Computing the precession of the perihelion of Mercury with Fortran 90

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There are some problems in physics for which the solution is particularly hard to be analitically derived. However some of them can be easy to be solved numerically. Planets orbit can be one of those cases, especially if we want the relativistic correction to be considered.

Of particular interest is the case of Mercury as historically it has been part of the evidence in favour of general relativity. Specifically, Kepler's laws predict for Mercury to follow an elliptical orbit with the Sun in one of the two foci. Moreover the effect of the other planets in the Solar system make this ellipse to preceed with time. This means that the semi-major axes changes angle with time.

In the seventeenth century, the value of Mercury's perihelion precession was already well known with an observational value of about 566 arcseconds/century. Further calculation prooved that the effect of the other planets (mainly Jupiter) can account for a precession of 523 arcseconds/century. This imply a mismatch of 43 arcsecond/century that need to be understood.

Fortunately enough, in 1917 Einstein came with his General Relativity (GR) and with a prediction for Mercury's precession of exactly 43 arcseconds/century! The details of GR's equations are not trivial but if we want to obtain this prediction numerically we just need to know how to correct the force of gravity[1], that is:

$$F_G \approx \frac{GM_S M_M}{r^2}\left(1+\frac{\alpha}{r^2}\right) \tag{1}$$

where M_S and M_M are the mass of the Sun and Mercury respectively, $G = 6.67\times 10^{-11}\,\frac{m^3}{kg\,s^2}$ is the gravitational constant and the correction α, that goes like $\propto r^{-4}$, has a value of $\alpha \approx 1.1\times 10^{-8} AU^2$.

From dynamics, we know that the force is related to the acceleration of a body (let's say Mercury) like this:

$$F = M_M \times a \tag{2}$$

Hence, the equations of motion can be derived just equating (1) and (2) and considering the attractive nature of gravity we add a minus sign:

$$\ddot{\vec{x}} = -\frac{GM_S}{r^2}\left(1+\frac{\alpha}{r^2}\right) \tag{3}$$

Now, if we want to extend the problem in a bidimensional space, we just need to consider the projection of the acceleration on the two axes:

$$\begin{cases} \ddot{x} = -\frac{GM_S}{r^2}\left(1+\frac{\alpha}{r^2}\right)\cos\vartheta \\ \ddot{y} = -\frac{GM_S}{r^2}\left(1+\frac{\alpha}{r^2}\right)\sin\vartheta \end{cases} \tag{4}$$

and using $\cos\vartheta = x/r$ and $\sin\vartheta = y/r$, we obtain:

$$\begin{cases} \ddot{x} = -\frac{GM_S}{r^3}\left(1+\frac{\alpha}{r^2}\right) x \\ \ddot{y} = -\frac{GM_S}{r^3}\left(1+\frac{\alpha}{r^2}\right) y \end{cases} \tag{5}$$

[1] For the case of Mercury it's enough to stop at the first order correction as in (1).

Now we have a system of two differential equation of the second order but we can use the following change of variables to make it a system of four differential equation of the first order:

$$\begin{cases} \dot{x} = v_x \\ \dot{v}_x = \ddot{x} \end{cases} \quad \begin{cases} \dot{y} = v_y \\ \dot{v}_y = \ddot{y} \end{cases} \tag{6}$$

$$\Rightarrow \begin{cases} \dot{v}_x = -\frac{GM_S\,x}{r^3}\left(1+\frac{\alpha}{r^2}\right) \\ \dot{x} = v_x \\ \dot{v}_y = -\frac{GM_S\,y}{r^3}\left(1+\frac{\alpha}{r^2}\right) \\ \dot{y} = v_y \end{cases} \tag{7}$$

We have reduced the system to 4 first order differential equations. This means that to make the system works we need to supply 4 initial conditions (i.e. $x_0, y_0, v_{x,0}, v_{y,0}$).

If we move to the usual notation for derivatives, it's easier to visualise what's going on in terms of differential variations:

$$\begin{cases} \frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{GM_S\,x}{r^3}\left(1+\frac{\alpha}{r^2}\right) \\ \frac{\mathrm{d}x}{\mathrm{d}t} = v_x \\ \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{GM_S\,y}{r^3}\left(1+\frac{\alpha}{r^2}\right) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = v_y \end{cases} \Rightarrow \begin{cases} \mathrm{d}v_x \; = -\frac{GM_S\,x}{r^3}\left(1+\frac{\alpha}{r^2}\right)\mathrm{d}t \\ \mathrm{d}x \; = v_x\,\mathrm{d}t \\ \mathrm{d}v_y \; = -\frac{GM_S\,y}{r^3}\left(1+\frac{\alpha}{r^2}\right)\mathrm{d}t \\ \mathrm{d}y \; = v_y\,\mathrm{d}t \end{cases} \tag{8}$$

In principle all of the differential quantities must tend to zero but we discretise them (e.g $\mathrm{d}v_x \;\; \rightarrow \Delta v_x = v_{x,f} - v_{x,i}$) so that we can obtain a numerical solution. This means we will have a certain precision increasing for smaller interval values.

$$\begin{cases} \Delta v_x = -\frac{GM_S\,x}{r^3}\left(1+\frac{\alpha}{r^2}\right)\Delta t \\ \Delta x = v_x \cdot \Delta t \\ \Delta v_y = -\frac{GM_S\,y}{r^3}\left(1+\frac{\alpha}{r^2}\right)\Delta t \\ \Delta y = v_y \cdot \Delta t \end{cases} \Rightarrow \begin{cases} v_{x,i+1} - v_{x,i} = -\frac{GM_S\,x}{r^3}\left(1+\frac{\alpha}{r^2}\right)\Delta t \\ x_{i+1} - x_i = v_x \cdot \Delta t \\ v_{y,i+1} - v_{y,i} = -\frac{GM_S\,y}{r^3}\left(1+\frac{\alpha}{r^2}\right)\Delta t \\ y_{i+1} - y_i = v_y \cdot \Delta t \end{cases} \tag{9}$$

The only choice we need to do is which values of velocity we want to use in the right-hand side of the equations: we can either use the old velocities v_i or the new velocities v_{i+1}. This is the only difference that classifies respectively the Euler's algorithm and the **Euler-Cromer's algorithm**. We decide to use the latter approach as it has better performance in the conservation of energy and this reveals crucial in the computation of planetary orbits. Thus, the equations that are going to feed our computer are:

$$\boxed{\begin{cases} v_{x,i+1} = v_{x,i} - \frac{GM_S\,x}{r^3}\left(1+\frac{\alpha}{r^2}\right)\Delta t \\ x_{i+1} = x_i + v_{x,i+1} \cdot \Delta t \\ v_{y,i+1} = v_{y,i} - \frac{GM_S\,y}{r^3}\left(1+\frac{\alpha}{r^2}\right)\Delta t \\ y_{i+1} = y_i + v_{y,i+1} \cdot \Delta t \end{cases}} \tag{10}$$

where Δt is an aribtrary small constant value that will determine the precision of our computation.

We must be careful to the units we want to use. It turns out to be usefull using Astronomical Units (AU) for distances (AU $\approx 1.5 \times 10^{11} m$) and years for time (1 yr $= 3.2 \times 10^7 s$). In this way, the obiquitous quantity GM_S becomes simply $GM_S = 4\pi^2 \mathrm{AU}^3/\mathrm{yr}^2$. This approximation seems magic but it simply comes from the fact that the Earth's velocity is $(2\pi r)/(1yr)$ and the Earth orbit is nearly circular so that we can equate the force of gravity to the centripetal force.

All of this stuff is included in the following Fortran lines:

```
!time from the beginning of the orbit
t = t + dt
!first update velocities
v_x = v_x - ((4.*Pi**2.*x*dt)/r**3.)*(1.+alpha/r**2.)
v_y = v_y - ((4.*Pi**2.*y*dt)/r**3.)*(1.+alpha/r**2.)
!and then update positions
x = x + v_x*dt
```

```
y = y + v_y*dt
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Some effort must be done to select the correct initial condition to put Mercury in its own elliptical orbit around the Sun. Luckily, we can use the conservation of kinetic and potential energy:

$$-\frac{GM_S M_M}{r_1} + \frac{1}{2} M_M v_1^2 = -\frac{GM_S M_M}{r_2} + \frac{1}{2} M_M v_2^2 \tag{11}$$

and since the force of gravity is a central force, we have also the conservation of angular momentum:

$$\vec{r}_1 \times \vec{v}_1 = \vec{r}_2 \times \vec{v}_2 \Rightarrow b_1\, v_1 = b_2\, v_2 \tag{12}$$

where b is the minor axes. This is easily to visualise if we choose r_1 to be the farther position from the sun and to make it coincides with the x axes ($y = 0$). In this case $r_1 = (1 + e)a$, whith e being the eccentricity of the orbit and a the semi-major axes. Since someone has already observed these values for the orbit of Mercury, $e = 0.206$ and $a = 0.39\text{AU}$, we also know $r_1 = 0.47\text{AU}$. This can be our **initial condition for the position** if we set:

$$\boxed{\begin{cases} x_0 = 0.47 \text{ AU} \\ y_0 = 0 \end{cases}} \tag{13}$$

The velocity now can be derived combining (11) and (12), obtaining:

$$v_1 = \sqrt{2GM_S \left[\frac{b^2}{a^2(1+e)^2 - b^2}\right] \left[\frac{1}{\sqrt{e^2a^2 + b^2}} - \frac{1}{a + ea}\right]} = \sqrt{\frac{GM_S(1-e)}{a(1+e)}} = 8.2 \text{ AU/yr} \tag{14}$$

At the perihelion the radial velocity must be null so that the entire modulus of the velocity must be tangent. Hence, according to the initial position for the position we have the following **initial condition for the velocity**:

$$\boxed{\begin{cases} v_{x,0} = 0 \\ v_{y,0} = 8.2 \text{ AU/yr} \end{cases}} \tag{15}$$

The only thing left before running our code is to choose a value for α. Unfortunately the actual value predicted by Einstein ($\alpha = 1.1 \times 10^{-8}\text{AU}^2$) produces too small effects to be directly detected in a computer simulation. However we can run the code several times for greater values of α and then extrapolate the value of the precession for the Einstein's value. An example of the orbits with those initial conditions and $\alpha = 0.005\text{AU}^2$ is given in Figure 1.

The position of the perihelion can be estimated studying the maxima of the radius analysed as a function of time (Figure 2). We can then compute the angle of the perihelion[2]:

$$\vartheta_p = \arccos(x/r) \tag{16}$$

and when we have these values after some orbits we can plot them as a function of time and realise that they can be interpolated by a straight line (Figure 3).

The slope of the (t, ϑ) relation is the rate of precession and is the quantity we are interested in. As already anticipated, we now need to obtain this slope for multiple values of alpha. That is what is shown in Figure 4. As we see from the figure, it has been possible to perform a linear fit and I did it using the least square method (implemented in the code). However we must be careful because the point we want to extrapolate ($\alpha = 1.1 \times 10^{-8}\text{AU}^2$) is really close to zero and it suffers to any small deviation due to numerical errors introduced in the simulation. Since we know that with $\alpha = 0$ there is no precession of the orbit, we can constrain the linear fit to pass through the origin. That is the meaning of the green line in Figure 4. With that constrain we can compute the prediction of the fit for the Einstein's α and using the data shown in Figure 4, I find a value of:

$$\boxed{\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}t}\right|_{\text{Mercury}} = 44.42 \text{ arcsec/century}} \tag{17}$$

in agreement with the ≈ 43 arcsec / century expected analitically[3].

[2] Alternatively we can also use the $\vartheta_p = \arctan(y/x)$ or $\vartheta_p = \arcsin(y/r)$.

[3] Note that the Fortran 90 code and the jupyter notebook I used to create the plot is available on https://github.com/giorgiomanzoni/mercurio

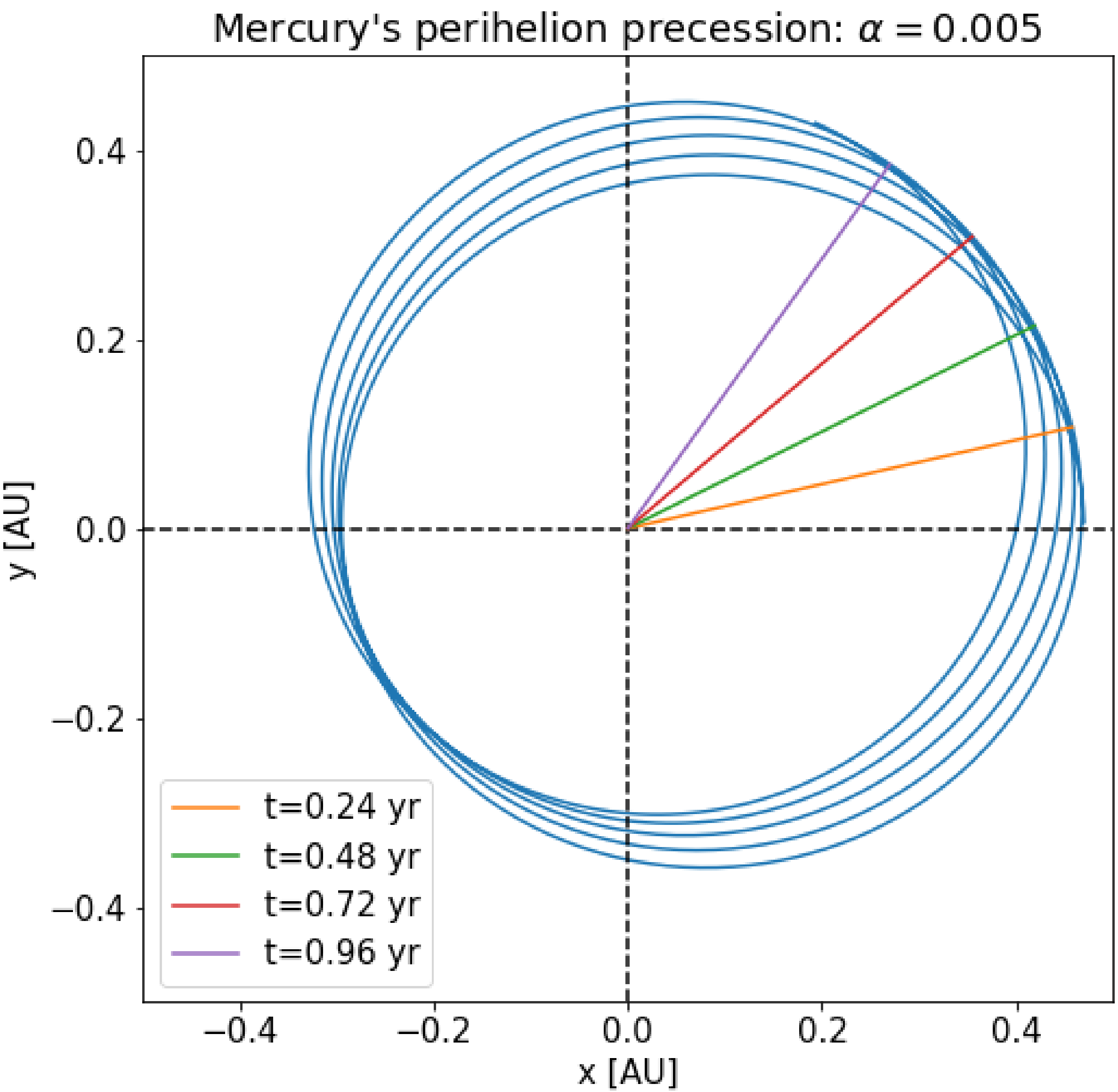

Figure 1: Orbit of mercury with initial condition as given by 13 and 15 using a time step $\Delta t = 0.0008$ yr and $\alpha = 0.005\mathrm{AU}^2$. The position of the perihelion starts laying on the x axes and then preceed as shown by the colour lines.

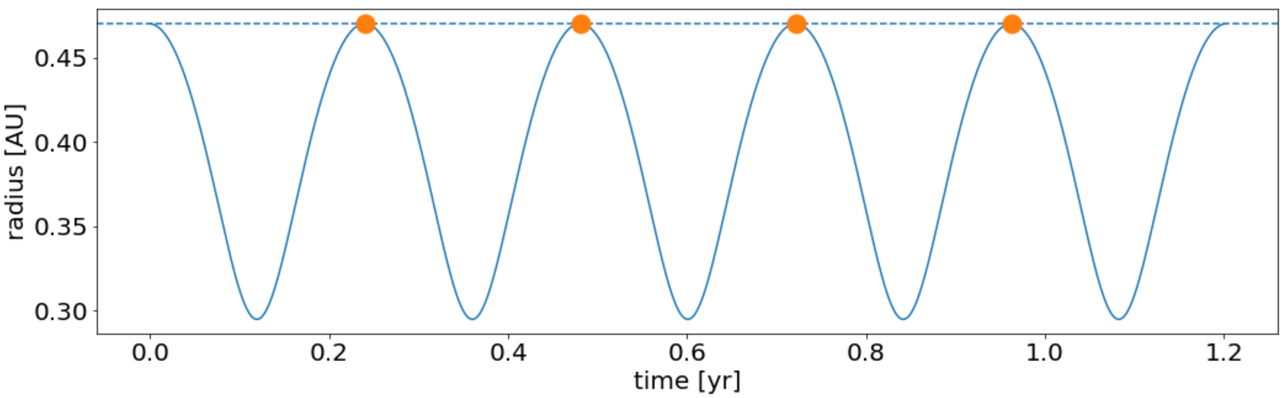

Figure 2: Radius of the position of Mercury as a function of time. The maxima coincide with the perihelion and can be used to study the precession of the orbit.

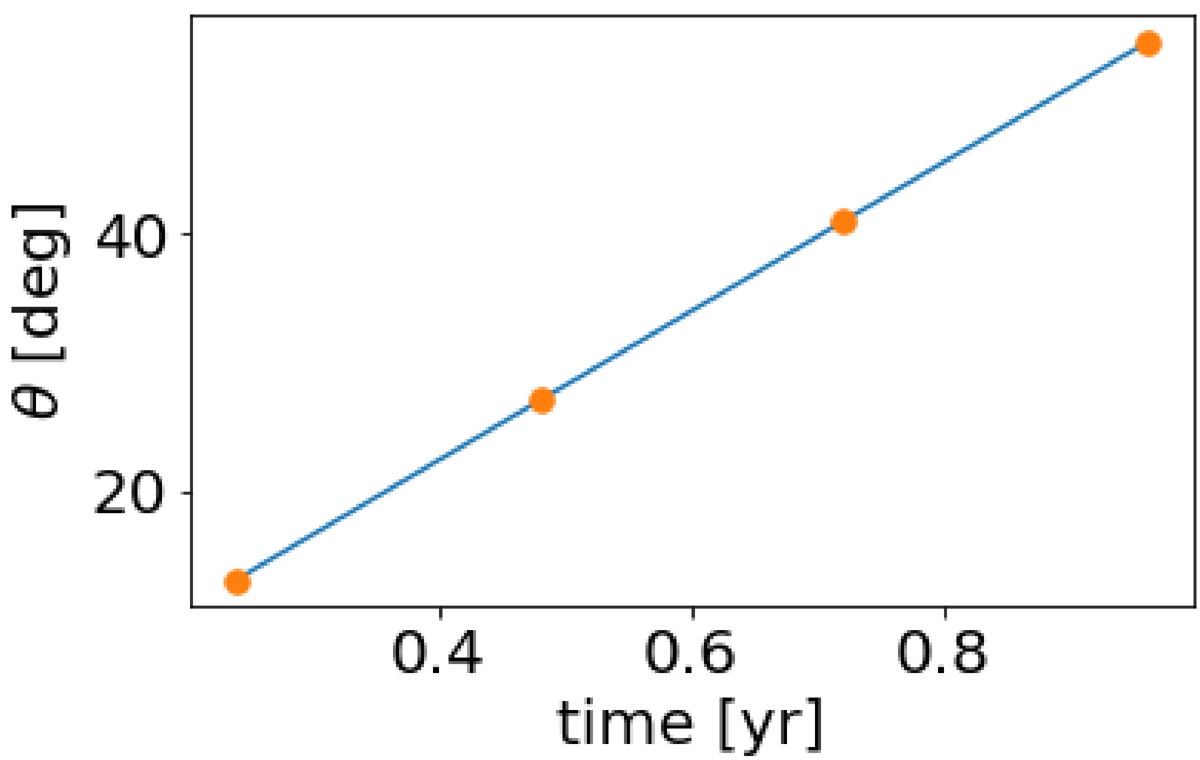

Figure 3: Position of the perihelion as a function of time for the example shown in the previous figures ($\alpha = 0.005$). In this partiocular case the slope is 57.86 deg/yr and the intercept -0.75 deg.

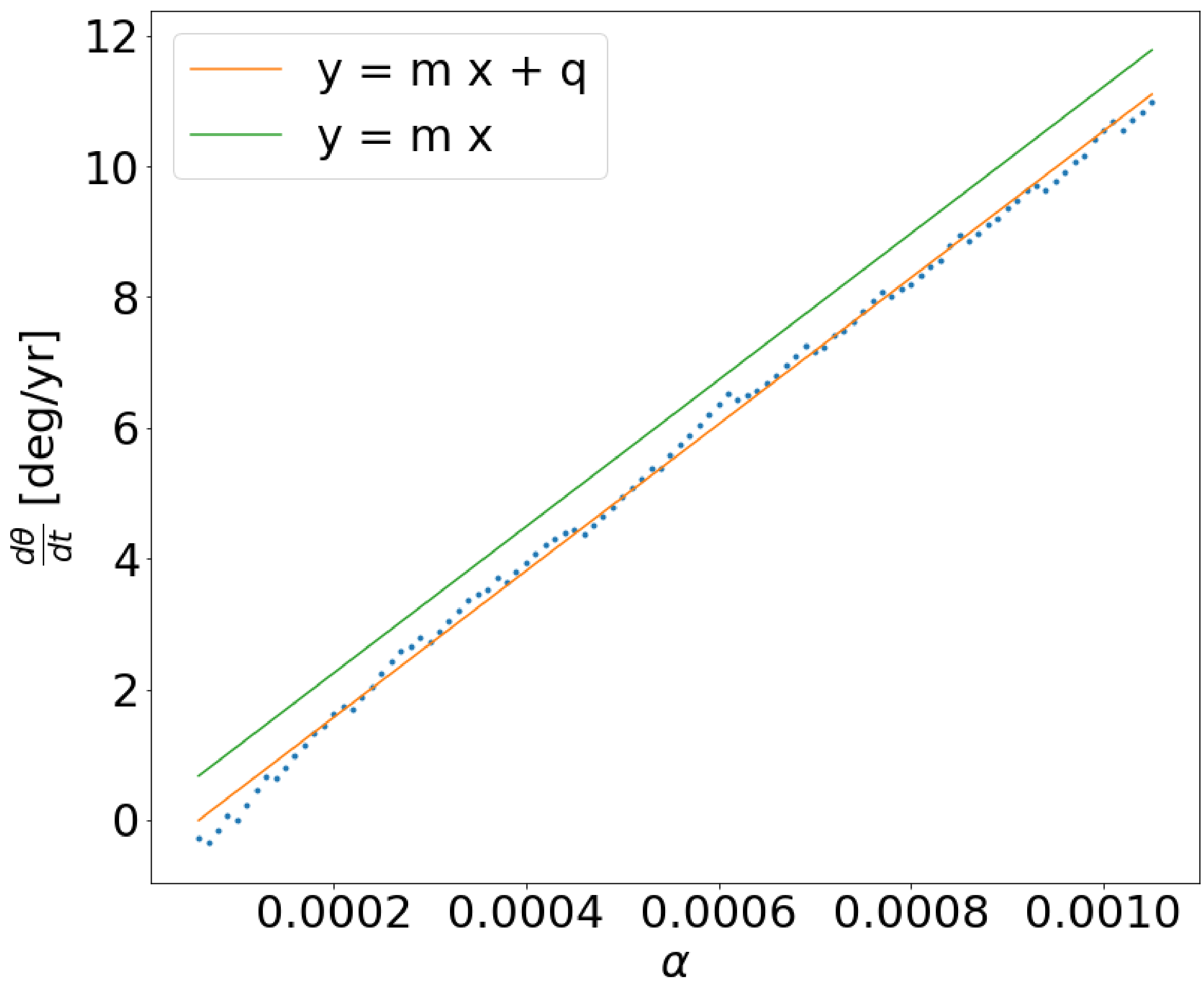

Figure 4: Precession rate as a function of α. Least square linear fit with (green line) and without (orange line) the constraint in the origin.