Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Control Systems

Andrey Polyakov

***Abstract*—Two types of nonlinear control algorithms are presented for uncertain linear plants. Controllers of the first type are stabilizing polynomial feedbacks that allow to adjust a guaranteed convergence time of system trajectories into a prespecified neighborhood of the origin independently on initial conditions. The control design procedure uses block control principles and finite-time attractivity properties of polynomial feedbacks. Controllers of the second type are modifications of the second order sliding mode control algorithms. They provide global finite-time stability of the closed-loop system and allow to adjust a guaranteed settling time independently on initial conditions. Control algorithms are presented for both single-input and multi-input systems. Theoretical results are supported by numerical simulations.**

***Index Terms*—Finite-time stability, polynomial feedback, second-order sliding mode control.**

I. Introduction

Finite-time stability and stabilization problems have often been a subject of research [4], [13], [19], [21]. The control theory provides many systems that exhibit finite-time convergence to the equilibrium. Frequently such systems appear in observation problems when finite-time convergence of the observed states to the real ones is required [3]. The high-order sliding mode control algorithms also provide finite-time convergence to the origin [16], [17], [20], [22]. Typically such controllers have mechanical and electromechanical applications [2], [5], [11].

The technical note deals with an extension *global finite-time stability* concept that is related to possible predefining of guaranteed convergence (settling) time *independently on initial conditions*. The corresponding property is called in this technical note by *fixed-time stability*. Such phenomenon was discovered in [6], [10], [18], and [23], where observers with predetermined finite convergence time have been developed. The present technical note mostly addresses the control design problem for linear plants providing *fixed-time convergence to the given set*. The developed control design procedure requires only controllability of the system, i.e., $\mathrm{rank}[B, AB, \ldots, A^{n-1}B] = n$. Control algorithms presented for *fixed-time stabilization of the origin* are restricted by the case $\mathrm{rank}[B, AB] = n$.

The proposed control laws in this technical note are of polynomial form. Polynomial state feedback control systems have considerable attention in nonlinear control [8]. This class of control systems appears in models of a wide range of applications such as chemical processes, electronic circuits and mechatronics, biological systems, etc. This technical note studies a special property of polynomial feedbacks, which is expressed in fixed-time attraction of solutions of the closed-loop system into any selected neighborhood of the origin.

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The author was with the Laboratory of Adaptive and Robust Control Systems, Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia. He is now with Non-A INRIA - LNE, Parc Scientifique, Villeneuve d'Ascq 59650, France (e-mail: andrey.polyakov@inria.fr).

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Usually finite-time stability is closely related with homogeneity property of the system. While being asymptotically stable and homogeneous of negative degree, the system is shown to approach the equilibrium point in finite time [16], [20]. The concept of *homogeneity in bi-limit* introduced in [1] generalizes this property providing that an asymptotically stable system is *fixed-time stable* if it is homogeneous of negative degree in 0-limit and homogeneous of positive degree in ∞-limit. Unfortunately, homogeneous approach does not allow to adjust or even estimate the settling time. To overcome this problem the technical note introduces a special modification of the so-called "nested" (terminal) second order sliding mode control algorithm [17] that provides fixed-time stability of the origin and allows to adjust the global settling time of the closed-loop system.

All control algorithms presented in the technical note are robust with respect to system disturbances and plant parameters variations in the case when the, so-called, *matching condition* [24] holds. It assumes that to guarantee successful elimination of system uncertainties or external disturbances they should act in the same subspace as an admissible control.

II. Finite-Time Stability and Some Further Extensions

Consider the following:

$$\dot{x} = g(t, x), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function, which can be discontinuous. The solutions of (1) are understood in the sense of Filippov [12]. Assume the origin is an equilibrium point of (1).

Definition 1 ([4], [20]): The origin of (1) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibria at some finite time moment, i.e., $x(t, x_0) = 0, \forall t \geq T(x_0)$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function.

The finite-time stability property may exhibit homogeneous systems with negative degree [16], [20]. Any solution of the system $\dot{x} = -x^{1/3}$, $x \in \mathbb{R}$ converges to the origin in finite time $T(x_0) := (3/2)\sqrt[3]{|x_0|^2}$.

Definition 2: The origin of (1) is said to be *fixed-time* stable if it is globally finite-time stable and the settling-time function $T(x_0)$ is bounded, i.e., $\exists T_{\max} > 0 : T(x_0) \leq T_{\max}, \forall x_0 \in \mathbb{R}^n$.

The origin of $\dot{x} = -x^{1/3} - x^3, x \in \mathbb{R}$ is fixed-time stable, since it is globally finite-time stable and $x(t, x_0) = 0$ for $\forall t \geq 2.5$ and $\forall x_0 \in \mathbb{R}$.

Definition 3: The set M is said to be globally *finite-time* attractive for (1) if any solution $x(t, x_0)$ of (1) reaches M in some finite time moment $t = T(x_0)$ and remains there $\forall t \geq T(x_0), T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function.

Definition 4: The set M is said to be *fixed-time* attractive for (1) if it is globally finite-time attractive and the settling-time function $T(x_0)$ is globally bounded by some number $T_{\max} > 0$.

Denote by $D^*\varphi(t)$ the upper right-hand derivative of a function $\varphi(t), D^*\varphi(t) := \lim\limits_{h \to +0} \sup(\varphi(t+h) - \varphi(t)/h)$.

Lemma 1: If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that 1) $V(x) = 0 \Rightarrow x \in M$; 2) any solution $x(t)$ of (1) satisfies the inequality $D^*V(x(t)) \leq -(\alpha V^p(x(t)) + \beta V^q(x(t)))^k$ for some $\alpha, \beta, p, q, k > 0 : pk < 1, qk > 1$ then the set $M \subset \mathbb{R}^n$ is globally fixed-time attractive for (1) and $T(x_0) \leq (1/\alpha^k(1-pk)) + (1/\beta^k(qk-1)), \forall x_0 \in \mathbb{R}^n$.

Proof: Due to 2) we have $D^*V(x(t)) \leq -\alpha^k V^{pk}(x(t))$ if $V(x(t)) \leq 1$ and $D^*V(x(t)) \leq -\beta^k V^{qk}(x(t))$ for $V(x(t)) > 1$. Hence, for any $x(t)$ such that $V(x(0)) > 1$ the last inequality guarantees $V(x(t)) \leq 1$ for $t \geq (1/\beta^k(qk-1))$ and for any $x(t)$ such that $V(x(t_0)) \leq 1$ we derive $V(x(t)) = 0$

for $t \geq t_0 + (1/\alpha^k(1-pk))$. Therefore, $V(x(t)) = 0$ for $\forall t \geq T_{\max} := (1/\alpha^k(1-pk)) + (1/\beta^k(qk-1))$ and $\forall x(t)$—solution of (1). The condition 1) implies $x(t) = 0$ for $\forall t \geq T_{\max}$. ■

If the condition 1) is replaced by $V(x) = 0 \Leftrightarrow x = 0$ then the set M is also invariant. If $M = \{0\}$ Lemma 1 helps to analyze fixed-time stability of the origin.

III. Problem Statement and Basic Assumptions

Consider the control system of the form

$$\dot{x} = Ax + Bu + f(t,x) \tag{2}$$

where $x \in \mathbb{R}^n$ is the vector of system states, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times m}$ is the matrix of control gains, $u \in \mathbb{R}^m$ is the vector of control inputs, and the function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ describes systems uncertainties.

We study (2) under the following classical *assumptions*: **1**. The pair (A, B) is controllable. **2**. The uncertain function $f(t,x)$ satisfies the matching condition: $f(t,x) = B\gamma(t,x)$, where $\gamma(t,x)$ is a function bounded by a known function $\gamma_0(t,x)$, i.e., $\|\gamma(t,x)\| \leq \gamma_0(t,x)$ for $\forall t > 0$ and $\forall x \in \mathbb{R}^n$.

Denote $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}, \|x\| := \max\limits_{i=1,2,\ldots,n} |x_i|$.

This technical note addresses two following problems: **1**) to design a feedback control $u = u(t,x)$ for (2), which provides the fixed-time attractivity property of the given ball B_r for a predefined global settling-time estimate $T_{\max}$; **2**) to design a feedback control $u = u(t,x)$, which guarantees fixed-time stability of the origin of the closed-loop system (2) for a predefined global settling-time estimate $T_{\max}$.

IV. Fixed-Time Controllers for Single Input Systems

A. Fixed-Time Attractivity

Consider the case $m = 1$. The linear transformation $y = Gx$, $G = [A^{n-1}b, A^{n-2}b, \ldots, Ab, b]^{-1}$ brings (after reordering) (2) to the Brunovsky form

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \ldots, \dot{y}_n = \sum_{i=1}^{n} a_i y_i + u + \gamma, \quad a_i \in \mathbb{R}. \tag{3}$$

Introduce the nonlinear coordinate transformation $s = \Phi(y)$: $s_i = y_i + \varphi_i, i = 1,2,\ldots,n, \varphi_1 = 0, \varphi_{i+1} =, \alpha_i(y_i + \varphi_i) + \beta_i(y_i+\varphi_i)^3 + \sum_{k=1}^{i}(\partial\varphi_i/\partial y_k)y_{k+1}$, $\alpha_i, \beta_i \in \mathbb{R}_+$. The inverse transformation $y = \Phi^{-1}(s)$ is $y_i = s_i + \psi_i, i = 1,2,\ldots,n, \psi_1 = 0$, $\psi_{i+1} =, -\alpha_i s_i - \beta_i s_i^3 + \sum_{k=1}^{i}(\partial\psi_i/\partial s_k)(-\alpha_k s_k - \beta_k s_k^3 + s_{k+1})$. Equation (3) is equivalent to

$$\begin{cases} \dot{s}_1 = -\alpha_1 s_1 - \beta_1 s_1^3 + s_2, \\ \dot{s}_2 = -\alpha_2 s_2 - \beta_2 s_2^3 + s_3, \\ \ldots \\ \dot{s}_n = \sum\limits_{i=1}^{n} a_i y_i + \sum\limits_{k=1}^{n-1} \frac{\partial\varphi_n}{\partial y_k} y_{k+1} + u + \gamma(t,y) \end{cases} \tag{4}$$

Let the feedback control u has the form

$$u = -\alpha_n s_n - \beta_n s_n^3 - \sum_{i=1}^{n} a_i y_i - \sum_{k=1}^{n-1} \frac{\partial\varphi_n}{\partial y_k} y_{k+1}. \tag{5}$$

Lemma 2: Let 1) $\varepsilon_i > 0$, $i = 1,2,\ldots,n$, and $T_{\max} > 0$ are arbitrary numbers; 2) $\alpha_1 = \overline{\alpha}_1 + (\varepsilon_2/\varepsilon_1), \ldots, \alpha_{n-1} = \overline{\alpha}_{n-1} + (\varepsilon_n/\varepsilon_{n-1})$, $\alpha_n = \overline{\alpha}_n + (\gamma_0(t,y)/\varepsilon_n)$, $\overline{\alpha}_i \geq 0$; 3) $\beta_i \geq (\overline{\alpha}_i/\varepsilon_i^2/\exp(2\overline{\alpha}_i(T_{\max}/n)) - 1)$, $i = 1,2,\ldots,n$. Then any solution $s(t)$ of (4), (5) satisfies inequalities $|s_i(t)| \leq \varepsilon_i$ for $\forall t \geq T_{\max}$.

Proof: Denote $V_i(t) = |s_i(t)|, i = 1,2,\ldots,n$. The inequality $D^*V_n(t) \leq -\alpha_n V_n(t) - \beta_n V_n^3(t) + \gamma_0$ holds for $\forall t > 0$ and $\dot{V}_n(t) \leq -\overline{\alpha}_n V_n(t) - \beta_n V_n^3(t)$ for $t > 0 : V_n(t) \geq \varepsilon_n$. Hence we derive $(\beta_n/\overline{\alpha}_n)V_n^2(t)+1 \leq \left(1 - (\beta_n V_n^2(0)/\beta_n V_n^2(0) + \overline{\alpha}_n)e^{-2\overline{\alpha}_n t}\right)^{-1}$ and $V_n(t) \leq \varepsilon_n$ for $\forall t \geq T_{\max}/n$. The similar considerations for $i = n-1$ give $V_i(t) \leq \varepsilon_i$ for $\forall t \geq (n-i+1)T_{\max}/n$. The last step ($i = 1$) provides $|s_i(t)| < \varepsilon_i$ for $\forall t > T_{\max}$. ■

Corollary 1: If $\varepsilon_i = \varepsilon > 0$ and $\overline{\alpha}_i = 1$, $i = 1,2,\ldots,n$ then the inequality $\|s(t)\| \leq \varepsilon$ for all $t > T_{\max}$, where $s(t)$ is an arbitrary solution of (4), (5) with control parameters $\alpha_i = 2, i = 1,2,\ldots,n-1$; $\alpha_n(t,x) = 1 + \gamma_0(t,x)/\varepsilon$ and $\beta_i = \beta := \varepsilon^{-2}(\exp(2T_{\max}/n) - 1)^{-1}, i = 1,2,\ldots,n$.

This corollary presents control design algorithm for (4) providing the fixed-time attractivity property of the ball B_ε.

Denote a set of polynomials of the order k by $\mathbb{P}^k$.

Corollary 2: If conditions of Corollary 1 hold, then there exist polynomials $p_i \in \mathbb{P}^i, i = 1,2,\ldots,n$ with nonnegative coefficients such that $|y_i(t)| \leq p_i(q)\varepsilon, \forall t > T_{\max}$, where $q = (\exp(2T_{\max}/n) - 1)^{-1}$ and $y(t)$ is an arbitrary solution of (3), (5).

Proof: Let $z_i > 0, i = 1,2,\ldots,n$ and $\tilde{\psi}_1 = 0$, $\tilde{\psi}_{i+1}(z_1,\ldots,z_{i+1}) = 2z_i + \beta z_i^3 + \sum_{k=1}^{i}(\partial\tilde{\psi}_i/\partial z_k)(2z_k + \beta z_k^3 + z_{k+1})$. Obviously, $|\psi_i(s_1,\ldots,s_i)| \leq \tilde{\psi}_i(|s_1|,\ldots,|s_i|)$. It can be shown $\tilde{\psi}_i(\varepsilon,\ldots,\varepsilon) = \varepsilon\sum_{j=0}^{i-1}\mu_{ij}(\beta\varepsilon^2)^j$, where $\mu_{ij} \geq 0$ are numbers. So, $\|s(t)\| \leq \varepsilon$ implies $|y_i(t)| \leq p_i(q)\varepsilon$, where $p_i(q) := 1 + \sum_{j=0}^{i-1}\mu_{ij}q^j$, $\beta\varepsilon^2 = q$. ■

The proof of Corollary 2 gives an algorithm of construction of the polynomials $p_i(q) \in \mathbb{P}^i$ that are needed for adjustment of control parameters to guarantee convergence of all solutions of the original system (2) into the prespecified neighborhood of the origin. The explicit form of the polynomial can be obtained using the recursive formulas for $\tilde{\psi}_i$ calculated in symbolic computation packages such as Mathematica. In particular $p_1(q) = 1, p_2(q) = 3 + q$.

Theorem 1: If $m = 1$ and the control u has (5) with parameters $\alpha_i = 2$, $i = 1,2,\ldots,n-1$, $\alpha_n(t,x) = 1 + \gamma_0(t,x)/\varepsilon$, $\beta_i = (q/\varepsilon^2), i = 1,2,\ldots,n$, $\varepsilon = r/(\|G^{-1}\|_\infty p_n(q))$, $q = (\exp(2T_{\max}/n) - 1)^{-1}$, where $r > 0$ and $T_{\max} > 0$, the polynomial $p_n \in \mathbb{P}^n$ is introduced in Corollary 2, then the ball B_r is the globally fixed-time attractive set of the closed-loop system (2) and $T(x_0) \leq T_{\max}$ for $\forall x_0 \in \mathbb{R}^n$.

B. Fixed-Time Stability

Let, for (2), the control $u_a(t,x) := u(t,x)$ provides fixed-time attractivity property of a ball B_r. The combination of this controller with a control law $u_f(x)$ providing local finite-time stability of the origin for (2) gives us a hybrid control algorithm $u_{fx}(t,x) = \begin{cases} u_a(t,x) & \text{for } x \notin B_r \\ u_f(t,x) & \text{for } x \in B_r \end{cases}$, which can afford fixed-time stability of the origin for (2). Design procedures of local finite-time controllers can be found in [14] and [16]. However, the existed controllers for local finite-time stabilization do not provide explicit algorithms for adjusting the control parameters to predefine the local settling time estimate (except "twisting" second-order sliding mode system [22]). The hybrid control scheme may produce chattering regimes [9] around the boundary of the ball B_r that slow down the convergence process. The nonhybrid controller providing *fixed-time stability of the origin* for (2) is more preferable. In this technical note such controller is designed only for linear plants satisfying *the additional assumption*: $\mathrm{rank}[B, AB] = n$. This condition restricts the class of controllable systems, but it still covers a lot of real-life control systems. For $m = 1$ the assumption implies $n = 2$.

Introduce the involution operation without loss of the number's sign by $z^{[q]} = |z|^q \mathrm{sign}[z]$, $z, q \in \mathbb{R}$.

Theorem 2: Let $T_{\max} > 0$, $m = 1$, $n = 2$ and the control $u = u_{fx}(t, Gx)$ has the form $u_{fx}(t,y) = -a_1y_1 - a_2y_2 - (\alpha_1 + 3\beta_1 y_1^2 + 2\gamma_0(t,y)/2)\mathrm{sign}[s] - (\alpha_2 s + \beta_2 s^{[3]})^{[1/2]}$, $s = y_2 + (y_2^{[2]} + \alpha_1 y_1 + \beta_1 y_1^{[3]})^{[0.5]}$, $(\alpha_1/2) = \alpha_2 = (\beta_1/2) = \beta_2 = (64/T_{\max}^2)$. Then the origin of the closed-loop system (2) is globally fixed-time stable and $T(x_0) \le T_{\max}, \forall x_0 \in \mathbb{R}^n$.

Proof: Show $s = 0$ is a sliding surface of (3)

$$D^*|s(t)| = \left[\dot{y}_2(t) + \frac{|y_2(t)|\,\dot{y}_2(t) + \frac{\alpha_1 + 3\beta_1 y_1^2(t)}{2} y_2(t)}{\left|y_2^{[2]}(t) + \alpha_1 y_1(t) + \beta_1 y_1^{[3]}(t)\right|^{1/2}}\right]\mathrm{sign}[s(t)].$$

Since $(\alpha_2 s + \beta_2 s^{[3]})^{[1/2]}\mathrm{sign}[s] = (\alpha_2|s| + \beta_2|s|^3)^{1/2}$ we have $\dot{y}_2\mathrm{sign}[s] = (a_1y_1 + a_2y_2 + u_{fx}(y) + \gamma(t,x))\mathrm{sign}[s] = -(\alpha_1 + 3\beta_1y_1^2/2) - (\alpha_2|s| + \beta_2|s|^3)^{1/2} - (\gamma_0 - \gamma\mathrm{sign}[s])$ for $s \neq 0$. Hence, $D^*|s| \le -(\alpha_2|s| + \beta_2|s|^3)^{1/2}$ and $|s(t)| = 0$ for $\forall t \ge T_s = (2/\sqrt{\alpha_2}) + (2/\sqrt{\beta_2}) = T_{\max}/2$. This implies $2y_2^{[2]} + \alpha_1 y_1 + \beta_1 y_1^3 = 0$ and $\dot{y}_1 = -((\alpha_1/2)y_1 + (\beta_1/2)y_1^3)^{[1/2]}$. Hence, $y_1 = 0$ for all $t \ge (T_{\max}/2) + (2\sqrt{2}/\sqrt{\alpha_1}) + (2\sqrt{2}/\sqrt{\beta_1}) = T_{\max}$. Finally remark that $y_1 = 0$ and $s = 0$ imply $y_2 = 0$. ■

For $\beta_1 = 0$ the switching line $s = 0$ coincides with the sliding surface of the finite-time "nested" controller [17]. The polynomial terms $\beta_1 y_1^{[3]}$ and $\beta_2 s^{[3]}$ are required for the fixed-time stability.

V. Fixed-Time Controllers for Multi Input Systems

A. Block Decomposition

Introduce the notations: $\mathrm{rown}(W)$ and $\mathrm{coln}(W)$ are numbers of rows and columns of the matrix W; $\ker(W)$ and $\mathrm{range}(W)$ are the null space and the column space of W; $\mathrm{nul}(W)$ is the matrix with columns defining the orthonormal basis of $\ker(W)$.

To adapt the fixed-time control design scheme developed for the single-input case we decompose the original multi input system (2) to a block from [7]. The required coordinate transformation can be constructed using matrices provided by Algorithm 1.

Algorithm 1

I. $A_0 = A, B_0 = B, T_0 = I_n, k = 0$. **II**. While $\mathrm{rank}(B_k) < \mathrm{rown}(A_k)$ do $A_{k+1} = B_k^{\perp} A_k (B_k^{\perp})^T$, $B_{k+1} = B_k^{\perp} A_k \tilde{B}_k$, $T_{k+1} = \binom{B_k^{\perp}}{\tilde{B}_k}$, $k = k+1$, where $B_k^{\perp} := (\mathrm{nul}(B_k^T))^T$, $\tilde{B}_k := (\mathrm{nul}(B_k^{\perp}))^T$.

This algorithm can be easily realized in a computational software system such as MATLAB.

Lemma 3: If the pair (A, B) is controllable then **1)** Algorithm 1 terminates after finite number of steps $k \le n-1$; **2)** the matrices $T_i \in \mathbb{R}^{\mathrm{rown}(B_i)\times \mathrm{rown}(B_i)}$, $i = 1, 2, \ldots, k$ are orthogonal; **3)** the orthogonal coordinate transformation $y = Gx$

$$G = \begin{pmatrix} T_k & 0 \\ 0 & I_{w_k} \end{pmatrix} \cdots \begin{pmatrix} T_2 & 0 \\ 0 & I_{w_2} \end{pmatrix} T_1, \quad w_i := n - \mathrm{rown}(T_i) \tag{6}$$

reduces (2) to the block form

$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = A_{21}y_1 + A_{22}y_2 + A_{23}y_3 \\ \ldots \\ \dot{y}_k = A_{k1}y_1 + \ldots + A_{kk}y_k + A_{kk+1}(u+\gamma) \end{cases} \tag{7}$$

where $y = (y_1^T, \ldots, y_k^T)^T, y_i \in \mathbb{R}^{n_i}, n_i := \mathrm{rank}(B_{k-i}), A_{kk+1} = \tilde{B}_0 B_0$ and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ are blocks of the matrix GAG^T such that $\mathrm{rank}(A_{ii+1}) = n_i, i = 1, 2, \ldots, k$.

Proof: **I**. Denote $B_0 = B, A_0 = A, r_0 = \mathrm{rank}(B_0)$. Controllability of (A_0, B_0) implies $r_0 > 0$. For $r_0 = n$ Algorithm 1 stops after initialization showing that the transformation is not required. **II**. Let $r_0 < n$. In this case $\mathbb{R}^n = \mathrm{range}(B_0) \oplus \ker(B_0^T)$. Columns of the matrix $\mathrm{nul}(B_0^T) \in \mathbb{R}^{n\times(n-r_0)}$ define the orthonormal basis for $\ker(B_0^T) = \mathrm{range}(\mathrm{nul}(B_0^T))$. Since $\mathrm{range}(B_0)$ is an orthogonal complement to $\ker(B_0^T)$ then $\mathrm{range}(B_0) = \ker(\mathrm{nul}(B_0^T)^T)$. So, the block matrix T_1 is square and orthogonal. Since $B_0^T\mathrm{nul}(B_0^T) = 0$ then the coordinate transformation $(\tilde{y}_1^T\ \tilde{y}_2^T)^T = T_1 x, \tilde{y}_1 \in \mathbb{R}^{n-r_0}, \tilde{y}_2 \in \mathbb{R}^0$ gives

$$\dot{\tilde{y}}_1 = A_1\tilde{y}_1 + A_2\tilde{y}_2, \quad \dot{\tilde{y}}_2 = \tilde{A}_{21}\tilde{y}_1 + \tilde{A}_{22}\tilde{y}_2 + \tilde{B}_0B_0(u+\gamma)$$

where $A_1, A_2, \tilde{A}_{21}, \tilde{A}_{22}$—blocks of matrix $T_1AT_1^T$.

Controllability of the pair (A_0, B_0) implies controllability of (A_1, B_1) [24], so Algorithm 1 can be continued for A_1 and B_1. Since $\mathrm{rown}(A_{i+1}) = \mathrm{rown}(A_i) - \mathrm{rank}(B_i)$ and $\mathrm{rank}(B_i) > 0$ then Algorithm 1 terminates in finite steps and the transformation $y = Gx$ reduces (2) to the block form (7). Finally $\mathrm{rank}(A_{i\,i+1}) = \mathrm{rank}(\tilde{B}_{k-i}B_{k-i}) = \mathrm{rank}(T_{k-i}B_{k-i}) = n_i$. ■

This technical note considers the restricted robust problem statement assuming that uncertainties and disturbances satisfy the matching condition. The control design for more general case can be done using the robust analysis of block controllability forms [15].

B. Fixed-Time Attractivity

Further considerations are presented for (2) transformed to the block form (7). Since $\mathrm{rank}(A_{i\,i+1}) = n_i$ then $A_{i\,i+1}A_{i\,i+1}^T$ is invertible and $A_{ii+1}^+ = A_{ii+1}^T(A_{ii+1}A_{ii+1}^T)^{-1}$ is the right inverse matrix of $A_{i\,i+1}$. Denote $z^{[p]} = (z_1^{[p]}, \ldots, z_k^{[p]})^T$ for $z \in \mathbb{R}^k$.

By analogy to single input case introduce the nonlinear coordinate transformation $s = \Phi(y), s = (s_1, \ldots, s_k)^T, s_i \in \mathbb{R}^{n_i}$: $s_i = y_i + \varphi_i$, $i = 1, 2, \ldots, k$, $\varphi_1 = 0$, $\varphi_{i+1} = A_{i\,i+1}^+(\alpha_i(y_i + \varphi_i) + \beta_i(y_i + \varphi_i)^{[3]} + \sum_{j=1}^{i} A_{ij}y_j + \sum_{r=1}^{i}(\partial\varphi_i/\partial y_r)\sum_{j=1}^{r+1}A_{rj}y_j)$, $\alpha_i, \beta_i > 0, i = 1, 2, \ldots, k-1$. The inverse transformation $y = \Phi^{-1}(s)$ is: $y_i = s_i + \psi_i, \psi_{i+1} = A_{ii+1}^+(\sum_{k=1}^{i}(\partial\psi_i/\partial s_k)(A_{i\,i+1}s_{k+1} - \alpha_k s_k - \beta_k s_k^{[3]}) - \alpha_i s_i - \beta_i s_i^{[3]} - \sum_{j=1}^{i} A_{ij}(s_j + \psi_j))$, $i = 1, 2, \ldots, k$, $\psi_1 = 0$. Then (7) is equivalent to

$$\begin{cases} \dot{s}_1 = -\alpha_1 s_1 - \beta_1 s_1^3 + A_{12}s_2 \\ \dot{s}_2 = -\alpha_2 s_2 - \beta_2 s_2^3 + A_{23}s_3 \\ \ldots \\ \dot{s}_k = \xi(y_1, \ldots, y_k) + A_{kk+1}(u+\gamma) \end{cases} \tag{8}$$

where $\xi(y_1, \ldots, y_k) := \sum_{i=1}^{k} A_{ki}y_i + \sum_{i=1}^{k-1}(\partial\varphi_k/\partial y_i)\sum_{j=1}^{i+1}A_{ij}y_j$. So, the feedback control providing fixed-time attractivity property to (8) has the form

$$u = -A_{kk+1}^+\left(\alpha_k s_k + \beta_k s_k^3 + \xi(y_1, \ldots, y_k)\right). \tag{9}$$

Let $\varepsilon, T_{max} \in \mathbb{R}_+$ and $\alpha_i = 1 + \|A_{ii+1}\|_\infty, i = 1, 2, \ldots, k-1, \alpha_k = \varepsilon + \|A_{kk+1}\|_\infty\gamma_0(t,y)/\varepsilon$ and $\beta_i = \beta := q/\varepsilon^2$, $i = 1, 2, \ldots, k$, $q = (\exp(2T_{\max}/k) - 1)^{-1}$ then repeating the proof of Lemma 2 we can show that $\|s(t)\|_\infty \le \varepsilon, \forall t \ge T_{\max}$.

Show that there exist polynomials $p_i \in \mathbb{P}^i, i = 1, 2, \ldots, k$ with nonnegative coefficients such that $\|y_i(t)\| \le p_i(q)\varepsilon$ for $\forall t > T_{\max}$, where $y(t)$ is an arbitrary solution of (7), (9).

Denote the vector-modulus and the matrix-modulus by $[]\cdot[]$, i.e., $[]v[] = (|v_1|, \ldots, |v_r|)^T$ for the vector $v = (v_1, \ldots, v_r)^T$

and $[]W[] = \{|w_{ij}|\}$ for the matrix $W = \{w_{ij}\}$. The inequalities $[]v_1[] \preceq []v_2[]$, $[]W_1[] \preceq []W_2[]$ are understood in a component-wise sense.

Let $\tilde{\psi}_{i+1} = []A_{i\,i+1}^{+}[](\alpha_i z_i + \beta z_i^3 + \sum_{j=1}^{i}[]A_{ij}[](z_j + \tilde{\psi}_j) + \sum_{k=1}^{i}(\partial\tilde{\psi}_i/\partial z_k)(\alpha_k z_k + \beta z_k^3 + []A_{ii+1}[]z_{k+1}))$, $\tilde{\psi}_1 = 0$, $z_i \in \mathbb{R}_+^{n_i}$, $i = 1, 2, \ldots, k$. Following the proof of Corollary 2 we can see that $[]\psi_i(s_1,\ldots,s_i)[] \preceq \tilde{\psi}_i([]s_1[],\ldots,[]s_i[])$, $\tilde{\psi}_i(\varepsilon\bar{e}_{n_1},\ldots,\varepsilon\bar{e}_{n_i}) = \varepsilon\sum_{j=0}^{i}(\beta\varepsilon^2)^j M_{ij}\bar{e}_{n_j}$, where $\bar{e}_r = (1,\ldots,1)^T \in \mathbb{R}_+^r$ and $M_{ij} \in \mathbb{R}_+^{n_i\times n_j}$. Hence it directly follows that $\|y_i\| \le \|s_i\| + \|\psi_i(s_1,\ldots,s_i)\| \le \varepsilon + \|\tilde{\psi}_i(\varepsilon\bar{e}_{n_1},\ldots,\varepsilon\bar{e}_{n_i})\| = \varepsilon + \varepsilon\sum_{j=0}^{i}(\beta\varepsilon^2)^j\|M_{ij}\|_\infty = p_i(q)\varepsilon$.

The polynomials $p_i(q)$ can be also calculated using some symbolic computation software. For instance, $p_1(q) = 1$ and $p_2(q) = 1 + \|A_{12}^{+}\|_\infty(1 + \|A_{12}\|_\infty + q + \|A_{11}\|_\infty)$. We have just proven the theorem for the multi input case.

Theorem 3: If the control u has the form (9) with parameters $\alpha_i = 1 + \|A_{i\,i+1}\|_\infty, i = 1, 2, \ldots, k-1, \alpha_k = 1 + (\gamma_0\|A_{k\,k+1}\|_\infty\|G\|_1 p_k(q)/r)$, $\beta_i = (q\|G\|_1^2 p_k^2(q)/r^2), i = 1, 2, \ldots, k$, $q = (\exp((2T_{\max}/k)) - 1)^{-1}$, where $r > 0$ and $T_{\max} > 0$, then the ball B_r is the globally fixed-time attractive set of the closed-loop system (2) with the settling-time function bounded by $T_{\max}$.

C. Fixed-Time Stability

Here we assume $\mathrm{rank}[B, AB] = n$. Then the transformation $y = Gx$ with G defined by (6) brings (2) to the form

$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = A_{21}y_1 + A_{22}y_2 + A_{23}(u + \gamma) \end{cases} \tag{10}$$

$y_1 \in \mathbb{R}^{n_1}$, $y_2 \in \mathbb{R}^{n_2}$, $n_2 = \mathrm{rank}(B) = \mathrm{rank}(A_{23})$, $n_1 = n - n_2 = \mathrm{rank}(A_{12}) \le n_2$. Denote $A_{12}^{\perp} = \mathrm{nul}(A_{12})^T$.

Theorem 4: Let the controller $u(t, y), y = (y_1^T, y_2^T)^T \in \mathbb{R}^n$ has the form

$$u = -A_{23}^{+}\left(u_{eq}(y) + u_d(t, y) + u_p(y)\right) \tag{11}$$

where $u_p = A_{12}^{+}(\alpha_2 s_1 + \beta_2 s_1^{[3]})^{[1/2]} + (A_{12}^{\perp})^{+}(\alpha_3 s_2 + \beta_3 s_2^{[3]})^{[1/2]}$, $u_{eq} = A_{12}^{+}((A_{11}^2 + A_{12}A_{21})y_1 + (A_{11}A_{12} + A_{12}A_{22})y_2) + (A_{12}^{\perp})^{+}A_{12}^{\perp}(A_{21}y_1 + A_{22}y_2)$, $s_2 = A_{12}^{\perp}y_2$, $u_d = (\alpha_1 + 3\beta_1\|y_1\|_\infty^2 + 2\|A_{12}A_{23}\|_\infty\gamma_0(t, y)/2)A_{12}^{+}\mathrm{sign}[s_1] + \gamma_0(t, y)\|A_{12}^{\perp}A_{23}\|_\infty(A_{12}^{\perp})^{+}\mathrm{sign}[s_2]$, $s_1 = A_{11}y_1 + A_{12}y_2 + ((A_{11}y_1 + A_{12}y_2)^{[2]} + \alpha_1 y_1 + \beta_1 y_1^{[3]})^{[1/2]}$ with $0.5\alpha_1 = \alpha_2 = 4\alpha_3 = 0.5\beta_1 = \beta_2 = 4\beta_3 = 64\,\mathrm{T}_{\max}^{-2}$, $T_{\max} > 0$. Then the origin of the closed-loop system (10) is globally fixed-time stable with the settling-time function bounded by $T_{\max}$.

Proof: First of all remark that $A_{12}(A_{12}^{\perp})^{+} = 0$ and $A_{12}^{\perp}A_{12}^{+} = 0$. Denote $z = A_{11}y_1 + A_{12}y_2$. In this case, (10) can be rewritten in the form

$$\dot{y}_1 = z, \quad \dot{z} = -A_{12}u_d - A_{12}u_p + A_{12}A_{23}\gamma$$
$$\dot{s}_2 = -A_{12}^{\perp}u_d - A_{12}^{\perp}u_p + A_{12}^{\perp}A_{23}\gamma.$$

Following the proof of Theorem 2, we similarly show $y_1(t) = 0$ and $z(t) = 0$ for all $t \ge T_{\max}$. Hence $A_{12}y_2(t) = 0$ for all $t \ge T_{\max}$. Since $\dot{s}_2 = -\gamma_0(t, y)\|A_{12}^{\perp}A_{23}\|_\infty\mathrm{sign}[s_2] - (\alpha_3 s_2 + \beta_3 s_2^{[3]})^{[1/2]} + A_{12}^{\perp}A_{23}\gamma$ then $s_2(t) = A_{12}^{\perp}y_2(t) = 0$ for all $t \ge T_{\max}$. ■

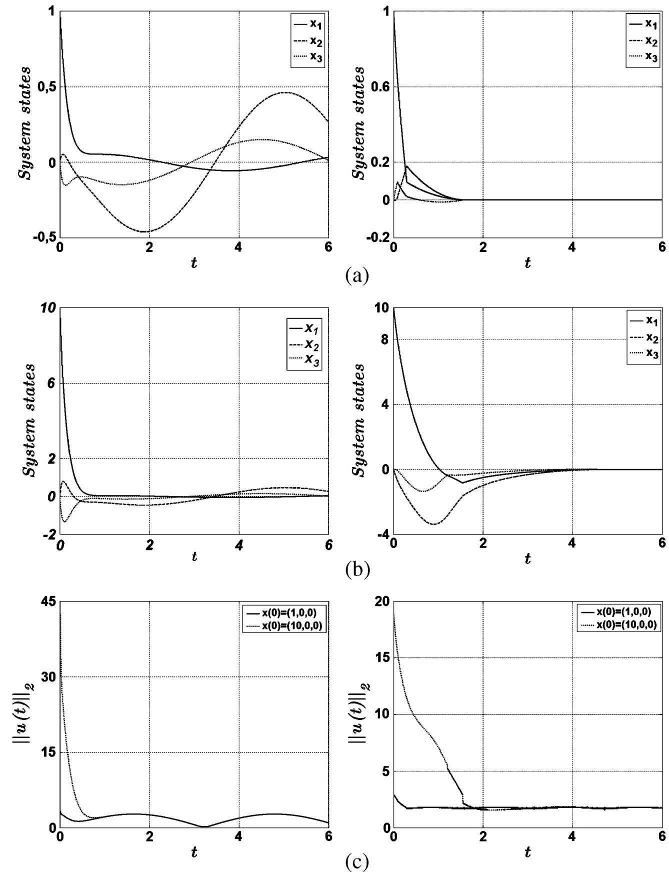

Fig. 1. Simulation results for fixed-time attraction (on the left side) and fixed-time stabilization (on the right side). (a) $x(0) = (1, 0, 0)$. (b) $x(0) = (10, 0, 0)$. (c) Control magnitude.

VI. Numerical Example

Consider as an example, (2) with

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 0 & 3 \\ 0 & -1 & 4 \end{bmatrix},\ B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & -3 \end{bmatrix},\ f = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}\sin(t)$$
$$G = \begin{bmatrix} 0.4286 & 0.8571 & 0.2857 \\ -0.8571 & 0.4857 & -0.1714 \\ -0.2857 & -0.1714 & 0.9429 \end{bmatrix}.$$

The transformation $y = Gx$ brings the system to the block from (10) with $y_1 \in \mathbb{R}$, $y_2 \in \mathbb{R}^2$, $\gamma(t, x) = (1, 1)^T\sin(t)$, $\gamma_0(t, x) = 1$. Using Theorem 3 the fixed-time attracting controller is designed for this system in the form (9) with $T_{\max} = 6$, $r = 1$, and $\beta_1 = \beta_2 = 0.032$, $\alpha_1 = 6.3918$, $\alpha_2 = 13.333$. The parameters $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 0.5$, $\alpha_3 = \beta_3 = 0.25$ of the fixed-time stabilizing controller (11) are taken from Theorem 4 for $T_{\max} = 8$. All simulations results are presented on Fig. 1. The last controller exhibits very small conservatism showing the settling-time ≈7.2 for $x(0) = (10^7, 0, 0)$.

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Complete Statistical Characterization of Discrete-Time LQG and Cumulant Control

Fucai Qian, Jianjun Gao, and Duan Li

***Abstract*—As the performance index of the linear-quadratic-Gaussian (LQG) problem is governed by the noncentral generalized Chi-square distribution, solely controlling the expected value of the performance index, as the traditional LQG theory aims at, is insufficient to deliver a satisfactory solution in some situations. While the risk sensitive control does control a specific weighting sum of various moments of the performance index, the single degree of freedom in adjusting the weighting coefficients in this specific weighting sum of various moments of the performance index often prevents the risk sensitive control from generating a desired pattern of high order moment-distribution. We achieve in this note the complete statistical characterization of the performance index for the discrete-time LQG formulation. More specifically, we derive a recursive relationship to obtain cumulants of various orders of the performance index successively. Parameterized in feedback gain, the optimal feedback control law can be computed off-line by solving a static polynomial optimization problem, thus serving two design goals: i) To shape the probability density function (pdf) of the performance index to attain, at least approximately, a given desired pattern by regulating cumulants of various orders, and ii) to improve the performance measure of an incumbent solution (generated by the risk sensitive control, for example) by adjusting the levels of cumulants of various orders.**

***Index Terms*—Cumulant control, linear-quadratic Gaussian (LQG) stochastic control, noncentral generalized Chi-square distribution, polynomial optimization.**

I. Introduction

We consider in this note the following discrete-time stochastic linear system:

$$x_{k+1} = Ax_k + Bu_k + \xi_k, \quad k = 0, 1, \cdots, N-1 \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state with a known initial state x_0, $u_k \in \mathbb{R}^m$ is the control, A and B are matrices of $n \times n$ and $n \times m$ dimensions, respectively, and $\{\xi_k \in \mathbb{R}^n\}|_{k=0}^{N-1}$ is a sequence of white Gaussian random noises with zero mean and covariance matrix $\{\Omega_k \in \mathbb{R}^{n\times n}\}|_{k=0}^{N-1}$. Our target is to control the following quadratic performance index:

$$J = x_N' Q_N x_N + \sum_{k=0}^{N-1} \left(x_k' Q_k x_k + u_k' R_k u_k\right) \tag{2}$$

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F. C. Qian is with the School of Automation and Information Engineering, Xi'an University of Technology, Xi'an Shaanxi 710048, China (e-mail: fcqian@xaut.edu.cn).

J. J. Gao is with the Department of Automation, School of Electronic Information and Electrical Engineering, Shanghai Jiao Tong University, Hong Kong, China (e-mail: jjgao@se.cuhk.edu.hk).

D. Li is with Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Hong Kong, China (e-mail: dli@se.cuhk.edu.hk).

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