Assignment 2

Ilaria Ronconi Student number: 10541302
ilaria.ronconi@mail.polimi.it
Gioele Greco Student number: 10551402
gioele.greco@mail.polimi.it

July 21, 2021

Weak fromulation

We have to write the weak formulation of the problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - c^2\dfrac{\partial^2 u}{\partial x^2}(x,t) + \sigma(x)(\tilde{c}d\frac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t}) = g(x,t) & \in \Omega \times (0,T], \\ u(x,0) = u_0(x) & \in \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) & \in \Omega, \\ u(-\delta L, t) = 0; & t \in (0,T], \\ u(L+\delta L, t) = 0 & t \in (0,T] \end{cases} \tag{1}$$

We take a test function $v \in C_0^1(0,L)$, with $C_0^1(0,L) = \{v : (0,L) \to \mathbb{R} : v \in C^1(0,L), v(0) = v(L) = 0\}$, and we multiply it for both side of (1) and integrate.

$$\int_\Omega \ddot{u}vdx - c^2\int_\Omega u''vdx - \int_{-\delta L}^0 \alpha(cu' + \dot{u})vdx + \int_L^{L+\delta L} \alpha(cu' + \dot{u})vdx = \int_\Omega gvdx \tag{2}$$

The problem is well posed if:

- $g \in L^2(0,L)$
- $v, v' \in L^2(0,L)$
- $u' \in L^2(0,L)$

with

$$L^2(\Omega) = \left\{ v : (\Omega) \to \mathbb{R} \quad s.t. \quad \sqrt{\int_\Omega v^2(x)dx} < +\infty \right\}$$

so the problem became:

For any $t \in (0,T]$, find $u = u(t) \in H_0^1(0,L)$ such that:
$\int_\Omega \ddot{u}vdx + \int_\Omega c^2u'v'dx - \int_{-\delta L}^0 \alpha(cu' + \dot{u})vdx + \int_L^{L+\delta L} \alpha(cu' + \dot{u})vdx = \int_\Omega gvdx$

where:
$H^1(0,L) = \{v : (0,L) \to \mathbb{R}$ such that $v, v' \in L^2(0,L)\}$
$H_0^1(0,L) = \{v \in H^1(0,L)$ such that $v(0) = v(L) = 0\}$

We have now to discretize the domain Ω in $\Omega_m = [\bar{x}_{m-1}, \bar{x}_m]$ elements of length $h_m = \bar{x}_m - \bar{x}_{m-1}$, such that:

$$[-\delta L, L+\delta L] = \bigcup_{m=1}^{M} \Omega_m$$

The equation becomes:

$$\sum_{m=1}^{M}\int_{\Omega_m} \ddot{u}v dx + \sum_{m=1}^{M}\int_{\Omega_m} c^2 u'v' dx - \sum_{m=1}^{B}\int_{\Omega_m} \alpha(cu'+\dot{u})v dx + \sum_{m=M-B}^{M}\int_{\Omega_m} \alpha(cu'+\dot{u})v dx = \sum_{m=1}^{M}\int_{\Omega_m} gv dx \tag{3}$$

with B integer number.

Galerkin formulation

For the Galerkin formulation we use the same formulation but for
$u_h \in V_h^N(\Omega) = \{v \in C^0(\Omega) : v|_{\Omega_m} \in \mathbb{P}^N \quad m = 1, ..., M \quad v(0) = v(L) = 0\}$

The Galerkin formulation is:

For any $t \in (0,T]$ find $u_h^N = u_h^N(t) \in V_h^N \subset H_0^1(\Omega)$ s.t.

$$\sum_{m=1}^{M}\int_{\Omega_m} \ddot{u}_h^N v_h^N dx + \sum_{m=1}^{M} c^2 \int_{\Omega_m} {u'_h}^N {v'_h}^N dx - \sum_{m=1}^{B}\int_{\Omega_m} \alpha(c{u'_h}^N + \dot{u}_h^N) v_h^N dx +$$
$$\sum_{m=M-B}^{M}\int_{\Omega_m} \alpha(c{u'_h}^N + \dot{u}_h^N) v_h^N dx = \sum_{m=1}^{M}\int_{\Omega_m} g_h^N v_h^N dx \qquad \forall v_h \in V_h^N \tag{4}$$

Algebraic formulation

Now we use the reference domain method with a normalize domain of interval (-1,1) to write u_h as a linear combination of basis function.The basis function $\phi_0, ..., \phi_m$ are related to legendre polynomials, which are orthogonal polynomials and are based on non-equispaced nodal points $x_0, ..., x_{n+1}$, known as GLL points.

$$u_h(x) = \sum_{m=1}^{M-1} u_m^\Gamma(t)\phi_m^*(x) + \sum_{m=1}^{M}\sum_{i=1}^{N-1} u_i^{(m)}(t)\tilde{\phi}_i^{(m)}(x) \tag{5}$$

where the first part is obtained by glueing togheter two basis function, and the second part is related to the internal nodes with respect to Ω_m.

The (5) can be written also as:

$$u(x,t) = \sum_{j=1}^{n} u_j(t)\phi_j(x) \tag{6}$$

with n related to the polynomial degree and M related to the number of global elements, so the total points inside the mesh can be calculated as $n = nM - 1$. Supposing that $v_h = \phi_i$, we have:

$$\sum_{j=1}^{n} \ddot{u}_j(t) \int_\Omega \phi_j(x)\phi_i(x)dx + \sum_{j=1}^{n} c^2 u_j(t) \int_\Omega \phi_j'(x)\phi_i'(x)dx + \sum_{j=1}^{n} -\alpha c u_j(t) \int_\Omega \phi_j'(x)\phi_i(x)dx + \\ + \sum_{j=1}^{n} \alpha \dot{u}_j(t) \int_\Omega \phi_j(x)\phi_i(x)dx = \int_\Omega g_n \phi_i(x)dx \tag{7}$$

we call:

- $\int_\Omega \phi_i(x)\phi_j(x)dx = m_{ij}$
- $\int_\Omega c^2\phi_j'(x)\phi_i'(x)dx = a_{ij}$
- $\int_\Omega -\alpha c\phi_j'(x)\phi_i(x)dx = c_{\sigma ij}$
- $\int_\Omega \alpha\phi_i(x)\phi_j(x)dx = d_{\sigma ij}$
- $\int_\Omega g(x,t)\phi_i(x)dx = f_i$

so we have:

$$\sum_{j=1}^{n} \ddot{u}_j(t)m_{ij} + \sum_{j=1}^{n} u_j(t)a_{ij} + \sum_{j=1}^{n} u_j(t)c_{\sigma ij} + \sum_{j=1}^{n} \dot{u}_j(t)d_{\sigma ij} = f_i \tag{8}$$

and in matrix form:

$$M\ddot{\mathbf{u}}(t) + D\dot{\mathbf{u}}(t) + (A + C)\mathbf{u}(t) = \mathbf{F}(t) \tag{9}$$

MATLAB implementation

We have now to implement (9), exploiting the *leap-frog* scheme with spectral element method (SEM). Considering the data we were given in the homework, $L = 2$, $c = 1$, $f = 0$, $v_0 = 0$, $u_0(x) = e^{5(x-1)^2}$ and $T = 3$, to compute the behaviour of the solution at different δL and α, we implement the main MATLAB function `C_main1D(TestName,nRef,alfa,dL)`.

We choose polynomial's order as $N = 3$ and the mesh size as $nRef = 5$. The total global elements are 2^{nRef}, with N points each given by the Legendre polynomials basis function. If we are looking for an absorbing condition in the PML, we need to choose the δL size properly, because if it is too small, the PML may not act, and we can have some reflection waves, like for Dirichelet boundary condition with attenuation. If we consider δL too big, we are using lot of elements and computing power to calculate nothing, because, if the PML has the absorbing property, the wave will not propagates long in the space. The time integration interval dt and the mesh size h have to be chosen properly. The condition of stability is the Courant Fedrich Lewy condition (CFL):

$$dt \leq cost\frac{h}{c} \tag{10}$$

Considering $c = 1$, $nRef$ has to be the smallest possible to have high dt, but it has to be much higher to reduce the error. In our case we found that $cost = 0.12$ works well for the considered problem.

We impose the Dirichlet boundary conditions in $x = -\delta L$ and $x = \delta L + L$, and we create the C_σ and D_σ matrix similarly to the quadrature formula for the M and A matrix. D_σ matrix, described in (7), is the same to the mass matrix M multiplied for α. The C matrix, also described in (7), is instead more complex and it can be assumed:

$$C_\sigma = \sum_{q=1}^{nqn} J_{\Omega loc} \omega_d \tilde{c}(x_q) \sigma(x_q) \phi_i'(x_q) \phi_j(x_q) \tag{11}$$

Finally, we add the function `C_L2_norm` to calculate the L^2 norm of u_h in the domain $(0, L)$ at the final observation time T, that compute the pointwise integral:

$$||u_h||_{L^2_{(0,L)}} = \sqrt{\int_0^L |u(x,T)|^2 dx} \tag{12}$$

From the PML we expect high absorbing condition, which means low energy residual inside the domain, so we have to look for the result of (12) to be lowest possible. We define also $\eta = 1 - \frac{||u_{h(t=3)}||_{L^2_{(0,L)}}}{||u_{h(t=dt)}||_{L^2_{(0,L)}}}$ that is the ratio of lost energy. With $nRef = 5$ the threeshold for dt is $dt = 0.0075$, if we choose $dt = 0.007$, the results for different δL and α are:

δL	α	$\|\|u_h\|\|_{L^2_{(0,L)}}$	$\eta\%$
2^{-3}	1	3.0329	38
2^{-3}	10	3.2665	33
2^{-3}	100	1.7398	64
2^{-3}	1000	glitch	none
2^{-2}	1	3.4377	26
2^{-2}	10	2.1915	53
2^{-2}	100	0.48455	89.5
2^{-2}	1000	glitch	none
2^{-1}	1	3.2676	23
2^{-1}	10	0.06125	98.5
2^{-1}	100	0.5782	86.5

So the best configuration is for big δL and $\alpha \leq 10$. For high values of α the PML acts like a rigid wall, so the energy remain in the domain because the wave are partially reflected. For $\alpha \to \infty$ the wave is insanely amplified.

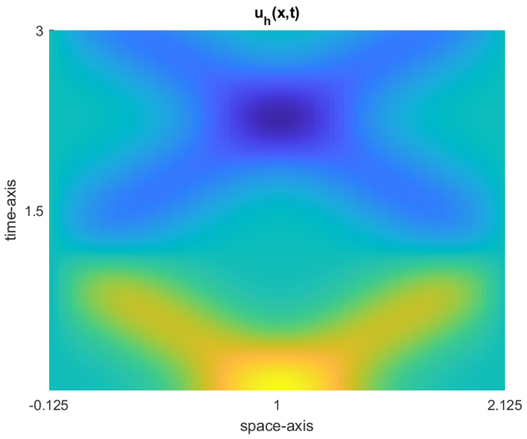
Absorbing boundary condition

Solving the same problem, without use the PML, but imposing the absorbing boundary condition on the end of the domain (0,L):

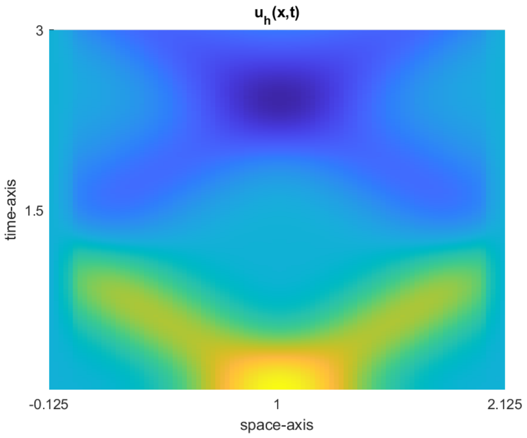
$$\begin{cases} \left[\frac{1}{c}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\right](0,t) = 0; \\ \left[\frac{1}{c}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\right](L,t) = 0; \end{cases} \tag{13}$$

Using the same parameters and the leap-frog too, we found $||u_h||_{L^2_{(0,L)}} = 0.03546$ and $\eta = 99\%$, so the results are comparable only with some configurations.

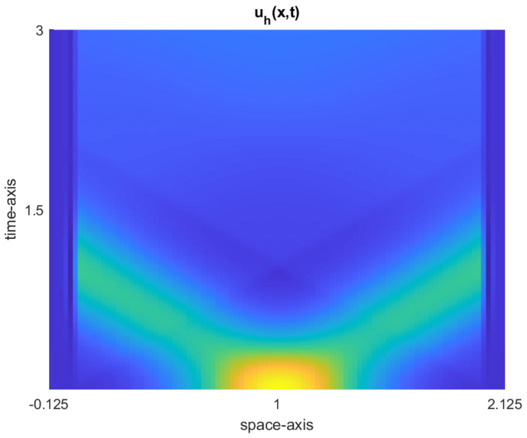
[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-3}$ and $\alpha = 1$]



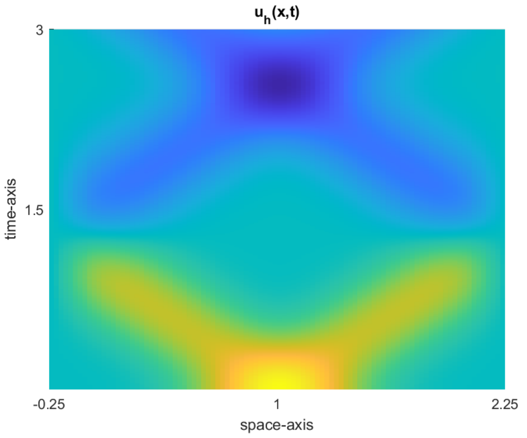
[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-3}$ and $\alpha = 10$]



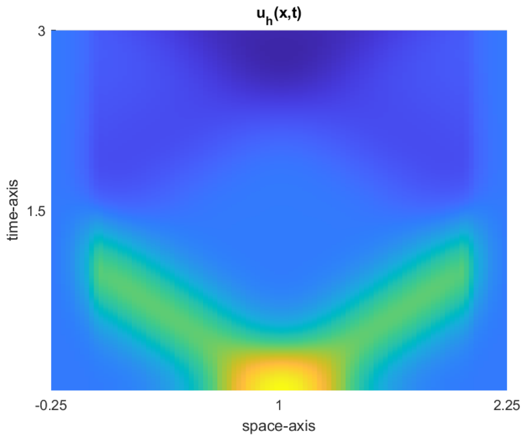
[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-3}$ and $\alpha = 100$]



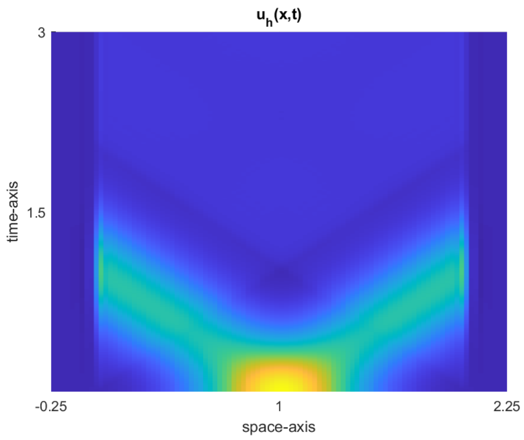
[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-2}$ and $\alpha = 1$]



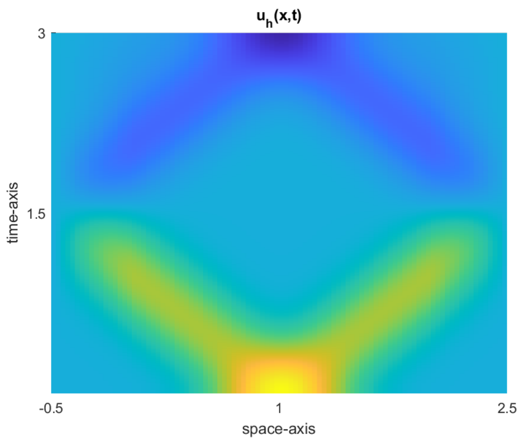
[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-2}$ and $\alpha = 10$]



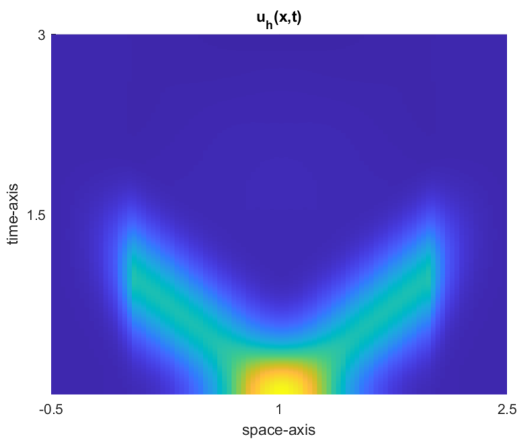
[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-2}$ and $\alpha = 100$]



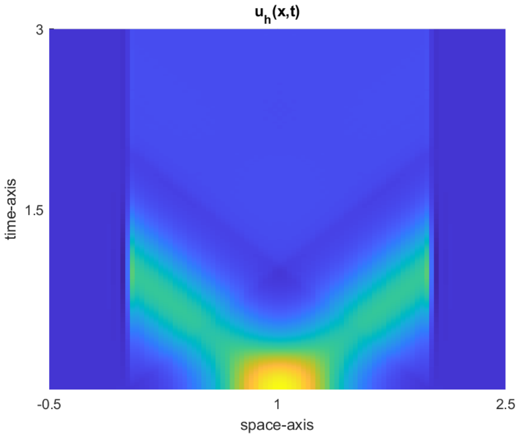
[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-1}$ and $\alpha = 1$]



[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-1}$ and $\alpha = 10$]



[Color plot of $u_h(x,t)$ in time with $\delta L = 2^{-1}$ and $\alpha = 100$]



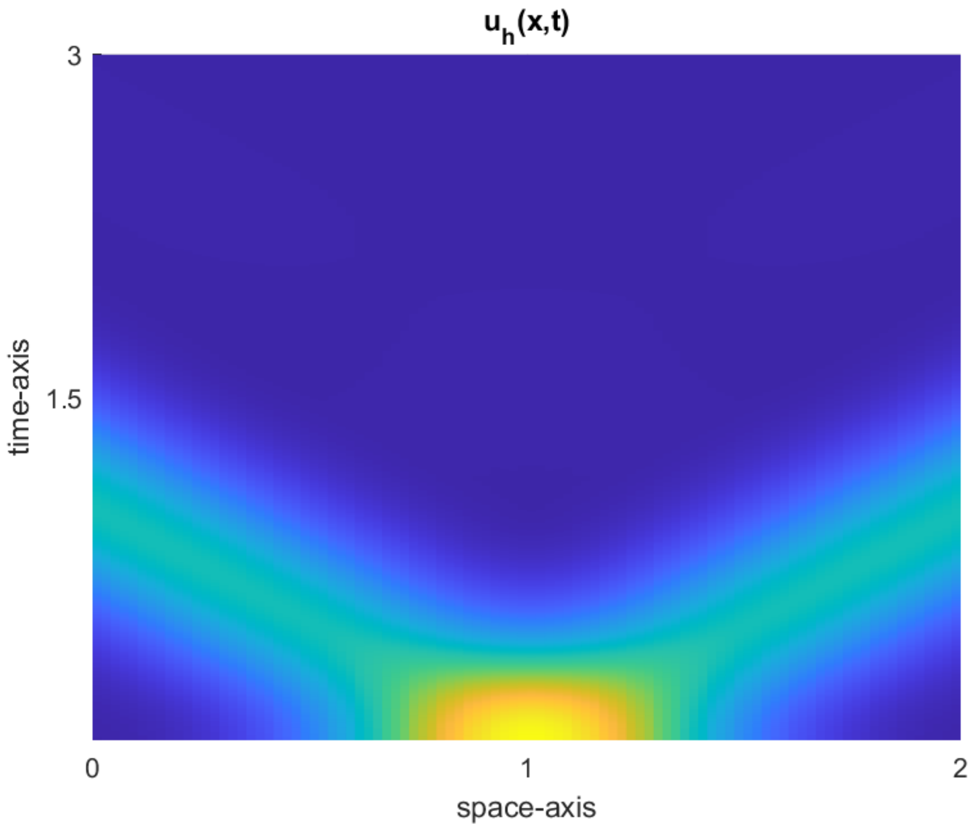


Figure 2: Results of the absorbing boundary condition