# 1 Carr-Madan Method Implementation

Consider $X_t$ such that the underlying process is given by $S_t = S_0 e^{rt+X_t}$.

The characteristic function of $X_t$ is

$$\phi_t(v) = \phi_{X_t}(v) = \mathbb{E}[e^{ivX_t}] = \int_{\mathbb{R}} e^{ivx} f_X(x)\, dx = \text{Fourier}(f_X),$$

where $f_X$ is the PDF and Fourier refers to the Fourier operator.

In the risk-neutral measure, $\phi_t(-i) = 1$.

Considering the risk-neutral measure for GBM (Black-Scholes):

$$\phi_t(v) = e^{-\sigma^2/2\, ivT - \sigma^2/2\, v^2 T}.$$

The inverse Fourier transform is given by:

$$F^{-1}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) e^{-ixv}\, dv,$$

where $v$ can be real or complex.

For $f \in L^2(\mathbb{R})$, $F^{-1}Ff = f$, but this also holds in other cases. Let $k = \log K$ be the log strike, and assume without loss of generality $t = 0$.

The price of a call option is given by

$$C(k) = e^{-rT}\mathbb{E}[(e^{rT+X_T} - e^k)^+].$$

$C(k)$ is not integrable (it tends to a positive constant as $k \to \infty$). The idea instead is to compute the Fourier transform of the (modified) time value of the option, that is, the function

$$z_T(k) = e^{-rT}\mathbb{E}[(e^{rT+X_T} - e^k)^+] - (1 - e^{k-rT})^+.$$

$z_T(k)$ is integrable as $k \to \infty$.

Let $\xi_T(v)$ denote the Fourier transform of $z_T(k)$:

$$\xi_T(v) = Fz_T(v) = \int_{-\infty}^{\infty} e^{-ivk} z_T(k)\, dk.$$

It can be expressed in terms of the characteristic function of $X_T$. First, since the discounted process is a martingale:

$$z_T(k) = e^{-rT}\int_{-\infty}^{\infty} \rho_T(x)\, dx\, (e^{rT+x} - e^k)(1_{k\le x+rT} - 1_{k\le rT}).$$

$$\xi_T(v) = e^{-rT}\int_{-\infty}^{\infty} \rho_T(x)\, dx \int_{-\infty}^{\infty} e^{ivk}(e^{rT+x} - e^k)(1_{k\le x+rT} - 1_{k\le rT})\, dk.$$

$$= e^{-rT}\int_{-\infty}^{\infty} \rho_T(x)\, dx \int_{x+rT}^{rT} e^{ivk}(e^k - e^{rT+x})\, dk.$$

$$= \int_{-\infty}^{\infty} \rho_T(x)\, dx \left\{ \frac{e^{ivrT}(1-e^x)}{iv(1+iv)} - \frac{e^{x+ivrT}}{iv(iv+1)} + \frac{e^{(iv+1)x+ivrT}}{iv(iv+1)} \right\}.$$

The first term in braces disappears due to the martingale condition and, after computing the other two, we get:

$$\xi_T(v) = e^{ivrT}\frac{\phi_T(v-i) - 1}{iv(1+iv)}.$$

The martingale condition guarantees that the numerator is 0 for $v = 0$ and the fraction has a finite limit as $v \to 0$.

Option prices can now be found by inverting the Fourier transform:

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \xi_T(v)\, dv.$$

Then,

$$C(k) = z_T(k) + (1 - e^{k-rT})^+.$$

To use the FFT, we compute the discrete Fourier transform (DFT) of vector $X$. For an input vector $x$ of length $N$, the DFT is a length $N$ vector $X$:

$$X(k) = \sum_{n=1}^{N} x(n) \exp\left(-i\frac{2\pi(k-1)(n-1)}{N}\right), \quad 1 \leq k < N.$$

For the FFT, we compute the inverse DFT:

$$x(n) = \frac{1}{N} \sum_{k=1}^{N} X(k) \exp\left(i\frac{2\pi(k-1)(n-1)}{N}\right), \quad 1 \leq n \leq N.$$

We need to compute $z_T(k)$, but we can reduce it to

$$z_T(k) = \frac{1}{\pi} \int_{0}^{\infty} e^{-ivk} \xi_T(v)\, dv.$$

To compute the integral, we use a quadrature formula:

$$z_T(k) \approx \frac{1}{\pi} \int_{0}^{A(N-1)/N} e^{-ivk} \xi_T(v)\, dv = \frac{1}{\pi} \sum_{j=0}^{N-1} \omega_j \eta e^{-i\eta jk} \xi_T(\eta j).$$

where $w_0 = w_{N-1} = 0.5$, and $w_j = 1$ otherwise, which is the trapezoidal formula with nodes $j\eta$, with $\eta = \frac{A}{N}$. Now we consider the following grid for the log strike $k_l = -\lambda N/2 + \lambda l$, with $\lambda = \frac{2\pi}{N\eta}$ and $l = 0, \ldots, N-1$:

$$z_T(k_l) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} \omega_j \eta e^{-i\eta j(-\lambda N/2 + \lambda l)} \xi_T(\eta j).$$

$$= \frac{1}{\pi} \sum_{j=0}^{N-1} \omega_j \eta e^{i\eta j\lambda N/2} e^{-i\eta j\lambda l} \xi_T(\eta j).$$

$$= \frac{1}{\pi} \sum_{j=0}^{N-1} \omega_j \eta e^{ij\pi} e^{-ijl\frac{2\pi}{N}} \xi_T(\eta j).$$

$$= \frac{1}{\pi} \text{FFT}\left(\{\omega_j \eta e^{ij\pi} \xi_T(\eta j)\}_{j=0}^{N-1}\right).$$