Linear Response Functions

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1 Motivation and Definitions

Consider the two linear response functions χ and χ_T outlined as follows.

$$\chi(t) = \frac{\exp -i\omega_0 t}{\sqrt{2\pi}} \left(iT^* a \int_{-\infty}^{\infty} \frac{\exp -i\omega'(t-\tau)}{1 - Ra \exp i\omega'\tau} d\omega \right) \tag{1}$$

$$\chi_T(t) = \frac{\exp -i\omega_0 t}{\sqrt{2\pi}} \left(R \int_{-\infty}^{\infty} \frac{\exp -i\omega' t}{1 - Ra \exp i\omega'\tau} d\omega - a \int_{-\infty}^{\infty} \frac{\exp -i\omega'(t-\tau)}{1 - Ra \exp i\omega'\tau} d\omega \right) \tag{2}$$

Where $\omega' = \omega - \omega_0$ (thus $d\omega' = d\omega$). Note the commonality of this integral:

$$I(a,b,c) = \int_{-\infty}^{\infty} \frac{\exp -iaz}{1 - b \exp icz} dz \tag{3}$$

Thus we may re-express the linear response functions as follows.

$$\chi(t) = \frac{\exp -i\omega_0}{\sqrt{2\pi}} iT^* a I(t-\tau, Ra, \tau) \tag{4}$$

$$\chi_T(t) = \frac{\exp -i\omega_0}{\sqrt{2\pi}} (RI(t, Ra, \tau) - aI(t-\tau, Ra, \tau)) \tag{5}$$

Thus if we can solve $I(a,b,c)$, we retrieve both functions for free. To do this we will use contour integration in the complex plane, the Residue Theorem, and Jordan's Lemma. First though, we define some terms to alleviate notation.

$$\hat{f}(z) = (1 - b \exp icz)^{-1} \tag{6}$$

$$f(z) = \hat{f}(z) \exp -iaz \tag{7}$$

Thus, avoiding notation on the functional dependence of I on a, b and c, we have:

$$I = \int_{-\infty}^{\infty} f(z) dz \tag{8}$$

2 Contour

We will expand f to the complex plane trivially, and consider a contour of integration to be the path $A_r = \{rx \in \mathbb{C} | x \in [-1, 1]\}, r \in \mathbb{R}^+$. Of course this implies the following.

$$I = \lim_{r \to \infty} \int_{A_r} f(z) dz \tag{9}$$

We may now consider that A_r as defined above is the top part of a semicircle of radius r, which we will label Ω_r. The Semicircular part will be labeled ${}_r = \{re^{i\theta} \in \mathbb{C} | \theta : 2\pi \to \pi\}$. Clearly $\Omega_r = A_r \cup C_r$, and by the properties of piece-wise continuous curves we have the following.

$$\oint_{\Omega_r} f(z) dz = \int_{A_r} f(z) dz + \int_{C_r} f(z) dz \tag{10}$$

Where the symbol on the left indicates that the integral is clockwise so as to start at $-r$ go to r, and then follow C_r, back to $-r$. Equivalently:

$$\int_{A_r} f(z) dz = \oint_{\Omega_r} f(z) dz - \int_{C_r} f(z) dz = -\oint_{\Omega_r} f(z) dz - \int_{C_r} f(z) dz \tag{11}$$

Thus we may characterize I as the following

$$I = -\lim_{r \to \infty} \oint_{\Omega_r} f(z) dz - \lim_{r \to \infty} \int_{C_r} f(z) dz \tag{12}$$

3 The Residue Theorem

3.1 Setup

Let $int(\Omega_r)$ be the simply connected region enclosed by Ω_r. Moreover, let $Poles(f)$ be the poles of f. The Residue Theorem states that if $S_{f,r} = Poles(f) \cap int(\Omega_r)$ and $|S_{f,r}| \le |\mathbb{N}|$ (that is: there is at most a countable number of poles of f in the region) then we have:

$$\oint_{\Omega_r} f(z) dz = 2\pi i \sum_{p \in S_{f,r}} Res[f, p] \tag{13}$$

Moreover, since all dependence on r is in $S_{f,r}$ we may express the limit as before, notationally at least, as

$$\lim_{r \to \infty} \oint_{\Omega_r} f(z) dz = 2\pi i \sum_{p \in \lim_{r \to \infty} S_{f,r}} Res[f, p] = 2\pi i \sum_{p \in S_{f,\infty}} Res[f, p] \tag{14}$$

Note: that $S_{f,\infty}$ denotes those poles of f which are in the lower half of the plane. Our first goal here, is to find those.

3.2 Poles

Recall that $f(z) = \hat{f}(z) \exp -iaz$. Of course $\exp(-iaz)$ is holomorphic and so all poles arise from $\hat{f}(z)$. Now $\hat{f}(z) = (1 - b \exp icz)^{-1}$ Since the argument of the inverse function is also holomorphic, we now that the zeroes of $1 - b \exp icz$ are the poles of $\hat{f}(z)$ and thereby $f(z)$. Moreover, by multiplicity, all these poles are simple (a note to hold on to for the section on residues). Thus we come to the following.

$$Poles(f) = \{p \in \mathbb{C} | 1 - b \exp icp = 0\} \tag{15}$$

What follows is a solution to the equation on the right

$$\begin{aligned}
1 - b \exp icp &= 0 \\
1 &= b \exp icp \\
b^{-1} &= \exp icp \\
\log b^{-1} &= \log \exp icp \\
-\log b &= icp + i2n\pi \\
-(\log b + i2n\pi) &= icp \\
-\frac{\log b + i2n\pi}{ic} &= p \\
i\frac{\log b + i2n\pi}{c} &= p \\
p &= \frac{-2n\pi + i \log b}{c}
\end{aligned}$$

This solution holds $\forall n \in \mathbb{Z}$, and so we write the following.

$$Poles(f) = \left\{ p_n = \frac{-2n\pi + i \log b}{c} \in \mathbb{C} | \forall n \in \mathbb{Z} \right\} \tag{16}$$

For these poles to be inside the lower half of the plane we require $Im(p_n) < 0$ which implies that $b \in (0, 1)$. If b is negative the solution is undefined, and if its greater than 1 than there are no poles in the region and thus the integral is trivially 0. Thus we suppose b is in the desired interval, and so all poles are in the desired region. That is:

$$S_{f,\infty} = Poles(f) \tag{17}$$

Of course as $Poles(f)$ is indexed by $\mathbb{Z}$, the two have the same cardinally (and therefore $S_{f,\infty}$ does as well).

To conclude, there are a countably infinite number of poles. All simple, all known, and all in the desired region.

3.3 Residues

$\forall n \in \mathbb{Z}$, what is $Res[f, p_n]$? Let's find out. Since all these poles are simple we may use the following method:

$$
\begin{aligned}
Res[f, p_n] &= \lim_{z \to p_n} (z - p_n) f(z) \\
&= \lim_{z \to p_n} \frac{(z - p_n) \exp -iaz}{1 - b \exp icz} \\
&= \lim_{z \to p_n} \frac{\exp -iaz - ia(z - p_n) \exp -iaz}{-icb \exp icz} \\
&= \lim_{z \to p_n} i \frac{1 - ia(z - p_n)}{bc} \exp(-i(a + c)z) \\
&= i(bc)^{-1} \exp(-i(a + c)p_n) \\
&= i(bc)^{-1} \exp\left(-i(a + c) \frac{-2n\pi + i \log b}{c}\right) \\
&= i(bc)^{-1} \exp\left(i \frac{(a + c)2n\pi}{c}\right) \exp\left(-i^2 \frac{(a + c) \log b}{c}\right) \\
&= i(bc)^{-1} \exp\left(i \frac{(a + c)2n\pi}{c}\right) \exp \log b^{\frac{a+c}{c}} \\
&= i(bc)^{-1} b^{\frac{a+c}{c}} \exp\left(i \frac{(a + c)2n\pi}{c}\right) \\
&= ib^{\frac{a+c}{c} - 1} c^{-1} \exp\left(i \frac{(a + c)2n\pi}{c}\right) \\
&= ib^{\frac{a}{c}} c^{-1} \exp\left(i \frac{(a + c)2n\pi}{c}\right)
\end{aligned}
$$

3.4 Conclusion

All together we have the following

$$
\begin{aligned}
\lim_{r \to \infty} \oint_{\Omega_r} f(z) dz &= 2\pi i \sum_{p \in S_{f,\infty}} Res[f, p] \\
&= 2\pi i \sum_{n=-\infty}^{\infty} Res[f, p_n] \\
&= 2\pi i \sum_{n=-\infty}^{\infty} ib^{\frac{a}{c}} c^{-1} \exp\left(i \frac{(a + c)2n\pi}{c}\right) \\
&= -2\pi b^{\frac{a}{c}} c^{-1} \sum_{n=-\infty}^{\infty} \exp\left(i \frac{(a + c)2n\pi}{c}\right)
\end{aligned}
$$

And so to finish this section we have the following equation

$$\lim_{r\to\infty}\oint_{\Omega_r} f(z)dz = -2\pi b^{\frac{a}{c}}c^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{(a+c)2n\pi}{c}\right) \tag{18}$$

4 Jordan's Lemma

4.1 Goal

For our purposes the lemma states the following

$$0 \le \left|\int_{C_r} f(z)dz\right| \le \frac{\pi}{a}\max_{z\in C_r}|\hat{f}(z)| \tag{19}$$

Since $C_r = \{z = re^{i\theta} \in \mathbb{C}|\theta \in [\pi, 2\pi]\}$ and since the absolute value, and the maximum functions are continuous with respect to r we have the following:

$$0 \le \left|\lim_{r\to\infty}\int_{C_r} f(z)dz\right| \le \frac{\pi}{a}\max_{\theta\in[\pi,2\pi]}|\lim_{r\to\infty}\hat{f}(re^{i\theta})| \tag{20}$$

The goal is clear, if we can show that the right hand side is 0, then the middle is zero, and by the fact that the absolute value function is a norm, if it is zero, its argument must be zero. Let us now try to bound the right hand side.

4.2 Bounding

Consider the following:

$$|\hat{f}(re^{i\theta})| = \left|(1 - b\exp icre^{i\theta})^{-1}1\right| = \left|1 - b\exp icre^{i\theta}\right|^{-1} \tag{21}$$

By the reverse triangle inequality we have the following:

$$\begin{aligned}
\left|1 - b\exp icre^{i\theta}\right| &\ge \left||1| - |b\exp icre^{i\theta}|\right| \\
&\ge |1 - b|\exp icr(\cos\theta + i\sin\theta)|| \\
&\ge \left|1 - b|\exp(icr\cos\theta)||\exp(i^2cr\sin\theta)|\right| \\
&\ge |1 - b\exp(-cr\sin\theta)|
\end{aligned}$$

Thus we have the following inequality

$$0 \le |\hat{f}(re^{i\theta})| = \left|1 - b\exp icre^{i\theta}\right|^{-1} \le \left|(1 - b\exp(-cr\sin\theta))^{-1}\right| \tag{22}$$

Applying the limit

$$0 \le |\lim_{r\to\infty}\hat{f}(re^{i\theta})| \le \left|\lim_{r\to\infty}(1 - b\exp(-cr\sin\theta))^{-1}\right| \tag{23}$$

A similar argument presents itself here: if we can show that the right hand side is 0, then we may conclude that the middle is zero.

Consider that $\forall\theta \in [\pi, 2\pi]$, we have the following inequality.

$$
\begin{aligned}
-1 \le \sin\theta & & \le 0 \\
cr \ge -cr\sin\theta & & \ge 0 \\
\exp cr \ge \exp\left(-cr\sin\theta\right) & & \ge 1 \\
1 - b\exp cr \le 1 - b\exp\left(-cr\sin\theta\right) & & \le 1 - b \\
\left(1 - b\exp cr\right)^{-1} \ge \left(1 - b\exp\left(-cr\sin\theta\right)\right)^{-1} & & \ge \frac{1}{1-b}
\end{aligned}
$$

We take a quick pause here to recall that $b \in (0,1)$ and thus $1-b > 0$. Leading to:

$$
\begin{aligned}
\left(1 - b\exp cr\right)^{-1} \ge \left(1 - b\exp\left(-cr\sin\theta\right)\right)^{-1} & & > 0 \\
\lim_{r\to\infty}\left(1 - b\exp cr\right)^{-1} \ge \lim_{r\to\infty}\left(1 - b\exp\left(-cr\sin\theta\right)\right)^{-1} & & > 0 \\
0 \ge \lim_{r\to\infty}\left(1 - b\exp\left(-cr\sin\theta\right)\right)^{-1} & & > 0
\end{aligned}
$$

Which, via a variant of the Squeeze theorem, we may conclude that

$$
\lim_{r\to\infty}\left(1 - b\exp\left(-cr\sin\theta\right)\right)^{-1} = 0 \tag{24}
$$

$\forall\theta \in [\pi, 2\pi]$. Moreover, we may restate an above inequality with this added information.

$$
0 \le |\lim_{r\to\infty}\hat{f}(re^{i\theta})| \le \left|\lim_{r\to\infty}\left(1 - b\exp\left(-cr\sin\theta\right)\right)^{-1}\right| = |0| = 0 \tag{25}
$$

Again invoking squeeze theorem we have that

$$
|\lim_{r\to\infty}\hat{f}(re^{i\theta})| = 0 \tag{26}
$$

4.3 Conclusion

We may now return to the original purpose of these boundings:

$$
0 \le \left|\lim_{r\to\infty}\int_{C_r} f(z)dz\right| \le \frac{\pi}{a}\max_{\theta\in[\pi,2\pi]}|\lim_{r\to\infty}\hat{f}(re^{i\theta})| = \frac{\pi}{a}\max_{\theta\in[\pi,2\pi]}(0) = 0 \tag{27}
$$

Which finally allows us to succeed in our goal of declaring:

$$
\left|\lim_{r\to\infty}\int_{C_r} f(z)dz\right| = 0 \tag{28}
$$

Which, for reasons discussed above leaves us with

$$
\lim_{r\to\infty}\int_{C_r} f(z)dz = 0 \tag{29}
$$

5 Solving the Integral

Sections 3 and 4 have allowed us to make the following argument:

$$
\begin{aligned}
I(a,b,c) &= -\lim_{r\to\infty}\oint_{\Omega_r} f(z)dz - \lim_{r\to\infty}\int_{C_r} f(z)dz \\
&= -\left(-2\pi b^{\frac{a}{c}} c^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{(a+c)2n\pi}{c}\right)\right) - (0) \\
&= 2\pi b^{\frac{a}{c}} c^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{(a+c)2n\pi}{c}\right)
\end{aligned}
$$

6 The Linear Response Functions

Return again to χ

$$
\begin{aligned}
\chi(t) &= \frac{\exp -i\omega_0 t}{\sqrt{2\pi}} iT^* a I(t-\tau, Ra, \tau) \\
&= \frac{\exp -i\omega_0 t}{\sqrt{2\pi}} iT^* a 2\pi (Ra)^{\frac{t-\tau}{\tau}} \tau^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{(t-\tau+\tau)2n\pi}{\tau}\right) \\
&= iT^*\frac{\sqrt{2\pi}}{\tau} R^{\frac{t}{\tau}-1} a^{\frac{t}{\tau}}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{t2n\pi}{\tau} - i\omega_0 t\right) \\
&= iT^*\frac{\sqrt{2\pi}}{R\tau}(Ra)^{\frac{t}{\tau}}\sum_{n=-\infty}^{\infty}\exp i\left(\frac{2n\pi}{\tau} - \omega_0\right) t
\end{aligned}
$$

Now turn to χ_T

$$
\begin{aligned}
\chi_T(t) &= \frac{\exp -i\omega_0 t}{\sqrt{2\pi}}(RI(t,Ra,\tau) - aI(t-\tau,Ra,\tau)) \\
&= \frac{\exp -i\omega_0 t}{\sqrt{2\pi}}\left(R2\pi(Ra)^{\frac{t}{\tau}}\tau^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{(t+\tau)2n\pi}{\tau}\right) - a2\pi(Ra)^{\frac{t-\tau}{\tau}}\tau^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{(t-\tau+\tau)2n\pi}{\tau}\right)\right) \\
&= \frac{\sqrt{2\pi}\exp -i\omega_0 t}{\tau}(Ra)^{\frac{t}{\tau}}\left(R\sum_{n=-\infty}^{\infty}\exp\left(i\frac{t2n\pi}{\tau}\right)\exp\left(i2n\pi\right) - a(Ra)^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{t2n\pi}{\tau}\right)\right) \\
&= \frac{\sqrt{2\pi}\exp -i\omega_0 t}{\tau}(Ra)^{\frac{t}{\tau}}\left(R\sum_{n=-\infty}^{\infty}\exp\left(i\frac{t2n\pi}{\tau}\right) - R^{-1}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{t2n\pi}{\tau}\right)\right) \\
&= \frac{\sqrt{2\pi}\exp -i\omega_0 t}{\tau}(Ra)^{\frac{t}{\tau}}(R-R^{-1})\sum_{n=-\infty}^{\infty}\exp\left(i\frac{t2n\pi}{\tau}\right) \\
&= (R^2-1)\frac{\sqrt{2\pi}}{R\tau}(Ra)^{\frac{t}{\tau}}\sum_{n=-\infty}^{\infty}\exp i\left(\frac{2n\pi}{\tau}-\omega_0\right)t \\
&= (R^2-R^2-T^{*2})\frac{\sqrt{2\pi}}{R\tau}(Ra)^{\frac{t}{\tau}}\sum_{n=-\infty}^{\infty}\exp i\left(\frac{2n\pi}{\tau}-\omega_0\right)t \\
&= -T^{*2}\frac{\sqrt{2\pi}}{R\tau}(Ra)^{\frac{t}{\tau}}\sum_{n=-\infty}^{\infty}\exp i\left(\frac{2n\pi}{\tau}-\omega_0\right)t
\end{aligned}
$$

By assumption of the integration: $t, t-\tau, \tau \geq 0$ (that is $t \geq \tau \geq 0$). Now, let us define a couple terms to ease notation. Firstly, let $k = (Ra)^{\frac{1}{\tau}}$, which by the assumption of the integral is in $(0,1)$. Secondly let ω_n be given by the following.

$$\omega_n = \frac{2n\pi}{\tau} - \omega_0 \tag{30}$$

And thirdly, define the function σ as follows.

$$\sigma(t) = \frac{\sqrt{2\pi}}{R\tau}k^t \sum_{n=-\infty}^{\infty}\exp i\omega_n t \tag{31}$$

Thus by the above two solutions we have the linear response functions succinctly given as

$$\chi(t) = iT^*\sigma(t) \tag{32}$$

$$\chi_T(t) = -T^{*2}\sigma(t) \tag{33}$$

Granted it was a bit of an abuse of notation and terminology to indicate that σ is a function, as technically speaking it is proportional to a divergent series. That said, if we spend a bit longer on this topic we will have a richer way to apply these linear response functions.

7 δ and σ

7.1 On the Formality of δ-Distributions

For any interval I define the set of all integrable complex valued functions on I as follows

$$L^1_{\mathbb{C}}(I) = \left\{ f : I \to \mathbb{C} \middle| \int_I f(t)dt \in \mathbb{C} \right\} \tag{34}$$

Then, we define a sequence of functions $\{\phi_n \in L^1_{\mathbb{C}}(I)\}_{\forall n \in \mathbb{N}}$ with the following properties being satisfied $\forall n \in \mathbb{N}$, $\forall a \in I$, and $\forall f \in L^1_{\mathbb{C}}(I)$.

$$\int_I \phi_n(t)dt = 1$$

$$\lim_{n\to\infty} \int_I \phi_n(t-a)f(t)dt = f(a)$$

Then we define a Dirac-delta $\delta = \lim_{n\to\infty} \phi_n$. Moreover, we can define the following notation.

$$\int_I \delta(t-a)f(t)dt = \lim_{n\to\infty} \int_I \phi_n(t-a)f(t)dt = f(a) \tag{35}$$

Now, taking $I = \mathbb{R}$, and defining a similar sequence $\{\Phi^T_{N,n} \in L^1_{\mathbb{C}}(I)\}_{\forall N,n \in \mathbb{N}}$ such that $\forall N, n \in \mathbb{N}$ $\Phi^T_{N,n}(t) = \sum_{k=-N}^N \phi_n(t - kT)$. By linearity we may note the following property:

$$\int_I \Phi^T_{N,n}(t)f(t)dt = \sum_{k=-N}^{N} \int_I \phi_n(t-kT)f(t)dt$$

Similarly, we define the Dirac-comb of period T as $\Delta_T = \lim_{n\to\infty} \lim_{N\to\infty} \Phi^T_{N,n}$. And once again, we may define the following notation:

$$\int_I \Delta_T(t)f(t)dt = \lim_{n\to\infty} \lim_{N\to\infty} \sum_{k=-N}^{N} \int_I \phi_n(t-kT)f(t)dt \tag{36}$$

We can play with the limits on the right hand side to write that a little more nicely

$$
\begin{aligned}
\int_I \Delta_T(t) f(t) dt &= \lim_{n\to\infty} \lim_{N\to\infty} \sum_{k=-N}^{N} \int_I \phi_n(t-kT) f(t) dt \\
&= \lim_{N\to\infty} \sum_{k=-N}^{N} \lim_{n\to\infty} \int_I \phi_n(t-kT) f(t) dt \\
&= \lim_{N\to\infty} \sum_{k=-N}^{N} \int_I \delta(t-kT) f(t) dt \\
&= \sum_{k=-\infty}^{\infty} \int_I \delta(t-kT) f(t) dt
\end{aligned}
$$

7.2 Fourier Series

Consider the Fourier Series decomposition of Δ_T. This is a valid construction as the Dirac-comb is a periodic Dirac-delta.

$$
\begin{aligned}
\Delta_T(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T} \exp\left(i\frac{2\pi n}{T}t\right) \int_0^T \Delta_T(t) \exp\left(i\frac{2\pi n}{T}t\right) dt \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{T} \exp\left(i\frac{2\pi n}{T}t\right) \sum_{k=-\infty}^{\infty} \int_0^T \delta(t-kT) \exp\left(i\frac{2\pi n}{T}t\right) dt \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{T} \exp\left(i\frac{2\pi nt}{T}\right) exp\left(i\frac{2\pi n}{T}(0T)\right) \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \exp\left(i\frac{2\pi n}{T}t\right)
\end{aligned}
$$

In the language of distributions we can formally express the above analysis of generalized functions as follows.

Consider the N^{th} Dirichlet Kernel of period T: D_N defined as

$$
D_N(t) = \frac{1}{T} \sum_{n=-N}^{N} \exp i\left(\frac{2\pi n}{T}t\right) \tag{37}
$$

It can be shown that $\{D_N\}_{N\in\mathbb{N}} \to \Delta_T$, in the sense expressed in the prior subsection.

7.3 Connection to σ

It is now our goal to connect Δ_T with σ

$$\begin{aligned}
\Delta_\tau(t) &= \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\exp i\left(\frac{2n\pi}{\tau}\right)t \\
\exp(-i\omega_0 t)\Delta_\tau(t) &= \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\exp i\left(\frac{2n\pi}{\tau}-\omega_0\right)t \\
&= \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\exp i\omega_n t \\
\frac{\sqrt{2\pi}}{R}k^t\exp(-i\omega_0 t)\Delta_\tau(t) &= \frac{\sqrt{2\pi}}{R\tau}k^t\sum_{n=-\infty}^{\infty}\exp i\omega_n t \\
\sigma(t) &= \frac{\sqrt{2\pi}}{R}k^t\exp(-i\omega_0 t)\Delta_\tau(t)
\end{aligned}$$

7.4 Conclusion

Thusly, we may express four equations which fully encapsulate the linear response functions completely.

Suppose $Ra \in (0,1)$, and $t \geq \tau \geq 0$ then:

$$k = (Ra)^{\frac{1}{\tau}} \tag{38}$$

$$\sigma(t) = \frac{\sqrt{2\pi}}{R}k^t\exp(-i\omega_0 t)\Delta_\tau(t) \tag{39}$$

$$\chi(t) = iT^*\sigma(t) \tag{40}$$

$$\chi_T(t) = -T^{*2}\sigma(t) \tag{41}$$

In particular this means that both χ and $\chi(t)$ are undefined for negative t.

8 Application of the Functions

Note the equations

$$E_1[E_0](t) = \int_{-\infty}^{t}\chi_T(t-t')E_0(t')dt' \tag{42}$$

$$E_3[E_0](t) = \int_{-\infty}^{t}\chi(t-t')E_0(t')dt' \tag{43}$$

Assuming $t \geq 0$. When $t' \leq 0$, $t - t' \geq 0$, so the response functions are well defined in that region. Moreover when $t' \in (0,t)$, $t - t' \geq 0$ and so the function is still well defined.

$$
\begin{aligned}
E_1[E_0](t) &= \int_{-\infty}^{t} \chi_T(t-t')E_0(t')dt' \\
&= \int_{-\infty}^{t} (R^2-1)\sigma(t-t')E_0(t')dt' \\
&= (R^2-1)\int_{-\infty}^{t} \sigma(t-t')E_0(t')dt'
\end{aligned}
$$

$$
\begin{aligned}
E_3[E_0](t) &= \int_{-\infty}^{t} \chi(t-t')E_0(t')dt' \\
&= \int_{-\infty}^{t} iT^*\sigma(t-t')E_0(t')dt' \\
&= iT^*\int_{-\infty}^{t} \sigma(t-t')E_0(t')dt'
\end{aligned}
$$

Note the recurrence of the integral $\int_{-\infty}^{t}\sigma(t-t')E_0(t')dt'$. So we only need to solve this integral to obtain both.

$$
\begin{aligned}
\int_{-\infty}^{t} \sigma(t-t')E_0(t')dt' &= \int_{-\infty}^{t} \frac{\sqrt{2\pi}}{R}k^{t-t'}\exp\left(-i\omega_0(t-t')\right)\Delta_\tau(t-t')E_0(t')dt' \\
&= \frac{\sqrt{2\pi}}{R}k^t\exp -i\omega_0 t\sum_{n=-\infty}^{\infty}\int_{-\infty}^{t} k^{-t'}\exp\left(i\omega_0 t'\right)\delta(t-t'-n\pi)E_0(t')dt' \\
&= \frac{\sqrt{2\pi}}{R}k^t\exp -i\omega_0 t\sum_{n=-\infty}^{\infty}\int_{-\infty}^{t} k^{-t'}\exp\left(i\omega_0 t'\right)\delta(t'-(t-n\tau))E_0(t')dt'
\end{aligned}
$$

For $n < 0$, $t - n\tau \notin (-\infty, t]$ and thus integrating over $\delta(t' - (t - n\pi))$ results in 0 for all of these n. So we continue with the calculation.

$$
\begin{aligned}
\int_{-\infty}^{t} \sigma(t-t')E_0(t')dt' &= \frac{\sqrt{2\pi}}{R}k^t\exp -i\omega_0 t\sum_{n=0}^{\infty} k^{-(t-n\tau)}\exp\left(i\omega_0(t-n\tau)\right)E_0(t-n\tau) \\
&= \frac{\sqrt{2\pi}}{R}k^{t-t}\exp -i\omega_0(t-t)\sum_{n=0}^{\infty} k^{n\tau}\exp -i\omega_0 n\tau E_0(t-n\tau) \\
&= \frac{\sqrt{2\pi}}{R}\sum_{n=0}^{\infty} k^{n\tau}\exp\left(-in\omega_0\tau\right)E_0(t-n\tau) \\
&= \frac{\sqrt{2\pi}}{R}\sum_{n=0}^{\infty} (Ra)^{\frac{n\tau}{\tau}}\exp\left(-i\phi n\right)E_0(t-n\tau) \\
&= \frac{\sqrt{2\pi}}{R}\sum_{n=0}^{\infty} (Ra)^{n}(\exp -i\phi)^n E_0(t-n\tau) \\
&= \frac{\sqrt{2\pi}}{R}\sum_{n=0}^{\infty} \left(\frac{Ra}{\exp i\phi}\right)^{n} E_0(t-n\tau)
\end{aligned}
$$

And thusly, we may finish with the following conclusions:

$$E_1[E_0](t) = -T^{*2}\frac{\sqrt{2\pi}}{R}\sum_{n=0}^{\infty}\left(\frac{Ra}{\exp i\phi}\right)^n E_0(t - n\tau) \tag{44}$$

$$E_3[E_0](t) = iT^{*}\frac{\sqrt{2\pi}}{R}\sum_{n=0}^{\infty}\left(\frac{Ra}{\exp i\phi}\right)^n E_0(t - n\tau) \tag{45}$$