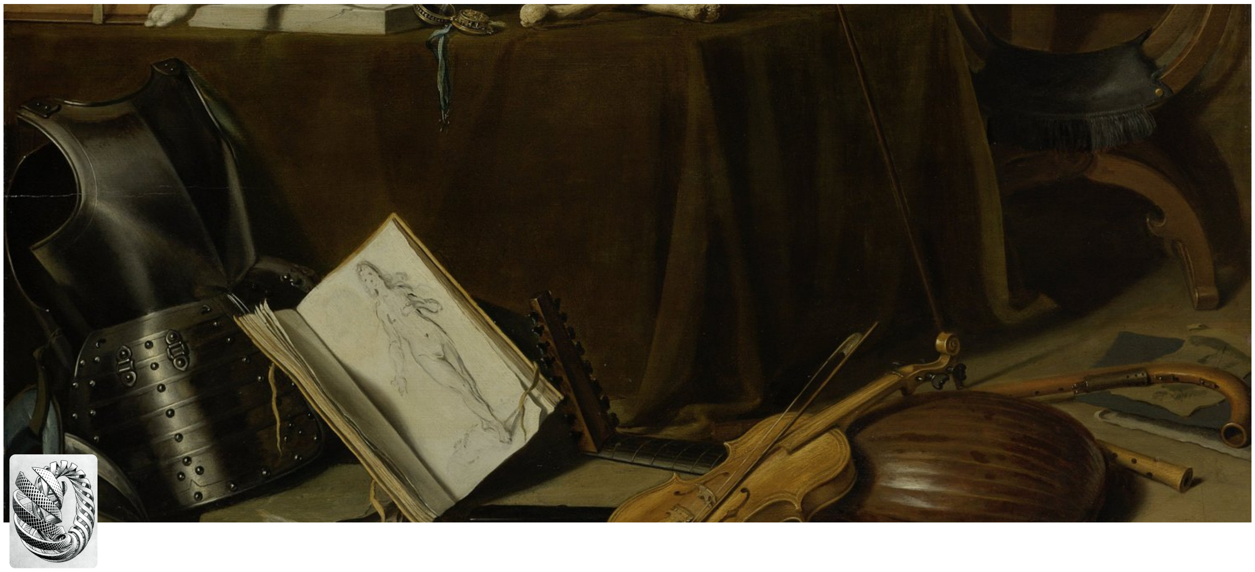


PI, Ch 2: Constructing Manifolds

In this chapter we will explore the category of manifolds. That is the collection of all manifolds and the maps between them which preserve the structure of a manifold. Additionally we explore a particularly important kind of manifold, a bundle, as well as the category thereof.

Category of Manifolds

Setup:

For this chapter $(M, \tau_M, \mathcal{A}_M)$ is assumed to be a real analytic manifold. Its charts are given by $\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^m$. Additionally so is $(N, \tau_N, \mathcal{A}_N)$ with charts $\rho_\beta : S_\beta \to T_\beta \subseteq \mathbb{R}^n$.

Recall: Analytic Maps and Diffeomorphisms

We already know that the morphism in this category are the (analytic) maps $\Psi \in C^\omega(M, N)$ with bijective analytic maps being called diffeomorphisms.

A map $\Psi : M \to N$ is called **analytic** if $\forall p \in M : \forall \phi_\alpha \in \mathcal{A}_M(p)$ and $\forall \rho_\beta \in \mathcal{A}_N(\Psi(p))$, if $\Psi_\alpha = \Psi|_{U_\alpha}$, (where $\Psi_\alpha : U_\alpha \to \Upsilon_\alpha \subseteq N$) and $\tilde{\rho}_{\alpha\beta} = \rho_\beta|_{\Upsilon_\alpha \cap S_\beta}$, then the following map $\Psi_{\alpha\beta}$

$$\Psi_{\alpha\beta} = \tilde{\rho}_{\alpha\beta} \circ \Psi_\alpha \circ \phi_\alpha^{-1} : V_\alpha \subseteq \mathbb{R}^m \to T_\beta \subseteq \mathbb{R}^n$$

Being called **the representation of Ψ with respect to charts ϕ_α, ρ_β ,** is analytic as a function from (an open subset of) $\mathbb{R}^m$ to (an open subset of) $\mathbb{R}^n$. That is $\Psi_{\alpha\beta} \in C^\omega(V_\alpha, T_\beta)$.

Moreover, if Ψ is bijective and its inverse is analytic then it is called a **diffeomorphism**.

💡 Is Ψ is a diffeomorphism of M and N, then $\Psi_{\alpha\beta}$ is a diffeomorphism of V_α and T_β

💡 Obviously every diffeo is a homeo

Def: *Submanifold and Embeddings*

A **submanifold** $\tilde{M} \subseteq_{\mathcal{M}fd} M$ is a sub-topological space which also happens to be a real analytic manifold.

An **embedding** is a analytic map, which is a diffeomorphism between its domain and its range.

The inclusion map $id|_{\tilde{M}}$ imbeds $\tilde{M}$ (or by composition any diffeomorphic space) into M.

Example: *Submanifold*

The first is that $S^n \subseteq \mathbb{R}^{n+1}$, or $SU(2), SL(2,\mathbb{C}) \subseteq GL(2,\mathbb{C})$. Since $S^3 \cong \ SU(2)$ it follows that the latter imbeds into $\mathbb{R}^4$.

Def: *Product Manifold*

The **product manifold** $M \times N$ is a product topological space, with a **product atlas** containing **product charts.** Product charts are simply $\chi_{\alpha\beta} = (\phi_\alpha, \rho_\beta) : U_\alpha \times S_\beta \to V_\alpha \times T_\beta$, such that we have $\forall (p,q) \in U_\alpha \times S_\beta : \chi_{\alpha\beta}(p,q) = (\phi_\alpha(p), \rho_\beta(q))$ and this is a chart as $V_\alpha \times T_\beta \subseteq \mathbb{R}^{m+n}$. The product atlas contains all product charts, and any other compatible charts that can be defined.

Example: *The Cylinder*

The cylinder (of infinite length) $C = S^1 \times \mathbb{R}$

Example: *The Torus*

The n-Torus $T^n = (S^1)^{n+1}$, has a natural pair of coordinates representing the angle from each of its member circles.

Bundles

💡 Here, our definitions are technically for **analytic bundles,** but one could replace our definition of a manifold as an **analytic manifold** to any other type and define that type of bundle.

Def: *Bundles*

Bundles, are a generalization of a product manifold. If E, M are manifolds and $\pi \in C^\omega(E, M)$ then the triplet (E, π, M) is called a **bundle**. Moreover, $fib(p) = preim_\pi\{p\}$ is called **the fibre at p.** The space E is called the **total space** (and sometimes in bad terminology the bundle) and the space M is called the **base space.** A bundle in which $\forall p \in M : fib(p)$ is a manifold diffeomorphic to F, is called a **fibre bundle with typical fibre F.**

Example: *Trivial (Fibre) Bundle is a Product Manifold*

A trivial (fibre) bundle is a bundle whose total space is diffeomorphic to the product of its base space with its typical fibre. e.g. $E = M \times F$. It has the natural projection map $\pi = proj^1$. Obviously C can be thought of as a trivial fibre bundle of S^1 with fibre $\mathbb{R}$. Alternatively T^1 can be thought of as a trivial fibre bundle of S^1 with fibre S^1.

Example: *Vector Bundles*

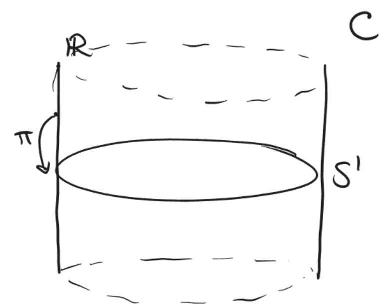
A V- vector bundle over a manifold $M$is defined as follows. Let $\forall p \in M : V_p$ be a real vector space isomorphic to V with dimension n . It was shown in the last chapter to be a manifold with a global chart $\xi_\alpha : V_p \to \mathbb{R}^n$ induced by some choice of basis.

Next define $VM = \bigcup_{\forall p \in M} V_p$. Now VM is a topological space with the disjoint union topology.

For every chart $\phi_\alpha \in \mathcal{A}(p)$ and every chart $\xi_\alpha \in \mathcal{A}_{V_p}$, we get a chart $\chi_{\alpha\beta} : VM \to V_\alpha \times \mathbb{R}^n$ with the definition $\chi_{\alpha\beta}(X) = (\phi_\alpha(p), \xi_\beta(X)) \; \forall X \in V_p$. Hence VM is a manifold.

Lastly, the projection map $\pi : VM \to M$ is simply defined by $\pi(X) = p \; \forall X \in V_p$. And so (VM, π, M) is a fibre bundle with typical fibre V for any vector space V and is called a V**-vector bundle over M.**

Notably since $\mathbb{R}$ is a vector space C can also be thought of as a $\mathbb{R}$-vector bundle over S^1.



Intuition: *A Low Dimensional Vector Bundle*

Def: *Sections*

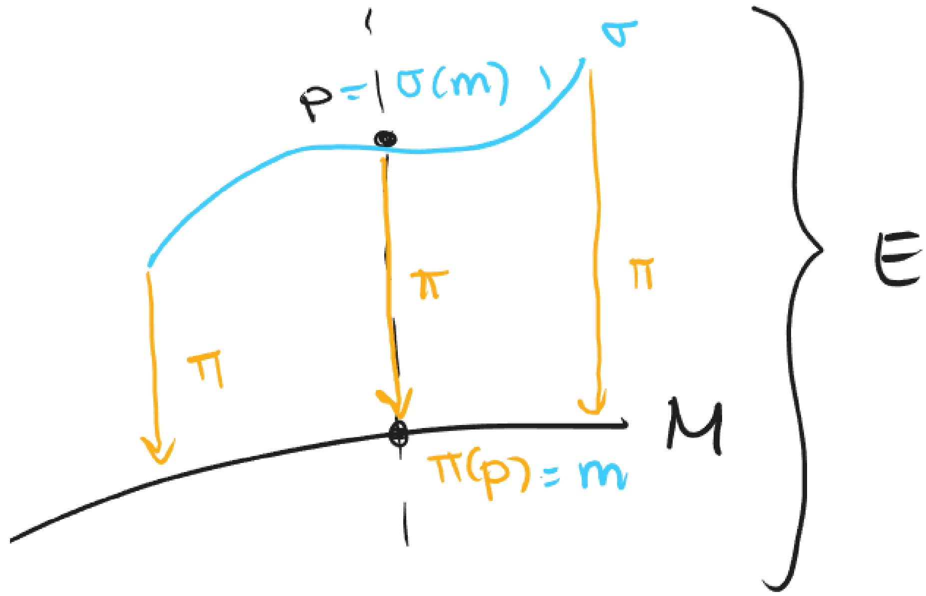
If (E, π, M) is a bundle, then a section of this bundle is $\sigma \in C^\omega(M, E)$ such that $\pi \circ \sigma = id_M$. We will call the set of all sections $\Gamma(E, \pi, M) = \{\sigma \in C^\omega(M, E) | \pi \circ \sigma = id_M\}$

💡 On a trivial fibre bundle, we are allowed to think of a section as $s : M \to F$ and take $\sigma_s(p) = (p, s(p))$, as without loss of generality

$$(proj^1 \circ \sigma_s)(p) = proj^1(p, s(p)) = p$$

However, we cannot generically think of a section as a map from a manifold to its typical fibre.

Intuition: *A Section*



Category of Bundles

Setup

Let $(E, \pi, M), (D, \chi, N)$ be bundles and let $\tilde{E} \subseteq E, \tilde{M} \subseteq M$ throughout this section

Def: *Bundle-morphism*

A **bundle-morphism** $\varrho : (E, \pi, M) \to (D, \chi, N)$ is a pair $\varrho = (f, g)$ where the functions are $f \in C^\omega(E, D)$ & $g \in C^\omega(M, N)$, s.t. the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{f} & D \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \chi} \\ M & \xrightarrow[g]{} & N \end{array}$$

$$\implies g \circ \pi = \chi \circ f$$

A **bundle isomorphism** $\varrho = (f, g)$ where f, g are diffeomorphisms and (f^{-1}, g^{-1}) is also a bundle-morphism called the **inverse bundle-morphism** ϱ^{-1} **.**

Def: *Sub-bundles*

A **sub-bundle** of (E, π, M) is a triplet $(\tilde{E}, \pi|_{\tilde{E}}, \tilde{M})$ such that $im_\pi \tilde{E} = \tilde{M}$. In particular every submanifold of M generates a sub-bundle by defining $\tilde{E} = preim_\pi \tilde{M}$. This is called the **restricted sub-bundle** to $\tilde{M}$

Def: *Local Bundle Isomorphism*

if $\forall p \in M : \forall U \in \mathcal{N}(p)$ the restricted bundle generated by $U,\ (\tilde{E}, \pi, U)$ is bundle isomorphic to the bundle (D, χ, N) then we say that (E, π, M) is **locally bundle isomorphic** to (D, χ, N).

If a bundle is locally isomorphic to a trivial bundle it is called **locally trivial.**

💡 There are obvious parallels to a local homeomorphism or chart here.

Example: *The Mobius Strip*

The Mobius strip is locally trivial whereas the cylinder is trivial, but both are $\mathbb{R}$ vector bundles over S^1.

Def: *Pullback Bundle*

Lastly, if N is a mfd with no given bundle structure, and $\Psi : N \to M$, where (E, π, M) is a given bundle, then we may "pullback" the bundle via the following construction

Define $D_\Psi = \{(q, e) \in N \times E | \Psi(q) = \pi(e)\}$, then we simply get the following commutative diagram.

$$\begin{array}{ccc} D_\Psi & \xrightarrow{proj^2} & E \\ {\scriptstyle proj1}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\Psi} & M \end{array}$$

And hence the **pullback bundle** $(D_\Psi, proj^1, N)$ is one such that $(proj^2, \Psi)$ is a bundle-morphism.

To summarize here, a bundle is just two manifolds glued together in some way. In the symplest way like a cylinder is just a product, but we may complicate bundles to make them more general.
