Summary Midterm Exam

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10 Introduction:

Vector plus Vector

$$\begin{pmatrix}1\\0.3\\7\end{pmatrix}+\begin{pmatrix}1\\0.6\\-2\end{pmatrix}=\begin{pmatrix}2\\0.9\\5\end{pmatrix}$$

Number times Vector

$$5\begin{pmatrix}1\\0.3\\-7\end{pmatrix}=\begin{pmatrix}5\\1.5\\-35\end{pmatrix}$$

Matrix and Vector multiplication:

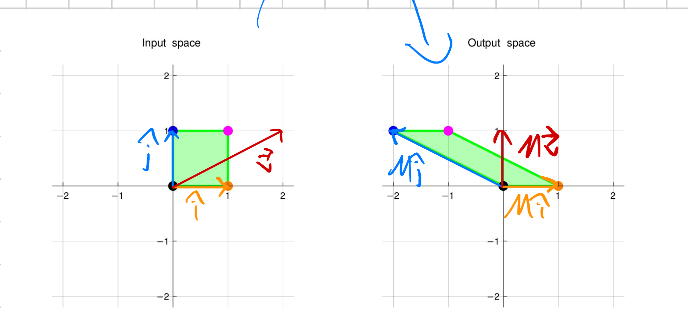
$$\begin{pmatrix}-2 & 1 & 0 & 0\\1 & -2 & 1 & 0\\0 & 1 & -2 & 1\end{pmatrix}\begin{pmatrix}1\\1\\3\\7\end{pmatrix}=1\begin{pmatrix}-2\\1\\0\end{pmatrix}+1\begin{pmatrix}1\\-2\\1\end{pmatrix}+3\begin{pmatrix}0\\1\\-2\end{pmatrix}+7\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$=\begin{pmatrix}1\cdot -2+1\cdot 1+3\cdot 0+7\cdot 0\\1\cdot 1+1\cdot -2+3\cdot 1+7\cdot 0\\1\cdot 0+1\cdot 1+3\cdot -2+7\cdot 1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

11 Matrix as linear map

M → What is M?



$$\hat{i}=\begin{pmatrix}1\\0\end{pmatrix}\rightarrow M\hat{i}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$\hat{j}=\begin{pmatrix}0\\1\end{pmatrix}\rightarrow M\hat{j}=\begin{pmatrix}-2\\1\end{pmatrix}$$

$$\vec{v}=\begin{pmatrix}2\\1\end{pmatrix}=2\cdot\hat{i}+1\cdot\hat{j}$$

$$\Longrightarrow M\vec{v}=2\cdot M\hat{i}+1\cdot M\hat{j}$$

$$M\vec{v}=2\cdot\begin{pmatrix}1\\0\end{pmatrix}+1\cdot\begin{pmatrix}-2\\1\end{pmatrix}\Longrightarrow M=\begin{pmatrix}M\hat{i} & M\hat{j}\end{pmatrix}$$

$$M\vec{v}=\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\qquad M=\begin{pmatrix}1 & -2\\0 & 1\end{pmatrix}$$

Where $M\hat{i}$ "lands"

Where $M\hat{j}$ "lands"

Shear matrix

A square matrix[1] with
- along the diagonal[2]: all 1s
- away from the diagonal: all 0s except for at most one entry

is called a shear matrix. It is actually a special kind of shear matrix, but we only consider this special kind for the moment.

Examples:

$$\begin{pmatrix}1&-2\\0&1\end{pmatrix} \quad \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&5&0&1\end{pmatrix}$$

shear matrix

[1] square matrix: $d \times d$ matrix for some d

[2] diagonal: upper left to lower right corner

Scaling matrix

A square matrix with
- along the diagonal: all equal to the same real number λ
- away from the diagonal: all 0s

is called a scaling matrix.

Examples:

$$\begin{pmatrix}1.5&0\\0&1.5\end{pmatrix} \quad \begin{pmatrix}0.1&0&0&0\\0&0.1&0&0\\0&0&0.1&0\\0&0&0&0.1\end{pmatrix}$$

Identity matrix

$\mathbb{1}$

The square matrix with
- along the diagonal: all 1s
- away from the diagonal: all 0s

is called the identity matrix.

Only matrices that return the same vector if multiplied with a vector

Example.

2x2 id. mat. $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ 4x4 id. mat. $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

The $d \times d$ identity matrix is typically denoted $\mathbb{1}_d$, or simply $\mathbb{1}$.

Diagonal matrix

A square matrix with
- along the diagonal: no condition
- away from the diagonal: all 0s

is called a diagonal matrix.

Examples:

$$\begin{pmatrix}0.5&0\\0&1.5\end{pmatrix} \quad \begin{pmatrix}2&0&0\\0&-4&0\\0&0&5.2\end{pmatrix} \quad \begin{pmatrix}3.1&0&0&0\\0&0&0&0\\0&0&12.1&0\\0&0&0&-2\end{pmatrix}$$

$\Rightarrow$ diagonal matrices

Permutation matrix

A square matrix with
- in every row: one 1, otherwise all 0s
- in every column: one 1, otherwise all 0s

is called a permutation matrix.

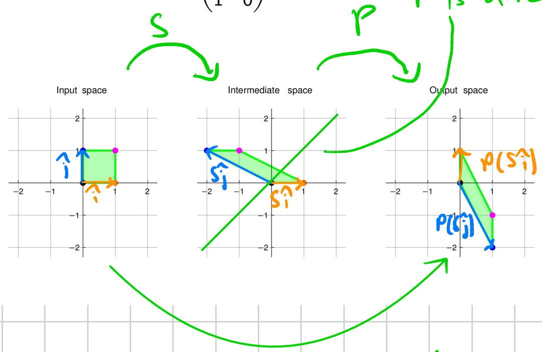
Examples:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \quad \begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix}$$

12 Matrix Multiplication

Applying **first** the shear matrix $S = \begin{pmatrix}1&-2\\0&1\end{pmatrix}$, **then** the permutation matrix $P = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, yields:

P is a reflection



$C \rightarrow$ directly applied

$C = \begin{pmatrix} P(S\hat{i}) & P(S\hat{j}) \end{pmatrix}$

$C = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

$\Rightarrow C = PS \neq SP$

$(AB)C = A(BC) = ABC$

$AB \neq BA$

Matrix multiplication:

$A = \begin{pmatrix} a_a & b_a \\ c_a & d_a \end{pmatrix} \quad B = \begin{pmatrix} a_b & b_b \\ c_b & d_b \end{pmatrix}$

For multiplication step by step

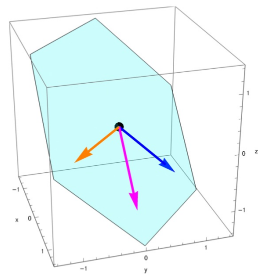
$$\Rightarrow AB = \begin{pmatrix} \underbrace{a_b \begin{pmatrix} a_a \\ c_a \end{pmatrix} + c_b \begin{pmatrix} b_a \\ d_a \end{pmatrix}}_{\text{first row}} \\ \underbrace{b_b \begin{pmatrix} a_a \\ c_a \end{pmatrix} + d_b \begin{pmatrix} b_a \\ d_a \end{pmatrix}}_{\text{second row}} \end{pmatrix}$$

$$AB = \begin{pmatrix} A\begin{pmatrix} a_b \\ c_b \end{pmatrix} & A\begin{pmatrix} b_b \\ d_b \end{pmatrix} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_b a_a + c_b b_a & b_b a_a + d_b b_a \\ a_b c_a + c_b d_a & b_b c_a + d_b d_a \end{pmatrix}$$

13 Span Supspace Basis Dimension

The set of all linear combinations of $\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ is still the same plane through the origin:



$$\Rightarrow \text{span} = \left\{ \overset{v_1}{\begin{pmatrix}1\\0\\-1\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}1\\-1\\0\end{pmatrix}}, \overset{v_3}{\begin{pmatrix}0\\1\\-1\end{pmatrix}} \right\}$$

$$\Rightarrow \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

- zero vector $\begin{pmatrix}0\\0\\0\end{pmatrix}$ is not affecting the span
- span: all linear combinations of k vectors
- basis: 1. $\text{span}\{v_1, \dots v_k\} = U$

  2. None of the k vectors is in the span of the other k-1 vectors.

  $\Rightarrow$ removing any one of the vectors would result in not beeing U

  $\Rightarrow$ k vectors need to be linearly independent

$\Rightarrow$ v_3 is a linear comb. of v_1 and v_3

$$\Rightarrow \text{span} = \left\{ \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\}$$

is the basis, since v_1 and v_2 are linearly independent

Question

$-a_1 + 0 + 0 = -2 \Rightarrow a_1 = 2$

$0 - a_2 + 0 = 7 \Rightarrow a_2 = -7$

$0 + 0 - a_3 = -10 \Rightarrow a_3 = 10$

$U \ni \begin{pmatrix}5\\-2\\7\\-10\end{pmatrix}$

Question. In $\mathbb{R}^4$, consider the set U of all

$a_1 + a_2 + a_3 = 5$

$2 + (-7) + 10 = 5$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

$U \not\ni \begin{pmatrix}5\\2\\7\\-1\end{pmatrix}$

for which $x_1 + x_2 + x_3 + x_4 = 0$. Is this a subspace?

Answer. Yes, it is equal to

$a_1\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} + a_3\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$

$$\text{span}\left\{ \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \right\} \overset{\text{right side contained in left side}}{\subseteq} = U \overset{\text{left side contained in right side}}{\supseteq}$$

because this span is in U, and every element in U is in this span.

$$\text{span}\left\{ \begin{pmatrix}x_1\\-x_2\\0\\0\end{pmatrix}, \begin{pmatrix}x_1\\0\\-x_3\\0\end{pmatrix}, \begin{pmatrix}x_1\\0\\0\\-x_4\end{pmatrix} \right\}$$

where all vectors in span result in $x_1 + x_2 + x_3 + x_4 = 0$

$$\Rightarrow \text{span}\left\{ \begin{pmatrix}x_1\\-2x_2\\0\\0\end{pmatrix}, \begin{pmatrix}x_1\\0\\-3x_3\\0\end{pmatrix}, \begin{pmatrix}x_1\\0\\0\\-4x_4\end{pmatrix} \right\}$$

where all vectors in span result in $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

$$\Rightarrow \text{span}\left\{ \begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-4\\0\\0\\1\end{pmatrix} \right\}$$

14 Row reduction (also known as Gaussian elimination)

Key fact 1 about rref:
It exists and there is an algorithm

Given a matrix A with real entries, there exists an rref B of the same size as A, such that the span of the rows of A equals the span of the rows of B. Such a B can be obtained from A, using the algorithm discussed in the example, using only the following elementary row operations that do not change the span:

- add multiple of one row to another row:
  $R_i \leftarrow R_i + aR_j$ with $i \neq j$
- multiply a row by a nonzero number:
  $R_i \leftarrow aR_i$ with $a \neq 0$
- swap two rows:
  $R_i \leftrightarrow R_j$

Key fact 2 about rref:
The nonzero rows are linearly independent

Given an rref, then the nonzero rows are linearly independent, meaning none of them is a linear combination of the others.

This follows from the structure of the pivot columns.

Key fact 3 about rref:
It is canonical

canonical means that the alg. produces a basis that depends only on the span of the input vectors

Weak version: If a matrix is simplfied to rref, then no matter which and how many elementary row operations were used, the result is always the same. In this sense, every matrix A has a unique reduced row echelon form. We denote it by

$$\mathrm{rref}(A)$$

Strong version: Suppose A and A' are two matrices such that the span of the rows of A is equal to the span of the rows of A'. Then

$$\mathrm{rref}(A) = \mathrm{rref}(A')$$

Actually, they can differ in the number of all-zero rows. One could define rref(A) to be the matrix after the all-zero rows are dropped, but conventionally, that is not done.

$$\mathrm{span}\left\{ \begin{pmatrix}1\\0\\2\\1\\3\end{pmatrix}, \begin{pmatrix}1\\0\\2\\-2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\5\\0\\6\end{pmatrix}, \begin{pmatrix}1\\2\\4\\1\\3\end{pmatrix} \right\}$$

What dimension has this span?

- $\cancel{\mathbb{R}^0}$ → more than one vector
- $\cancel{\mathbb{R}^1}$ → $\mathbb{R}_3$ needs to consist of at least 3 vectors
- $\cancel{\geq \mathbb{R}^5}$ → consists of 4 vectors

⇒ between $\mathbb{R}^2$ and $\mathbb{R}^4$

put span into matrix
→ vectors become cols in matrix

nice pivot

$$\begin{pmatrix} \boxed{1} & 0 & 2 & 1 & 3 \\ 1 & 0 & 2 & -2 & 3 \\ 2 & 1 & 5 & 0 & 6 \\ 1 & 2 & 4 & 1 & 3 \end{pmatrix}$$

1. Clean C_1:

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_1$

$R_4 \leftarrow R_4 - R_1$

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{pmatrix}$$

2. Then $C_2, C_3 \ldots$

$$\mathrm{rref} = \begin{pmatrix} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3 pivots
⇒ dim 3

basis of span: $\left\{ \begin{pmatrix}1\\0\\2\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} \right\}$

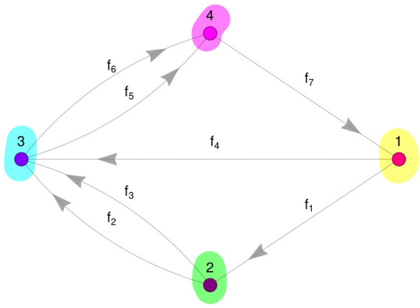
Checklist for rref:

- All zero rows at the bottom
- Above all zero rows:
  - The first nonzero entry in each row is a 1, called pivot
  - The pivot columns give the id. matrix

15 Homogeneous linear equations and Nullspace

Cont., traffic flow equations

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_7 \end{pmatrix}$$



vertex 1:	$-f_1 - f_4 + f_7 = 0$
vertex 2:	$f_1 - f_2 - f_3 = 0$
vertex 3:	$f_2 + f_3 + f_4 - f_5 - f_6 = 0$
vertex 4:	$f_5 + f_6 - f_7 = 0$

Cont., matrix form

linear comb. of unknowns $\Rightarrow$ linear homogeneous

all 0s

$$\begin{aligned} -f_1 - f_4 + f_7 &= 0 \\ f_1 - f_2 - f_3 &= 0 \\ f_2 + f_3 + f_4 - f_5 - f_6 &= 0 \\ f_5 + f_6 - f_7 &= 0 \end{aligned}$$

- 4 equations
- 7 unknowns

is equivalent to

edge 1 — starts at first vertex and goes to second vertex

edge 7

$Af = 0$

$$\begin{array}{c} \text{vertex 1} \\ \\ \\ \text{vertex 4} \end{array} \underbrace{\begin{pmatrix} \boxed{-1} & 0 & 0 & -1 & 0 & 0 & 1 \\ \boxed{1} & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \boxed{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & -1 \end{pmatrix}}_{= A,\ \text{known as incidence matrix}} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix} = 0$$

edge 5 — starts at third vertex and goes to 4th vertex

$$Af = 0 \iff \text{rref}(A)\, f = 0$$

pivot cols

pivot knowns

non pivot unknowns

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{\text{rref}(A)} \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix}}_{f} = 0$$

$\Rightarrow$ pivot knowns can be expressed through pivot unknowns

$$\Rightarrow \underbrace{\begin{aligned} -f_1 - f_4 + f_7 &= 0 \\ f_1 - f_2 - f_3 &= 0 \\ f_2 + f_3 + f_4 - f_5 - f_6 &= 0 \\ f_5 + f_6 - f_7 &= 0 \end{aligned}}$$

$$\begin{aligned} f_1 &= -f_4 + f_7 \\ f_2 &= -f_3 - f_4 + f_7 \\ f_5 &= -f_6 + f_7 \end{aligned}$$

> Non-pivot vars f_3, f_4, f_6, f_7 are free and uniquely determine pivot vars f_1, f_2, f_5

$\Rightarrow$ Solution to $Af = 0$:

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix} = \begin{pmatrix} -f_4 + f_7 \\ -f_3 - f_4 + f_7 \\ f_3 \\ f_4 \\ -f_6 + f_7 \\ f_6 \\ f_7 \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} f_3 & f_4 & f_6 & f_7 \end{array} \\ \underbrace{\begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{nullspace } S} \end{array} \begin{pmatrix} f_3 \\ f_4 \\ f_6 \\ f_7 \end{pmatrix}$$

id. mat.

- A system of equations is called linear homogeneous if and only if it can be written as

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad Ax = 0$$

for some matrix A that does not depend on $x_1 \dots x_n$

- Set of all x with $Ax = 0$ is called nullspace of A or kernel of A. It is a subspace

$\Rightarrow$ Constructing solution x of $Ax=0$ is equivalent to constructing basis of the nullspace A

$\Rightarrow$ $Ax = 0 \rightarrow AS = 0 \rightarrow x = Su$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ↳ non pivot vars

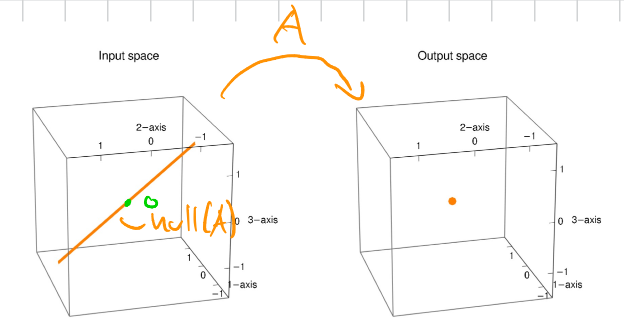
$\Rightarrow$ S solution operator of $Ax=0$.
If S has 0 cols $\rightarrow x=0$ only solution to $Ax=0$

16 Columnspace Rank

$$\begin{aligned} x + y + z &= 0 \\ y + 2z &= 0 \\ x - z &= 0 \end{aligned} \Rightarrow A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

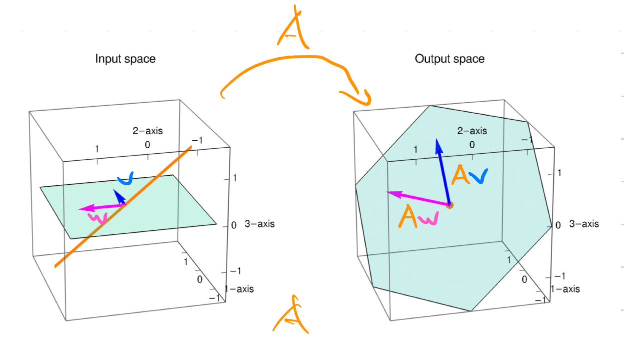
$$\rightarrow S = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

↳ basis null(A)
1 dim

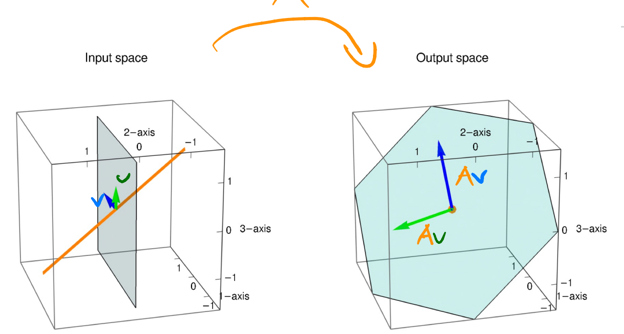


Two linearly independent vectors / inputs w and v that intersect with null(A) at the origin

$$w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



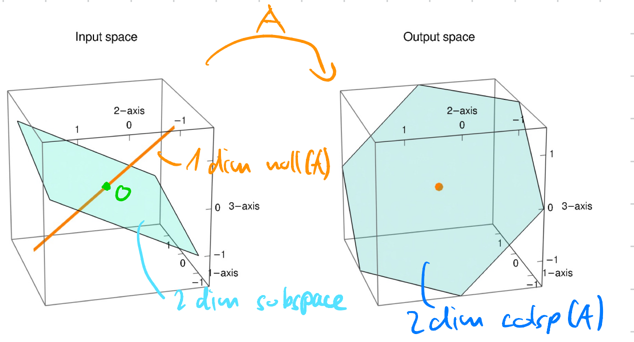
$$\text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \xrightarrow{A} \text{span}\left\{ A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$



$$u = \begin{pmatrix} 0 \\ 0 \\ .5 \end{pmatrix} \qquad \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ .5 \end{pmatrix} \right\}$$

$$\xrightarrow{A} \text{span}\left\{ A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, A\begin{pmatrix} 0 \\ 0 \\ .5 \end{pmatrix} \right\}$$

$\Rightarrow$ Output is the same $\Rightarrow$ colsp



Any 2-dim subspace that intersects with null(A) at the origin. The result of mapping A onto the subspace is the columspace, colsp(A)

$\Rightarrow$ $\boxed{\text{colsp}(A) \text{ is the span of all columns of } A}$

$\Rightarrow$ $\dim(\text{colsp}(A)) = \dim(\text{input space}) - \dim(\text{null}(A))$

$$\begin{aligned} \text{rank}(A) &= \text{number of pivots in rref}(A) \\ &= \dim(\text{input space}) - \dim(\text{null}(A)) \\ &= \dim(\text{colsp}(A)) \end{aligned}$$

$\Rightarrow$ if A is $m \times n$ then $\text{rank}(A) \le m$ and $\text{rank}(A) \le n$

The basis of a colsp can be obtained by computing the rref of the transpose. Ex:

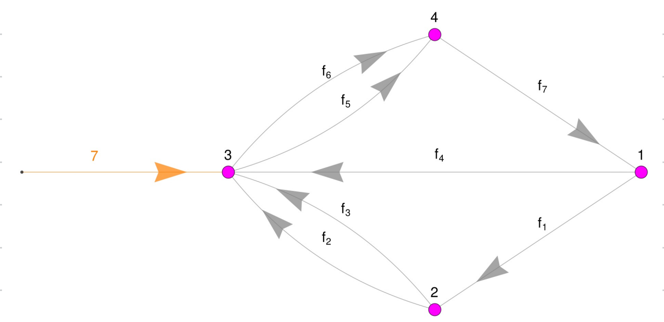
$$A = \begin{pmatrix} 1 & 5 & 34 & 233 \\ 1 & 8 & 55 & 377 \\ 2 & 13 & 89 & 610 \\ 3 & 21 & 144 & 987 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 \\ 34 & 55 & 89 & 144 \\ 233 & 377 & 610 & 987 \end{pmatrix}$$

$\Rightarrow$ compute $\text{rref}(A^T)$

$$\text{rref}(A^T) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$\Rightarrow$ basis of $\text{colsp}(A)$ $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$

17 Inhomogeneous linear equations



$$3: \; f_2 + f_3 + f_4 - f_5 - f_6 + 7 = 0$$
$$f_2 + f_3 + f_4 - f_5 - f_6 = -7$$

A solution f exist to $Af = b$ if b is in the colsp of A

$\Rightarrow$ Check $\text{rank}(A) = 3$, hence $\text{colsp}(A)$ is 3-dim subspace of $\mathbb{R}^4$! b might be in there

Cont. $f_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + f_2 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \ldots + f_7 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A \text{ (incidence matrix)}} \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix}}_{f \to \text{unknown}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ -7 \\ 0 \end{pmatrix}}_{b}$$

$A, b \to$ known

This is an inhomogeneous linear system.

Does a solution exist? * $\Leftrightarrow$ is $b \in \text{colsp}(A)$?

Compute all solutions!

* $Af = 0$, then $f = 0$ is obviously a solution

To get the solution augment the matrix A with b

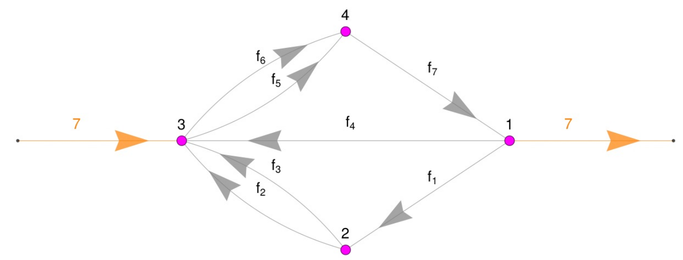
$$(A\,|\,b) = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & -1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ -1 \end{pmatrix} = 0$$

$\Rightarrow$ linear homogeneous system, compute rref$(A\,|\,b)$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ -1 \end{pmatrix} = 0$$

- Last col is pivot col
  - $\Rightarrow$ 4th equation is unsatisfiable
  - $\Rightarrow$ no solution, system is inconsistent

$1 \cdot (-1) = -1 \neq 0$



Total traffic into system is zero.
Ad-hoc observation: Solution exists using only f_4.

But again, we want to run our systematic method.

1: $-f_1 - f_4 + f_7 - 7 = 0$

$-f_1 - f_4 + f_7 = 7$

3: $f_2 + f_3 + f_4 - f_5 - f_6 + 7 = 0$

$f_2 + f_3 + f_4 - f_5 - f_6 = -7$

$\Rightarrow \; b = \begin{pmatrix} 7 \\ 0 \\ -7 \\ 0 \end{pmatrix}$

$\Rightarrow$ $Af = b$ $\rightarrow$ Augment A with b, add -1 to f

$\rightarrow (A\,|\,b)\begin{pmatrix} f \\ -1 \end{pmatrix} = 0 \rightarrow$ rref$(A\,|\,b)$

$$\begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & -1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_7 \\ -1 \end{pmatrix} = 0 \;\rightarrow \text{rref}\; \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 & -7 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_7 \\ -1 \end{pmatrix} = 0$$

- Last col not pivot col

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_7 \end{pmatrix} = \begin{pmatrix} -7 \\ -7 \\ 0 \end{pmatrix} \Rightarrow$$

<u>One</u> solution: $f = \begin{pmatrix} -7 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ All solutions: Adding element in null(A)

$f_* + \text{null}(A)$

Summary of the method

Suppose a matrix A and a vector b are given, with real entries. Consider the linear inhomogeneous system for the unknown x:

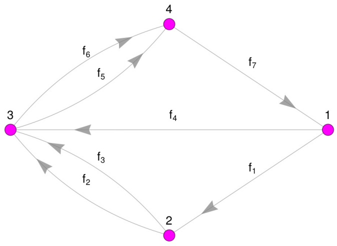
$$Ax = b$$

Theorem. *Consider the rref of the augmented matrix:*

$$\mathrm{rref}(A|b)$$

- *If the last column is a pivot column, then no solution x exists.*
- *If the last column is not a pivot column, then:*
  - *There is unique solution x_* that uses only components corresponding to pivot columns.*
  - *The space of all solutions x is $x_* + \mathrm{null}(A)$.*

In the first case, the system is called inconsistent, in the second case, consistent.



Suppose you measure $f_2 = 10$, $f_4 = -2$, $f_6 = 7$.
Want to determine f.

$$\left.\begin{aligned} -f_1 - f_4 + f_7 &= 0 \\ f_1 - f_2 - f_3 &= 0 \\ f_2 + f_3 + f_4 - f_5 - f_6 &= 0 \\ f_5 + f_6 - f_7 &= 0 \end{aligned}\right\} \text{junctions}$$

$$\left.\begin{aligned} f_2 &= 10 \\ f_4 &= -2 \\ f_6 &= 7 \end{aligned}\right\} \text{measurements}$$

inhom. lin. sys.

Since 3 values are already known, there is another option.

Option 2. Simply replace known values:

$$\begin{aligned} -f_1 + f_7 &= -2 \\ f_1 - f_3 &= 10 \\ f_3 - f_5 &= -1 \\ f_5 - f_7 &= -7 \end{aligned}$$

20 Square matrix and inverse

Square matrix: num of rows equals num of cols

A linear systems with as many equations as unknowns such as

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_3 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26 \end{aligned}$$

leads to a square matrix:

$$\underbrace{\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 39 \\ 34 \\ 26 \end{pmatrix}}_{b} \Rightarrow (A \mid b)$$

The augmented matrix is

$$(A|b) = \left(\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right)$$

"usually" if A is square, then $\mathrm{rref}(A) = \mathbb{1}$

$\boxed{\mathbb{1}} \to$ id. mat.

id. mat. to the left

Its rref is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{37}{4} \\ 0 & 1 & 0 & \frac{17}{4} \\ 0 & 0 & 1 & \frac{11}{4} \end{array}\right)$$

$= x$ then $Ax = b$

Generally true: When the rref of a square matrix is $\mathbb{1}$, then the rref of the augmented matrix is $(\mathbb{1}|\text{solution})$.

id. mat.

Doing this gives one solution, to get general solution we can use the standard basis vectors:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Augment A with the standard basis vectors $(A \mid Av_1\ Av_2\ Av_3)$ which gives $(A \mid \mathbb{1})$

$\Rightarrow$ Computing the rref results in $(\mathbb{1} \mid v_1\ v_2\ v_3)$ or in other words the general solution

$$\mathrm{rref}\left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) = (\mathbb{1}|v_1 v_2 v_3)$$

$$\mathrm{rref}\begin{pmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \underbrace{\begin{array}{ccc} \frac{7}{12} & -\frac{1}{3} & -\frac{1}{12} \\ -\frac{5}{12} & \frac{2}{3} & -\frac{1}{12} \\ \frac{1}{12} & -\frac{1}{3} & \frac{5}{12} \end{array}}_{A^{-1}}\right) \qquad A^{-1} = \begin{pmatrix} \frac{7}{12} & -\frac{1}{3} & -\frac{1}{12} \\ -\frac{5}{12} & \frac{2}{3} & -\frac{1}{12} \\ \frac{1}{12} & -\frac{1}{3} & \frac{5}{12} \end{pmatrix}$$

$\Rightarrow Ax = b \to x = A^{-1}b$

$$\underbrace{\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}}_{=A} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

is $x = b_1 v_1 + b_2 v_2 + b_3 v_3$.

Proof.

$$\begin{aligned} Ax &= A(b_1 v_1 + b_2 v_2 + b_3 v_3) \\ &= b_1 A v_1 + b_2 A v_2 + b_3 A v_3 = b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \end{aligned}$$

Inverse matrix

Definition and Theorem.
A square matrix A is said to be invertible if and only if $\operatorname{rref}(A) = \mathbb{1}$. In this case, define a new matrix A^{-1} by

$$\operatorname{rref}(A|\mathbb{1}) = (\mathbb{1}|A^{-1})$$

called the inverse matrix of A.

- *It is the unique matrix that satisfies $AA^{-1} = \mathbb{1}$.*
- *It is the unique matrix that satisfies $A^{-1}A = \mathbb{1}$.*
- *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*

Idea of Proof. The definition relies on the uniqueness of rref. For $AA^{-1} = \mathbb{1}$ see the example above. Our definition of A^{-1} implies $\operatorname{rref}(A^{-1}|\mathbb{1}) = (\mathbb{1}|A)$. Hence $\operatorname{rref}(A^{-1}) = \mathbb{1}$ and $(A^{-1})^{-1} = A$. Hence $A^{-1}A = A^{-1}(A^{-1})^{-1} = \mathbb{1}$. Uniqueness: If B satisfies $AB = \mathbb{1}$ then $B = A^{-1}AB = A^{-1}\mathbb{1} = A^{-1}$, using the associativity of matrix multiplication. If C satisfies $CA = \mathbb{1}$ then $C = CAA^{-1} = \mathbb{1}A^{-1} = A^{-1}$.

Properties of invertible square matrix

Theorem. *For a real $n \times n$ matrix A, the following are equivalent:*

- *$\operatorname{rref}(A) = \mathbb{1}$.*
- *A is invertible.*
- *The nullspace $\operatorname{null}(A)$ contains only the zero vector.*
- *The column space $\operatorname{colsp}(A)$ is all of $\mathbb{R}^n$.*
- *For every $b \in \mathbb{R}^n$ there is a unique $x \in \mathbb{R}^n$ such that $Ax = b$.*

This follows from our construction of nullspace and columnspace using rref; the solution of the inhomogeneous problem using the rref of the augmented matrix.

Inverse of AB

Theorem. *Suppose A and B are square matrices of the same size. Then AB is invertible if and only if A and B are invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. Applying AB means first applying B and then applying A; therefore, to undo AB, we must first undo what A did, then undo what B did, which is $B^{-1}A^{-1}$. As a computation:

$$(B^{-1}A^{-1})(AB) = B^{-1}(AA^{-1})B = B^{-1}B = \mathbb{1}$$

Why $(AB)^{-1}$ cannot possibly be $A^{-1}B^{-1}$

An $n \times n$ matrix maps $\mathbb{R}^n$ to $\mathbb{R}^n$, but conceptually, imagine that these spaces are unrelated. The input could be volume, the output weight. Let us use color to distinguish different types.

$A : \mathbb{R}^n \to \mathbb{R}^n$
$B : \mathbb{R}^n \to \mathbb{R}^n$
Output space of B matches input space of A so that AB is defined.

$A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ because it undoes what A does
$B^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ because it undoes what B does

Note that $A^{-1}B^{-1}$ is undefined; the types do not match.
By contrast, the types match just fine for $B^{-1}A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$.

Given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are symbols. Compute the row reduction

$$\operatorname{rref}(A|\mathbb{1})$$

and derive a formula for A^{-1}. State all assumptions that you make.

$\Rightarrow$

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

Multiply the first row by $\frac{1}{a}$, assuming $a \neq 0$:

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

Add $-c$ times the first row to the second row:

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{pmatrix}$$

Simplify:

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{pmatrix}$$

Multiply the second row by $\frac{a}{ad-bc}$, assuming $a \neq 0$ and $ad - bc \neq 0$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Add $-b/a$ times the second row to the first row:

$$\begin{pmatrix} 1 & 0 & \frac{1}{a} + \frac{bc/a}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Simplify:

$$\boxed{\begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}}$$

Therefore $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, valid when $ad - bc \neq 0$ (do not need $a \neq 0$).

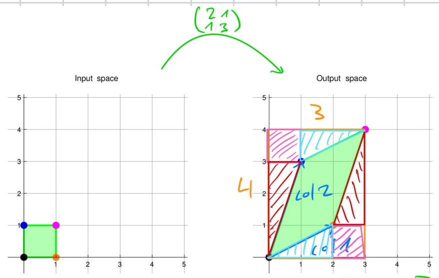
21 Determinant

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The denominator is the determinant:

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

Matrix is invertible if and only if the determinant is $\neq 0$.



$12 - 2 - 3 - 2 = 5$
see next slide

$$\det\left(\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\right) = 2 \cdot 3 - 1 \cdot 1 = 5 \text{ equals area of the parallelogram}$$

$a = 2$, $b = 1$, $c = 1$, $d = 3$

$|ad - bc|$

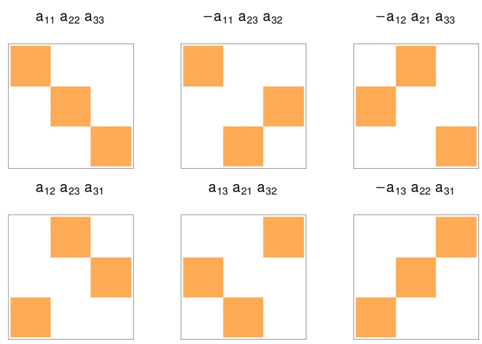
To generally get the area after applying linear map:

area = area in input space · det

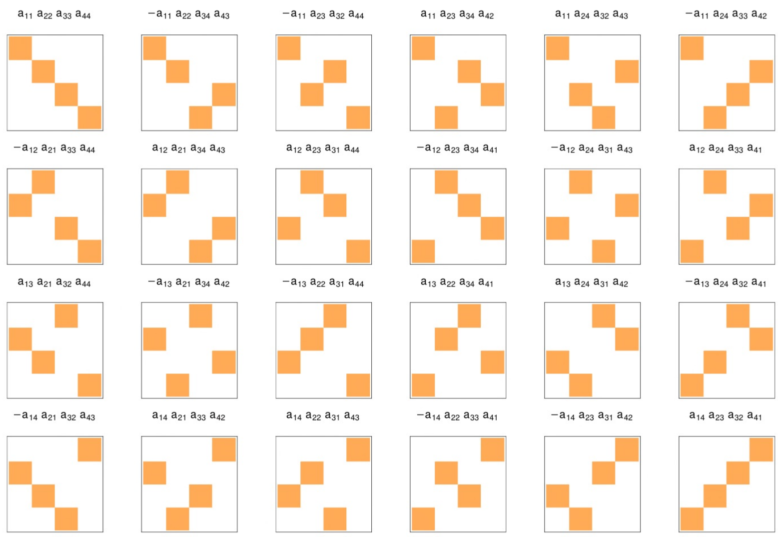
Inverse of 3×3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\underbrace{\boxed{aei - ahf - dbi + dhc + gbf - gec}}_{\text{def}}} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

$\Rightarrow$



For 4×4 matrix:

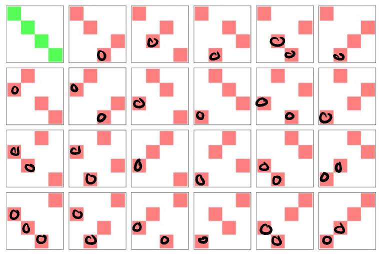


Def for upper/lower triangular mat.

$$\det\left(\begin{pmatrix} \lambda_1 & * & * & \dots & * & * \\ 0 & \lambda_2 & * & \dots & * & * \\ 0 & 0 & \lambda_3 & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{n-1} & * \\ 0 & 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}\right) = \lambda_1\lambda_2\lambda_3\cdots\lambda_{n-1}\lambda_n$$

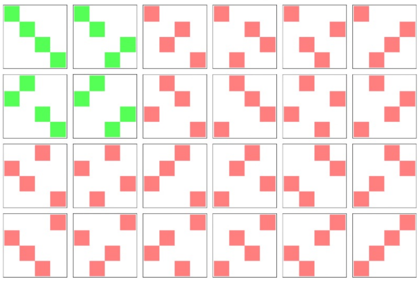
Since these entries have a 0, their corresponding boxes don't contribute to the def:

$$\det\left(\begin{pmatrix} 3 & -6 & 7 & 11 \\ 0 & 5 & 1 & 0.2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}\right) = 3 \cdot 5 \cdot (-1) \cdot 2 = -30$$



$$\det\left(\begin{pmatrix} 3 & 4 & -2 & 3 \\ 7 & 11 & 5 & 9 \\ 0 & 0 & 19 & 2 \\ 0 & 0 & 13 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3 & 4 \\ 7 & 11 \end{pmatrix}\right) \det\left(\begin{pmatrix} 19 & 2 \\ 13 & 1 \end{pmatrix}\right)$$

$$= 5 \cdot (-7) = -35$$



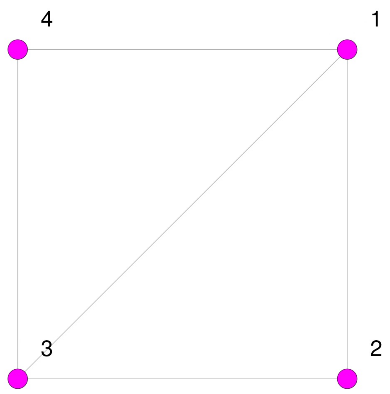
If A, D are square matrices, then

$$\det\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right) = \det(A)\det(D)$$

$\Rightarrow$ B is irrelevant

22 Matrix powers and adjacency matrix

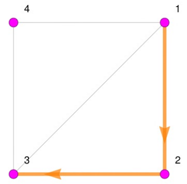
$A^0 = \mathbb{1}$

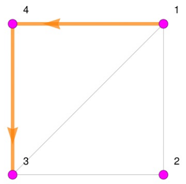


Adjacency matrix of this undirected graph:

$$A = \begin{array}{c} \begin{pmatrix} 0 & \boxed{1} & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 & v_3 \end{matrix} \end{array} \begin{array}{l} \text{vertex 1} \\ \text{vertex 2} \\ \text{vertex 3} \\ \text{vertex 4} \end{array}$$

Tells that vertex 1 is connected with vertex 2





$$A^2 = \begin{array}{cc} & \begin{array}{cccc} 1 & & & \end{array} \\ \begin{array}{c} \\ \\ 3 \\ \\ \end{array} & \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \end{array}$$

3 possibilities to go from 1 to 1 with 2 steps

2 possibilities to go from 1 to 3 with 2 steps

Entry 31 in A^2 counts the number of walks of length 2 from vertex 1 to vertex 3.

Terminology:

A walk of length n is any chain $v_0 \to v_1 \to \ldots \to v_n$ of vertices such that there is an edge from v_0 is v_1, from v_1 to v_2, and so on.

A path is a walk where all $n+1$ vertices are distinct.

Example:

$1 \to 2 \to 3 \to 2$ is a walk of length 3, not a path

$1 \to 4 \to 3 \to 2$ is a walk and a path of length 3

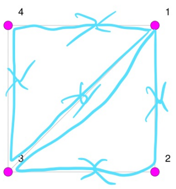
The trace of a square matrix is the sum of all diagonal entries:

$$\operatorname{tr}(A^2) = \operatorname{tr} \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} = 3+2+3+2 = 10$$

The trace of A^2 counts the attached edges with a walk of 2 from vertex i back to vertex i

$$\Rightarrow \operatorname{tr}(A^2) = 2 \cdot \#\text{edges}$$

$$\#\text{edges} = \frac{1}{2} \operatorname{tr}(A^2)$$

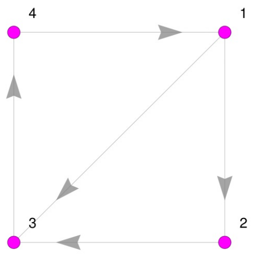


$$\operatorname{tr}(A^3) = \operatorname{tr} \begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{pmatrix} = 4+2+4+2 = 12$$

The trace of A^3 counts the same as the trace of A^2, however instead of edges it counts triangles with a walk of 3

$$\Rightarrow \operatorname{tr}(A^3) = 6 \cdot \#\text{triangles}$$

$$\#\text{triangles} = \frac{1}{6} \cdot \operatorname{tr}(A^3)$$



For a directed graph these principles stay the same, the only thing that differs is what the adjacency matrix tells.

The adjacency matrix of this directed graph is

$$A = \begin{pmatrix} \boxed{0 \; 0 \; 0 \; 1} \\ 1 \; 0 \; 0 \; 0 \\ \boxed{1 \; 1 \; 0 \; 0} \\ 0 \; 0 \; 1 \; 0 \end{pmatrix}$$

vertex 1 is only connected with vertex 4

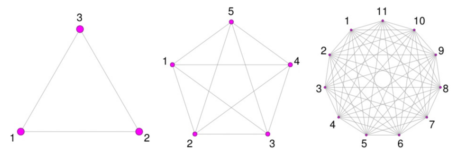
vertex 3 is only connected with vertex 1 and vertex 2 a.)

$n = 3$

```python
A = np.array([[0, 1, 1], [1, 0, 1], [1, 1, 0]])
matrix_power(A, 2)
```
✓ 0.0s

```
array([[2, 1, 1],
       [1, 2, 1],
       [1, 1, 2]])
```

Problem 2



The complete graph on n vertices is the undirected graph where all edges are present; the cases $n = 3$, $n = 5$ and $n = 11$ respectively are shown above. The adjacency matrix A_n can be described as follows: It is the $n \times n$ matrix all whose diagonal entries are equal to 0, all whose off-diagonal entries are equal to 1.

a. Give a similar description of $(A_n)^2$.
b. Give a similar description of $(A_n)^3$.
c. Compute the trace $\mathrm{tr}((A_n)^2)$. How many edges are there?
Check. For $n = 5$, your answer should give 10 edges.
d. Compute the trace $\mathrm{tr}((A_n)^3)$. How many triangles are there?
Check. For $n = 3$, your answer should give 1 triangle.

$n = 5$

```python
A = np.array(
    [
        [0, 1, 1, 1, 1],
        [1, 0, 1, 1, 1],
        [1, 1, 0, 1, 1],
        [1, 1, 1, 0, 1],
        [1, 1, 1, 1, 0],
    ]
)
matrix_power(A, 2)
```
✓ 0.0s

```
array([[4, 3, 3, 3, 3],
       [3, 4, 3, 3, 3],
       [3, 3, 4, 3, 3],
       [3, 3, 3, 4, 3],
       [3, 3, 3, 3, 4]])
```

$n = 11$

```python
A = np.array(
    [
        [0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],
        [1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1],
        [1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1],
        [1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1],
        [1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1],
        [1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1],
        [1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1],
        [1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1],
        [1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1],
        [1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1],
        [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
    ]
)
matrix_power(A, 2)
```
✓ 0.0s

```
array([[10,  9,  9,  9,  9,  9,  9,  9,  9,  9,  9],
       [ 9, 10,  9,  9,  9,  9,  9,  9,  9,  9,  9],
       [ 9,  9, 10,  9,  9,  9,  9,  9,  9,  9,  9],
       [ 9,  9,  9, 10,  9,  9,  9,  9,  9,  9,  9],
       [ 9,  9,  9,  9, 10,  9,  9,  9,  9,  9,  9],
       [ 9,  9,  9,  9,  9, 10,  9,  9,  9,  9,  9],
       [ 9,  9,  9,  9,  9,  9, 10,  9,  9,  9,  9],
       [ 9,  9,  9,  9,  9,  9,  9, 10,  9,  9,  9],
       [ 9,  9,  9,  9,  9,  9,  9,  9, 10,  9,  9],
       [ 9,  9,  9,  9,  9,  9,  9,  9,  9, 10,  9],
       [ 9,  9,  9,  9,  9,  9,  9,  9,  9,  9, 10]])
```

diagonal: $n-1$

off diagonal: $n-2$

b.)

$n = 3$

```python
A = np.array([[0, 1, 1], [1, 0, 1], [1, 1, 0]])
matrix_power(A, 3)
```
✓ 0.0s

```
array([[2, 3, 3],
       [3, 2, 3],
       [3, 3, 2]])
```

$n = 5$

```python
A = np.array(
    [
        [0, 1, 1, 1, 1],
        [1, 0, 1, 1, 1],
        [1, 1, 0, 1, 1],
        [1, 1, 1, 0, 1],
        [1, 1, 1, 1, 0],
    ]
)
matrix_power(A, 3)
```
✓ 0.0s

```
array([[12, 13, 13, 13, 13],
       [13, 12, 13, 13, 13],
       [13, 13, 12, 13, 13],
       [13, 13, 13, 12, 13],
       [13, 13, 13, 13, 12]])
```

$n = 11$

```python
A = np.array(
    [
        [0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],
        [1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1],
        [1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1],
        [1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1],
        [1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1],
        [1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1],
        [1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1],
        [1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1],
        [1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1],
        [1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1],
        [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
    ]
)
matrix_power(A, 3)
```
✓ 0.0s

```
array([[90, 91, 91, 91, 91, 91, 91, 91, 91, 91, 91],
       [91, 90, 91, 91, 91, 91, 91, 91, 91, 91, 91],
       [91, 91, 90, 91, 91, 91, 91, 91, 91, 91, 91],
       [91, 91, 91, 90, 91, 91, 91, 91, 91, 91, 91],
       [91, 91, 91, 91, 90, 91, 91, 91, 91, 91, 91],
       [91, 91, 91, 91, 91, 90, 91, 91, 91, 91, 91],
       [91, 91, 91, 91, 91, 91, 90, 91, 91, 91, 91],
       [91, 91, 91, 91, 91, 91, 91, 90, 91, 91, 91],
       [91, 91, 91, 91, 91, 91, 91, 91, 90, 91, 91],
       [91, 91, 91, 91, 91, 91, 91, 91, 91, 90, 91],
       [91, 91, 91, 91, 91, 91, 91, 91, 91, 91, 90]])
```

- diagonal: $n=5$

$A_{11} = 0 \cdot 4 + 1 \cdot 3 + 1 \cdot 3 + 1 \cdot 3 + 1 \cdot 3$

$= 0 + 4 \cdot 3$

$= (n-1)(n-2) = 12$

Check $n=11 \Rightarrow (11-1)(11-2) = 10 \cdot 9 = 90$

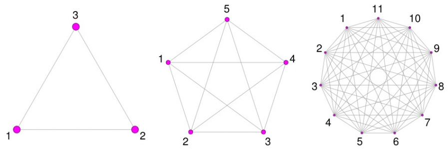
- off diagonal: $n=5$

$A_{21} = 1 \cdot 4 + 0 \cdot 3 + 1 \cdot 3 + 1 \cdot 3 + 1 \cdot 3$

$= 4 + 0 + 3 \cdot 3 = 4 + 3^2$

$= (n-1) + (n-2)^2 = 13$

Problem 2



The complete graph on n vertices is the undirected graph where all edges are present; the cases $n = 3$, $n = 5$ and $n = 11$ respectively are shown above. The adjacency matrix A_n can be described as follows: It is the $n \times n$ matrix all whose diagonal entries are equal to 0, all whose off-diagonal entries are equal to 1.

a. Give a similar description of $(A_n)^2$.

b. Give a similar description of $(A_n)^3$.

c. Compute the trace $\mathrm{tr}((A_n)^2)$. How many edges are there?
Check. For $n = 5$, your answer should give 10 edges.

d. Compute the trace $\mathrm{tr}((A_n)^3)$. How many triangles are there?
Check. For $n = 3$, your answer should give 1 triangle.

c) $\mathrm{tr}((A_n)^2) = n \cdot (n-1)$

$= \underline{n^2 - n}$

$\#\text{edges} = \frac{1}{2}\,\mathrm{tr}((A_n)^2) = \underline{\frac{n^2-n}{2}}$

$n = 5 \Rightarrow \mathrm{tr}(A^2) = 5^2 - 5$

$= 25 - 5 = 20$

$\#\text{edges} = \frac{20}{2} = 10$

d.) $\mathrm{tr}((A_n)^2) = \underline{n \cdot [(n-1)(n-2)]}$

$\#\text{triangles} = \frac{1}{6}\,\mathrm{tr}((A_n)^3) = \underline{\frac{n \cdot [(n-1)(n-2)]}{6}}$

$n = 3 \Rightarrow \mathrm{tr}(A^3) = 3 \cdot ((3-1)(3-2))$

$= 3 \cdot 2 \cdot 1 = 6$

$\#\text{triangles} = \frac{6}{6} = 1$

23 Block multiplication and inversion

Given 4 matrices A, B, C, D can build new matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

provided sizes fit, as in

Operations can be done blockwise:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} Av + Bw \\ Cv + Dw \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{1} & X \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & Y \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & X+Y \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{1} & Z \\ 0 & \mathbb{1} \end{pmatrix}^n = \begin{pmatrix} \mathbb{1} & nZ \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BK^{-1}CA^{-1} & -A^{-1}BK^{-1} \\ -K^{-1}CA^{-1} & K^{-1} \end{pmatrix}$$

$$K = D - CA^{-1}B$$

To invert block matrix:

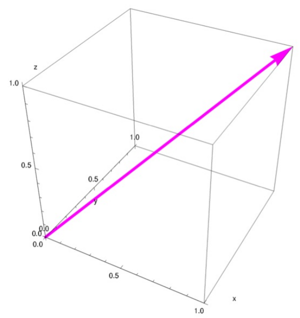
- A and D are square
- A and K are invertible

30 Dot product

$v = \begin{pmatrix} 2 \\ 5 \\ 4 \\ -2 \end{pmatrix}$ $\Rightarrow$ norm is square root of sum of squares of all entries (Also known as 2-norm or Euclidean norm)

$\Rightarrow \sqrt{2^2 + 5^2 + 4^2 + (-2)^2} = \sqrt{49} = 7$

norm represents distance in physical space



$$\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

There are other norms:

- 1-norm: sum of absolute values:

  $\|v\|_{1\text{-norm}} = |2| + |5| + |4| + |-2| = 13$

  $\hookrightarrow$ also known as Manhattan distance

- p-norm: $\left\| \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \right\|_{p\text{-norm}} = \sqrt[p]{|v_1|^p + \ldots + |v_d|^p}$

- 2-norm is the default

  $\Rightarrow$ When speaking about norm, 2-norm is meant

Dot product

$$\begin{pmatrix}2\\5\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}9\\1\\-2\\3\end{pmatrix} = 2\cdot 9+5\cdot 1+4\cdot(-2)+(-2)\cdot 3 = \underline{9}$$

1x4 4x1 1x1

$$\begin{pmatrix}2 & 5 & 4 & -2\end{pmatrix}\begin{pmatrix}9\\1\\-2\\3\end{pmatrix} = (9)$$

Unlike length, the dot product is not part of everyday language, but it is mathematically natural and powerful. For a start, it can be used to give an equivalent definition of the 2-norm:

$$\|v\| = \sqrt{v\cdot v}$$

$\wr\wr$ sum of squares of entries

The Cauchy-Schwarz inequality

Fact. *For all vectors v, w one has*

$$\underbrace{-\overset{-4\quad 5}{\|v\|\cdot\|w\|}}_{-20} \le v\cdot w \le \underbrace{\overset{4\quad 5}{\|v\|\cdot\|w\|}}_{20}$$

Furthermore, assuming $v \neq 0$ and $w \neq 0$:

- *$v\cdot w = \|v\|\cdot\|w\|$ if and only if v and w are fully aligned, meaning $v = kw$ for some number $k > 0$.* → pointing in same direction
- *$v\cdot w = -\|v\|\cdot\|w\|$ if and only if v and w are fully anti-aligned, meaning $v = -kw$ for some number $k > 0$.* → pointing in diffeent direction

Quantifying alignment

Given the Cauchy-Schwarz inequality, one regards

$$-1 \le \frac{v\cdot w}{\|v\|\|w\|} \le 1$$

as a measure of how strongly $v \neq 0$ and $w \neq 0$ are aligned.

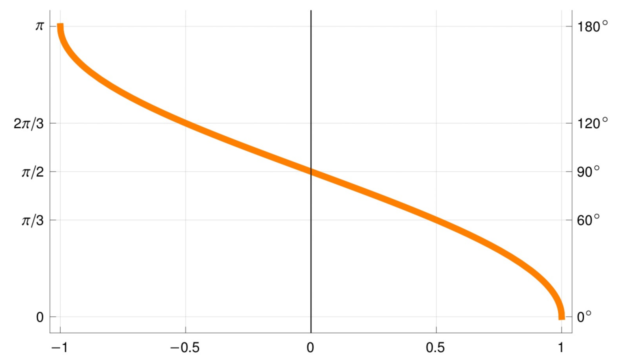
If $v \neq 0$ and $w \neq 0$ angle can be defined

$$\boxed{\angle(v,w) = \arccos\frac{v\cdot w}{\|v\|\|w\|}}$$

1	fully aligned	0 deg
between 0 and 1	partially aligned	
0	orthogonal	90 deg
between -1 and 0	partially anti-aligned	
-1	fully anti-aligned	180 deg

$$v\cdot w = \|v\|\|w\|\cos\angle(v,w)$$

- Generalizes concept of angle to all dimensions



$\cos 0° = 1$
$\arccos 1 = 0°$

$\cos 90° = 0$
$\arccos 0 = 90°$ ⇒ $\boxed{\text{orthogonal: } v\cdot w = 0,\ \angle(v,w) = \frac{\pi}{2} = 90°}$

$\cos 180° = -1$
$\arccos(-1) = 180°$