Construction of Continuous and Piecewise Affine Feedback Stabilizers for Nonlinear Systems

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***Abstract*—In this article, two methods for constructing continuous and piecewise affine (CPA) feedback stabilizers for nonlinear systems are presented. First, a construction based on a piecewise affine interpolation of Sontag's "universal" formula is developed. Stability of the corresponding closed-loop system is verified *a posteriori* by means of a CPA control Lyapunov function and subsequently solving a feasibility problem. Second, we develop a procedure for computing CPA feedback stabilizers via linear programming, which allows for the optimization of a control-oriented criterion in the synthesis procedure. Stability conditions are *a priori* specified in the linear program, which removes the necessity for *a posteriori* verification of closed-loop stability. We illustrate the developed methods via two application-inspired examples considering the stabilization of an inverted pendulum and the stabilization of a healthy equilibrium of the hypothalamic-pituitary-adrenal axis.**

***Index Terms*—Computational methods, Lyapunov methods, stability of nonlinear systems, stabilization.**

I. Introduction

A RESULT by Artstein [1] showed that the existence of a smooth control Lyapunov function (CLF) for a system that is affine in the input is equivalent to the existence of a feedback stabilizer that is continuous, except possibly at the origin. This equivalence was proven constructively by Sontag in [2], which resulted in a "universal" formula for a CLF-based feedback stabilizer. The stabilizer from Sontag [2] was adapted in [3] to the case with bounded controls, providing a feedback stabilizer that takes values in the unit ball, given the existence of an appropriate CLF. The controller was further generalized in [4] for bounded or positive input restrictions.

An approach for computing CLFs by partitioning the state space was introduced in [5], where a continuous and piecewise affine (CPA) CLF is constructed along with a switching control strategy via a mixed integer linear program (LP), for systems that admit a smooth CLF. This approach has been generalized in [6], for systems that do not necessarily admit a smooth CLF, among which Artstein's circles [1] and the nonholonomic integrator [7], cf., [8]. Furthermore, the recursive procedure in [9] provides a method for the construction of rational CLFs accompanying polynomial feedback stabilizers.

CLFs are powerful tools for stabilization, but the corresponding universal formulas can yield complex expressions. For control engineering practice, it is desirable to achieve simple, e.g., piecewise constant or affine, expressions for a feedback stabilizer. The challenge is to parameterize and compute such stabilizers.

In this article, we consider the computation of a feedback stabilizer, which is CPA on a simplicial subdivision of the state space. Since the feedback stabilizer is affine on each region, the implementation of such a control law is simple compared to the "universal" formula in [2]. In fact, since the gradient is constant on each simplicial region, the application of the resulting controllers can be achieved via so-called lookup tables. The efficiency of CPA feedback laws has already been substantiated within the field of explicit model predictive control (see, e.g., [10] and [11]). We distinguish two cases for the proposed construction. First, we consider the CPA interpolation of the "universal" controller as a candidate stabilizing feedback law. Verification of closed-loop stability is performed *a posteriori* via a feasibility problem, involving a finite number of inequalities. Second, we provide an algorithm for computing a CPA feedback law using CPA CLFs via a single LP, allowing for optimality in the synthesis through a control-oriented objective. No *a posteriori* verification of closed-loop stability for the latter feedback is required.

We will illustrate both methods first for the stabilization of an inverted pendulum and provide a comparison with a common controller for robot manipulators. Then, we will demonstrate the stabilization of a healthy equilibrium of a neuroendocrine system called the hypothalamic-pituitary-adrenal (HPA) axis [12], via the LP-based controller synthesis. This application is inspired by automated treatment and the implementation of the stabilizer could be carried by a pharmaceutical supply of cortisol from a biomedical point of view.

The developed control synthesis procedures are compare with other methods in the literature as follows. The stability conditions in this article are related to the CPA approach for computing Lyapunov functions (LFs) for nonlinear differential

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inclusions [13]. The methods in [5] and [6] are partially build on the results in [13] for the synthesis of a CLF together with a piecewise constant control, leading to a mixed integer LP. We note that a CPA CLF synthesized according to Baier and Hafstein [5] or Baier *et al.* [6] can serve as a basis for the linear programming based construction of a CPA stabilizer, as described in Section III of this article. The piecewise affine parameterization for control has also been considered in [14], wherein an automated controller synthesis procedure was proposed for a class of piecewise affine hybrid systems on simplices. This procedure has been extended in [15], for the indirect synthesis of a CPA controller for nonlinear systems that are affine w.r.t. the control input, via a hybrid abstraction of the system dynamics.

The remainder of this article is organized as follows. In Section II, we provide some preliminaries and concepts from nonsmooth Lyapunov theory. Section III-A provides a method for the construction and verification of a "universal" CPA feedback stabilizer. In Section III-B, we present an algorithm for the construction of a CPA feedback law via linear programming. The proposed methods are analyzed and compared via two illustrative examples in Section IV. Conclusions are summarized in Section V.

II. Preliminaries

A. Notation and Definitions

The sets of nonnegative and positive integers and nonnegative and positive reals are denoted by $\mathbb{N}$, $\mathbb{N}_{>0}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$, respectively. Given $a \in \mathbb{N}$ and $b \in \mathbb{N}$ such that $a < b$, we denote $\mathbb{N}_{[a:b]} := \{a, a+1, \dots, b-1, b\}$. A set $\mathcal{S} \subseteq \mathbb{R}^n$ is called proper if it contains the origin in its interior. Every closed and bounded subset of $\mathbb{R}^n$ is called a compact set. Given a compact set $\mathcal{S} \subset \mathbb{R}^n$, we denote the boundary and interior of $\mathcal{S}$ by $\partial\mathcal{S}$ and $\mathcal{S}^\circ$, respectively. A function $z : \mathcal{S} \to \mathbb{R}$, with proper set $S \subseteq \mathbb{R}^n$, is called positive definite if $z(0) = 0$ and $z(x) > 0$ for all $x \in \mathcal{S} \setminus \{0\}$. For $x \in \mathbb{R}^n$ and $p \in \mathbb{N}_{>0}$, we define the norm $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$. Moreover, $\|x\|_\infty := \max_{i \in \mathbb{N}_{[1:n]}} |x_i|$. Finally, the open ball centered at 0 of radius $\varepsilon \in \mathbb{R}_{>0}$ is denoted by $\mathbb{B}_\varepsilon$, i.e., $\mathbb{B}_\varepsilon := \{x \in \mathbb{R}^n \mid \|x\|_2 < \varepsilon\}$.

Consider the nonlinear continuous-time, time-invariant, system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^n$, $u \in \mathbb{U} \subseteq \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ are twice continuously differentiable functions and $f(0) = 0$, i.e., the origin is an equilibrium for (1).

For the computation of the feedback stabilizer $u = k(x)$, $k : \mathbb{R}^n \to \mathbb{R}^m$, we will consider CPA functions, both for the CLF and feedback stabilizer itself. Thereto, we introduce the definition of a proper triangulation and such CPA functions. Let $v_0, \dots, v_m \in \mathbb{R}^n$ be a collection of vectors. The vectors in the collection $v_0, \dots, v_m$ are said to be affinely independent if $\sum_{i=1}^m \lambda_i(v_i - v_0) = 0$ implies $\lambda_i = 0$ for all $i \in \mathbb{N}_{[1:m]}$.

Definition 2.1: Given a collection of affinely independent vectors $v_0, \dots, v_n$, an n-simplex in $\mathbb{R}^n$ is defined as its convex hull, i.e.,

$$\mathrm{co}\{v_0, \dots, v_n\} := \left\{ \sum_{i=0}^n \lambda_i v_i \,\middle|\, \lambda_i \in [0,1],\ \sum_{i=0}^n \lambda_i = 1 \right\}.$$

Given a simplex $\mathfrak{S} \subset \mathbb{R}^n$, the diameter of $\mathfrak{S}$ is defined by

$$\mathrm{diam}(\mathfrak{S}) := \max_{\alpha,\beta \in \mathfrak{S}} \|\alpha - \beta\|_2.$$

Definition 2.2: Let $\mathfrak{M}$ be a proper subset of $\mathbb{R}^n$ and let $\mathfrak{T} = \{\mathfrak{S}_1, \dots, \mathfrak{S}_z\}$, where $\mathfrak{S}_\nu$, $\nu \in \mathbb{N}_{[1:z]}$ is an n-simplex. Suppose $\mathfrak{M} = \cup_{\nu \in \mathbb{N}_{[1:z]}} \mathfrak{S}_\nu$. If for all $\mathfrak{S}_a, \mathfrak{S}_b \in \mathfrak{T}$, $a \neq b$, we have $\mathfrak{S}_a \cap \mathfrak{S}_b = \emptyset$ or $\mathfrak{S}_a \cap \mathfrak{S}_b$ is a common face of $\mathfrak{S}_a$ and $\mathfrak{S}_b$, then $\mathfrak{T}$ is called a triangulation of $\mathfrak{M}$. If, additionally, the origin is a vertex of a simplex $\mathfrak{S}_\nu \in \mathfrak{T}$, then $\mathfrak{T}$ is called a proper triangulation of $\mathfrak{M}$.

For a triangulation, as defined earlier, we define the set containing all vertices of simplices in the triangulation by

$$\mathcal{V}_\mathfrak{T} = \left\{ x \in \mathbb{R}^n \,|\, x \text{ is a vertex of } \mathfrak{S}_\nu \in \mathfrak{T},\ \nu \in \mathbb{N}_{[1:z]} \right\}.$$

For all $x \in \mathfrak{M}$, we define the active index set $I_\mathfrak{T}(x) := \{\nu \in \mathbb{N}_{[1:z]} \,|\, x \in \mathfrak{S}_\nu\}$.

Consider the map $Z_\mathcal{V} : \mathcal{V}_\mathfrak{T} \to \mathbb{R}_{\geq 0}$. The map $Z_\mathcal{V}$ can be extended to a unique piecewise affine map $Z : \mathfrak{M} \to \mathbb{R}_{\geq 0}$. Indeed, for each simplex $\mathfrak{S}_\nu \in \mathfrak{T}$, $\nu \in \mathbb{N}_{[1:z]}$, the unique affine interpolation that interpolates $Z_\mathcal{V}$ at the vertices $v_0, \dots, v_n$ of $\mathfrak{S}_\nu$ is given by

$$Z_\nu(x) = \sum_{j=0}^n \lambda_j(x) Z_\mathcal{V}(v_j) = a_\nu^\top x + b_\nu$$

where for each $x \in \mathfrak{S}_\nu$

$$x = \sum_{j=0}^n \lambda_j(x) v_j,\ \sum_{j=0}^n \lambda_j(x) = 1,\ \lambda_j(x) \geq 0 \quad \forall j \in \mathbb{N}_{[0:n]}.$$

The CPA function is now given by $Z(x) := Z_\nu(x) = a_\nu^\top x + b_\nu$ for all $x \in \mathfrak{S}_\nu$. Note that $Z(x) = Z_\mathcal{V}(x)$ for all $x \in \mathcal{V}_\mathfrak{T}$. Furthermore, note that $\nabla Z_\nu = a_\nu$.

In the sequel, we will compute a CPA feedback stabilizer $u = k(x)$, $k(x) = k_\nu(x)$ for all $x \in \mathfrak{S}_\nu$, where $k_\nu(x) : \mathfrak{S}_\nu \to \mathbb{R}^m$ is affine for each $\nu \in \mathbb{N}_{[1:z]}$ and $k : \mathfrak{M} \to \mathbb{R}^m$ is Lipschitz. The closed-loop system becomes a switched system given by

$$\dot{x} = f_\nu(x) := f(x) + g(x)k_\nu(x), \quad x \in \mathfrak{S}_\nu. \tag{2}$$

Note that the resulting closed-loop system is a switched ordinary differential equation and is a special case of a differential inclusion. Indeed, we can associate it to the differential inclusion

$$\dot{x} \in F(x) := \mathrm{co}\{f_\nu(x) \,|\, \nu \in I_\mathfrak{T}(x)\}. \tag{3}$$

For a further discussion on this special case of differential inclusions, we refer the reader to [13, Example 2.3]. Furthermore, note that although system (2) is a switched system, the right-hand side of (2) is still continuous, in fact Lipschitz, on $\mathfrak{M}$, due to the Lipschitz continuity of k, f, and g. Note also that, therefore, $F(x)$ always reduces to a singleton. On a common face of two simplices $\mathfrak{S}_a$ and $\mathfrak{S}_b$, we have $\dot{x} \in \mathrm{co}\{f_a(x), f_b(x)\} = \{f_a(x)\} = \{f_b(x)\}$ for all $x \in \mathfrak{S}_a \cap \mathfrak{S}_b$. We obviously have $\dot{x} \in \{f_\nu(x)\}$ for all $x \in \mathfrak{S}_\nu^\circ$.

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B. Concepts From Nonsmooth Lyapunov Theory

In the sequel, we will consider (control) LFs, which are continuously differentiable, but for the computation and verification of the feedback stabilizer, we also consider CPA functions, which are not smooth, in general, but merely continuous or Lipschitz. We will make use of an alternative calculus for nonsmooth analysis, known as Clarke's generalized gradient. For a Lipschitz continuous function V, Clarke's generalized gradient is given by [16, Th. 2.5.1]

$$\partial_{\mathrm{Cl}} V(x) = \mathrm{co}\left\{\lim_{i\to\infty} \nabla V(x_i) \,|\, x_i \to x,\ \nabla V(x_i) \text{ exists}\right\}.$$

We define a CLF, respectively, strong LF in the sense of generalized gradients in a similar fashion as in [5] below.

Definition 2.3: A positive definite and Lipschitz continuous function $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$, with $\mathcal{S} \subset \mathbb{R}^n$ being some proper and compact subset, is called a CLF in the sense of generalized gradients for (1), if the condition

$$\forall x \in \mathcal{S},\ \exists u \in \mathbb{U} : \sup_{\zeta \in \partial_{\mathrm{Cl}} V(x)} \langle \zeta, f(x) + g(x)u \rangle \leq -\alpha(x) \quad (4)$$

holds true, where $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lipschitz continuous and positive definite function.

Definition 2.4: A positive definite and Lipschitz continuous function $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$, with $\mathcal{S} \subset \mathbb{R}^n$ being some proper and compact subset, is called a strong LF in the sense of generalized gradients for (3), if the condition

$$\forall x \in \mathcal{S} : \sup_{\zeta \in \partial_{\mathrm{Cl}} V(x)} \sup_{p \in F(x)} \langle \zeta, p \rangle \leq -\alpha(x) \quad (5)$$

holds true, where $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lipschitz continuous and positive definite function.

Considering the structure of a CPA LF and the closed-loop system (2), we state an alternative sufficient condition for (5), as derived in [13] for general differential inclusions. The condition we state differs from the one in [13] in the sense that the domains of the vector fields of the switched system (2) coincide with the simplices in the triangulation for the CLF, i.e., the CLF and feedback stabilizer utilize a common triangulation. The proof for Proposition 2.1 relies mainly on the proof in [13, Prop. 3.6], but we include the details for completeness. For a CPA function $V : \mathcal{S} \to \mathbb{R}$ on a triangulation $\mathfrak{T}$, Clarke's generalized gradient is given by [13]

$$\partial_{\mathrm{Cl}} V(x) = \mathrm{co}\left\{\nabla V_\nu \,|\, \nu \in I_{\mathfrak{T}}(x)\right\}. \quad (6)$$

Proposition 2.1: Consider system (2) and let $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ be a positive definite CPA function on the triangulation $\mathfrak{T}$ of a subset $\mathcal{S} \subset \mathbb{R}^n$. Then, the inequality

$$\langle \nabla V_\nu, f_\nu(x) \rangle \leq -\alpha(x) \quad \forall x \in \mathcal{S},\ \nu \in I_{\mathfrak{T}}(x) \quad (7)$$

implies condition (5).

Proof: Due to the continuity of the right-hand side of (2), condition (7) implies that for each $x \in \mathcal{S}$

$$\langle \nabla V_\nu, f_\mu(x) \rangle \leq -\alpha(x) \quad \forall (\nu, \mu) \in I_{\mathfrak{T}}(x) \times I_{\mathfrak{T}}(x).$$

From (6), we have that every $\zeta \in \partial_{\mathrm{Cl}} V(x)$ can be written as a convex combination

$$\zeta = \sum_{\nu \in I_{\mathfrak{T}}(x)} \alpha_\nu \nabla V_\nu$$

where $\alpha_\nu \in \mathbb{R}_{\geq 0}$ and $\nu \in I_{\mathfrak{T}}(x)$, such that $\sum_{\nu \in I_{\mathfrak{T}}(x)} \alpha_\nu = 1$. From (3), we obtain directly that for every $p \in F(x)$

$$p = \sum_{\mu \in I_{\mathfrak{T}}(x)} \beta_\mu f_\mu(x)$$

where $\beta_\mu \in \mathbb{R}_{\geq 0}$ and $\mu \in I_{\mathfrak{T}}(x)$, such that $\sum_{\mu \in I_{\mathfrak{T}}(x)} \beta_\mu = 1$. But then since

$$\langle \nabla V_\nu, f_\mu(x) \rangle \leq -\alpha(x) \quad \forall x \in \mathcal{S},\ \nu, \mu \in I_{\mathfrak{T}}(x)$$

we have for all $x \in \mathcal{S}$

$$\begin{aligned}\langle \zeta, p \rangle &= \left\langle \sum_{\nu \in I_{\mathfrak{T}}(x)} \alpha_\nu \nabla V_\nu, \sum_{\mu \in I_{\mathfrak{T}}(x)} \beta_\mu f_\mu(x) \right\rangle \\ &= \sum_{\nu \in I_{\mathfrak{T}}(x)} \alpha_\nu \sum_{\mu \in I_{\mathfrak{T}}(x)} \beta_\mu \langle \nabla V_\nu, f_\mu(x) \rangle \\ &\leq -\alpha(x) \quad \forall \zeta \in \partial_{\mathrm{Cl}} V(x),\ p \in F(x)\end{aligned}$$

which directly proves the assertion. ■

III. Main Results

A. "Universal" CPA Stabilizers

In this section, we consider the construction of a CPA feedback law, via a "universal" controller introduced by Sontag in [2] for a given CLF, on a region $\mathcal{S} \setminus \mathbb{B}_\varepsilon$, where $\varepsilon \in \mathbb{R}_{>0}$ is an arbitrarily small constant. We verify the feasibility of a finite number of suitable conditions on the vertices contained in the vertex set $\mathcal{V}_{\mathfrak{T}}$ of a triangulation $\mathfrak{T}$. The conditions will ensure that V is a strong LF for the closed-loop system (2) on $\mathcal{S} \setminus \mathbb{B}_\varepsilon$ with the CPA feedback, and thus also a CLF on $\mathcal{S} \setminus \mathbb{B}_\varepsilon$ for (1). Feasibility of the conditions will be guaranteed for a suitable triangulation $\mathfrak{T}$, which we consider to be fixed *a priori*. For simplicity, we consider the open-loop system (1) for the scalar input case, i.e., $m = 1$, for the remainder of this article. The following results can be extended to the multi-input case *mutatis mutandis*, considering the "universal" controller for multi-input systems [2].

1) Construction: The "universal" controller is a map $\bar{k} : \mathbb{R}^n \to \mathbb{R}$ that is smooth on $\mathbb{R}^n \setminus \{0\}$ and exists whenever a smooth CLF $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ exists for the system (1) with $u \in \mathbb{U} = \mathbb{R}$ (see also [2, Th. 1]): $u = \bar{k}(x) := 0$ if $\langle \nabla W(x), g(x) \rangle = 0$ and

$$u = \bar{k}(x) := -\frac{\langle \nabla W(x), f(x) \rangle + \sqrt{\langle \nabla W(x), f(x) \rangle^2 + \langle \nabla W(x), g(x) \rangle^4}}{\langle \nabla W(x), g(x) \rangle} \quad (8)$$

otherwise. In [2], the vector fields f and g, as well as the CLF W, are assumed to be smooth. Therefore, the condition that W

is a strong LF for the closed-loop system $\dot{x} = f(x) + g(x)\bar{k}(x)$ is simplified into

$$\langle \nabla W(x), f(x)\rangle + \langle \nabla W(x), g(x)\bar{k}(x)\rangle \leq -\alpha(x). \quad (9)$$

Condition (9) is easily verified to hold with input $u = \bar{k}(x)$ for

$$\alpha(x) = \sqrt{\langle \nabla W(x), f(x)\rangle^2 + \langle \nabla W(x), g(x)\rangle^4}.$$

Based on the preceding "universal" controller by Sontag, we propose to construct a CPA variant from this controller (8). For this, consider the vertex set $\mathcal{V}_{\mathfrak{T}}$ for some triangulation $\mathfrak{T}$ of $\mathcal{S}$ and consequently the mapping $k_{\mathcal{V}} : \mathcal{V}_{\mathfrak{T}} \to \mathbb{R}$, defined by

$$k_{\mathcal{V}}(\xi) := \bar{k}(\xi) \quad \forall \xi \in \mathcal{V}_{\mathfrak{T}}.$$

Then, we construct the controller $k : \mathcal{S} \to \mathbb{R}$ via the unique piecewise affine interpolation of $k_{\mathcal{V}}$, such that

$$k(x) := k_\nu(x) \quad \forall x \in \mathfrak{S}_\nu := \mathrm{co}\{v_0, v_1, \ldots, v_n\}$$

$$k_\nu(x) := \sum_{i=0}^{n} \lambda_i(x) k_{\mathcal{V}}(v_i) = \langle \nabla k_\nu, x\rangle + \kappa_\nu \quad (10)$$

where $\lambda_i(x) \geq 0$, $i \in \mathbb{N}_{[0:n]}$, are the barycentric coordinates with $\sum_{i=0}^{n} \lambda_i(x) = 1$, such that $x = \sum_{i=0}^{n} \lambda_i(x) v_i$ for all $x \in \mathfrak{S}_\nu$. Furthermore, $\nabla k_\nu \in \mathbb{R}^n$ is the gradient of k on the simplex $\mathfrak{S}_\nu$ and $\kappa_\nu \in \mathbb{R}$ is some appropriate constant.

Remark 3.1: Online implementation of the proposed controller k in a practical setup requires solving a point-location problem, which is to find $\mathfrak{S}_\nu \in \mathfrak{T}$ such that the state $x \in \mathfrak{S}_\nu$. This problem can be solved online with dedicated computationally efficient algorithms (see, e.g., [17] and [18]).

2) Verification: Although the original controller (8) is a stabilizing feedback for system (1), given a CLF W, its CPA counterpart (10) is not guaranteed to render the closed-loop system stable. Therefore, we propose to verify condition (5) via a CPA interpolation V of W on the triangulation $\mathfrak{T}$. Then, condition (5) is simplified into

$$\langle \nabla V_\nu, f_\nu(x)\rangle \leq -\alpha(x) \quad \forall x \in \mathcal{S},\ \nu \in I_{\mathfrak{T}}(x)$$

by Proposition 2.1, which turns out to be easily verifiable as stated in the following proposition.

Proposition 3.1: Let $f_\mu : \mathfrak{S}_\nu \to \mathbb{R}^n$ be a twice continuously differentiable function on $\mathfrak{S}_\nu = \mathrm{co}\{v_0, v_1, \ldots, v_n\}$ and let $V : \mathfrak{S}_\nu \to \mathbb{R}$ be an affine function such that $V(x) = \nabla V_\nu^\top x + c_\nu$, with ∇V_ν the gradient of V on $\mathfrak{S}_\nu$ and $c_\nu \in \mathbb{R}$. If the inequality

$$\langle \nabla V_\nu, f_\mu(v_i)\rangle + nB_\nu h_\nu^2 \|\nabla V_\nu\|_1 \leq -\rho\|v_i\|_2 \quad (11)$$

holds for all $i \in \mathbb{N}_{[0:n]}$, with $\rho \in \mathbb{R}_{>0}$, $h_\nu := \mathrm{diam}(\mathfrak{S}_\nu)$ and the upper bound

$$B_\nu \geq \max_{r,s,i \in \mathbb{N}_{[1:n]}} \max_{\xi \in \mathfrak{S}_\nu} \left| \frac{\partial^2 f_{\mu,i}}{\partial \xi_r \partial \xi_s}(\xi) \right|$$

holds, with $f_{\mu,i}$ the ith entry of f_μ, then the inequality

$$\langle \nabla V_\nu, f_\mu(\xi)\rangle \leq -\rho\|\xi\|_2$$

holds for all $\xi \in \mathfrak{S}_\nu$.

A proof for the aforementioned proposition can be found in [13, Cor. 4.3].

The discussed verification procedure then results into a feasibility problem consisting of a finite number of inequalities, as stated in the following problem.

Problem 3.1: Given the CPA feedback (10), verify for every vertex v_i of every simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}$, $v_i \neq 0$

$$\langle \nabla V_\nu, f_\nu(v_i)\rangle + nB_\nu h_\nu^2 \|\nabla V_\nu\|_1 \leq -\rho\|v_i\|_2 \quad (12)$$

where $f_\nu(x) = f(x) + g(x)k_\nu(x)$, $\rho \in \mathbb{R}_{>0}$, $h_\nu = \mathrm{diam}(\mathfrak{S}_\nu)$ and

$$B_\nu \geq \max_{r,s,j \in \mathbb{N}_{[0:n]}} \max_{\xi \in \mathfrak{S}_\nu} \left| \frac{\partial^2 f_{\nu,j}}{\partial \xi_r \partial \xi_s}(\xi) \right|.$$

Note that we do not demand condition (12) to hold close to the origin in the feasibility problem. That is, we restrict to a subset $\mathfrak{T}^\star := \{\mathfrak{S} \in \mathfrak{T} \,|\, 0 \notin \mathfrak{S}\} \subset \mathfrak{T}$. Hence, the triangulation should be chosen such that $\mathfrak{T}^\star$ is not empty.

Proposition 3.2: Let $W : \mathcal{S} \to \mathbb{R}_{\geq 0}$ be a twice continuously differentiable CLF for (1) and let $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ be the CPA interpolation of W on the triangulation $\mathfrak{T}$. If the decrease conditions (12) are feasible, then the following hold:

1) V is a CLF in the sense of generalized gradients for (1);
2) V is a strong LF in the sense of generalized gradients for (3)

on $\mathcal{S}^\star := \cup_{\mathfrak{S}_\nu \in \mathfrak{T}^\star} \mathfrak{S}_\nu$.

Proof: Since W is a CLF for (1), it is positive definite, by definition. But then V must also be positive definite since $W(0) = V(0) = 0$ and $V(x) > 0$ for all $x \in \mathcal{S} \setminus \{0\}$. Indeed, consider an arbitrary simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}$, then

$$V(x) = \sum_{i=0}^{n} \lambda_i(x) W(v_i) \quad \forall x \in \mathfrak{S}_\nu$$

with $\sum_{i=0}^{n} \lambda_i(x) = 1$ and $\lambda_i(x) \geq 0$, $i \in \mathbb{N}_{[0:n]}$, which reveals the assertion directly.

Next, we show that the decrease condition holds for V. Consider an arbitrary simplex $\mathfrak{S}_\nu := \{v_0, \ldots, v_n\} \in \mathfrak{T}^\star = \{\mathfrak{S} \in \mathfrak{T} \,|\, 0 \notin \mathfrak{S}\} \subset \mathfrak{T}$. It then follows by condition (12) that $\langle \nabla V_\nu, f_\nu(x)\rangle \leq -\rho\|x\|_2 =: -\alpha(x)$, for all $x \in \mathfrak{S}_\nu$ via Proposition 3.1, which implies $\langle \nabla V_\nu, f_\nu(x)\rangle \leq -\alpha(x)$ for all $x \in \mathcal{S}^\star$ and $\nu \in I_{\mathfrak{T}}(x)$. Indeed, pick an arbitrary $\xi \in \mathcal{S}^\star$, then we need $\langle \nabla V_\nu, f_\nu(\xi)\rangle \leq -\alpha(\xi)$ for all $\nu \in I_{\mathfrak{T}}(\xi)$. Take an arbitrary $\mu \in I_{\mathfrak{T}}(\xi)$, then we have $\langle \nabla V_\mu, f_\mu(x)\rangle \leq -\alpha(x)$ for all $x \in \mathfrak{S}_\mu$. But then also $\langle \nabla V_\mu, f_\mu(\xi)\rangle \leq -\alpha(\xi)$ since $\xi \in \mathfrak{S}_\mu$. Since ξ and μ were arbitrary, we conclude that $\langle \nabla V_\nu, f_\nu(x)\rangle \leq -\alpha(x)$ for all $x \in \mathcal{S}^\star$ and $\nu \in I_{\mathfrak{T}}(x)$.

Then, via Proposition 2.1, condition (5) is implied for all $x \in \mathcal{S}^\star$. Consequently, condition (4) is satisfied on $\mathcal{S}^\star$ for $u = k(x)$. ■

Remark 3.2: If V is a CLF and strong LF in the sense of generalized gradients for (1) and (3) on $\mathcal{S}^\star$, it also is on a subset $\mathcal{S} \setminus \mathbb{B}_\varepsilon \subset \mathcal{S}^\star$ for all $\varepsilon > 0$ so that $\mathbb{B}_\varepsilon \supset \cup_{\mathfrak{S}_\nu \in \mathfrak{T} \setminus \mathfrak{T}^\star} \mathfrak{S}_\nu$. The size of $\varepsilon > 0$ thus depends on $\mathrm{diam}(\mathfrak{S}_\nu)$, $\mathfrak{S}_\nu \in \mathfrak{T} \setminus \mathfrak{T}^\star$.

In the next corollary, we will show that if there exists a twice continuously differentiable CLF for (1), then Problem 3.1 is feasible for a suitable triangulation, i.e., there exists a

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triangulation such that the CPA function $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is a CLF in the sense of generalized gradients for (1) and a strong LF in the sense of generalized gradients for (2) on $\mathcal{S} \setminus \mathbb{B}_\varepsilon$, with the CPA feedback $k : \mathcal{S} \to \mathbb{R}$.

Corollary 3.1: Let $W : \mathcal{S} \to \mathbb{R}_{\geq 0}$ be a twice continuously differentiable CLF for system (1). Then, there exists a triangulation $\mathfrak{T}$ such that the CPA interpolation $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ of W on $\mathfrak{T}$ satisfies the conditions (12) in Problem 3.1, together with the CPA feedback $k : \mathcal{S} \to \mathbb{R}$.

Proof: The proof largely resembles the proof for the existence of a CPA LF for differential inclusions, given in [13, Th. 4.6]. We provide a sketch of the proof and refer the reader to the proof in [13, Th. 4.6] for details.

Since $W : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is a CLF for (1), there exists a $u \in \mathbb{U}$ for each $x \in \mathcal{S}$ such that

$$\langle \nabla W(x), f(x) \rangle + \langle \nabla W(x), g(x) \rangle u \leq -\alpha(x)$$

for some positive definite $\alpha : \mathcal{S} \to \mathbb{R}_{\geq 0}$. For $u = \bar{k}(x)$ as in (8), we know that one can take

$$\alpha(x) := \sqrt{\langle \nabla W(x), f(x) \rangle^2 + \langle \nabla W(x), g(x) \rangle^4}.$$

Given that $\mathcal{S}$ is a compact set and $\alpha(x)$ is a continuous positive definite function, one can always take a sufficiently small $\rho \in \mathbb{R}_{>0}$ such that $\alpha(x) \geq \rho \|x\|_2$, for all $x \in \mathcal{S}$, $\|x\|_2 \geq \varepsilon > 0$.

Take an arbitrary but fixed simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}^\star := \{\mathfrak{S} \in \mathfrak{T} \,|\, 0 \notin \mathfrak{S}\} \subset \mathfrak{T}$. For all $i \in \mathbb{N}_{[0:n]}$, we have

$$\begin{aligned}
\langle \nabla V_\nu, f_\nu(v_i) \rangle &= \langle \nabla W(v_i) + \nabla V_\nu - \nabla W(v_i), f_\nu(v_i) \rangle \\
&= \langle \nabla W(v_i), f_\nu(v_i) \rangle + \langle \nabla V_\nu - \nabla W(v_i), f_\nu(v_i) \rangle \\
&\leq -\alpha(v_i) + \langle \nabla V_\nu - \nabla W(v_i), f_\nu(v_i) \rangle \\
&\leq -\rho \|v_i\|_2 + \|\nabla V_\nu - \nabla W(v_i)\|_2 \|f_\nu(v_i)\|_2 \\
&\leq -\rho \|v_i\|_2 + nAh \left(\frac{1}{2} X^\star n^{\frac{1}{2}} h + 1 \right) L \|v_i\|_2
\end{aligned}$$

where the last inequality follows from the proof in [13, Th. 4.6], with

$$A := \max_{\xi \in \mathcal{S}} \max_{r,s \in \mathbb{N}_{[1:n]}} \left| \frac{\partial^2 W}{\partial \xi_r \partial \xi_s}(\xi) \right|, \quad h := \max_{\mathfrak{S}_\nu \in \mathfrak{T}^\star} \mathrm{diam}(\mathfrak{S}_\nu)$$

and the common Lipschitz constant L of f_ν, $\mathfrak{S}_\nu \in \mathfrak{T}^\star$. Furthermore, the matrix $X^\star := \max_{\mathfrak{S}_\nu \in \mathfrak{T}^\star} \|X_\nu^{-1}\|$, where for every $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}^\star$

$$X_\nu := (v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0)^\top.$$

Thus, the constraints

$$\langle \nabla V_\nu, f_\nu(v_i) \rangle + nB_\nu h_\nu^2 \|\nabla V_\nu\|_1 \leq -\frac{1}{2} \rho \|v_i\|_2$$

are satisfied whenever h satisfies

$$nAh \left(\frac{1}{2} X^\star n^{\frac{1}{2}} h + 1 \right) L \|v_i\|_2 + nBh^2 C \leq -\frac{1}{2} \rho \|v_i\|_2$$

where $B := \max_{\mathfrak{S}_\nu \in \mathfrak{T}^\star} B_\nu$ and $C := \max_{\mathfrak{S}_\nu \in \mathfrak{T}^\star} \|\nabla V_\nu\|_1$. Since we omit a small neighborhood around the origin, we have $\|v_i\|_2 \geq \varepsilon$ for some $\varepsilon > 0$ and the constraints (12) are fulfilled when h is small enough, i.e., the condition

$$nAh \left(\frac{1}{2} X^\star n^{\frac{1}{2}} h + 1 \right) L\varepsilon + nBh^2 C \leq \frac{1}{2} \rho \varepsilon$$

is satisfied. Such $h > 0$ exists, since the triangulation can be chosen such that $hX^* \leq R$ for some fixed $R > 0$, cf., [19, Th. 2.7] and [13, Remark 4.7].

Since the latter proof holds for every arbitrary simplex $\mathfrak{S}_\nu \in \mathfrak{T}^\star$, we have shown that there exists a simplex diameter h, or equivalently a triangulation $\mathfrak{T}$, such that the conditions (12) in Problem 3.1 are satisfied. ∎

While Proposition 3.1 provides a way to verify the decrease of the CPA function V with the feedback k, it can be rather conservative, due to the addition of an interpolation error term. Alternatively, one can also turn the verification problem into a finite number of constrained optimization problems, as discussed in [20]. The method in [20] exploits constrained optimization of $\langle \nabla V_\nu, f_\nu(x) \rangle$ on each simplex $\mathfrak{S}_\nu \in \mathfrak{T}$, with the sensible assumption that the cost function is either concave or convex on the simplex $\mathfrak{S}_\nu$. To keep this section consistent with the following section on linear programming based feedback computation, we will use Proposition 3.1 for verification, but note that the method from Steentjes *et al.* [20] can be used for *a posteriori* verification on simplices intersecting with the excluded region $\mathbb{B}_\varepsilon$, such that V can be guaranteed to be a CLF in the sense of generalized gradients on $\mathcal{S}$, rather than on $\mathcal{S} \setminus \mathbb{B}_\varepsilon$. Alternatively, one can make use of a so-called fanlike triangulation close to the origin, introduced in [21] for the computation of CPA LFs in two dimensions. This technique was exploited further in [22] and the implementation of such a triangulation is discussed in [23].

B. Linear Programming Based CPA Stabilizers

In this section, we consider the computation of a CPA feedback $k : \mathcal{S} \to \mathbb{U}$ for some compact subset $\mathbb{U} \subset \mathbb{R}$, via an LP. In contrast to the previous section, the decrease conditions for the CLF are specified via a finite number of inequalities in the LP, which removes the need to verify these conditions *a posteriori*.

1) Algorithm: Prior to the construction of the feedback stabilizer, we construct a candidate CPA CLF in the sense of generalized gradients $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ on a triangulation $\mathfrak{T}$, with $\mathcal{S} \subset \mathbb{R}^n$ some subset, via the CPA interpolation of a given continuously differentiable candidate CLF $W : \mathcal{S} \to \mathbb{R}_{\geq 0}$. Then, we compute a CPA feedback $k : \mathcal{S} \to \mathbb{R}$ such that condition (5) holds for the closed-loop system defined by (2). A sufficient condition for inequality (5) to hold is

$$\langle \nabla V_\nu, f(x) + g(x) k_\nu(x) \rangle \leq -\alpha(x) \quad \forall x \in \mathcal{S},\ \nu \in I_{\mathfrak{T}}(x)$$

via Proposition 2.1. Then, by Proposition 3.1, it suffices to satisfy

$$\langle \nabla V_\nu, f(v_i) + g(v_i) k_\nu(v_i) \rangle + nB_\nu h_\nu^2 \|\nabla V_\nu\|_1 \leq -\alpha(v_i)$$

for every simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}$, with $v_i \neq 0$ for all $i \in \mathbb{N}_{[0:n]}$, and $\alpha(x) := \rho \|x\|_2$, $\rho \in \mathbb{R}_{>0}$. We propose to compute a map $k_{\mathcal{V}} : \mathcal{V}_{\mathfrak{T}} \to \mathbb{R}$, with $\mathcal{V}_{\mathfrak{T}}$ being the vertex set for the triangulation $\mathfrak{T}$. That is, we compute values $k_{\mathcal{V}}(v_i) = k_\nu(v_i)$ for every vertex v_i of every simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}$, such that the linear constraints

$$\langle \nabla V_\nu, f(v_i) + g(v_i)k_{\mathcal{V}}(v_i)\rangle + nB_\nu h_\nu^2 \|\nabla V_\nu\|_1 \le -\alpha(v_i)$$

are satisfied.

Note that the upper bounds B_ν on the second derivatives of the closed-loop vector field $f_\nu(x) = f(x) + g(x)k_\nu(x)$ are defined as in Proposition 3.1, i.e.,

$$B_\nu \ge \max_{r,s,i\in\mathbb{N}_{[1:n]}} \max_{\xi\in\mathfrak{S}_\nu} \left| \frac{\partial^2 f_i(\xi)}{\partial\xi_r\partial\xi_s} + \frac{\partial^2 g_i(\xi)k_\nu(\xi)}{\partial\xi_r\partial\xi_s} \right| \tag{13}$$

where f_i and g_i denote the scalar components of f and g, respectively. The right-hand side of (13) will always exist whenever f and g are twice continuously differentiable on $\mathfrak{S}_\nu$, since k_ν is affine on $\mathfrak{S}_\nu$. However, k_ν is not known in advance and we thus need an upperbound on the right-hand side of (13) that we can compute, as provided in the following lemma.

Lemma 3.1: Let the closed-loop vector field $f_\nu : \mathfrak{S}_\nu \to \mathbb{R}^n$ be twice continuously differentiable. Furthermore, let the bounds on k_ν and the scalar components of the gradient of k_ν be given by

$$-K_\nu \le k_\nu(v_i) \le K_\nu \quad \forall i \in \mathbb{N}_{[0:n]}$$

and

$$-K_{\nabla,\nu} \le \nabla k_{\nu,i} \le K_{\nabla,\nu} \quad \forall i \in \mathbb{N}_{[1:n]}$$

respectively, for $K_\nu,\ K_{\nabla,\nu} \in \mathbb{R}_{\ge 0}$. Then

$$\left|\frac{\partial^2 f_{\nu,i}}{\partial\xi_r\partial\xi_s}(\xi)\right| \le \left|\frac{\partial^2 f_i}{\partial\xi_r\partial\xi_s}(\xi)\right| + \left|\frac{\partial^2 g_i}{\partial\xi_r\partial\xi_s}(\xi)\right| K_\nu + \left(\left|\frac{\partial g_i}{\partial\xi_r}(\xi)\right| + \left|\frac{\partial g_i}{\partial\xi_s}(\xi)\right|\right) K_{\nabla,\nu}$$

for all $\xi \in \mathfrak{S}_\nu$, with $f_{\nu,i}$, f_i, and g_i, $i \in \mathbb{N}_{[1:n]}$ the scalar components of f_ν, f, and g, respectively.

Proof: First, note that via the triangle inequality, we have

$$\left|\frac{\partial^2 f_{\nu,i}}{\partial\xi_r\partial\xi_s}(\xi)\right| \le \left|\frac{\partial^2 f_i}{\partial\xi_r\partial\xi_s}(\xi)\right| + \left|\frac{\partial^2 g_i(\xi)k_\nu(\xi)}{\partial\xi_r\partial\xi_s}\right|.$$

Then, since $\frac{\partial^2 k_\nu}{\partial\xi_r\partial\xi_s}(\xi) = 0$ for all $\xi \in \mathfrak{S}_\nu$, it follows that

$$\begin{aligned}\left|\frac{\partial^2 g_i(\xi)k_\nu(\xi)}{\partial\xi_r\partial\xi_s}\right| &= \left|\frac{\partial^2 g_i(\xi)}{\partial\xi_r\partial\xi_s}k_\nu(\xi) + \frac{\partial g_i(\xi)}{\partial\xi_r}\nabla k_{\nu,s} + \frac{\partial g_i(\xi)}{\partial\xi_s}\nabla k_{\nu,r}\right| \\ &\le \left|\frac{\partial^2 g_i(\xi)}{\partial\xi_r\partial\xi_s}\right| K_\nu + \left(\left|\frac{\partial g_i}{\partial\xi_r}(\xi)\right| + \left|\frac{\partial g_i}{\partial\xi_s}(\xi)\right|\right) K_{\nabla,\nu}\end{aligned}$$

where we used the product rule and triangle inequality for the equality and inequality, respectively, and the assertion follows. ■

Remark 3.3: For open-loop systems for which the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is constant on each simplex $\mathfrak{S}_\nu \in \mathfrak{T}$, i.e., $g(\xi) = g_\nu$ for all $\xi \in \mathfrak{S}_\nu$ for some $g_\nu \in \mathbb{R}^n$, the upper bound in Proposition 3.1 can be set as

$$B_\nu \ge \max_{r,s,i\in\mathbb{N}_{[1:n]}} \max_{\xi\in\mathfrak{S}_\nu} \left|\frac{\partial^2 f_i}{\partial\xi_r\partial\xi_s}(\xi)\right|$$

which follows directly by Lemma 3.1.

We now state our algorithm for the construction of a CPA feedback $k : \mathcal{S} \to \mathbb{R}$, which stabilizes the system (1) such that the CPA interpolation $V : \mathcal{S} \to \mathbb{R}_{\ge 0}$ of the CLF $W : \mathcal{S} \to \mathbb{R}_{\ge 0}$ is a CLF and a strong LF in the sense of generalized gradients for systems (1) and (2), respectively, on $\mathcal{S} \setminus \mathbb{B}_\varepsilon$.

Algorithm 3.1: Let $W : \mathcal{S} \to \mathbb{R}_{\ge 0}$ be a CLF for the open-loop system (1) and let $V : \mathcal{S} \to \mathbb{R}_{\ge 0}$ be the CPA interpolation of W on the triangulation $\mathfrak{T}$ of $\mathcal{S} \subset \mathbb{R}^n$. Let the variables for the LP be $k_{\mathcal{V}}(v_i)$ for every vertex v_i of each simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}$. The linear constraints for the LP are then constructed as follows.

1) For each vertex v_i of each simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}$ set

$$\check{k}_{\nu,i} \le k_{\mathcal{V}}(v_i) \le \hat{k}_{\nu,i}$$

for some $\check{k}_{\nu,i},\ \hat{k}_{\nu,i} \in \mathbb{U}$ such that $\check{k}_{\nu,i} \le \hat{k}_{\nu,i}$.

2) For every simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T},\ v_i \neq 0$, set the constraints

$$\langle \nabla V_\nu, f(v_i) + g(v_i)k_{\mathcal{V}}(v_i)\rangle + nB_\nu h_\nu^2 \|\nabla V_\nu\|_1 \le -\alpha(v_i)$$

for each $i \in \mathbb{N}_{[0:n]}$, where B_ν and h_ν are chosen as in Proposition 3.1.

3) For every simplex $\mathfrak{S}_\nu := \mathrm{co}\{v_0, \ldots, v_n\} \in \mathfrak{T}$, we introduce the additional variables c_ν and require the following constraints on each component $\nabla k_{\nu,i}$ of ∇k_ν:

$$-c_\nu \le \nabla k_{\nu,i} \le c_\nu \quad \forall i \in \mathbb{N}_{[1:n]}.$$

If the LP problem with the constraints 1)–3) has a feasible solution, then a value $k_{\mathcal{V}}(\xi)$ for all $\xi \in \mathcal{V}_{\mathfrak{T}}$ has been computed. We consequently compute the unique CPA map $k : \mathcal{S} \to \mathbb{U}$ from $k_{\mathcal{V}} : \mathcal{V}_{\mathfrak{T}} \to \mathbb{U}$ by interpolation.

Remark 3.4: Algorithm 3.1 provides an alternative to the procedure for the construction of a CPA feedback stabilizer in Section III-A and to the original "universal" controller from Sontag [2] on a compact set, and thus implicitly an alternative proof for Artstein's result [1, Th. 5.1], except for an arbitrarily small region around the origin. Indeed, since constraint 2) in Algorithm 3.1 is feasible for a suitable triangulation by Corollary 3.1, the LP in Algorithm 3.1 is always feasible whenever the upper and lower bounds in constraint 1) are omitted or set such that they are not limiting.

Since Algorithm 3.1 features a feasibility problem, the objective function for the LP is free. This is an advantage to the "universal" (CPA) stabilizer in Section III-A from a control perspective, with possible objectives as minimizing input variation or maximizing performance, for example. Related to the LP, one can set the objective such that the average or maximum gradient of the feedback k is minimized, corresponding to the minimization of

$$\sum_{\mathfrak{S}_\nu\in\mathfrak{T}} \|\nabla k_\nu\|_\infty \quad \text{or} \quad \max_{\mathfrak{S}_\nu\in\mathfrak{T}} \|\nabla k_\nu\|_\infty$$

respectively, or such that average decrease of the CLF at the vertices w.r.t. the trajectories is maximized, i.e., the objective function

$$\sum_{\mathfrak{S}_\nu \in \mathfrak{T}} \sum_{i=0}^{n} \langle \nabla V_\nu, g(v_i) k_{\mathcal{V}}(v_i) \rangle$$

is minimized. A tradeoff between minimizing input variation and maximizing performance can be achieved via the linear combination

$$\gamma \sum_{\mathfrak{S}_\nu \in \mathfrak{T}} \|\nabla k_\nu\|_\infty + (1-\gamma) \sum_{\mathfrak{S}_\nu \in \mathfrak{T}} \sum_{i=0}^{n} \langle \nabla V_\nu, g(v_i) k_{\mathcal{V}}(v_i) \rangle$$

where $\gamma \in [0, 1]$.

Remark 3.5: The preceding cost functions with the conditions in Algorithm 3.1 result in an LP. Nonlinear cost functions, such as quadratic ones, can be used as well in combination with Algorithm 3.1; this will not influence the feasibility of the problem.

2) Complexity Analysis: Recall the cardinality z of $\mathfrak{T}$, i.e., the number of simplices in the triangulation $\mathfrak{T}$ of $\mathcal{S} \subset \mathbb{R}^n$. In the absence of a cost function (no additional decision variables), the number of optimization variables in the LP from Algorithm 3.1 is at most equal to the cardinality of $\mathcal{V}_{\mathfrak{T}}$, which is upper bounded by $z(n+1)$. In Algorithm 3.1, condition 1) yields at most $2z(n+1)$, condition 2) yields at most $z(n+1)$, and condition 3) yields at most $2zn$ linear inequalities. Thus, Algorithm 3.1 amounts to solving an LP with at most $z(n+1)$ variables and $z(5n+3)$ inequalities, i.e., an LP with $O(zn)$ variables and constraints. The straightforward generalization to the multivariate input case leads to an LP with $O(znm)$ optimization variables and constraints, with input cardinality $m \in \mathbb{N}$. We note, however, that the required number of simplices z in the triangulation $\mathfrak{T}$ may grow considerably for increased state dimension n, depending on the system dynamics.

IV. Numerical Examples

A. Stabilization of an Inverted Pendulum

Consider a pendulum with mass m, inertia about the axis of rotation J, and a distance l between its center of mass and the axis of rotation. The pendulum is subject to gravity with acceleration g and a torque τ applied to the axis of rotation. A model for such a pendulum is [24]

$$J\ddot{q} + mgl \sin(q) = \tau \tag{14}$$

with q and $\dot{q}$ the angular position and angular velocity of the pendulum, respectively. Considering the numerical values $J = 1$ and $mgl = 1$, we rewrite the model in state-space form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) + u \end{cases} \tag{15}$$

where $x_1 = q$, $x_2 = \dot{q}$, and $u = \tau$. The uncontrolled pendulum ($\tau = 0$) has stable equilibria $(2n\pi,\, 0)$, $n \in \mathbb{Z}$, corresponding to a downward position and unstable equilibria $((2n-1)\pi,\, 0)$, $n \in \mathbb{Z}$, corresponding to an upright position. The objective is to stabilize the pendulum in the upright position, i.e., to stabilize $x^* = (q^*, \dot{q}^*) = (\pi, 0)$, via a feedback $u \in \mathbb{U}$.

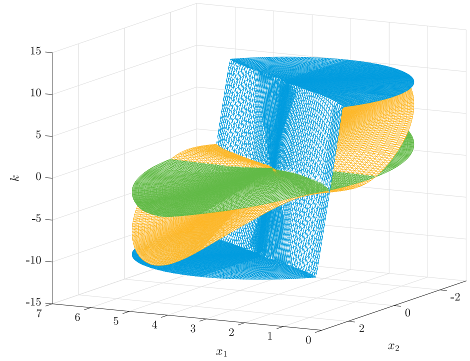

Fig. 1. CPA stabilizers $k_{\mathrm{LP},1}$ (blue), $k_{\mathrm{LP},2}$ (green), and $k_{\mathrm{LP},3}$ (yellow) for the inverted pendulum computed via linear programming. The "universal" CPA stabilizer k is equal to $k_{\mathrm{LP},3}$.

Note that although (15) is not a system of the form in (1), as its desired equilibrium is not located at the origin, we can still apply the theory in this article by applying the change of variable $\xi := x - x^*$. We consider the set $\mathcal{S} \subset \mathbb{R}^n$ as a polytope approximating a circular disc with radius π and a center at x^* and create a proper triangulation $\mathfrak{T}$ of $\mathcal{S}$. Let the candidate CLF $W(x) = x^\top P x$, where $P = \begin{pmatrix} 0.675 & 0.09 \\ 0.09 & 0.675 \end{pmatrix} > 0$, is such that

$$\exists K \in \mathbb{R}^{1\times 2}: \quad (A + BK)^\top P + P(A + BK) < 0$$

with $A = \frac{\partial f}{\partial x}(x^*)$ and $B = g(x^*) = \mathrm{col}(0, 1)$. Let $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ be the CPA interpolation of W on the triangulation $\mathfrak{T}$.

First, we construct a CPA feedback $k(x)$ via the procedure discussed in Section III-A. It is found that Problem 3.1 is feasible for $\rho := 10^{-5}$ and, therefore, V is a CLF in the sense of generalized gradients for (15) and a strong LF in the sense of generalized gradients for (2) on $\mathcal{S}^\star := \cup_{\mathfrak{S}_\nu \in \mathfrak{T}^\star} \mathfrak{S}_\nu$, where

$$\mathfrak{T}^\star := \{\mathfrak{S}_\nu \in \mathfrak{T} \,|\, \mathfrak{S}_\nu \cap \{x^*\} = \emptyset\}$$

and, consequently, also on $\mathcal{S} \setminus (\mathbb{B}_\varepsilon + \{x^*\}) \subset \mathcal{S}^\star$, with $\varepsilon = 8 \cdot 10^{-3}$ so that $\mathbb{B}_\varepsilon + \{x^*\}$ contains the union of simplices intersecting $\{x^*\}$. The resulting CPA stabilizer k is depicted in Fig. 1 in yellow.

Next, we compute three feedback stabilizers via linear programming, as described in Section III-B, for three different control objectives. We set the bounds $|k_\nu(\xi)| \leq 10$ for all $\xi \in \mathcal{V}_{\mathfrak{T}}$, which will not be limiting since V is a CLF in the sense of generalized gradients on $\mathcal{S}^\star$ with $\mathbb{U} := \{u \in \mathbb{R} \,|\, |u| \leq 10\}$, c.f., Fig. 1.

For the first objective, we minimize the cost function

$$\gamma \max_{\mathfrak{S}_\nu \in \mathfrak{T}} \|\nabla k_\nu\|_\infty + (1-\gamma) \sum_{\mathfrak{S}_\nu \in \mathfrak{T}} \sum_{i=0}^{n} \langle \nabla V_\nu, g(v_i) k_{\mathcal{V}}(v_i) \rangle \tag{16}$$

for $\gamma = \frac{999}{1000}$, which represents a tradeoff between minimizing torque variation and maximizing the average decrease of V w.r.t. to the trajectories. The LP problem, consisting of 7923 optimization variables and 108 304 constraints, was solved in 1.29 s on a modern PC using MOSEK Optimization Suite [25].

[Review Copy Only]

[MR4308458. Date: October 29, 2021 Sent to Professor Francesca Maria Ceragioli for reviewing purposes only]

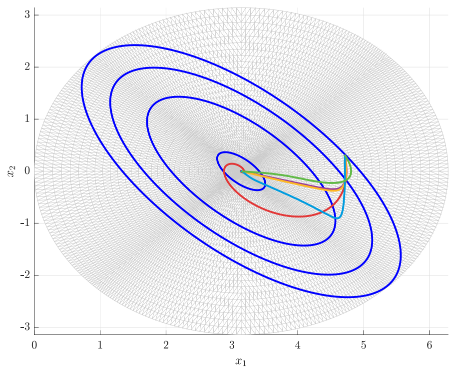

Fig. 2. Trajectories for the pendulum in closed loop with a PD controller with gravity compensation k_{PD} for $K_d = 1$ (red), $K_d = 4$ (purple), CPA stabilizers $k_{\text{LP},1}$ (blue), $k_{\text{LP},2}$ (green), and $k_{\text{LP},3}$ (yellow), together with the triangulation $\mathfrak{T}$ of $\mathcal{S}$ and level sets of V (dark blue).

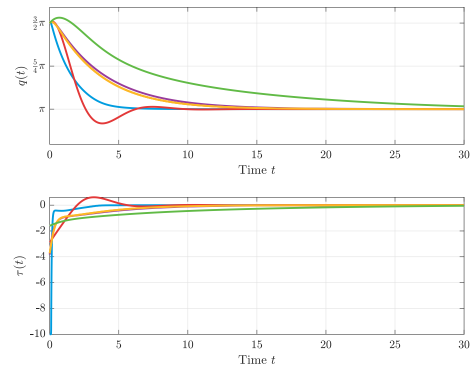

Fig. 3. Pendulum angle $q(t)$ and applied torque $\tau(t)$, for $u = k_{\text{PD}}(x)$ with $K_d = 1$ (red) and $K_d = 4$ (purple), $u = k_{\text{LP},1}$ (blue), $u = k_{\text{LP},2}$ (green), and $u = k_{\text{LP},3}$ (yellow) for the pendulum (14) in closed loop with the corresponding controllers.

Fig. 1 shows the resulting CPA feedback $k_{\text{LP},1}$ corresponding to (16) in blue. For the second objective, we minimize the cost function

$$\gamma \max_{\mathfrak{S}_\nu \in \mathfrak{T}} \|\nabla k_\nu\|_\infty + (1-\gamma) \max_{x \in \mathcal{V}_{\mathfrak{T}}} |k_{\mathcal{V}}(x)| \tag{17}$$

for $\gamma = \frac{1}{100}$, which represents a tradeoff between minimizing torque variation and minimizing the maximum torque magnitude that is applied to the pendulum. Fig. 1 shows the resulting stabilizer, $k_{\text{LP},2}$ in green. Finally, the third objective is to minimize the difference between the LP-based stabilizer and Sontag's "universal" formula $\bar{k}$, i.e., to minimize

$$\sum_{x \in \mathcal{V}_{\mathfrak{T}}} |k_{\mathcal{V}}(x) - \bar{k}(x)|. \tag{18}$$

The resulting CPA stabilizer $k_{\text{LP},3}$ is equal to the CPA stabilizer $k(x)$ computed via the procedure in Section III-A, as depicted in Fig. 1 in yellow.

For a performance comparison, we consider the well-known PD-plus-gravity-compensation controller for the position control of robot manipulators [26]

$$k_{\text{PD}}(x) = K_p(x_1^* - x_1) + K_d(x_2^* - x_2) + \sin(x_1) \tag{19}$$

which asymptotically stabilizes the state x^* of (15) for constants K_p, $K_d > 0$ [24]. The control law (19) is equal to the Lyapunov-based control law for a reaction-wheel pendulum in [27, Sect. 4.3.2] for an appropriate choice of K_p, K_d, and reduces to the passivity-based state feedback for a pendulum in [28, Example 7.2.2] for $K_p = 1$ and $K_d = 0$. Fig. 2 shows the trajectories for (15) initialized in $(\frac{3}{2}\pi,\ \frac{1}{10}\pi)$, in closed loop with the CPA controllers $u = k_{\text{LP},i}(x), i \in \{1, 2, 3\}$, and $u = k_{\text{PD}}(x)$ for $K_d = 1$ and $K_d = 4$, with $K_p = 1$.

We observe that the controller k_{PD} with $K_d = 1$ yields an overshoot for the pendulum angle, shown in Fig. 3, whereas k_{PD} with $K_d = 4$ achieves a performance that is similar to the CPA stabilizer $k = k_{\text{LP},3}$. Controller $k_{\text{LP},1}$, corresponding to (16), yields a considerably faster convergence without overshoot, by exploiting the available torque range specified by $\mathbb{U}$. Stabilizer $k_{\text{LP},2}$, corresponding to (17), yields the smallest maximum torque magnitude, but yields a slower convergence. Note that $k_{\text{LP},i}(x) \in \mathbb{U}$ for all $x \in \mathcal{S}$, whereas $k_{\text{PD}}(x) \notin \mathbb{U}$ for some $x \in \mathcal{S}$, with $K_d = 4$. The results demonstrate the freedom to optimize a desired criterion for the LP-based controller while guaranteeing stability and feasibility of control inputs.

B. Control of the Cortisol Level Within the HPA Axis

We consider the control of the HPA axis, described by [12]

$$\begin{cases} \dot{x}_1 = \left(1 + \zeta \frac{x_3^\alpha}{1+x_3^\alpha} - \psi \frac{x_3^\gamma}{x_3^\gamma + \tilde{c}_3^\gamma}\right) - \tilde{w}_1 x_1 \\ \dot{x}_2 = \left(1 - \rho \frac{x_3^\alpha}{1+x_3^\alpha}\right) x_1 - \tilde{w}_2 x_2 \\ \dot{x}_3 = x_2 - \tilde{w}_3 x_3. \end{cases} \tag{20}$$

The HPA axis is a physiological system that regulates the concentrations of the hormones CRH (x_1), ACTH (x_2), and cortisol (x_3). Maintenance of a basal cortisol concentration is important, since low-cortisol and high-cortisol depressions are correlated with various diseases [12]. The analysis in [12] revealed that possible biomarkers for individuals sensitive to stress are the parameters $\tilde{w}_1 = 4.79$, $\tilde{w}_2 = 0.964$, $\tilde{w}_3 = 0.251$, $\tilde{c}_3 = 0.464$, $\psi = 1$, $\rho = 0.5$, $\zeta = 4$, $\gamma = 5$, and $\alpha = 5$, such that the model (20) admits two stable equilibria $x_{E,1}^* = (0.12, 0.12, 0.48)$ and $x_{E,3}^* = (0.78, 0.43, 1.72)$ corresponding to depressed cortisol concentrations and one unstable equilibrium $x_{E,2}^* = (0.22, 0.20, 0.80)$ corresponding to a basal cortisol concentration [12].

The goal is to redirect trajectories from $x_{E,1}^*$ to the healthy equilibrium $= x_{E,2}^*$, via a CPA feedback stabilizer u such that $\dot{x} = f(x) + g(x)u$, with $g(x) = (0\ 0\ 1)^\top$. This implies that the control input is the rate of pharmaceutical removal or supply of cortisol from circulation, which has been shown to be a plausible strategy for controlling the HPA axis [29], [30].

As in Section IV.A, we apply a change of variable: $\xi := x - x_{E,2}^*$. We consider the candidate CLF $W(x) = x_1^2 + x_2^2 + \frac{1}{4}x_3^2$ and the CPA interpolation $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ of W on a triangulation $\mathfrak{T}$ of the hyperrectangle $\mathcal{S} = [-0.22, 0.22]^2 \times [-0.44, 0.44] + \{x_{E,2}^*\}$ with a center at $x_{E,2}^*$. Following the procedure outlined in Section III-A, we construct a "universal" CPA feedback stabilizer $k : \mathcal{S} \to \mathbb{R}$. We find that Problem 3.1 is not feasible

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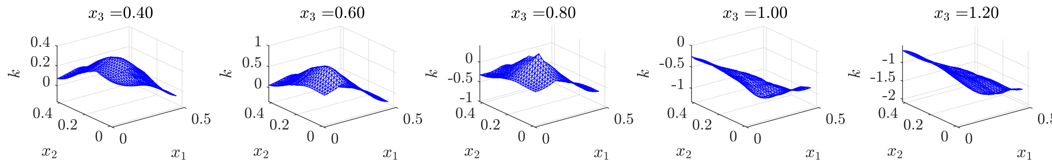

Fig. 4. Constructed CPA stabilizer for the HPA axis for various values of the cortisol concentration x_3.

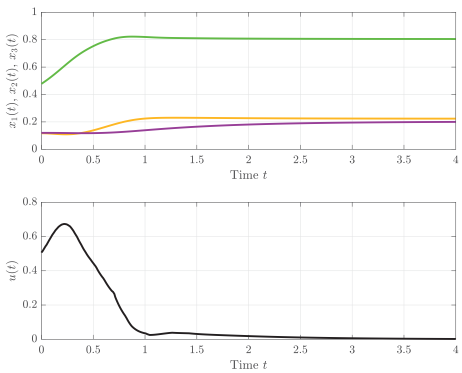

Fig. 5. Input $u(t) = k_{\text{LP}}(x(t))$ and states $x_1(t)$ (yellow), $x_2(t)$ (purple), and $x_3(t)$ (green) for the HPA axis (20) initialized in $x^*_{E,1}$.

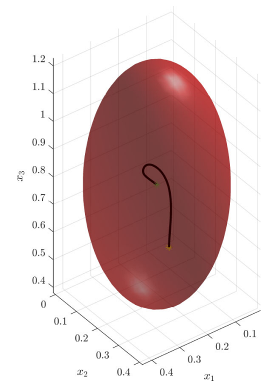

Fig. 6. Level set $\{x \in \mathcal{S} \,|\, V(x) = 0.045\}$ (red), equilibria $x^*_{E,1}$ (yellow), $x^*_{E,2}$ (green), and a trajectory for the HPA axis (20) initialized in $x^*_{E,1}$ (black) with $u = k_{\text{LP}}(x(t))$.

for the considered triangulation and, hence, we cannot conclude that k stabilizes the equilibrium $x^*_{E,2}$.

Next, we consider the computation of a feedback stabilizer via linear programming. The same triangulation and $V(x)$ are utilized. We set the bounds $|k_\nu| \leq 10$ for all $\xi \in \mathcal{V}_{\mathfrak{T}}$. The objective for the LP is free and we set it such that the objective function

$$\max_{\mathfrak{S}_\nu \in \mathfrak{T}} \|\nabla k_\nu\|_\infty$$

is minimized. The resulting LP problem, consisting of 23 807 optimization variables and 1 275 840 constraints, was solved in 5.08 s using MOSEK Optimization Suite [25] for $\rho = 10^{-10}$. Hence, we conclude that V is a CLF in the sense of generalized gradients for (1) and a strong LF in the sense of generalized gradients for (2) on $\mathcal{S}^\star := \cup_{\mathfrak{S}_\nu \in \mathfrak{T}^\star} \mathfrak{S}_\nu$, where

$$\mathfrak{T}^\star := \{\mathfrak{S}_\nu \in \mathfrak{T} \,|\, \mathfrak{S}_\nu \cap \{0\} = \emptyset\}.$$

Fig. 4 visualizes the computed CPA feedback $k_{\text{LP}}(x)$ for various values of the cortisol concentration x_3.

Fig. 5 shows the simulation results for the HPA axis in closed loop with $u = k_{\text{LP}}(x)$, initialized in the low-cortisol depressed equilibrium, i.e., $x(0) = x^*_{E,1}$. A level set of V is displayed in Fig. 6, together with the simulated closed-loop trajectory in the state space. We observe that the state is steered toward the healthy equilibrium $x^*_{E,2}$ and, hence, the level of cortisol is increased to its basal value. From a biomedical point of view, the positive control input corresponds with a pharmaceutical supplement of cortisol, analogous to the pharmaceutical removal of cortisol obtained for a high-cortisol depressed state in [29]. The advantage of the CPA feedback $u = k_{\text{LP}}(x)$ with respect to this application is represented by the ability to efficiently compute control inputs via lookup tables for treatment. Additionally, medical professionals can analyze the control before implementation because of its explicit nature, which was advocated for the correction of a dysfunctioning HPA axis in [31].

V. Conclusion

We have proposed two procedures for the construction of CPA feedback stabilizers for general nonlinear systems affine in the input. For the verification of "universal" CPA control laws, a feasibility problem consisting of a finite number of inequalities was proposed. Given a twice continuously differentiable CLF, the verification of the CPA stabilizer is feasible for a suitable triangulation. A numerical method for computing CPA feedback stabilizers with *a priori* stability guarantees was presented. This method allows for optimizing a user specified cost function. We illustrated the synthesis and performance of the proposed methods for an inverted pendulum and the HPA axis. For the pendulum, a "universal" and three LP-based stabilizers for various control objectives were obtained, which were shown to achieve competitive performance in comparison with a PD-plus-gravity-compensation controller.

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