PII, Ch 7: Lie Groups and Their Lie Algebras

Now that the foundations of differential geometry are out of the way we can move towards an understanding of gauge theory. To get there we need to begin with this chapter on Lie groups. Here I assume at least in passing an understanding of Lie algebras. The more important factor in this chapter is understanding Lie groups in terms of the language we set up in Part I. This means charts, pushforwards and pullbacks, flows. We will additionally see that in the case of a matrix group (as we are used to dealing with in physics) this topic reduces to a very simple algebraic exercise.

Left Invariant Vector Fields and The Lie Algebra $\mathfrak{g}$ of G

Def: *Lie Groups*

A **Lie Group** G is a group which is4 also a smooth manifold, and whose product and inverse functions are also smooth. In a Lie Group we have two clearly defined auto-diffeomorphisms called **left multiplication** by g, l_g and **conjugacy** by g, α_g.

$$\begin{aligned} l_g &: G \to G \\ h &\to gh \end{aligned} \qquad\qquad \begin{aligned} \alpha_g &: G \to G \\ h &\to ghg^{-1} \end{aligned}$$

Both functions are incredibly important but for now we focus on l_g.

Def: *Left Invariant Vector Fields*

We call a vector field $X \in \Gamma^1_0 G$ to be **left invariant** if $X \in \mathrm{Left}(G) = \bigcap_{g\in G} \mathrm{Inv}_{l_g} G$. That is $\forall g \in G$ we have that $\forall f \in \Gamma^0_0 G$, $l_g^*(X \triangleright f) = X \triangleright l_g^* f$. We know from a previous section that $\mathrm{Left}(G)$ is a (real) Lie Subalgebra of $\Gamma^1_0 G$, but we intend to prove that it is also a finite dimensional subalgebra.

Theorem: $\mathrm{Left}(G) \cong_{VS} T_{\mathbf{1}} G$

Proof: *By Construction*

Lets define the map $\eta : T_{\mathbf{1}} G \to \mathrm{Left}(G)$ as follows; $\eta(A)$ is that vector field which $\forall g \in G$ satisfies $\eta(A)|_g = l_{g*}|_{\mathbf{1}} A$.

Is this map well defined? Since l_g are all smooth and $l_g \circ l_h = l_{gh}$ this is in fact a smooth section of the vector bundle. But is it left invariant? Consider

$$\begin{gathered} [l_h^*(\eta(A) \triangleright f)]|_g = [\eta(A) \triangleright f]_{hg} = \eta(A)|_{hg} \triangleright f = l_{hg*}|_{\mathbf{1}} A \triangleright f = l_{h*}|_g l_{g*}|_{\mathbf{1}} A \triangleright f = l_{h*}|_g \eta(A)|_g \triangleright f = [\eta(A) \triangleright l_h^* f]|_g \\ \implies l_h^*(\eta(A) \triangleright f) = \eta(A) \triangleright l_h^* f. \\ \eta(A) \in \mathrm{Left}(G) \end{gathered}$$

Here we used the fact that $l_{hg*}|_{\mathbf{1}} = l_{h*}|_g l_{g*}|_{\mathbf{1}}$. So it is well defined.

Is it linear? Yes because l_{g*} is a linear function .

Is it injective? Yes since l_g is a bijection it follows that $\ker l_{g*}|_{\mathbf{1}} = \{0\}$.

Is it surjective? Consider the vector field $X \in \text{Left}(G)$. Take $\eta(X|_{\mathbf{1}})|_g = l_{g*}|_{\mathbf{1}} X|_{\mathbf{1}} = (l_{g*}X)|_{\mathbf{1}}$. Using left invariance we have $l_{g*}X = X \circ l_g$ and so $(l_{g*}X)|_{\mathbf{1}} = X|_{l_g \mathbf{1}} = X|_g$ Hence for any vector field $X|_g = \eta(X|_{\mathbf{1}})|_g$. That is for any $X \in \text{Left}(G)$ there exists $A \in T_{\mathbf{1}}G$ such that $\eta(A) = X$ just choose $A = X|_{\mathbf{1}}$. E.g.

$$\eta(X|_{\mathbf{1}}) = X$$

QED.

Corollary:

We have in this process defined the functions η, η^{-1} given by the following $\eta(A)|_g = l_{g*}|_{\mathbf{1}} A$ and $\eta^{-1}(X) = X|_{\mathbf{1}}$

Now we know that $\text{Left}(G)$ has the structure of a Lie Algebra, we may inherit that structure to $T_{\mathbf{1}}G$ by using η as follows

Def: *Lie Algebra $\mathfrak{g}$ of a Lie Group G*

The **Lie Algebra $\mathfrak{g}$ of a lie Group** G is $\mathfrak{g} = T_{\mathbf{1}}G$ equipped with the bracket $[\_,\_] : \mathfrak{g}^2 \to \mathfrak{g}$ given as follows: $\forall A, B \in \mathfrak{g}$ we have

$$[A, B] = \eta^{-1}([\eta(A), \eta(B)])$$

Where the bracket $[\eta(A), \eta(B)]$ is taken in $\text{Left}(G)$.

> 💡 We change our terminology somewhat. The left invariant vector field $\eta(A)$ will be written X^A and will be called the **vector field generated by A**. In this context $[A, B] = [X^A, X^B]|_{\mathbf{1}}$.

Exponential Flow of X^A

We begin with a theorem about integral curves of left invariant vector fields

Theorem:

$$\begin{gathered}\forall \gamma_{X^A} \in \text{Curves}(\mathbf{1}), \forall s \in \mathbb{R} : \gamma^s_{X^A} \in \text{Curves}(\gamma_{X^A}(s)) \\ \implies \gamma^s_{X^A}(t) = \gamma_{X^A}(s)\gamma_{X^A}(t) = \gamma_{X^A}(s+t)\end{gathered}$$

Proof:

The fact that $\gamma^s_{X^A}(t) = \gamma_{X^A}(s)\gamma_{X^A}(t)$ is obvious, it comes from the fact that X^A is left invariant, and so is in particular invariant under left multiplication by $\gamma_{X^A}(s)$. Then the fact that $l_g(\gamma_{X^A}(t)) \in \text{Curves}(g)$ follows directly from a lemma in the section introducing integral curves. The latter part is a little harder to prove, but not impossible. $\gamma_{X^A}(s+t)|_{t=0} = \gamma_{X^A}(s)$ and so it is also in $\text{Curves}(\gamma^A_X(s))$ we need only show that it is integral to X^A and this follows from the fact that it satisfies the differential equation

$$\frac{d}{dt}(\phi^\mu \circ \gamma_{X^A}(s+t)) = \frac{d}{d(s+t)}(\phi^\mu \circ \gamma_{X^A}(s+t))\frac{d(s+t)}{dt} = X^A|^\mu_{\gamma_{X^A}(s+t)}$$

And by the uniqueness of solutions to ODEs with initial conditions it follows that the three are equal.

Corollary

$$\gamma_{X^A}(s)\gamma_{X^A}(t) = \gamma_{X^A}(t)\gamma_{X^A}(s)$$
$$\gamma_{X^A}(0) = \mathbf{1} \implies \gamma_{X^A}(t)^{-1} = \gamma_{X^A}(-t)$$

One last note that we can point out

$$\frac{d}{dt}(\phi^\mu \circ \gamma_{X^A})(0) = X^A|_{\mathbf{1}}^\mu = A^\mu$$

We compare this to the definition of e^{tA} in $\mathbb{R}$

$$\frac{d}{dt}e^{tA}|_0 = A$$

All of these properties lead to the following definition

Def: *Exponential of a Lie Algebra*

The **exponential map** for a lie group G is $\exp : \mathfrak{g} \to G$ which is given as follows, $\exp(A) = \gamma_{X^A}(1)$.

💡 An important note is that $\exp$ is not usually a bijection. This is because as $\mathfrak{g}$ is first of all a vector space meaning that topologically it is non compact. Whereas G may be compact. Hence as $\exp$ is smooth and compactness is a topological invariant it follows that $\exp$ is generically not an injection. Moreover if $g = \exp(A)$ then g is connected to the identity by γ_{X^A}. Not every $g \in G$ has this property as G may be disconnected. So in fact $\operatorname{rng} \exp = G^+$ what we call the path connected subgroup. This means that any lie algebra $\mathfrak{g}$ may be the lie algebra of several different lie groups $G_1, G_2, \ldots$. Which lie groups and algebras correspond would be in a different chapter.

Corollary: *Properties of the Exponential Map*

$$\exp(tA) = \gamma_{X^A}(t)$$
$$\exp(0) = \mathbf{1}$$
$$\exp(tA)\exp(sA) = \exp((s+t)A) = \exp(sA)\exp(sA)$$
$$\exp(tA)^{-1} = \exp(-tA)$$

Lemma: *Baker-Campbell-Haussdorff Formula*

The following is stated but not proven (as it is tedious)

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \ldots\right)$$

Def: *Adjoint Map*

Recall the definition of $\alpha_g(h) = ghg^{-1}$ as a diffeomorphism. Note that in particular $\alpha_g(\mathbf{1}) = g\mathbf{1}g^{-1} = gg^{-1} = \mathbf{1}$, and so the map $\alpha_{g*}|_{\mathbf{1}} : \mathfrak{g} \to \mathfrak{g}$, so the conjugacy defines a clear automorphism on the lie algebra. This automorphism is called the **adjoint map** $\mathrm{Ad}_g = \alpha_{g*}|_{\mathbf{1}}$. Importantly this defines a map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ which is a representation of G called its **adjoint representation**.

Theorem : *Adjoint-Exponential Theorem*

$$\mathrm{Ad}_{\exp(tA)} = e^{t\mathrm{ad}_A}$$

Proof:

Note that $e^{t\mathrm{ad}_A}$ satisfies the following ODE $\frac{d}{dt}e^{t\mathrm{ad}_A}|_0 = \mathrm{ad}_A$ with the initial condition $e^{0\mathrm{ad}_A} = I$ the identity transformation on $\mathfrak{g}$. By the uniqueness theorems if $\mathrm{Ad}_{\exp(tA)}$ also satisfies these conditions they must be equal. One can check this rather quickly.

Maurer-Cartan Form

There is another natural mapping, this one is from $\Gamma^1_0 G \to \mathfrak{g}$ and as such lives in $\Omega^1 G \otimes \mathfrak{g}$, it is called the Maurer-Cartan Form

Def: *Maurer-Cartan Form*

The **Maurer-Cartan Form** $\Xi \in \Omega^1 G \otimes \mathfrak{g}$ is a $\mathfrak{g}$ values 1 form on G defined in the following way

$$\Xi|_g \triangleright X = (l_{g^{-1}*} X)|_g$$

Since $l_{g^{-1}*} X|_g \in T_{g^{-1}g} G = T_{\mathbf{1}} G = \mathfrak{g}$ it follows that Ξ is lie algebra valued and hence well defined.

Lemma: *MC Form on Left Invariant Vector Fields*

Take $X^A \in \text{Left}(G)$, then

⁍

Proof:

For any $g \in G$ we have

$$\Xi|_g \triangleright X^A = (l_{g^{-1}*} X^A)|_g = X^A \circ l_{g^{-1}}|_g = X^A|_{\mathbf{1}} = A$$

> 💡 Now this chapter ends on the Maurer-Cartan form. At this point it's use is unclear but we will need it when we later tackle gauge transformation behaviours, so we'll leave it at that.

So in this chapter we gave our first real example of a manifold, a Lie group. We then explored the various maps, and algebras associated with this group in a good bit of detail. In the subsequent chapter we will use this manifold to construct others.
