PI, Ch 5: Operators on Fields

The goal of this chapter is rather straightforward. What are the maps between tensor fields? This will take us to understand pushforwards, and their better cousins pullbacks, and additionally this will lead to a differential homomorphism to relate tensor fields on different manifolds to each other. Then we will ask “Are there any natural derivatives we can define from no other structure but this?”. This will bring us into the notion of integral curves, flows, and eventually something called the Lie derivative.

Pushforwards

Def: (Pointwise Pushforward)

Let M, N be manifolds, and let $\psi : M \to N$ be an analytic map. Then $\forall p \in M$ we can consider the **pointwise pushforward of** ψ **at the point** p to be the following induced map:

$$\psi_*|_p : T_pM \to T_{\psi(p)}N$$
$$\forall f \in \Gamma^0_0 N : \psi_*|_p(X|_p) \triangleright f = X|_p \triangleright (f \circ \psi)$$

Now the question arises: can this be generalized to a global pushforward? Naively we could say that $\forall X \in \Gamma^1_0 M$ let the vector field $\psi_*(X) \in \Gamma^1_0 N$ be defined such that at every point $q = \psi(p) : \exists p \in M$ we have

$$\psi_*(X)|_q = \psi_*|_p(X_p)$$

Immediately there is a problem. To do this we assumed that ψ was a surjection (which generically is not true). Moreover, what if ψ was noninjective? Then there would be two distinct points p_1, p_2 such that $\psi(p_1) = \psi(p_2) = q$. By the naïve definition would we write

$$\psi_*(X)|_q = \psi_*|_{p_1}(X_{p_1}) \ \text{ or } \ \psi_*|_{p_2}(X_{p_2})\ ?$$

So it follows that the only way to get a global generalization of the pushforward is if the function was invertible (and thereby a diffeomorphism). Under such a case we can always say write :

$$\psi_*(X)|_q = \psi_*|_{\psi^{-1}(q)}(X_{\psi^{-1}(q)})$$

Def: ***Global Pushforward of a Diffeomorphism***

Let M, N be manifolds of dimension m, n respectively, and let $\psi : M \to N$ be a **diffeomorphism**. Then the **pushforward of** ψ is the linear map $\psi_* : \Gamma^1_0 M \to \Gamma^1_0 N$ such that $\forall X \in \Gamma^1_0 M, \forall q \in N$ we have

$$\psi_*(X)|_q = \psi_*|_{\psi^{-1}(q)}(X_{\psi^{-1}(q)})$$

As a bit of notation we omit writing brackets everywhere so $\psi_* X$

Lemma: ***Components of a Pushforward***

Let $t \in \mathcal{A}_M(p)$ and let $\phi \in \mathcal{A}_N(\psi(p))\, \forall p \in M$. Then we can show that:

$$\psi_* \frac{\partial}{\partial t^\mu} = \frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu} \frac{\partial}{\partial \phi^\rho} \circ \psi$$

Proof:

Consider the following

$$\psi_* \frac{\partial}{\partial t^\mu} \triangleright f = \frac{\partial}{\partial t^\mu} \triangleright (f \circ \psi) = \partial_\mu (f \circ \psi \circ t^{-1}) \circ t$$

Lets expand the inside as

$$\partial_\mu (f \circ \phi^{-1} \circ \phi \circ \psi \circ t^{-1}) \circ t$$

Notice that the map $\hat{\psi} = \phi \circ \psi \circ t^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ and the map $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$, so we have a case of the chain rule:

$$\partial_\mu (f \circ \phi^{-1} \circ \hat{\psi}) \circ t = [\partial_\rho (f \circ \phi^{-1}) \circ \hat{\psi} \circ t] \partial_\mu (\text{proj}^\rho \circ \hat{\psi}) \circ t$$

Note that $\hat{\psi} \circ t = \phi \circ \psi$ so the first term becomes$\partial_\rho (f \circ \phi^{-1}) \circ \hat{\psi} \circ t = \partial_\rho (f \circ \phi^{-1}) \circ \phi \circ \psi$, or equivalently

$$\left(\frac{\partial}{\partial \phi^\rho} \triangleright f \right) \circ \psi = \left(\frac{\partial}{\partial \phi^\rho} \circ \psi \right) \triangleright f$$

Additionally we know that that $\text{proj}^\rho \circ \hat{\psi} = \phi^\rho \circ \psi \circ t^{-1}, \partial_\mu (\text{proj}^\rho \circ \hat{\psi}) \circ t = \partial_\mu (\phi^\rho \circ \psi \circ t^{-1}) \circ t$. Or equivalently

$$\frac{\partial}{\partial t^\mu} \triangleright (\phi^\rho \circ \psi)$$

For simplicity we'll write this second term as $\frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu}$, which in total means that what we have is

$$\psi_* \frac{\partial}{\partial t^\mu} \triangleright f = \frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu} \left(\frac{\partial}{\partial \phi^\rho} \circ \psi \right) \triangleright f$$

Or generically

$$\psi_* \frac{\partial}{\partial t^\mu} = \frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu} \left(\frac{\partial}{\partial \phi^\rho} \circ \psi \right)$$
$$\psi_* \frac{\partial}{\partial t^\mu} = \left[\left(\frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu} \circ \psi^{-1} \right) \frac{\partial}{\partial \phi^\rho} \right] \circ \psi$$

QED

> 💡 This expression allows us to see something very clear: $\forall X \in \Gamma^1_0 M, \exists Y \in \Gamma^1_0 N$ such that $\psi_* X = Y \circ \psi$. Therefore the map $\psi_* X \circ \psi^{-1} = Y \in \Gamma^1_0 N$ establishes a homomorphism between $\Gamma^1_0 M$ and $\Gamma^1_0 N$. We call this map the differential homomorphism

Def: *Differential Homomorphism*

If ψ is a **diffeomorphism**, then the map $H_\psi : \Gamma^1_0 M \to \Gamma^1_0 N$ is defined as follows. $\forall X \in \Gamma^1_0 M : H_\psi(X) = \psi_* X \circ \psi^{-1}$ is called the **differential homomorphism induced by** ψ **(for vector fields).** So in coordinates like the above example

$$H_\psi \frac{\partial}{\partial t^\mu} = \left(\frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu} \circ \psi^{-1} \right) \frac{\partial}{\partial \phi^\rho}$$

> 💡 So why would someone care about this? Well we now have the ability to compare a vector field on one manifold with a vector field on another via this homomorphism! Also this is clearly the differential geometry version of the chain rule.

Pullbacks

Now that we've explored how the vectors transform, is there some analogous transformation for the co-vectors? Yes!

Def: *Pullback*

Let M, N be manifolds of dimension m, n respectively, and let $\psi : M \to N$ be an analytic map. Then $\forall p \in M$ we can consider the **pointwise pullback of** ψ **at the point** p to be the following induced map:

$$\psi^*|_p : T^*_{\psi(p)} N \to T^*_p M$$
$$\forall X \in T_p M : \psi^*|_p(\omega|_{\psi(p)}) \triangleright X = \omega|_{\psi(p)} \triangleright (\psi_*|_p X|_p)$$

Now again the question can be asked? Can this be extended globally? Again; lets come up with a naïve definition. Suppose that $\psi^*(\omega) \in \Gamma^0_1 M$ such that $\forall p \in M$

$$\psi^*(\omega)|_p = \psi^*|_p(\omega|_{\psi(p)}) = \psi^*|_p(\omega \circ \psi)|_p$$

Unlike in the previous section, there are no ambiguities about ψ being a surjection or an injection. The above is always well defined. So yes it can be extended globally in this way. Let M, N be manifolds, and let $\psi : M \to N$ be an **analytic map**. Then **the pullback of** ψ is the map $\psi^* : \Gamma^1_0 N \to \Gamma^1_0 M$ as follows $\forall p \in M : \psi^*(\omega)|_p = \psi^*|_p(\omega|_{\psi(p)})$, and again we change the notation to write $\psi^* \omega$.

> 💡 Note that for any ψ, $\psi^* \omega$ is always defined. In some sense the pullback is the **more fundamental map**

> 💡 In this notation we can note that if we take ψ as a diffeomorphism then $\forall X \in \Gamma^1_0 M$ we have $H_\psi X \in \Gamma^1_0 N$ such that $H_\psi X = \psi_* X \circ \psi^{-1}$ so it follows that $\psi_* X = H_\psi X \circ \psi$. Therefore
>
> $$\psi^* \omega \triangleright X = (\omega \circ \psi) \triangleright \psi_* X = (\omega \circ \psi) \triangleright (H_\psi X \circ \psi) = (\omega \triangleright H_\psi X) \circ \psi$$

Lemma: *Components of the Pullback*

Let $t \in \mathcal{A}_M(p)$ and let $\phi \in \mathcal{A}_N(\psi(p)) \, \forall p \in M$. Then we can show that:

$$\psi^* d\phi^\gamma = \frac{\partial(\phi^\gamma \circ \psi)}{\partial t^\mu} dt^\mu$$

Proof:

Recall that

$$H_\psi \frac{\partial}{\partial t^\mu} = \left(\frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu} \circ \psi^{-1} \right) \frac{\partial}{\partial \phi^\rho}$$

Then use the face that $\psi^* d\phi^\gamma \triangleright \frac{\partial}{\partial t^\mu} = (d\phi^\gamma \triangleright H_\psi \frac{\partial}{\partial t^\mu}) \circ \psi$. And so we get

$$\left(d\phi^\gamma \triangleright \left(\frac{\partial(\phi^\rho \circ \psi)}{\partial t^\mu} \circ \psi^{-1} \right) \frac{\partial}{\partial \phi^\rho} \right) \circ \psi = \left(\frac{\partial(\phi^\gamma \circ \psi)}{\partial t^\mu} \circ \psi^{-1} \right) \circ \psi = \frac{\partial(\phi^\gamma \circ \psi)}{\partial t^\mu}$$

But we know that $\psi^* d\phi^\gamma = (A^\gamma_\nu dt^\nu) \circ \psi$. But if we evaluate $\triangleright \frac{\partial}{\partial t^\mu}$ on the right hand side we simply get $A^\gamma_\mu \circ \psi$. And so we conclude that $A^\gamma_\nu = \frac{\partial(\phi^\gamma \circ \psi)}{\partial t^\mu}$. And though we did this for a diffeomorphism ψ we easily could have done this pointwise and then reconstituted the original function at the end. Ergo for any analytic map ψ we have

$$\psi^* d\phi^\gamma = \frac{\partial(\phi^\gamma \circ \psi)}{\partial t^\mu} dt^\mu$$

QED.

> 💡 If we have a metric then we will have a way of expressing $\psi^* d\phi^\gamma$ as a vector field. Then we can see that the above will become the form of the chain rule like for the $H_\psi \frac{\partial}{\partial t^\mu}$, even if ψ is not a diffeomorphism. This is naively what is called the chain rule. This is structure we don't yet have.

Def: *Differential Homomorphism (for Co-vector Fields)*

If ψ is a **diffeomorphism**, then the map $H_\psi : \Gamma^0_1 M \to \Gamma^0_1 N$ is defined as follows. $\forall \omega \in \Gamma^0_1 M : H_\psi(\omega) = (\psi^{-1})^* \omega$ and is called the **differential homomorphism induced by ψ (for co-vector fields)**

> 💡 Note that unlike with vectors $\psi^* \omega$ is already a map from $p \to T^*_p M$, and hence lives in $\Gamma^0_1 M$. So we can simply say that the differential homomorphism for co-vectors is defined by the pullback! **Except**, we cant. Because for vector fields the differential homomorphism was a map from $\Gamma^1_0 M \to \Gamma^1_0 N$. We would like out new differential homomorphism to go from $\Gamma^0_1 M \to \Gamma^0_1 N$. However ψ^* does the opposite. This is why we use $(\psi^{-1})^*$ to define the differential homomorphism!

Def: *Differential Homomorphism (of any Tensor Field)*

While were at it why might as well define it for scalar fields. If ψ is a **diffeomorphism**, then the map $H_\psi : \Gamma^0_0 M \to \Gamma^0_0 N$is defined as follows: $\forall f \in \Gamma^0_0 M : H_\psi(f) = f \circ \psi$ and is called the differential homomorphism induced by ψ (for scalar fields).

So in summary if $f \in \Gamma^0_0 M, X \in \Gamma^1_0 M, \omega \in \Gamma^0_1 M$ then we have

$$\begin{gathered} H_\psi f = f \circ \psi \\ H_\psi X = \psi_* X \circ \psi^{-1} \\ H_\psi \omega = (\psi^{-1})^* \omega \end{gathered}$$

Lastly since we have a notion of $\otimes$ for tensor fields we could simply define the differential homomorphism for tensor fields as follows. $\forall T \in \Gamma^s_r M$ we know that

$$\exists X_i \in \Gamma^1_0 M, \omega_j \in \Gamma^0_1 M : T = \left(\bigotimes_{i=1}^{s} X_i \right) \otimes \left(\bigotimes_{j=1}^{r} \omega_j \right)$$

So lets simply define H_ψ to be a homomorphism with respect to the tensor product. If ψ is a **diffeomorphism**, then the map $H_\psi : \Gamma^s_r M \to \Gamma^s_r N$ is defined as follows

$$H_\psi(T) = \left(\bigotimes_{i=1}^{s} H_\psi(X_i) \right) \otimes \left(\bigotimes_{j=1}^{r} H_\psi(\omega_j) \right)$$

And is called the **differential homomorphism induced by** ψ

> 💡 Now, provided we have a diffeomorphism $\psi : M \to N$ we can always compare (r, s) tensors fields!

Invariant Vector Fields

Let $\psi : M \to M$ be analytic (not necessarily a diffeomorphisms). Then it follows that for any function $f \in \Gamma^0_0 M$ and any vector field $X \in \Gamma^1_0 M$ we have the function $X \triangleright f$. Since this is a function we can act on it with the pullback of ψ to get the new function $\psi^*(X \triangleright f)$. Alternatively f itself can be pulled back so we might consider $X \triangleright \psi^* f$. Generically there should be no reason these two should be the same, which motivates the following definition

Def: ψ - *Invariant Vector Fields*

Let $\psi : M \to M$ be analytic , then a **vector field** $X \in \Gamma^1_0 M$ is called ψ**-invariant** if and only if $\forall f \in \Gamma^0_0 M$ we have

$$\psi^*(X \triangleright f) = X \triangleright \psi^* f$$

The set of all ψ-invariant vector fields is called $\mathrm{Inv}_\psi M = \{X \in \Gamma^1_0 M : \psi^*(X \triangleright f) = X \triangleright \psi^* f\}$.

💡 We can note that locally $\psi^*(X \triangleright f)|_p = (X \triangleright f)|_{\psi(p)} = X|_{\psi(p)} \triangleright f$. Similarly $(X \triangleright \psi^* f)|_p = \psi_*|_p X|_p \triangleright f$. Since this is true for all f then a vector field is left invariant if and only if

$$X|_{\psi(p)} = \psi_*|_p X|_p$$
$$(X \circ \psi)|_p = \psi_*|_p X|_p$$

Moreover if ψ happens to be a diffeomorphism then we can write the even more clear version of this condition

$$X \circ \psi = \psi_* X$$
$$X = \psi_* X \circ \psi^{-1} = H_\psi X$$

So for diffeomorphisms ψ invariance is equivalent to being an eigenvector for the differential homomorphism with unit eigenvalue

$$H_\psi X = X$$

Lemma: $\mathrm{Inv}_\psi M \subseteq_{\mathcal{L}ie\mathcal{A}} \Gamma^1_0 M$

Proof:

Suppose $X, Y \in \mathrm{Inv}_\psi M$ and $a \in \mathbb{R}$. Then we have that for any $f \in \Gamma^0_0 M : \psi^*(Z \triangleright f) = Z \triangleright \psi^* f$ for $Z = X, Y$. We begin with the closure of addition

$$\psi^*[(X+Y) \triangleright f] = \psi^*(X \triangleright f + Y \triangleright f) = \psi^*(X \triangleright f) + \psi^*(Y \triangleright f)$$

Using the linearity of $\triangleright$ and ψ^*. Next using the invariance property

$$\psi^*(X \triangleright f) + \psi^*(Y \triangleright f) = X \triangleright \psi^* f + Y \triangleright \psi^* f$$

And finally with invariance

$$\psi^*[(X+Y) \triangleright f] = (X+Y) \triangleright \psi^* f$$

Similarly we use linearity twice to show that $\psi^*[(aX) \triangleright f] = (aX) \triangleright \psi^* f$

. Lastly we need to show the closure of the bracket. By definition $[X, Y] \triangleright = X \triangleright \circ Y \triangleright - Y \triangleright \circ X \triangleright$. Thus we have

$$\psi^*([X,Y] \triangleright f) = \psi^*((X \triangleright \circ Y \triangleright - Y \triangleright \circ X \triangleright)(f)) = \psi^*((X \triangleright (Y \triangleright f) - Y \triangleright (X \triangleright f))$$

Now using the linearity of ψ^* we express the terms as follows

$$\psi^*((X \triangleright (Y \triangleright f) - Y \triangleright (X \triangleright f)) = \psi^*((X \triangleright (Y \triangleright f)) - \psi^*(Y \triangleright (X \triangleright f))$$

And now since $Z \triangleright f$ is just a function we can use the invariance property twice

$$\psi^*((X \triangleright (Y \triangleright f)) - \psi^*(Y \triangleright (X \triangleright f)) = X \triangleright \psi^*(Y \triangleright f) - Y \triangleright \psi^*(X \triangleright f))$$
$$X \triangleright (Y \triangleright \psi^* f) - Y \triangleright (X \triangleright \psi^* f) = (X \triangleright \circ Y \triangleright - Y \triangleright \circ X \triangleright)(\psi^* f)$$
$$[X,Y] \triangleright \psi^* f$$

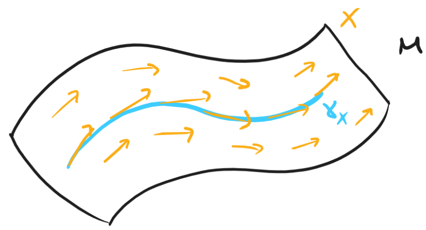
Ergo $\psi^*([X,Y] \triangleright f) = [X,Y] \triangleright \psi^* f$. We have that $\mathrm{Inv}_\psi M$ is in fact closed under all three operations and so is a Lie subalgebra of the vector field algebra. QED.

Corollary:

Integral Curves, Exponential Flows

It follows from abstract algebra then that if $\Psi = \{\psi : M \to M\}$ is some subset of analytic functions on M then $\cap_{\psi\in\Psi}\text{Inv}_\psi M$ is also a Lie subalgebra; the algebra of vector fields which are ψ invariant for every $\psi \in \Psi$.

For any $X \in \Gamma^1_0 M$ we could imagine a curve $\gamma \in \text{Curves}M$ satisfying the following pseudo equation $\frac{d\gamma_X}{dt}(t) = X|_{\gamma(t)}$. That is X is a vector field such that at the point $\gamma(t)$ the velocity of vector of the γ is given by x. The problem is that such an equation can't actually be written because we dont know what it means to take a derivative even with a single real variable in M, and even if we did we would be sure that the result would not be in $T_{\gamma(t)}M$. Luckily we have access to local bases which remedies this problem.



Def: *Integral Curves*

A curve γ_X on M is called **an integral curve of** $X \in \Gamma^1_0 M$ if and only if $\forall t \in M$; for every chart $\phi \in \mathcal{A}(\gamma(t))$ over the neighborhood U we have the local components $X = X^\mu \frac{\partial}{\partial \phi^\mu}$ and this equation is satisfied.

$$\frac{d}{dt}(\phi^\mu \circ \gamma|_U)(t) = X^\mu|_{\gamma(t)}$$

💡 Now if the set is $\mathbb{R}^n$ with a canonical global chart then this reduces to the equation $\frac{d\gamma_X}{dt}(t) = X|_{\gamma(t)}$

Lemma:

Let $X \in \Gamma^1_0 M$, then $\forall p \in M : X|_p = D_{\gamma_X}|_p$ for $\gamma_X \in \text{Curves}(p)$. The proof of this is very clear.

Lemma:

Let $X \in \text{Inv}_\psi M$ for some diffeomorphism ψ and $\gamma_X \in \text{Curves}(p)$ then $\tilde{\gamma}_X = \psi \circ \gamma_X \in \text{Curves}(\psi(p))$

Proof

Consider the differential equation using $\tilde{\gamma}_X$ as follows

$$\frac{d}{dt}(\phi^\mu \circ \tilde{\gamma}_X)(t) = \frac{d}{dt}(\phi^\mu \circ \psi \circ \gamma_X)(t) = \frac{d}{dt}(\phi^\mu \circ \psi \circ \phi^{-1} \circ \phi \circ \gamma_X)(t)$$

Using multivariate chain rule this breaks into

$$\partial_\nu(\phi^\mu \circ \psi \circ \phi^{-1})(\phi(\gamma_X(t))\frac{d}{dt}(\phi^\nu \circ \gamma_X)(t)$$
$$\frac{\partial(\phi^\mu \circ \psi)}{\partial \phi^\nu}\Big|_{\gamma_X(t)} X^\nu|_{\gamma_X(t)} = \frac{\partial(\phi^\mu \circ \psi)}{\partial \phi^\nu} d\phi^\nu(X)\Big|_{\gamma_X(t)} = \psi_* X|_{\gamma_X(t)}$$

But now using left invariance $\psi_* X|_{\gamma_X(t)} = X|_{\psi \circ \gamma_X(t)} = X|_{\tilde{\gamma}_X(t)}$. But then $\tilde{\gamma}_X$ satisfies the ODE

$$\frac{d}{dt}(\phi^\nu \circ \tilde{\gamma}_X)(t) = X^\mu|_{\tilde{\gamma}_X(t)}$$

With initial condition $\tilde{\gamma}_X(0) = \psi(\gamma_X(0)) = \psi(p)$. Which is equivalent to saying $\tilde{\gamma}_X$ is well defined as an integral curve and is based at $\psi(p)$. QED.

Lemma:

$\gamma_X(st) = \gamma_{sX}(t)$

Proof:

We do this via the chain rule

$$\frac{d}{dt}(\phi^\mu \circ \gamma_X(st)) = \frac{d}{d(st)}(\phi^\mu \circ \gamma_X(st))\frac{d(st)}{dt} = sX^\mu|_{\gamma_X(st)}$$

QED.

So pictorially this is very intuitive and inline with multivariable calculus, we have a curve whose derivative is the vector field X. In some sense this is the simplest kind of ODE one could define on an arbitrary manifold. But how can we find solutions to such ODEs?

Def: *Flows*

An analytic map $\psi : M \times \mathbb{R} \to M$ is called a **flow** if and only if $\forall t \in \mathbb{R}; \psi^t = \psi(-, t) : M \to M$ is a diffeomorphism with

$$\psi^0 = \mathrm{id}_M$$
$$\psi^t \circ \psi^s = \psi^{t+s}$$
$$(\psi^t)^{-1} = \psi^{-t}$$

> 💡 That is to say a flow is a smooth action of the additive group $\mathbb{R}$ M. An important subset of flows are the so-called exponential flows of a vector field X

Def: *Exponential Flows*

Let $X \in \Gamma^1_0 M$. Then ψ_X is **an exponential flow for** X if and only if $\forall p \in M$; for every chart $\phi \in \mathcal{A}(p)$ over the neighborhood U we have the local components $X = X^\mu \frac{\partial}{\partial \phi^\mu}$ and this equation is satisfied.

$$\frac{d}{dt}(\phi^\mu \circ \psi_X(p, -)|_U)(t) = X^\mu|_p$$

In a slight abuse of notion (for clarity) this is written as

$$\frac{d}{dt}(\phi^\mu \circ \psi^t_X) = X^\mu$$

💡 Let $\gamma_X \in \text{Curves}(p)$, then $\gamma_X(t) = \psi_X^t(p)$. This follows directly from the definition of both integral curves and exponential flows. Furthermore this means that $\psi_X^t \circ \gamma_X(s) = \gamma_X(s+t)$ the flow evolves the curve. But this means that for any given t, the integral curve $\gamma_X(s+t)$ is equivalent to another integral curve $\gamma_X^t \in \text{Curves}(\psi_X^t(p))$. So we use this in the following . Alternatively this means that $\gamma_X \in \text{Curves}(p)$ is $\gamma_X(t) = \psi_X^t(p)$.

Lemma: Well Definition

It is clear to see that exponential flows satisfy all the axioms of flow.

Lemma: $X \in \text{Inv}_{\psi_X^t} M$

Proof:

$$(\psi_X^t{}_* X \triangleright f)|_p = X|_p \triangleright (f \circ \psi_X^t) = D_{\gamma_X}|_p \triangleright (f \circ \psi_X^t) = D(f \circ \psi_X^t \circ \gamma_X)(0) = D(f \circ \gamma_X^t)(0) = D_{\gamma_X^t}|_{\gamma_X^t(0)} \triangleright f$$

But $\gamma_X^t(0) = \psi_X^t(p)$ so we get $D_{\gamma_X^t}|_{\gamma_X^t(0)} = D|_{\gamma_X^t}|_{\psi_X^t(p)} = X|_{\psi_X^t(p)}$ as γ_X^t is just a curve based at the point to which we are tangent. From here we have

$$(\psi_X^t{}_* X \triangleright f)|_p = (X \triangleright f)|_{\psi_X^t(p)}$$

Since this is true for every function we settle on $\psi_X^t{}_* X = X \circ \psi_X^t$. We note from earlier that this is equivalent to being ψ_X^t invariant.

Corollary

$$(\psi_X^t{}_* X \triangleright f)|_p = (X \triangleright \psi_X^t{}^* f)|_p = D(\psi_X^t{}^* f|_p)$$

The last part of the inequality is clear.

Lemma:

Take $X \in \text{Inv}_\varrho M,$ then $\varrho \circ \psi_X^t$.

Proof

we already know that $\tilde{\gamma}_X \in \text{Curves}(\varrho(p))$, it follows then that since $\psi_X^t(q) = \sigma_X(t)$ for $\sigma_X \in \text{Curves}(q)$ it then follows that $\psi_X^t(\varrho(p)) = \tilde{\gamma}_X(t)$ for $\gamma_X \in \text{Curves}(p)$. Finally this means thata

$$\begin{aligned} \psi_X^t(\varrho(p)) = \tilde{\gamma}_X(t) = \varrho(\gamma_X(t)) = \varrho(\psi_X^t(p)) \\ \implies \psi_X^t \circ \varrho = \varrho \circ \psi_X^t \end{aligned}$$

Lie Derivatives

Def: *Lie Transport by a Vector Field*

Let $X \in \Gamma_0^1 M$, then for any $t \in \mathbb{R}$ the exponential flow ψ_X^t is a diffeomorphism. This means that its differential homomorphism $H_{\psi_X^t}$ is well defined. From this we define the **Lie transport operator** for the vector field X. $L_X^t = H_{\psi_X^t} - \mathbf{1}$. This operator measures the difference in a tensor field at a point p and at the point along the integral curve $\gamma_X(t) \in \text{Curves}(p)$. From this it is quick to define the **Lie derivative** operator $\mathcal{L}_X$ which measures the infinitesimal rate of change along integral curves of X

$$\mathcal{L}_X = \lim_{t \to 0} \frac{L_X^t}{t}$$

Corollary:

As a vector field is invariant under its own flow it has eigenvalue 1 for its flows differential homomorphism. Which is to say $L_X^t X = H_{\psi_X^t} X - \mathbf{1}X = X - X = 0$. And hence $\mathcal{L}_X X = 0$.

Theorem: *Some Properties of the Lie Derivative*

$$\mathcal{L}_X (T + S) = \mathcal{L}_X T + \mathcal{L}_X S$$
$$\mathcal{L}_X (T \otimes S) = \mathcal{L}_X T \otimes S + T \otimes \mathcal{L}_X S$$
$$\mathcal{L}_X a = 0 \ \ \forall a \in \mathbb{R}$$
$$\mathcal{L}_X f = X \triangleright f \ \ \forall f \in \Gamma_0^0 M$$
$$\mathcal{L}_X Y = \mathrm{ad}_X Y \ \ \forall Y \in \Gamma_0^1 M$$
$$\mathcal{L}_X \omega \triangleright Y = X \triangleright (\omega \triangleright Y) - \omega \triangleright (\mathcal{L}_X Y)$$
$$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$$

Proof:

I proved all of these from first principles, but since the proofs are rather long I will only say here that to prove $\mathcal{L}_X f, \mathcal{L}_X Y, \mathcal{L}_X \omega \triangleright Y$ you must expand things with a Taylor series using

$$\frac{d}{dt}(\phi^\mu \circ \psi_X^t) = X^\mu$$

💡 So we can think of the Lie derivative as the generalization of the adjoint of the Lie algebra of vector fields to all tensor fields.

So we have pullbacks of co-vector fields as our primary operator on tensor fields. In the case when the map is invertible we have pushforwards of vector fields, allowing for the definition of the differential homomorphism. One such case is in an exponential flow (the generalization of an integral curve). The corresponding differential homomorphism allows for the definition of the Lie derivative. In the process we also introduced the notion of a vector field being invariant to some map, and we saw some properties of such invariant fields.
