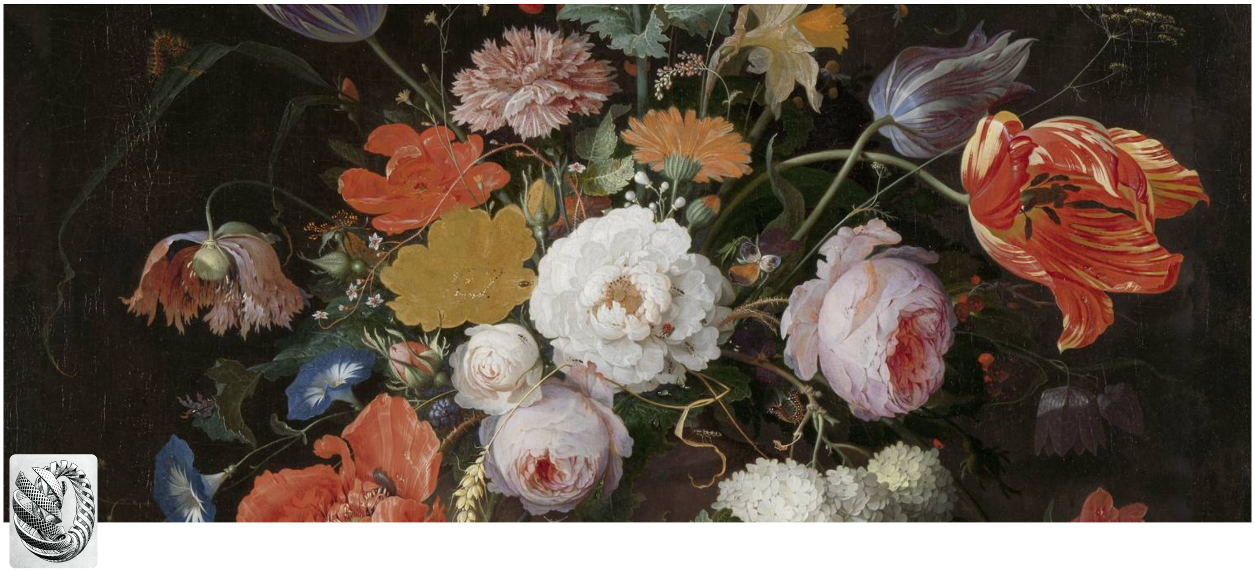

Conclusion

If we wish to bring all that we've discussed into a more tangible form, and answer the questions from the beginning we would do so as follows.

Given the elementary notions of a manifold M, or a bundle, and on it some of the fundamental constructions like tangent bundles and importantly ΩM we set out to define a group action for a Lie group G, thereby getting a principal bundle. As an example $\pi : M \times G \to M$ being a trivial G fiber bundle, under right action by G in the obvious way. On this bundle lies a connection 1 form $\omega \in \Omega^1 P \otimes \mathfrak{g}$.

Next we restrict attention to a local section $\sigma : U \to U \times G$, that is a gauge. For this choice of gauge ω is represented by a gauge potential $A = \sigma^* \omega$. We may not ever know the actual connection form ω or even the gauge σ, but if we know A and posit a gauge transformation $g : U \to G$ it follows that any other gauge potential is expressible as $\mathrm{Ad}_{g^{-1}} A + g^* \Xi_g$.

Now completely separate from this we have an associated bundle whose fibre F is some representation space for G.

Sections v of this bundle are locally given by functions $s : P \to F$ called gauge representatives satisfying the following behaviour under a gauge transformations $g^{-1} \triangleright s$. If we think of gauges like bases in linear algebra, then gauge representatives are the components. Importantly like vectors under a change of bases, sections are gauge invariant.

There is a natural derivative acting on these gauge representatives that transforms as they do. That is it is gauge covariant, defined as follows

$$\nabla_X s = ds \triangleright X + \tilde{\rho}[A \triangleright X](s)$$

This gauge covariance is desired as it leaves the section $\nabla_X s$ represents gauge invariant like the section s represents.

As an example take $M = \mathbb{R}^4$, $G = U(1)$ and $F = \mathbb{C}$. Then gauge potentials are $A \in \Omega^1 \mathbb{R}^4 \otimes \mathfrak{u}(1) = \mathbb{R}^4 \otimes i\mathbb{R} = i\mathbb{R}^4$. In physics the gauge maps are taken to be the imaginary part of A, but that is just convention. Moreover gauge transformations are maps $g : \mathbb{R}^4 \to U(1)$. That is $g(x) = e^{i\alpha(x)}$. Thus we know that under a gauge transformation

$$A \to g^{-1}Ag + g^{-1}dg = A + g^{-1}g(id\alpha) = A + id\alpha$$

Which we recognize from scalar electrodynamics. Now we have our gauge representatives $\varphi : \mathbb{R}^4 \to \mathbb{C}$ satisfying

$$\varphi \to g^{-1}\varphi = e^{-i\alpha}\varphi$$

Additionally we get the gauge covariant derivative of these representatives

$$\nabla_{\frac{\partial}{\partial x^\mu}} \varphi = d\varphi \triangleright \frac{\partial}{\partial x^\mu} + \left(A \triangleright \frac{\partial}{\partial x^\mu}\right)(\varphi) = \partial_\mu \varphi + A_\mu \varphi = (\partial_\mu + A_\mu)\varphi$$

Which we again recognize. In some sense we've answered all of the questions we set out to, but we are left with another one. Was all of this worth it to rederive something we already know? Setting aside reasons about the importance of rigour, I would argue that there is a greater prize here. Namely consider instead what happens if instead of $U(1)$ we choose $SO(1,3)$ and instead of $\mathbb{C}$ we choose $\mathbb{R}^4$, then instead of the electrodynamic covariant derivative we instead get the covariant derivative from general relativity (there are additional restrictions involving the metric and the relation between these kinds of associated bundles and the tangent bundles we originally constructed here but that is beyond the scope of this argument). The convention there would be to write A as Γ and F as R, which are called the Christoffel symbols and Riemann Curvature Tensor respectively. What if instead we choose $SU(3)$ and pick $\mathbb{C}^3$? The new theory is one of chromodynamics. At all three levels the math is the same.

But in the process of generalizing these theories under the same math we have acquired a whole slew of new theories to add to our mathematical toolbelt for use completely outside of gauge theory.

Like I mentioned in the introduction there are a number of ways to go from here. Firstly there is the matter of integration, but setting that aside, one might wonder how to take these tools and apply them to spinor electrodynamics and chromodynamics. While I don't quite know how specifically this works yet I am certain it involves the fact that we use a representation space F and not just the fundamental representation space (in much the same way $\mathbb{R}^3$ is a representation space for $SU(2)$ despite the fact that it is not the fundamental representation space $\mathbb{C}^2$).

Additionally we have Riemannian geometry wherein a metric naturally induces a connection called a Levi-Civita connection using the isometry group of the metric. Metrics also allow for the definition of the gauge-invariant scalar sometimes written $\mathrm{tr}(F^{\mu\nu}F_{\mu\nu})$ and hence allow for a full Gauge Theoretic Lagrangian (that being the true domain of Yang-Mills theory). In a completely different branch is Hamiltonian mechanics which in its mathematical form is symplectic geometry, utilizing groups like $Sp(n)$.

In the end, the goal was a success. I went from first principles to understand where the gauge behaviour we are given heuristically in class comes from. In the process however I gained a rich understanding of the formalism of differential geometry, one which I may apply to a wide range of problems in the future as a result of this project.