Matrix powers

A square matrix has matrix powers. For example,

$$A^5 = AAAAA$$

which, as a linear map, corresponds to applying A five times:

$A^0 = \mathbb{1}$

$$A^5 v = A(A(A(A(Av))))$$

Matrix powers are meaningful in some applications, but not in all applications. For example, if the input vector of A has units of volume, and the output vector contains weights, then applying A several times is not meaningful.

```
>>> import numpy as np
>>> import time

>>> n = 5000
>>> A = np.random.randn(n,n)
>>> v = np.random.randn(n)

>>> t1 = time.time()
>>> w1 = ( A @ A @ A @ A @ A ) @ v  #  or np.linalg.matrix_power(A,5) @ v
>>> t1 = time.time()-t1
```

5'000 x5'000 (5 times)

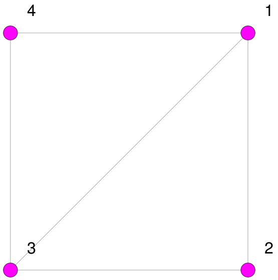
```
>>> t2 = time.time()
>>> w2 = A @ ( A @ ( A @ ( A @ ( A @ v ) ) ) )
>>> t2 = time.time()-t2
```

5'000 x1

```
>>> # results are identical up to numerical errors
>>> np.allclose(w1,w2,rtol=1e-10)
True

>>> # but second method is 100 times faster
>>> (t1,t2)
(6.3, 0.064)
```

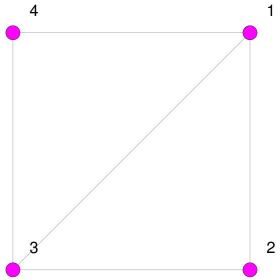
Example



A graph can represent

- a computer network
- a public transport network
- a social graph
- ...

Cont.



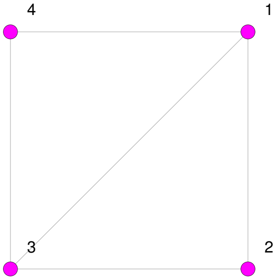
The adjacency matrix of this undirected graph is

edge 4 → no connected edge 2

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

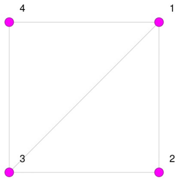
$A^T = A$ (i.e. A is symmetric)

Cont.



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Cont.

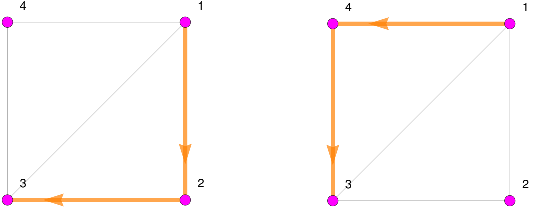


$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 & \cdots \\ 0\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 0 & \cdots \\ 0\cdot 1 + 1\cdot 1 + 1\cdot 0 + 1\cdot 1 & \cdots \\ 0\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 0 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Cont.



1

$$A^2 = 3\begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

3 possibilities to go from 1 to 1 with 2 steps

2 possibilities to go from 1 to 3 with 2 steps

Entry 31 in A^2 counts the number of walks of length 2 from vertex 1 to vertex 3.

Terminology

A walk of length n is any chain $v_0 \to v_1 \to \ldots \to v_n$ of vertices such that there is an edge from v_0 is v_1, from v_1 to v_2, and so on.

A path is a walk where all $n + 1$ vertices are distinct.

Example:

$1 \to 2 \to 3 \to 2$ is a walk of length 3, not a path

$1 \to 4 \to 3 \to 2$ is a walk and a path of length 3

Fact 1

Fact 1. *Given an undirected graph with*

- *no loop*
- *at most one edge between any two vertices*

Enumerate the vertices and denote by A the adjacency matrix. Then for every integer $n \geq 1$, entry ij in the matrix power A^n is the number of walks of length n from vertex j to i:

$$(A^n)_{ij} \;=\; \#(\textit{walks of length } n \textit{ from } j \textit{ to } i)$$

The case $n = 1$ is clear. But why is this true for $n > 1$?

Proof of Fact 1 – **important!**

Denote by $w_{n,ij}$ the number of walks of length n from j to i.

Decomposing every walk of length n into two parts,
the first $n-1$ steps and the last step, we have

$$w_{n,ij} = \sum_{k=1}^{\#\text{vertices}} w_{1,ik} w_{n-1,kj}$$

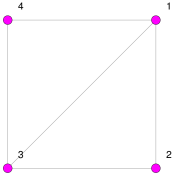
Since $w_{1,ij} = A_{ij}$, and assuming we already know, by induction, that $w_{n-1,kj} = (A^{n-1})_{kj}$, then this identity says that

$$w_{n,ij} = \sum_{k=1}^{\#\text{vertices}} A_{ik}(A^{n-1})_{kj} = (A^n)_{ij}$$

Example, resumed

tr(M) = trace (M) = sum of diagonal entries

square



The trace of a square matrix is the sum of all diagonal entries:

$$\operatorname{tr}(A^2) \;=\; \operatorname{tr}\begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \;=\; 3+2+3+2 \;=\; 10$$

What does $\operatorname{tr}(A^2)$ count?

Cont.

$(A^2)_{ii}$ is the number of walks of length 2 from vertex i back to vertex i, meaning the number of edges attached to vertex i.

Since every edge is attached to two vertices, we get

$$\mathrm{tr}(A^2) = \sum_{i=1}^{\#\text{vertices}} (A^2)_{ii} = 2 \cdot \#\text{edges}$$

$\#\text{edges} = \frac{1}{2}\,\mathrm{tr}(A^2)$

Fact 2

Fact 2. *Under the assumptions of Fact 1, we have*

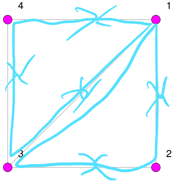
$$\#\textit{edges} = \frac{1}{2}\,\mathrm{tr}(A^2)$$

Similarly,

$$\#\textit{triangles} = \frac{1}{6}\,\mathrm{tr}(A^3)$$

By definition, a triangle is a closed walk of length 3 but with the direction and the initial vertex forgotten. Forgetting the direction gives a factor 2, forgetting the initial vertex gives a factor 3, and $2 \cdot 3 = 6$. (Equivalently, a triangle is a subset of 3 vertices where every vertex is connected to every other by an edge.)

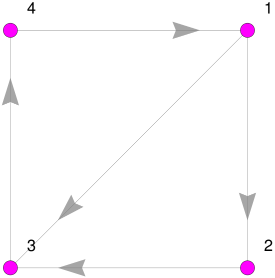
Example, resumed



$$\mathrm{tr}(A^3) \;=\; \mathrm{tr}\begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{pmatrix} \;=\; 4+2+4+2 \;=\; 12$$

giving $\frac{1}{6}\,\mathrm{tr}(A^3) = 2$ triangles.

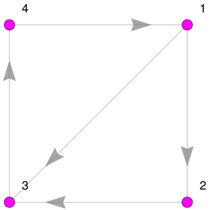
New example



The adjacency matrix of this directed graph is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Cont.

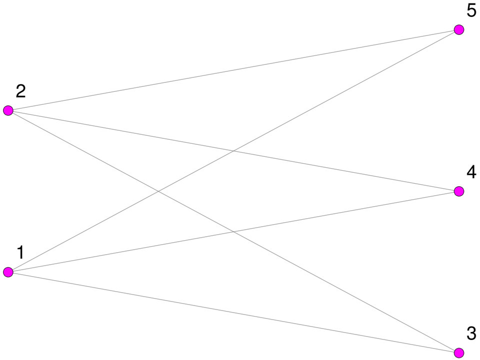


A^n still counts walks, but streets are now one-way. Example:

$$A^{10} = \begin{pmatrix} \mathbf{3} & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

What are the 3 walks of length 10 from vertex 1 to vertex 1?

Problem 1



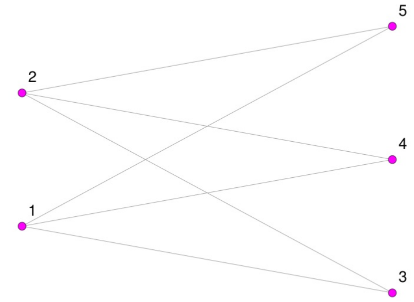
a. Find the adjacency matrix A of this undirected graph.

b. Compute A^2 and A^3 by hand.

c. From the 51-entry of A^3, determine how many walks of length 3 there are from vertex 1 to vertex 5.

d. Make a list of all the walks in c.

Problem 1



a. Find the adjacency matrix A of this undirected graph.

b. Compute A^2 and A^3 by hand.

c. From the 51-entry of A^3, determine how many walks of length 3 there are from vertex 1 to vertex 5.

d. Make a list of all the walks in c.

a.)

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

b.)

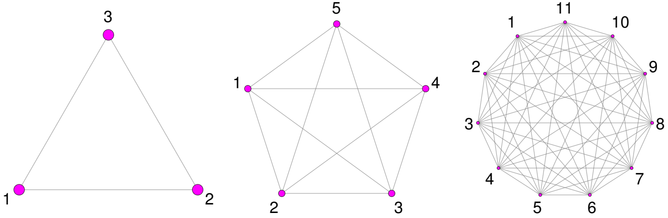
$$A^2_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot 0 + 0\cdot 0 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 \\ 0\cdot 0 + 0\cdot 0 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 \\ 1\cdot 0 + 1\cdot 0 + 0\cdot 1 + 0\cdot 1 + 0\cdot 1 \\ 1\cdot 0 + 1\cdot 0 + 0\cdot 1 + 0\cdot 1 + 0\cdot 1 \\ 1\cdot 0 + 1\cdot 0 + 0\cdot 1 + 0\cdot 1 + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A^2_3 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 1 + 0\cdot 1 + 1\cdot 0 + 1\cdot 0 + 1\cdot 0 \\ 0\cdot 1 + 0\cdot 1 + 1\cdot 0 + 1\cdot 0 + 1\cdot 0 \\ 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 \\ 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 \\ 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} \quad \Rightarrow \quad A^3 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

Problem 2



The complete graph on n vertices is the undirected graph where all edges are present; the cases $n = 3$, $n = 5$ and $n = 11$ respectively are shown above. The adjacency matrix A_n can be described as follows: It is the $n \times n$ matrix all whose diagonal entries are equal to 0, all whose off-diagonal entries are equal to 1.

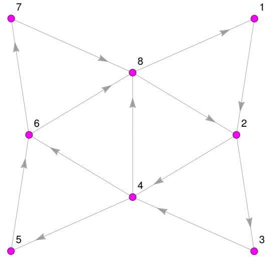
a. Give a similar description of $(A_n)^2$.

b. Give a similar description of $(A_n)^3$.

c. Compute the trace $\mathrm{tr}((A_n)^2)$. How many edges are there?
   Check. For $n = 5$, your answer should give 10 edges.

d. Compute the trace $\mathrm{tr}((A_n)^3)$. How many triangles are there?
   Check. For $n = 3$, your answer should give 1 triangle.

Problem 3 **[peer assessment]**



Solve this problem in Python+NumPy.

a. Define the adjacency matrix A of this directed graph.

b. Compute A^3. **Check:** Should be $\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}$.

c. Compute $\mathrm{tr}(A^4)$, the number of all closed walks of length 4.

d. Compute the number of walks of length 10 from vertex 1 to vertex 1.

e. Compute the number of walks of length 10 from vertex 1 to vertex 1 that never use the $4 \to 8$ edge.

f. Compute the number of walks of length 10 from vertex 1 to vertex 1 that use the $4 \to 8$ edge at least once.

Solution to Problem 1

a.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

b.

$$A^2 = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 6 & 6 & 6 \\ 0 & 0 & 6 & 6 & 6 \\ 6 & 6 & 0 & 0 & 0 \\ 6 & 6 & 0 & 0 & 0 \\ 6 & 6 & 0 & 0 & 0 \end{pmatrix}$$

c. $(A^3)_{51} = 6$.

d.

$$1 \to 3 \to 1 \to 5$$
$$1 \to 4 \to 1 \to 5$$
$$1 \to 5 \to 1 \to 5$$
$$1 \to 3 \to 2 \to 5$$
$$1 \to 4 \to 2 \to 5$$
$$1 \to 5 \to 2 \to 5$$

Solution to Problem 2

a. $(A_n)^2$ is the $n \times n$ matrix

- all whose diagonal entries are equal to $n - 1$
- all whose off-diagonal entries are equal to $n - 2$

b. $(A_n)^3$ is the $n \times n$ matrix

- all whose diagonal entries are equal to $(n - 1)(n - 2)$
- all whose off-diagonal entries are equal to $n - 1 + (n - 2)^2$

c. $\mathrm{tr}((A_n)^2) = n(n - 1)$, therefore

$$\#\text{edges} = \frac{n(n-1)}{2}$$

d. $\mathrm{tr}((A_n)^3) = n(n - 1)(n - 2)$, therefore

$$\#\text{triangles} = \frac{n(n-1)(n-2)}{6}$$