Assignment 3

Ilaria Ronconi Student number: 10541302
ilaria.ronconi@mail.polimi.it
Gioele Greco Student number: 10551402
gioele.greco@mail.polimi.it

July 17, 2021

1 Discontinuos Galerkin formulation

Let us consider the following wave propagation problem in $(0, L)(0, T]$:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + \frac{\partial F(u)}{\partial x}(x,t) = 0 \quad in \ \ (0,L)(0,T] \\ u(x,0) = u_0(x) \quad \ in(0,L). \end{cases} \tag{1}$$

We are talkong about the Riemann problem that for $F(u) = \frac{u^2}{2}$ falls in a particular case called Burger's equation. So, decomposing the function partial derivative, we can write that

$$\frac{\partial F(u)}{\partial x} = \frac{\partial F(u)}{\partial u}\frac{\partial u}{\partial x} = u\frac{\partial u}{\partial x}, \tag{2}$$

so the eq(1) begin

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3}$$

that remind the Navier-Stokes equation without pressure and viscosity terms. We have $u_0(x)$ that is a given function regular enough. The boundary conditions are, for a regular enough function ϕ that regulate the following inflow boundary for $x = 0$:

$$u(0,t) = \phi(t) \ \ t \in (0,T]. \tag{4}$$

So we may have a discontinuity between the boundary and the domain itself. If u_0 is discontinuos, the discontinuosity propagates with a finite velocity in the domain. To avoid the Gibbs phenomena, we can smooth the step regularising u_0, this is good for linear problem, in this case we have a non linear problem. So for u_L the value of u on the left of the discontinuity, and for u_R the value of u on the right of the discontinuity can be:

$$\begin{cases} u_L > u_R \longrightarrow \ shock \ wave \\ u_L < u_R \longrightarrow \ rarefaction \ wave \end{cases} \tag{5}$$

To treat this kind of problem we need to use the discontinuos Galerkin formulation that create elements that are not constrained each other. The first thing that we need to do is to discretize the domain. Calling $_h$ the mesh of the domain $\Omega = (0, L)$, made by elements k_j:

$$W_h = \{v_h \in L^2(0,L) : v_h|_{k_j} \in \mathbb{P}^r \ \forall k_j \in_h\} \tag{6}$$

The solution goes from $u(x,t)$ to $u_h(x_j,t)$ in W_h, so, using the Gudunov method, the problem in the Galerkin formulation becomes: $\forall t \in (0,T]\ \ \forall v_h \in \mathbb{P}^0$ find $u_h(t) \in W_h : \forall j = 0,1,...,m-1$

$$\int_{x_j}^{x_{j+1}} \frac{\partial u_h}{\partial t} v_h dx + \int_{x_j}^{x_{j+1}} \frac{\partial F(u_h)}{\partial x} v_h dx = 0. \tag{7}$$

We call the interval $I_j = (x_j, x_{j+1})$ and we solve per part the second integral

$$\int_{x_j}^{x_{j+1}} \frac{\partial F(u_h)}{\partial x} v_h dx = -\int_{x_j}^{x_{j+1}} F(u_h) \frac{\partial v_h}{\partial x} v_h dx - [F(u_h)v_h]_{x_j}^{x_{j+1}} \tag{8}$$

For $H_{j+1} = F(u_h(x_{j+1}^{left}))$ and $H_j = F(u_h(x_j^{right}))$, where H_j is the non linear flux in the point x_j and depends on the value of u_h in x_j, assuming $x_0 = 0$ is the inflow point, for $j = 0$ we have the eq.(4). So we can write that

$$H(u_h, t) = H(u_h(x_j^{left}, t), u_h(x_j^{right}, t)). \tag{9}$$

Considering $\mathbb{P}^0$ so r=0, if $u_h = u_h^{(j)} \in I_j$, if we consider the test function $v_h = 1 \in I_j$, so in the first term of eq.(7) can be grouped $\int_{I_j} dx = h_j$ then eq.(7) becomes:

$$h_j \frac{\partial u_h^{(j)}}{\partial t} = H(u_h^{(j-1)}, u_h^{(j)}, t) - H(u_h^{(j)}, u_h^{(j+1)}, t) \tag{10}$$

That shows the function has to be monotone w.r.t. the arguments, so we are in the case of the Gudunov flux:

$$H(v,w) = \begin{cases} \min_{x\in[v,w]} F(u) & v \le w \\ \max_{x\in[w,v]} F(u) & v > w \end{cases} \tag{11}$$

Even if $F(u) = \frac{u^2}{2}$ is a strictly convex function, we have that, for $H(v,w)$ discretized to $H({u_h^{(j+1)}}_{left}, {u_h^{(j+1)}}_{right}, t)$, the two possibilities are the same and goes to be $F(u_h(x_{j+1}^{left}, t))$, in the case of the third term of the eq. 10. As the same for the second term we have $F(u_h(x_j^{right}, t))$.
Finally the equation can be written:

$$h_j \frac{\partial u_h^{(j)}}{\partial t} + F(u_h(x_{j+1}^{left})) - F(u_h(x_j^{right})) = 0 \tag{12}$$

that, for $f(u) = \frac{u^2}{2}$ becomes the Burger's equation:

$$h_j \frac{\partial u_h^{(j)}}{\partial t} + u_h^2(x_{j+1}^{left}, t)/2 - 0u_h^2(x_j^{right}, t)/2 = 0 \tag{13}$$

Now if we collect the terms that are note derived in time in one element $\mathcal{L}(u_h^j)$:

$$\mathcal{L}(u_h^j) = -\frac{1}{2h_j}(u_h^2(x_j^{right}, t) + u_h^2(x_{j+1}^{left}, t) \ \ \forall I_j \tag{14}$$

So $\dot{u}_h^j = \mathcal{L}(u_h^j)\ \ \forall I_j$, also for $j = 0$, that is the inflow condition. So the following system consider the whole domain:

$$\begin{cases} M\dot{\mathbf{u}}_h = \mathcal{L}(\mathbf{u}_h) \\ \mathbf{u}_h(0) = \mathbf{u}_h^0(x) \end{cases} \tag{15}$$

where M is the sum of $m_j = \int_{I_j} v_h(x)u_h(x)dx$ elements.

2 Runge-Kutta scheme

To solve the system 15, we can use the Runge-Kutta scheme. We compute the third order that solve the n+1 time instant for the value at n, as written in the assignment. The results are discussed in the following section.

3 Critical time and speed of propagation of the discontinuity

The critical time is the instant in which appears the shock wave. That time can also be seen as the instant in which the characteristic curve of the solution intersects the characteristic line of the shock wave. This occurs only if $u_{left} > u_{right}$. It can be calculated as:

$$t_c = -\frac{1}{inf_x \; u_0'(x)F''(u_0(x))} \tag{16}$$

That, considering our conditions,

$$\begin{cases} u_0(x) = e^{-(x-3)^2} \longrightarrow u_0'(x) = -2(x-3)u_0(x), \\ F(u) = \frac{u^2}{2} \longrightarrow F'(u) = u \longrightarrow F''(u) = 1, \end{cases} \tag{17}$$

To find the minimum of $u_0'(x)$ we need to compute the roots of its second derivative:

$$u_0''(x) = -2u_0(x) - 2(x-3)u_0'(x) \longrightarrow u_0''(x) = -(2-4(x-3)^2)e^{-(x-3)^2} \tag{18}$$

$$u_0''(x) = (4x^2 - 24x + 34)e^{-(x-3)^2} \longrightarrow x_{1,2} = 3 \pm \frac{1}{\sqrt{2}}$$

Now computing $u_0'(x_1) = -\sqrt{2}e^{-\frac{1}{2}} = -\sqrt{\frac{2}{e}}$ and $u_0'(x_2) = +\sqrt{2}e^{-\frac{1}{2}} = \sqrt{\frac{2}{e}}$ we observe that $u_0'(x_1)$ is the minimum value, so finally:

$$t_c = \sqrt{\frac{e}{2}} \simeq 1.166[s], \tag{19}$$

After the time t_c the shock wave propagate with velocity σ, if there aren't any external forces, like in this case, $sigma$ can be calculated as:

$$\sigma \simeq \frac{x(max(u(t_{final}))) - x(max(u(t_c))}{t_{final} - t_c}.$$

That graphically returns $\sigma \simeq 5.3 - 4.13 - 1.166 \simeq 0.65$.

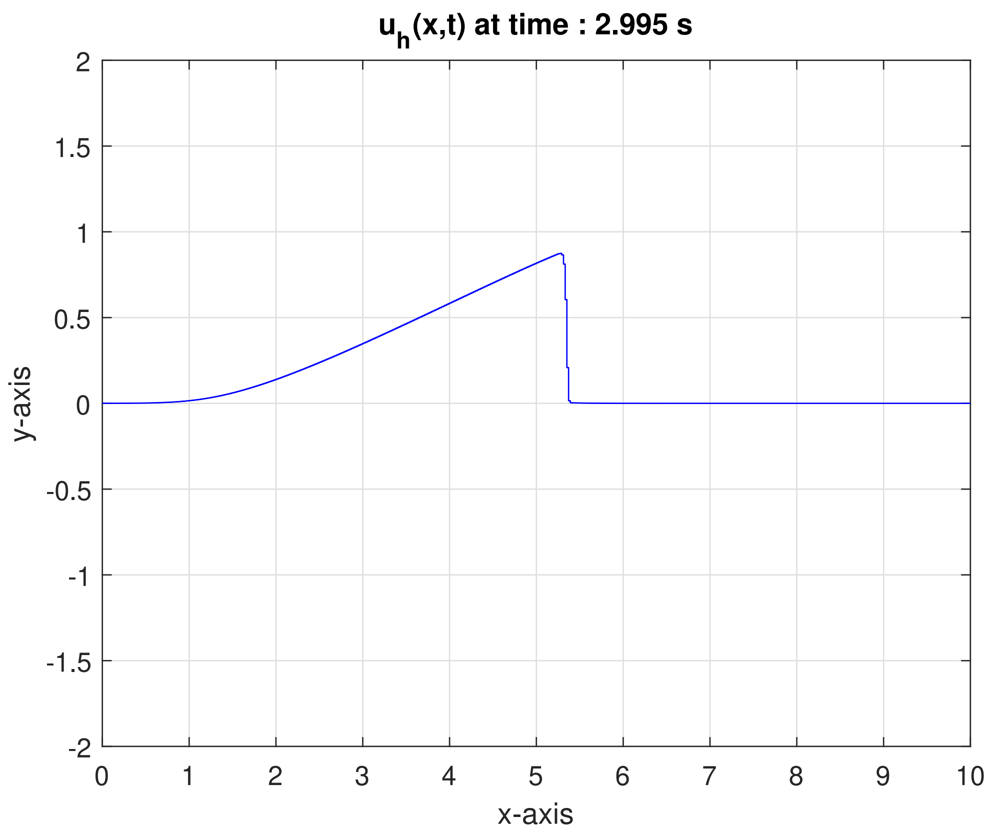

Figure 1: Results at the end of the computation for T=3[s]

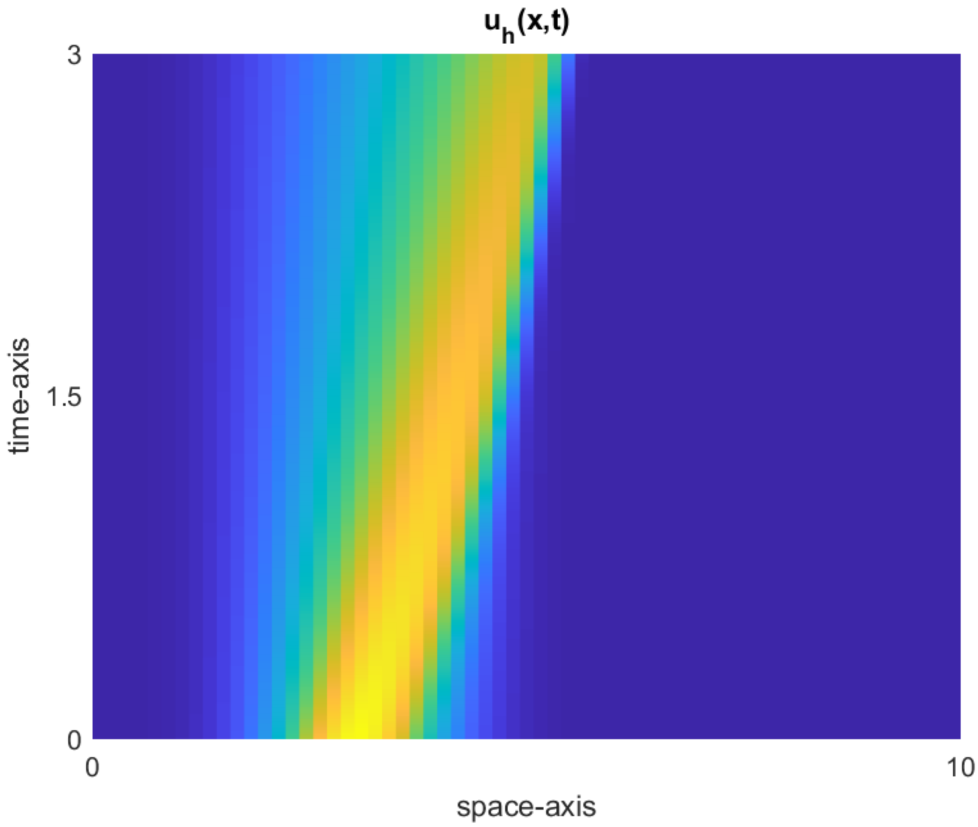

Figure 2: The plot of the solution time (y axis), space (x axis), intensity (color axis). With $h \simeq 0.02$ so for $nRef = 6$. The shock wave is not so clear.

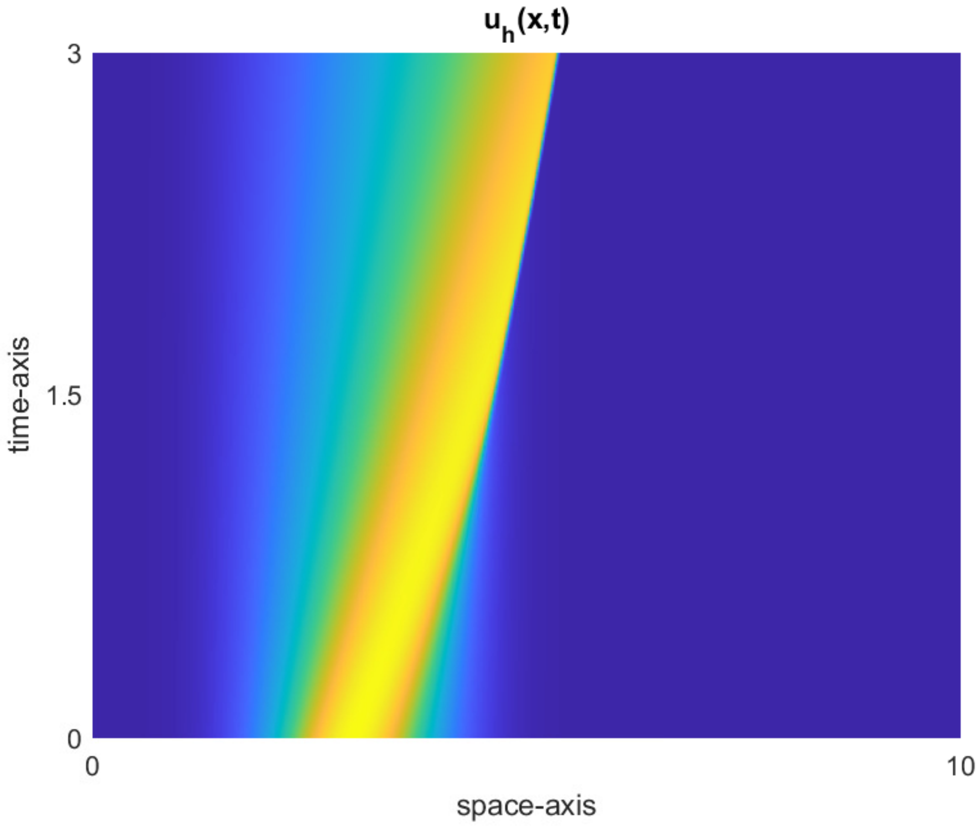

Figure 3: The plot of the solution time (y axis), space (x axis), intensity (color axis). With $h \simeq 0.002$ so for $nRef = 9$. The shock wave is evident from 1,17[s] to the end.