PII, Ch 10 : Gauge Theory

In this final chapter, we use all the setup from the prior 4 chapters to discuss our main topic. In actuality at this point, the chapter is relatively brief as so many previous chapters were used to set it up. The end result will be to answer the following questions. What is a gauge? What are gauge transformations? What about gauge potentials and field strengths. How does a gauge interact with the fields like in scalar electrodynamics? And what about gauge covariant derivatives? All of this follows.

Introduction to Gauges

Def: *Gauge (Local Section)*

Since we'd like our principle bundles to be nontrivial and have sections we need to make the definition less restrictive, let Ube a neighbourhood in Mthen a the bundle may be restricted to U call this $(P|_U, \pi|_U, U)$ and because the group action occurs fibre-wise the result is still a principle bundle, then a **gauge** is a **local section** is a map $\sigma \in \Gamma(P|_U, \pi|_u, U)$. Sometimes local sections are called local trivializations because there existence means that $P|_U$ is isomorphic to $U \times G$. In practice though a local section is a map $\sigma : U \to P$ such that $\forall x \in U : \pi(\sigma(x)) = x$.

Def: *Gauge Potential & Field Strength*

Given any gauge σ on a neighbourhood $U \subset M$ it and given any V valued form on P, $\varrho \in \Omega^k P \otimes V$, it follows that the pullback of ϱ by σ, $\rho = \sigma^* \varrho \in \Omega^k U \otimes V$. That is the pullback of a form on the total space by a gauge is a local form on the base space. Since we practically work with the base space M locally (on U) it turns out that ρ is more often the term we encounter. We have two special cases, given any gauge σ the pullback of the connection form $A = \sigma^* \omega$ is called the **potential field of the gauge** σ , or simply the **gauge potential or gauge field**, moreover the pullback of the curvature form $F = \sigma^* \Omega$ is called the **field strength of the gauge** σ or simply the **field strength**.

Corollary:

Since the pullback is linear it follows immediately that

$$F = dA + A \times A$$

Def: *Gauge Transformations*

Let σ_1 and σ_2 be two local sections over U. Then $\sigma_1(x), \sigma_2(x) \in \mathrm{fib}(x)$. Ergo for every $x \in U$ we must have a g such that $\sigma_2(x) = \sigma_1(x) \triangleleft g$ and this g depends on x. Hence we define the **gauge transformation** $g_{12} : U \to G$ is that map such that $\forall x \in U$ we have

$$\sigma_2(x) = \sigma_1(x) \triangleleft g_{12}(x)$$

Theorem: *Transformation Behaviour of Gauge Field and Strength*

Given a gauge transformation g, between two sections σ_1, σ_2 if we define $A_i = \sigma_i^* \omega$ and $F_i = \sigma_i^* \Omega$, it follows that

$$A_2 = \mathrm{Ad}_{g^{-1}} A_1 + g^* \Xi_g$$
$$F_2 = \mathrm{Ad}_{g^{-1}} F_1$$

Which is often written for matrix groups as

$$A_2 = g^{-1} A_1 g + g^{-1} dg$$
$$F_2 = g^{-1} F_1 g$$

Proof:

To prove this we needed to test it on several vectors like in the previous chapter. As a result the proof was omitted but the sketch is the same.

Corollary

In the case of Abelian groups and when $g = \exp(\alpha)$ where $\alpha : U \to \mathfrak{g}$ it then follows that

$$A_2 = A_1 + d\alpha$$

Which is how gauge transformations are first introduced in the case of $G = U(1)$ for electrodynamics.

Associated Bundles

Def: *Equivalence Relation on $P \times F$*

Given a principal G-bundle and a set F which is equipped with a left G-action $\triangleright : G \times F \to F$ then we may define **an equivalence relation** on $P \times F$ as follows

$$\forall (p,f),(p',f') \in P \times F : (p,f) \cong (p',f') \iff \exists g \in G : p' = p \triangleleft g, f' = g^{-1} \triangleright f$$

Proof: *Well Definition*

Reflexivity: as $\mathbf{1} = \mathbf{1}^{-1}, p = p \triangleleft \mathbf{1}, f = \mathbf{1} \triangleright f$, hence $(p,f) \cong (p,f)$. Symmetry $(p,f) \cong (p',f')$ then $p' = p \triangleleft g \implies p = p' \triangleleft g^{-1}$ and similarly for f such that $(p',f') \cong (p,f)$. Transitivity $p'' = p' \triangleleft g', p' = p \triangleleft g \implies p'' = p \triangleleft gg'$ and similarly for $f$such that $(p,f) \cong (p'',f'')$. QED.

Def: *Associated Total Space P_F*

We have $P_F = P \times F / \cong$ which then allows us to define a new notation for the elements of P_F: $[p,f] = [(p,f)]_{\cong}$. Note that we have a well defined projection map $\pi_F : P_F \to M$ given by $\pi_F([p,f]) = \pi(p)$.

Proof: *Well Definition of π_F*

Note that $\pi_F([p,f]) = \pi_F([p \triangleleft g, g^{-1} \triangleright f])$ because the arguments are the same, this would imply then that $\pi(p) = \pi(p \triangleleft g)$. And we know this is true because the action is closed in the fibre of the principal bundle. QED.

Def: *Associated Bundle of P by F and $\triangleright$*

So given a principle bundle P and a set F with a left G action $\triangleright$, we call (P_F, π_F, M) the **associated bundle of P by F and $\triangleright$.**

Def: ***Associated Bundle Morphisms***

Given a principle G-bundle map (u, ϕ) from $(P, \pi, M) \to (P', \pi', M')$ there is an **associated fibre bundle morphism** from$P_F \to P'_F$ (same fibre) given as $(\tilde{u}, \phi)$ defined as follows

$$\tilde{u}([p, f]) = [u(p), f]$$

> 💡 $(\tilde{u}, \phi)$ is clearly a bundle morphism with an additional requirement, so associated bundles may be isomorphic as bundles but not as associated bundles

Theorem:

Sections of an Associated Bundle are equivalent to equivariant functions $\varphi : P \to F$. That is there exists a bijection $\chi : \Gamma P_F \to C^\infty_G(P, F)$

Proof:

for any$\phi \in C^\infty(P, F)$ construct $v \in \Gamma P_F$ as follows. For all $x \in M$ take any $p \in \mathrm{fib}(x)$ and then map

$$v(x) = [p, \phi(p)]$$

Is this well defined? Take a different $p' \in \mathrm{fib}(p)$, then it follows that there exists exactly one $g \in G$ such that $p' = p \triangleleft g$ and

$$v(x) = [p \triangleleft g, \phi(p \triangleleft g)] = [p \triangleleft g, g^{-1} \triangleright \phi(p)] = [p, \phi(p)]$$

So this is well defined. Moreover $(\pi_F \circ v)(x) = \pi(p) = x$ so it is a section.

Def: ***Gauge Representation of a Section***

Let $s : U \to F$ and let σ be a gauge on U. Then it follows that for any $v \in \Gamma P_F$ that we can identify be a section of the associated bundle. We know that there exists a unique function ϕ

$$v(x) = [p, \phi(p)]$$

where we can choose any $p \in \mathrm{fib}(x)$. Well in particular we can choose $\sigma(x) \in \mathrm{fib}(x)$ se we can express

$$v(x) = [\sigma(x), (\phi \circ \sigma)(x)] = [\sigma(x), \sigma^* \phi(x)]$$

We call $s = \sigma^* \phi : U \to F$ the **gauge representation of** v .

Theorem: *Gauge Transformation Behaviour of Sections and their Representation*

Let $s_i = \sigma_i^* \phi_v$ and $v_i = [\sigma_i, s_i]$ and let $g = g_{12}$, then under the gauge transformation we have

$$s_2(x) = g(x)^{-1} \triangleright s_1(x)$$
$$v_2(x) = v_1(x)$$

Sections are gauge invariant (as they should be)

Proof:

and we note that under a gauge transformation

$$\sigma_2(x) = \sigma_1(x) \triangleright g(x)$$

Hence

$$s_2(x) = \sigma_2^* \chi_v(x) = \chi_v(\sigma_2(x)) = \chi_v(\sigma_1(x) \triangleleft g(x))$$
$$g(x)^{-1} \triangleright \chi_v(\sigma_1(x)) = g(x)^{-1} \triangleright \sigma_1^* \chi_v(x) = g(x)^{-1} \triangleright s_1(x)$$

Which proves the first part, for the second part simply note that

$$v_2(x) = [\sigma_2(x), s_2(x)] = [\sigma_1(x) \triangleleft g(x), g(x)^{-1} \triangleright s_1(x)]$$
$$= [\sigma_1(x), s_1(x)] = v_1(x)$$

💡 Hence we may think of a section of the principle bundle as a collection of local gauges σ_i and gauge representations s_i over U_i such that in any intersection we satisfy the requisite gauge transformation behaviour much in the same way we used to think of them as a bunch of local bases and coordinates satisfying Jacobian transformation behaviour.

Gauge Derivative

From now on we assume that F is a vector space and that $\triangleright$ is a group representation Π. It then follows that there exists a lie algebra representation $\rho = \Pi_*|_{\mathbf{1}}$, and so

Lemma:

$\forall \phi \in C^\infty_G(P, F)$, we have

$$d\phi \triangleright X^A = -\tilde{\rho}[\omega](\phi) \triangleright X^A$$

Proof:

$$d\phi \triangleright X^A = X^A \triangleright \phi = \frac{d}{dt}(\phi(p \triangleleft \exp(tA))|_0$$

> 💡 If F is a vector space such that $\triangleright$ is a representation Π, then there exists a lie algebra representation $\rho = \Pi_*|_{\mathbf{1}}$ on F, then
>
> $$\mathcal{D}\phi = d\phi + \tilde{\rho}[\omega](\phi)$$
>
> Hence if we take the pullback of this by a gauge
>
> $$\sigma^*\mathcal{D}\phi = d(\sigma^*\phi) + \tilde{\rho}[\sigma^*\omega](\sigma^*\phi) = ds + \tilde{\rho}[A](s)$$
>
> Motivating the following definition

Def: *Gauge Derivative*

If v has a principle representation ϕ and a σ-gauge representation s then $\forall X \in \Gamma^1_0 M$ we define the **gauge (covariant) derivative** in the direction as X as follows

$$\begin{gathered}\nabla_X : C^\infty(U, F) \to C^\infty(U, F) \\ s \to \nabla_X s = ds \triangleright X + \tilde{\rho}[A \triangleright X](s)\end{gathered}$$

In other words if $s = \sigma^*\phi$ we simply define $\nabla_X s = (\sigma^*\mathcal{D}\phi) \triangleright X$. Hence ∇ depends on the gauge. Moreover under a gauge transformation $s \to g^{-1} \triangleright s$ we can show that $\nabla_X s \to g^{-1} \triangleright \nabla_X s$. This is why it is called covariant because the derivative of $\nabla_X s$ transforms the same way s does.

Proof: *of Covariance*

Proving covariance is trivial, simply write s and A under a gauge transformation and compute $\nabla_X s$ to show it transforms covariantly.

Def: *Extension of the Definition to Sections*

Given the fact that the derivative is covariant we may extend the definition of the **covariant derivative** to the associated bundle

$$\nabla_X : \Gamma P_F \to \Gamma P_F$$
$$v = [\sigma, s] \to \nabla_X v = [\sigma, \nabla_X s]$$

> 💡 The covariance implies that under a gauge transformation $\nabla_X v$ remains unchanged, that is it is gauge invariant

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Finally in this last chapter we built on the fact that principal bundles cannot have a global section without being trivial to define a local section, or gauge. From this we were able to pullback the connection and curvature forms to define the gauge potential and field strength. We then introduced how gauges change, and explored how potentials and field strengths change. There is a great analogy to changes of bases with the potential representing the components and the connection form representing the vector. Finally we finalized everything with an application of all of these connections. Sections of an associated bundle can be expressed as local functions or gauge representatives, on which a gauge covariant derivative exists. Putting the main goal of this project to rest. From first principles we derived what gauges are and how they work.
