# Linear Kalman Filtering Algorithm With Noisy Control Input Variable

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***Abstract*—This brief focuses on the development of a linear Kalman filtering algorithm when the control input variable is corrupted by noises. The noisy input is considered in the derivation process of the Kalman filter, and an extra term is included in the covariance matrix of the one step error. A bias estimation is naturally generated by the input noise. To reduce the bias, a new cost function of the state estimation error with a regularization term is proposed to obtain the Kalman gain matrix. Simulation results in the context of discrete time state estimation demonstrate that the proposed algorithm can achieve excellent estimation performance in terms of the steady-state misalignment under noisy input environments.**

***Index Terms*—Linear Kalman filter, noisy control variable, discrete time state estimation.**

## I. Introduction

The traditional linear kalman filtering (LKF) algorithm as a powerful tool has been widely used in many engineering areas owing to its computational simplicity and elegance [1]–[3], especially for discrete-time system state estimation (DTSSE) [4], [5]. Most nonlinear KF algorithms, such as extend KF (EKF), unscented KF (UKF) and so on [6]–[8], are also developed based on the basic idea of LKF. Therefore, it is still an important topic to study the performance of the LKF under different conditions.

The state noise and observation noise are usually considered in traditional KF. The LKF can provide reliable estimates when the control input vector as system input is well measured. In many practical situations, however, not only the system uncertainty is from the process and measurement noises, but also the control input vector may corrupted by additive noise, leading to system instability. In such cases, it will render the estimates unreliable when the traditional LKF algorithm is applied to the DTSSE problem with noisy input. Recently, the influence of the input on the performance of LKF has been investigated in the literature. Yong *et al.* [4] proposed a unified filter for simultaneous input and state estimation of linear discrete-time stochastic systems. The problem of state estimation for a linear system with unknown input was discussed in [9]. Hsieh [10] proposed a system reformation approach for unbiased minimum-variance input and state estimation for systems with unknown inputs. The problem of simultaneous input and state estimation has been concerned in [11]–[13] for a class of linear discrete-time systems with missing measurements and correlated noises. However, these studies do not consider the case of the input corrupted by noises, which will affect both the system and the output. It is well established that noisy inputs in state estimation problem may cause bias in the estimates obtained by common KFs. In recent years, several effective adaptive filtering algorithms, such as total least-squares [14], [15], bias-compensated least mean square (LMS)-type approaches [16]–[20], and bias-compensated recursive least squares algorithms [21], [22] have been developed to solve the bias caused by the input noise. To the best of our knowledge, however, the LKF with noisy input has not been investigated in the existing literatures. Thus, in this brief our goal is to solve the state estimation problem with control input corrupted by noises, and a novel LKF algorithm with noisy input (denoted as LKFwNI) is proposed.

We first demonstrate that the parameter estimate produced by standard LKF algorithm is biased when the control input variable is corrupted by noises. Second, we reformulate this estimation problem using a new cost function and develop a robust LKF that can produce unbiased estimate.

The remainder of this brief is organized as follows. In Section II, we present the traditional LKF algorithm. In Section III, the LKF algorithm with noisy control input variable is derived. In Section IV, simulations are carried out to verify the performance of the new algorithm. Section V concludes this brief.

## II. Review of the Linear Kalman Filter

Consider a linear discrete time system, whose discrete-time state space representation is expressed as

$$x_k = \mathbf{A}x_{k-1} + \mathbf{B}u_{k-1} + w_{k-1} \tag{1}$$

$$y_k = \mathbf{C}x_k + v_k \tag{2}$$

Manuscript received August 11, 2018; accepted October 23, 2018. Date of publication October 31, 2018; date of current version June 26, 2019. This work was supported in part by the National Natural Science Foundation of China under Grant 91648208, in part by the National Natural Science Foundation-Shenzhen Joint Research Program under Grant U1613219, and in part by the Natural Science Basic Research Plan in Shaanxi Province of China under Grant 2017JM6033. This brief was recommended by Associate Editor S. Wang. *(Corresponding author: Wentao Ma.)*

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Digital Object Identifier 10.1109/TCSII.2018.2878951

TABLE I
LINEAR KALMAN FILTERING ALGORITHM

| |
|---|
| Initialization: |
| $\hat{x}_0 = E[x_0]$ |
| $\mathbf{P}_0 = E\left[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T\right]$ |
| Prior estimation: |
| $\hat{x}_{k\|k-1} = \mathbf{A}\hat{x}_{k-1} + \mathbf{B}u_{k-1}$ |
| $\mathbf{P}_{k\|k-1} = \mathbf{A}\mathbf{P}_{k-1}\mathbf{A}^T + \mathbf{Q}_{k-1}$ |
| Posterior estimation: |
| $\mathbf{K}_k = \mathbf{P}_{k\|k-1}\mathbf{C}^T(\mathbf{C}\mathbf{P}_{k\|k-1}\mathbf{C}^T + \mathbf{R}_k)^{-1}$ |
| $\hat{x}_k = \hat{x}_{k\|k-1} + \mathbf{K}_k(y_k - \mathbf{C}\hat{x}_{k\|k-1})$ |
| $\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_{k\|k-1}$ |

where $x_k \in R^n$ and $y_k \in R^m$ are the state vector and the measurement vector at time index $k$, $u_k \in R^n$ stands for a control variable (i.e., the input of the system). $\mathbf{A} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times n}$ and $\mathbf{C} \in R^{m\times n}$ are the state matrix, control matrix, and measurement matrix, respectively. $w_{k-1} \in R^n$ and $v_k \in R^m$ are the process noise and measurement noise, which are mutually uncorrelated, zero-mean, white random signals with known covariance matrices $\mathbf{Q}_k$ and $\mathbf{R}_k$, respectively. For the state estimate $\hat{x}_{k-1}$a t time $k$-1, its error covariance matrix can be represented by:

$$\mathbf{P}_{k-1} = E\left[(x_{k-1} - \hat{x}_{k-1})(x_{k-1} - \hat{x}_{k-1})^T\right] \tag{3}$$

From (1),we obtain the one step state prediction at time $k$ as

$$\hat{x}_{k|k-1} = \mathbf{A}\hat{x}_{k-1} + \mathbf{B}u_{k-1} \tag{4}$$

The covariance matrix of the one step prediction error is given by

$$\mathbf{P}_{k|k-1} = E\left[(x_{k-1} - \hat{x}_{k|k-1})(x_{k-1} - \hat{x}_{k|k-1})^T\right] \tag{5}$$

where $E[]$ denotes the expectation operator. Then, the linear kalman filtering algorithm can be described in Table I.

## III. LINEAR KALMAN FILTER WITH NOISY INPUT

The control variable is usually used as the input of the system. In practice, the input signal is often corrupted by some noises, and the performance of the traditional LKF will degrade when the system meets the noisy input. In this section, a new LKF algorithm with noisy control variable will be derived. The noisy input is denoted as

$$\widehat{u}_{k-1} = u_{k-1} + v_{in,k-1} \tag{6}$$

where $v_{in,k-1} \in R^n$ stands for the input noise vector which is a zero-mean white Gaussian noise with variance $\sigma^2_{v_{in}}$, and is independent of $u_{k-1}$. The linear model (6) differs from those used in the previous works about LKF, and it is assumed that the actual input vector is available. There are many practical situations, however, where the systems only have access to noisy measurements of the input data. We use (6) to model such disturbance in the control variable, and to investigate the effect of the noise $v_{in,k-1}$.

The measurement set at time $k$ is denoted by $Y_k = \{y_i\}_{i=1}^k$. We obtain the priori estimate of the state as

$$\hat{x}_{k|k-1} = E[x_k|Y_{k-1}] \tag{7}$$

Furthermore, the state posteriori estimation can be represented as

$$\hat{x}_k = E[x_k|Y_k] \tag{8}$$

According to (1) and (6), we get the one step prediction of the state at time $k$ as

$$\hat{x}_{k|k-1} = \mathbf{A}\hat{x}_{k-1} + \mathbf{B}\widehat{u}_{k-1} \tag{9}$$

In addition, the state prediction error is

$$\begin{aligned}\widehat{e}_{k|k-1} &= x_k - \hat{x}_{k|k-1} \\ &= x_k - \mathbf{A}\hat{x}_{k-1} - \mathbf{B}\widehat{u}_{k-1} \\ &= \mathbf{A}(x_{k-1} - \hat{x}_{k-1}) + w_{k-1} - \mathbf{B}v_{in,k-1}\end{aligned} \tag{10}$$

Combining (1) and (10), the covariance matrix $\widehat{\mathbf{P}}_{k|k-1}$of the one step error can be computed as

$$\begin{aligned}\widehat{\mathbf{P}}_{k|k-1} &= E[\widehat{e}_{k|k-1}\widehat{e}^T_{k|k-1}] \\ &= E\begin{bmatrix}(\mathbf{A}(x_{k-1} - \hat{x}_{k-1}) + \mathbf{B}(u_{k-1} - \widehat{u}_{k-1}) + w_{k-1})\cdot \\ (\mathbf{A}(x_{k-1} - \hat{x}_{k-1}) + \mathbf{B}(u_{k-1} - \widehat{u}_{k-1}) + w_{k-1})^T\end{bmatrix} \\ &= \mathbf{A}E\left[(x_{k-1} - \hat{x}_{k-1})(x_{k-1} - \hat{x}_{k-1})^T\right]\mathbf{A}^T \\ &\quad + \mathbf{B}E\left[n_{k-1}n^T_{k-1}\right]\mathbf{B}^T + E\left[w_{k-1}w^T_{k-1}\right] \\ &= \mathbf{A}\widehat{\mathbf{P}}_{k-1}\mathbf{A}^T + \sigma^2_{v_{in}}\mathbf{B}\mathbf{B}^T + \mathbf{Q}_{k-1}\end{aligned} \tag{11}$$

In this brief, we assume that the input noise variance is known a priori. Now, we derive the correction module as follows. Similar to LKF, we have

$$\hat{x}_k = \hat{x}_{k|k-1} + \widehat{\mathbf{K}}_k i_k \tag{12}$$

where $\widehat{\mathbf{K}}_k$ is the kalman gain matrix, and $i_k$ denotes a correction term given by

$$i_k = y_k - \mathbf{C}\hat{x}_{k|k-1} \tag{13}$$

Then, we define the state estimate error as $\widehat{e}_k = x_k - \hat{x}_k$, and its covariance matrix is

$$\widehat{\mathbf{P}}_k = E\left[\widehat{e}_k\widehat{e}^T_k\right] \tag{14}$$

Combining (10) (12) (13) and (14), we have

$$\begin{aligned}\widehat{e}_k &= x_k - \hat{x}_k \\ &= x_k - \hat{x}_{k|k-1} - \widehat{\mathbf{K}}_k(\mathbf{C}x_k + v_k - \mathbf{C}\hat{x}_{k|k-1}) \\ &= (I - \widehat{\mathbf{K}}_k\mathbf{C})\widehat{e}_{k|k-1} - \widehat{\mathbf{K}}_k v_k\end{aligned} \tag{15}$$

According to (14) and (15), we obtain the covariance of the state estimate error

$$\begin{aligned}\widehat{\mathbf{P}}_k &= E\left[\left((\mathbf{I} - \mathbf{K}_k\mathbf{C})\widehat{e}_{k|k-1} - \mathbf{K}_k v_k\right)\left((\mathbf{I} - \mathbf{K}_k\mathbf{C})\widehat{e}_{k|k-1} - \mathbf{K}_k v_k\right)^T\right] \\ &= (\mathbf{I} - \mathbf{K}_k\mathbf{C})E\left[\widehat{e}_{k|k-1}\widehat{e}^T_{k|k-1}\right](\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k E[v_k v^T_k]\mathbf{K}^T_k \\ &\quad - (\mathbf{I} - \mathbf{K}_k\mathbf{C})E\left[\widehat{e}_{k|k-1}v^T_k\right]\mathbf{K}^T_k - \mathbf{K}_k E[v_k\widehat{e}^T_{k|k-1}](\mathbf{I} - \mathbf{K}_k\mathbf{C})^T\end{aligned} \tag{16}$$

Since the state prediction error $\widehat{e}_{k|k-1}$ is independent of $v_k$, we have

$$\begin{aligned}\widehat{\mathbf{P}}_k &= (\mathbf{I}-\widehat{\mathbf{K}}_k\mathbf{C})E\left[\widehat{e}_{k|k-1}\widehat{e}_{k|k-1}^T\right](\mathbf{I}-\widehat{\mathbf{K}}_k\mathbf{C})^T \\ &\quad + \widehat{\mathbf{K}}_k E[v_k v_k^T]\widehat{\mathbf{K}}_k^T \\ &= (\mathbf{I}-\widehat{\mathbf{K}}_k\mathbf{C})\widehat{\mathbf{P}}_{k|k-1}(\mathbf{I}-\widehat{\mathbf{K}}_k\mathbf{C})^T + \widehat{\mathbf{K}}_k\mathbf{R}_k\widehat{\mathbf{K}}_k^T \end{aligned} \tag{17}$$

Due to the second term in the left of the equation (11), a biased estimation will be produced by the tradition LKF. Therefore, to reduce the bias caused by the input noise, inspired by [17], we define the following cost function.

$$\begin{aligned}\mathbf{J}_k &= E\left[\widehat{e}_{1,k}^2+\widehat{e}_{2,k}^2+\cdots+\widehat{e}_{n,k}^2\right]-\sigma_{v_{in}}^2||\widehat{\mathbf{K}}_k||^2 \\ &= E\left[\widehat{e}_k\widehat{e}_k^T\right]-\sigma_{v_{in}}^2||\widehat{\mathbf{K}}_k||^2 \\ &= E\left[tr(\widehat{e}_k\widehat{e}_k^T)\right]-\sigma_{v_{in}}^2||\widehat{\mathbf{K}}_k||^2 \\ &= tr(E\left[\widehat{e}_k\widehat{e}_k^T\right])-\sigma_{v_{in}}^2||\widehat{\mathbf{K}}_k||^2 \\ &= tr(\widehat{P}_k)-\sigma_{v_{in}}^2||\widehat{\mathbf{K}}_k||^2\end{aligned} \tag{18}$$

For a diagram matrix $\mathbf{\Lambda}$, we have

$$\frac{\partial tr(\mathbf{B\Lambda B}^T)}{\partial \mathbf{B}} = 2\mathbf{B\Lambda} \tag{19}$$

From (17) and using (19), the derivative of the proposed cost function (18) with respect to the kalman gain matrix $\widehat{\mathbf{K}}_k$ is

$$\frac{\partial \mathbf{J}_k}{\partial \widehat{\mathbf{K}}_k} = 2(I-\widehat{\mathbf{K}}_k\mathbf{C})\widehat{\mathbf{P}}_{k|k-1}(-\mathbf{C}^T)+2\widehat{\mathbf{K}}_k\mathbf{R}_k-2\sigma_{v_{in}}^2\widehat{\mathbf{K}}_k \tag{20}$$

Letting the gradient in (20) be equal to zero, we have

$$\widehat{\mathbf{K}}_k = (\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T)(\mathbf{C}\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T+\mathbf{R}_k-\sigma_{v_{in}}^2)^{-1} \tag{21}$$

To get a simple form of the covariance matrix of the state estimate error, we denote

$$\mathbf{T}_k = \mathbf{C}\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T+\mathbf{R}_k-\sigma_{v_{in}}^2 \tag{22}$$

Substituting (21) into (17), we have

$$\begin{aligned}\widehat{\mathbf{P}}_k &= (\mathbf{I}-\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}\mathbf{C})\widehat{\mathbf{P}}_{k|k-1}(\mathbf{I}-\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}\mathbf{C})^T \\ &\quad + \widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}\mathbf{R}_k\mathbf{T}_k^{-1}\mathbf{C}\widehat{\mathbf{P}}_{k|k-1} \\ &= \widehat{\mathbf{P}}_{k|k-1}-2\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}\mathbf{C}\widehat{\mathbf{P}}_{k|k-1} \\ &\quad + \widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}(\mathbf{C}\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T+\mathbf{R}_k)\mathbf{T}_k^{-1}\mathbf{C}\widehat{\mathbf{P}}_{k|k-1} \\ &= \widehat{\mathbf{P}}_{k|k-1}-2\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}\mathbf{C}\widehat{\mathbf{P}}_{k|k-1} \\ &\quad + \widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}\mathbf{T}_k\mathbf{T}_k^{-1}\mathbf{C}\widehat{\mathbf{P}}_{k|k-1} \\ &= \widehat{\mathbf{P}}_{k|k-1}-\widehat{\mathbf{P}}_{k|k-1}\mathbf{C}^T\mathbf{T}_k^{-1}\mathbf{C}\widehat{\mathbf{P}}_{k|k-1}\end{aligned} \tag{23}$$

Using (21)-(23), the covariance matrix of the state estimate error can be simply expressed as

$$\begin{aligned}\widehat{\mathbf{P}}_k &= \widehat{\mathbf{P}}_{k|k-1}-\widehat{\mathbf{K}}_k\mathbf{C}\widehat{\mathbf{P}}_{k|k-1} \\ &= (\mathbf{I}-\widehat{\mathbf{K}}_k\mathbf{C})\widehat{\mathbf{P}}_{k|k-1}\end{aligned} \tag{24}$$

Table II summarizes the proposed LKFwNI algorithm.

*Remark:* From Table I, we know that the LKFwNI will perform close to the traditional LKF algorithm when $\sigma_{v_{in}}^2 = 0$, i.e., the control vector is noise free. The bias, which is normally expressed in terms of the second-order statistics of the control input noises, is removed from the solution by subtraction in (21). In addition, only two extra terms are introduced in the covariance matrix of the one step error and kalman gain matrix in comparison to the original LKF, hence the computational complexity of the new algorithm is acceptable relative to the benefits it brings.

TABLE II
LINEAR KALMAN FILTERING ALGORITHM WITH NOISY INPUT

| |
|---|
| Initialization: |
| $\hat{x}_0 = E[x_0]$ |
| $\widehat{\mathbf{P}}_0 = E\left[(x_0-\hat{x}_0)(x_0-\hat{x}_0)^T\right]$ |
| Prior estimation: |
| $\hat{x}_{k\|k-1} = \mathbf{A}\hat{x}_{k-1}+\mathbf{B}\widehat{u}_{k-1}$ |
| $\widehat{\mathbf{P}}_{k\|k-1} = \mathbf{A}\widehat{\mathbf{P}}_{k-1}\mathbf{A}^T+\sigma_{v_{in}}^2\mathbf{BB}^T+\mathbf{Q}_{k-1}$ |
| Posterior estimation: |
| $\widehat{\mathbf{K}}_k = (\widehat{\mathbf{P}}_{k\|k-1}\mathbf{C}^T)(\mathbf{C}\widehat{\mathbf{P}}_{k\|k-1}\mathbf{C}^T+\mathbf{R}_k-\sigma_{v_{in}}^2)^{-1}$ |
| $\hat{x}_k = \hat{x}_{k\|k-1}+\widehat{\mathbf{K}}_k(y_k-\mathbf{C}\hat{x}_{k\|k-1})$ |
| $\widehat{\mathbf{P}}_k = (\mathbf{I}-\widehat{\mathbf{K}}_k\mathbf{C})\widehat{\mathbf{P}}_{k\|k-1}$ |

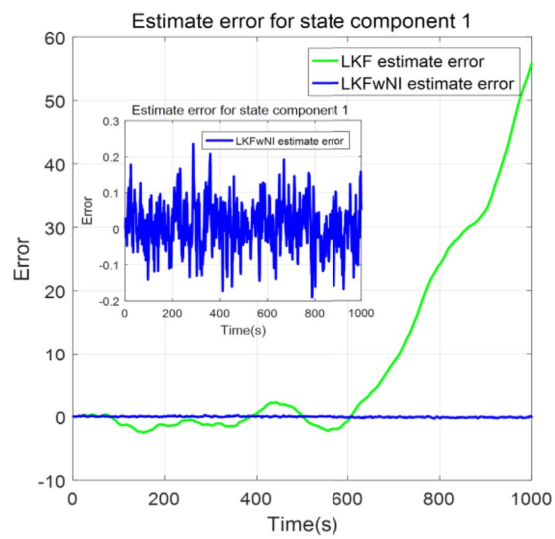


Fig. 1. Estimate error for state component 1.

## IV. SIMULATION RESULTS

To demonstrate the effectiveness of the proposed LKFwIN algorithm given in Section III, we consider two or three dimensional control input variable in discrete-time state estimation. All simulation results are obtained by averaging over 100 independent trials. The performance of the proposed algorithm is compared with the traditional LKF.

The mean square error (MSE), maximum absolute error (MAE), Root MSE (RMSE) are calculated according to

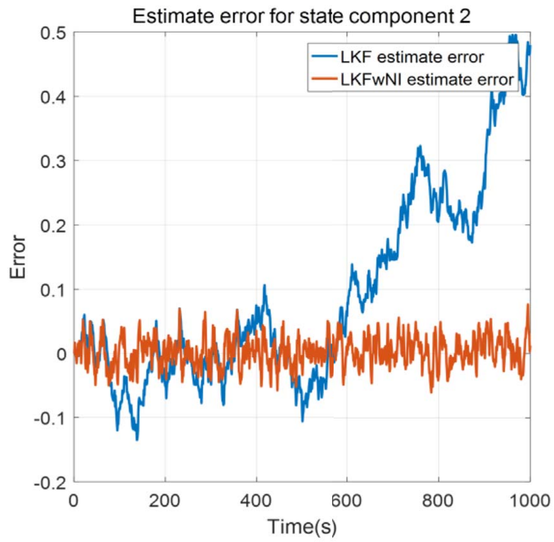


Fig. 2. Estimate error for state component 2.

the formula shown in (25).

$$
\begin{aligned}
MSE_{x_i} &= \frac{1}{MN}\sum_{j=1}^{M}\sum_{k=1}^{N}(x_i(k)-\hat{x}_i(k)\ )^2 \quad (i=1,2,\cdots n)\\
MAXE_{x_i} &= Max\frac{1}{M}\sum_{j=1}^{M}|x_i(k)-\hat{x}_i(k)| \quad (i=1,2,\ldots,n)\\
RMSE_{x_i} &= \sqrt{MSE}
\end{aligned}
\tag{25}
$$

where $M = 100$ is the number of monte-carlo trials, N is the number of discrete time points, $x_i(k)$ and $\hat{x}_i(k)$ are the $i$-th entry of the true state vector and its estimated value obtained in the simulations.

### A. Numerical Example 1

A system formulated by (1) and (2) is considered with the following system matrices.

$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = [0.5]$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\mathbf{Q} = \mathbf{0}_{2\times 2}$, $\mathbf{R} = [1]$, the error covariance matrix is initialized at $\mathbf{P}_0 = \widehat{\mathbf{P}}_0 = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$, and the initial guess for state $\hat{x}_0 = \begin{bmatrix} 95 \\ 1 \end{bmatrix}$. The input in this example is generated by a Gaussian random sequence with zero mean and unite variance, and it is corrupted by a zero-mean uncorrelated Gaussian noise with variance 0.05. The number of discrete time points N is set at 1000.

We first investigate the performance of the proposed algorithm in terms of the estimate error. Fig. 1 and 2 show the estimate error of the two components of the state vector, respectively. One can observe that the error of traditional LKF algorithm has large fluctuation, while the proposed algorithm shows small fluctuation. The main reason is that the bias caused by the input noise is cancelled by the additional term in (21). The simulation results of each component in the state vector by using the evaluation criterions in (25) are given in Table III.

TABLE III
PERFORMANCE OF LKF VS. LKFWIN

| | LKF | | LKFwNI | |
|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_1$ | $x_2$ |
| MSE | 313.6693 | 0.0317 | 0.0040 | 0.00056 |
| RMSE | 17.7107 | 0.1781 | 0.0634 | 0.0238 |
| MAXE | 55.8053 | 0.4955 | 0.2360 | 0.0765 |

TABLE IV
PERFORMANCE OF LKF VS. LKFWIN

| | LKF | | | LKFwNI | | |
|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_1$ | $x_2$ | $x_3$ |
| MSE | 0.0638 | 0.1806 | 0.0245 | 0.0576 | 0.1531 | 0.0233 |
| RMSE | 0.1920 | 0.2399 | 0.4250 | 0.3913 | 0.1566 | 0.1527 |
| MAXE | 4.0800 | 7.6800 | 1.7600 | 3.7600 | 6.8800 | 1.9200 |

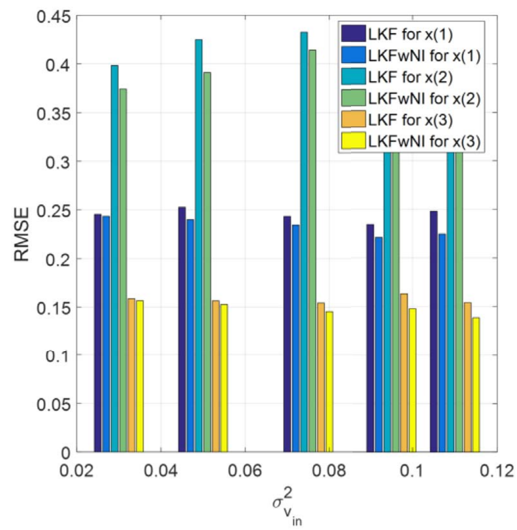


Fig. 3. RMSEs with different noise powers.

### B. Numerical Example 2

In the second example, we consider a system with the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ \mathbf{Q} = \mathbf{0}_{3\times 3},$$

$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the error covariance matrix is initialized at $\mathbf{P}_0 = \widehat{\mathbf{P}}_0 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the initial guess for state $\hat{x}_0 = \begin{bmatrix} 35 \\ 0 \\ 1 \end{bmatrix}$.

The control input and input noise are coincide with those in the first example, except now the dimension is three. N is 800 in this example.

First, the MSE, RMSE, and MAXE results are computed to evaluate the performance of the LKF and LKFwNI. The results are given in Table IV. We can see that the proposed LKFwNI algorithm obtain better performance than the LKF algorithm.

Second, we perform simulations under different levels of noise power in terms of noise variance to further illustrate the merits of the new method. The noise variances are set at 0.03, 0.05, 0.075, 0.095, and 0.11, respectively. The RMSE is used to evaluate the performance of the proposed LKFwNI in comparison to the traditional LKF, and the results are plotted in Fig. 3. One can observe that the RMSE results of the LKFwIN algorithm are lower than the traditional LKF for all components of the state vector.

## V. Conclusion

In this brief, the noisy control input variable has been taken into account in the linear kalman filtering (LKF) algorithm. Due to the noise added into the control variable, a bias estimate is naturally generated by the traditional LKF for the state estimation. Therefore, the noisy control variable was considered in the derivation of a new LKF and the bias caused by the input noise was analyzed first in this brief. For reducing the bias, a novel cost function was proposed to obtain an effective kalman gain matrix with which the new algorithm can overcome the bias estimation problem. Numerical simulations verified that the proposed algorithm can achieve lower steady-state misalignment when compared with the traditional LKF algorithm.

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