LINEARIZATION OF DISCRETE TIME NONLINEAR SYSTEMS*

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ABSTRACT

We characterize the equivalence of single-input, single-output, discrete-time nonlinear systems to linear ones, via a state coordinate change and with or without feedback. Four cases are distinguished by allowing or disallowing feedback as well as by including the output map or not; the interdependence of these problems is analyzed. An important feature which distinguishes these discrete-time problems from the corresponding problem in continuous-time is that the state coordinate transformation is here directly computable as a higher composition of the system and output maps.

1. INTRODUCTION

The problem of linearization of control systems by state feedback and coordinate change has attracted a great deal of attention in the literature and consists of determining a state feedback law as well as new coordinates in the state space such that the resulting closed loop system becomes a linear one.

Many authors have studied linearization [9], [11],[21] and approximate linearization [13] by state feedback and coordinate change for nonlinear continuous time systems. More recently J. W. Grizzle [5] and H. G. Lee and S. I. Marcus [15] have addressed and solved the above questions in the discrete time case. A more restrictive problem, linearization by state coordinate change only, has been studied by A. J. Krener [12] and H. J. Sussmann [23] and W. Dayawansa et.al. [2] for continuous time systems without outputs and by L. Hunt et.al. [8] for continuous time systems with outputs.

Another interesting problem consists of the choice of a state feedback law with the objective of making the input-dependent part of the output of the closed-loop system linear in the new input. This problem has been solved by A. Isidori and A. Ruberti [10] for continuous time affine systems and by H. G. Lee and S. I. Marcus [16] for discrete time systems by using a structure algorithm which is a generalized version of Silverman's structure algorithm [20]. In this paper, we discuss the following four linearization problems for single-input discrete time systems: (i) linearization of a system without output by state coordinate change; (ii) linearization of a system without output by state coordinate change and feedback; (iii) linearization of an input-output system by state coordinate change; and (iv) linearization of an input-output system by state coordinate change and feedback.

Problem (ii) was solved by H. G. Lee and S. I. Marcus [15]. We show that problem (i) has an important role in solving problem (iv), by showing that problem (iv) is solvable if and only if problem (i) is solvable for the closed-loop system with the fixed feedback which solves the input-output linearization problem.

Throughout this paper Σ_1 will denote a single-input discrete time nonlinear system, whose state dynamics are of interest only, i.e.,

$$\Sigma_1: \quad x(t+1) = f(x(t),u(t)) \tag{1}$$

while Σ_2 will refer to a system with outputs

$$\Sigma_2: \quad \begin{aligned} x(t+1) &= f(x(t),u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ and f and h are smooth functions with $f(0,0) = 0$ and $h(0) = 0$.

Variants of the definitions that follow are standard in the literature.

Definition 1: Two discrete time systems are said to be state equivalent if there exists a smooth state coordinate change around $0 \in \mathbb{R}^N$ which transforms the one into the other.

Definition 2: A discrete time nonlinear system of the form Σ_1 or Σ_2 is said to be linearizable by state coordinate change if it is state equivalent to a reachable or minimal linear system respectively.

An important attribute of discrete time systems which are linearizable by state coordinate change is that the inverse of the linearizing transformation is directly computable from the state dynamics. Hence, in this respect, they differ considerably from continuous time systems for which the linearizing transformation is

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obtained as a solution of a set of partial differential equations.

Definition 3: A discrete time system is said to be linearizable by state coordinate change and feedback if there exists a smooth nonlinear feedback $u = \gamma(x,v)$ such that the closed-loop system is linearizable by state coordinate change.

In Section 2 we define our notation and review some background material. In Section 3 we examine the linearization problems defined above. Various examples are provided in Section 4. Proofs can be found in [22]. Also, we show in [22] that our approach can be applied to solve the linearization problem for a single-input, single-output continuous time affine nonlinear system by state coordinate change and feedback.

2. PRELIMINARIES AND DEFINITIONS

Let M be an N-dimensional differentiable manifold and π: $B \to M$ a vector bundle over M. A discrete time control system (on B) is a 4-tuple (B,M,f,h), where f: $B \to M$ and h: $M \to \mathbb{R}$ are smooth maps. With $B_x = \pi^{-1}(x)$ denoting the fiber over $x \in M$ we define, for each nonnegative integer k, the k^{th} product-bundle B^k by $B^k = \bigcup_{y \in M} (\underbrace{B_y \times \ldots \times B_y}_{k \text{ times}})$.

Then for each $k \geq 0$, B^k, endowed with the product topology on the fibers, is a smooth vector bundle over the base space M. Note that B^k may be viewed as a vector bundle over B^{k-1} and let π: $B^k \to B^{k-1}$ denote the obvious projection. Also, B^0 denotes the zero-section $M \times \{0\}$, which is isomorphic to, and therefore identified with, M.

The problem of linearization being local in nature, we are going to restrict our attention to the case where $B = V \times \mathbb{R}$ is a local vector bundle and $V \subset M$ is an open neighborhood of M. The system dynamics relative to an input sequence are described in a natural manner if one defines a map F: $B^k \to B^{k-1}$ by $F(x,u_1,\ldots,u_k) = (f(x,u_k),u_1,\ldots,u_{k-1})$ for $k > 0$, while if $x \in M \simeq B^0$ we define $F(x) = f(x,0)$.

Note that F restricted on $B \simeq B^1$ is identical with the system map f. The k^{th} composition F^k of the map F is well-defined by the above, and the restriction of F^k on B is interpreted as an "impulse response" of the system and will be denoted by $\hat{f}^k$, which is also defined in Section 3. Now consider B^N and the map F^N: $B^N \to B^0 \simeq M$. Let $\Psi_x(F_x)$ be the restriction of $F^N(F^{N+1})$ on the fiber of $B^N(B^{N+1})$ above $x \in M$, i.e., $\Psi_x(u_1,u_2,\ldots,u_N) = f(f(\ldots(f(x,u_N),\ldots),u_2),u_1)$ and $F_x(u_1,u_2,\ldots,u_{N+1}) = f(f(\ldots(f(x,u_{N+1}),\ldots),u_2),u_1)$. Clearly then Ψ_x: $\mathbb{R}^N \to M$, while F_x: $\mathbb{R}^{N+1} \to M$.

Definition 4: The characteristic number of the system (2) is defined as the smallest integer ρ such that $\frac{\partial}{\partial u}(h \circ \hat{f}^\rho) \not\equiv 0$. Thus $t=\rho$ is the first instant of time at which the output is affected by the input applied at time t=0.

Definition 5: Let h: $M \to Q$ be smooth and h_* be surjective. Let X be a smooth vector field on M. Then $h_*(X)$ is said to be a well-defined smooth vector field on Q if $h_*(X_p) = h_*(X_q)$ whenever $h(p) = h(q)$.

The above definition extends to distributions as well, and the following lemma in [4] gives a useful characterization.

Lemma 1: Let h: $M \to Q$ be a smooth function, from the smooth manifold M to the smooth manifold Q, such that h_*: $TM \to TQ$ is surjective. Let $K := \ker h_*$, and let D be an involutive distribution on M such that $D \cap K$ has constant dimension. Then $h_* D$ is a well-defined involutive distribution on Q if and only if $D + K$ is involutive.

The following lemma gives a necessary and sufficient condition for $h_*(X)$ to be a well-defined vector field when h_* is surjective.

Lemma 2: Let h: $M \to Q$ be a smooth function, such that h_*: $T_pM \to T_{h(p)}Q$ is surjective and X a smooth vector field on M. Then the vector field Y defined pointwise by $Y_{h(p)} = h_*(X_p)$ is a well-defined vector field on some open neighborhood of h(p) if and only if $[X, \ker h_*] \subset \ker h_*$ on some open neighborhood of p.

The proof of the following lemma is straightforward.

Lemma 3: Consider the following single-input single-output reachable linear system

$$z(t+1) = Az(t) + bu(t)$$

$$y(t) = cz(t), \quad z \in \mathbb{R}^N$$

with $c \neq 0$. Then there exists $1 \times N$ matrix F such that (c,A+bF) is observable.

3. MAIN RESULTS

We begin this section by characterizing the systems of the form (1) which are linearizable by state coordinate change.

Theorem 4: The discrete time system in (1) is (locally) linearizable by state coordinate change if and only if (i) $(\Psi_0)_*|_{u=0}$ is an isomorphism; (ii) $[\frac{\partial}{\partial u_i}, \ker (F_0)_*] \subset \ker (F_0)_*$ for $1 \leq i \leq N+1$. Furthermore, $T = (\Psi_0)^{-1}$ is a linearizing coordinate change.

Consider now the map F: $B^k \to B^{k-1}$ which in local fiber respecting coordinates may be expressed as $f(x,u_1,\ldots,u_k) = (f(x,u_k),u_1,\ldots,u_{k-1})$ and let $f^k = f(f(\ldots f(x,_1,u_1),\ldots),u_{k-1}),u_k)$. Note that condition (i) of Theorem 4 is equivalent, in local coordinates, to the statement that $\{(\frac{\partial f}{\partial u})(0,0), (\frac{\partial f}{\partial x})(0,0)(\frac{\partial f}{\partial u})(0,0),\ldots,(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})(0,0)\}$ are linearly independent. The latter implies that $[(\frac{\partial f}{\partial x})(0,0) \vdots (\frac{\partial f}{\partial u})(0,0)]$ has full rank and hence that the map f^k is onto an open neighborhood of $0 \in M$. It can be easily seen that if $(F_0)_*(\frac{\partial}{\partial u_i})$ are well-defined, then

$(F_0)_*(\frac{\partial}{\partial u_i}) = f^i_*(\frac{\partial}{\partial u_i})$. If $(\frac{\partial f}{\partial x})_{(0,0)}$ is nonsingular, then the impulse response $\hat{f}^i: B \to M$ maps onto an open neighborhood of $0 \in M$. It is the case though that $\hat{f}^i_*(\frac{\partial}{\partial u})$ might be well-defined while $(F_0)_*(\frac{\partial}{\partial u_i})$ is not. The following theorem resolves this discrepancy.

Theorem 5: Suppose that $(\frac{\partial f}{\partial x})_{(0,0)}$ is nonsingular Then $[\frac{\partial}{\partial u_i}, \ker (F_0)_*] \subset \ker (F_0)_*$, $i=1,\dots,N+1$ if and only if (a) $\hat{f}^i_*(\frac{\partial}{\partial u})$ is well-defined for $i=1,\dots,N+1$; (b) $[\hat{f}^i_*(\frac{\partial}{\partial u}), f^j_*(\frac{\partial}{\partial u})] = 0$ for $1 \le i,j \le N+1$.

Necessary and sufficient conditions for local linearization by state coordinate change and feedback are obtained in [15]. An equivalent characterization analogous to Theorem 4 follows.

Theorem 6: Σ_1 is locally linearizable by state coordinate change and feedback if and only if
(i) $(\Psi_0)_*|_{u=0}$ is an isomorphism;
(ii) $[\frac{\partial}{\partial u_i}, \ker (F_0)_*] \subset \ker (F_0)_* + \mathrm{span}\,\{\frac{\partial}{\partial u_i}\}$ for $1 \le i \le N-1$.

Remark: Obviously, the conditions of Theorem 4 imply those of Theorem 6.

We now turn to the problem of linearization of a system with outputs. Consider the system Σ_2 in (2) and suppose that it is linearizable by a coordinate change. Then, of course, the state dynamics must satify the hypotheses of Theorem 4. On the other hand, since Ψ_0 is the inverse of the linearizing transformation of the state dynamics, it is evident that $h \circ \Psi_0$ must be a linear map. These two requirements are, obviously, sufficient for linearization as well.

Let us note that the linearizability of Σ_2 is preserved if one replaces $h(x)$ by $h \circ f(x,0)$, for if $z = T(x)$ is a linearizing transformation then $h \circ f(T^{-1}(z),0) = (h \circ T^{-1}) \circ (T \circ f(T^{-1}(z),0))$ which is merely a composition of two linear maps. Let us define $\hat{h}^i(x,u) \equiv h \circ \hat{f}^i(x,u)$, $i=0,1,\dots$, and $\hat{H}(x,u) = (h,\hat{h}^1,\dots,\hat{h}^{N-1})^T$. Clearly $\hat{H}: M \times \mathbb{R} \to \mathbb{R}^N$. Let us denote by H_0 the restriction of $\hat{H}$ on M, i.e., $H_0(x) = \hat{H}(x,0)$. It follows, by induction on the argument above, that for Σ_2 to be linearizable the map $H_0 \circ \Psi_0$ mapping $\mathbb{R}^N$ into $\mathbb{R}^N$ must be linear.

We state the following theorem.

Theorem 7: Σ_2 is locally linearizable by state coordinate change if and only if (i) $\{(\frac{\partial f}{\partial u})_{(0,0)}, (\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \dots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\}$ are linearly independent, (ii) $\{(\frac{\partial h}{\partial x})_{x=0}, (\frac{\partial h}{\partial x})_{x=0}(\frac{\partial f}{\partial x})_{(0,0)}, \dots, (\frac{\partial h}{\partial x})_{x=0}(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}\}$ are linearly independent, (iii) $\frac{\partial}{\partial u}(h \circ \hat{f}^i) =$ constant for $1 \le i \le N$, and (iv) $h \circ \hat{f}^N(x,0) \in \mathrm{span}_{\mathbb{R}}\,\{h(x), h \circ f(x,0), \dots, h \circ \hat{f}^{N-1}(x,0)\}$.

The observability assumption (ii) implies that H_0 must be a local diffeomorphism and hence that $H_0 \circ \Psi_0$ is a linear isomorphism if the system is linearizable. On the other hand, the map $H_0 \circ F_0$: $\mathbb{R}^{N+1} \to \mathbb{R}^N$ must also be linear, or in other words, the vector fields $(H_0 \circ F_0)_* \frac{\partial}{\partial u_i}$ must be constant vector fields in $\mathbb{R}^N$, for all $i=1,\dots,N+1$. It is clear that this latter condition is sufficient for the vector fields $(F_0)_* \frac{\partial}{\partial u_i}$, $i=1,\dots,N+1$ to be well defined. Theorem 7 can then be rephrased in a rather compact form which we state as a proposition.

Proposition 1: The system Σ_2 is linearizable by a state coordinate change if and only if the map $H_0 \circ F_0$ is linear and onto $\mathbb{R}^N$.

We now turn to the problem of including feedback.

Theorem 8: Σ_2 is locally linearizable by state coordinate change and feedback if and only if (i) $\{(\frac{\partial f}{\partial u})_{(0,0)}, (\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \dots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\}$ are linearly independent, (ii) $(\frac{\partial}{\partial u}(h \circ \hat{f}^\rho))_{(0,0)} \neq 0$, and (iii) $\tilde{f}^i_*(\frac{\partial}{\partial v_1})$ is a well-defined smooth vector field on an open neighborhood of $0 \in \mathbb{R}^N$ for $1 \le i \le N+1$, where $\tilde{f}(x,v) = f(x,g(x,v))$ and $h \circ \hat{f}^\rho(x,g(x,v)) = v$.

Remark: $u = g(x,v)$ in Theorem 8 is the feedback which solves the input-output linearization problem [16]. Thus Theorem 8 shows that Σ_2 is locally linearizable by state coordinate change and feedback if and only if the closed-loop state equation is linearizable by state coordinate change (i.e., the conditions of Theorem 4 are satisfied). Another interesting fact is that the closed-loop input-output system does not have to satisfy condition (ii) of Theorem 7. This is due to the fact that we can apply a feedback, which is linear in the new state, in order to make the closed-loop system satisfy condition (ii) of Theorem 7 without destroying linearity of the output map. This idea can be applied to single-input single-output continuous time affine systems as discussed in [22].

Corollary 9: Suppose that the following conditions hold: (i) $\rho = N$, (ii) $\{(\frac{\partial f}{\partial u})_{(0,0)}, (\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \dots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\}$ are linearly independent, (iii) $(\frac{\partial}{\partial u}(h \circ \hat{f}^\rho))_{(0,0)} \neq 0$. Then, Σ_2 is linearizable by state coordinate change and feedback. Furthermore, there exist a state coordinate change $z(x)$ and a feedback $v(x,u)$ such that

$$z(t+1) = Az(t) + bv(t)$$
$$y(t) = cz(t)$$
where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad (N\times N \text{ matrix}),$$

$b = (\ 0\ \dots\ 0\ 1\ )^T$, $(N\times 1$ matrix)
and $c = (\ 1\ 0\ \dots\ 0\ )$ $(1\times N$ matrix).

4. EXAMPLES

In this section we present various examples of the theory.

Example 1: Consider

$$\Sigma: \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) - u(t)^2 \\ u(t) \end{bmatrix} = f(x(t),u(t))$$

Then

$$\Psi_x(u_1,u_2) = f(f(x,u_2),u_1) = \begin{bmatrix} u_2 - u_1^2 \\ u_1 \end{bmatrix}$$

$$\Psi_0(u_1,u_2) = \begin{bmatrix} u_2 - u_1^2 \\ u_1 \end{bmatrix}$$

$$F_x(u_1,u_2,u_3) = f(f(f(x,u_3),u_2),u_1) = \begin{bmatrix} u_2 - u_1^2 \\ u_1 \end{bmatrix}$$

$$F_0(u_1,u_2,u_3) = \begin{bmatrix} u_2 - u_1^2 \\ u_1 \end{bmatrix}$$

Thus $\ker (F_0)_* = \text{span} \{\frac{\partial}{\partial u_3}\}$.

Since $((\Psi_0)_*)_{u=0}$ is nonsingular and $[\frac{\partial}{\partial u_i}, \ker (F_0)_*] \subset \ker (F_0)_*$ for $1 \le i \le 3$, conditions (i) and (ii) of Theorem 4 are satisfied. Hence Σ is locally linearizable by state coordinate change. Furthermore, a desired state coordinate transformation can be obtained by solving the following equations:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Psi_0(\eta_1,\eta_2) = \begin{bmatrix} \eta_2 - \eta_1^2 \\ \eta_1 \end{bmatrix}$$

Solving the above, we obtain $\eta_1 = x_2$ and $\eta_2 = x_1 + x_2^2$ and hence

$$z = T(x) = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2^2 \end{bmatrix}$$

transforming the system into

$$z(t+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

Example 2: Consider

$$\Sigma: \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ (1 + x_1(t))u(t) \end{bmatrix} = f(x(t),u(t))$$

Then

$$\Psi_x(u_1,u_2) = f(f(x,u_2),u_1) = \begin{bmatrix} (1+x_1)u_2 \\ (1+x_2)u_1 \end{bmatrix}$$

$$\Psi_0(u_1,u_2) = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$$

$$F_x(u_1,u_2,u_3) = f(f(f(x,u_3),u_2),u_1) = \begin{bmatrix} (1+x_2)u_2 \\ (1+(1+x_1)u_3)u_1 \end{bmatrix}$$

$$F_0(u_1,u_2,u_3) = \begin{bmatrix} u_2 \\ (1+u_3)u_1 \end{bmatrix}$$

Thus $\ker (F_0)_* = \text{span} \{u_1 \frac{\partial}{\partial u_1} - (1+u_3) \frac{\partial}{\partial u_3}\}$.

Since $((\Psi_0)_*)_{u=0}$ is nonsingular, condition (i) of Theorem 4 is satisfied. Note that $[\frac{\partial}{\partial u_i}, u_1\frac{\partial}{\partial u_1} - (1+u_3)\frac{\partial}{\partial u_3}] = \frac{\partial}{\partial u_1} \notin \ker (F_0)_*$. Therefore, Σ is not locally linearizable by state coordinate change. However, since $[\frac{\partial}{\partial u_1}, \ker (F_0)_*] \subset \ker (F_0)_* + \text{span} \{\frac{\partial}{\partial u_1}\}$, Σ is locally linearizable by state coordinate change and feedback by Theorem 6.

Example 3: Consider

$$\Sigma: \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) + (x_3(t) - x_1(t)^2 + u(t))^2 \\ x_3(t) - x_1(t)^2 + u(t) \\ x_1(t) + 2x_2(t)(x_3(t) - x_1(t)^2 \\ \quad + u(t))^2 + (x_3(t) - x_1(t)^2 + u(t))^4 \end{bmatrix}$$

$$y(t) = h(x(t)) = x_1(t) - x_2(t)^2$$

Since $(\frac{\partial f}{\partial u})(0,0) = (\ 0\ 1\ 0\ )^T$, $(\frac{\partial h}{\partial x})_{x=0} = (\ 1\ 0\ 0\ )$, and

$$(\frac{\partial f}{\partial x})(0,0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

it is easy to see that conditions (i) and (ii) of Theorem 7 are satisfied. Now, note that

$$h \circ f(x,u) = x_2$$
$$h \circ \hat{f}^2(x,u) = x_3 - x_1^2 + u$$
$$h \circ \hat{f}^3(x,u) = x_1 - x_2^2$$

Therefore, condition (iii) is satisfied. Clearly,

$h\circ\hat{f}^3(x,0) = x_1-x_2^2 \in \underset{\mathbb{R}}{\text{span}}\ \{h(x), h\circ f(x,0), h\circ\hat{f}^2(x,0)\}$

$= \underset{\mathbb{R}}{\text{span}}\ \{x_1-x_2^2, x_2, x_3-x_1^2\}$. It follows from the proof of Theorem 7 that a desired state coordinate change may be obtained as follows:

$$z_1 = T_1(x) = h(x) = x_1-x_2^2$$

$$z_2 = T_2(x) = h\circ f(x,0) = x_2$$

$$z_3 = T_3(x) = h\circ\hat{f}^2(x,0) = x_3-x_1^2$$

Applying the above transformation, it is easy to show that the new states satisfy

$$z(t+1) = Az(t) + bu(t)$$

$$y(t) = cz(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad b = (\ 0\ 1\ 0\ )^T, \text{ and } c = (\ 1\ 0\ 0\ )$$

Since Σ is linearizable by state coordinate change, Σ is also linearizable by state coordinate change and feedback.

Example 4: Consider

$$\Sigma: \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t)+(u(t)+u(t)^2)^2 \\ u(t)+u(t)^2 \\ x_1(t)+2x_2(t)(u(t)+u(t)^2)^2+(u(t)+u(t)^2)^4 \end{bmatrix} = f(x(t),u(t))$$

$$y(t) = x_1(t)-x_2(t)^2 = h(x(t))$$

Note that

$(\frac{\partial f}{\partial u})(0,0) = (\ 0\ 1\ 0\ )^T$, $(\frac{\partial h}{\partial x})_{x=0} = (\ 1\ 0\ 0\ )$, and $(\frac{\partial f}{\partial x})(0,0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Therefore, condition (i) of Theorem 7 is satisfied. However, since $h\circ\hat{f}^2(x,u) = u+u^2$, condition (iii) of Theorem 7 is not satisfied. Thus Σ is not locally linearizable by state coordinate change. On the other hand, the conditions of Theorem 8 are satisfied with ρ=2. We apply the following feedback law which solves the input-output linearization problem:

$$h\circ\hat{f}^2(x,u) = v.$$

Then

$$\tilde{f}(x,v) = \begin{bmatrix} x_2+v^2 \\ v \\ x_1+2x_2v^2+v^4 \end{bmatrix}$$

Note that

$$F_0(v_1,v_2,v_3,v_4) = \tilde{f}(\tilde{f}(\tilde{f}(\tilde{f}(0,v_4),v_3),v_2),v_1) = \begin{bmatrix} v_2+v_1^2 \\ v_1 \\ v_3+v_2^2+2v_2v_1^2+v_1^4 \end{bmatrix}$$

and

$$\ker\,(F_0)_* = \text{span}\ \{\frac{\partial}{\partial v_4}\}.$$

Since $[\frac{\partial}{\partial v_i}, \ker\,(F_0)_*] \subset \ker\,(F_0)_*$ for $1 \le i \le 4$, $f_*^i(\frac{\partial}{\partial v_1})$ is a well-defined vector field for $1 \le i \le 4$. Hence Σ is locally linearizable by state coordinate change and feedback.

5. CONCLUSIONS

We dealt, in a geometric framework, with problems of linearization of discrete-time nonlinear systems. As is well known, all (geometric) characterizations of linearizability of continuous-time systems involve hypotheses on the involutivity of vector fields. The assumptions of the theorems in this work do not ask for the verification of, but rather automatically imply, involutivity; thus, they are simpler, from a calculational point of view, than the corresponding conditions in [4] which require the calculation of differential maps of vector fields in order to verify involutivity.

Another interesting feature in discrete-time linearization problems, in comparison with the continuous time case, is that if a discrete-time system is linearizable by state coordinate change only, then a linearizing transformation is obtained as a higher composition of the system and output maps. It is not yet known whether the required feedback, if needed, can assume a similar simple form.

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