Hence, s_0 is a pole of $G(s)$ if and only if $D(s_0) + kN(s_0) = 0$.

∎

In addition, it is straightforward to realize that the poles of $H(s)$ coincide with the poles of $T(s)$. Applying the Argument Principle to the rational function $H(s)$ leads to the following conclusions:

number of the poles of G *with positive real part*
$=$ *number of poles of* T *with positive real part* $+ \, Q$

where Q denotes the number of times the curve $H(-\mathrm{i}\,t)$ encircles the origin in the counterclockwise sense, while the parameter t moves from $-\infty$ to $+\infty$.

On the other hand, it is evident that $-Q$ represents the number of times the curve $T(-\mathrm{i}\,t)$ encircles the point of coordinates $(-\frac{1}{k}, 0)$ of the complex plane in clockwise sense, while the parameter t moves from $-\infty$ to $+\infty$. The following statement resumes the conclusions.

Proposition 8.6 *The static output feedback $-ky$ stabilizes in BIBO (and so also in internal) sense the SISO system (8.1) if the number of times the Nyquist diagram of its transfer function $T(s)$ encircles the point $(-\frac{1}{k}, 0)$ in clockwise sense while the parameter t moves from $-\infty$ to $+\infty$, is equal to the number of poles of the given system lying in the open right half of the complex plane.*

In practical applications, one draws the Nyquist diagram of the given system, and then checks whether there exists a region $\mathcal{D}$ encircled by the diagram the required number of times. If this region exists and intersects the negative real axis, the system is stabilizable. A stabilizing feedback is provided by any value of k such that $(-\frac{1}{k}, 0) \in \mathcal{D}$.

8.5 Disturbance decoupling

In this last section we discuss an important application which involves both frequency domain and time domain techniques. Consider the system

$$\begin{cases} \dot{x} = Ax + Gd \\ y = Cx \end{cases} \tag{8.22}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$, $d \in \mathbf{R}^q$. The input $d(t) : [0, +\infty) \to \mathbf{R}^q$ is now interpreted as a disturbance. In other words, $d(t)$ is a unknown and undesired input; we just assume that it is piecewise continuous and right continuous, in order to guarantee existence of solutions. For each initial state x_0, the variation of constants formula yields

$$y(t, x_0, d(\cdot)) = Ce^{tA}x_0 + \int_0^t Ce^{(t-\tau)A}Gd(\tau)\, d\tau$$

which reduces to

$$y_0(t) = Ce^{tA}x_0$$

when $d(t) = 0$ for each $t \geq 0$. The function $y_0(t)$ is called the *uncorrupted output signal.* It may happen that $y(t) = y_0(t)$ even for not vanishing disturbances $d(t)$.

Example 8.7 Clearly, the output of the (not completely observable) system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 + d \\ \dot{x}_2 = x_2 \\ \\ y = x_2 \end{cases}$$

is not affected by the disturbance.

∎

Definition 8.5 *Let us denote, as before, by $y_0(t)$ the uncorrupted output, that is the output corresponding to some initial state x_0 and the vanishing input $d(t) = 0$. We say that the system is* disturbance decoupled *if we have $y(t, x_0, d(\cdot)) = y_0(t)$ for each $t \geq 0$, each initial state x_0 and each input $d(t)$.*

Proposition 8.7 *The following statements are equivalent:*

(i) the system (8.22) is disturbance decoupled;

(ii) the impulse response matrix $W(t) = Ce^{tA}G$ vanishes for $t \geq 0$ (and hence, being a real analytic function, for each $t \in \mathbf{R}$);

(iii) the transfer matrix $T(s) = C(sI - A)^{-1}G$ vanishes for $s \in \mathbf{C}$;

(iv) for each integer $k \geq 0$, one has $CA^kG = 0$.

Proof The equivalences *(i)* $\iff$ *(ii)* $\iff$ *(iii)* are straightforward. Thus, we focus on the statement *(iv)*, and we will prove that it is equivalent to *(ii)*. Assume first that the identity

$$W(t) = Ce^{tA}G = 0 \tag{8.23}$$

holds for each $t \in \mathbf{R}$. To begin with, the substitution $t = 0$ yields $CG = 0$. Coming back to (8.23) and taking the derivative, we obtain

$$CAe^{tA}G = 0 \tag{8.24}$$

for $t \in \mathbf{R}$, which implies $CAG = 0$ by the substitution $t = 0$. We repeat the procedure, taking now the derivative of (8.24) and letting again $t = 0$. This time we obtain $CA^2G = 0$. Continuing in this way, we conclude finally that $CA^kG = 0$ for each integer $k \geq 0$. The converse implication is immediate, since

$$W(t) = Ce^{tA}G = \sum_{k=0}^{\infty} \frac{t^k}{k!} CA^k G$$

for each $t \in R$.

∎

Remark 8.8 According to the Cayley-Hamilton Theorem, it is sufficient to check condition *(iv)* of Proposition 8.7 for $k = 0, \ldots, n-1$.

∎

Next we establish a necessary and sufficient condition.

Definition 8.6 *Let A be a real matrix of dimensions $n \times n$. A subspace V of $\mathbf{R}^n$ is said to be an* algebraic (o geometric) invariant *for A if $AV \subseteq V$.*

The subspace V is said to be a dynamic invariant *for A if from $x_0 \in V$ it follows $e^{tA}x_0 \in V$ for each $t \geq 0$ (and hence for each $t \in \mathbf{R}$).*

Proposition 8.8 *The subspace V is an* algebraic invariant *for A if and only if it is a* dynamic invariant *for A.*

Proof Let V be an algebraic invariant. For each $x_0 \in V$, we clearly have $Ax_0 \in V$, $A^2x_0 \in V$, and so on. Hence, $e^{tA}x_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x_0$ belongs to V. On the other hand, let V be a dynamic invariant, and let $x_0 \in V$. Then, for each $t \neq 0$, we also have

$$\frac{e^{tA}x_0 - x_0}{t} \in V \ .$$

Taking the limit for $t \to 0$, we get $Ax_0 \in V$.

∎

Theorem 8.4 *The given system is disturbance decoupled if and only if there exists a subspace V of $\mathbf{R}^n$ which is an algebraic invariant for A, and such that* $\operatorname{im} G \subseteq V \subseteq \ker C$.

Proof Assume that the system is disturbance decoupled. Let us introduce a matrix H, whose columns coincide with the columns of the matrices $G, AG, A^2G, \dots, A^{n-1}G$, in this order. The matrix H can be interpreted as a linear map from $\mathbf{R}^{n\times q}$ in $\mathbf{R}^n$. Let $V = \operatorname{im} H$. By the Cayley-Hamilton Theorem, V is an algebraic invariant. The inclusion $\operatorname{im} G \subseteq V$ is obvious, while the other one $V \subseteq \ker C$ follows from Proposition 8.7, *(iv)*.

To prove the converse, we first remark that if a subspace V is an algebraic invariant and $\operatorname{im} G \subseteq V$, then clearly $\operatorname{im}(A^k G) \subseteq V$ for each positive integer k. As a consequence, since $V \subseteq \ker C$, we also have $CA^kGx = 0$ for each integer $k \geq 0$ and each $x \in \mathbf{R}^n$. The conclusion follows, using again Proposition 8.7, *(iv)*.

∎

There is an other characterization of disturbance decoupled systems. By means of a linear change of coordinates, we can put the system in the observability canonical form

$$\begin{cases} \dot z_1 = A_{11}z_1 + A_{12}z_2 + G_1 d \\ \dot z_2 = A_{22}z_2 + G_2 d \\ \\ y = C_2 z_2 \end{cases} \tag{8.25}$$

where $z_1 \in \mathbf{R}^{n-r}$, $z_2 \in \mathbf{R}^r$ for some nonnegative integer $r \leq n$, and the subsystem

$$\begin{cases} \dot z_2 = A_{22}z_2 + G_2 d \\ \\ y = C_2 z_2 \end{cases} \tag{8.26}$$

is completely observable. The case $r = 0$ is trivial, so we can assume $r > 0$.

Theorem 8.5 *The system (8.22) is disturbance decoupled if and only if $G_2 = 0$, where G_2 is the matrix appearing in (8.25).*

Proof The sufficient part is evident (to be formal, it can be easily obtained as an application of Theorem 8.4). Let us prove the necessary part.

Assume that the system is disturbance decoupled. Taking into account the form (8.25), for each integer $k \geq 0$, we see that CA^k can be written as a row block matrix $(0 \mid C_2A_{22}^k)$, where 0 denotes here a block of $n - r$ zero columns. From this, it easily follows that $CA^kG = C_2A_{22}^kG_2$ for each integer $k \geq 0$. Since the subsystem (8.26) is completely observable, the matrix

$$M = \begin{pmatrix} C_2 \\ C_2A_{22} \\ \dots \\ C_2A_{22}^{r-1} \end{pmatrix}$$

has a maximal rank i.e., $\operatorname{rank} M = r$. Now, assume by contradiction that $v = G_2 d \neq 0$ for some $d \in \mathbf{R}^q$ (note that $v \in \mathbf{R}^r$ and that, necessarily, $d \neq 0$). The vector $Mv \in \mathbf{R}^{p \times r}$ is a linear combination of the r linearly independent columns of M, so that being $v \neq 0$, we also have $Mv \neq 0$. But

$$Mv = \begin{pmatrix} C_2 G_2 d \\ C_2 A_{22} G_2 d \\ \dots \\ C_2 A_{22}^{r-1} G_2 d \end{pmatrix} = \begin{pmatrix} C_2 G_2 \\ C_2 A_{22} G_2 \\ \dots \\ C_2 A_{22}^{r-1} G_2 \end{pmatrix} d \tag{8.27}$$

with $d \neq 0$. On the other hand, the disturbance decoupling assumption implies

$$C_2 G_2 = C_2 A_{22} G_2 = \ldots = C_2 A_{22}^{r-1} G_2 = 0 \ . \tag{8.28}$$

Clearly, (8.27) and (8.28) are in contradiction. Therefore, we must have $G_2 d = 0$ for each $d \in \mathbf{R}^q$, and this means that $G_2 = 0$.

∎

If the given system is not disturbance decoupled, we can try to achieve this property by the use of a suitable feedback law. In other words, we add a control term in the system equation

$$\begin{cases} \dot{x} = Ax + Bu + Gd \\ y = Cx \end{cases} \tag{8.29}$$

where with the usual notation $u \in \mathbf{R}^m$, and we ask whether it is possible to find a static state feedback of the form $u = Fx$ such that the closed-loop system

$$\begin{cases} \dot{x} = (A + BF)x + Gd \\ y = Cx \end{cases}$$

is disturbance decoupled. The conditions for answering this question rest on the introduction of a new notion of invariance, concerning the state equation

$$\dot{x} = Ax + Bu \ . \tag{8.30}$$

Definition 8.7 *A subspace $V \subseteq \mathbf{R}^n$ is said to be a* strong controlled invariant *for the system (8.30) if for each $x_0 \in V$ and each admissible input $u(t) : [0, +\infty) \to \mathbf{R}^m$ we have $x(t, x_0, u(\cdot)) \in V$ for each $t \geq 0$.*

Apart from the modified terminology, the definition above coincides with the notion already introduced in Section 5.3.2.

Definition 8.8 *A subspace $V \subseteq \mathbf{R}^n$ is said to be a* weak controlled invariant *for the system (8.30) if for each $x_0 \in V$ there exists an admissible input $u(t) : [0, +\infty) \to \mathbf{R}^m$ such that $x(t, x_0, u(\cdot)) \in V$ for each $t \geq 0$.*

Example 8.8 The subspace $V = \{(x_1, x_2) : x_2 = 0\} \subseteq \mathbf{R}^2$ is a weak controlled invariant, but not a strong controlled invariant, for the system

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = u \ . \end{cases}$$

Note that this system is completely controllable.

∎

The weak controlled invariant subspaces can be characterized in the following way.

Proposition 8.9 *The following statements are equivalent.*

(i) V is a weak controlled invariant;

(ii) $AV \subseteq V + \operatorname{im} B$;

(iii) there exists a matrix F with n columns and m rows such that $(A + BF)V \subseteq V$.

Proof First we prove that *(i)* $\Longrightarrow$ *(ii)*. Let $x_0 \in V$ and let $u(t) : [0, +\infty) \to \mathbf{R}^m$ be an input such that $x(t, x_0, u(\cdot)) \in V$ for each $t \geq 0$. Without loss of generality, we can extend continuously $u(t)$ on a small interval $(-\varepsilon, 0)$, so that $x(t, x_0, u(\cdot))$ can be considered of class C^1 at $t = 0$. Then

$$\lim_{t \to 0^+} \frac{x(t, x_0, u(\cdot)) - x_0}{t} = \dot{x}(0) \in V$$

that is $Ax_0 + Bu(0) \in V$, or $Ax_0 \in V - Bu(0)$.

Next we prove that *(ii)* $\Longrightarrow$ *(iii)*. Let $\dim V = k \leq n$. Let $e_1, \ldots, e_n$ be a basis of $\mathbf{R}^n$, such that its first k elements $e_1, \ldots, e_k$ constitute a basis of V. Then for each $i = 1, \ldots, k$ one has $Ae_i = g_i + Bu_i$ for some $g_i \in V$ and some $u_i \in \mathbf{R}^m$. Let us chose other vectors $u_{k+1}, \ldots, u_n \in \mathbf{R}^m$ in arbitrary way, and define the matrix F by the relations $Fe_j = -u_j$, for $j = 1, \ldots, n$. Then we have, for $i = 1, \ldots, k$,

$$(A + BF)e_i = Ae_i + BFe_i = g_i + Bu_i - Bu_i = g_i \in V \ .$$

Finally we prove that *(iii)* $\Longrightarrow$ *(i)*. Let $x_0 \in V$ and let $x(t)$ be the solution of the closed loop system

$$\begin{cases} \dot{x} = (A + BF)x \\ x(0) = x_0 \ . \end{cases} \tag{8.31}$$

Of course, $x(t)$ is also a solution of the problem

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ x(0) = x_0 \end{cases}$$

where $u(t) = Fx(t)$. The proof is completed, by noticing that V is a dynamic invariant with respect to system (8.31).

∎

We are finally able to state the main result of this section.

Theorem 8.6 *System (8.29) can be rendered disturbance decoupled by means of a linear feedback if and only if there exists a subspace* $V \subseteq \mathbf{R}^n$ *which is a weak controlled invariant for the system (8.30) and such that* $\operatorname{im} G \subseteq V \subseteq \ker C$*.*

Proof Let us prove first the necessary part. So let F be a matrix such that the system

$$\begin{cases} \dot{x} = (A + BF)x + Gd \\ y = Cx \end{cases} \tag{8.32}$$

is disturbance decoupled. According to Theorem 8.4, there exists an algebraic invariant subspace V, such that $\operatorname{im} G \subseteq V \subseteq \ker C$. This implies that $(A + BF)V \subseteq V$, and this in turn means that V is a weak controlled invariant, by Proposition 8.9.

Then we prove the sufficient part. If V is weak controlled invariant, then by Proposition 8.9 there exists F such that $(A + BF)V \subseteq V$. Together with the inclusions $\operatorname{im} G \subseteq V \subseteq \ker C$, this implies finally that the system (8.32) is disturbance decoupled by Theorem 8.4.

∎