PII, Ch 8: Bundles with Lie Group Actions

Roughly the argument in this chapter is as follows. What if we had the action of a Lie group on a manifold. This isn't new, we see it all the time like when a subgroup $GL(n)$ acts on $\mathbb{R}^n$. However now we'll define such actions for more complicated manifolds. In the process we will define principal bundles.

Actions of A Lie Group on a Manifold

Def: *Actions*

Recall that a **right (resp. left) action** of a group G on a manifold M is a map $\triangleleft : M \times G \to M$ that satisfies a list of properties

$$
\begin{gathered}
p \triangleleft \mathbf{1} = p \\
(p \triangleleft g) \triangleleft h = p \triangleleft gh
\end{gathered}
$$

For left actions we have $\triangleright : G \times M \to M$ satisfying the same properties from the left. If M is a manifold and G a lie group we also require $\triangleleft$ to be smooth.

Def: ***Stabilizer of an Action***

The **stabilizer of a point** $p \in M$ is the set $\mathrm{stab}(p)$ defined as follows

$$\mathrm{stab}(p) = \{g \in G | p \triangleleft g = p\}$$

Lemma: ***The Stabilizer is a Subgroup***

$$\mathrm{stab}(p) \subset_{\mathcal{G}rp} G$$

Proof:

$$\begin{gathered} g, h \in \mathrm{stab}(p) \implies p = p \triangleleft g = p \triangleleft h \\ p \triangleleft (gh) = (p \triangleleft g) \triangleleft h = p \triangleleft h = p \implies gh \in \mathrm{stab}(p) \end{gathered}$$

QED.

Def: ***Free Actions***

An action is free if and only if $\forall p \in M : \mathrm{stab}(p) = \{\mathbf{1}\}$, that is if all stabilizers are trivial.

Def: ***Orbit of an Action***

The **orbit of a point** $p \in M$ is the set $\mathrm{orb}(p) = \{q \in M | \exists g \in G : q = p \triangleleft g\}$.

Corollary:

Lemma: ***Orbits of Free Actions***

If an action is free then the $\forall q \in \mathrm{orb}(p), \exists! g \in G : q = p \triangleleft g$.

Proof:

Suppose there were two distinct g_1, g_2 then:

$$\begin{gathered} p \triangleleft g_1 = p \triangleleft g_2 \\ (p \triangleleft g_1) \triangleleft g_2^{-1} = p \triangleleft \mathbf{1} \\ p \triangleleft (g_1 g_2^{-1}) = p \\ \implies g_1 g_2^{-1} \in \mathrm{stab}(p) \end{gathered}$$

But the action is free so we are forced into $g_1 g_2^{-1} = \mathbf{1} \implies g_1 = g_2$, which is a contradiction. Hence there exists only 1.

Theorem:

Under a free action $\text{orb}(p) \cong_{\mathcal{L}ie\mathcal{G}} G$

Proof:

First we need to define a group structure on $\text{orb}(p)$, we can do this as

$$q_1 q_2 = q \iff q_i = p \triangleleft g_i, q = p \triangleleft g_1 g_2$$

Thus multiplication is well defined as $g_1, g_2, g_1 g_2$ are unique to q_1, q_2 and q. From this associativity follows trivially, the existence of an identity is clear as $p = p \triangleleft \mathbf{1}$ so $pq = q$, and inverses is simply $q^{-1} = p \triangleleft g^{-1}$ where $q = p \triangleleft g$. Now the map $\varphi : p \triangleleft g \to g$ is clearly a surjection as we have already constructed the domain element for any g. Proving it is an injection takes some more effort but we do it as follows;

$$\text{ker}\varphi = \{q \in \text{orb}(p) | \varphi(q) = \mathbf{1}\}$$

But $\varphi(q) = \mathbf{1} \iff q = p \triangleleft \mathbf{1} = p$ and hence $\ker \varphi = \{p\}$ which is trivial, so the map is also an injection and therefore an isomorphism of groups. Lastly $\text{orb}(p)$ is a sub-manifold of M and so it follows by the smoothness of the map $\triangleleft$ that φ is also two ways continuous, ergo the map is actually a lie group homomorphism. QED.

Def: *Equivalence Relation of an Action*

Assuming $\triangleleft$ a right G action on M we define the following **equivalence relation** $\forall q_i \in M$: $q_1 \equiv q_2 \iff q_1 \in \text{orb}(q_2)$. That this is an equivalence relation does still need to be proved but it is very straight forward.

Lemma:

The above is well defined as an equivalence relation and furthermore $[p] = \text{orb}(p)$. The proof is omitted as it is trivial.

Def: *Quotient Space*

As a result we define $M/G = M_{\equiv} = \bigcup_{p\in M} \operatorname{orb}(p)$ is called the **quotient space** and furthermore we get the canonical projection map $\operatorname{orb}: p \to \operatorname{orb}(p)$

Def: *Equivariant Maps*

The morphism of manifolds with action is called an **equivariant map**. Let $\triangleleft_i$ be a right G_i action on M_i, then an equivariant map (ρ, ψ) with $\rho : G_1 \to G_2$ a lie group homomorphism and $\psi : M_1 \to M_2$ satisfies

$$\psi(p \triangleleft_1 g) = \psi(p) \triangleleft_2 \rho(g)$$

Which effectively means the natural diagram commutes. Sometimes we are only interested in equivariant maps with given ρ, such $\psi's$ are called ρ-equivariant maps.

Corollary: *Self-Equivariance*

If $G_1 = G_2$ then a natural choice is $\rho = \mathrm{id}_G$ and so a map ψ is self-equivariant if

$$\psi(p \triangleleft_1 g) = \psi(p) \triangleleft_2 g$$

Principle Bundles

Just as $T_pM \cong_{\mathcal{VS}} \mathbb{R}^m$ which allowed us to construct the bundle of $\bigcup_{p\in M} T_pM$ with fibre $\mathbb{R}^m$ we have $\operatorname{orb}(p) \cong_{\mathcal{Lie}\mathcal{G}} G$ and so we would like to consider the bundle of $M/G = \bigcup_{p\in M} \operatorname{orb}(p)$ with fibre G. To do this we realize that we already have a bundle $(M, \varrho, M/G)$ which has fibre G.

Def: *Principle Bundles*

For any Lie Group G , if (P, π, M) is a bundle and $\triangleleft$ is a free right G action on P which is closed fibre-wise, with fibre-wise transitivity is called **a principle G bundle.** These last two conditions are as follows : $\forall x \in M$

$$\forall p \in \operatorname{fib}(x) \implies p \triangleleft g \in \operatorname{fib}(x)$$
$$\forall p, q \in \operatorname{fib}(x), \exists g \in G : p = q \triangleleft g$$

In some sense this is the natural way to extend actions to fibre bundles. Although we still need to show that a principle bundle is a fibre bundle.

Lemma:

$$\pi(p) = \pi(q) \iff \exists g \in G : q = p \triangleleft g$$

Proof:

We begin with the backwards proof. Suppose $q = p \triangleleft g$ then by closure we have that $q, p \in \mathrm{fib}(x)$ and hence $\pi(p) = \pi(q)$. Now lets prove the forwards statement by contradiction. Suppose $q \neq p \triangleleft g$ and that $\pi(p) = \pi(q)$. By the final statement we know that $q, p \in \mathrm{fib}(x)$ but by transitivity we know that $\exists g \in G : q = p \triangleleft g$, which violates the first assumption. Therefore we have proved the forward statement by contradiction. QED.

Theorem: ***Principle G Bundles are Fibre Bundles with Typical Fibre G.***

More precisely $\forall p \in P : \mathrm{fib}(\pi(p)) = \mathrm{orb}(p) \cong_{\mathcal{L}ie\mathcal{G}} G$.

Proof:

This follows immediately from the above lemma, since $\mathrm{fib}(\pi(p)) = \{q \in P : \pi(q) = \pi(p)\} = \{q \in P : \exists g \in G, q = p \triangleleft g\} = \mathrm{orb}(p) \cong_{\mathcal{L}ie\mathcal{G}} G$. And since π is surjective $\forall x \in M$ we can write $x = \pi(p)$ so we have $\mathrm{fib}(x) \cong_{\mathcal{L}ie\mathcal{G}} G$. Moreover since all of the fibres have an isomorphic structure it follows that a principle bundle is a fibre bundle. QED.

Def: ***Principle Bundle Morphisms***

A **map between principle G_i bundles** $(P_i, \triangleleft_i, \pi_i, M_i)$ is a triple (ρ, u, f) such that (u, f) is a bundle-morphism, $\rho : G_1 \to G_2$ is a homomorphism and u is ρ-equivariant. That is

$$f \circ \pi_1 = \pi_2 \circ u$$
$$p \triangleleft_1 g = u(p) \triangleleft_2 \rho(g)$$

In a principle bundle isomorphism f, u, ρ are all diffeomorphisms

Lemma:

Let P_1 and P_2 be the total spaces of two principle G bundles over M, then if there exists a smooth map $u : P_1 \to P_2$ satisfying the following properties then they are principle bundle isomorphic

$$
\begin{gathered}
\pi_1 = \pi_2 \circ u \\
u(p \triangleleft_1 g) = u(p) \triangleleft_2 g
\end{gathered}
$$

Proof:

Since the two base spaces are the same $f = \mathrm{id}_M$ and $\rho = \mathrm{id}_G$ which are obviously diffeomorphisms. And we note that if u satisfies the above properties then clearly

$$
\begin{gathered}
f \circ \pi_1 = \pi_2 \circ u \\
p \triangleleft_1 g = u(p) \triangleleft_2 \rho(g)
\end{gathered}
$$

as f and ρ are identity maps. Hence there exists a principle bundle isomorphism. we need only use these properties to show that u is a invertible. We begin by showing injectivity. Suppose $p, q \in P_1$ and that $u(p) = u(q)$ then we can take the following

$$
\pi_2(u(p)) = \pi_2(u(q)) \implies \pi_1(p) = \pi_1(q) \implies q = p \triangleleft_1 g
$$

But then $u(q) = u(p \triangleleft_1 g) = u(p) \triangleleft_2 g$. Which means that $u(p) = u(p) \triangleleft_2 g$. But since the action is free it must be that $g = \mathbf{1}$. Which means that $q = p$. QED for injectivity. Next for surjectivity, $\forall p' \in P_2$ take $p \in \mathrm{fib}(\pi_2(p'))$ this makes sense as the base space is the same, and so then it follows that $\pi_1(p) = \pi_2(p')$ then use the fact that $\pi_1(p) = \pi_2(u(p))$ to get $\pi_2(u(p)) = \pi_2(p')$ so $u(p) \in \mathrm{fib}(\pi_2(p'))$ which means that $u(p) \triangleleft_2 g = p'$. Then we have that $u(p \triangleleft_1 g) = p'$ and but $p \triangleleft_1 g = q$ so then we have $u(q) = p'$. QED.

Def: *Trivial Principle Bundles*

The following bundle $(M \times G, \mathrm{proj}^1, M)$ can be equipped with the right G action on $M \times G$ as follows $(x, g) \blacktriangleleft h = (x, gh)$. This action is free and fiber-wise transitive so this bundle is a principle G bundle. Any principle G bundle over M is called **trivial** if it is principle bundle isomorphic to this bundle. By the above theorem this means that there exists a map $u : P \to M \times G$ satisfying

$$
\begin{gathered}
\pi = \mathrm{proj}^1 \circ u \\
u(p \triangleleft g) = u(p) \blacktriangleleft g
\end{gathered}
$$

Theorem: ***A Principle Bundle with Global Sections is Trivial***

Let $\Gamma(P, \pi, M) \neq \emptyset$, then $(P, \triangleleft, \pi, M)$ is trivial.

Proof:

Suppose $\sigma \in \Gamma(P, \pi, M)$ then $\sigma(x) \in \operatorname{fib}(x)$ by definition. In particular then $\sigma(\pi(p)) \in \operatorname{fib}(\pi(p)) = \operatorname{orb}(p)$ and hence $\exists g \in G$ such that $\sigma(\pi(p)) \triangleleft g = p$. Importantly the value of g depends smoothly on p so define $\chi : P \to G$ satisfying

$$\sigma(\pi(p)) \triangleleft \chi(p) = p$$

Now lets act on both sides with an arbitrary element g so

$$\sigma(\pi(p)) \triangleleft \chi(p)g = p \triangleleft g$$

But by the above we know that $p \triangleleft g = \sigma(\pi(p \triangleleft g)) \triangleleft \chi(p \triangleleft g)$, combining this with the fact that $\pi(p \triangleleft g) = \pi(p)$ it follows that

$$\sigma(\pi(p)) \triangleleft \chi(p)g = \sigma(\pi(p)) \triangleleft \chi(p \triangleleft g)$$

Lastly by freedom we see that $\chi(p)g = \chi(p \triangleleft g)$. So to summarize, if there exists a map σ then there exists a map χ with the property $\chi(p)g = \chi(p \triangleleft g)$. Now define the map $u : P \to M \times G$ as follows, $u(p) = (\pi(p), \chi(p))$. Then we have that $\operatorname{proj}^1(u(p)) = \pi(p)$ by construction and

$$\begin{gathered} u(g \triangleleft g) = (\pi(p \triangleleft g), \chi(p \triangleleft g)) \\ = (\pi(p), \chi(p)g) = (\pi(p), \chi(p)) \blacktriangleleft g \\ = u(p) \blacktriangleleft g \end{gathered}$$

Since both conditions are satisfied it follows by the previous lemma that the two bundles are isomorphic.

So the first half of the chapter explains that there is a natural way a lie group can act on a manifold (in the form of a free action) and explores the fact that the orbits are isomorphic to the group. The second half uses that notion to design bundles whose fibres are the group itself (isomorphically) and then goes into detail on which of those bundles are trivial. Far from being a mathematical point, triviality will naturally lead to the requirement of a gauge.
