Advanced Algorithms and Computational Models (module A)

Communities

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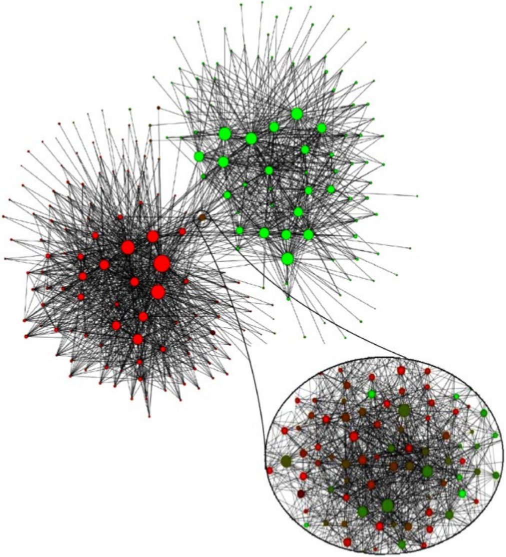
2019-2020

Introduction

- Belgium appears to be the model bicultural society: 59% of its citizens are Flemish, speaking Dutch and 40% are Walloons who speak French
- As multiethnic countries break up all over the world, we must ask: How did this country foster the peaceful coexistence of these two ethnic groups since 1830?
- Is Belgium a densely knitted society, where it does not matter if one is Flemish or Walloon?
- Or we have two nations within the same borders, that learned to minimize contact with each other?

Introduction

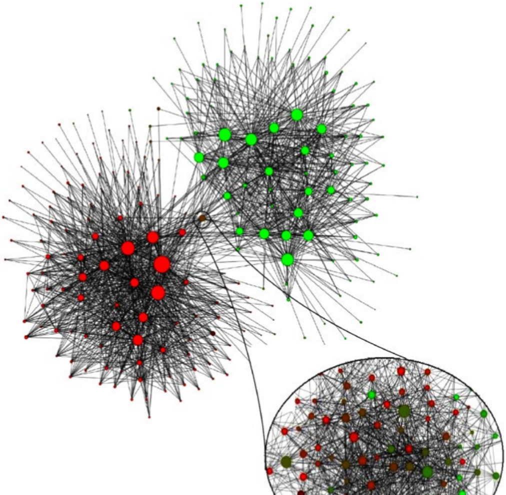
Communities in Belgium



Introduction

Definition

A **community** is a group of nodes that have a higher likelihood of connecting to each other than to nodes from other communities



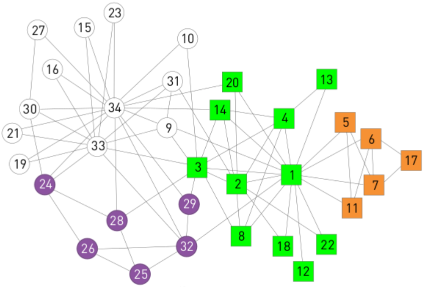
Introduction

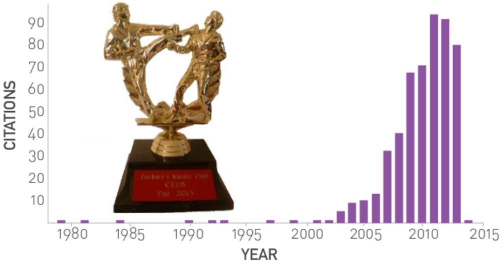
An example: social networks

- Social networks easily contain communities
- For example, the employees of a company are more likely to interact with their coworkers than with employees of other companies
- Therefore, work places appear as densely interconnected communities within the social net- work
- A social network that has received particular attention in the context of community detection is known as Zachary's Karate Club, capturing the links between 34 members of a karate club. The interest in the dataset is driven by a singular event: a conflict between the president and the instructor split the club into two

Introduction

Zachary's Karate Club





Introduction

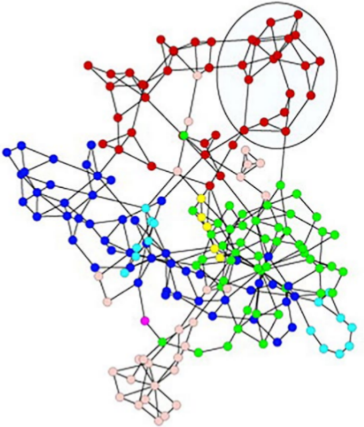
An example: biological networks

- Communities play a fundamental role in our understanding of how specific biological functions are encoded in cellular networks

- For example, proteins that are involved in the same disease tend to interact with each other: each disease can be linked to a well-defined neighbourhood of the cellular network

Introduction

A metabolic network



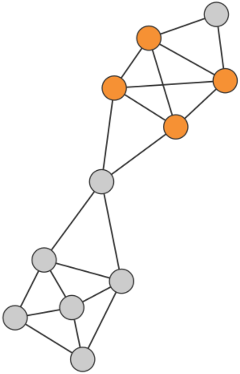
Basics of Communities

- One of the first studies on community structure defined a community as group of individuals whose members all know each other
- In graph theoretic terms this means that a community is a **complete subgraph** or a **clique**
- This is wrong: triangles are frequent in networks, while larger cliques are rare
- Moreover, requiring that a community to be a complete subgraph may be too restrictive

Basics of Communities

Cliques

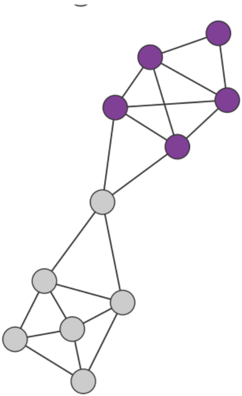
A *clique* corresponds to a complete subgraph. The highest order clique of this net- work is a square



Basics of Communities

Strong communities

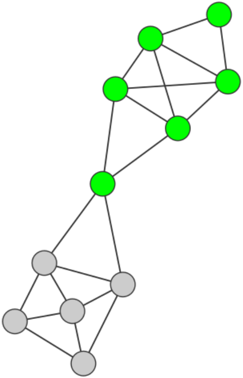
A *strong community* is a connected subgraph whose nodes have more links to other nodes in the same community that to nodes that belong to other communities



Basics of Communities

Weak communities

A *weak community* is a connected subgraph whose nodes' internal degree exceeds their external degree



Basics of Communities

Number of communities

- How many ways the nodes of a network can be grouped into communities?

- An approximate answer is provided by the simplest community finding problem, called *graph bisection*

- The idea consists in dividing a network into two non-overlapping subgraphs such that the number of links between the nodes in the two groups (*cut size*) is minimized

- The graph bisection problem can be solved by inspecting all possible divisions into two groups and choosing the one with the smallest cut size

Basics of Communities

Number of communities

- The computational cost of this approach is roughly given by the number of distinct ways we can partition a network of N nodes into groups N_1 and N_2 nodes is

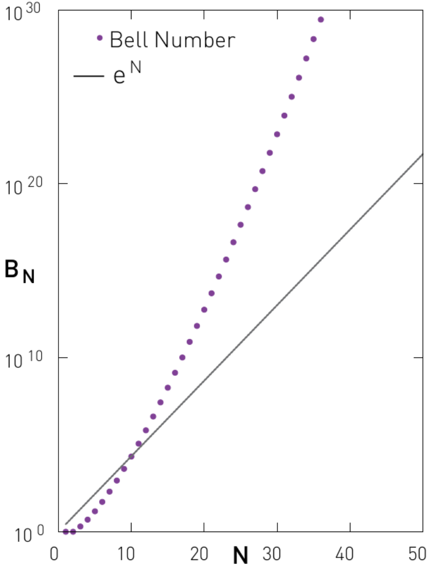
$$\frac{N!}{N_1!N_2!}$$

- In the (very simple) case of a network with 10 nodes which is bisected into two subgraphs of $N_1 = N_2 = 5$, 252 bisections must be checked to find the one with the smallest cut size

- In the (simple) case of a network with 100 nodes which is bisected into two subgraphs of $N_1 = N_2 = 50$, 10^{29} bisections must be checked

Basics of Communities

Number of communities



Hierarchical Clustering

Introduction

- Hierchical clustering is used to unveil the community structure of large networks in polynomial time
- Its starting point is a *similarity matrix*, whose elements x_{ij} express the distance of node i from node j
- Similarity matrix is then used to iteratively identify groups of nodes with high similarity
- Two procedures can be used: *agglomerative algorithms*, which merge nodes with high similarity into the same community, while *divisive algorithms* isolate communities by removing low similarity links that tend to connect communities
- Both procedures produce a *dendrogram*, a hierchical tree that helps in predicting possible communities

Hierarchical Clustering

Agglomerative procedures: the Ravasz algorithm

- Define the similarity matrix
- Decide group similarity
- Apply hierarchical clustering
- Dendrogram

Hierarchical Clustering

The Ravasz algorithm: define the similarity matrix

- Similarity should be high for node pairs belonging to the same community and low for node pairs belonging to different communities
- Nodes that connect to each other and share neighbors likely belong to the same community, hence their x_{ij} should be large
- The topological overlap matrix is defined as

$$x_{ij}^0 = \frac{J(i,j)}{min(k_i, k_j) + 1 - \Theta(A_{ij})}$$

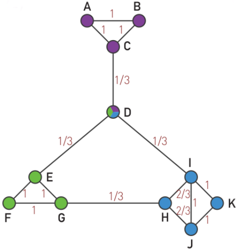
- $J(i,j)$ is the number of common neighbors of nodes i and j
- $\Theta(x)$ is the Heaviside step function, zero for $x \leq 0$ and one for $x > 0$
- $min(k_i, k_j)$ is the smaller of the degrees k_i and k_j

Hierarchical Clustering

The Ravasz algorithm: define the similarity matrix

$$x_{ij}^0 = \frac{J(i,j)}{min(k_i, k_j) + 1 - \Theta(A_{ij})}$$

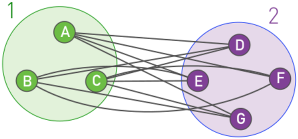
- $x_{ij}^0 = 1$ if nodes i and j are linked and share the same neighbors (nodes A and B)
- $x_{ij}^0 = 0$ if nodes i and j do not have common neighbors
- Members of the same dense local network neighborhood have high topological overlap



Hierarchical Clustering

The Ravasz algorithm: decide group similarity

- As nodes are merged into small communities, the similarity of two communities must be measured
- The Ravasz algorithms used the *average cluster similarity*
- The similarity of two communities is defined as the average of x_{ij} over all node pairs i and j that belong to distinct communities



Hierarchical Clustering

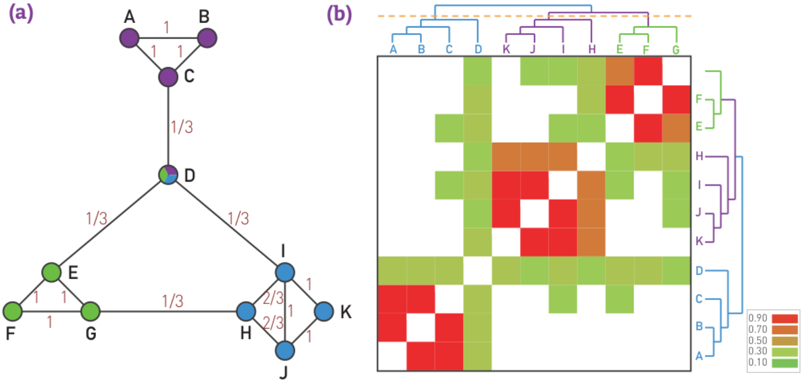
The Ravasz algorithm: apply hierchical clustering

The Ravasz algorithm uses the following procedure to identify the communities:

1. Assign each node to a community of its own and evaluate x_{ij} for all node pairs
2. Find the community pair or the node pair with the highest similarity and merge them into a single community
3. Calculate the similarity between the new community and all other communities
4. Repeat Steps 2 and 3 until all nodes form a single community

Hierarchical Clustering

The Ravasz algorithm: Dendrogram



Hierarchical Clustering

Divisive procedures: the Girvan-Newman algorithm

Divisive procedures systematically remove the links connecting nodes that belong to different communities, eventually breaking a network into isolated communities

- Define centrality
- Hierarchical clustering

Hierarchical Clustering

The Girvan-Newman algorithm: define centrality

- In divisive algorithms x_{ij}, called *centrality*, selects node pairs that are in different communities
- Therefore x_{ij} must be high (or low) if nodes i and j belong to different communities and small if they are in the same community
- The most widely used centrality (in this context) is the *link betweenness*, defining x_{ij} as the number of shortest paths that go through the link (i, j)
- Links connecting different communities are expected to have large x_{ij}, while links within a community have small x_{ij}

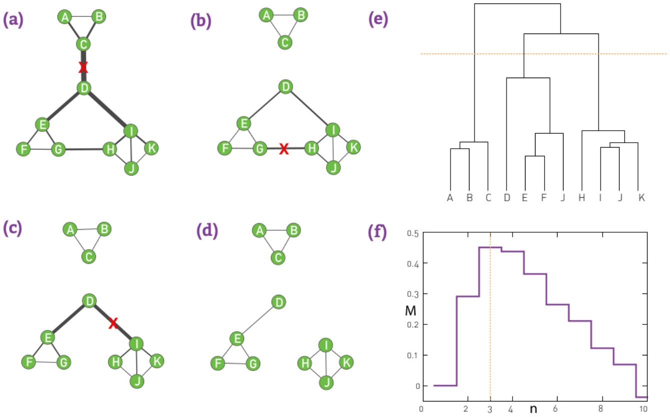
Hierarchical Clustering

The Girvan-Newman algorithm: Hierarchical clustering

1. Compute the centrality x_{ij} of each link
2. Remove the link with the highest centrality. In case of a tie, choose one link randomly
3. Recalculate the centrality of each link for the altered network
4. Repeat Steps 2 and 3 until all links are removed

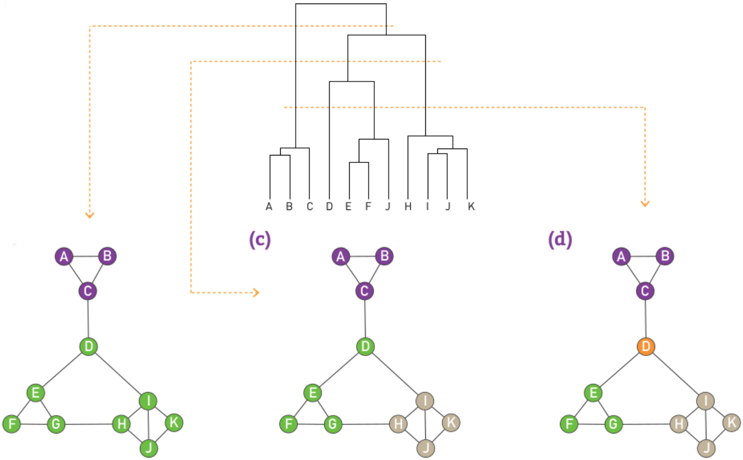
Hierarchical Clustering

The Girvan-Newman algorithm: Hierarchical clustering



Hierarchical Clustering

Ambiguity



Modularity

Introduction

- Consider a network with N nodes, L links
- Consider also a partition into n_c communities
- Each partition has N_c nodes connected to each other by L_c links
- If L_c is larger than the **expected** number of links between the N_c nodes, then the nodes of the subgraph C_c could be part of a true community

Modularity

Introduction

- We therefore measure

$$M_c = \frac{1}{2L} \sum_{(i,j) \in C_c} (A_{ij} - p_{ij})$$

  which expresses the difference between the real number of links and the **expected** number of links if the subgraph were randomly wired

- p_{ij} can be estimated in the case of random networks, in which

$$p_{ij} = \frac{k_i k_j}{2L}$$

Modularity

Introduction

- The expression

$$M_c = \frac{1}{2L} \sum_{(i,j) \in C_c} (A_{ij} - p_{ij})$$

can then be simplified as

$$M_c = \frac{L_c}{L} - \left(\frac{k_c}{2L}\right)^2$$

Modularity

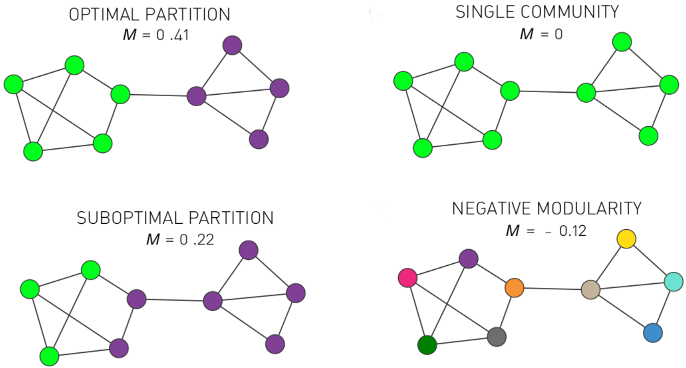
Introduction

- This concept can be generalized to a full network
- Consider the complete partition that breaks the network into n_c communities
- Then

$$M = \sum_{c=1}^{n_c} \left[\frac{L_c}{L} - \left(\frac{k_c}{2L} \right)^2 \right]$$

- which expresses the difference between the local link density and the expected link density summed over all communities

Modularity


OPTIMAL PARTITION
M = 0 .41
SINGLE COMMUNITY
M = 0
SUBOPTIMAL PARTITION
M = 0 .22
NEGATIVE MODULARITY
M = – 0.12

Modularity

The greedy algorithm

- Partitions with higher modularity correspond to partitions that more accurately capture the communities
- Therefore it is reasonable to conclude that it the partition with maximum modularity corresponds to the optimal community structure
- The greedy algorithm (Newman) finds partitions with close to maximal M

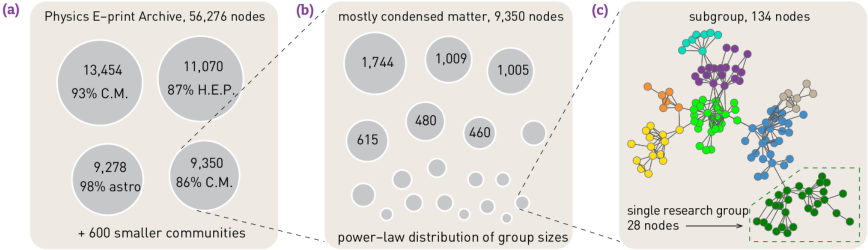
Modularity

The greedy algorithm

1. Assign each node to a community of its own, starting with N communities of single nodes
2. Inspect each community pair connected by at least one link and compute the modularity difference $\Delta \mathrm{M}$ obtained if we merge them. Identify the community pair for which $\Delta \mathrm{M}$ is the largest and merge them. Note that modularity is always calculated for the full network
3. Repeat Step 2 until all nodes merge into a single community, recording M for each step
4. Select the partition for which M is maximal

Modularity

The greedy algorithm


(a)
Physics E–print Archive, 56,276 nodes
13,454
93% C.M.
11,070
87% H.E.P.
9,278
98% astro
9,350
86% C.M.
+ 600 smaller communities
(b)
mostly condensed matter, 9,350 nodes
1,744
1,009
1,005
615
480
460
power–law distribution of group sizes
(c)
subgroup, 134 nodes
single research group
28 nodes