Assignment 3

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1 Discontinuos Galerkin formulation

Let us consider the following wave propagation problem in $(0, L) \times (0, T]$:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + \frac{\partial F(u)}{\partial x}(x,t) = 0 \quad in \;\; (0,L) \times (0,T] \\ u(x,0) = u_0(x) \quad in (0,L). \end{cases} \tag{1}$$

This is a Riemann problem and in the case of $F(u) = \frac{u^2}{2}$ we are in a particular case known as Burger's equation. Decomposing the function partial derivative, we can write that

$$\frac{\partial F(u)}{\partial x} = \frac{\partial F(u)}{\partial u}\frac{\partial u}{\partial x} = u\frac{\partial u}{\partial x}, \tag{2}$$

so the eq(1) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3}$$

This result is very similar to the Navier-Stokes equation, but without pressure and viscosity terms.
We were given $u_0(x) = e^{-(x-3)^2}$, a regular enough function, and the boundary conditions

$$u(0,t) = \phi(t) \;\; t \in (0,T] \tag{4}$$

where ϕ is a regular enough function that regulate the inflow boundary for $x = 0$.
We know that in case of a non-linear hyperbolic equation, there may be some discontinuity. In particular the discontinuity propagates with a finite velocity inside domain and in the point of the discontinuity shows up the Gibbs phenomena.
In this case we have a non linear problem, so the solution can be:

$$\begin{cases} u_L > u_R \longrightarrow \;\; shock\; wave \\ u_L < u_R \longrightarrow \;\; rarefaction\; wave \end{cases} \tag{5}$$

where u_L is the value of u on the left of the discontinuity, and u_R is the value of u on the right of the discontinuity.
To treat this kind of problem we need to use the discontinuos Galerkin formulation: it creates elements that are not constrained to each other. The first step is to discretize the domain. Calling T_h the mesh of the domain $\Omega = (0, L)$, made by elements k_j, we define a discrete domain:

$$W_h = \{v_h \in L^2(0,L) : v_h|_{k_j} \in \mathbb{P}^r \; \forall k_j \in T_h\} \tag{6}$$

The solution $u(x, t)$ is approximated in to a discrete solution $u_h(x_j, t)$ in the domain W_h and exploiting the Gudunov method, the Galerkin formulation of the problem becomes:

$\forall t \in (0,T]$ $\forall v_h \in \mathbb{P}^0$ find $u_h(t) \in W_h$ such that: $\forall j = 0,1,...,m-1$

$$\int_{x_j}^{x_{j+1}} \frac{\partial u_h}{\partial t} v_h dx + \int_{x_j}^{x_{j+1}} \frac{\partial F(u_h)}{\partial x} v_h dx = 0. \tag{7}$$

We call the interval $I_j = (x_j, x_{j+1})$ and we solve by part the second integral

$$\int_{x_j}^{x_{j+1}} \frac{\partial F(u_h)}{\partial x} v_h dx = -\int_{x_j}^{x_{j+1}} F(u_h) \frac{\partial v_h}{\partial x} v_h dx + [F(u_h)v_h]_{x_j}^{x_{j+1}} \tag{8}$$

We now have that $H_{j+1} = F(u_h(x_{j+1}^{left}))$ and $H_j = F(u_h(x_j^{right}))$, where H_j is the non linear flux in the point x_j and depends on the value of u_h in x_j. If we also assume that $x_0 = 0$ is the inflow point, for $j = 0$ we have the eq. (4). We can write:

$$H(u_h, t) = H(u_h(x_j^{left}, t), u_h(x_j^{right}, t)). \tag{9}$$

If we want to define an admissible numerical flux which will lead us to an entropic solution, it has to be consistent with the flux in the case of constant function and to be a perturbation of a monotone scheme.
So let's now consider $u_h \in \mathbb{P}^0$, $u_h = u_h^{(j)} \in I_j$, $v_h = 1 \in I_j$, the first term of eq.(7) is $\int_{I_j} dx = h_j$. Consider also an admissible flux, like the Gudunov flux:

$$H(v,w) = \begin{cases} \min_{x\in[v,w]} F(u) & v \leq w \\ \max_{x\in[w,v]} F(u) & v > w \end{cases} \tag{10}$$

then eq.(7) is reduced to:

$$h_j \frac{\partial u_h^{(j)}}{\partial t} = H(u_h^{(j-1)}, u_h^{(j)}, t) - H(u_h^{(j)}, u_h^{(j+1)}, t) \tag{11}$$

Since $F(u) = \frac{1}{2}u^2$ is a strictly convex function, we have:

$$h_j \frac{\partial u_h^{(j)}}{\partial t} + F(u_h(x_{j+1}^{left})) - F(u_h(x_j^{right})) = 0 \tag{12}$$

and if we substitute $f(u) = \frac{u^2}{2}$ it becomes the Burger's equation:

$$h_j \frac{\partial u_h^{(j)}}{\partial t} + \frac{u_h^2}{2}(x_{j+1}^{left}, t) - \frac{u_h^2}{2}(x_j^{right}, t) = 0 \tag{13}$$

Now if we collect the terms that are not derived in time in one element $\mathcal{L}(u_h^j)$:

$$\mathcal{L}(u_h^j) = -\frac{1}{2h_j}(u_h^2(x_j^{right}, t) + u_h^2(x_{j+1}^{left}, t) \quad \forall I_j \tag{14}$$

we can rewrite the problem in the following system:

$$\begin{cases} M\dot{\mathbf{u}}_h = \mathcal{L}(\mathbf{u}_h) \\ \mathbf{u}_h(0) = \mathbf{u}_h^0(x) \end{cases} \tag{15}$$

where M is the sum of $m_j = \int_{I_j} v_h(x)u_h(x)dx$ elements.

2 Runge-Kutta scheme

To solve the system 15, we can use the Runge-Kutta scheme. We compute the third order that solve the n+1 time instant for the value at n, as written in the assignment. The results are discussed in the following section.

3 Critical time and speed of propagation of the discontinuity

The critical time is the instant in which appears the shock wave. That time can also be seen as the instant in which characteristic curve of the solution intersects the characteristic line of the shock wave. This occurs only if $u_{left} > u_{right}$. It can be calculated as:

$$t_c = -\frac{1}{inf_x \; u_0'(x)F''(u_0(x))} \tag{16}$$

In our case we have

$$\begin{cases} u_0(x) = e^{-(x-3)^2} \longrightarrow u_0'(x) = -2(x-3)u_0(x), \\ F(u) = \frac{u^2}{2} \longrightarrow F'(u) = u \longrightarrow F''(u) = 1, \end{cases} \tag{17}$$

and to find the minimum of $u_0'(x)$ we need to compute the roots of its second derivative:

$$u_0''(x) = -2u_0(x) - 2(x-3)u_0'(x) \longrightarrow u_0''(x) = -(2 - 4(x-3)^2)e^{-(x-3)^2} \tag{18}$$

$$u_0''(x) = (4x^2 - 24x + 34)e^{-(x-3)^2} \longrightarrow x_{1,2} = 3 \pm \frac{1}{\sqrt{2}}$$

Now computing $u_0'(x_1) = -\sqrt{2}e^{-\frac{1}{2}} = -\sqrt{\frac{2}{e}}$ and $u_0'(x_2) = +\sqrt{2}e^{-\frac{1}{2}} = \sqrt{\frac{2}{e}}$ we observe that $u_0'(x_1)$ is the minimum value, so finally:

$$t_c = \sqrt{\frac{e}{2}} \simeq 1.166[s], \tag{19}$$

After the time t_c the shock wave propagate with velocity σ, if there aren't any external forces, like in this case, σ can be calculated as:

$$\sigma \simeq \frac{x(max(u(t_{final}))) - x(max(u(t_c))}{t_{final} - t_c}.$$

That graphically returns $\sigma \simeq \frac{5.3-4.1}{3-1.166} \simeq 0.65$.

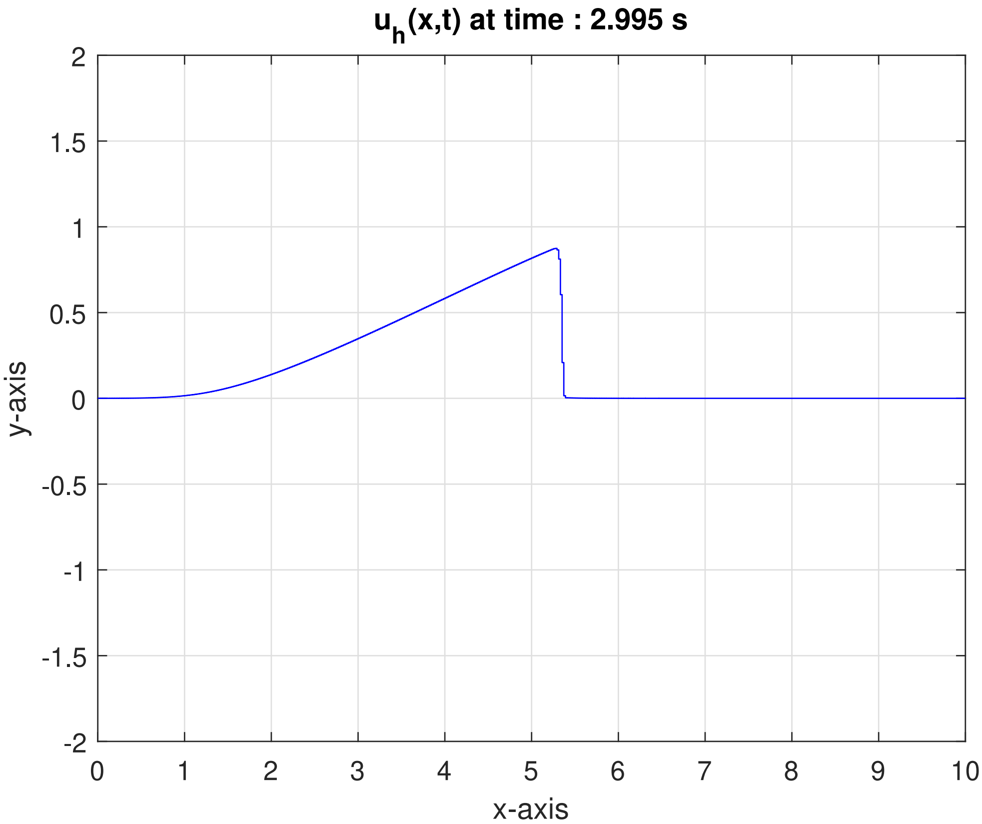

Figure 1: Results at the end of the computation for T=3[s]

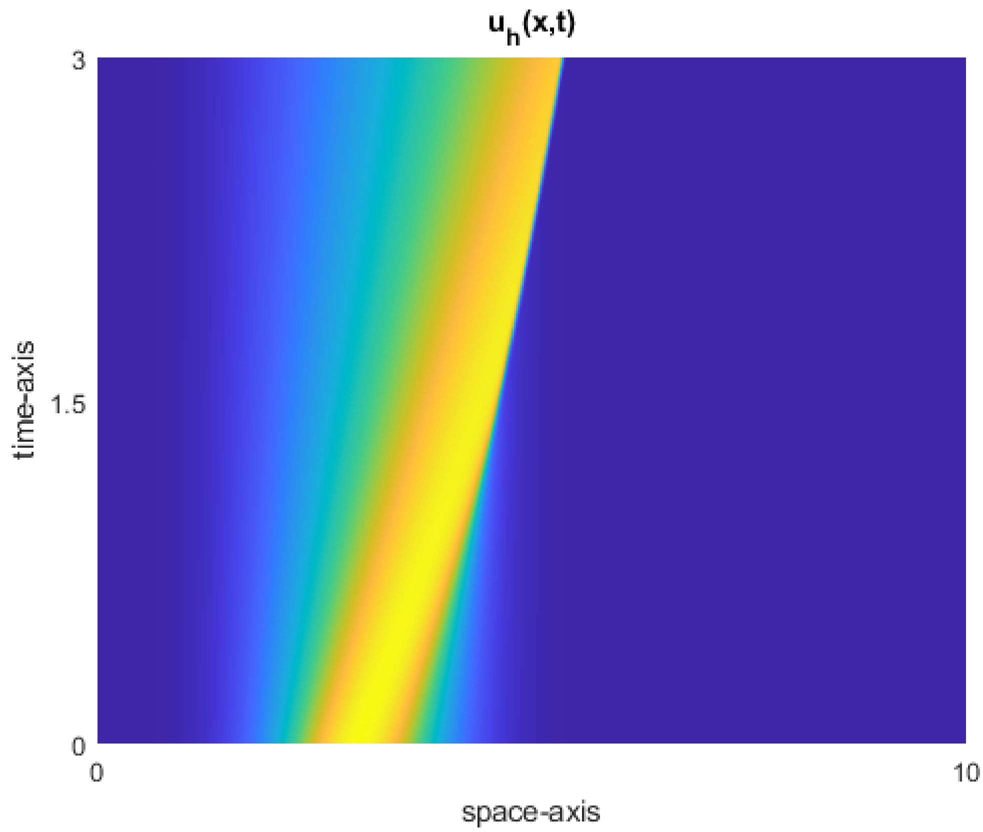

Figure 2: Plot of the solution time (y axis), space (x axis), intensity (color axis). With $h \simeq 0.02$ so for $nRef = 6$.