Peer Assessment Gion Rubitschung

Problem 3 **[peer assessment]**

Let V be the 4×3 matrix with orthonormal columns

$$v_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \qquad v_2 = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} \qquad v_3 = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

Let

$$w_1 = \begin{pmatrix}-9\\6\\2\end{pmatrix} \qquad w_2 = \begin{pmatrix}1\\4\\-8\end{pmatrix}$$

Compute:

a. $\|w_1\|$ and $\|w_2\|$ and the dot product $w_1 \cdot w_2$.

b. $\|Vw_1\|$ and $\|Vw_2\|$ and the dot product $(Vw_1) \cdot (Vw_2)$.

c. The 4×4 matrix P that orthogonally projects onto span$\{v_1, v_2, v_3\}$.

d. $P\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

a.) $\|w_1\| = \sqrt{(-9)^2 + 6^2 + 2^2}$

$= \sqrt{81 + 36 + 4} = \sqrt{121} = \pm 11$

$\|w_2\| = \sqrt{1^2 + 4^2 + (-8)^2}$

$= \sqrt{1 + 16 + 64} = \sqrt{81} = \pm 9$

$w_1 \cdot w_2 = -9 \cdot 1 + 6 \cdot 4 + 2 \cdot (-8) = -9 + 24 - 16 = -1$

b.) $V = \frac{1}{2}\begin{pmatrix}1&1&1\\1&1&-1\\1&-1&1\\1&-1&-1\end{pmatrix}$

$\|Vw_1\| = \left\| \frac{1}{2}\begin{pmatrix}1&1&1\\1&1&-1\\1&-1&1\\1&-1&-1\end{pmatrix}\begin{pmatrix}-9\\6\\2\end{pmatrix} \right\| = \left\| \begin{pmatrix}-\frac{1}{2}\\-\frac{5}{2}\\-\frac{13}{2}\\-\frac{17}{2}\end{pmatrix} \right\|$

$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{5}{2}\right)^2 + \left(-\frac{13}{2}\right)^2 + \left(-\frac{17}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{25}{4} + \frac{169}{4} + \frac{289}{4}} = \sqrt{\frac{484}{4}} = \sqrt{121} = \pm 11$

Fact (reformulated). *A matrix V has orthonormal columns if and only if it preserves the dot product, meaning for all vectors x, y:*

$$(Vx) \cdot (Vy) = x \cdot y$$

$\Rightarrow \|Vw_1\| = \|w_1\| = \pm 11$

$\|Vw_2\| = \|w_2\| = \pm 9$

$(Vw_1) \cdot (Vw_2) = w_1 \cdot w_2 = -1$

c.) $P = VV^T = \frac{1}{2}\begin{pmatrix}1&1&1\\1&1&-1\\1&-1&1\\1&-1&-1\end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\end{pmatrix}$

$P = \frac{1}{4}\begin{pmatrix}3&1&1&-1\\1&3&-1&1\\1&-1&3&1\\-1&1&1&3\end{pmatrix}$

d.) $P\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3\\1\\1\\-1\end{pmatrix}$