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ScienceDirect
Systems & Control Letters 56 (2007) 272–276
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

On the reachability in any fixed time for positive continuous-time linear systems

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Received 22 May 2006; received in revised form 13 September 2006; accepted 7 October 2006
Available online 4 December 2006

Abstract

This paper deals with the reachability of continuous-time linear positive systems. The reachability of such systems, which we will call here the strong reachability, amounts to the possibility of steering the state in any fixed time to any point of the positive orthant by using nonnegative control functions. The main result of this paper essentially says that the only strongly reachable positive systems are those made of decoupled scalar subsystems. Moreover, the strongly reachable set is also characterized.

Keywords: Positive linear systems; Reachability

1. Introduction

Positive linear systems appear naturally in the modeling of a lot of practical problems in economics, engineering and biology [8,1]. The reachability of such systems has been studied in the past but mainly for discrete-time systems [5,6,10], see [2–4] for more recent references in which it appears that this problem is now well solved and understood.

It seems that the reachability of continuous-time systems is harder and was much less studied, see [9] for a classical reference and [7] for a more recent state of the art. A common (and surprising) feature of these works is that they use as a definition of reachability the possibility of steering the state in any point of the positive orthant in *finite* time. In this paper we consider the possibility of steering the state in any point of the positive orthant in any fixed time. This property is referred to in this paper as the *strong reachability*. This seems to us to be more in the spirit of the original definition of reachability although the two definitions coincide in the case where the input function is not restricted to be nonnegative. In the case of positive systems, however, the two definitions are not equivalent. The example 13 of [7] shows a system that is reachable in finite time but not reachable in any fixed time (for some parameters choice).

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doi:10.1016/j.sysconle.2006.10.021

The main result of this paper is essentially that the only strongly reachable positive systems are those made of decoupled scalar subsystems. For nonstrongly reachable positive systems, a complete characterization of the set of strongly reachable states is given.

The paper is organized as follows. In Section 2 we review some well-known material on positive systems and their graph representation. In Section 3 we prove that the scalar systems are strongly reachable and give the necessary and sufficient conditions for strong reachability. Section 4 provides the strongly reachable set.

2. Preliminaries

2.1. Linear positive systems: definitions and basic results

In this paper we denote by $\mathbb{R}_+$ the set of nonnegative real numbers. A $p \times q$ matrix L is said to be *positive* if $L_{ij} \geqslant 0$ for $i = 1, \ldots, p$, $j = 1, \ldots, q$, and there exists indices i', j' such that $L_{i'j'} > 0$.

We study continuous-time linear time-invariant systems of the following form:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where the matrices A and B are real valued matrices of respective dimensions $n \times n$ and $n \times m$. The matrix B is a positive matrix and A is of Metzler type that is, $A_{ij} \geqslant 0$ for $i = 1, \dots, n$, $j = 1, \dots, n$ and $i \neq j$. The initial state $x(0) \in \mathbb{R}^n_+$ and the control input is a nonnegative piecewise continuous function $u(t) \in \mathscr{U} = \mathbb{R}^m_+$. The resulting state is therefore $x(t) \in \mathscr{X} = \mathbb{R}^n_+$.

Definition 1. The positive system (1) is said to be strongly reachable if and only if for any $T > 0$ and any $x_f \in \mathbb{R}^n_+$ there exists a nonnegative piecewise continuous function $u(t)$ that moves the state of the system from $x(0) = 0$ to $x(T) = x_f$.

It must be clear that the hard constraint here is the nonnegativity of the input function. If this constraint is dropped the reachability condition remains the Kalman criterion i.e. $\mathrm{rank}[B, AB, \dots, A^{n-1}B] = n$.

The following result is easy to prove and often used in the theory of linear positive systems.

Proposition 1. *The positive linear system* (1) *is strongly reachable if and only if for any $T > 0$ and any $i \in \{1, \dots, n\}$ there exists a nonnegative piecewise continuous function $u(t)$ that moves the state from* $x(0) = 0$ to $x(T) = e_i$ *where e_i is the ith unit basis vector*.

2.2. Graph of a positive system

We associate with a positive linear system of form (1) the so-called influence graph $G(A, B)$ [7]. The *vertex* set V of the graph $G(A, B)$ is given by $X \cup U$ with $X = \{x_1, \dots, x_n\}$ the set of state vertices and $U = \{u_1, \dots, u_m\}$ the set of input vertices. Hence, V consists of $n+m$ vertices. The *edge set* E of the graph $G(A, B)$ is composed of $E_A \cup E_B$ with $E_A = \{(x_j, x_i) | A_{ij} \neq 0\}$, $E_B = \{(u_j, x_i) | B_{ij} > 0\}$. For an edge (v, v'), v is called the *initial vertex* and v' the *final vertex*.

We say that there exists a *path* from vertex w_0 to vertex w_t if there are vertices $w_1, \dots, w_{t-1} \in V$ such that $(w_{i-1}, w_i) \in E$ for $i = 1, 2, \dots, t$. We call the vertex w_0 the *initial vertex* of the path and w_t the *final vertex*.

3. Strongly reachable systems

3.1. Scalar positive systems are strongly reachable

We consider a scalar positive system as in (1) with one input and one state ($m = n = 1$):

$$\dot{x}(t) = \alpha x(t) + \beta u(t), \tag{2}$$

where $\beta > 0$.

Denote by $H(t)$ the unit step function. With $x(0) = 0$ and $u(t) = H(t)$ we get for system (2) the response $x(t) = (\beta/\alpha)(1 - \mathrm{e}^{\alpha t})$ for $\alpha \neq 0$ and $x(t) = \beta t$ for $\alpha = 0$. Then, for any $T > 0$ and any $x_f \in \mathbb{R}_+$, using the input function

$$u(t) = \frac{\alpha x_f}{\beta(1 - \mathrm{e}^{\alpha T})} H(t) \quad \text{if } \alpha \neq 0 \tag{3}$$

and

$$u(t) = \frac{x_f}{\beta T} H(t) \quad \text{if } \alpha = 0, \tag{4}$$

we get $x(T) = x_f$ and the system is therefore strongly reachable. This proves the following.

Proposition 2. *The scalar system defined in* (2) *is strongly reachable.*

3.2. The general case

We consider again a system of type (1) with n states, m inputs and the associated graph $G(A, B)$. A piecewise continuous function $f(t)$ is called positive on $[0, T]$ if $f(t) \geqslant 0$ for any t in $[0, T]$ and there exists $t_1 \in [0, T[$ such that $f(t_1) > 0$. The following proposition generalizes a result of [7]:

Proposition 3. *Consider the input vector $\bar{u}_i(t) = [0, \dots, 0, u_i(t), 0, \dots, 0]^{\mathrm{T}}$ where the ith component $u_i(t)$ is a piecewise continuous positive function on* $[0, T]$. *Then $x_j(t)$ is a positive function on* $[0, T]$ *and* $x_j(T) > 0$, *if and only if there exists a path from the input vertex u_i to the state vertex x_j in $G(A, B)$.*

Proof. In Theorem 9 of [7] it is proved that there exists $t_1 \in [0, T[$ such that $x_j(t_1) > 0$ if and only if there exists a path from the input vertex u_i to the state vertex x_j in $G(A, B)$. Now let us prove that this implies $x_j(T) > 0$.

$$x(T) = \mathrm{e}^{A(T-t_1)} x(t_1) + \int_{t_1}^{T} \mathrm{e}^{A(T-t_1-\tau)} B u(\tau) \, \mathrm{d}\tau. \tag{5}$$

A can be written as $A = A' - \alpha I_n$ where α is a positive real number, I_n is the identity matrix of order n and A' is a positive matrix. In Eq. (5) the integral is nonnegative. Then let us consider the first part $\mathrm{e}^{A(T-t_1)} x(t_1) = \mathrm{e}^{-\alpha(T-t_1)} \mathrm{e}^{A'(T-t_1)} x(t_1)$. $\mathrm{e}^{A'(T-t_1)} = I_n + A'(T - t_1) + A'^2(T - t_1)^2/2 + \cdots$ is a positive matrix with, in particular, positive diagonal terms. Therefore $x_j(T) > 0$. $\square$

We can then state the general result on strong reachability.

Theorem 1. *A continuous-time positive system of type* (1) *with graph $G(A, B)$ is strongly reachable if and only if*:

1. *There exists a set of distinct input vertices $\{u_{i_1}, \dots, u_{i_n}\}$ such that there is an edge (u_{i_j}, x_j) for $j = 1, \dots, n$, and this edge is the unique edge of $G(A, B)$ with begin vertex u_{i_j}.*
2. *There is no edge (x_j, x_l) for $j = 1, \dots, n$, $l = 1, \dots, n$, $j \neq l$ in $G(A, B)$.*

Proof (*Necessity*). We consider the strong reachability of the jth basis vector e_j.

The input vector can be written as

$$u(t) = \sum_{1}^{m} \bar{u}_i(t) \quad \text{with } \bar{u}_i(t) = [0, \dots, 0, u_i(t), 0, \dots, 0]^{\mathrm{T}},$$

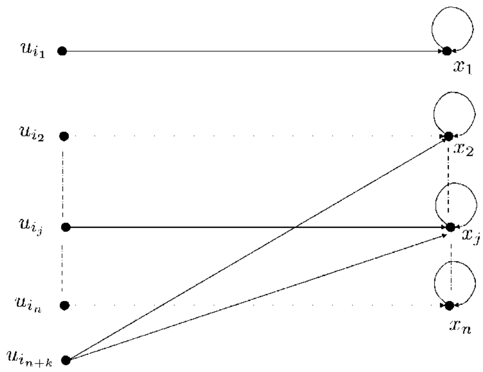

Fig. 1. Graph of a strongly reachable positive system.

where $u_i(t)$ is the ith component of $\bar{u}_i(t)$. The corresponding state vector $x(t)$ can be decomposed accordingly as $x(t) = \sum_1^m \bar{x}_i(t)$ where $\bar{x}_i(t)$ is the state vector corresponding to the input vector $\bar{u}_i(t)$. From this decomposition if we want $x(T) = e_j$ we must have at least one $\bar{x}_i(t)$ whose jth component is positive and all the $\bar{x}_i(t)$'s must have the other components equal to zero. Let $\bar{x}_k(t)$ be such that its jth component is positive. This implies that $u_k(t)$ is a positive function and from Proposition 3 that there exists a path from the input u_k to x_j in $G(A, B)$. Moreover, the other components of $\bar{x}_k(t)$ being zero, there must exist no other path with initial vertex u_k and terminal vertex $x_l \neq x_j$. These two requirements on the graph, namely a path from some input vertex u_k to x_j and no path from u_k to x_l for $l \neq j$ in $G(A, B)$, imply that:

- there is an edge (u_k, x_j),
- there is no edge (u_k, x_p) with $p \neq j$,
- there is no edge (x_j, x_q) with $q \neq j$.

This set of requirements gives points 1 and 2 of the theorem.

Sufficiency: The conditions of the theorem imply a one-to-one correspondence between the state vertices and a set of n input vertices. The system is then composed of a set of decoupled scalar systems as illustrated in Fig. 1. From Proposition 2 each x_j is strongly reachable using input u_{i_j}. Then each basis vector is strongly reachable and from Proposition 1 the whole system is strongly reachable. □

Theorem 1 can be reformulated in another way as follows.

Theorem 2. *A continuous-time positive system of type* (1) *is strongly reachable if and only if after a possible reordering of the inputs the matrix A is diagonal, the matrix B can be written as $B = (D, B_1)$ where D is an order n diagonal positive matrix with positive diagonal entries and B_1, which exists only if $m > n$, is an arbitrary $n \times (m - n)$ positive matrix.*

As a conclusion of this section we may note that the strong reachability requires an excess of the number of inputs over the number of states and a very special structure for matrices A and B. Therefore, most linear systems will not be strongly reachable and it is then of interest to characterize the set of strongly reachable states. This will be done in the next section.

4. The strongly reachable set

In this section we will use the following notations. $\|.\|$ denotes the Euclidian norm, int(.) and cl(.) denote, respectively, the interior and the closure of a given set in the induced topology.

Define the strongly reachable set as $\mathscr{R} = \{x_f \in \mathbb{R}^n_+ \mid$ for $x(0) = 0$ and any $T > 0$ there exists a nonnegative piecewise continuous function $u(t)$ such that $x(T) = x_f\}$.

A first property of the strongly reachable set is easily obtained.

Proposition 4. *$\mathscr{R}$ is a convex cone.*

Proof. Follows directly from the linearity and the positivity of the system. □

In order to prove the basic results of this section, the following lemma is needed:

Lemma 1. *For any $T > 0$ and any piecewise continuous nonnegative function $v : [0, T] \to \mathbb{R}^n_+$, the following property holds*:

$$\int_0^T \|v(\tau)\| \,\mathrm{d}\tau \leqslant \gamma \left\| \int_0^T v(\tau)\,\mathrm{d}\tau \right\| \tag{6}$$

for some $\gamma > 0$.

Proof. This is a direct consequence of the nonnegativity of v and of classical arguments on norms equivalence in Euclidian spaces. □

Proposition 5. *The strongly reachable set is contained in the cone $\mathscr{C}(B)$ generated by the columns of B.*

Proof. Consider a strongly reachable point x_f. This means that for all $T > 0$ there exists some control profile $u_T(\cdot)$ such that $x_f = x(T) = \int_0^T \mathrm{e}^{A(T-\tau)} B u_T(\tau)\,\mathrm{d}\tau$. x_f can be written as $x_f = x_1(T) + x_2(T)$ where $x_1(T) = \int_0^T B u_T(\tau)\,\mathrm{d}\tau$ and $x_2(T) = \int_0^T (I - \mathrm{e}^{A(T-\tau)}) B u_T(\tau)\,\mathrm{d}\tau$. On the other hand, there is some sufficiently small $T_0 > 0$ such that for all $T \leqslant T_0$ and all $\tau \in [0, T]$, we have $\|(I - \mathrm{e}^{A(T-\tau)})\| \leqslant \alpha(T-\tau) \leqslant \alpha T$ for some $\alpha > 0$. Therefore,

$$\begin{aligned}\|x_2(T)\| &\leqslant \int_0^T \|I - \mathrm{e}^{A(T-\tau)}\| \cdot \|Bu(\tau)\|\,\mathrm{d}\tau \\ &\leqslant \alpha T \int_0^T \|Bu(\tau)\|\,\mathrm{d}\tau,\end{aligned} \tag{7}$$

and by Lemma 1, there exists some $\gamma > 0$ such that

$$\|x_2(T)\| \leqslant (\gamma\alpha T) \cdot \left\| \int_0^T Bu(\tau)\,\mathrm{d}\tau \right\| = (\gamma\alpha T) \cdot \|x_1(T)\|. \tag{8}$$

Since this must be true for all $T \in [0, T_0]$ and $x_1(T) \in \mathscr{C}(B)$, it follows that $x_f \in \mathrm{cl}(\mathscr{C}(B))$ and since $\mathscr{C}(B)$ is closed $x_f \in \mathscr{C}(B)$. □

Proposition 6. *The strongly reachable set contains the interior of* $\mathscr{C}(B)$.

Proof. Let $x_f \in \mathrm{int}(\mathscr{C}(B))$. Take some arbitrary $T > 0$. Consider A_d and B_d as the discretized versions of the system matrices for some sampling period $\tau \leqslant T$. Recall that

$$B_\mathrm{d} := \int_0^\tau \mathrm{e}^{A(\tau-\sigma)} B \,\mathrm{d}\sigma = \tau \cdot B + \delta B(\tau), \quad \lim_{\tau \to 0} \frac{\delta B(\tau)}{\tau} = 0.$$

Let us consider the following iteration:

1. Take $u^{(0)} = 0$.
2. For all $i \geqslant 1$, define $u^{(i)} \geqslant 0$ to be the solution (if any) of the following constrained optimization problem:

$$\min_{v \in \mathbb{R}_+^m} \|v - u^{(i-1)}\|^2 \quad \text{under}\, \tau \cdot Bv = x_f - \delta B \cdot u^{(i-1)}. \tag{9}$$

We shall prove the following results about the above iteration:

1. There exists a sufficiently small τ such that the iteration is defined on $\mathbb{R}_+^m$ for all i.
2. The iterates satisfy the following inequality for some $h \in [0, 1[$:

$$\|u^{(i)} - u^{(i-1)}\| \leqslant h \cdot \tau \cdot \|u^{(i-1)} - u^{(i-2)}\|. \tag{10}$$

This would lead to the result since under these conditions, the iterations converge to some value u^* for which one has by definition

$$[\tau B + \delta B(\tau)]u^* = x_f, \tag{11}$$

and hence, the control function defined by

$$u(\sigma) := \begin{cases} 0 & \text{for } \sigma \leqslant T - \tau, \\ u^* & \text{for } \sigma \in [T - \tau, T] \end{cases} \tag{12}$$

achieves the requirement $x(T) = x_f$.

Proof of 1: Note that $u^{(1)}$ is well defined since $x_f \in \mathrm{int}(\mathscr{C}(B))$ by assumption. For $u^{(2)}$, it is a solution of

$$\tau B u^{(2)} = x_f - \delta B(\tau) \cdot u^{(1)}, \tag{13}$$

but for τ sufficiently small, $x_f - \delta B(\tau) \cdot u^{(1)}$ is still in $\mathrm{int}(\mathscr{C}(B))$ and the iteration is defined on $\mathbb{R}_+^m$ for $i = 2$. Again, for sufficiently small τ, $u^{(3)}$ is well defined. The problem is to prove that this process can be repeated indefinitely while the desired target $x_f - \delta B(\tau)u^{(i)}$ remains in $\mathrm{int}(\mathscr{C}(B))$. To prove this, the inequality (10) will be used. Indeed, if this inequality holds, then one can write

$$\forall i \geqslant 2 \quad \|u^{(i)} - u^{(i-1)}\| \leqslant (h\tau)^{i-1} \|u^{(1)}\|,$$

and therefore, for all $i \geqslant 2$, one has

$$\|u^{(i)} - u^{(1)}\| \leqslant \left[\sum_{k=2}^{i} (h\tau)^{k-1}\right] \|u^{(1)}\| = \underbrace{\frac{h\tau[1 - (h\tau)^{i-1}]}{1 - h\tau}}_{\mathrm{O}(h\tau)} \|u^{(1)}\|, \tag{14}$$

and for sufficiently small τ, one has

$$\forall i \geqslant 2 \quad \|u^{(i)} - u^{(1)}\| \leqslant \mathrm{d}(u^{(1)}, \delta\mathscr{C}(B)), \tag{15}$$

where $\delta\mathscr{C}(B)$ denotes the boundary of $\mathscr{C}(B)$. This proves that for sufficiently small τ, $x_f - \delta B(\tau) \cdot u^{(i)}$ is in $\mathrm{int}(\mathscr{C}(B))$. Therefore, all we have to do is to prove inequality (10).

Proof of inequality (10):

$$\begin{aligned} \tau B[u^{(i)} - u^{(i-1)}] &= x_f - \delta B(\tau)u^{(i-1)} - \tau B u^{(i-1)} \\ &= \underbrace{\tau B u^{(i-1)} + \delta B(\tau) u^{(i-2)}}_{x_f} - \delta B(\tau) u^{(i-1)} \\ &\quad - \tau B u^{(i-1)} \\ &= \delta B(\tau)[u^{(i-2)} - u^{(i-1)}], \end{aligned} \tag{16}$$

and therefore

$$\|u^{(i)} - u^{(i-1)}\| \leqslant \frac{1}{\tau \cdot \underline{\sigma}^{1/2}(B)} \|\delta B(\tau)\| \cdot \|u^{(i-1)} - u^{(i-2)}\|,$$

where $\underline{\sigma}^{1/2}(B)$ is the smallest nonzero singular value of B. Therefore, inequality (10) follows from the property $\delta B(\tau)\tau = o(\tau)$. □

The results of this section can be summarized in the following theorem.

Theorem 3. *Consider the linear positive system defined in* (1). *The strongly reachable set* $\mathscr{R}$ *of this system is a convex cone, moreover,* $\mathrm{cl}(\mathscr{R}) = \mathscr{C}(B)$, *the cone generated by the columns of B.*

Define now the almost strongly reachable set as $\mathscr{R}_\mathrm{a} = \{x_f \in \mathbb{R}_+^n \mid$ for $x(0) = 0$, any $\varepsilon > 0$ and any $T > 0$ there exists a nonnegative piecewise continuous function $u(t)$ such that $\|x_f - x(T)\| \leqslant \varepsilon\}$.

The two following results easily follow from Theorem 3.

Corollary 1. *The almost strongly reachable set is* $\mathscr{C}(B)$, *the cone generated by the columns of B.*

Proposition 7. *A linear positive system defined in* (1) *is almost strongly reachable if and only if there is a set of columns of B* $(b_{i_1}, b_{i_2}, \ldots, b_{i_n})$ *such that* $b_{i_j} = \alpha_j e_j$, *for* $i = 1, \ldots, n$, *where* e_j *is the jth basis vector and* $\alpha_j > 0$.

Proof. This is a direct consequence of the fact that an edge of a simplex cannot be generated by positive combination of the other edges. □

Remark 1. To illustrate the results of this paper let us consider the case $m = n$. Denote again by $\mathscr{R}$ the strongly reachable set and by $\mathscr{R}_{\mathrm{a}}$ the almost strongly reachable set. From Theorem 3 and its corollaries it follows that the system is almost reachable if and only if B is a diagonal nonsingular positive matrix (up to a permutation). From Theorem 2 the system is strongly reachable, i.e. $\mathscr{R} = \mathbb{R}^n_+$, if and only if besides the previous condition the matrix A is also diagonal.

5. Conclusion

In this paper the problem of reachability for continuous-time positive systems is studied in the case where we ask for this reachability in any fixed time. The condition for such a reachability are given together with the set of states which are reachable in this sense.

The results of this paper may appear to be "intuitively obvious" since reachability in any fixed time induces input functions which are approximates of impulse distributions. Since only the Dirac distribution can be approached by non-negative functions, it is then natural that the set of strongly reachable states is in the cone generated by the columns of B.

Anyway, we believe that the results of this paper constitute a useful step in the complete solution of the reachability problem for positive systems. In particular they reinforce the feeling that only very particular positive systems are reachable and that this will remain true even when allowing reachability in finite time. The example 13 in [7] is also supporting this feeling.

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