PI, Ch 3: Differential Calculus at a Point

This chapter is all about scalar fields and their derivatives (w.r.t a particular direction) at a point. It goes in depth on the proofs for all of the algebraic properties of the scalar fields , as well as the properties of the derivative operators. Lastly it is about the natural basis to work in, and how things change when the basis does. This section is also full of proofs, so take this as a warning for long proofs.

Scalar Fields and Curves.

We begin by looking at two types of analytic maps

Def: *Scalar Fields*

A **scalar field** on M is a map $f \in C^{\omega}(M, \mathbb{R})$, for convinence we will denote $C^{\omega}(M, \mathbb{R}) = \Gamma^0_0 M$. This notation will make sense in the next chapter.

Lemma: $\Gamma^0_0 M$ *is (pointwise) a Commutative and Unital Ring*

That is if we have the operations $\forall f, g \in \Gamma_0^0 M: \ f+g, fg \in \Gamma_0^0 M$ via pointwise definition

$$\forall p \in M : (f+g)(p) = f(p) + g(p), \ \ (fg)(p) = f(p)g(p)$$

Then the algebraic structure is a CU ring.

Proof:

CANI for addition derives from CANI on real addition, and CAN for nonzero multiplication derives from CAN on nonzero real multiplication.

> 💡 CANI and ADDU are abbreviations for the axioms of a vector space. C: commutativity, A: associativity N: existence of a neutral element I: existence of inverses all for the addition, and ADDU corresponds to the relation between addition and scalar multiplication with D: distributivity (of both kinds) and U: the property that 1 is the identity for scalar multiplication.

Notably, not every nonzero function has an inverse: suppose $f(q) = 0$ and suppose $\exists f^{-1} \in \Gamma_0^0 M$ then $|\lim_{p\to q} f^{-1}(p)| \to \infty$, ergo $f^{-1} \notin \Gamma_0^0 M$. So generically the scalar fields do not form an algebraic field.

Corollary: $\Gamma_0^0 M$ *is a Real Algebra*

This follows by defining the constant function $\bar{\lambda} \in \Gamma_0^0 M : \forall p \in M \ , \bar{\lambda}(p) = \lambda$, then scalar multiplication by λ is simply multiplication by the associated constant function.

Corollary: *Composition Identities*

$\forall f, g \in \Gamma_0^0 M, \ \forall \Psi \in C^\omega(N, M)$ and $\forall \alpha, \beta \in \mathbb{R}$ we have

$$\begin{gathered}(\alpha f + \beta g) \circ \Psi = \alpha f \circ \Psi + \beta g \circ \Psi \\ (fg) \circ \Psi = (f \circ \Psi)(g \circ \Psi)\end{gathered}$$

Which follow directly from the pointwise definition of the addition and multiplication.

Def: *Curves*

A **curve** on M is a map $\gamma \in C^\omega((-\epsilon, \epsilon), M)$ where $\epsilon \in \mathbb{R} \cup \{\pm\infty\}$. For any point $p \in M$ the set of all **curves centered at** p is

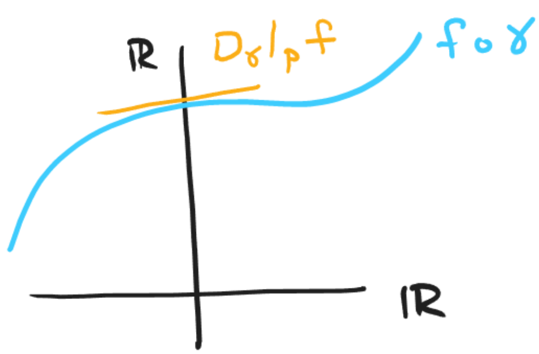
$$\text{Curves}(p) = \{\gamma \in C^\omega((-\epsilon, \epsilon), M) | \text{rng}\gamma \in \mathcal{N}(p) \text{ and } \gamma(0) = p\}$$

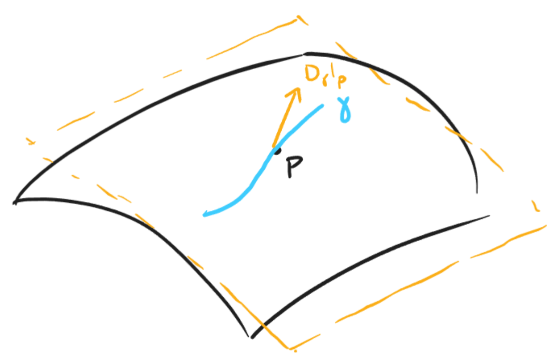
Directional Derivatives

Def: *Pointwise Directional Derivative by a Curve*

$\forall p \in M$ we define **the directional derivative at** p **by a curve** $\gamma \in \text{Curves}(p)$ to be $D_\gamma|_p$ defined as follows.

$$\begin{aligned} D_\gamma|_p : \Gamma^0_0 M &\to \mathbb{R} \\ f &\mapsto D_\gamma f|p \\ \text{s.t. } D_\gamma f|p &= D[f \circ \gamma](0) \end{aligned}$$





Lemma: *Directional Derivatives are Linear*

$\forall p \in M,\ \forall \gamma \in \text{Curves}(p),\ \forall f, g \in \Gamma^0_0 M$ and $\forall \alpha, \beta \in \mathbb{R}$, we have

$$D_\gamma(\alpha f + \beta g)|p = \alpha D_\gamma f|p + \beta D_\gamma g|p$$

Proof

$$D_\gamma(\alpha f + \beta g)|p$$
$$D[(\alpha f + \beta g) \circ \gamma](0)$$
$$D[\alpha f \circ \gamma + \beta g \circ \gamma](0)$$
$$\alpha D[f \circ \gamma](0) + \beta D[g \circ \gamma](0)$$
$$\alpha D_\gamma f|p + \beta D_\gamma g|p$$

Lemma: *This Identity (the "Almost Leibniz Law")*

$\forall p \in M,\ \forall \gamma \in \text{Curves}(p),\ \forall f, g \in \Gamma^0_0 M$ we have

$$D_\gamma(fg)|p = g(p) D_\gamma f|p + f(p) D_\gamma g|p$$

Proof

$$D_\gamma(fg)|p$$
$$D[(fg) \circ \gamma](0)$$
$$D[(f \circ \gamma)(g \circ \gamma)](0)$$
$$(g \circ \gamma)(0) D[(f \circ \gamma)](0) + (f \circ \gamma)(0) D[(g \circ \gamma)](0)$$
$$g(\gamma((0)) D[(f \circ \gamma)](0) + f(\gamma((0)) D[(g \circ \gamma)](0)$$
$$g(p) D_\gamma f|p + f(p) D_\gamma g|p$$

The Tangent Space

Def: *The Tangent Space*

$\forall p \in M$ **the tangent space** to M at p is $T_p M = \{D_\gamma|p : \Gamma^0_0 M \to \mathbb{R} | \forall \gamma \in \text{Curves}(p)\}$ the set of all directional derivatives to a curve for any curve. It may be equipped with addition and scalar multiplication pointwise. That is $\forall D_\gamma|_p, D_\delta|_p \in T_p M, \forall \alpha \in \mathbb{R}$ we have

$$\forall f \in \Gamma^0_0 M : (D_\gamma|_p + D_\delta|_p) f = D_\gamma f|_p + D_\delta f|_p,\ (\alpha D_\gamma|p) f = \alpha D_\gamma f|p$$

Lemma: *Well Definition*

It remains to be shown that these operations are closed in $T_p M$ as the result of a sum or a scaling might not be the directional derivative to a curve. Thus we must find curves which represent the above operations.

Proof: ***Addition is Closed***

Suppose $\phi \in \mathcal{A}(p)$ with domain $U$and call

$$\sigma = \phi^{-1} \circ (\phi \circ \gamma + \phi \circ \delta - \phi(p)\overline{\mathbf{1}})$$

> 💡 Where $\mathbf{1}$ is the vector in $\mathbb{R}^m$ containing ones in each entry, and $\overline{\lambda}$ is the constant function for the vector $\lambda \in \mathbb{R}^m$. Note that here addition makes sense as we are dealing with functions from $\phi \circ \gamma + \phi \circ \delta - \phi(p)\overline{\mathbf{1}} : \mathbb{R} \to \mathbb{R}^m$, and so we have naive "pointwise addition of vector fields".

Its not hard to convince yourself that $\sigma \in \text{Curves}(p)$, just plug in $\sigma(0) = p$. Now taking $\forall f \in \Gamma_0^0 M$ we get

$$D_\sigma|pf$$
$$D_\sigma f|p = D[f \circ \sigma](0)$$
$$D[(f \circ \phi^{-1}) \circ (\phi \circ \gamma + \phi \circ \delta - \phi(p)\overline{\mathbf{1}})](0)$$

Now $f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}$ and as previously mentioned $(\phi \circ \gamma + \phi \circ \delta - \phi(p)\overline{\mathbf{1}}) : \mathbb{R} \to \mathbb{R}^m$ so we need to use the chain rule

> 💡 if $\chi : \mathbb{R}^m \to \mathbb{R}^n$, and $\xi : \mathbb{R}^n \to \mathbb{R}^k$ then
>
> $$\partial_\nu[\text{proj}^\rho \circ \chi \circ \xi] = (\partial_\mu[\text{proj}^\rho \circ \chi] \circ \xi)\partial_\nu[\text{proj}^\mu \circ \xi]$$

but for us $\nu, \rho \in \{1\}$ so we may call $\chi = f \circ \phi^{-1}, \xi = \phi \circ \gamma + \phi \circ \delta - \phi(p)\overline{\mathbf{1}}$, then

$$D[\chi \circ \xi](0) = (\partial_\mu[\chi] \circ \xi)(0) D[\text{proj}^\mu \circ \xi](0)$$

Now

$$(\partial_\mu[\chi] \circ \xi)(0)$$
$$(\partial_\mu[\chi])(\xi(0))$$
$$\partial_\mu[f \circ \phi^{-1}](\phi(p))$$

and

$$\begin{aligned} \mathrm{proj}^\mu \circ \xi = \phi^\mu \circ \gamma + \phi^\mu \circ \delta - \phi^\mu(p)\overline{\mathbf{1}} \\ D[\mathrm{proj}^\mu \circ \xi] = D[\phi^\mu \circ \gamma + \phi^\mu \circ \delta - \phi^\mu(p)\overline{\mathbf{1}}] \\ = D[\phi^\mu \circ \gamma] + D[\phi^\mu \circ \delta] - \phi^\mu(p)D[\overline{\mathbf{1}}] \end{aligned}$$

by linearity. Since $\overline{\mathbf{1}}$ is a constant function $D[\overline{\mathbf{1}}] = 0$.

$$\begin{gathered} D[\mathrm{proj}^\mu \circ \xi] = D[\phi^\mu \circ \gamma] + D[\phi^\mu \circ \delta] \\ D[\mathrm{proj}^\mu \circ \xi](0) = D[\phi^\mu \circ \gamma](0) + D[\phi^\mu \circ \delta](0) \end{gathered}$$

Putting these terms together we get

$$\begin{gathered} D[\chi \circ \xi](0) \\ \partial_\mu[f \circ \phi^{-1}](\phi(p))(D[\phi^\mu \circ \gamma](0) + D[\phi^\mu \circ \delta](0)) \\ \partial_\mu[f \circ \phi^{-1}](\phi(p))D[\phi^\mu \circ \gamma](0) + \partial_\mu[f \circ \phi^{-1}](\phi(p))D[\phi^\mu \circ \delta](0) \end{gathered}$$

But we can clearly see that

$$\begin{gathered} \partial_\mu[f \circ \phi^{-1}](\phi(p))D[\phi^\mu \circ \gamma](0) \\ \partial_\mu[f \circ \phi^{-1}](\phi \circ \gamma)(0)D[\mathrm{proj}^\mu \circ \phi \circ \gamma](0) \\ D[f \circ \phi^{-1} \circ \phi \circ \gamma](0) \\ D[f \circ \gamma](0) \\ D_\gamma f|p \\ D_\gamma|pf \end{gathered}$$

And identically for δ. All in all this means that

And since this holds for all functions f, it follows that

$$D_\gamma|_p + D_\delta|_p = D_\sigma|_p$$

And so there is a curve whose directional derivative corresponds to the sum of the directional derivatives of any two curves. e.g. addition is closed.

Proof: Scaling is Closed

This one is easier. Let $\hat{\lambda}(t) = \lambda t : \forall \lambda, t \in \mathbb{R}$. Then taking

It is still easy to see that $\sigma \in \mathrm{Curves}(p)$. Now $\forall f \in \Gamma^0_0 M$ we get

$$D_\sigma|_p f = D[f \circ \sigma](p) = D[f \circ \gamma \circ \hat{\alpha}](0)$$

Now employ the one dimensional chain rule

$$D[f \circ \gamma \circ \hat{\alpha}](0) = D[f \circ \gamma](\hat{\alpha}(0))D[\hat{\alpha}](0)$$

clearly $D[\hat{\alpha}] = \overline{\alpha}$, and so $D[\hat{\alpha}](0) = \overline{\alpha}(0) = \alpha$, whereas $\hat{\alpha}(0) = 0$.

$$\begin{gathered} D[f \circ \gamma \circ \hat{\alpha}](0) = D[f \circ \gamma](0)\alpha = \alpha D_\gamma f|_p \\ \implies D_\sigma|_p f = (\alpha D_\gamma|_p) f \end{gathered}$$

And so there is a curve whose directional derivative corresponds to the scaling of the directional derivative of any given curve.

Corollary: *The Tangent Space is a Real Vector Space*

$T_p M$ is a $\mathbb{R}$ vector space under these operations, since CANI for addition comes from CANI on $\mathbb{R}$ and same for ADDU.

A Basis of The Tangent Space

Def: *Chart induced Curve*

Let $\phi \in \mathcal{A}(p)$, then $\tilde{\phi}_\nu \in \mathrm{Curves}(p)$ is the **curve induced by the chart** ϕ defined by

$$\phi^\mu \circ \tilde{\phi}_\nu = \hat{\delta}^\mu_\nu - \phi(p)\overline{1}$$

Intuition:

Def: *Chart Induced Directional Derivative*

The μ th directional derivative induced by the chart $\phi \in \mathcal{A}(p)$ is

$$\left.\frac{\partial}{\partial\phi^\mu}\right|_p = D_{\tilde{\phi}_\mu}|_p$$

Lemma: *This Identity*

$\forall f \in \Gamma^0_0 M$ we have

$$\left.\frac{\partial}{\partial\phi^\mu}\right|_p f = \partial_\mu[f \circ \phi^{-1}](\phi(p))$$

Proof:

$$\begin{gathered}\left.\frac{\partial}{\partial\phi^\mu}\right|_p f = D_{\tilde{\phi}_\mu}|_p f = D_{\tilde{\phi}_\mu} f|_p = D[f \circ \tilde{\phi}_\mu](0) \\ D[f \circ \phi^{-1} \circ \phi \circ \tilde{\phi}_\mu](0) \\ \partial_\nu[f \circ \phi^{-1}](\phi(\tilde{\phi}_\mu(0))D[\mathrm{proj}^\nu \circ \phi \circ \tilde{\phi}_\mu](0) \\ \partial_\nu[f \circ \phi^{-1}](\phi(p))D[\phi^\nu \circ \tilde{\phi}_\mu](0)\end{gathered}$$

Now

$$\begin{gathered}D[\phi^\nu \circ \tilde{\phi}_\mu] \\ D[\hat{\delta}^\mu_\nu - \phi(p)\overline{1}] \\ D[\hat{\delta}^\mu_\nu] - \phi(p)D[\overline{1}] \\ \overline{\delta^\nu_\mu} - \phi(p) * 0 \\ \overline{\delta^\nu_\mu}\end{gathered}$$

Which when evaluated at 0 gives δ^ν_μ. Hence

$$\begin{gathered}\left.\frac{\partial}{\partial\phi^\mu}\right|_p f = \partial_\nu[f \circ \phi^{-1}](\phi(p))\delta^\nu_\mu \\ \left.\frac{\partial}{\partial\phi^\mu}\right|_p f = \partial_\mu[f \circ \phi^{-1}](\phi(p))\end{gathered}$$

Lemma: ***A basis of the tangent space***

$\left\{ \frac{\partial}{\partial\phi^\mu}\Big|_p \right\}_\mu$ is a basis for T_pM.

Proof:

$\forall\gamma \in \text{Curves}(p)$, and $\forall f \in \Gamma^0_0 M$ we have

$$\begin{gathered} D_\gamma|_p f = D_\gamma f|_p = D[f\circ\gamma](0) \\ D[(f\circ\phi^{-1})\circ(\phi\circ\gamma)](0) \\ \partial_\mu[f\circ\phi^{-1}](\phi(\gamma(0))D[\text{proj}^\mu\circ\phi\circ\gamma](0) \\ \partial_\mu[f\circ\phi^{-1}](\phi(p))D[\phi^\mu\circ\gamma](0) \\ \frac{\partial f}{\partial\phi^\mu}\bigg|_p D_\gamma\phi^\mu|_p \\ D_\gamma|_p\phi^\mu \left.\frac{\partial}{\partial\phi^\mu}\right|_p f \end{gathered}$$

And since this holds for all functions we get

$$D_\gamma|_p = D_\gamma|_p\phi^\mu \left.\frac{\partial}{\partial\phi^\mu}\right|_p$$

Which expresses every element in T_pM as a span of this set. Lastly, note that if we want the left hand side to be zero, then by definition $D_\gamma|_p = 0 \implies D_\gamma|_p\phi^\mu = 0$ and so the set is also linearly independent.

Corollary:

$$D_\gamma|_p = D_\gamma|_p\phi^\mu \left.\frac{\partial}{\partial\phi^\mu}\right|_p$$

Corollary:

The tangent space has the same dimension as the underlying manifold. That is $\dim T_pM = \dim M = m$

Change of Basis

Theorem: ***Local Jacobian***

Suppose we have two charts $\phi, \psi \in \mathcal{A}(p)$, and for the sake of simplicity suppose they have a common domain (if not we can restrict both to the intersecting domain and obtain two new charts satisfying this property). Then it follows that we get two bases for T_pM being

$$\left.\frac{\partial}{\partial \phi^\mu}\right|_p, \left.\frac{\partial}{\partial \psi^\mu}\right|_p$$

Then it follows that we may change the basis as

$$\left.\frac{\partial}{\partial \phi^\mu}\right|_p = \left.\frac{\partial \psi^\nu}{\partial \phi^\mu}\right|_p \left.\frac{\partial}{\partial \psi^\mu}\right|_p$$

Proof

$\forall f \in \Gamma_0^0 M$ we get

$$\begin{gathered}
\left.\frac{\partial}{\partial \phi^\mu}\right|_p f = \partial_\mu[f \circ \phi^{-1}](\phi(p)) \\
\partial_\mu[(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})](\phi(p)) \\
\partial_\nu[f \circ \psi^{-1}](\psi(\phi^{-1}(\phi(p))))\partial_\mu[\text{proj}^\nu \circ \psi \circ \phi^{-1}](\phi(p)) \\
\partial_\nu[f \circ \psi^{-1}](\psi(p))\partial_\mu[\psi^\nu \circ \phi^{-1}](\phi(p)) \\
\left.\frac{\partial f}{\partial \psi^\nu}\right|_p \left.\frac{\partial \psi^\nu}{\partial \phi^\mu}\right|_p \\
\left.\frac{\partial \psi^\nu}{\partial \phi^\mu}\right|_p \left.\frac{\partial}{\partial \psi^\mu}\right|_p f
\end{gathered}$$

And since this holds for all functions we get

$$\left.\frac{\partial}{\partial \phi^\mu}\right|_p = \left.\frac{\partial \psi^\nu}{\partial \phi^\mu}\right|_p \left.\frac{\partial}{\partial \psi^\nu}\right|_p$$

Corollary

In a coordinatization, the components of a vector X^μ transform via

$$\tilde{X}^\nu = \frac{\partial \psi^\nu}{\partial \phi^\mu}\bigg|_p X^\mu$$

$$X^\mu = \frac{\partial \phi^\mu}{\partial \psi^\nu}\bigg|_p \tilde{X}^\nu$$

$$\implies \frac{\partial \psi^\nu}{\partial \phi^\mu}\bigg|_p \frac{\partial \phi^\mu}{\partial \psi^\rho}\bigg|_p = \delta^\nu_\rho$$

Def: *Local Jacobian Matrix*

Hence, we get the notion that $\forall p \in M, \forall \phi_\alpha, \phi_\beta \in \mathcal{A}(p)$ s.t. $U_\alpha = U_\beta$; we have a linear transformation (AKA a matrix) $J_{\alpha\beta}|_p \in \mathrm{GL}(\mathbb{R}, n)$ called the **Jocabian of the charts ϕ_α, ϕ_β at the point p,** given by

$$J_{\alpha\beta}|_p(X)^\nu = \frac{\partial \phi_\beta^\nu}{\partial \phi_\alpha^\mu}\bigg|_p X^\mu$$

Often we may write $\frac{\partial \phi_\beta}{\partial \phi_\alpha}\Big|_p$ for the Jacobian

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In conclusion a tangent vector is a differential operator that acts on scalar fields at a point, and it requires a curve to point in. The set of tangent vectors is a vector space and has a natural basis given a chart), and changing basis induces a Jacobian.
