PI, Ch 6: Forms

As the last chapter of part I, this chapter begins to actually introduce properly new topics that we can use later. Firstly we talk about forms (specific kinds of tensor fields) and operators on them. Namely the exterior derivative and pullback. Then we extend the definition to the notion of valued forms which will be invaluable in gauge theory.

An Introduction to Forms

We should dwell for a moment on the notion of generalizations. In prior section we defined ▷ as the action of a vector field on a function pointwise and ► as action of tensor contraction pointwise and in the last section we simplified notation by just referring to both objects as ▷. The problem is that ▷ cant be generalized from vector fields to vectorial tensor fields (no co-vector components), whereas ► can. Actually ► can even be extended to something will call valued fields. This leads to the fact that the pushforward cannot be generalized but the pullback can. In fact the exterior derivative can as well. Now since ► is the action of "eating" a generic tensor in practice the only thing we ever eat are vectors multilinearly. Here's an example: a linear transformation eats a vector linear. An inner product eats two vectors bilinearly (symmetrically). A symplectic product does the same antisymmetrically. Valued fields are in some sense the generalization of multilinearity of functions not just to the $\mathbb{R}$, but to any vector space. And the objects which eat vectors multilinearly are co-vectors, or tensor products of co-vectors, or symmetric/antisymmetric products of co-vectors etc. So instead of always having to say the bundle of k-many (symmetric/antisymmetric/etc.) tensor products of the co-vectors T_p^*M. We simply say the bundle of $k-$forms. For antisymmetric k-forms this bundle has a locally finite dimension so we will talk about the bundle of antisymmetric forms which will inherit the wedge product pointwise to become something we will call the Grassmann algebra. All this to say in this section we are referring to forms.

Def: k-*Forms*

The k-**Form Bundle** on M is the vector bundle $T^{*k}M = \bigcup_{p\in M} T_p^* M^{\otimes k}$. It has **symmetric** and **antisymmetric sub-bundles** $\mathrm{Sym}^k M$ and $\bigwedge^k M$ each with fibers $\mathrm{Sym}^n(T_p^*M)$ and $\bigwedge^n(T_p^*M)$ respectively. A k-**Form** is a section of one of these three bundles $T^{*k}M, \mathrm{Sym}^k M, \bigwedge^k M$ (the latter two are **symmetric and antisymmetric k-forms** respectively). We will call the sections of these bundles $F^kM = \Gamma_k^0 M, S^kM, \Omega^k M$ respectively.

Now since the antisymmetric spaces are finite (there are only 2^m of them) it follows that we should define the fiber-wise direct sum of these bundles as the result will have a finite local basis. So the antisymmetric form bundle

which will have sections in $\Omega M = \bigoplus_n \Omega^k M$ called antisymmetric forms (with the degree not specified)

Def: *Wedge Product on ΩM and the Grassmann Algebra*

We may inherit the **wedge (exterior) product** to ΩM from $\bigwedge(T_p^* M)$ pointwise such that

$$\forall \omega, \sigma \in \Omega M : (\omega \wedge \sigma)|_p = \omega|_p \wedge \sigma|_p$$

And by this the end result is that $(\Omega M, \wedge)$ is an algebra called the **Grassmann algebra** of M. As an example if $\omega, \sigma \in \Omega^1 M$ then we have $\forall X, Y \in \Gamma_0^1 M$

$$(\omega \wedge \sigma) \triangleright (X, Y) = \omega(X)\sigma(Y) - \sigma(X)\omega(Y)$$

Lemma: *Graded Local Basis*

Moreover we know that the local basis of ΩM is simply given

$$\begin{gathered} d\phi^{\mu_1} \wedge \dots \wedge d\phi^{\mu_k} \\ \text{s.t.,} \ \mu_1 < \dots < \mu_k \end{gathered}$$

Which we abbreviate using the multi-index notation. Hence for $\omega \in \Omega M$ we have

$$\omega = \omega_{\mathrm{M}} d\phi^{\mathrm{M}}$$

Exterior Derivative and Pullbacks of Forms

The exterior derivative as it exists $d : \Gamma_0^0 M \to \Gamma_1^0 M$ can now be extended in the following way by realizing that $\Omega^i M = \Gamma_i^0 M$ for $i = 0, 1$.

Def: *Exterior Derivative*

The general coordinate free definition of the **exterior derivative** is $\forall \omega \in \Omega^k M$ define $d_k : \Omega^k M \to \Omega^{k+1} M$ and then $d \in \mathrm{End}(\Omega M)$ as $d|_{\Omega^k M} = d_k$ as follows

$$d\omega \triangleright (X_i)_{i=1}^{k+1} = \sum_{i=1}^{k+1} (-1)^{i+1} X_i \triangleright \left(\omega \triangleright (X_j)_{j=1, j \neq i}^{k+1}\right) + \sum_{l=1, l<i}^{k+1} (-1)^{i+l} \omega \triangleright ([X_i, X_l], X_j)_{j=1, j \neq i, l}^{k+1}$$

💡 So for 0 forms

$$df \triangleright X = X \triangleright f + 0 = X \triangleright f$$

As we had before. Whereas for 1 forms

$$d\omega \triangleright (X, Y) = X \triangleright (\omega \triangleright (Y)) - Y \triangleright (\omega \triangleright (X)) - \omega \triangleright ([X, Y])$$

Lets drop the action notation to see whats going on:

$$d\omega = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = \{X, Y\}_\omega - \omega([X, Y])$$

Where $\{,\}_\omega$ is just a notational trick to compare the two terms. The exterior derivative here is the difference between the form on the bracket of the 2-form-bracket. And so on and so forth.

Obviously this gets tedious to right down , but there is a local formula one can use

Lemma: *Exterior Derivative Locally*

If $\omega \in \Omega M$ such that $\omega = f_I d\phi^I$ then:

$$d\omega = \frac{\partial f_I}{\partial \phi^\mu} d\phi^\mu \wedge d\phi^I$$

Proof:

Not given. it's simple but long.

Def: *Pullback of Forms*

The notion of pullback can also be given in this way. Let ω be any k-form (symmetric or antisymmetric or neither). Then the **pullback** of ω by any analytic map ψ is the form $\psi^*\omega$ given by the following identity

$$(\psi^*\omega \triangleright (X_i)_{i=1}^k)|_{\psi(p)} = \omega \triangleright ((\psi_*|_{\psi(p)} X_i|_{\psi(p)})_{i=1}^k)$$

For antisymmetric forms $\omega \in \Omega M$ just define ψ^* as being linear by the grade

Lemma: $[d, \psi^*] = 0$ *on* ΩM

$\forall \omega \in \Omega M$ we have

$$d\psi^*\omega = \psi^* d\omega$$

Valued Forms

Recall that for any two vector bundles P, Q we have a notion of the vector bundle $P \otimes Q$ where the sections $\sigma \in \Gamma(P \otimes Q)$ are given by a linear combination of $\omega|_p \otimes \gamma|_p$ where $\omega \in \Gamma P$ and $\gamma \in \Gamma Q$. This is how we constructed $T^r_s(TM)$ and how we defined tensor products of fields two sections ago. Here we use it for a different purpose. Let V be an arbitrary vector space. We can also think of it as a trivial vector bundle over M with typical fiber V. Then a tensor product section $\omega \in \Gamma V$is given by $\forall p \in M : \omega|_p = v \in V$ a constant vector. And the tensor product $V \otimes P$ simply has sections which are isomorphic to $V \otimes \Gamma P$. In particular if P is one of the form bundles above then ΓP can be thought of as the bundle of functions which multilinearly eat some number of vectors and spit out a real number. If we then take $V \otimes \Gamma P$ the result is now not a real number but an element of V. That is it is vector-valued

Def: *Valued Forms*

Let V be an arbitrary vector space. Then the **V-valued form bundle** is $V \otimes P$ for $P = T^{*k}M, \mathrm{Sym}^k M, \bigwedge^k M, \bigwedge M$. Its sections are **$V$-valued forms** which live in $V \otimes \Gamma P$ for $\Gamma P = F^k M, S^k M, \Omega^k M, \Omega M$. For the rest of this chapter we will assume that V-valued forms are antisymmetric.

💡 How can we extend the definitions we have for $\triangleright, d, \psi^*$ to these forms? Well to do this we'll need to see what these forms look like. Lets make it easy. Say $M = \mathbb{R}^2$ and we have the global canonical coordinates x^i. Then ΩM has a basis $\{1, dx, dy, dx \wedge dy\}$. Now lets suppose that V has a basis $\{a_i\}$. Then an arbitrary element of $V \otimes \Omega M$ looks like (omitting writing tensor products everywhere)

$$\omega = f^i a_i + g^j a_j dx + h^k a_k dy + e^l a_l dx \wedge dy$$

Where f^i, g^j, h^k, e^l are all $\Gamma^0_0 M$. So the first question would be, how can we allow such an object to act on a vector field? Simply let the co-vector terms do it for you. That is

$$(g^j_\nu a_j dx^\nu) \triangleright X = g^j_\nu a_j (dx^\nu \triangleright X) = X^\mu \delta^\nu_\mu g^j_\nu a_j = X^\mu g^j_\mu a_j = \tilde{g}^j_\mu a_j$$

And the result is vector valued as we want it to. What about exterior derivatives how could they possibly act? In the same way, just let them operate on the co-vector term so

$$d(g^j_\mu a_j dx^\mu) = \frac{\partial g^j_\mu}{\partial x^\nu} a_j dx^\nu \wedge dx^\mu$$

And finally what about the pullback? Same thing. In general the fact that $\triangleright, d, \psi^*$ don't care about where forms take there values means we can define all three in the following way for generic V-valued forms.

Def: *Extensions of Definitions to Valued Forms*

Extend the definitions $\triangleright, d, \psi^*$ in the following ways.

$$(a \otimes \omega) \triangleright X = a \otimes (\omega \triangleright X)$$
$$d(a \otimes \omega) a \otimes d\omega$$
$$\psi^*(a \otimes \omega) = a \otimes \psi^* \omega$$

And then extend these linearly over all tensor products.

Algebra Valued Forms

Suppose that the vector space in which the forms take their value is also an algebra. Does this extend anything nicely for antisymmetric forms in the Grassmann algebra? Well instead of taking the tensor products of vector spaces we take the tensor product of algebras wherein

$$(a \otimes x)(b \otimes y) = ab \otimes xy$$

Following bi-linearity and the like. For us this translates to the following

Def: *Valued Grassmann Algebra*

Let A be an algebra, then the set of antisymmetric A valued forms $A \otimes \Omega M$ has **a product** $\times$ given by the following: $\omega = a \otimes \alpha$ and $\sigma = b \otimes \beta$ for $a, b \in A$ and $\alpha, \beta \in \Omega M$ then

$$\omega \times \sigma = ab \otimes \alpha \wedge \beta$$

And this product is extended bilinearly. The algebra $A \otimes \Omega M$ with $\times$ is called the Valued Grassmann Algbera. For convenience we also define $\omega \Delta \sigma = [a, b] \otimes \alpha \wedge \beta$. Which since both $[,]$ & $\wedge$ are anticommuting we get $\sigma \Delta \omega = \omega \Delta \sigma$.

💡 Lets work locally and again omit tensor products. So for $\omega = f^i_\mu a_i dx^\mu$ and $\sigma = g^i_\nu a_j dx^\nu$ for multi-indices μ, ν and a basis a_j for A (where f, g are functions for every index). Then we observe that

$$\sigma \times \omega = f^i_\mu g^j_\nu a_j a_i dx^\nu \wedge dx^\mu = f^i_\mu g^j_\nu (a_j a_i - a_i a_j + a_i a_j) dx^\nu \wedge dx^\mu = \sigma \times \omega = f^i_\mu g^j_\nu a_j a_i dx^\nu \wedge dx^\mu$$
$$f^i_\mu g^j_\nu ([a_j, a_i] + a_i a_j) dx^\nu \wedge dx^\mu$$
$$f^i_\mu g^j_\nu [a_j, a_i] dx^\nu \wedge dx^\mu + f^i_\mu g^j_\nu a_i a_j dx^\nu \wedge dx^\mu$$
$$f^i_\mu g^j_\nu [a_j, a_i] dx^\nu \wedge dx^\mu - f^i_\mu g^j_\nu a_i a_j dx^\mu \wedge dx^\nu$$
$$\sigma \Delta \omega + (-1)^{|\mu| + |\nu|} \omega \times \sigma$$
$$\implies \sigma \times \omega = \sigma \Delta \omega + (-1)^{|\mu| + |\nu|} \omega \times \sigma$$

All to say that this new product on is not necessarily anticommutative. If A is abelian like $\mathbb{R}$ then we get the usual Grassmann rules $\sigma \times \omega = (-1)^{|\mu| + |\nu|} \omega \times \sigma$. But if the algebra is nontrivial (like a Lie algebra) then the product might obey some strange properties.

Lemma:

$$\sigma \times \omega = \sigma \Delta \omega + (-1)^{|\mu| + |\nu|} \omega \times \sigma$$

So lets assume A is an anticommutative algebra (like a Lie Algebra). Then the commutator in the above definition is actually 2 times the product. So in that case we have that

$$\omega \Delta \sigma = 2 \omega \times \sigma$$

And so the commutation rule becomes

$$\sigma \times \omega = \sigma \Delta \omega + (-1)^{|\mu| + |\nu|} \omega \times \sigma = (2 + (-1)^{|\mu| + |\nu|}) \omega \times \sigma$$
$$\implies \sigma \times \omega = (2 + (-1)^{|\mu| + |\nu|}) \omega \times \sigma$$

In particular for 1 forms $\sigma \times \omega = \omega \times \sigma$. That is $\omega \times \omega \neq 0$, as it would normally be for a Grassmann algebra.

Corollary:

In an anticommutative algebra the product of 1 forms is commutative.

$$\sigma \times \omega = \omega \times \sigma$$

💡 What could this product look like? Lets again assume that you have the ability to write $\omega = A_\nu dx^\nu$ where for each ν we have $A_\nu = f^i_\nu a_i$ as before. Then the product if these are Lie Algebra valued is

$$\omega \times \omega = [A_\mu, A_\nu] dx^\mu \wedge dx^\nu$$

Which if we feed this two form the vectors ∂_μ (using the shorthand convention) then

$$(\omega \times \omega) \triangleright (\partial_\rho, \partial_\sigma) = [A_\rho, A_\sigma]$$

This should be very familiar to anyone who knows anything about gauge theories.

Def: *Local Representations of Algebra Valued Forms*

Are there any other ways in which Valued forms can interact? There's one more. If V is a vector space which is a representation space for the algebra A through the representation ϱ then there is a function called **a local representation** $\tilde{\varrho} : A \otimes \Omega M \to \mathrm{End}(V \otimes \Omega M)$ as follows:

$$\tilde{\varrho}[A_\mu \otimes dx^\mu](v_\nu \otimes dx^\nu) = \varrho[A_\mu](v_\nu) dx^\mu \wedge dx^\nu$$

In conclusion, a form can be thought of as a (usually antisymmetric) function of vector fields. The value that the form takes is irrelevant so long as it lives in a vector space pointwise. Such forms have a natural product (given an algebra), exterior derivative and pullback. All of these will become very relevant in the next part.
