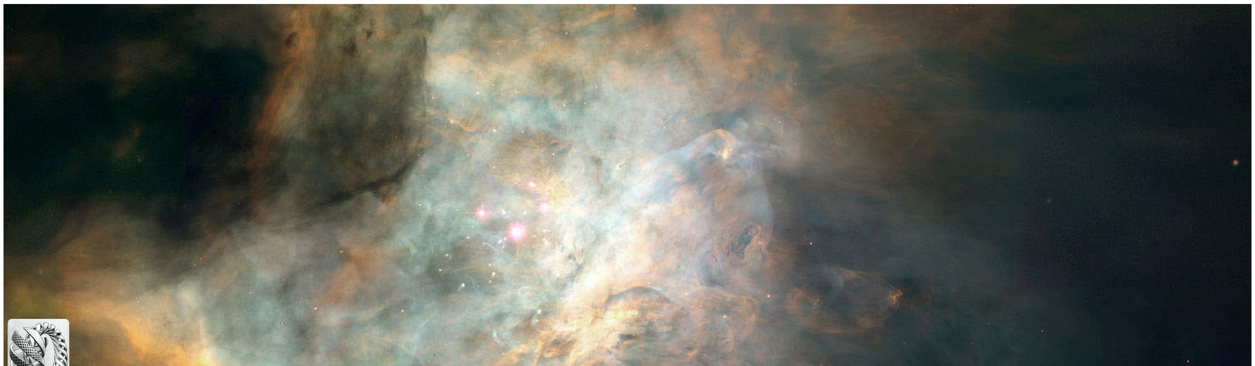

PI, Ch 4: Global Differential Calculus

In this chapter we take the tangent space we established in the last chapter and use it to define vector fields on the tangent bundle. We then generalize this to all types of tensor fields. Additionally we ask what about 2nd derivatives? A question which leads naturally to a Lie algebra structure for the vector fields.

The Tangent Bundle & Vector Fields

Def: *Tangent Bundle*

In the last section we showed that $\forall p \in M : \exists T_p M \cong \mathbb{R}^m$ a fin dim vector space. Thus by an example in section 2 on constructing manifolds it follows that

$$TM = \bigcup_{p \in M} T_p M$$

and $\pi(D_\gamma|_p) = p$, then (TM, π, M) is a vector bundle on M. Typically we denote this by TM and call it the **tangent bundle**.

Now we aren't particularly concerned with the tangent bundle as much as we are its sections. smooth maps from the manifold to a vector space at every point. Naively, we may think of these as **vector fields.** Moreover, we add some notation $\Gamma^1_0 M = \Gamma(TM, \pi, M)$

Theorem: $\Gamma^1_0 M$ *is a* $\Gamma^0_0 M$ *Module*

By defining all operations pointwise, using the fact that $\forall \sigma \in \Gamma^1_0 M, \forall p \in M : \sigma(p) \in T_p M$, and the fact that $T_p M$ is a $\mathbb{R}$ vector space, as well as the fact that $\forall f \in \Gamma^0_0 M, \forall p \in M : f(p) \in \mathbb{R}$. We get the following.

$$\begin{gathered}\forall \sigma, \delta \in \Gamma^1_0 M, \forall f \in \Gamma^0_0 M, \forall p \in M, \\ (\sigma + \delta)(p) = \sigma(p) + \delta(p) \\ (f\sigma)(p) = f(p)\sigma(p)\end{gathered}$$

💡 We specifically defined scalar field multiplication from the left. This is because we will use right multiplication to mean something completely different.

Proof

$\Gamma^0_0 M$ is a ring, and CANI ADDU are inherited from $T_p M$, hence it is a module. But the scalar fields do not necessarily form a division ring let alone a field so generically we have that...

Corollary: ***Vector Fields Do Not Always Have a Basis***

What are Vector Fields?

💡 Lets see if we can reason out what a vector field is

$$\sigma \in \Gamma^1_0 M$$
$$\sigma : M \to TM \text{ s.t.} \pi \circ \sigma = \mathrm{id}_M$$

this means that $\sigma(p) \in T_p M$. In other words, there exists a curve $\tilde{\sigma}(p) \in \mathrm{Curves}(p)$, s.t.

$$\sigma(p) = D_{\tilde{\sigma}(p)}|_p$$

and if $\phi \in \mathcal{A}(p)$ with domain U

$$\sigma(p) = D_{\tilde{\sigma}(p)}|_p \phi^\mu \left.\frac{\partial}{\partial \phi^\mu}\right|_p$$

And further, if we restrict our consideration to $\Gamma^1_0 U$, then we may easily define

$$\frac{\partial}{\partial \phi^\mu} \in \Gamma^1_0 U$$
$$\frac{\partial}{\partial \phi^\mu}(p) = \left.\frac{\partial}{\partial \phi^\mu}\right|_p$$

similarly

$$D_{\tilde{\sigma}} \phi^\mu \in \Gamma^0_0 U$$
$$D_{\tilde{\sigma}} \phi^\mu(p) = D_{\tilde{\sigma}(p)}|_p \phi^\mu$$

Then it follows that

Result: ***Local Vector Fields***

The above note can be summarized as on any chart $\phi_\alpha \in \mathcal{A}(p)$, we mean $\sigma_\alpha = \sigma|_{U_\alpha} \in \Gamma^1_0 U_\alpha$, we have $\forall f^\mu_\alpha \in \Gamma^0_0 U_\alpha$, we get

$$\sigma_\alpha = f^\mu_\alpha \frac{\partial}{\partial \phi^\mu_\alpha}$$

This means that $\Gamma^1_0 U_\alpha$ is a $\Gamma^0_0 U_\alpha$ vector space with a basis $\left\{\frac{\partial}{\partial \phi^\mu_\alpha}\right\}_\mu$.

💡 Now if there are two completely overlapping charts $\phi_\alpha, \phi_\beta \in \mathcal{A}(p)$ s.t. $U_\alpha = U_\beta$, it follows that the same local vector field can be represented two completely different ways,

$$\sigma_\alpha = f_\alpha^\mu \frac{\partial}{\partial \phi_\alpha^\mu}, \sigma_\beta = f_\beta^\nu \frac{\partial}{\partial \phi_\beta^\nu},$$

But the two should be equal as they are the same field. So, let us go to a point and see if there is some way of expressing one as the other

$$\begin{aligned} \sigma_\alpha(p) = f_\alpha^\mu(p) \left.\frac{\partial}{\partial \phi_\alpha^\mu}\right|_p = f_\alpha^\mu(p) \left.\frac{\partial \phi_\beta^\nu}{\partial \phi_\alpha^\mu}\right|_p \left.\frac{\partial}{\partial \phi_\beta^\nu}\right|_p = f_\beta^\nu(p) \left.\frac{\partial}{\partial \phi_\beta^\nu}\right|_p \\ \implies f_\beta^\nu(p) = f_\alpha^\mu(p) \left.\frac{\partial \phi_\beta^\nu}{\partial \phi_\alpha^\mu}\right|_p = \left.\frac{\partial \phi_\beta}{\partial \phi_\alpha}\right|_p (f_\alpha(p))^\nu \end{aligned}$$

And hence it stands to reason that similar to the partial derivative fields, we may extend the local Jacobian to a field

Def: ***Jacobian Matrix***

$\phi_\alpha, \phi_\beta \in \mathcal{A}(p)$ s.t. $U_\alpha = U_\beta = U$, we have $J_{\alpha\beta} \in \mathrm{GL}(\Gamma_0^0 U, n)$, s.t.

Or we may represent the matrix with $\frac{\partial \phi_\beta}{\partial \phi_\alpha}$.

Result: ***What are Vector Fields?***

Suppose that ϕ_α is some chart, and $f_\alpha \in \Gamma_0^0 U_\alpha^n$ then a vector field $\sigma \in \Gamma_0^1 M$ satisfies

$$\sigma|_{U_\alpha} = \sigma_\alpha$$

$$\sigma_\alpha = f_\alpha^\mu \frac{\partial}{\partial \phi_\alpha^\mu}$$

and for any two completely overlapping charts ϕ_α, ϕ_β with domain U

$$f_\beta = \frac{\partial \phi_\beta}{\partial \phi_\alpha}(f_\alpha)$$

💡 We could think of this as the practical definition of vector fields, to avoid talking about the coordinate free description.

What do Vector Fields do?

We know that vectors (at a point) can act on a scalar field by taking the der

Vectors can act on a scalar field in the following way

Recall: ***Vector Action***

Any vector at a point $p \in M$ is $X|_p = D_\gamma|_p \; \exists \gamma \in \text{Curves}(p)$. Then $\forall f \in \Gamma^0_0 M$ we have

$$X|pf = D[f \circ \gamma](0)$$

Def: *Vector Fields as Operators on Scalar Fields*

Well we can use this **action** to define an operator $X\triangleright \in L(\Gamma^0_0 M), \forall X \in \Gamma^1_0 M$ as follows

$$\begin{aligned} X\triangleright : \Gamma^0_0 M &\to \Gamma^0_0 M \\ f &\mapsto X \triangleright f \\ \text{s.t. } (X \triangleright f)(p) &= X|_p f \end{aligned}$$

Effectively we can think of $\triangleright : \Gamma^1_0 M \times \Gamma^0_0 M \to \Gamma^0_0 M$, and what we wish to show is that it is
(1) $\triangleright$ is $\Gamma^0_0 M$ linear in the first entry and
(2) $\triangleright$ is an $\mathbb{R}$ derivation in the second entry

Lemma: $\triangleright$ *is* $\Gamma^0_0 M$ *Linear in the First Entry*

$$\begin{gathered} \forall f, g \in \Gamma^0_0 M, \forall X, Y \in \Gamma^1_0 M \\ (X + Y) \triangleright f = X \triangleright f + Y \triangleright f \\ (gX) \triangleright f = g(X \triangleright f) \end{gathered}$$

Proof:

Simply evaluate the left-hand side of both at any point, and then use the $\mathbb{R}$-linearity of vector operations at any point, then remove the point dependence.

Lemma: $\triangleright$ *is an* $\mathbb{R}$ *Derivation in the Second Entry (Leibniz's Law)*

$$\begin{gathered} \forall a \in \mathbb{R}, \forall f, g \in \Gamma^0_0 M, \forall X \in \Gamma^1_0 M \\ X \triangleright (f + g) = X \triangleright f + X \triangleright g \\ X \triangleright (af) = a(X \triangleright f) \\ X \triangleright (fg) = (X \triangleright f)g + f(X \triangleright g) \end{gathered}$$

Proof:

Simply evaluate the left-hand side of both at any point, and then use the $\mathbb{R}$-linearity and pseudo-Leibniz Law of $D_\gamma|_p \; \forall \gamma \in \text{Curves}(p)$, then remove the point dependence.

Co-vector Fields

Now since $\Gamma^1_0 M$ is a module, it follows that $\Gamma^1_0 M^*$ must exist. However **this is not what we mean by the set of co-vector fields.**

Def: *Cotangent Space, Cotangent Bundle, and its Sections*

Instead we go back to the tangent space at a point T_pM. This is an $\mathbb{R}$ vector space, and so $T_p^*M = T_pM^*$ must exist and have a basis, and so we can define the bundle T^*M as we defined the bundler TM, which has a set of sections which we call $\Gamma^0_1 M$, the set of **co-vector fields**.

> 💡 We begin our exploration of these so-called **co-vector fields** by defining an object called the **exterior derivative**. Note, the exterior derivative will be redefined more than once. Each time the previous definition can be re-contextualized as a special case of the new definition. We begin here.

Def: *Exterior Derivative and Action of the Image of the Derivative on Vector Fields*

The exterior derivative $d : \Gamma^0_0 M \to \Gamma^0_1 M$ as follows

$$\begin{aligned} d : \Gamma^0_0 M &\to \Gamma^0_1 M \\ f &\mapsto df \end{aligned}$$

We know this is well defined by the first Lemma above since it follows that df is $\Gamma^0_0 M$-linear and so lives in $\Gamma^0_1 M$. Now what we wish to show is that every co-vector is expressible using this derivative. This is all well and good but we'd like to know what this does. To do this we define an action of these co-vector fields on vector fields as follows. For now we restrict our attention to $\text{rng } d$. $\forall \omega \in \text{rng } d$, $\exists f \in \Gamma^0_0 M$ such that $\omega = df$, we define

$$\begin{aligned} \omega \blacktriangleright : \Gamma^1_0 M &\to \Gamma^0_0 M \\ X &\mapsto \omega \blacktriangleright X \\ \text{s.t. } \omega \blacktriangleright X &= df \blacktriangleright X = X \triangleright f \end{aligned}$$

Now we can show that $\Gamma^0_1 M = \text{rng } d$. To do this we work locally.

> 💡 Remember that <u>locally</u> ($\forall p \in M, \exists \phi \in \mathcal{A}(p)$, s.t. $\text{dom}\phi = U$) we always have $\Gamma^1_0 U$ is a $\Gamma^0_0 U$ vector space and we can use Jacobians to go between any two local pictures. It follows then that $\Gamma^0_1 U$ is also a $\Gamma^0_0 U$ vector space, and we can use an inverse Jacobian to go between any two local pictures.

Lemma: *Identity for Exterior Derivative of Component Functions*

Locally we consider the functions $\phi^\nu \in \Gamma^0_0 U$

$$d\phi^\nu \blacktriangleright \frac{\partial}{\partial \phi^\mu} = \bar{\delta}^\nu_\mu$$

Proof:

$$\begin{aligned} d\phi^\nu \blacktriangleright \frac{\partial}{\partial \phi^\mu} &= \frac{\partial}{\partial \phi^\mu} \triangleright \phi^\nu \\ \implies \forall p \in U, \left(\frac{\partial}{\partial \phi^\mu} \triangleright \phi^\nu \right)(p) &= \frac{\partial}{\partial \phi^\mu}\Big|_p \phi^\nu \\ = \partial_\mu(\phi^\nu \circ \phi^{-1})(\phi(p)) &= \partial_\mu(\text{proj}^\nu \circ \phi \circ \phi^{-1})(\phi(p)) \\ &= \partial_\mu(\text{proj}^\nu)(\phi(p)) = \delta^\mu_\nu \end{aligned}$$

> 💡 Hence if we can show that $\{d\phi^\nu\}_\nu$ is a basis for $\Gamma^0_1 U$ then it follows that $\text{im}_d U = \Gamma^0_1 U$, and then since all dimensions are the same locally as globally $\text{rng } d = \Gamma^0_1 M$ and it has a local basis $\{d\phi^\nu\}_\nu$.

Lemma & Proof: *Dual Basis Fields*

Note that since $d\phi^\nu \blacktriangleright \frac{\partial}{\partial\phi^\mu} = \bar{\delta}^\nu_\mu$, it follows that $d\phi^\nu|_p\left(\frac{\partial}{\partial\phi^\mu}|_p\right) = \delta^\nu_\mu$. So $\{d\phi^\nu|_p\}$ is the dual basis in T^*_pM to $\left\{\frac{\partial}{\partial\phi^\mu}|_p\right\}$ in T_pM. Hence $\{d\phi^\nu\}_\nu$ is a basis for $\Gamma^0_1 U$.

Corollary: *Inverse Jacobians*

Since we are now working with the dual basis fields we can use a lemma from introductory linear algebra to conclude that if $\omega = f_\nu d\phi^\nu_\alpha = g_\mu d\phi^\mu_\beta$, then the component functions transform with the inverse Jacobian

$$f_\nu = \frac{\partial\phi^\mu_\beta}{\partial\phi^\nu_\alpha} g_\mu$$

Tensor Fields in General

Def: *Tensor Space, Bundle, and Fields*

Define the (r, s) **tensor tangent space** of M at p to be $T^s_r(T_pM) = T_pM^{\otimes r} \otimes T^*_pM^{\otimes s}$. As T_pM is a fin. dim. m real vector space, it follows that this construction is well defined, and is a fin $(r+s)m$ dim real vector space with a basis, namely

$$\left\{\left(\bigotimes_{i=1}^{r} \frac{\partial}{\partial\phi^{\mu_i}}|_p\right) \otimes \left(\bigotimes_{j=1}^{s} d\phi^{\nu_i}|_p\right)\right\}$$

This vector space induces a vector **bundle** $T^s_r M$ in the usual way vector bundles are induced, and the section of these bundles are the sets $\Gamma^s_r M = \Gamma(T^s_r M)$ of so called **tensor fields**.

Def: *Tensor Field Product*

Now the existence of tensor fields allows us to define **tensor field products** as pointwise tensor products

$$\begin{gathered} \tilde{\otimes} : \Gamma^p_q M \times \Gamma^s_r M \to \Gamma^{q+s}_{p+r} M \\ \text{s.t. } (T\tilde{\otimes}S)(p) = T(p) \otimes S(p) \end{gathered}$$

From this point we omit the tilde in the tensor field product.

> 💡 Note that this product is actually more general than we've used it here. In general let V and W be the fibres of two vector bundles over the same base space $\pi_V : P_V \to M$, and $\pi_W : P_W \to M$ we can define the tensor products of any two sections $\sigma \in \Gamma(P_V)$ and $\delta \in \Gamma(P_W)$ as $(\delta\tilde{\otimes}\sigma)|_p = \delta|_p \otimes \sigma|_p$, we call the bundle of which $\delta\tilde{\otimes}\sigma$ is a section $P_{V\otimes W}$. In particular if P_W is just a trivial bundle sections in $\Gamma(P_{V\otimes W}) \cong \Gamma(P_V) \otimes W$ (where again the product is local). This construction will be used later.

Def: *Local Tensor Basis*

Thankfully this allows us to re-express the original basis as

$$\left\{D\phi^{\underline{\nu}}_{\underline{\mu}}|_p\right\} = \left\{\left[\left(\bigotimes_{i=1}^{r} \frac{\partial}{\partial\phi^{\mu_i}}\right) \otimes \left(\bigotimes_{j=1}^{s} d\phi^{\nu_i}\right)\right]|_p\right\}$$

And since $T^s_r(T_pM)$ has this basis for every point, it follows that there is a **local basis of tensor fields** which we may describe as

$$\left\{D\phi^{\underline{\nu}}_{\underline{\mu}}\right\}$$

💡 Now lets generalize what we know about transformation behaviour. Every chart ${}_\alpha\phi$ we get a different local basis $D_\alpha \phi^{\underline{\nu}}_{\underline{\mu}}$, pointwise:

$$D_\alpha \phi^{\underline{\nu}}_{\underline{\mu}}|_p = \left(\bigotimes_{i=1}^{r} \frac{\partial}{\partial_\alpha \phi^{\mu_i}}|_p\right) \otimes \left(\bigotimes_{j=1}^{s} d_\alpha \phi^{\nu_i}|_p\right)$$
$$= \left(\bigotimes_{i=1}^{r} \frac{\partial_\beta \phi^{\tilde{\mu}_i}}{\partial_\alpha \phi^{\mu_i}}|_p \frac{\partial}{\partial_\beta \phi^{\tilde{\mu}_i}}|_p\right) \otimes \left(\bigotimes_{j=1}^{s} \frac{\partial_\alpha \phi^{\nu_j}}{\partial_\beta \phi^{\tilde{\nu}_j}}|_p d_\beta \phi^{\tilde{\nu}_j}|_p\right)$$

Pulling the coefficients out

$$\prod_{i,j=1}^{r,s} \frac{\partial_\beta \phi^{\tilde{\mu}_i}}{\partial_\alpha \phi^{\mu_i}}|_p \frac{\partial_\alpha \phi^{\nu_j}}{\partial_\beta \phi^{\tilde{\nu}_j}}|_p \left(\bigotimes_{i=1}^{r} \frac{\partial}{\partial_\beta \phi^{\tilde{\mu}_i}}|_p\right) \otimes \left(\bigotimes_{j=1}^{s} d_\beta \phi^{\tilde{\nu}_j}|_p\right)$$
$$= \left(\prod_{i,j=1}^{r,s} \frac{\partial_\beta \phi^{\tilde{\mu}_i}}{\partial_\alpha \phi^{\mu_i}} \frac{\partial_\alpha \phi^{\nu_j}}{\partial_\beta \phi^{\tilde{\nu}_j}}\right)|_p D_\beta \phi^{\underline{\tilde{\nu}}}_{\underline{\tilde{\mu}}}|_p$$

Def: *Jacobian Field(s)*

We have the generic transformation behaviour. Lets define the most general (r,s) **Jacobian (scalar) field** (from ${}_\alpha\phi \to {}_\beta\phi$ **).**

$$\mathrm{Jac}_r^s(\alpha;\beta)^{\underline{\tilde{\mu}},\underline{\nu}}_{\underline{\mu},\underline{\tilde{\nu}}} = \prod_{i,j=1}^{r,s} \frac{\partial_\beta \phi^{\tilde{\mu}_i}}{\partial_\alpha \phi^{\mu_i}} \frac{\partial_\alpha \phi^{\nu_j}}{\partial_\beta \phi^{\tilde{\nu}_j}}$$

Corollary: *Transformation Behaviour of Tensor Fields*

Giving the generic transformation laws

$$D_\alpha \phi^{\underline{\nu}}_{\underline{\mu}} = \mathrm{Jac}_r^s(\alpha;\beta)^{\underline{\nu},\underline{\tilde{\mu}}}_{\underline{\mu},\underline{\tilde{\nu}}} D_\beta \phi^{\underline{\tilde{\nu}}}_{\underline{\tilde{\mu}}}$$

And hence by linear independence if $T = {}_\alpha f^{\underline{\mu}}_{\underline{\nu}} D_\alpha \phi^{\underline{\nu}}_{\underline{\mu}} = {}_\beta f^{\underline{\tilde{\mu}}}_{\underline{\tilde{\nu}}} D_\beta \phi^{\underline{\tilde{\nu}}}_{\underline{\tilde{\mu}}} \in \Gamma^s_r M$, then it follows that

$${}_\beta f^{\underline{\tilde{\mu}}}_{\underline{\tilde{\nu}}} = \mathrm{Jac}_r^s(\alpha;\beta)^{\underline{\nu},\underline{\tilde{\mu}}}_{\underline{\mu},\underline{\tilde{\nu}}}\, {}_\alpha f^{\underline{\mu}}_{\underline{\nu}}$$

💡 Which allows us to define $\Gamma^s_r M$ in terms of the local transformation behaviour to avoid the coordinate free expressions.

Def: *Contraction of Fields*

Recall that an (r,s) tensor can act on (p,q) tensor where $p \leq s, q \leq r$ in a process called contraction. As before we'll define ▶ as the generic **action of contraction**, and express as in elementary linear algebra how the contraction is occurring through the use of dashes. So if we take $(0,2)$ tensor field and feed it a co-vector field in the second slot, we could write that (locally) as follows

$$T \blacktriangleright (\_, v) = T^{\nu\mu} v_\rho \frac{\partial}{\partial \phi^\nu} \otimes \left(\frac{\partial}{\partial \phi^\mu} \blacktriangleright d\phi^\rho\right) = T^{\nu\mu} v_\mu \frac{\partial}{\partial \phi^\nu}$$

💡 At this point we are fully prepared to look at differential operators on tensor fields (which themselves are differential operators)

By convention we omit the ▶

Lie Bracket

Remark: *2nd Order Differentiation?*

Now that we have explored all possible tensor fields built out of T_pM we can ask the question, why only differentiate once? That is surely treated as maps $X\triangleright, Y\triangleright : \Gamma^0_0 M \to \Gamma^0_0 M$ under composition the product $X \triangleright \circ Y\triangleright : \Gamma^0_0 M \to \Gamma^0_0 M$ corresponds to an order 2 differentiation? Why not look at vectors of that form. The answer is because they aren't vectors. Lets imagine they were vectors. Then we'd be able to write

$$X \triangleright \circ Y\triangleright = XY\triangleright, \ \ \exists XY \in \Gamma^1_0 M$$

However if we expand the left hand side by applying it to a function we get (we write everything here in shorthand notation to save time)

$$XY \triangleright f = (X^\mu \partial_\mu) \triangleright (Y^\nu \partial_\nu \triangleright f) = X^\mu \partial_\mu \triangleright (Y^\nu \partial_\nu f) = X^\mu (\partial_\mu Y^\nu \partial_\nu f - Y^\nu \partial_\nu \partial_\mu f) = X^\mu (\partial_\mu Y^\nu \partial_\nu - Y^\nu \partial_\nu \partial_\mu) \triangleright f$$
$$\implies XY = X^\mu (\partial_\mu Y^\nu \partial_\nu - Y^\nu \partial_\nu \partial_\mu)$$

The first term $X^\mu \partial_\mu Y^\nu \partial_\nu \in \Gamma^1_0 M$ but there is no term like $X^\mu Y^\nu \partial_\nu \partial_\mu$. Moreover it doesn't even transform like a vector in the way we talk about in physics as its components pick up 2 factors of the Jacobian. So we're left in a pickle. Fortunately we can take advantage of the fact that $\partial_\nu \partial_\mu = \partial_\mu \partial_\nu$ to extract a vector which does exist out of this whole mess.

Def: *Lie Bracket of Vector Fields*

$$[\_,\_] : \Gamma^1_0 M^2 \to \Gamma^1_0 M$$
$$(X,Y) \to [X,Y]$$

such that we have $[X,Y]\triangleright = [X\triangleright, Y\triangleright]$ treated as the commutator on the group with composition as multiplication. Hence in components we get

$$[X,Y]^\nu = X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu$$

Because of the properties of $\triangleright$ it turns out that $[\_,\_]$ is $\mathbb{R}$ bilinear, anticommutative and satisfies the Jacobi Identity (not hard to check). And so we conclude that $\Gamma^1_0 M$ is now a Lie Algebra (unfortunately it is not a finite dimensional Lie algebra, we'll have to work a little harder to get that). But as a consequence we do now have a notion of ad_X. In subsequent chapters we'll study the Lie Derivative of X which will generalize the adjoint to act on any tensor field.

So to summarize, a vector field is just a smooth assignment of a vector to a point in the sense of a section of a bundle. the set of vector fields is not a vector spaces and so doesn't have a basis but it does have a local basis satisfying local Jacobian rules. There's no vector field version of a second derivative but there is one for the Lie bracket of two vector fields which is the closest we can get to a second derivative. This formalism can be generalized to co-vector fields and tensor field of arbitrary rank, and in the process so can the notion of contraction and tensor product.
