Assignment 2: Causal Inference

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1 Structure of the Network

The problem modelled is a generic trip by car, influenced by factors such as the strike of public transport, road works or even the weather, and the delay that comes with it. The causal diagram that model this problem includes the following variables:

- `Weather`: weather during the journey that should be *sunny* or *rainy*.
- `Strike`: *true* if a strike of public transport takes place, *false* otherwise.
- `RushHour`: *true* if the time it's rush hour, *false* otherwise.
- `RoadConditions`: condition of the road floor, that depends on the weather, and should be *dry* or *wet*.
- `Mood`: mood of the driver, dependent on the weather, that is *good* or *bad*.
- `RoadWorks`: *true* if there are road maintenance works in progress, *false* otherwise. This variable depends on the conditions of the road.
- `Speed`: driving velocity, that should be *slow* or *fast*, dependent on the road condition, if it's rush hour and if there are road works.
- `Danger`: danger incurred during the trip, that should be *low* or *high*, dependent on the driving velocity, the road conditions and if it's rush hour.
- `Accident`: *true* if an accident occurred during the trip, *false* otherwise. It is influenced by the danger, the mood of the driver and if there is a strike of the public transport.
- `Delay`: *true* if the trip is delayed, *false* otherwise.

The objective of the network is highlight how weather and humour impacts on travel safety. The graph could provide valuable indications about the correlation between the driver humour and a delayed trip, or for example between the weather and the risk of an accident and also on how the road conditions influences car crashes.

Each node is connected by an arrow to one or more other nodes upon which it has a causal influence. Most of the arcs orientation are self-explaining. An exception was made for the `Mood` variable, that influences the accident risk and is caused only by the weather and not for example by the delay.

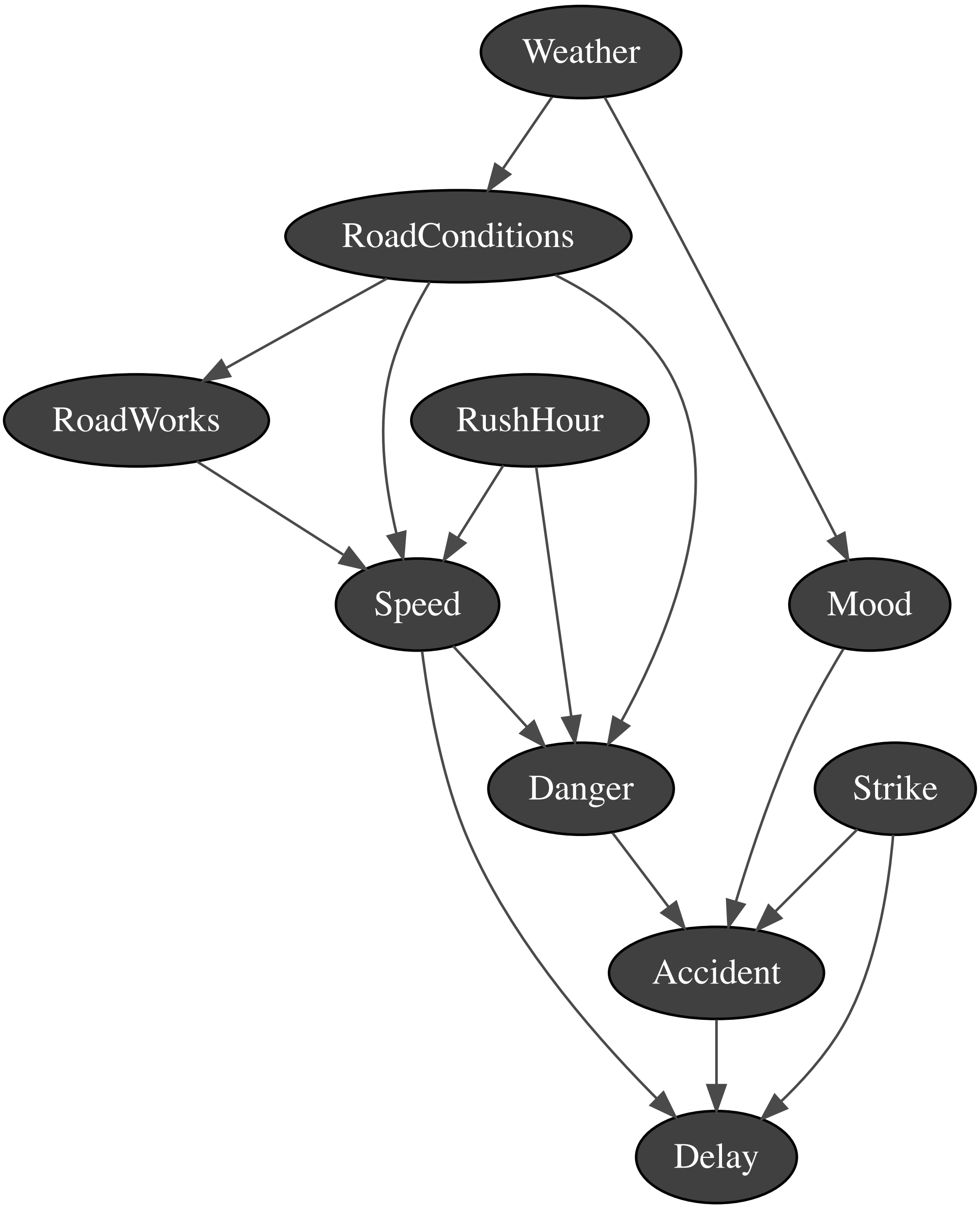

Figure 1: Bayesian Network

An arc can be inverted if and only if no v-structures, i.e. colliders in which the parents are not adjacent, are generated or destroyed in doing so.

In the model there are only two arrow that can be inverted, and these are the one that goes from the variable `Weather` to `Mood` and the one that goes from the variable `Weather` to `RoadConditions`. In fact, even inverting the two arrows no v-structured are created, therefore the three graphs that are generated by the reversions of the two arcs are equivalent and not distinguishable by any statistical test.

Instead, the arc from `RoadWorks` to `RoadConditions` cannot be reverted since doing this a v-structure is created. Hence, the arc could only be turned on condition of turn also the one between `Weather` and `RoadConditions`.

D-Separation

D-Separation tells when two variables are d-separated along a path (blocked), that means independent and when they are d-connected along a path (unblocked) or likely dependent. They are actually independent if they are d-separated along all possible paths. They are likely dependent if there is at least one unblocked path connecting them.

A path is blocked by a set of nodes if and only if the path contains a chain of nodes or a fork such that the middle node is in the set of nodes or if the path contains a collider such that the collision node and every descendant are not in the given set of nodes.

- X: `RushHour`, Y: `RoadConditions`:
  Conditioning on one of the variables `Strike`, `RoadWorks` `Weather` or `Mood`, `RushHour` and `RoadConditions` are d-separated since the path from these two variables are all blocked. It is possible to write

  $$Y = \alpha + r_x X + r_a A + \epsilon \quad \text{with } r_a = 0$$

  where A for example is `Strike`. They are not actually independent since they are not d-separated along all the paths. For example, conditioning on one of the variables `Danger`,`Speed`, `Accident` and `Delay`, the path from X to Y contains a collider in which the evidence is a collision node. In fact, all the variables d-connected are dependent in the real problem, for example `RushHour`, `RoadConditions` and `Danger`, while `RushHour`, `RoadConditions` and `Mood` are independent.

- X: `RushHour`, Y: `Strike`:
  Conditioning on one of the variables `Danger`, `RoadWorks`, `Weather`, `Speed` or `Mood`, `RushHour` and `Strike` are d-separated since the path from these two variables are all blocked. Instead, they are not d-separated along the paths that condition on variables `Accident` and `Delay` since in both the cases the paths contain a collider in which the given is the collision node. Hence, they are not independent. In the real problem, in fact, the d-connected variables are dependent, for example `RushHour`, `Strike` and `Danger`, while `RushHour`, `Strike` and `RoadWorks` are independent.

- X: `Speed`, Y: `Accident`:
  Are not d-separated given any of the variables of the domain. "Danger", "Mood", "RoadConditions"

- X: `RoadConditions`, Y: `Strike`:
  In this case the same considerations made for the variables `RushHour` and `Strike` in example 2 apply.

- X: `Speed`, Y: `Mood`:
  Conditioning on one of the variables `RoadConditions` or `Weather`, `Speed` and `Mood` are d-separated since the path from these two variables are all blocked. Instead, they are not d-separated along the paths that condition on all the other variables, so they are not independent. The d-connected variables, for example `Speed`, `Mood` and `Danger`, are dependent in the real problem, while `Speed`, `Mood` and `RoadConditions` are not dependent.

2 Conditional Probability Tables

The Conditional Probability Tables (CPTs) of the variables of the model, that show all possible inputs and outcomes with their associated probabilities, are filled sometimes using information retrieved from online survey, other times are estimated based on common sense. In case of the variable `Weather`, the prior probability is difficult to estimate because it is dependent on different factors, such as the location. For this reason I decided to use some information retrieved online[1] about the average precipitation in Lugano. Also the probabilities associated to the variable `Strike` are obtained from internet and referred to the number of strike of the public transport in Lugano.

Strike	
False	True
0.9600	0.0400

Strike CPT

Weather	
Sun	Rain
0.7000	0.3000

Weather CPT

RushHour	
False	True
0.8000	0.2000

RushHour CPT

	RoadConditions	
Weather	Dry	Wet
Sun	0.7000	0.3000
Rain	0.2000	0.8000

RoadConditions CPT

	Mood	
Weather	Good	Bad
Sun	0.7000	0.3000
Rain	0.4000	0.6000

Mood CPT

	RoadWorks	
RoadConditions	False	True
Dry	0.3000	0.7000
Wet	0.8000	0.2000

RoadWorks CPT

			Speed	
RoadConditions	RushHour	RoadWorks	Slow	Fast
Dry	False	False	0.1500	0.8500
		True	0.7500	0.2500
	True	False	0.8000	0.2000
		True	0.8500	0.1500
Wet	False	False	0.6000	0.4000
		True	0.9000	0.1000
	True	False	0.8000	0.2000
		True	0.9500	0.0500

Speed CPT

			Danger	
RoadConditions	RushHour	Speed	Low	High
Dry	False	Slow	0.9500	0.0500
		Fast	0.3000	0.7000
	True	Slow	0.4500	0.5500
		Fast	0.1500	0.8500
Wet	False	Slow	0.4500	0.5500
		Fast	0.1500	0.8500
	True	Slow	0.3500	0.6500
		Fast	0.0500	0.9500

Danger CPT

[1] `https://www.climatestotravel.com/climate/switzerland`

			Accident	
Danger	**Strike**	**Mood**	**False**	**True**
Low	False	Good	0.9500	0.0500
		Bad	0.4500	0.5500
	True	Good	0.8500	0.1500
		Bad	0.3000	0.7000
High	False	Good	0.4000	0.6000
		Bad	0.2000	0.8000
	True	Good	0.1500	0.8500
		Bad	0.0500	0.9500

Accident CPT

			Delay	
Speed	**Strike**	**Accident**	**False**	**True**
Slow	False	False	0.7500	0.2500
		True	0.0500	0.9500
	True	False	0.3500	0.6500
		True	0.0100	0.9900
Fast	False	False	0.8500	0.1500
		True	0.0500	0.9500
	True	False	0.6000	0.4000
		True	0.0100	0.9900

Delay CPT

3 Causal Inference

Causal Effect

Given the graph, and a pair of variables `X: Speed` and `Y: Accident`. In this case, `A: RoadConditions` and `B: RushHour` are the backdoor variables. Then the causal effect of `X` on `Y` is given by the following formula:

$$\mathrm{P}(Y = y \mid do(X = x)) = \sum_{A} \sum_{B} \mathrm{P}(Y = y \mid X = x, A = a, B = b)\mathrm{P}(A = a, B = b) \tag{1}$$

where a, b, y and x range over all the combinations of values that the associated variable can take.

Calculating the cases in which $\texttt{Y} : \{y_0 = \texttt{false}, y_1 = \texttt{true}\}$ given $\texttt{X} : \{x_0 = \texttt{slow}, x_1 = \texttt{fast}\}$:

$$\mathrm{P}(y_0 \mid do(x_0)) = \sum_{A} \sum_{B} \mathrm{P}(y_0 \mid x_0, A = a, B = b)\mathrm{P}(A = a, B = b) = 0.6062$$

$$\mathrm{P}(y_1 \mid do(x_0)) = \sum_{A} \sum_{B} \mathrm{P}(y_1 \mid x_0, A = a, B = b)\mathrm{P}(A = a, B = b) = 0.3938$$

$$\mathrm{P}(y_0 \mid do(x_1)) = \sum_{A} \sum_{B} \mathrm{P}(y_0 \mid x_1, A = a, B = b)\mathrm{P}(A = a, B = b) = 0.4053$$

$$\mathrm{P}(y_1 \mid do(x_1)) = \sum_{A} \sum_{B} \mathrm{P}(y_1 \mid x_1, A = a, B = b)\mathrm{P}(A = a, B = b) = 0.5947$$

Confounders

Given the variables `X: Speed` and `Y: Accident`, they do not have any confounder. In fact, the possible confounders can be identified searching for the parents of `X`, that are `RoadWorks`, `RoadConditions` and `RushHour`, but none of this variables is a parent of `Y`, so they do not have any confounder.

Randomised Controlled Study

With this specific problem is not possible to perform randomised controlled study on every variable. For example, we cannot influence the weather, so randomise variables affecting `RoadConditions` or `Mood` is impossible.

By the way, if we are interested in performing a randomised controlled study to disentangle the causal effect between `X`: `Speed` and `Y`: `Accident` from their correlation it is necessary to fix or vary randomly the variable `X`. In this case is possible to fix the driving velocity by introducing for example a speed limit.

Average Causal Effect

The Average Causal Effect (ACE) of X on Y is computed for both the possible values of the variable `Accident`, that are $\texttt{Y} : \{y_0 = \texttt{false}, y_1 = \texttt{true}\}$

$$\begin{aligned} \text{ACE} &= \text{P}(y_0 \mid do(x_0)) - \text{P}(y_0 \mid do(x_1)) \approx -0.189 \\ \text{ACE} &= \text{P}(y_1 \mid do(x_0)) - \text{P}(y_1 \mid do(x_1)) \approx 0.189 \end{aligned} \tag{2}$$

C-Specific Effect

Given a new pair of variable such that `X`: `Danger` and `Y`: `Delay`, and chosen `C`: `RoadConditions`, the C-Specific Effect is given by:

$$P(Y = y|do(X = x), C = c) = \sum_z P(Y = y|X = x, C = c, Z = z)P(Z = z|C = c) \tag{3}$$

The set Z identified, such that $C \cup Z$ satisfy the backdoor criterion, includes the variable `Speed`. Defined $\texttt{X} : \{x_0 = \texttt{low}, x_1 = \texttt{high}\}$, $\texttt{Y} : \{y_0 = \texttt{false}, y_1 = \texttt{true}\}$ and $\texttt{C} : \{c_0 = \texttt{dry}, c_1 = \texttt{wet}\}$ it is possible to compute the C-Specific Effect for all the possible realisation of the variables as follows:

$$P(y_0|do(x_0), c_0) = \sum_z P(y_0|x_0, c_0, z)P(z|c_0) = 0.6150$$

$$P(y_1|do(x_0), c_0) = \sum_z P(y_1|x_0, c_0, z)P(z|c_0) = 0.3850$$

$$P(y_0|do(x_1), c_0) = \sum_z P(y_0|x_1, c_0, z)P(z|c_0) = 0.2866$$

$$P(y_1|do(x_1), c_0) = \sum_z P(y_1|x_1, c_0, z)P(z|c_0) = 0.7134$$

$$P(y_0|do(x_0), c_1) = \sum_z P(y_0|x_0, c_1, z)P(z|c_1) = 0.5633$$

$$P(y_1|do(x_0), c_1) = \sum_z P(y_1|x_0, c_1, z)P(z|c_1) = 0.4367$$

$$P(y_0|do(x_1), c_1) = \sum_z P(y_0|x_1, c_1, z)P(z|c_1) = 0.2662$$

$$P(y_1|do(x_1), c_1) = \sum_z P(y_1|x_1, c_1, z)P(z|c_1) = 0.7338$$

		Delay	
Danger	RoadConditions	False	True
Low	Dry	0.6150	0.3850
	Wet	0.5633	0.4367
High	Dry	0.2866	0.7134
	Wet	0.2662	0.7338

CPT C-Specific Effect

The minimal set of variables that must be measured in order to estimate the c-specific effect of X on Y includes `Danger`, `Delay`, `Speed` and `RoadConditions`.

Conditional Intervention

Given `X`: `Danger` and `Y`: `Delay`, and chosen `C`: `RoadConditions`, the Conditional Intervention in which we are interested is:

$$P(Y = y|do(X = g(C)))$$

where

$$g(C) := \left\{ \begin{array}{ll} low, & \text{if } C = dry \\ high, & \text{if } C = wet \end{array} \right\}$$

Then the conditional intervention is computed using

$$P(Y = y|do(X = g(C))) = \sum_c P(Y = y|do(X = g(C)), C = c)P(C = c) \tag{4}$$

Mediation and Controlled Direct Effect

Given a new pair of variable such that `X`: `RushHour` and `Y`: `Danger`, the variable `M`: `Speed` is a mediation variable between `X` and `Y`. The Controlled Direct Effect is computed with the following formula:

$$\text{CDE} = P(Y = y|\text{do}(X = x), \text{do}(M = m)) - P(Y = y|\text{do}(X = x'), \text{do}(M = m)) \tag{5}$$

Since there are not any spurious path between X and Y, the formula can be rewritten as follows:

$$\text{CDE} = P(y|x, \text{do}(m)) - P(y|x', \text{do}(m))$$

Then, applying the backdoor criterion it is possible to bock the spurious path between M and Y, in particular using `Z`: `RoadConditions` as backdoor variable. We should consider the variable `W`: `Weather` since is parent of Z. Finally we obtain

$$\text{CDE} = \left[\sum_{z,w} P(y|x, m, z)P(z|w)P(w)\right] - \left[\sum_{z,w} P(y|x', m, z)P(z|w)P(w)\right]$$

Defined X : $\{x_0 = \texttt{true}, x_1 = \texttt{false}\}$, Y : $\{y_0 = \texttt{low}, y_1 = \texttt{high}\}$, M : $\{m_0 = \texttt{slow}, m_1 = \texttt{fast}\}$ the CME are:

$$\text{CDE} = P(y_0|\text{do}(x_0), \text{do}(m_0)) - P(y_0|\text{do}(x_1), \text{do}(m_0)) = 0.1050 - 0.2330 = -0.128$$

$$\text{CDE} = P(y_1|\text{do}(x_0), \text{do}(m_0)) - P(y_1|\text{do}(x_1), \text{do}(m_0)) = 0.8950 - 0.7675 = 0.128$$

$$\text{CDE} = P(y_0|\text{do}(x_0), \text{do}(m_1)) - P(y_0|\text{do}(x_1), \text{do}(m_1)) = 0.405 - 0.725 = -0.31999999999999995$$

$$\text{CDE} = P(y_1|\text{do}(x_0), \text{do}(m_1)) - P(y_1|\text{do}(x_1), \text{do}(m_1)) = 0.595 - 0.275 = 0.31999999999999995$$

CDE of Y changing the value of X.

4 Simulation

Suppose that you can't measure some parents of variable X chosen in every point of "Causal Inference". Repeat the "Causal Inference" part of the exercise considering this new situation.

5 Comment on the Results

What kind of experience have you got with this model? E.g., is the causal model responding in a sensible way to your queries? What should be changed/modified to make it more realistic?