Bern University of Applied Sciences
Linear Algebra for Data Engineering, BZG1167
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Practice midterm exam – comparable in format and difficulty to midterm

Your name: ______________________________

Note:

- Number of problems: 16. Maximal score: 22. Guarantees:

Score	*Grade*
$\mathbf{12}$ or more	$\mathbf{50\%}$ or higher
$\mathbf{20}$ or more	$\mathbf{100\%}$

- No electronic devices.
- By default, answers do not have to be justified. However, in some cases a justification is explicitly required, meaning you have to explain how you obtained your result; in such cases, even correct answers may be given zero points if no or insufficient justification is provided.

1,5P

Problem 1 (1 point)**.** Let M be the 3×5 matrix that, when applied to a vector in $\mathbb{R}^5$, discards the second and fourth entry. That is,

1P

$$M \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} a \\ c \\ e \end{pmatrix}$$

Write down the matrix M.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ ✓

Problem 2 (1 point)**.** Compute the following matrix-vector multiplication:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x+3y \\ 4y+5z \end{pmatrix}$$

0,5P

$$1x + 0y + 0z = 1x$$
$$2x + 3y + 0z = 2x+3y$$
$$0x + 4y + 5z = 4y+5z$$

~✓

Problem 3 (1 point)**.** Compute the following matrix power:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}^2$$

Hint: Recall that for a square matrix M, the second matrix power M^2 is equal to the matrix-matrix product MM.

$$M^2 = \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 10 & 25 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}$$

f

0P

$M\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}+\frac{1}{2}\\ \frac{1}{2}+\frac{3}{2}\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$ $\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{3}{2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}+\frac{1}{2}\\ \frac{1}{2}+\frac{3}{2}\end{pmatrix}=\begin{pmatrix}\frac{4}{2}\\ \frac{4}{2}\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$ 1P

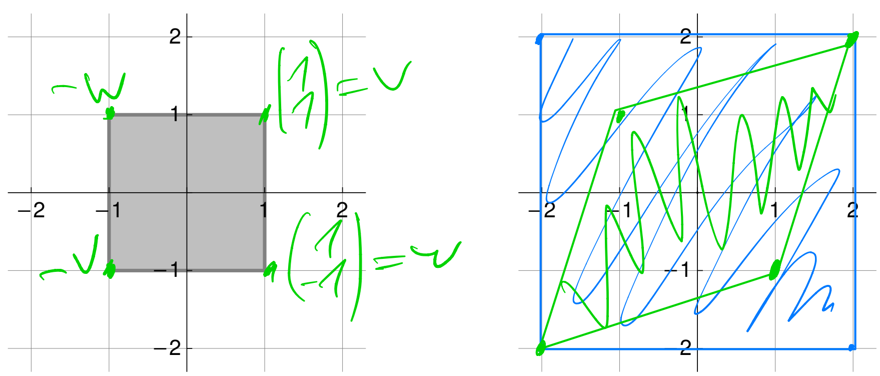
$M\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}-\frac{1}{2}\\ \frac{1}{2}-\frac{3}{2}\end{pmatrix}=\begin{pmatrix}\frac{2}{2}\\ -\frac{2}{2}\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$ f

Problem 4 (2 points)**.** The matrix $M=\begin{pmatrix}3/2 & 1/2\\ 1/2 & 3/2\end{pmatrix}$ defines a map $\mathbb{R}^2\to\mathbb{R}^2$.

a. When M is applied to every point of the shaded square in the input space $\mathbb{R}^2$, one obtains a polygon in the output space $\mathbb{R}^2$. Sketch the polygon.

0P

Input space Output space



b. *Justification required.*
Compute $\det M$. Compute the area of the polygon in the output space $\mathbb{R}^2$.

$\det(M)=\frac{3}{2}\cdot\frac{3}{2}-\frac{1}{2}\cdot\frac{1}{2}=\frac{9}{4}-\frac{1}{4}=\frac{8}{4}=2$ ✓ 21 Slide 1 (and 5) 0,5P

Since the determinant is equal to the area, the area is also 2 f $\Rightarrow A_{pol}=\text{Area square}\cdot|\det M|$
$=\underline{8}$ 16 Slide 9 0,5P

Problem 5 (1 point)**.** *Justification required.*

What is the biggest rank that a 7×4 matrix can have?

If A is $m\times n$, then $\operatorname{rank}(A)\le m$ and $\operatorname{rank}(A)\le n$

$\Rightarrow \underline{4}$ ✓✓

• Rank equals num of pivots in rref, hence rank ≤ 4

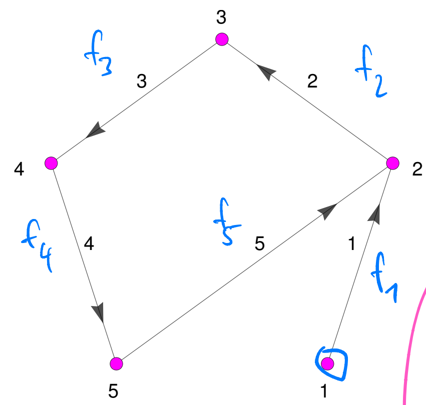
• Rank 4 is possible, e.g. $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$ → has 4 pivots → rank 4 !

$\Rightarrow$ 4 is largest

1P

- incidende mat. tells the "flow" of each vertex
- adjaency mat. tells if a vertex is connected with another vertex

Problem 6 (2 points)**.** Consider the following directed graph. Enumerations of the nodes and edges are given.



What is the difference between incidende mat. and adjaency mat.?

a. Write down the incidende matrix of this graph.

$$-f_1 = 0$$
$$f_1 - f_2 + f_5 = 0$$
$$f_2 - f_3 = 0$$
$$f_3 - f_4 = 0$$
$$f_4 - f_5 = 0$$

$$\Rightarrow A = \begin{array}{c} \text{edge1} \; \ldots \; \text{edge5} \\ \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array} \begin{array}{l} \text{vertex 1} \\ \vdots \\ \text{vertex 5} \end{array}$$

✓ 1P

b. *Justification required.*
Find a nonzero vector in the nullspace of the incidende matrix.

0P

1. rref of A with all zero rows dropped

$R_1 \leftarrow -R_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

For solution see 2slides later ⇓

$R_2 \leftarrow R_2 - R_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$R_2 \leftarrow -R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$R_3 \leftarrow R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad f$$

$R_3 \leftarrow -R_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$R_4 \leftarrow R_4 - R_3$

$R_4 \leftarrow -R_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$R_5 \leftarrow R_5 - R_4$

$R_5 \leftarrow R_5$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_2 \leftarrow R_2 + R_5$

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$$

$\Rightarrow S = 0$

$\Rightarrow$ Since the rref with all zero rows dropped of A yields an id. mat. the nullspace of A is zero and therefore there is no nonzero vector in the nullspace of A.

f

① nullspace A = nullspace rref (A)

rref A:
$$\begin{matrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{matrix} \quad \xrightarrow{\cdot(-1)} \quad \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{matrix}$$

$$\Rightarrow \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$\Rightarrow$ hence $S = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \leftarrow$ vector in nullspace of A

② $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is admissible traffic flow conf $\Rightarrow Av = \begin{pmatrix} -1 & \cdots & 0 \\ 1 & & 1 \\ 0 & & 0 \\ 0 & & 0 \\ 0 & & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$

1P

0P

Problem 7 (1 point). *Justification required.*

Suppose v_1, v_2, v_3 are three vectors in $\mathbb{R}^5$ and $U = \text{span}\{v_1, v_2, v_3\}$. Suppose $v_1 \neq 0$, $v_2 \neq 0$, $v_3 \neq 0$. What are the possible dimensions that U can have?

$\dim(U) = 3$

$\Rightarrow$ since there are 3 vectors in U, U cannot be over 3 dimensions.

$\dim(U) > 3$ not possible for 3 vectors

$\dim(U) = 0$ not possible since vectors are nonzero

Ex 1: $v_1 = v_2 = v_3$ $\dim(U) = 1$

2: $v_1 = v_2$, linearly independent v_3 $\dim(U) = 2$

3: v_1, v_2, v_3 linearly ind. $\dim(U) = 3$

Problem 8 (1 point). *Justification required.*

Compute the inverse of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

See 20 P3 and 21 from slide 1

Hint: Compute the reduced row echelon form of the augmented matrix $(A|\mathbb{1})$, *where* $\mathbb{1}$ *is the* 4×4 *identity matrix.*

1P

$$(A|\mathbb{1}) = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

1. Swap $R_1 \Leftrightarrow R_4$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

2. Swap $R_4 \Leftrightarrow R_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

3. Swap $R_3 \Leftrightarrow R_4$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ Applying the inverse matrix to the og. mat. the result must yield an id. mat.

$$\Rightarrow AA^{-1} = \underline{\mathbb{1}} \quad \checkmark$$

OP

OP

2

Problem 9 (2 points). Let x be Alice's current age. Let y be Bob's current age.

a. Bob is 20 years younger than Alice. In two years, Alice will be twice as old as Bob. Write this information as a linear system

Alice will be $x+2$
Bob will be $y+2$

$$\boxed{A\begin{pmatrix} x \\ y \end{pmatrix} = b}$$

for some 2×2 matrix A and some 2-component vector b.

$$y = x - 20$$
$$x+2 = 2(y+2)$$

i.e. $\boxed{\begin{aligned} x-y&=20 \\ x-2y&=2 \end{aligned}}$ so $A = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$, $b = \begin{pmatrix} 20 \\ 2 \end{pmatrix}$

$\boxed{\begin{aligned} Mv&=b \\ v&=M^{-1}b \end{aligned}}$

b. *Justification required.*
Compute A^{-1}. Then use it to compute x and y.

(can check $AA^{-1} = \mathbb{1}$)

$$A^{-1} = \underbrace{\frac{1}{1(-2)-1\cdot(-1)}}_{=-1}\begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = b \text{ so } \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}b = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 20 \\ 2 \end{pmatrix} = \begin{pmatrix} 38 \\ 18 \end{pmatrix}$$

OP

Problem 10 (1 point). Define the so-called Hamming matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

See 16 slide 14

Write down a matrix S whose columns are a basis of the nullspace of H.

$$S = \begin{pmatrix} -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

f H already is in rref
$\Rightarrow$ S must have size $7\times(7-3) = 7\times 4$
pivot cols

$$S = \begin{pmatrix} -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\text{Check } HS=0}$

3P

doesn't come in at all

See 1.6 Slide 11

Problem 11 (1 point). *Justification required.* 1P

A linear map $\mathbb{R}^9 \to \mathbb{R}^{18}$ has rank 6. Determine the dimension of the nullspace and the dimension of the column space.

Since the linear map has rank 6, the dim of the nullspace is also 6. $\dim(\text{nullsp}) = 6$ ✓

$\Rightarrow$ We know that the input space has the dim of 9, subtracting the dim of the nullsp gives the dim of colsp $\Rightarrow \dim(\text{colsp}) = 9 - 6 = 3$ ✓

Problem 12 (2 points). *Justification required.* 2P

Determine if $\begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \\ 16 \end{pmatrix}$ is in the span of the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix}$.

Hint: This is equivalent to asking if there exists a solution x to the linear inhomogeneous problem

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix}}_{5\times 3}\underbrace{x}_{3\times 1 \text{ must be}} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \\ 16 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 3 & 9 & 4 \\ 1 & 4 & 16 & 9 \\ 1 & 5 & 25 & 16 \end{pmatrix}}_{\text{rref with all zero rows dropped}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ -1 \end{pmatrix} = 0$$

$$1\begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix} - 2\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \\ 16 \end{pmatrix}$$

$\Rightarrow$ Solution exists

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_1$

$R_4 \leftarrow R_4 - R_1$

$R_5 \leftarrow R_5 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 4 \\ 0 & 3 & 15 & 9 \\ 0 & 4 & 24 & 16 \end{pmatrix}$$

$R_3 \leftarrow \frac{1}{2} R_3$

$R_4 \leftarrow \frac{1}{3} R_4$

$R_5 \leftarrow \frac{1}{4} R_5$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{pmatrix}$$

$R_1 \leftarrow R_1 - R_2$

$R_3 \leftarrow R_3 - R_2$

$R_4 \leftarrow R_4 - R_2$

$R_5 \leftarrow R_5 - R_2$

$$\begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

$R_1 \leftarrow R_1 + 2R_3$

$R_2 \leftarrow R_2 - 3R_3$

$R_4 \leftarrow R_4 - 2R_3$

$R_5 \leftarrow R_5 - 3R_3$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\Downarrow$

gives solution

$\Downarrow$

$\begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \\ 16 \end{pmatrix}$ is in span

1P

Problem 13 (1 point). *Justification required.*

1P

$$K = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Is v in the nullspace of K?

If v is in the nullspace, the linear combination of K and v needs to yield 0.

$$\Rightarrow Kv = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \checkmark$$

$\Rightarrow$ v is in the nullspace of K

Problem 14 (1 point). *Justification required.*

0P

Write down two invertible 2×2 matrices A and B for which

$$(AB)^{-1} \neq A^{-1}B^{-1}$$

$(AB)^{-1} = A^{-1}B^{-1}$

You are required to check that the two sides are in fact different!

$(AB)^{-1}$ says to multiply A and B first and then take the inverse from the result

f

$A^{-1}B^{-1}$ says to inverse A and B first and the multiply the results

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \quad \det(A) = 4-6 = -2 \quad \det(B) = 10-12 = -2$$

$$(AB)^{-1} = \begin{pmatrix} 10 & 13 \\ 18 & 29 \end{pmatrix}^{-1} \quad A^{-1}B^{-1} = \begin{pmatrix} \frac{1}{-2} & \frac{2}{-2} \\ \frac{3}{-2} & \frac{4}{-2} \end{pmatrix} \begin{pmatrix} \frac{2}{-2} & \frac{3}{-2} \\ \frac{4}{-2} & \frac{5}{-2} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^{-1} = A$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B^{-1} = B$$

different!

$\neq$

$$(AB)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^{-1}B^{-1} = A \cdot B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Problem 15 (2 points). *Justification required.*
Define two vectors that depend on a parameter $a > 0$:

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ a \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -a \end{pmatrix}$$

1P

a. Determine a such that u is orthogonal to v.

$u \perp v \Rightarrow u \cdot v = 0$

$1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + a \cdot -a = 0 \quad | \; TU$

$-a^2 + 3 = 0 \quad | + a^2$

$a^2 = 3$

$\Rightarrow \underline{\underline{a_1 = \sqrt{3} \quad a_2 = -\sqrt{3}}}$ ✓

b. Determine a such that the angle between u and v is equal to $\frac{\pi}{3}$ (60 degrees).
Hint: $\cos\frac{\pi}{3} = \frac{1}{2}$.

$\measuredangle(u,v) = 60° = \frac{\pi}{3} \Leftrightarrow \frac{u \cdot v}{\|u\| \cdot \|v\|} = \cos 60° = \frac{1}{2}$

$u \cdot v = 3 - a^2$

$\|u\| = \|v\| = \sqrt{3+a^2}$

$\Rightarrow \frac{3 - a^2}{3 + a^2} = \frac{1}{2} \quad | \cdot 2 \quad | \cdot (3+a^2)$

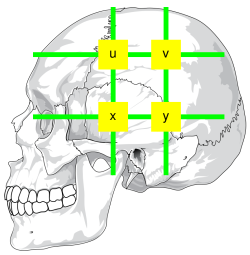
$2(3-a^2) = 3 + a^2 \quad | \; TU$

$6 - 2a^2 = 3 + a^2 \quad | + 2a^2 \quad | -3$

$3 = 3a^2 \quad | : 3$

$a^2 = 1 = a_1 \quad a_2 = -1$

Problem 16 (2 points). *Justification required.*



Consider the following linear inhomogeneous problem, derived from the toy tomography problem above:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \\ 4 \end{pmatrix}$$

a. Determine a solution with $v = 0$.

$1u + 0 + 1u + 0y = 2$ |TU

$u = 2$

$0u + 0 + 1x + 1y = 5 \Rightarrow$ not usable

$1u + 0 + 1x + 0y = 3$ |TU |u=2

$x + 2 = 3$ |−2

$x = 1$

$0u + 0 + 0x + 1y = 4$

$y = 4$

$\Rightarrow$ $u = 2$, $x = 1$, $y = 4$

b. Determine a solution with $x = 0$.

$1u + 1v + 0x + 0y = 2$

$v + v = 2 \Rightarrow$ not usable

$0u + 0v + 0 + 1y = 5$ |TU

$y = 5$

$1u + 0u + u + 0y = 3$ |TU

$u = 3$

$0u + 1v + 0 + 1y = 4$ |TU |y=5

$v + 5 = 4$ |−5 $\Rightarrow$ $v = -1$

$\Rightarrow$ $V = -1$, $V = 3$, $Y = 5$