PI, Ch 1: Charts, Coordinates, and Atlases

Part I, is focused on the very fundamentals of differential geometry. The first two chapters are very definition heavy. It isn't until chapter 6 when we start doing something properly new. Still the basics were covered as it is important given the goal of this project to have a rigorous understanding. That said, in this chapter we introduce what a manifold is and try to explain it intuitively. We then explore the important notion of coordinates with charts and atlases. Finally we talk about some kinds of manifolds we can get by restricting an atlas.

General Theory

Def: *Manifold*

A n-dimensional R **manifold,** or just a mfld is a paracompact Haussdorff topological space (M, τ) which is locally homeomorphic to R^n, where R can be any set.

Typically, we require $R = \mathbb{R}, \mathbb{C}$ and we will always deal with so called real and complex manifolds.

Def: *Charts and Atlases*

The definition of a manifold requires a couple terms, the very relevant one is a local homeomorphism, something I will recall now. Let $(X,\tau), (\tilde{X},\tilde{\tau})$ be topological spaces, then a X **is locally homeomorphic to** $\tilde{X}$ iff

$$\forall p \in X : \exists U \in \mathcal{N}(p) \ \wedge \ \exists V \in \tilde{\tau} : \ \phi \in \mathcal{H}omeo(U,V)$$

Moreover, if a set is locally homeomorphic to another set, then a local homeomorphism is called a **chart.** By convention, we index charts as follows $\phi_\alpha : U_\alpha \to V_\alpha$. A chart is called **global** iff $U_\alpha = X$, otherwise it is **local**.

Lastly, by the definition of a local homeo there is such an open set U_α for every point. Thus, it is implied that there exists a collection of charts with domains which cover the set, such a collection is called an **atlas**. That is:

$$\mathcal{A} = \left\{ \phi_\alpha | \bigcup_\alpha U_\alpha = X \right\}$$

💡 We will call the atlas containing a point p , $\mathcal{A}(p) = \{\phi_\alpha \in \mathcal{A} | U_\alpha \in \mathcal{N}(p)\}$

Def: *Coordinates of a Chart*

In our case, $V_\alpha \subset R^n$, thusly the map $\phi^\mu_\alpha = proj^\mu \circ \phi_\alpha$ is well defined $\forall \mu \in \{1, ...n\}$. These maps are called the **coordinate maps** of the chart. Additionally, $\forall p \in M : \phi^\mu_\alpha(p)$ are called the **coordinates of p with respect to the chart ϕ_α.**

Def: *Chart Transition Functions, and Compatibility*

Two charts ϕ_α, ϕ_β with non-disjoint domains $(U_\alpha \cap U_\beta = U_{\alpha\beta} \neq \emptyset)$ have **chart transition functions** $\phi_{\alpha\beta}, \& \ \phi_{\beta\alpha}$ defined where

$$\tilde{V}_\alpha = im_{\phi_\alpha} U_{\alpha\beta}$$
$$\tilde{\phi}_\alpha = \phi_\alpha|_{U_{\alpha\beta}}$$

such that the following commutes

$$\begin{array}{ccc} U_{\alpha\beta} & = & U_{\alpha\beta} \\ {\scriptstyle\tilde{\phi}_\alpha}\downarrow & & \downarrow{\scriptstyle\tilde{\phi}_\beta} \\ \tilde{V}_\alpha & \xrightarrow{\phi_{\alpha\beta}} & \tilde{V}_\beta \end{array}$$

That is

$$\phi_{\alpha\beta} = \tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$$

Moreover, two charts are said to be **$\mathcal{F}$-compatible**, iff their domains are disjoint or their domains are non disjoint and the chart transitions functions are $\phi_{\alpha\beta} \in \mathcal{F}(\tilde{V}_\alpha, \tilde{V}_\beta)$. Here the type of the compatibility extends to any and all function classes defined in the codomain.

Finally, compatibility extends to atlases. An **atlas $\mathcal{A}$ is of type $\mathcal{F}$** iff all elements of the atlas are $\mathcal{F}$-compatible

Def: *$\mathcal{F}$-Manifolds and Their Morphisms*

So by the above definition, we will say that an n-dimensional $R\,\mathcal{F}$-mfd is $(M, \tau, \mathcal{A})$ where $\mathcal{A}$ is a type $\mathcal{F}$ atlas with member charts $\phi_\alpha : U_\alpha \to V_\alpha \subset R^{\dim M}$.

Additionally, assuming M, N both $R\,\mathcal{F}$-mfds with potentially different dimension m, n resp., then an **$R\,\mathcal{F}-$mfd morphism** $\Psi : M \to N$ is one wherein for any point $p \in M$ and charts $\phi_\alpha \in \mathcal{A}_M(p)$, for the domain and $\phi_\beta \in \mathcal{A}_N(\Psi(p))$ for the codomain we have that the function $f \in \mathcal{F}(V_\alpha, V_\beta)$ where:

$$\Psi_\alpha = \Psi|_{U_\alpha} \;\wedge\; rng\Psi_\alpha = \tilde{U}_\alpha \subset U_\beta$$
$$\tilde{\phi}_\beta = \phi_\beta|_{\tilde{U}_\alpha}$$

and the following diagram commutes

$$\begin{array}{ccc} M \supseteq U_\alpha & \xrightarrow{\Psi_\alpha} & \tilde{U}_\alpha \subseteq U_\beta \subseteq N \\ {\scriptstyle\phi_\alpha}\downarrow & & \downarrow{\scriptstyle\tilde{\phi}_\beta} \\ R^m \supseteq V_\alpha & \xrightarrow{f} & V_\beta \subseteq R^n \end{array}$$

$$\implies f = \tilde{\phi}_\beta \circ \Psi_\alpha \circ \phi_\alpha^{-1}$$

💡 Note that where chart transition functions have a type as a function from subsets of a set R to itself, morphisms have a type as functions from subsets of R^m to R^n. This implicitly assumes that this type is well defined for these kinds of functions.

We denote that $\Psi : M \to N$ is an $R\,\mathcal{F}-$mfd morphism by

$$\Psi \in \mathcal{F}(M, N)$$

Types of Manifolds

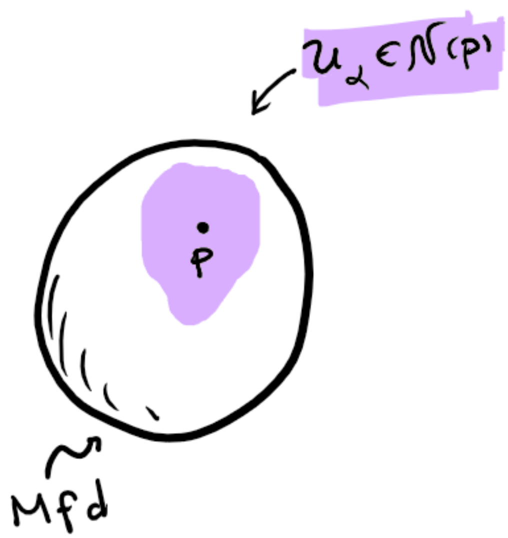
This has been very abstract, so lets bring it down to earth a touch. For our purposes all manifolds are for now real, and some types of functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ include

- **Continuous** (so f is C^0 or $f \in C^0(\mathbb{R}^m, \mathbb{R}^n)$)
  - In this case the definition of a continuous manifold includes nothing more than a manifold, except maybe prescribing some charts.
- **Differentiable** (so f is C^k or $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$)
  - These are nice, but it can be shown that a differentiable atlas can be restricted to the next type
- **Smooth** (so f is C^∞ or $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$)
  - That is the function can be differentiated arbitrarily many times. If the space has even dimension and the function happens to satisfy the Cauchy-Reimann equations then we would have the equivalent of a **complex smooth manifold aka a holomorphic manifold,** additionally if the functions have Laurent series expansions we could consider a **meromorphic manifold**
- **Analytic** (do f is C^ω or $f \in C^\omega(\mathbb{R}^m, \mathbb{R}^n)$)
  - For our part, we will work with **real analytic manifolds**, which have Taylor Series expansions. From now on, a mfd is a real analytic manifold. Importantly, a bijective real analytic manifold morphism is called a **diffeomorphism** or diffeo and denoted $C^\omega(M, N)$.

Analytic Manifolds

Lets take all this terminology we built up and explain it with some visualizations. Moreover we can define some more intuitive objects.

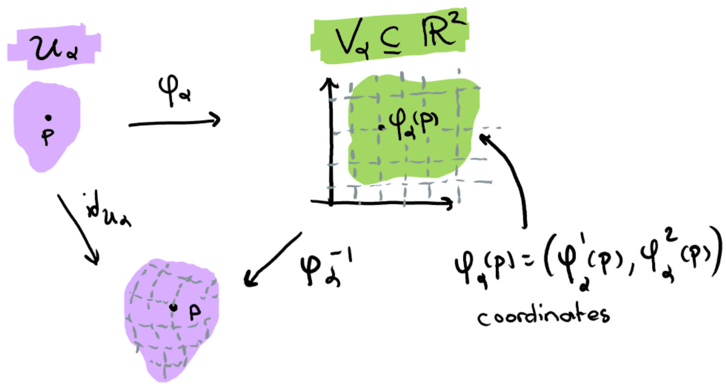
Intuition: ***Manifolds***



An example of a manifold

From this picture, we can think of a chart as taking axes from $\mathbb{R}^2$ and imprinting them on the manifold as in this picture

Intuition: ***Charts***



A chart

Intuition: *Chart Transition Maps*

And lastly, compatible charts are very simply illustrated by the following

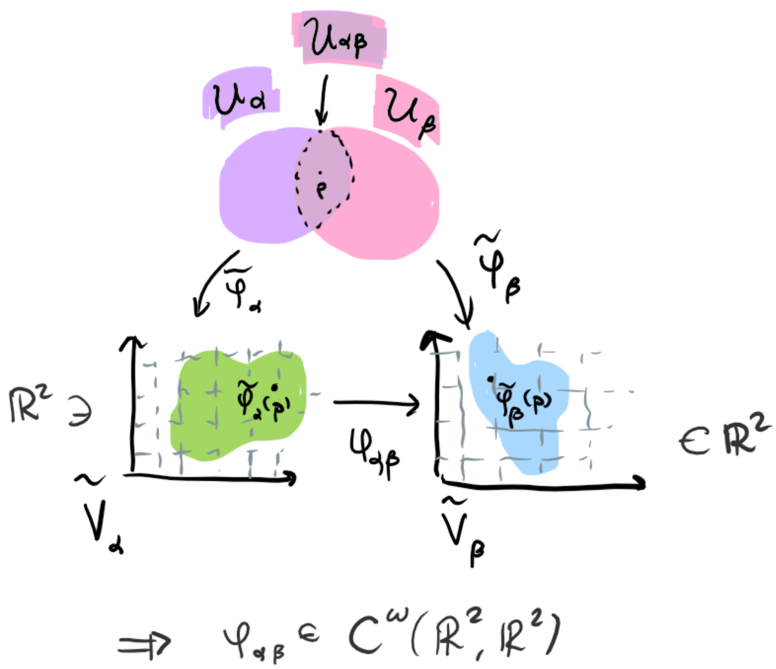


Chart transition maps are what in Cal III were naively called "vector fields"

In conclusion, manifolds are very simple objects: they are “surfaces” in higher dimensional space.