(ii) Alternatively by explicitly calculating solutions and noticing that the reachable set from the origin at any fixed instant is a line. □

Exercise 3.5.25 Refer to Exercise 3.2.12. Assume here that the ground speed is not constant but is instead a function of time, $c = c(t)$ (take c to be smooth, though much less is needed). Apply the rank criteria for time-varying linear systems to conclude that the system is controllable in any nontrivial interval in which c is not identically zero. □

3.6 Bounded Controls*

In this section, we deal with continuous-time time-invariant systems of the form $\dot{x} = Ax + Bu$ for which $\mathcal{U}$ is a subset of $\mathbb{R}^m$. Linear systems have, by definition, $\mathcal{U} = \mathbb{R}^m$, so we call such systems "linear systems with constrained controls". For instance, take the system $\dot{x} = -x + u$ ($n = m = 1$), with $\mathcal{U} = (-1, 1)$. The pair $(A, B) = (1, 1)$ is controllable, but the system with restricted controls is not, since it is impossible to transfer the state $x = 0$ to $z = 2$ (since $\dot{x}(t) < 0$ whenever $x(t) \in (1, 2)$).

In order to avoid confusion with reachability for the associated linear systems with unconstrained controls, in this section we will say that $z \in \mathbb{R}^n$ can be $\mathcal{U}$-*reached* from $x \in \mathbb{R}^n$ (or that x *can be* $\mathcal{U}$-*controlled to* z) *in time* T if there is some input $\omega : [0, T] \to \mathbb{R}^m$ so that $\phi(T, 0, x, \omega) = z$ and $\omega(t) \in \mathcal{U}$ for (almost) all $t \in [0, T]$. We define the reachable set in time $T \geq 0$

$$\mathcal{R}_{\mathcal{U}}^T(x) := \{z \in \mathbb{R}^n \ \mathcal{U}\text{-reachable from } x \text{ in time } T\}$$

and $\mathcal{R}_{\mathcal{U}}(x) := \bigcup_{T \geq 0} \mathcal{R}_{\mathcal{U}}^T(x)$. (Thus $\mathcal{R}_{\mathbb{R}^m}(x)$ is the same as what we earlier called $\mathcal{R}(x)$.)

In this section we will establish the following result.

Theorem 6 *Let $\mathcal{U}$ be a bounded neighborhood of zero. Then, $\mathcal{R}_{\mathcal{U}}(0) = \mathbb{R}^n$ if and only if*

(a) the pair (A, B) is controllable, and

(b) the matrix A has no eigenvalues with negative real part.

Observe that the necessity of controllability for the pair (A, B) is obvious, since $\mathcal{R}_{\mathcal{U}}(0) \subseteq \mathcal{R}(0)$.

We prove Theorem 6 after a series of preliminary results.

Lemma 3.6.1 Let $\mathcal{U} \subseteq \mathbb{R}^m$ and pick any two $S, T \geq 0$. Then

$$\mathcal{R}_{\mathcal{U}}^T(0) + e^{TA}\mathcal{R}_{\mathcal{U}}^S(0) = \mathcal{R}_{\mathcal{U}}^{S+T}(0)\,.$$

* This section can be skipped with no loss of continuity.

Proof. Pick

$$x_1 = \int_0^T e^{(T-\tau)A} B\omega_1(\tau)d\tau = \int_S^{S+T} e^{(S+T-\tau)A} B\omega_1(\tau - S)d\tau$$

and

$$x_2 = \int_0^S e^{(S-\tau)A} B\omega_2(\tau)d\tau$$

with the inputs ω_i $\mathcal{U}$-valued. Note that

$$e^{TA}x_2 = \int_0^S e^{(S+T-\tau)A} B\omega_2(\tau)d\tau\,.$$

Thus

$$x_1 + e^{TA}x_2 = \int_0^{S+T} e^{(S+T-\tau)A} B\omega(\tau)d\tau$$

where

$$\omega(s) = \begin{cases} \omega_2(\tau) & 0 \le \tau \le S \\ \omega_1(\tau - S) & S \le \tau \le S+T. \end{cases}$$

Note that $\omega(t) \in \mathcal{U}$ for all $t \in [0, S+T]$. Thus $\mathcal{R}^T_\mathcal{U}(0) + e^{TA}\mathcal{R}^S_\mathcal{U}(0) \subseteq \mathcal{R}^{S+T}_\mathcal{U}(0)$. The converse inclusion follows by reversing these steps. ∎

By induction on q we then conclude:

Corollary 3.6.2 Let $\mathcal{U} \subseteq \mathbb{R}^m$ and pick any $T \ge 0$ and any integer $q \ge 1$. Then

$$\mathcal{R}^T_\mathcal{U}(0) + e^{TA}\mathcal{R}^T_\mathcal{U}(0) + \ldots + e^{(q-1)TA}\mathcal{R}^T_\mathcal{U}(0) = \mathcal{R}^{qT}_\mathcal{U}(0)\,.$$

Proposition 3.6.3 (1) If $\mathcal{U} \subseteq \mathbb{R}^m$ is convex, then $\mathcal{R}_\mathcal{U}(0)$ is a convex subset of $\mathbb{R}^n$. (2) If (A, B) is controllable and $\mathcal{U} \subseteq \mathbb{R}^m$ is a neighborhood of $0 \in \mathbb{R}^m$, then $\mathcal{R}_\mathcal{U}(0)$ is an open subset of $\mathbb{R}^n$.

Proof. Convexity of each $\mathcal{R}^T_\mathcal{U}(0)$ follows from linearity of $\phi(T, 0, 0, u)$ on u and convexity of $\mathcal{U}$. This proves that the (increasing) union $\mathcal{R}_\mathcal{U}(0)$ is convex, when $\mathcal{U}$ is convex.

Assume now that (A, B) is controllable and $\mathcal{U} \subseteq \mathbb{R}^m$ is a neighborhood of 0. We first prove that, for each $T > 0$, $\mathcal{R}^T_\mathcal{U}(0)$ is a neighborhood of $0 \in \mathbb{R}^n$. Fix such a T. Pick a subset $\mathcal{U}_0 \subseteq \mathcal{U}$ which is a convex neighborhood of 0. The desired conclusion will follow if we show that $0 \in \mathbb{R}^n$ is in the interior of $\mathcal{R}^T_{\mathcal{U}_0}(0)$, since $\mathcal{R}^T_{\mathcal{U}_0}(0) \subseteq \mathcal{R}^T_\mathcal{U}(0)$. So without loss of generality, for the rest of this paragraph we replace $\mathcal{U}$ by $\mathcal{U}_0$ and hence assume that $\mathcal{U}$ is also convex. Pick any basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Let $e_0 := -\sum_{i=1}^n e_i$. For each $i = 0, \ldots, n$ there is an input ω_i, not necessarily $\mathcal{U}$-valued, so that

$$e_i = \phi(T, 0, 0, \omega_i)\,.$$

Let $\mu > 0$ be so that, with $\omega_i' := \frac{1}{\mu}\omega_i$, $i = 0, \ldots, n$, it holds for all i that $\omega_i'(t) \in \mathcal{U}$ for (almost) all $t \in [0, T]$. (There exists some such μ because $\mathcal{U}$ is a neighborhood of 0 and the ω_i are essentially bounded.) Thus $e_i' := \frac{1}{\mu}e_i = \phi(T, 0, 0, \omega_i') \in \mathcal{R}_{\mathcal{U}}^T(0)$ for each i.

Pick any $\varepsilon_1, \ldots, \varepsilon_n$ such that $|\varepsilon_i| \leq \frac{1}{2(n+1)}$ for all i. Then

$$\frac{\varepsilon_1}{\mu}e_1 + \ldots + \frac{\varepsilon_n}{\mu}e_n \;=\; \sum_{i=1}^{n}\left(\frac{1-\varepsilon}{n+1} + \varepsilon_i\right)e_i' + \frac{1-\varepsilon}{n+1}e_0'$$

with $\varepsilon = \sum_i \varepsilon_i$. This is a convex combination. Since all $e_i' \in \mathcal{R}_{\mathcal{U}}^T(0)$, and this set is convex, it follows that $\frac{\varepsilon_1}{\mu}e_1 + \ldots + \frac{\varepsilon_n}{\mu}e_n \in \mathcal{R}_{\mathcal{U}}^T(0)$ for all small enough $\varepsilon_1, \ldots, \varepsilon_n$, which shows that $\mathcal{R}_{\mathcal{U}}^T(0)$ is a neighborhood of 0.

Finally, we show that $\mathcal{R}_{\mathcal{U}}(0)$ is open. Pick any $S > 0$. By the previous discussion, there is some open subset $V \subseteq \mathcal{R}_{\mathcal{U}}^S(0)$ containing zero. Pick any $x \in \mathcal{R}_{\mathcal{U}}(0)$; we wish to show that some neighborhood of x is included in $\mathcal{R}_{\mathcal{U}}(0)$. Let $\omega : [0, T] \to \mathcal{U}$ be so that $x = \phi(T, 0, 0, \omega)$. The set $W := e^{TA}V$ is open, because e^{TA} is nonsingular. For each $y = e^{TA}v \in W$,

$$y + x = e^{TA}v + \phi(T, 0, 0, \omega) \in \mathcal{R}_{\mathcal{U}}^T(V)\,.$$

Thus $x + W$ is an open subset of $\mathcal{R}_{\mathcal{U}}^T(V) \subseteq \mathcal{R}_{\mathcal{U}}^{S+T}(0) \subseteq \mathcal{R}_{\mathcal{U}}(0)$ which contains x. ∎

For each eigenvalue λ of A and each positive integer k we let

$$J_{k,\lambda} := \ker\,(\lambda I - A)^k$$

(a subspace of $\mathbb{C}^n$) and the set of real parts

$$J_{k,\lambda}^{\mathbb{R}} := \mathrm{Re}\,(J_{k,\lambda}) = \{\mathrm{Re}\,v \mid v \in J_{k,\lambda}\}$$

(a subspace of $\mathbb{R}^n$). Observe that if $v \in J_{k,\lambda}$, $v = v_1 + iv_2$ with $v_j \in \mathbb{R}^n$, $j = 1, 2$, then $v_1 \in J_{k,\lambda}^{\mathbb{R}}$, by definition, but also the imaginary part $v_2 \in J_{k,\lambda}^{\mathbb{R}}$, because $(-iv)$ belongs to the subspace $J_{k,\lambda}$. We also let $J_{0,\lambda} = J_{0,\lambda}^{\mathbb{R}} = \{0\}$.

Let L be the sum of the various spaces $J_{k,\lambda}^{\mathbb{R}}$, with $\mathrm{Re}\,\lambda \geq 0$, and let M be the sum of the various spaces $J_{k,\lambda}^{\mathbb{R}}$, with $\mathrm{Re}\,\lambda < 0$. Each of these spaces is A-invariant, because if v is an eigenvector of A, and $v = v_1 + iv_2$ is its decomposition into real and imaginary parts, then the subspace of $\mathbb{R}^n$ spanned by v_1 and v_2 is A-invariant. From the Jordan form decomposition, we know that every element in $\mathbb{C}^n$ can be written as a sum of elements in the various "generalized eigenspaces" $J_{k,\lambda}$, so taking real parts we know that $\mathbb{R}^n$ splits into the direct sum of L and M. (In fact, L is the largest invariant subspace on which all eigenvalues of A have nonnegative real parts, and analogously for M.)

We will need this general observation:

Lemma 3.6.4 If C is an open convex subset of $\mathbb{R}^n$ and L is a subspace of $\mathbb{R}^n$ contained in C, then $C + L = C$.

Proof. Clearly $C = C + 0 \subseteq C + L$, so we only need prove the other inclusion. Pick any $x \in C$ and $y \in L$. Then, for all $\varepsilon \neq 0$:

$$x + y = \left(\frac{1}{1+\varepsilon}\right)[(1+\varepsilon)x] + \left(\frac{\varepsilon}{1+\varepsilon}\right)\left[\left(\frac{1+\varepsilon}{\varepsilon}\right)y\right].$$

Since C is open, $(1+\varepsilon)x \in C$ for some sufficiently small $\varepsilon > 0$. Since L is a subspace, $\left(\frac{1+\varepsilon}{\varepsilon}\right) y \in L \subseteq C$. Thus $x + y \in C$, by convexity. ∎

The main technical fact needed is as follows. Fix any eigenvalue $\lambda = \alpha + i\beta$ of A with real part $\alpha \geq 0$, and denote for simplicity $J_k^{\mathbb{R}} := J_{k,\lambda}^{\mathbb{R}}$.

Lemma 3.6.5 Assume that (A, B) is controllable and $\mathcal{U} \subseteq \mathbb{R}^m$ is a neighborhood of 0. Then $J_k^{\mathbb{R}} \subseteq \mathcal{R}_{\mathcal{U}}(0)$ for all k.

Proof. First replacing if necessary $\mathcal{U}$ by a convex subset, we may assume without loss of generality that $\mathcal{U}$ is a convex neighborhood of 0. We prove the statement by induction on k, the case $k = 0$ being trivial. So assume that $J_{k-1}^{\mathbb{R}} \subseteq \mathcal{R}_{\mathcal{U}}(0)$, and take any $\widetilde{v} \in J_{k,\lambda}$, $\widetilde{v} = \widetilde{v}_1 + i\widetilde{v}_2$. We must show that $\widetilde{v}_1 \in \mathcal{R}_{\mathcal{U}}(0)$.

First pick any $T > 0$ so that $e^{\lambda Tj} = e^{\alpha Tj}$ for all $j = 0, 1, \ldots$. (If $\beta = 0$ one may take any $T > 0$; otherwise, we may use for instance $T = \frac{2\pi}{|\beta|}$.) Next choose any $\delta > 0$ with the property that $v_1 := \delta\widetilde{v}_1 \in \mathcal{R}_{\mathcal{U}}^T(0)$. (There is such a δ because $\mathcal{R}_{\mathcal{U}}(0)$ contains 0 in its interior, by Proposition 3.6.3.) Since $v \in \ker(\lambda I - A)^k$, where $v = \delta\widetilde{v}$,

$$e^{(A-\lambda I)t}v = \left(I + t(A - \lambda I) + \frac{t^2}{2}(A - \lambda I)^2 + \ldots\right)v = v + w \quad \forall\, t\,,$$

where $w \in J_{k-1}$. Thus

$$e^{\alpha t}v = e^{\lambda t}v = e^{tA}v - e^{\lambda t}w = e^{tA}v - e^{\alpha t}w \quad \forall\, t = jT, j = 0, 1, \ldots. \tag{3.26}$$

Decomposing into real and imaginary parts $w = w_1 + iw_2$ and taking real parts in Equation (3.26),

$$e^{\alpha t}v_1 = e^{tA}v_1 - e^{\alpha t}w_1 \quad \forall\, t = jT, j = 0, 1, \ldots.$$

Now pick any integer $q \geq 1/\delta$. Then

$$\left(\sum_{j=0}^{q-1} e^{\alpha jT}\right) v_1 = \sum_{j=0}^{q-1} e^{jTA} v_1 + w'$$

where $w' = -\sum e^{\alpha jT}w_1$ belongs to the subspace $J_{k-1}^{\mathbb{R}}$. Applying first Corollary 3.6.2 and then Lemma 3.6.4, we conclude that

$$pv_1 \in \mathcal{R}_{\mathcal{U}}^{qT}(0) + J_{k-1}^{\mathbb{R}} \subseteq \mathcal{R}_{\mathcal{U}}(0)$$

where

$$p = \sum_{j=0}^{q-1} e^{\alpha j T} \geq \sum_{j=0}^{q-1} 1 = q \geq \frac{1}{\delta}\,.$$

(Here is precisely where we used that $\alpha \geq 0$.) Therefore $\delta p \widetilde{v}_1 = pv_1 \in R_{\mathcal{U}}(0)$. On the other hand, $\delta p \geq 1$ means that

$$\widetilde{v}_1 = \frac{1}{\delta p}\delta p \widetilde{v}_1 + \left(1 - \frac{1}{\delta p}\right) 0$$

is a convex combination. Since $\delta p \widetilde{v}_1$ and 0 both belong to $\mathcal{R}_{\mathcal{U}}(0)$, we conclude by convexity of the latter that indeed $\widetilde{v}_1 \in \mathcal{R}_{\mathcal{U}}(0)$. ∎

Corollary 3.6.6 Assume that (A, B) is controllable and $\mathcal{U} \subseteq \mathbb{R}^m$ is a neighborhood of 0. Then $L \subseteq \mathcal{R}_{\mathcal{U}}(0)$.

Proof. As before, we may assume without loss of generality that $\mathcal{U}$ is convex. We have that L is the sum of the spaces $J^{\mathbb{R}}_{k,\lambda}$, over all eigenvalues λ with real part nonnegative, and each of these spaces is included in $\mathcal{R}_{\mathcal{U}}(0)$. In general, if L_1 and L_2 are two subspaces of a convex set C, $L_1 + L_2 \subseteq C$ (since $x + y = \frac{1}{2}(2x) + \frac{1}{2}(2y)$), so the sum of the L's is indeed included in $\mathcal{R}_{\mathcal{U}}(0)$. ∎

The next result says that the reachable set from zero is a "thickened linear subspace":

Corollary 3.6.7 Assume that (A, B) is controllable and $\mathcal{U} \subseteq \mathbb{R}^m$ is a convex and bounded neighborhood of 0. Then there exists a set $\mathcal{B}$ such that $\mathcal{R}_{\mathcal{U}}(0) = \mathcal{B} + L$ and $\mathcal{B}$ is bounded, convex, and open relative to M.

Proof. We claim that $\mathcal{R}_{\mathcal{U}}(0) = (\mathcal{R}_{\mathcal{U}}(0) \bigcap M) + L$. One inclusion is clear from

$$\left(\mathcal{R}_{\mathcal{U}}(0) \bigcap M\right) + L \;\subseteq\; \mathcal{R}_{\mathcal{U}}(0) + L \;=\; \mathcal{R}_{\mathcal{U}}(0)$$

(applying Lemma 3.6.4). Conversely, any $v \in \mathcal{R}_{\mathcal{U}}(0)$ can be decomposed as $v = x + y \in M + L$; we need to show that $x \in R_{\mathcal{U}}(0)$. But $x = v - y \in R_{\mathcal{U}}(0) + L = R_{\mathcal{U}}(0)$ (applying the same Lemma yet again). This establishes the claim.

We let $\mathcal{B} := \mathcal{R}_{\mathcal{U}}(0) \bigcap M$. This set is convex and open in M because $\mathcal{R}_{\mathcal{U}}(0)$ is open and convex. We only need to prove that it is bounded.

Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the projection on M along L, that is, $P(x + y) = x$ if $x \in M$, $y \in L$. Observe that $PA = AP$ because each of L and M are A-invariant (so $v = x + y$, $Ax \in M$, $Ay \in L$, imply $PAv = Ax = APv$). Pick any $x \in \mathcal{R}_{\mathcal{U}}(0) \bigcap M$. Since $x \in \mathcal{R}_{\mathcal{U}}(0)$, there are some T and some ω so that

$$x = \int_0^T e^{(T-\tau)A} B\omega(\tau)\, d\tau\,.$$

On the other hand, since $x \in M$, $x = Px$. Thus:

$$x = Px = \int_0^T Pe^{(T-\tau)A} B\omega(\tau)\,d\tau = \int_0^T e^{(T-\tau)A} x(\tau)\,d\tau\,,$$

where $x(\tau) = PB\omega(\tau) \in M \bigcap PB(\mathcal{U})$ for all τ.

Since the restriction of A to M has all its eigenvalues with negative real part, there are positive constants $c, \mu > 0$ such that $\left\|e^{tA}x\right\| \leq ce^{-\mu t}\left\|x\right\|$ for all $t \geq 0$ and all $x \in M$. Since $PB(\mathcal{U})$ is bounded, there is then some constant c' such that, if x is also in $PB(\mathcal{U})$, $\left\|e^{tA}x\right\| \leq c'e^{-\mu t}$ for all $t \geq 0$. So, for x as above we conclude

$$\|x\| \leq c' \int_0^T e^{-\mu(T-\tau)}\,d\tau \leq \frac{c'}{\mu}(1 - e^{-\mu T}) \leq \frac{c'}{\mu}\,,$$

and we proved that $\mathcal{B}$ is bounded. ∎

Proof of Theorem 6

Assume first that $\mathcal{R}_{\mathcal{U}}(0) = \mathbb{R}^n$. We already remarked that the pair (A, B) must be reachable. If the eigenvalue condition (b) does not hold, then L is a proper subspace of $\mathbb{R}^n$ and $M \neq 0$. Enlarging $\mathcal{U}$ if necessary, we may assume that $\mathcal{U}$ is convex and bounded. Lemma 3.6.7 claims that $\mathbb{R}^n = \mathcal{R}_{\mathcal{U}}(0)$ is then a subset of $L + \mathcal{B}$, with bounded $\mathcal{B}$, a contradiction. Conversely, assume that (a) and (b) hold. By Corollary 3.6.6, $\mathbb{R}^n = L \subseteq \mathcal{R}_{\mathcal{U}}(0)$. ∎

We also say that the system Σ is $\mathcal{U}$-*controllable* if $\mathcal{R}_{\mathcal{U}}(x) = \mathbb{R}^n$ for all $x \in \mathbb{R}^n$.

Exercise 3.6.8 Let $\mathcal{U}$ be a bounded neighborhood of zero. Show:

1. Every state can be $\mathcal{U}$-controlled to zero if and only if (a) the pair (A, B) is controllable and (b) the matrix A has no eigenvalues with positive real part.

2. Σ is completely $\mathcal{U}$-controllable (that is, $\mathcal{R}_{\mathcal{U}}(x) = \mathbb{R}^n$ for all $x \in \mathbb{R}^n$) if and only if (a) the pair (A, B) is controllable, and (b) all eigenvalues of the matrix A are purely imaginary. ∎

3.7 First-Order Local Controllability

For nonlinear systems, the best one often can do regarding controllability notions is to characterize local controllability —and even that problem is not totally understood. In order to talk about local notions, a topology must be introduced in the state space. Even though the only nontrivial results will be proved for discrete-time and continuous-time systems of class $\mathcal{C}^1$, it is more natural to provide the basic definitions in somewhat more generality. A class called here *topological systems* is introduced, which captures the basic continuity properties needed. In Chapter 5, a few basic facts about stability are proved in the general context of such systems. From now on in this Chapter, and unless otherwise stated, *system* means *topological system* in the sense of Definition 3.7.1 below.