PII, Ch 9 : Connections

This chapter introduces the notion of vertical vector fields in a principal bundle and then asks what would a horizontal field be? In order to get there we must talk about how to connect several different tangent spaces horizontally which amounts to a special choice of one form on the principal bundle. From this comes a special derivative that keeps this connection in mind.

Vertical Spaces and Generated Vector Fields

Def: *Generated Vector Fields on a Principle Bundle*

Consider that for any $A \in \mathfrak{g}$ we have that $\exp(tA) \in G$, and hence we may define the following curve $\gamma(t) = p \triangleleft \exp(tA)$, by the fibre-wise closure of the action $\gamma(t)$ is a curve in the fibre, this allows us to define the following, **the vector field generated by** A, $X^A \in \Gamma^1_0 P$ as follows

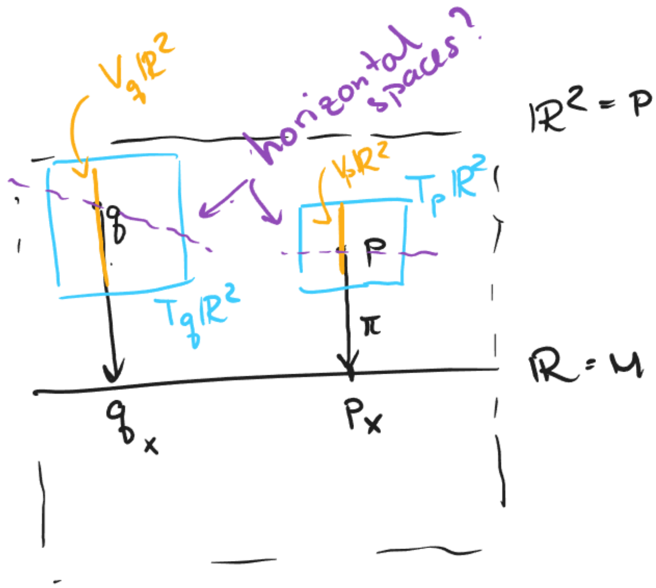
$$X^A|_p \triangleright f = \frac{d}{dt} f(p \triangleright \exp(tA))|_{t=0}$$

Def: *Vertical Spaces*

The existence of the surjection $\pi : P \to M$ induces a map $\pi_*|_p : T_pP \to T_{\pi(p)}M$. The we define $V_pP = \ker \pi_*|_p$, called the **vertical subspace** . Furthermore we denote the sub-bundle $VP = \bigcup_{p \in M} V_pP$. A vector field X is said to be vertical if

$$\forall p \in P : X|_p \in V_pP$$

We should picture the following in a simple example with $P = \mathbb{R}^2$and $M = \mathbb{R}^1$



Lemma: *Generated Vector Fields are Vertical*

Proof:

$$\pi_*|_p X^A|_p \triangleright f = X^A|_p \triangleright (f \circ \pi) = \frac{d}{dt} f(\pi(p \triangleright \exp(tA)))|_{t=0} = \frac{d}{dt} f(\pi(p))|_{t=0} = 0$$
$$\implies X^A|_p \in \ker \pi_*|_p$$

Lemma: *Vertical Vector Fields are Generated*

Suppose $\pi_*|_p X|_p = 0$, then it follows that $\frac{d}{dt} f(\pi(\gamma(t)) = 0$, well as $\gamma(t)$ is a curve in the fibre it follows that $\gamma(t) = \tilde{p} \triangleright g(t)$ where $\tilde{p} \in \mathrm{orb}(p)$ and we know that $g(t) = g_0 \exp(tA)$ then we can write choose $p = \tilde{p} \triangleleft g_0$. Such that we get $\gamma(t) = p \triangleright \exp(tA)$. Then we have that

$$X = X^A$$

Corollary

Since all vertical vector fields are generated and vice versa it follows that $i_p : \mathfrak{g} \to V_p P$ given by $i_p(A) = X^A|_p$ is a bijection $\forall p \in P$

Lemma: *An Identity For Vertical Fields*

$$\triangleleft g_* X^A = X^{\mathrm{Ad}_{g^{-1}} A}$$

Proof:

Omitted but was a quick computation.

Connections, Connected Derivative and Curvature

We would like to have a similar notion for what it would mean to have a horizontal space but to do that we would need to know what it means

Def: *Connection (Horizontal Spaces)*

A connection is a map $f : P \to \mathcal{P}(TP)$ such that $f(p) = H_p P \in \mathcal{P}(T_p P)$ and we have

$$H_p P \subset_{\mathcal{VS}} T_p P$$
$$V_p P + H_p P = T_p P$$

These two conditions induce the existence of the breakdown $\forall X \in \Gamma^1_0 P : X|_p = \mathrm{hor}(X)|_p + \mathrm{ver}(X)|_p$ such that $\mathrm{hor}(X)|_p \in H_p P$ and $\mathrm{ver}(X)|_p \in V_p P$. We have in addition that

$$\mathrm{hor}(X), \mathrm{ver}(X) \in \Gamma^1_0 P$$

> 💡 Evidently the choice of such an f is non-canonical and arbitrary. To concretize it a little we require that a connection on a principle bundle is compatible with the principle structure.

Corollary:

A field is horizontal if $X = \mathrm{hor}(X)$ and vertical if $X = \mathrm{ver}(X)$

Def: *Principle Connections*

If P is a principle G bundle then we a connection f is called **a principle G connection** if and only if

$$\forall g \in G : H_{p \triangleleft g} = \mathrm{im}_{\triangleleft g_*|_p}(H_p P)$$

For our purposes all connections will be principle.

Theorem: *A Principle Connection is equivalent to a $\mathfrak{g}$ Valued 1-Form*

A Principle Connection is equivalent to a Lie Algebra Valued 1 form $\omega \in \Omega^1 P \otimes \mathfrak{g}$ satisfying

$$\omega \triangleright X^A = A$$
$$(\triangleleft g^* \omega) \triangleright X = \mathrm{Ad}_{g^{-1}}(\omega \triangleright X)$$

Proof:

The forwards proof goes as follows, suppose we have a principle connection, then $\mathrm{ver}(X)$ exists, then we may define

$$\omega|_p \triangleright X = i_p^{-1}(\mathrm{ver}(X|_p)) \in \mathfrak{g}$$

Which intern defines ω. From this it follows $H_pP = \ker(\omega|_p \triangleright)$. Why? Because if $\omega \triangleright X = 0 \iff \mathrm{ver}(X) = 0$ as i is invertible. And if so $X = \mathrm{hor}(X)$ so naturally $X|_p = \mathrm{hor}(X)|_p \in H_pP$. Now that ω satisfies the first condition is trivial as X^A is vertical so $i_p^{-1}(\mathrm{ver}(X^A|_p)) = i_p^{-1}(X^A|_p) = A$. That ω satisfies the 2nd property is a little more difficult, we need to consider two cases. First when X is vertical, so $X = X^A$, then

$$((\triangleleft g^*\omega) \triangleright X^A)|_p = \omega|_{p\triangleleft g} \triangleright (\triangleleft g_* X^A|_p) = \omega|_{p\triangleleft g} \triangleright (X^{\mathrm{Ad}_{g^{-1}}A}|_{p\triangleleft g}) = \mathrm{Ad}_{g^{-1}}A = \mathrm{Ad}_{g^{-1}}(\omega \triangleright X^A)|_p$$

Since this holds for all p we have that $(\triangleleft g^*\omega) \triangleright X^A = \mathrm{Ad}_{g^{-1}}(\omega \triangleright X^A)$. Now if we should assume that the field is horizontal, then the argument progresses in much the same way until

$$((\triangleleft g^*\omega) \triangleright X)|_p = \omega|_{p\triangleleft g} \triangleright (\triangleleft g_* X|_p)$$

But now since X is horizontal it follows that $X|_p \in H_pP$ and hence $\triangleleft g_*|_p X|_p \in \mathrm{im}_{\triangleleft g_*}(H_pP)$ but by the axioms of a principle connection we know that this is equivalent to $H_{p\triangleleft g}P$ and hence $\triangleleft g_* X$ is again horizontal. Lets call it Y to save time. Hence we now have

$$\omega \triangleright Y|_{p\triangleleft g} = i_{p\triangleleft g}^{-1}(\mathrm{ver}(Y)|_{p\triangleleft g})$$

But as Y is horizontal the whole thing is 0

$$\omega \triangleright Y|_{p\triangleleft g} = 0 = \mathrm{Ad}_{g^{-1}}0 = \mathrm{Ad}_{g^{-1}}\omega \triangleright X$$

This completes the forwards proof. The backwards proof is quite simple. Given the two conditions of ω, we define the map ver as follows

$$\mathrm{ver}(X)|_p = i_p(\omega \triangleright X|_p)$$

We know this works because all vertical fields are generated $\mathrm{ver}(X^A)|_p = X^A|_p$ which is exactly what $i_p(\omega \triangleright X^A|_p) = i_p(A) = X^A|_p$. Next take $\mathrm{hor}(X) = X - \mathrm{ver}(X)$. We know both $\mathrm{hor}, \mathrm{ver}$ are smooth functions. Finally define $H_pP = \ker(\omega|_p \triangleright)$. Its clear from the 2nd property that $\forall g \in G : H_{p\triangleleft g} = \mathrm{im}_{\triangleleft g_*|_p}(H_pP)$ as we explained in the forwards proof. Hence any Lie algebra valued 1 form ω induces a principal connection. QED.

💡 For these reasons some authors write call a Lie algebra valued 1-form a connection. This feels kind of sneaky to me so I will separate the terminology, but I will call such 1-forms **connection 1-forms**. Moreover we will call a principle bundle with a chosen connection, **a principle bundle with connection**.

Given any connection 1-form we have the ability to define a new exterior derivative that takes the connection into account. We call this the connection's exterior derivative

Lemma: ***Brackets of Vector Fields***

If X_1, X_2 are horizontal fields and Y_1, Y_2 are vertical fields then $[X_1, X_2], [X_1, Y_1]$ are horizontal and $[Y_1, Y_2]$ is vertical

Def: ***Connected Derivative***

Let V be any vector space (or algebra), then $\mathcal{D}_\omega : \Omega^k P \otimes V \to \Omega^{k+1} P \otimes V$ which is defined as follows. $\forall \phi \in \Omega^k P \otimes V$ we have that $\mathcal{D}_\omega \phi$ is that $k+1$ form satisfying

$$\mathcal{D}_\omega \phi \triangleright (X_i)_{i=1}^{k+1} = d\phi \triangleright (\mathrm{hor}(X_i))_{i=1}^{k+1}$$

Given that the choice of connection is usually implies we omit writing the subscript ω every time.

Def: ***Curvature 2-Form***

We define the $\mathfrak{g}$-valued 2 form called the curvature form of the connection as follows

$$\Omega = \mathcal{D}\omega$$

Lemma:

$$\Omega = d\omega + \omega \times \omega$$

Proof:

The proof for this theorem is long, but roughly consist of evaluating $\mathcal{D}\omega$ and $d\omega + \omega \times \omega$ on vertical and horizontal fields, to show equality.

Theorem: ***Bianchi's Identity***

$$\mathcal{D}\Omega = 0$$

Proof:

Like the above proof, this proof evaluates on vertical and horizontal fields to show equality.

> 💡 It should be noted that Bianchi I can be re-expressed as $\mathcal{D}^2\omega = 0$, in contrast with $d^2\omega = 0$. No where the latter can be generalized into $d^2 = 0$ the former cannot be generalized to $\mathcal{D}^2 = 0$, and so Bianchi I stands as a special case where this property of d appears in $\mathcal{D}$.

In summary for this chapter, vertical vector fields are those generated by Lie algebra elements, and horizontal fields are those generated by 0 (roughly as they lie in the kernel of the connection form). All of these notions about giving connecting at first completely separate and distinct tangent spaces together with a 1 form causes the existence of a special derivative.