SISSA mathLab 15 2010-2025

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From Bifurcations to Machine Learning

Federico Pichi

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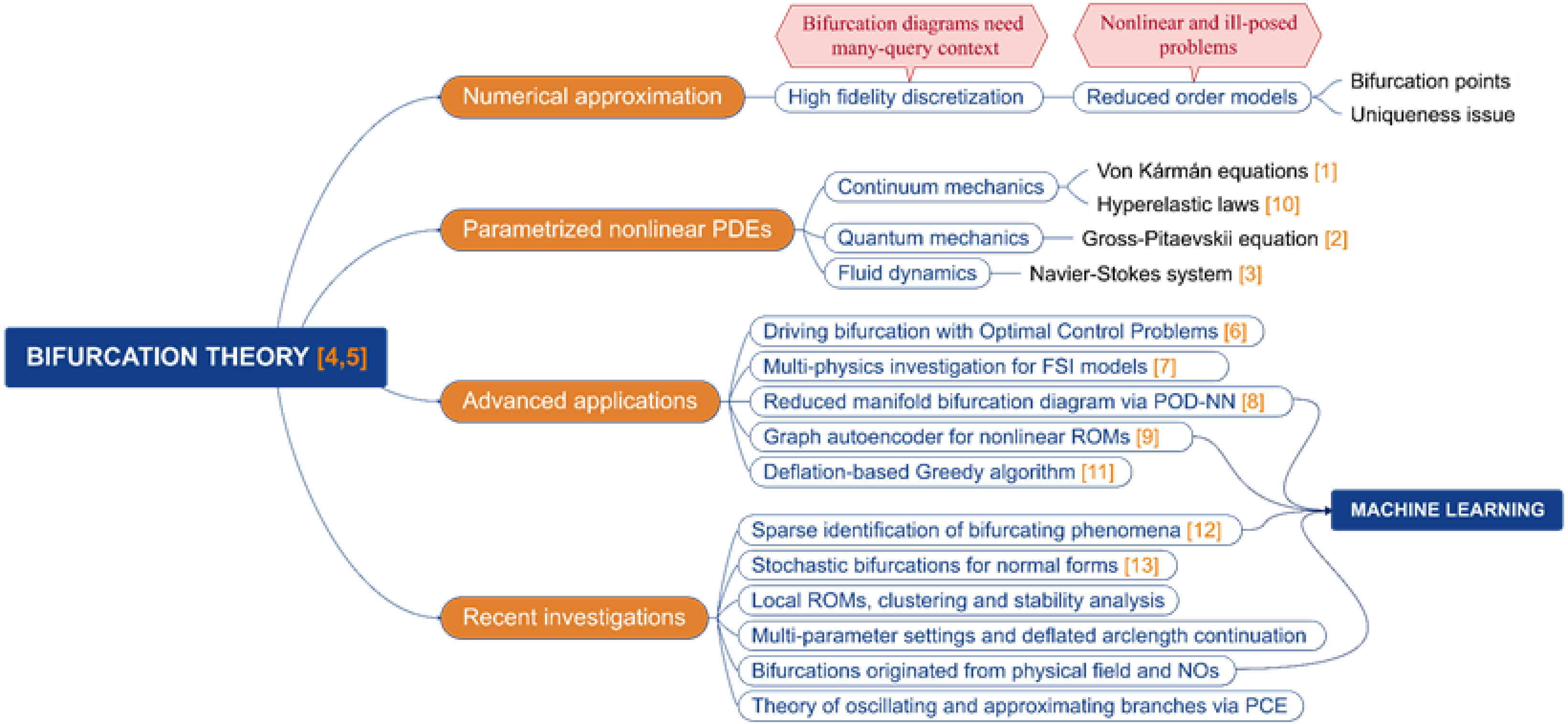
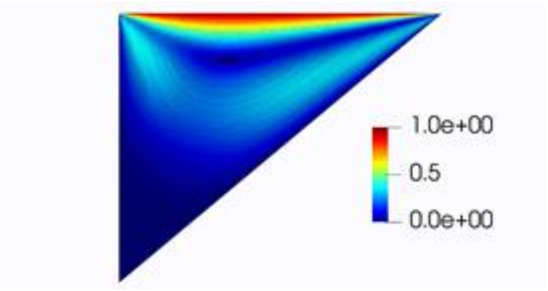
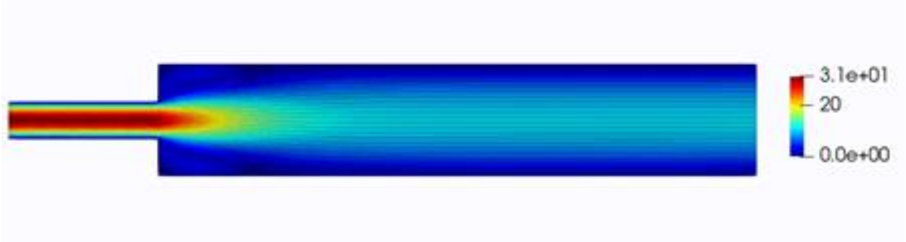
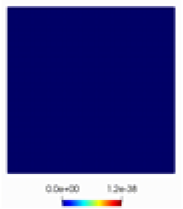
Exploring multiple directions

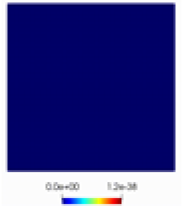


What are bifurcations?

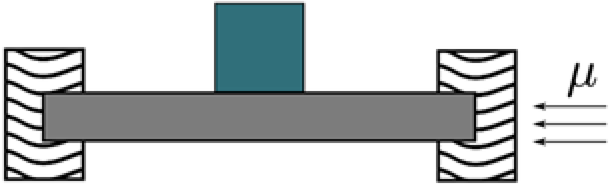








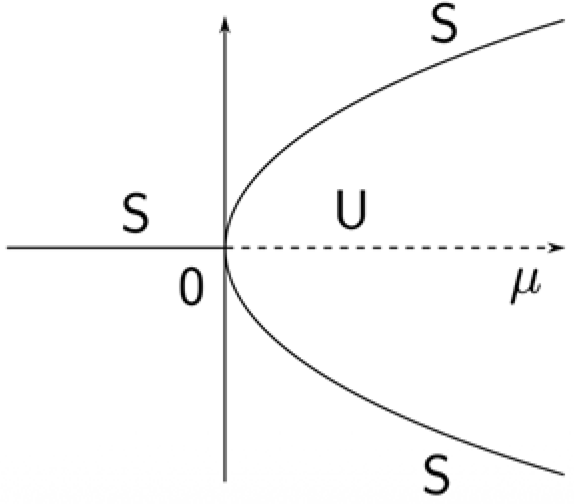
Example: A **beam** under compression **buckles** at a **critical load**



Example from ODEs

Landau's model describes **hydrodynamical instabilities**:

$$\frac{du}{dt} = \mu u - u^3$$



Supercritical pitchfork bifurcation

$$\mu^* = 0$$

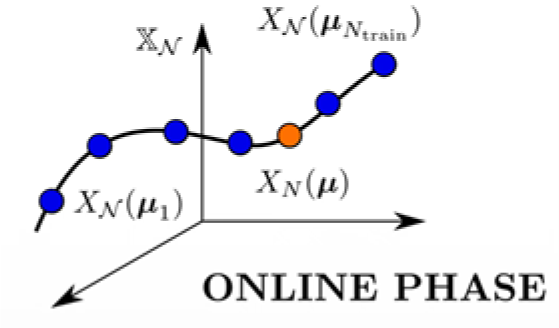
- $\mu \leq 0 \quad \rightarrow \quad \exists!$ solution $u \equiv 0$
- $\mu > 0 \quad \rightarrow \quad 3$ coexisting solutions

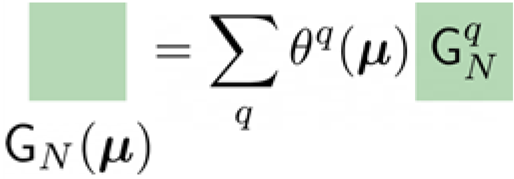
Projection-based Reduced Order Models (ROMs)

Common issues in the for bifurcating nonlinear PDEs context:

- Unbearable **computational cost** with nonlinear solver for **fine meshes** and **nested loops**
- Choice of the **initial guess** and **continuation method** to completely reconstruct the **bifurcation diagram**
- Identification of the **bifurcation points/curves/surfaces** in multi-parametric setting

$$\underbrace{G(X(\boldsymbol{\mu});\boldsymbol{\mu})=0}_{\text{Partial differential equation}} \rightsquigarrow \underbrace{G_{\mathcal{N}}(X_{\mathcal{N}}(\boldsymbol{\mu});\boldsymbol{\mu})=0}_{\text{High Fidelity approximation}} \rightsquigarrow \underbrace{G_N(X_N(\boldsymbol{\mu});\boldsymbol{\mu})=0}_{\text{Reduced Basis approximation}}$$





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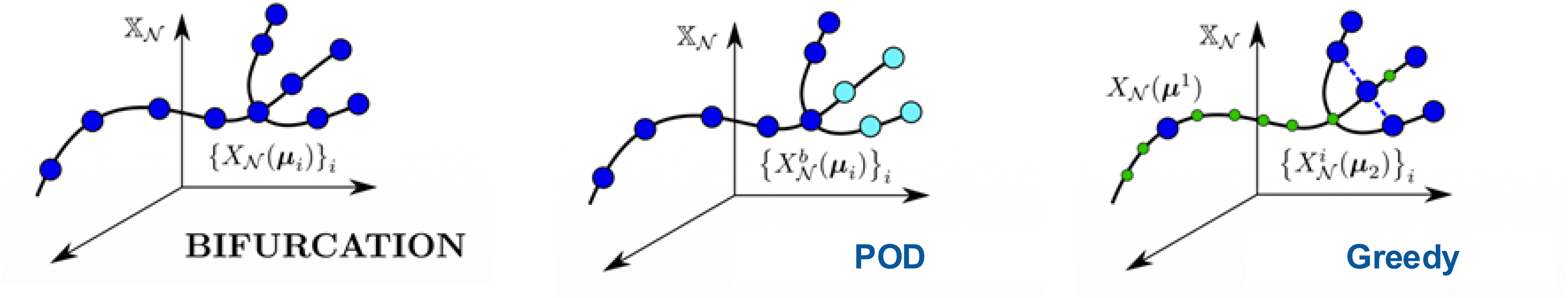
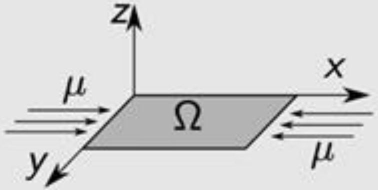


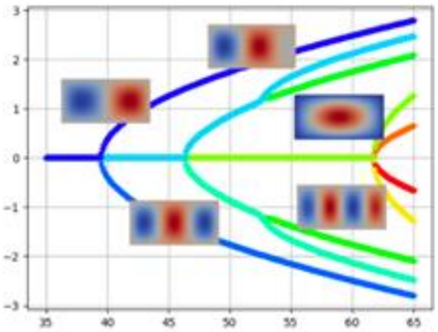
Bifurcation Problems

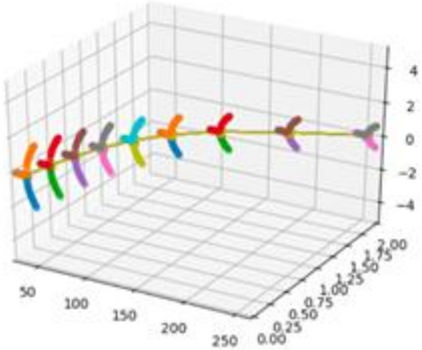
Von Kármán's buckling plate

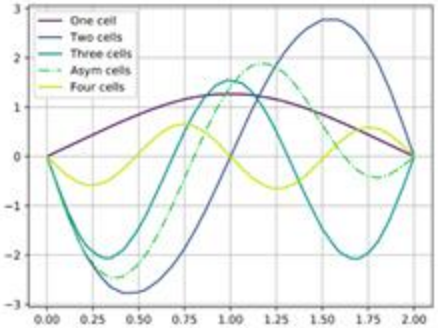
Efficient POD-based computation of multiple and secondary bifurcation branches in the multi-parametric setting coupled with spectral analysis.

$$\begin{cases} \Delta^2 u + \mu u_{xx} = [\phi, u] & \text{in } \Omega, \\ \Delta^2 \phi = -[u, u] & \text{in } \Omega, \end{cases}$$







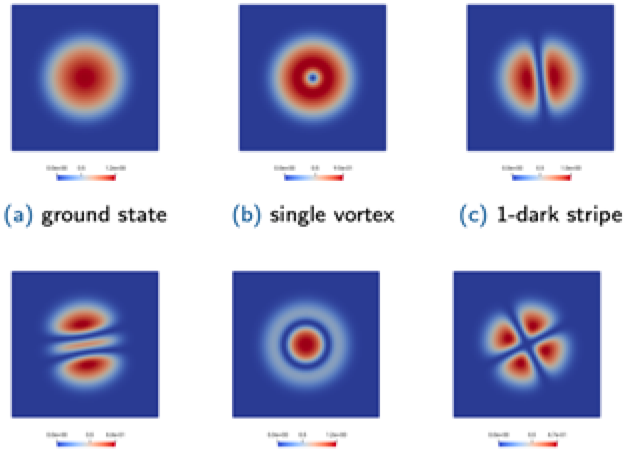


[1] FP, Rozza (2019). **"Reduced basis approaches for parametrized bifurcation problems held by non-linear Von Kármán equations".** Journal of Scientific Computing

Gross-Pitaevskii BEC equation

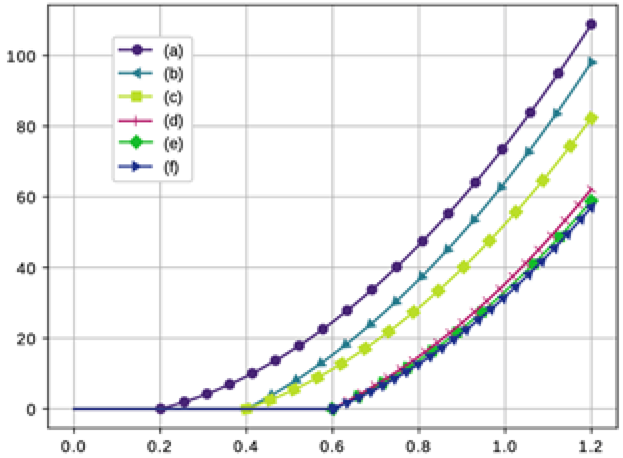
Branch-wise approach with **Hermite** guesses detecting **(D)EIM** instabilities.

$$-\frac{1}{2}\Delta\phi + |\phi|^2\phi + W(\Omega)\phi - \mu\phi = 0$$



(a) ground state (b) single vortex (c) 1-dark stripe

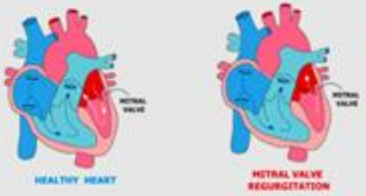
(d) 2-dark stripe (e) soliton ring (f) soliton cross



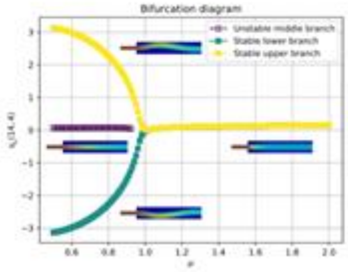
[2] FP, Quaini, Rozza (2020). "A **Reduced Order Modeling Technique to Study Bifurcating Phenomena: Application to the Gross-Pitaevskii Equation**". SISC

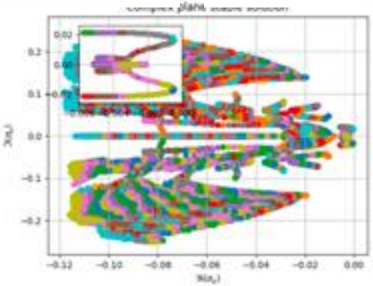
Navier-Stokes and the Coanda effect

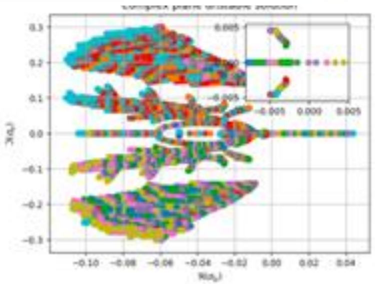
$$
\begin{cases}
-\mu\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = 0 & \text{in } \Omega,\\
\nabla\cdot\mathbf{v} = 0 & \text{in } \Omega,
\end{cases}
$$



Detection of **multiple wall-hugging** phenomena exploiting **deflation strategy**.







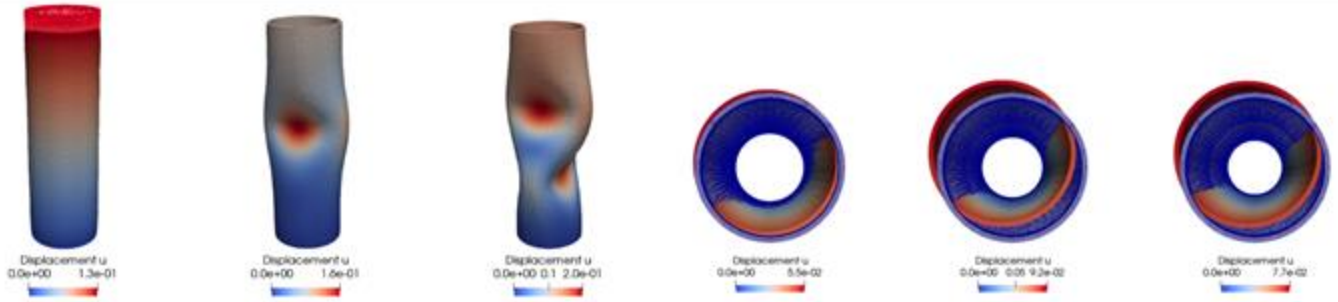
[3] Pintore, FP, Hess, Rozza, Canuto (2020). “**Efficient computation of bifurcation diagrams with a deflated approach to RB spectral element method**”. ACOM

Buckling of hyper-elastic beams

Question: How **BCs** and **geometrical parameters** affect the bifurcation?

How: **Speedup** for realistic **SVK 3D** models with **Dirichlet** and **Neumann** compression for Norwegian petroleum industry.

$$-\mathrm{div}\left(\frac{\partial\psi(F)}{\partial F}\right) = f \quad \text{in } \Omega, \qquad \text{where} \qquad \psi(F) = \lambda_1 \mathcal{E} : \mathcal{E} + \frac{\lambda_2}{2}(\mathrm{tr}(\mathcal{E}))^2$$



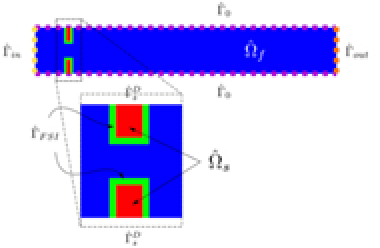
[10] FP, Rozza (2024). "**Reduced Order Models for the Buckling of Hyperelastic Beams**". RAMSES

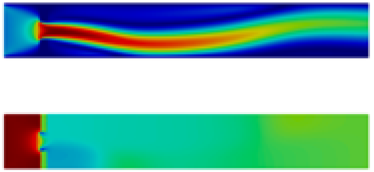
Multi-physics bifurcations

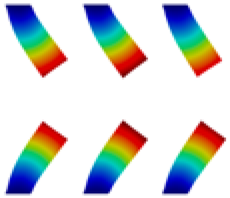
Question: How does the **elastic structure** influence the bifurcating behavior?

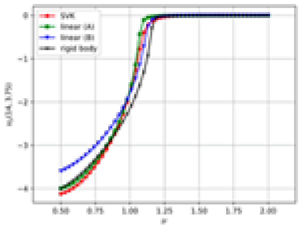
How: **ALE** coupled system with **nonlinear** constitutive relations for the leaflets.

$$\begin{cases} \rho_f J \nabla \mathbf{u}_f \mathbf{F}^{-1} \mathbf{u}_f - \operatorname{div}(J \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) \mathbf{F}^{-T}) = 0 \quad \text{in } \Omega_f, \\ \operatorname{div}(J \mathbf{F}^{-1} \mathbf{u}_f) = 0 \quad \text{in } \Omega_f, \\ \Delta \boldsymbol{d}_f = 0 \quad \text{in } \Omega_f, \\ -\operatorname{div} \mathbf{P}(\boldsymbol{d}_s) = 0 \text{ in } \Omega_s, \end{cases} \quad \text{with} \quad \begin{cases} \boldsymbol{d}_f = \boldsymbol{d}_s \\ \mathbf{u}_f = 0 \\ \mathbf{P}\boldsymbol{n} = (J \boldsymbol{\sigma}_f \mathbf{F}_{fluid}^{-T}) \boldsymbol{n} \end{cases} \text{on } \hat{\Gamma}_{FSI}$$









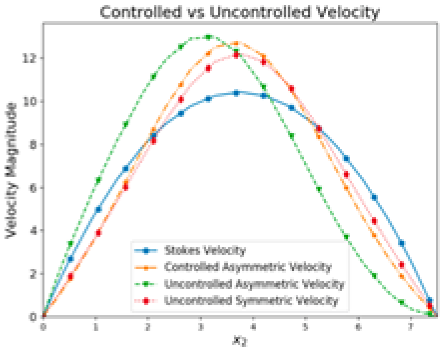
[7] Khamlich, FP, Rozza (2021). "**Model order reduction for bifurcating phenomena in Fluid-Structure Interaction problems**". IJNMF

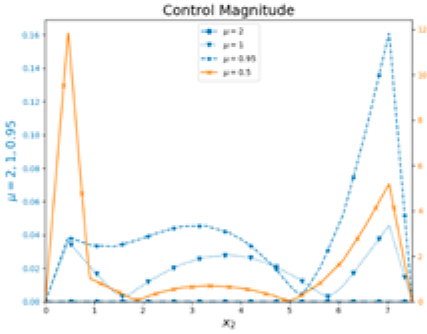
Advanced Applications and Machine Learning

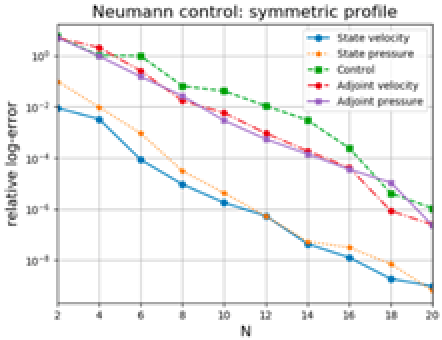
Optimal Control Problems to steer bifurcations

Questions:
- Can we influence the bifurcating behavior of the system?
- How do the stability properties of the system change?

Goal: Recover a desired v_d on Γ_{obs}
$\rightsquigarrow$ symmetric Stokes solution







[5] FP, Strazzullo, Ballarin, and Rozza (2022). “**Driving bifurcating parametrized nonlinear PDEs by optimal control strategies**”. ESAIM: M2AN

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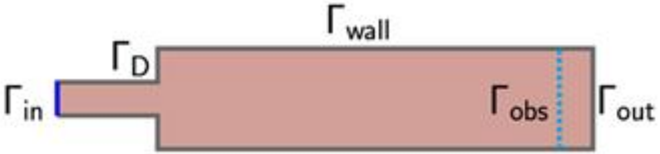

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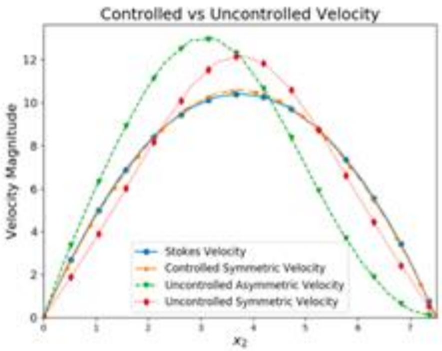

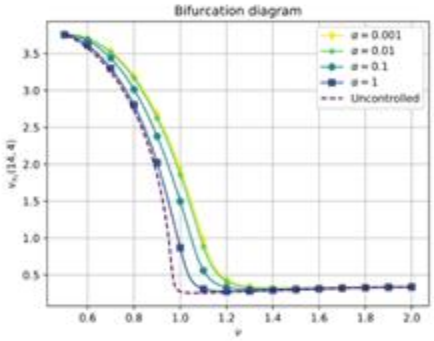

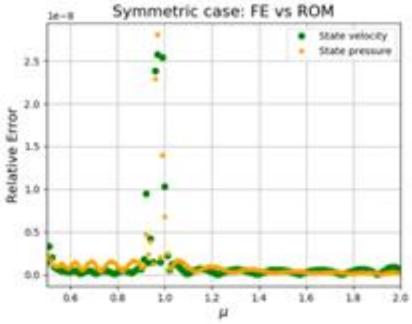

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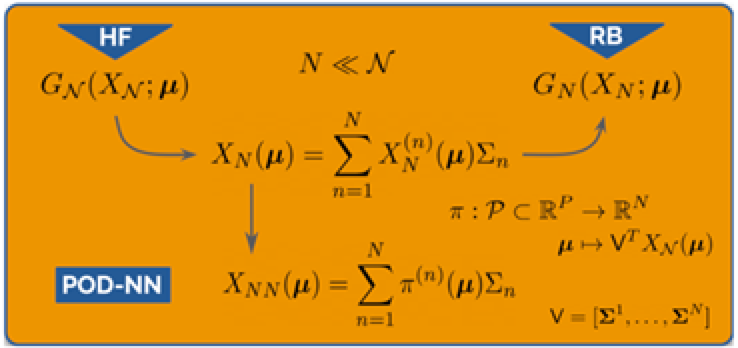
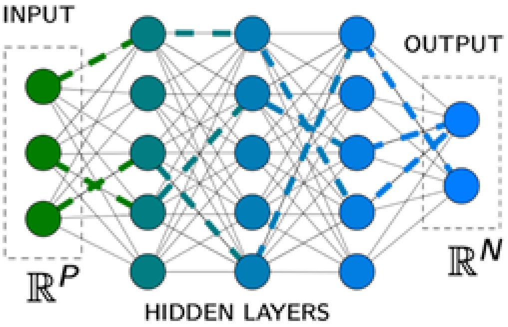
Data-driven and non-intrusive linear ROMs

Goal: Investigate efficiently complex **bifurcating** behavior in **real-time**.

How: Combining **ROMs** and **deep learning** of reduced coefficients.





POD-NN approach:
approximate $\pi : \mathcal{P} \subset \mathbb{R}^P \to \mathbb{R}^N$ such that $\boldsymbol{\mu} \mapsto \pi(\boldsymbol{\mu})$ from a training set given by the pairs $\{(\boldsymbol{\mu}^i, \mathsf{V}^T X_\mathcal{N}(\boldsymbol{\mu}^i))\}_{i=1}^{N_{tr}}$ obtained from the offline POD procedure.

[8] FP, Ballarin, Rozza, and Hesthaven (2022). "**An artificial neural network approach to bifurcating phenomena in computational fluid dynamics**". C&F

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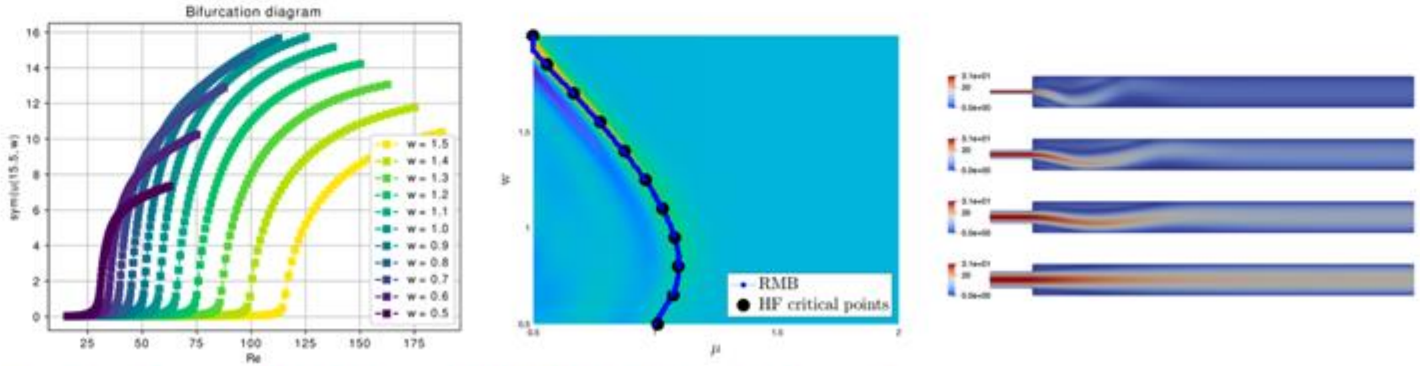


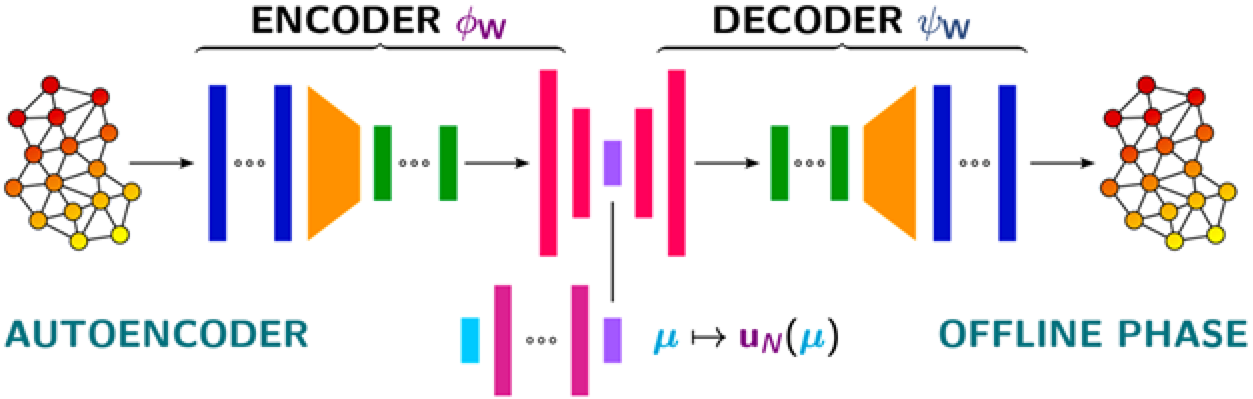
Reduced manifold based (RMB) bifurcation diagram:

Curvature-based detection tool exploiting the **non-smoothness** of the reduced manifold to track **critical points** evolution.

[8] FP, Ballarin, Rozza, and Hesthaven (2022). “**An artificial neural network approach to bifurcating phenomena in computational fluid dynamics**”. C&F

Nonlinear graph-based ROM

Graph Convolutional Autoencoder as efficient and **geometrically-consistent** nonlinear ROM for complex and **varying domains** with unstructured meshes.

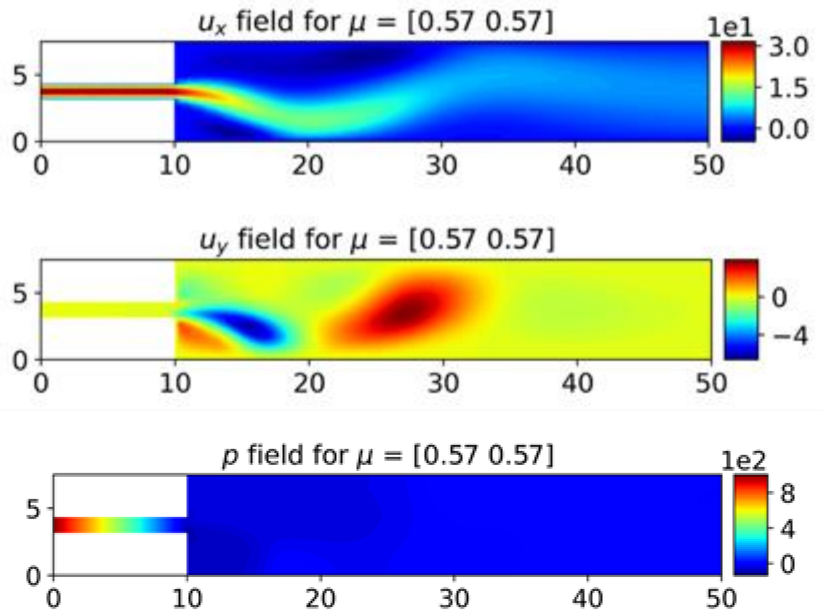


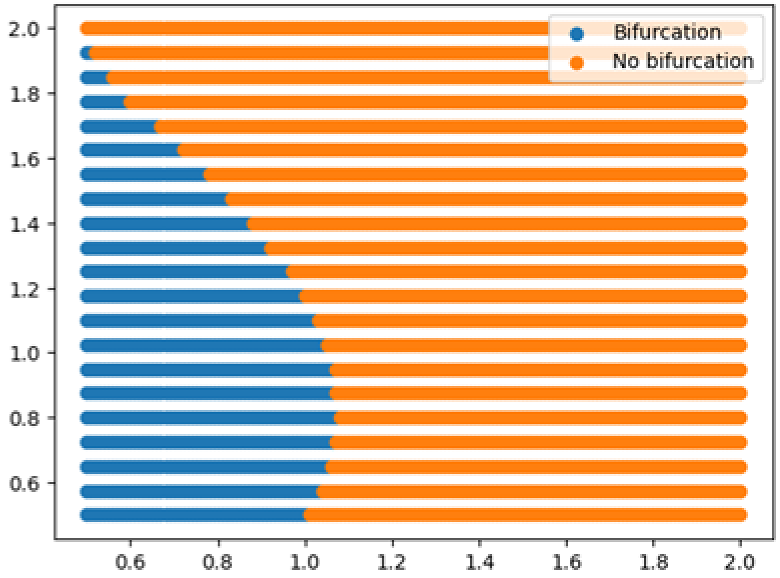
$$\underline{\textit{Dataset}}:\ \{u_{\mathcal{N}}(\mu^i), \Omega_{\mathcal{N}}(\mu^i)\}_{i=1}^{N_s} \quad \rightsquigarrow \quad \text{approximate } \tilde{u}_{\mathcal{N}}(\mu) = \psi_{\mathbf{W}}(\phi_{\mathbf{W}}(u_{\mathcal{N}}(\mu)))$$

$$\textbf{LOSS}: \quad \mathcal{L} = \frac{1}{N_{tr}} \sum_{i=1}^{N_{tr}} \|\mathbf{u}_N(\mu^i) - \tilde{\mathbf{u}}_N(\mu^i)\|_2^2 + \frac{1}{N_{tr}} \sum_{i=1}^{N_{tr}} \|\tilde{u}_{\mathcal{N}}(\mu^i) - u_{\mathcal{N}}(\mu^i)\|_2^2$$

[9] FP, Moya, Hesthaven (2024). "**A convolutional graph neural network approach to model order reduction to non-linear parametrized PDEs**". JCP

Nonlinear graph-based ROM





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Collaborators on bifurcating phenomena

- F. Ballarin
- C. Canuto
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- N. Kumar
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- A. Quaini
- G. Rozza
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- G. Venier
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- M. Hirsch
- D.B.P. Huynh
- G. Noselli
- O. Morrison
- L. Mosconi
- O. Mula
- D. Oberto
- A. Orunnukaran
- M. Ramadan
- S. Rathore
- F. Romor
- L. Scandurra
- C. Schwab
- G. Stabile
- N. Tonicello
- D. Torlo
- C. Valentino

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[2] Pichi, F., Quaini, A., Rozza, G., 2020. **A Reduced Order Modeling Technique to Study Bifurcating Phenomena: Application to the Gross--Pitaevskii Equation**. SIAM J. Sci. Comput. 42, B1115–B1135. https://doi.org/10.1137/20M1313106

[3] Pintore, M., Pichi, F., Hess, M., Rozza, G., Canuto, C., 2021. **Efficient computation of bifurcation diagrams with a deflated approach to reduced basis spectral element method**. Adv Comput Math 47, 1. https://doi.org/10.1007/s10444-020-09827-6

[4] Pichi, F., 2020. **Reduced order models for parametric bifurcation problems in nonlinear PDEs** (Ph.D. Thesis). SISSA. https://iris.sissa.it/handle/20.500.11767/114329

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Thank you for your attention!

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