a regular partition (induced by the number of elements in common with $\{1,2,3\}$) of the vertices into 4 groups.

Matrices Associated with Kneser (7, 3)

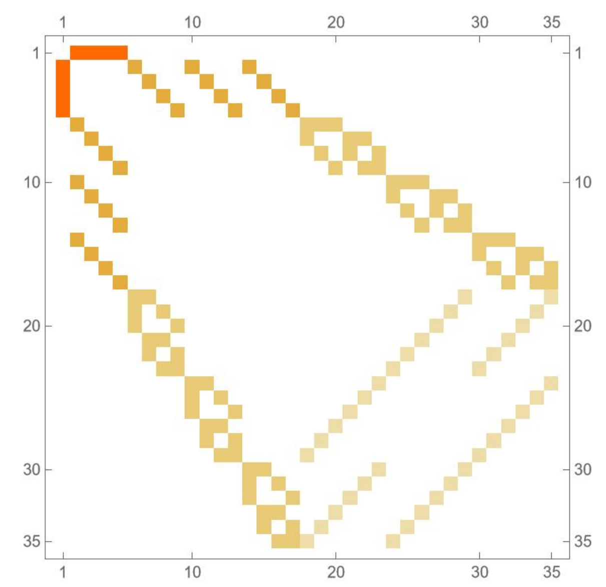


Fig. 2: Adjacency matrix of the Kneser (7, 3) graph.

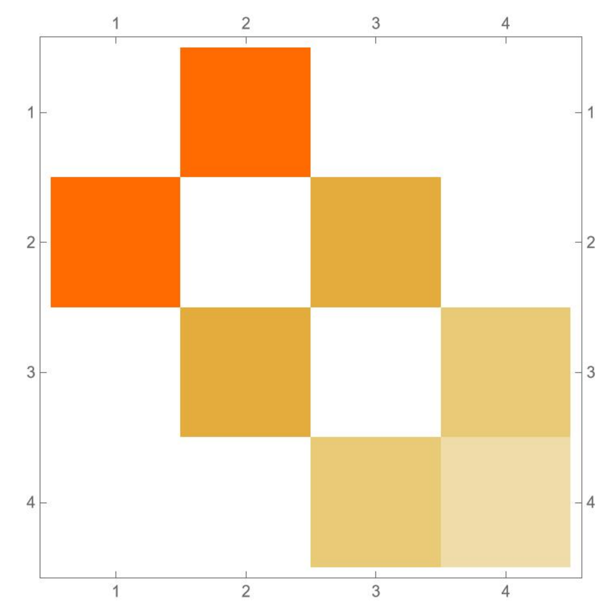


Fig. 3: Intersection probability Matrix of Kneser (7, 3).

Optimal Column Strategies on Kneser (7, 3)

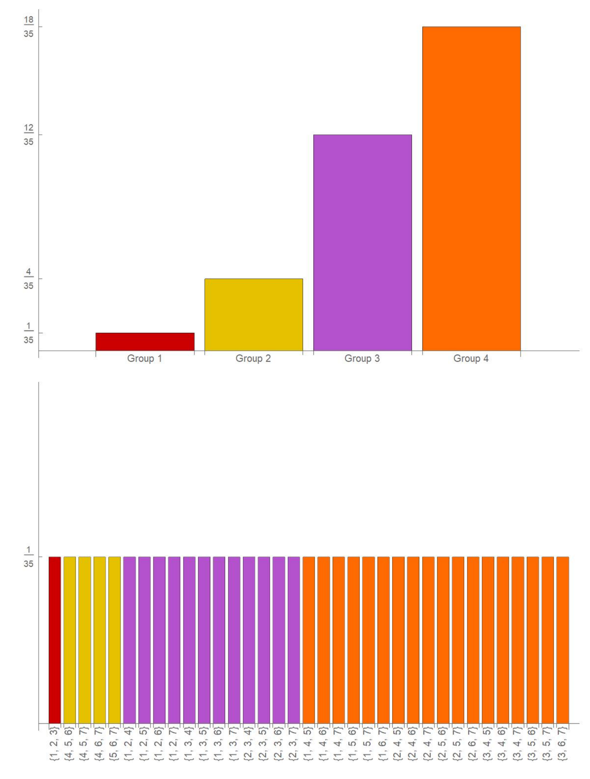


Fig. 6: Optimal strategy; $v = \frac{4}{35}$.

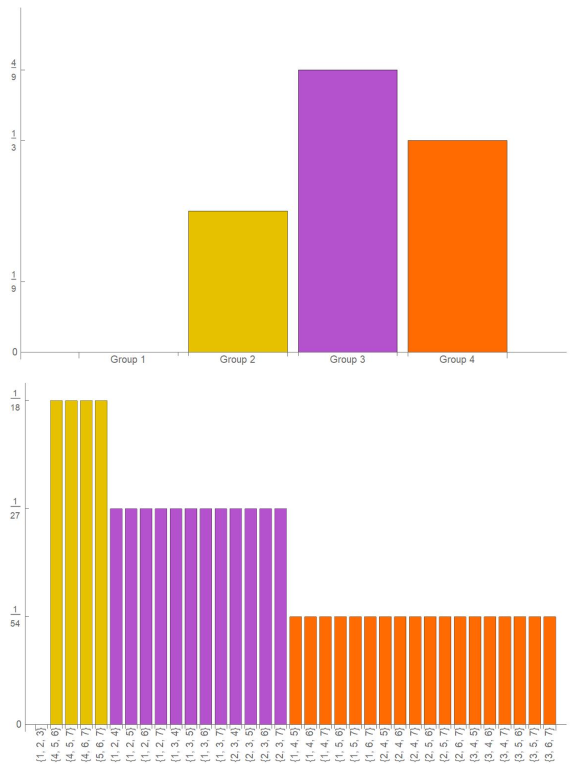


Fig. 7: Optimal strategy without first row; $v' = \frac{4}{36} = \frac{1}{9}$.

Results

Throughout this project we have used our method to prove the non-singularity of many graph families including:

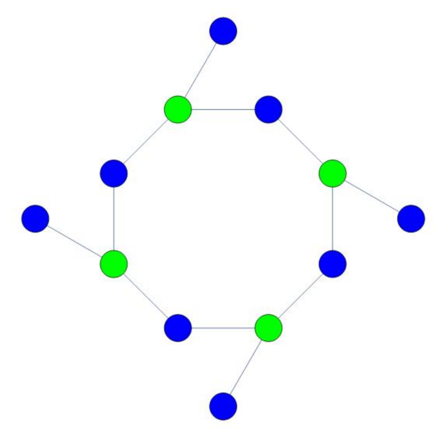
- Complete Graph
- Cycle Graph
- Rook's Graph
- Rook Complement Graph
- Andrásfai Graph
- Kneser Graph
- Cyclotomic Graph
(Check the QR code for more visuals/proofs!)

Open Questions

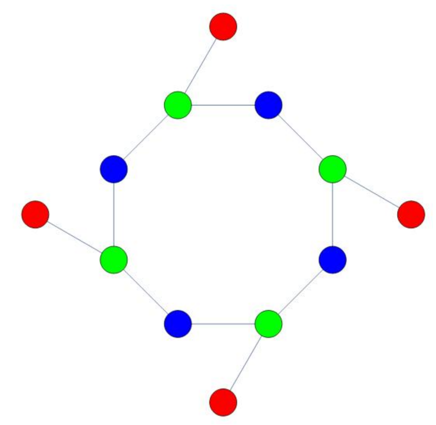
Can we completely determine the eigenvalues and eigenvectors of an adjacency matrix from an associated intersection or intersection probability matrix?

Can we classify vertices by the effect of their removal from the graph?

- Removing a vertex from row play, color-code those that reduce or maintain the value when removed.



- Removing a vertex from BOTH row and column play, color-code those that reduce, maintain, or increase the value when removed.



Which graph families correspond to matrix games whose optimal row strategies must use every row?

Acknowledgments

Our research was supported through the Frost Undergraduate Student Research Award from the Bill and Linda Frost Fund for Summer 2022.