Introduction

In the outline for this project, the stated goal was to gain “in depth understanding and knowledge of the foundations of vector or principal bundles over smooth manifolds and connections over them”. In that sense the project was a success, although the specifics we’re altered as the project took shape.

Firstly, the main sources we intended to draw from; “Foundations of Differential Geometry” by Koboyashi & Nomizu and “Geometry, Topology and Physics” by Nakahara proved to be more blunt and harder to breach than I originally expected. As a result the more pedagogical lecture series “Lectures on the Geometrical Anatomy of Theoretical Physics” by Schuller [1] was followed very closely, with some omissions. Any information gained outside of this lecture series was done directly in conversation with Prof. Bertola during out meetings.

The sections on set theory, topology, vector space and module theory were skipped as I was already familiar with these topics in enough depth to proceed to the main goal of this project. Additionally the comprehensive section on lie algebras was skipped because it was more tangential to the main goal. However I will be returning to this topic as soon as I can. Finally there were some topics that we’re skipped for brevity as they are more applications of the theory in the mathematical sense. These were mostly topics in algebraic topology: the fundamental group, deRham Co-homology, (and the Homology of a manifold) as well as Holonomy via parallel transport. That said the end of the lecture series additionally has applications in physics including quantum mechanics on curves spaces, spin structures, and kinematical and dynamical symmetries. These were also skipped for the same reason as the applications in algebraic topology, that being that the main goal of the project was to understand deeply the foundations of the theory. These many applications will come later I have no doubt.

Additionally in the application form the goal was also stated to include “a final report on the definition of Yang-Mills theory” using both the textbook “Quantum Field Theory and the Standard Model” by Schwartz as well as the original paper by Yang & Mills with an explanation of the general theory, scalar and spinor electrodynamics, and chromodynamics. However the explanation of the theory in the book by Schwartz proved to be too cursory for an in-depth understanding (as that textbook is more concerned with computations obtained by the theory as it is after all a physics textbook and not a math textbook) and the original paper much like the above two textbooks was to advanced at this stage. That said, while the goal changed to exclude the final report, what it instead included was a pedagogical approach to Gauge Theory (which for our purposes is Yang-Mills theory). Additionally this new approach was more cohesive with the original goal stated above then writing such a report as gauge theory and connections on principal bundles are concepts related very closely. In other words the two topics were folded in together.

There were some topics I would have like to have covered in addition to the ones I did, but time was constrained such that I couldn’t. Largest among these were as follow, integration on manifolds including the generalized Stoke’s theorem but also a rigorous definition for a field theoretic Lagrangian, as well as spinor theories (topics I worked on with Prof. Frank in my summer 2021 research). I have to imagine that these topics will come up again soon. In practice this means that the only gauge theories I worked with were the fundamental representation theories like $U(1)$ scalar and $SU(3)$ for non-spinor chromodynamics which I treated in a general sense. It did become clear though that this also included the $SO(1,3)$ theory which was equivalent to General Relativity, which was a fun surprise.

With all that out of the way, I will proceed with my report on Gauge Theory from first principles. Given the goal of this project, the format of this paper will read less like a typical paper one would publish, and more like a collection of class notes. That said it should always be kept in mind that the main goal is to move toward a deep understanding of gauge theory from a mathematical point of view. The pedagogical appraoch we will be taking separates into two parts.

Part I: Foundations of Differential Geometry

- Charts, Coordinates, and Atlases

- An introduction to the notion of a manifold and some very simple pictorial examples

- Constructing Manifolds
  - An analysis of the category of manifolds, the maps between them, and types of manifolds (most importantly bundles)
- Differential Calculus at a Point
  - In this chapter we zoom in to a point on a manifold and construct differential operators we call vectors acting on scalar fields at that point
- Global Differential Calculus
  - Generalization of the previous chapter to the whole manifold defining vector fields, and from that all kinds of tensor fields. Importantly here is the notion of a local basis field, and the introduction of the Lie Bracket.
- Operators on Fields
  - A look at functions of tensor fields, pushforwards, pullbacks, exponential flows and Lie Derivatives. Additionally we look at the notion of invariant vector fields.
- Forms
  - A restriction to a specific kind of tensor field, and special operators we may define on them (like the exterior derivative)

Part II: Application, Gauge Theory

- Lie Groups and Their Lie Algebras
  - An introduction to the notion of lie groups, and especially to the notion of an exponential map
- Bundles with Lie Group Actions
  - A specific kind of bundle on which a lie group acts in a particular way is called a principal bundle. It is explored in this chapter.
- Connections
  - The notion of connecting different parts of a principle bundle
- Gauge Theory

- Taking gauge allows one to use the principal bundle formalism without ever making reference to it. Here we additionally explore the mathematical generalization of the gauge field and field strength.

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[1] Frederic Schuller (March 12th 2016) *Lectures on Geometrical Anatomy of Theoretical Physics* [Playlist]. YouTube. https://www.youtube.com/playlist?list=PLPH7f_7ZlzxTi6kS4vCmv4ZKm9u8g5yic.