## Exercise 1

① incompressive two-layer fluid $\rho_1 > \rho_2$ *Giorgio Checola*

$$PE = g\int z\,\rho\, dV \qquad h(y) = \frac{1}{2}H + \gamma y$$

a)

$$PE = \int_0^H \int_{-L}^{L} g\rho z \overbrace{dy\,dz}^{\text{2D volume}} \longrightarrow \frac{PE}{g} = \int_{-L}^{L} \rho_1 \int_0^{h(y)} z\, dz\, dy + \int_{-L}^{L} \rho_2 \int_{h(y)}^{H} z\, dz\, dy =$$

$$= \int_{-L}^{L} \rho_1 \frac{\left(\frac{H^2}{4} + \gamma^2 y^2 + H\gamma y\right)}{2} dy + \int_{-L}^{L} \rho_2 \frac{\left(H^2 - \frac{H^2}{4} - \gamma^2 y^2 - H\gamma y\right)}{2} dy =$$

$$= \left[\rho_1 \frac{H^2}{8} y + \rho_1 \frac{\gamma^2 y^3}{6} + \rho_1 \frac{H\gamma y^2}{4}\right]_{-L}^{L} + \left[\rho_2 \frac{3}{8} H^2 y - \rho_2 \frac{\gamma^2 y^3}{6} - \rho_2 \frac{H\gamma y^2}{4}\right]_{-L}^{L} =$$

$$= 2\rho_1 \frac{H^2}{8} L + 2\rho_1 \frac{\gamma^2 L^3}{6} + \cancel{\rho_1 \frac{H\gamma L^2}{4}} + 2\rho_2 \frac{3}{8} H^2 L - 2\rho_2 \frac{\gamma^2 L^3}{6} - \cancel{\rho_2 \frac{H\gamma L^2}{4}} =$$

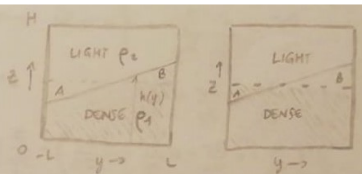
$$= \rho_1 \frac{H^2}{4} L + \rho_1 \frac{\gamma^2 L^3}{3} + \rho_2 \frac{3}{4} H^2 L - \rho_2 \frac{\gamma^2 L^3}{3} = \frac{H^2 L}{4}(\rho_1 + 3\rho_2) + \frac{1}{3}\gamma^2 L^3 (\rho_1 - \rho_2)$$

$$\Rightarrow PE = \rho_1 H^2 L \left(\frac{g}{4} + \frac{3}{4} g \frac{\rho_2}{\rho_1}\right) + \frac{1}{3}\gamma^2 L^3 \rho_1 \left(g - \frac{\rho_2}{\rho_1} g\right) =$$

$$= \rho_1 H^2 L \left(g - \frac{3}{4} g + \frac{3}{4} g \frac{\rho_2}{\rho_1}\right) + \frac{1}{3}\gamma^2 L^3 \rho_1 \left(\frac{\rho_1 - \rho_2}{\rho_1} g\right) =$$

$$= \rho_1 H^2 L \left(g - \frac{3}{4}\left(g - \frac{\rho_2}{\rho_1} g\right)\right) + \frac{1}{3}\gamma^2 L^3 \rho_1 g' = H^2 L \rho_1 \left(g - \frac{3}{4} g'\right) + \frac{1}{3}\rho_1 g' \gamma^2 L^3$$

$$\frac{PE_{\text{final state}}}{g} = \int_{-L}^{L} \rho_1 \int_0^{H/2} z\, dz\, dy + \int_{-L}^{L} \rho_2 \int_{\frac{1}{2}H}^{H} z\, dz\, dy =$$




$$= 2L\rho_1 \frac{H^2}{8} + 2L\rho_2 \left(\frac{H^2}{2} - \frac{H^2}{8}\right) =$$

$$= \rho_1 \frac{H^2}{4} L + \frac{3}{4}\rho_2 H^2 L = H^2 L \rho_1 \left(\frac{1}{4} + \frac{3}{4}\frac{\rho_2}{\rho_1}\right)$$

$$\Rightarrow PE_{\text{final state}} = H^2 L \rho_1 \left(\frac{g}{4} + \frac{3}{4}\frac{\rho_2}{\rho_1}\right) = H^2 L \rho_1 \left(g - \frac{3}{4} g \left(\frac{\rho_1 - \rho_2}{\rho_1}\right)\right) = H^2 L \rho_1 \left(g - \frac{3}{4} g'\right)$$

$$APE = PE_{\text{initial state}} - PE_{\text{final state}} = \frac{1}{3}\rho_1 g' \gamma^2 L^3$$

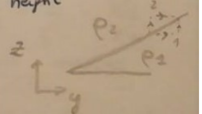
which is independent of the depth H because it is the amount of PE available to be converted to KE (available to drive motion). It is fundamental only the relation at the interface of the two fluid, not its height

b)

$$\tan\gamma \approx \gamma = \frac{dz}{dy}$$

approximation for small angles; definition of tangent

$$p_1 = p_2$$




$$\frac{\partial p_1}{\partial y} dy + \frac{\partial p_1}{\partial z} dz = \frac{\partial p_2}{\partial z} dz + \frac{\partial p_2}{\partial y} dy$$

$$\rightarrow \frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y} = \frac{dz}{dy}\left(\frac{\partial p_2}{\partial z} - \frac{\partial p_1}{\partial z}\right) \Rightarrow \frac{dz}{dy} = \frac{\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}}{\frac{\partial p_2}{\partial z} - \frac{\partial p_1}{\partial z}} = \frac{\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}}{g(\rho_1 - \rho_2)}$$

hydrostatic balance

$$u_g(p_2) - u_g(p_1) = -\frac{1}{f\rho_2}\frac{\partial p_2}{\partial y} + \frac{1}{f\rho_1}\frac{\partial p_1}{\partial y} = \frac{1}{f}\left(\frac{\partial p_1}{\partial y}\frac{1}{\rho_1} - \frac{\partial p_2}{\partial y}\frac{1}{\rho_2}\right)$$

assuming $\rho_1 \approx \rho_2 \longrightarrow \simeq \frac{1}{f}\frac{1}{\rho_1}\left(\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}\right) = \frac{1}{f}\frac{g(\rho_1 - \rho_2)}{\rho_1}\frac{dz}{dy} = \frac{1}{f} g' \gamma$ Margules relation lower layer at rest $\Rightarrow u_1 = 0,\ \dfrac{\partial p_1}{\partial y} = 0$

from the tangent definition $\dfrac{\partial^2 p}{\partial y^2} = \left(\dfrac{dz}{dy}\right)^2 g^2 (\rho_1-\rho_2)^2$

$$\Rightarrow KE = \int_{h(y)}^{H}\int_{-L}^{L} \frac{1}{2}\rho_2 u_2^2\, dy\, dz = \frac{1}{2}\rho_2 \int_{h(y)}^{H}\int_{-L}^{L} \frac{1}{f^2\rho_2^2}\frac{\partial^2 p}{\partial y^2}\, dy\, dz = \frac{1}{2}\frac{\rho_2}{f^2\rho_2^2}\int_{h(y)}^{H}\int_{-L}^{L} g^2(\rho_1-\rho_2)^2\left(\frac{dz}{dy}\right)^2 dy\, dz$$

zonal geostrophic velocity

$$= \frac{1}{2}\frac{\rho_2}{f^2} g'^2 \gamma^2 \int_{-L}^{L}\left(\frac{H}{2} - \gamma y\right) dy = \frac{1}{2}\rho_2 \frac{g'^2\gamma^2}{f^2}\left(2L\frac{H}{2} - \cancel{\gamma\frac{L^2}{2}} + \cancel{\gamma\frac{L^2}{2}}\right)$$

$$= \frac{1}{2}\rho_2 \frac{g'^2\gamma^2}{f^2} HL$$

c) $$\frac{APE}{KE} = \frac{\frac{1}{3}\rho_1 \cancel{g'}\cancel{\gamma^2} L^{\cancel{3}\,2}}{\frac{1}{2}\rho_2 \frac{g'^2\cancel{\gamma^2}}{f^2} H\cancel{L}} = \frac{2}{3}\frac{\rho_1}{\rho_2} L^2 \frac{1}{\frac{\sqrt{g'H}}{f}\frac{\sqrt{g'H}}{f}} = \frac{2}{3}\frac{\rho_1}{\rho_2}\frac{L^2}{L_R^2} \overset{\rho_1\approx\rho_2}{\simeq} \frac{2}{3}\frac{L^2}{L_R^2}$$

For the atmosphere circulation ($L \simeq 4000$ Km, $L_R = 700$ Km) $\dfrac{APE}{KE} \simeq 21.77\dfrac{\rho_1}{\rho_2} \simeq 21.77$ if assuming $\rho_1 \approx \rho_2$

d) hydrostatic balance and flat lower boundary

$$IE = c_v \cdot T \cdot m = c_v \int \rho T\, dV = \frac{c_v}{R}\int p\, dV \overset{=}{=} \frac{c_v}{R} g \int z\rho\, dV$$

$$IE + PE = TPE = \frac{c_v}{R} g \int z\rho\, dV + g\int z\rho\, dV = \frac{5}{2}\frac{\cancel{R}}{\cancel{R}} g \int z\rho\, dV + g\int z\rho\, dV = \frac{7}{2} g \int z\rho\, dV$$

total potential energy

$\rightarrow$ TPE is $\frac{7}{2}$ PE

## Exercise 2

(2)

a) $$\frac{Du}{Dt} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} - fv = 0 \longrightarrow \frac{\partial}{\partial y}\frac{Du}{Dt} + \cancel{\frac{1}{\rho_0}\frac{\partial}{\partial y}\frac{\partial p}{\partial x}} - \frac{\partial}{\partial y}(fv) = 0$$

$$\frac{Dv}{Dt} + \frac{1}{\rho_0}\frac{\partial p}{\partial y} + fu = 0 \longrightarrow \frac{\partial}{\partial x}\frac{Dv}{Dt} + \cancel{\frac{1}{\rho_0}\frac{\partial}{\partial x}\frac{\partial p}{\partial y}} + \frac{\partial}{\partial x}(fu) = 0$$

$\frac{\partial f}{\partial t} = 0$

$\Rightarrow$ I can add it

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \frac{\partial}{\partial x}\frac{Dv}{Dt} - \frac{\partial}{\partial y}\frac{Du}{Dt} + u\frac{\partial f}{\partial x} + f\frac{\partial u}{\partial x} + v\frac{\partial f}{\partial y} + f\frac{\partial v}{\partial y} = 0$$

$$\longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + f\left(\cancel{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}\right) = 0$$

$$\longrightarrow\left(\frac{\partial}{\partial t}\frac{\partial v}{\partial x} + u\frac{\partial}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial}{\partial y}\frac{\partial v}{\partial x} + w\frac{\partial}{\partial z}\frac{\partial v}{\partial x}\right) - \left(\frac{\partial}{\partial t}\frac{\partial u}{\partial y} + u\frac{\partial}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial}{\partial y}\frac{\partial u}{\partial y} + w\frac{\partial}{\partial z}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right) = 0$$

$$\Rightarrow \frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) = 0$$

b) $$\hat{z}\cdot\nabla\times\vec{u} = \hat{z}\cdot\left(\frac{\partial}{\partial x}\hat{i}, \frac{\partial}{\partial y}\hat{j}, \frac{\partial}{\partial z}\hat{k}\right)\times\left(u\hat{i}, v\hat{j}, w\hat{k}\right) =$$

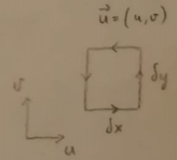
$$= \hat{z}\left(\frac{\partial v}{\partial x}\hat{k} - \frac{\partial w}{\partial x}\hat{j} - \frac{\partial u}{\partial y}\hat{k} + \frac{\partial w}{\partial y}\hat{i} + \frac{\partial u}{\partial z}\hat{j} - \frac{\partial v}{\partial z}\hat{i}\right) = \hat{z}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$ vertical component of the vorticity $\nabla\times\vec{u}$

$$\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f}_{\text{absolute vorticity}} = \underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\text{relative vorticity}} + \underbrace{f}_{\text{planetary vorticity}}$$

c) $$\text{circulation} = u\delta x + \left(v + \frac{\partial v}{\partial x}\right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y\right)\delta x - v\delta y$$

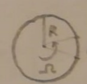
$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x\delta y$$




area enclosed $= \delta x \delta y$

$$\Longrightarrow \frac{\text{circulation}}{\text{area enclosed}} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Fluids in solid body rotation have no part in motion relative to any other part of the fluid. In this case we can consider a circle of radius R, angular velocity $\Omega$




$$\frac{\text{circulation}}{\text{area enclosed}} = \frac{\oint U\, dl}{\pi R^2} = \frac{\int_0^{2\pi}\Omega R\, d(\lambda R)}{\pi R^2} = \frac{2\pi R^2\Omega}{\pi R^2} = 2\Omega$$

d) $$\zeta = \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{f\rho}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{\rho f}\frac{\partial p}{\partial y}\right) = \frac{1}{\rho f}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right)$$

## Exercise 3

(3) $\langle T_e \rangle = 265\ K$

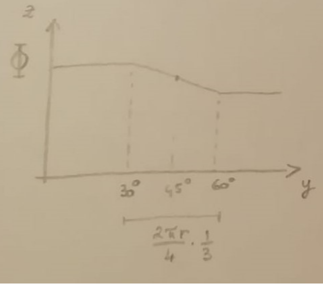
$\langle T_p \rangle = 235\ K$

$$\Phi_e - \cancel{\emptyset} = R \langle T_e \rangle \ln \frac{1000}{200} = 122406 \ \frac{m^2}{s^2}$$

$$\Phi_p - \cancel{\emptyset} = R \langle T_p \rangle \ln \frac{1000}{200} = 108548.54 \ \frac{m^2}{s^2}$$

$$\rightarrow \Phi_e - \Phi_p = 13857.46 \ \frac{m^2}{s^2}$$

$$\Delta z = \frac{13857.46}{g} = 1414 \ m = 1.4 \ Km$$




$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y} = -\frac{1}{10^{-4}} \frac{(-13857.46)}{\frac{2\pi r}{4} \cdot \frac{1}{3}} = 41.37 \ m/s$$

mean zonal geostrophic wind on the 200 hPa surface at 45° lat. in the winter hemisphere

## Exercise 4

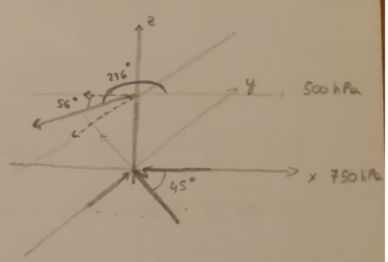
(4) $p_1 = 750\ hPa$

$p_2 = 500\ hPa$

$$\frac{\partial T}{\partial x} = -\frac{3K}{100\ Km} = -0.03\ K/Km$$

south east $\vec{V} = 20\ m/s$ at $45°$

$$\vec{V} = u_g \hat{\imath} + v_g \hat{\jmath} = -20 \cos 45\, \hat{\imath} + 20 \cos 45\, \hat{\jmath}$$
$$= -14.14\, \hat{\imath} + 14.14\, \hat{\jmath}$$




$$v_g(p_2) = v_g(p_1) + \frac{R}{f}\frac{\partial}{\partial x}\langle T \rangle \ln \frac{p_1}{p_2} =$$
$$= 14.14 - 34.9 = -20.76\ m/s$$

$$u_g(p_2) = u_g(p_1) - \cancel{\frac{R}{f}\frac{\partial}{\partial y}\langle T \rangle \ln \frac{p_1}{p_2}} = -14.14\ m/s$$

$$\vec{V}(p_2) = \sqrt{(-14.14)^2 + (-20.76)^2} = 25.12\ m/s$$

$$\theta = 180° + \arctan\left(\frac{-20.76}{-14.14}\right) \simeq 236°$$

$\rightarrow$ the geostrophic wind at 500 hPa is from the northeast