Gion Rubitschung D1P 30.09.23

12_Matrix_multiplication Problem 4

In [ ]:
```
import numpy as np
from numpy.linalg import matrix_power
```

Let M be the 5×5 matrix that shifts vector entries upwards, with the first entry becoming the last entry. That is, for all $a, b, c, d, e \in \mathbb{R}$:

$$M \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} b \\ c \\ d \\ e \\ a \end{pmatrix}$$

In [ ]:
```
M = np.array(
    [
        [0, 1, 0, 0, 0],
        [0, 0, 1, 0, 0],
        [0, 0, 0, 1, 0],
        [0, 0, 0, 0, 1],
        [1, 0, 0, 0, 0],
    ]
)
v = np.array([[1], [2], [3], [4], [5]])
print(M @ v)
```

```
[[2]
 [3]
 [4]
 [5]
 [1]]
```

For every integer $k \geq 1$, define the k-th matrix power by

$$M^k = \underbrace{MM \cdots M}_{k \text{ factors}}$$

For example, $M^1 = M$ and $M^2 = MM$ and $M^3 = MMM$.

In [ ]:
```
print("M 2-th:")
print(matrix_power(M, 2))
print("M 3-th:")
print(matrix_power(M, 3))
print("M 4-th:")
print(matrix_power(M, 4))
```

```
M 2-th:
[[0 0 1 0 0]
 [0 0 0 1 0]
 [0 0 0 0 1]
 [1 0 0 0 0]
 [0 1 0 0 0]]
M 3-th:
[[0 0 0 1 0]
 [0 0 0 0 1]
 [1 0 0 0 0]
 [0 1 0 0 0]
 [0 0 1 0 0]]
M 4-th:
[[0 0 0 0 1]
 [1 0 0 0 0]
 [0 1 0 0 0]
 [0 0 1 0 0]
 [0 0 0 1 0]]
```

a.

a. M^2 corresponds to applying M twice. Determine

$$M^2 \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$$

In [ ]:
```
M2 = matrix_power(M, 2)

print(M2 @ v)
```

```
[[3]
 [4]
 [5]
 [1]
 [2]]
```

b.

b. Determine the smallest integer $k \geq 1$ such that M^k is the identity matrix.

In [ ]:
```
M5 = matrix_power(M, 5)

print(M5)
```

```
[[1 0 0 0 0]
 [0 1 0 0 0]
 [0 0 1 0 0]
 [0 0 0 1 0]
 [0 0 0 0 1]]
```

c.

c. Determine the 19-th matrix power, M^{19}.

In [ ]:
```
M6 = matrix_power(M, 6)
M11 = matrix_power(M, 11)
M16 = matrix_power(M, 16)

print(np.array_equal(M, M6))
print(np.array_equal(M, M11))
print(np.array_equal(M, M16))
```

```
True
True
True
```

In [ ]:
```
M4 = matrix_power(M, 4)
M19 = matrix_power(M, 19)

print(M4)
print(M19)
print(np.array_equal(M4, M19))
```

```
[[0 0 0 0 1]
 [1 0 0 0 0]
 [0 1 0 0 0]
 [0 0 1 0 0]
 [0 0 0 1 0]]
[[0 0 0 0 1]
 [1 0 0 0 0]
 [0 1 0 0 0]
 [0 0 1 0 0]
 [0 0 0 1 0]]
True
```