ROBUST CONTROL VIA INTEGRAL ACTION

It is a known fact, also from basic control courses, that if the loop function has a pole in the origin then the regulated error converges to zero robustly.

As we shall see here this is true also for (a class of) nonlinear systems

STANDING ASSUMPTION: The error $\tilde{y}_r = y_r - y_r^*$ between the regulated output y_r and its target value y_r^* is measured

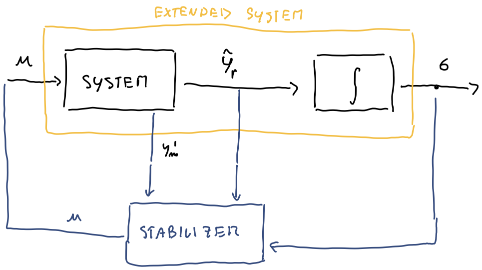
$$\hookrightarrow \quad y_m = \begin{pmatrix} \tilde{y}_r \\ y_m' \end{pmatrix} \qquad \text{for some } y_m'$$

Moreover, the number of inputs equals that of regulated outputs: $\boxed{n_u = n_r}$

GENERAL PARADIGM: we add an integrator (POLE IN THE ORIGIN) in series to the controlled system processing the error $\tilde{y}_r$ (in the multi-variable case it's one integrator for each component of $\tilde{y}_r$).

This produces an "EXTENDED SYSTEM".

Then, we stabilize the extended system



BIBLIOGRAFY: H. KHALIL, NONLINEAR SYSTEMS, Chap. 12.3

PRELIMINARY MATERIAL

Let us start considering the following SERIES interconnection:

$$u \longrightarrow \boxed{\begin{matrix}\dot{x} = Ax + Bu \\ y = Cx\end{matrix}} \xrightarrow{\;y\;} \boxed{\dot{\sigma} = y} \xrightarrow{\;\sigma\;}$$

QUESTION: when is this series stabilizable from u ?

The series satisfies:

$$\begin{pmatrix} \dot{x} \\ \dot{\sigma} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ \sigma \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \qquad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ \sigma \in \mathbb{R}^p)$$

RESULT. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ be such that (A, B) is stabilizable. Then

$$\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} B \\ 0 \end{pmatrix}$$

is stabilizable if

$$\operatorname{rank}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \#\text{ rows} = n + p \qquad \text{(NON-RESONANCE CONDITION)}$$

(proof from MODULE 1)

RESULT. Let $K = [K_1 \;\; K_2]$ $\quad (K_1 \in \mathbb{R}^{m \times n},\ K_2 \in \mathbb{R}^{m \times p})$ be such that

$$\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} K = \begin{pmatrix} A + BK_1 & BK_2 \\ C & 0 \end{pmatrix}$$

is Hurwitz. Then, K_2 is invertible.

Proof.

Suppose, by contradiction that K_2 is singular. Then

$$\exists\, w \in \mathbb{R}^p / \{0\},\ K_2 w = 0.$$

But this implies

$$\begin{bmatrix} A + BK_1 & BK_2 \\ C & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} BK_2 w \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ w \end{bmatrix}$$

$\Rightarrow$ 0 is an eigenvalue of $\begin{pmatrix} A + BK_1 & BK_2 \\ C & 0 \end{pmatrix}$, a contradiction.

STATE - FEEDBACK CASE : $\boxed{y'_m = x}$

RESULT. Given y_r^*, let (x^*, u^*) be a solution to the SOLVABILITY Eqs

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*, u^*) \end{cases}$$

and let $A = \frac{\partial f}{\partial x}(x^*, u^*)$, $B = \frac{\partial f}{\partial u}(x^*, u^*)$, $C_r = \frac{\partial h_r}{\partial x}(x^*)$. Suppose that (A, B) is stabilizable and

$$\operatorname{rank}\begin{pmatrix} A & B \\ C_r & 0 \end{pmatrix} = n_x + n_r$$

and let $K = [K_1 \;\; K_2]$ with $K_1 \in \mathbb{R}^{n_u \times n_x}$, $K_2 \in \mathbb{R}^{n_u \times n_r}$ be such that

$$\begin{pmatrix} A & 0 \\ C_r & 0 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} K = \begin{pmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{pmatrix}$$

is Hurwitz (always possible to find such K due to the NON-RESONANCE CONDITION).

Then, for every estimate $\hat{u}^*$ and $\hat{x}^*$ of u^* and x^* (even $\hat{u}^* = 0$, $\hat{x}^* = 0$) the controller

$$\begin{cases} \dot{\sigma} = \tilde{y}_r \\ u = \hat{u}^* + K_1(x - \hat{x}^*) + K_2 \sigma \end{cases}$$

has the form

$$\begin{cases} \dot{\xi} = g(\xi, y_m) \\ u = \gamma(\xi, y_m) \end{cases}$$

is such that there exists $\sigma^* \in \mathbb{R}^{n_r}$ s.t. the equilibrium (x^*, σ^*) is LAS for the closed-loop system.

PROOF.

We can write (see previous classes)

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} + f_{HOT}(x, u) \qquad (\tilde{x} := x - x^*, \;\; \tilde{u} = u - u^*)$$

$$\tilde{y}_r = C_r \tilde{x} + h_{r,HOT}(x, u)$$

Taking

$$\begin{cases} u = \hat{u}^* + K_1(x - \hat{x}^*) + K_2 \sigma \\ \dot{\sigma} = \tilde{y}_r = C_r \tilde{x} + h_{r,HOT}(x, u) \end{cases}$$

leads to

$$\dot{\tilde{x}} = A\tilde{x} + B\left(\hat{u}^* + K_1(x-\hat{x}^*) + K_2\sigma - u^*\right) + f_{HOT}(x,u)$$

$$\overset{!}{=} A\tilde{x} + BK_1\left(x \pm x^* - \hat{x}^*\right) + BK_2\sigma + B(\hat{u}^* - u^*) + f_{HOT}(x,u)$$

$$= (A+BK_1)\tilde{x} + BK_2\sigma + B\left[K_1(x^* - \hat{x}^*) + \hat{u}^* - u^*\right] + f_{HOT}(x,u)$$

$$\dot{\sigma} = C_r\tilde{x} + h_{r,HOT}(x,u)$$

Define

$$\sigma^* = K_2^{-1}\cdot\left[u^* - \hat{u}^* - K_1(x^* - \hat{x}^*)\right] \qquad (K_2 \text{ is invertible, see above})$$

and change coordinates

$$\sigma \longmapsto \tilde{\sigma} = \sigma - \sigma^* \qquad (\sigma = \tilde{\sigma} + \sigma^*)$$

Then, we obtain:

$$\dot{\tilde{x}} = (A+BK_1)\tilde{x} + BK_2\tilde{\sigma} + BK_2\sigma^* + B\underbrace{\left(K_1(x^* - \hat{x}^*) + \hat{u}^* - u^*\right)}_{= -K_2\sigma^*} + f_{HOT}(x,u)$$

$$\overset{!}{=} (A+BK_1)\tilde{x} + BK_2\tilde{\sigma} + f_{HOT}(x,u)$$

$$\dot{\tilde{\sigma}} = \dot{\sigma} - \underset{0}{\underset{\shortparallel}{\dot{\sigma}^*}} = \dot{\sigma} = C_r\tilde{x} + h_{r,HOT}(x,u)$$

Grouping the equations yields:

$$\begin{pmatrix}\dot{\tilde{x}}\\ \dot{\tilde{\sigma}}\end{pmatrix} = \underbrace{\begin{pmatrix}A+BK_1 & BK_2\\ C_r & 0\end{pmatrix}}_{\text{Hurwitz linear part}}\begin{pmatrix}\tilde{x}\\ \tilde{\sigma}\end{pmatrix} + \underbrace{\begin{pmatrix}f_{HOT}(x,u)\\ h_{r,HOT}(x,u)\end{pmatrix}}_{\text{H.O.T}}$$

$$\Downarrow$$

The result follows from Lyapunov's indirect method.

REMARKS

1) The control law works for every $\hat{u}^*, \hat{x}^*$ (also $\hat{u}^* = 0$ and $\hat{x}^* = 0$)

2) The equilibrium σ^* of σ is

$$\sigma^* = K_2^{-1} \cdot \left[u^* - \hat{u}^* - K_1 \left(x^* - \hat{x}^* \right) \right]$$

Thus, repeating the computations of the proof, we get

$$u = \hat{u}^* + K_1 (x - \hat{x}^*) + K_2 \sigma \qquad \text{(by definition)}$$

$$= u^* + K_1 \tilde{x} + K_2 \tilde{\sigma} \qquad \text{(using } \sigma^* \text{)}$$

← control law we had implemented if we knew x^* and u^*

⇒ $\sigma(t)$ converges to the value σ^* making up for our mistakes in estimating x^* and u^*

↳ related to ITERATIVE LEARNING when implemented in discrete time

3) The closed-loop system is

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\sigma}} \end{pmatrix} = \begin{pmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\sigma} \end{pmatrix} + \begin{pmatrix} f_{HOT}\left(x, u^* + K_1 \tilde{x} + K_2 \tilde{\sigma}\right) \\ h_{r,HOT}(x) \end{pmatrix}$$

Compared to the state-feedback law $u = u^* + K_1 \tilde{x}$ with perfect knowledge of u^* and x^*, here we have an additional term $K_2 \tilde{\sigma}$

↓

Transitories linked to σ (poor knowledge of u^* and x^*) may reduce the domain of attraction

4) The gains K_1 and K_2 may still depend on x^* and u^* as they are tuned on A, B, C_r

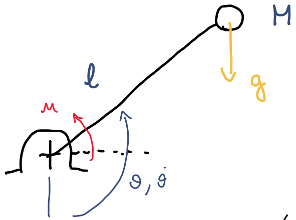
↳ Less critical as K_1 and K_2 must only ensure closed-loop stability and:

1) Stability is robust: small uncertainties in A and B are tollerated

2) Often the robustness margins are high

see examples

EXAMPLE: ACTUATED PENDULUM



GOAL: stabilize the pendulum to a given position $y_r^* = \theta^*$

pendulum equations: $\left(x_1 = \theta \;,\; x_2 = \dot\theta \right)$

$$
\begin{cases}
\dot x_1 = x_2 \\
\dot x_2 = -\dfrac{g}{\ell}\sin x_1 - \dfrac{\beta}{M\ell^2} x_2 + \dfrac{1}{M\ell^2} u \\
y_m = x \\
y_r = x_1
\end{cases}
$$

The control law found before (see previous notes) was

$$
u(t) = \underbrace{M g \ell \sin\theta^*}_{u^*} + K\left[x(t) - \underbrace{\begin{pmatrix} \theta^* \\ 0 \end{pmatrix}}_{x^*} \right]
$$

(uncertain)

The robustified controller is (pick $\hat u^* = 0$ and $\hat x^* = x^*$ as we know θ^*)

$$
\begin{cases}
\dot\sigma = x_1 - \theta^* \\
u = K_1 (x - \hat x^*) + K_2 \sigma
\end{cases}
$$

(NO EXPLICIT DEPENDENCY FROM u^*)

In which K_1 and K_2 must be such that

$$
\begin{pmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{pmatrix}, \quad
A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos\theta^* & -\frac{\beta}{M\ell^2} \end{pmatrix}, \quad
B = \begin{pmatrix} 0 \\ \frac{1}{M\ell^2} \end{pmatrix}, \quad
C_r = \begin{pmatrix} 1 & 0 \end{pmatrix}
$$

is Hurwitz

Let see what values of K_1 and K_2 work:

$$A_{CL} \doteq \begin{pmatrix} A+BK_1 & BK \\ C_r & 0 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{g}{\ell}\cos\theta^* + \frac{K_{11}}{Me^2} & -\frac{\beta}{Me^2} + \frac{K_{12}}{Me^2} & \frac{K_2}{Me^2} \\ 1 & 0 & 0 \end{bmatrix}$$

Let us change bles using

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \longrightarrow \left[T \begin{pmatrix} x \\ \sigma \end{pmatrix} = \begin{pmatrix} \sigma \\ x \end{pmatrix} \right.$$

$\downarrow$

we just swapped σ and x

Then

$$T A_{CL} T^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{K_2}{Me^2} & -\frac{g}{\ell}\cos\theta^* + \frac{K_{11}}{Me^2} & \frac{K_{12}-\beta}{Me^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix}$$

$$\begin{cases} \alpha_0 = -\frac{K_2}{Me^2} \\ \alpha_1 = \frac{g}{\ell}\cos\theta^* - \frac{K_{11}}{Me^2} \\ \alpha_2 = \frac{\beta - K_{12}}{Me^2} \end{cases}$$

Recall from Module 1 that

$$\varphi(A_{CL}) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \lambda^3$$

Let's find conditions on K_1 and K_2 ensuring that A_{CL} is Hurwitz in a robust way:

RESULT (ROUTH - HURWITZ CRITERION) the eigenvalues of A_{cl} have all negative real part

iff $\alpha_0, \alpha_1, \alpha_2 > 0$ and $\alpha_1\alpha_2 > \alpha_0$

↳ In our case, these conditions are equivalent to

$$\begin{cases} K_2 < 0 & \text{(I)} \\ K_{11} < Mlg\cos\theta^* & \text{(II)} \\ K_{12} < \beta \quad \leftarrow \text{always true if } K_{12} < 0 & \text{(III)} \\ \left(\frac{g}{l}\cos\theta^* - \frac{K_{11}}{Ml^2}\right)\left(\frac{\beta - K_{12}}{Ml^2}\right) > -\frac{K_2}{Ml^2} & \text{(IV)} \end{cases}$$

RESULT Fix $K_2 < 0$ and $K_{12} < 0$, and then choose $K_{11} < 0$ so that

$$|K_{11}| > Mgl + \frac{|K_2|}{|K_{12}|} Ml^2$$

(we do not know Mgl but we can easily find an upperbound)

Then, (I) - (IV) are satisfied

If $c > 0$ is such that $c > Mgl$ then the condition can be substituted by $|K_{11}| > c + \frac{|K_2|}{|K_{12}|}$

PROOF.

Conditions (I) and (III) are obviously true since K_{12} and K_2 have been chosen negative.

To prove (II) note that:

$$|K_{11}| > Mgl + \frac{|K_2|}{|K_{12}|} Ml^2 \Rightarrow K_{11} = -|K_{11}| < -Mgl \leq Mgl\cos\theta^* \leftarrow \text{since } \cos\theta^* \geq -1$$

Finally, we have

$$-K_{11} = |K_{11}| > Mgl + \frac{\overset{-K_2}{\overset{\shortparallel}{|K_2|}}}{\underset{\underset{-K_{12}}{\shortparallel}}{|K_{12}|}} Ml^2 \geq -Mgl\cos\theta^* + \frac{-K_2}{\beta - K_{12}} \cdot Ml^2 \Rightarrow \text{(IV)}$$

since $1 \geq -\cos\theta^*$

since $\beta - K_{12} \geq -K_{12}$

$$\Rightarrow \frac{1}{-K_{12}} \geq \frac{1}{\beta - K_{12}}$$

REMARKS

1. In this case, closed-loop stability can be guaranteed ROBUSTLY with arbitrarily large margin

   ↳ Indeed it is enough choose $K_{12}, K_2, K_{11} < 0$ such that $|K_{11}|$ is sufficiently large

   ⇒ The integral control law is ROBUST

2. The steady-state equilibrium of σ is

$$\sigma^* = \frac{1}{K_2} u^* = \frac{1}{K_2} Mg\ell \sin\theta^*$$

   ⇒ the integral action $K_2\sigma$ asymptotically gives the ideal feedforward control action needed to keep the pendulum in the right position

   → Such an action basically compensates the gravitational torque

   ↳ In robotics this relates to the "gravity-compensation" terms

3. By defining the "regulation error"

$$e(t) \doteq \vartheta(t) - \theta^* \qquad \left(= x_1(t) - \theta^*\right)$$

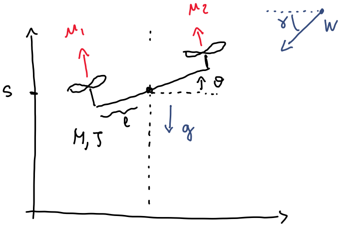
and noting that $\dot{e}(t) = \dot{\theta}(t) = x_2(t)$, the control law reads (let $\sigma(0)=0$)

$$\begin{cases} \dot{\sigma}(t) = x_1(t) - \theta^* = e(t) \\ u(t) = K_1(x(t) - x^*) + k_2\sigma(t) = K_{11}e(t) + K_{12}\dot{e}(t) + K_2\sigma(t) \end{cases}$$

$$\Rightarrow \quad u(t) = K_P e(t) + K_I \int_0^t e(s)\,ds + K_D \dot{e}(t) \qquad \left(K_P \doteq K_{11},\ K_I \doteq K_2,\ K_D \doteq K_{12}\right)$$

which is the equation of a $\boxed{\text{PID controller}}$

EXAMPLE: PLANAR DRONE



MODEL: (see previous notes)

$$\begin{cases} \dot{x} = \frac{1}{M}\cos x_2 \cdot (u_1 + u_2) - g - \frac{W}{M}\sin\gamma - \frac{\mu}{M} x_1 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\frac{\beta}{J} x_3 + \frac{\ell}{J}(u_2 - u_1) \end{cases}$$

GOAL: drive the vertical velocity $y_r \doteq x_1$ to $y_r^* = 0$

Previous control law (with $x_2^* = 0$)

$$u(t) = u^* + K(x(t) - x^*) = \begin{pmatrix} \frac{Mg + w\sin\gamma}{2} \\ \frac{Mg + w\sin\gamma}{2} \end{pmatrix} + K\,x(t)$$

HIGHLY UNCERTAIN

The integral control law is

$$\begin{cases} \dot{\sigma} = x_1 \\ u = K_1 x + K_2 \sigma \end{cases}$$

In which K_1 and K_2 are such that the following matrix is Hurwitz

$$A_{CL} \doteq \begin{pmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{pmatrix} = \begin{bmatrix} \frac{K_{11}^1 + K_{21}^1 - \mu}{M} & \frac{K_{12}^1 + K_{22}^1}{M} & \frac{K_{13}^1 + K_{23}^1}{M} & \frac{K_1^2 + K_2^2}{M} \\ 0 & 0 & 1 & 0 \\ \frac{\ell}{J}(K_{21}^1 - K_{11}^1) & \frac{\ell}{J}(K_{22}^1 - K_{12}^1) & \frac{\ell}{J}(K_{23}^1 - K_{13}^1) - \frac{\beta}{J} & \frac{\ell}{J}(K_2^2 - K_1^2) \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where $K_1 = \begin{pmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \end{pmatrix}$ $\quad K_2 = \begin{pmatrix} K_1^2 \\ K_2^2 \end{pmatrix}$

An example of a robust choice for K_1 and K_2 is

$$K_1 = -\frac{1}{2}\begin{pmatrix} \gamma_1 & -\gamma_2 & -\gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}, \quad K_2 = -\frac{1}{2}\begin{pmatrix} \gamma_4 \\ \gamma_4 \end{pmatrix} \quad \text{for } \underline{\underline{\text{any}}} \quad \gamma_1 > 0,\ \gamma_2 > 0,\ \gamma_3 > 0,\ \gamma_4 > 0$$

Indeed with this choice we have

$$A_{CL} = \begin{bmatrix} -\frac{\gamma_1+\mu}{M} & 0 & 0 & -\frac{\gamma_4}{M} \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma_2\frac{\ell}{J} & -\frac{\beta+\gamma_3\ell}{J} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Transforming A_{CL} with

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Yields:

$$T A_{CL} T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{\gamma_1+\mu}{M} & 0 & 0 & -\frac{\gamma_4}{M} \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma_2\frac{\ell}{J} & -\frac{\beta+\gamma_3\ell}{J} & 0 \end{bmatrix} T^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{\gamma_4}{M} & -\frac{\gamma_1+\mu}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\gamma_2\frac{\ell}{J} & -\frac{\beta+\gamma_3\ell}{J} \end{bmatrix}$$

$$\Rightarrow \varphi(A_{CL}) = \left(\lambda^2 + \left(\frac{\gamma_1+\mu}{M}\right)\lambda + \frac{\gamma_4}{M}\right)\left(\lambda^2 + \left(\frac{\beta+\gamma_3\ell}{J}\right)\lambda + \gamma_2\frac{\ell}{J}\right)$$

(ROUTH-HURWITZ) all eigenvalues have negative real parts since

$$\frac{\gamma_1+\mu}{M} > 0, \quad \frac{\gamma_4}{M} > 0, \quad \frac{\beta+\gamma_3\ell}{J} > 0, \quad \frac{\gamma_2\ell}{J} > 0$$

REMARK. Also in this case the integral control law is ROBUST since

- K_1 and K_2 can be choosen independentely from the model's parameters
- The integral action automatically compensates gravity and wind by producing u^*

REMARK. An OUTPUT-FEEDBACK solution where

$$y_m = \begin{pmatrix} y_r \\ y_m' \end{pmatrix}$$

and $y_m' \neq x$ can be obtained via the separation principle as before by relying on a Luemberger observer.