System Theory and Advanced Control

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Contents

Chapter 1

Dynamic system representation

1.1 system classification

Dynamic system := system with memory
continuous time:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y(t) &= h(x(t), u(t), t) \qquad y \in \mathbb{R}^p \end{aligned}$$

discrete time:

$$\begin{aligned} x(t+1) &= f(x(t), u(t), t) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y(t) &= h(x(t), u(t), t) \qquad y \in \mathbb{R}^p \end{aligned}$$

Stationary systems: vector fields $f(\cdot), h(\cdot)$ do not depend on t
Linear systems: vector fields $f(\cdot), h(\cdot)$ are linear
Autonomous systems: vector fields $f(\cdot), h(\cdot)$ do not depend on $(u(t), t)$

1.1.1 Linear systems

$$\left.\begin{array}{r} \dot{x}(t) \\ x(t+1) \end{array}\right\} = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

Stationary linear systems

$$\left.\begin{array}{r} \dot{x}(t) \\ x(t+1) \end{array}\right\} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Proper linear systems

$$\left.\begin{array}{r} \dot{x}(t) \\ x(t+1) \end{array}\right\} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Autonomous linear systems

$$\left.\begin{array}{r} \dot{x}(t) \\ x(t+1) \end{array}\right\} = Ax(t)$$
$$y(t) = Cx(t)$$

1.2 Phase portrait

1.3 Coordinate changes

A given State Space model implies a coordinate framework in $\mathbb{R}^n$ wrt which the state x is described. A coordinate change for the system is a function $\Phi(\cdot)$, invertible, such that

$$z(t) = \Phi(x(t))$$

z(t) being the system's state in the new coordinate framework.

1.3.1 Linear coordinate change

A linear coordinate change can be expressed by $T : \mathbb{R}^n \to \mathbb{R}^n, \quad z = Tx$ where T is an n x n nonsingular matrix.

$$\left.\begin{aligned} \dot{x}(t) \\ x(t+1) \end{aligned}\right\} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\left.\begin{aligned} \dot{z}(t) \\ z(t+1) \end{aligned}\right\} = \tilde{A}z(t) + \tilde{B}u(t)$$
$$y(t) = \tilde{C}z(t) + \tilde{D}u(t)$$

with

$$\tilde{A} = TAT^{-1} \qquad \tilde{B} = TB \qquad \tilde{C} = CT^{-1} \qquad \tilde{D} = D$$

1.3.2 non-linear coordinate change

Definition 1.3.1 (Diffeomorphism)**.** $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a global diffeomorphism if

- it is smooth (differentiable a sufficient number of times)
- there exists $\Phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n \quad : \quad \Phi^{-1}(\Phi(x)) = x \quad \forall x \in \mathbb{R}^n$

Local diffeomorphism: same but with $U \subset \mathbb{R}^n$ instead of $\mathbb{R}^n$

$$\dot{x} = f(x) + g(x)u \quad \text{input-affine system (linear wrt u)}$$
$$y = h(x)$$

becomes

$$\dot{z} = \tilde{f}(z) + \tilde{g}(z)u$$
$$y = \tilde{h}(x)$$

where

$$\dot{z} = \frac{d\Phi}{dx}\left(f(x) + g(x)u\right) \text{ therefore}$$
$$\dot{z} = \left.\frac{d\Phi}{dx}f(x)\right|_{x=\Phi^{-1}(z)} + \left.\frac{d\Phi}{dx}g(x)\right|_{x=\Phi^{-1}(z)} u$$

Hence

$$\tilde{f}(z) := \left.\frac{d\Phi}{dx}f(x)\right|_{x=\Phi^{-1}(z)}$$
$$\tilde{g}(z) := \left.\frac{d\Phi}{dx}g(x)\right|_{x=\Phi^{-1}(z)} \qquad \tilde{h}(z) := h(\Phi^{-1}(z))$$

Result. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function, $\bar{x} \in \mathbb{R}^n$ and suppose that $\left.\frac{d\Phi}{dx}\right|_{\bar{x}}$ is non singular. Then $\Phi(\cdot)$ is a local diffeomorphism at $\bar{x}$. If $\left.\frac{d\Phi}{dx}\right|_{\bar{x}} \quad \forall \bar{x} \in \mathbb{R}^n$, and $\Phi(\cdot)$ is radially unbounded, then $\Phi(\cdot)$ is a global diffeomorphism

1.3.3 Diagonalization of a matrix

Theorem 1.3.1. A matrix A is purely diagonalizable by a change of coordinates iff the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

Let $\{\lambda_i, \dots, \lambda_r\}$ be the set of eigenvalues of A, $\{v_{11}, \dots, v_{1g_1}, v_{21}, \dots, v_{rg_r}\}$ be the set of associated eigenvectors, and g_i be the associated geometric multiplicity. A can be diagonalized by choosing

$$T = \begin{bmatrix} v_{11} & \cdots & v_{1g_1} & v_{21} & \cdots & v_{rg_r} \end{bmatrix}$$

and we have

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} D_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & D_{\lambda_r} \end{pmatrix} \quad \text{with} D_{\lambda_i} = \begin{pmatrix} \lambda_i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i \end{pmatrix} \text{ of dimension } g_i \times g_i$$

Theorem 1.3.2 (Cayley-Hamilton theorem)**.** Each matrix is solution of its characteristic equation

$$\varphi(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I_n$$

complex conjugate eigenvalues

Suppose $\lambda_i = a_i + jb_i$ and $\lambda_{i+1} = a_i - jb_i$, $a_i, b_i \in \mathbb{R}$. Let $v_i = u_i + jw_i, u_i, w_i \in \mathbb{R}^n$ and v_{i+1} be the two linearly independent eigenvectors relative to λ_i, λ_{i+1}. We have that $v_{i+1} = u_i - jw_i$ and that u_i and w_i are linearly independent vectors also linearly independent from all the other eigenvectors.
Instead of

$$T^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_i & v_{i+1} & \cdots & v_n \end{bmatrix}$$

we can use

$$T^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & u_i & w_i & \cdots & v_n \end{bmatrix}$$

and the resulting matrix is

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & & & \cdots & 0 \\ 0 & \lambda_2 & \cdots & & & & \vdots \\ \vdots & 0 & \ddots & & & & \vdots \\ & \vdots & & \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix} & & & \\ & & & & & \ddots & \\ 0 & \cdots & & & & \cdots & \lambda_r \end{pmatrix}$$

1.3.4 Jordan form

Generalized eigenvectors

Definition 1.3.2 (generalized eigenvector)**.** We say a vector $v \in \mathbb{R}^n$ is a generalized eigenvector ($\mathrm{GED_k}$) of dimension k linked to λ (eigenvalue of A) if:

$$(\lambda I - A)^j \neq 0 \quad \forall j \in 0, \dots, k-1$$
$$(\lambda I - A)^k v = 0$$

Generalized eigenvectors are linearly independent, and for any $A \in \mathbb{R}^n x \mathbb{R}^n$, there are exactly n generalized eigenvectors. This means that it is always possible to form a base of $\mathbb{R}^n$ of generalized eigenvectors.

Let $\{v_{i1}, \dots, v_{i1}^{n_{i1}}, \dots, v_{ig_i}, \dots, v_{ig_i}^{n_{ig_i}}\}$ be the set of generalized eigenvectors associated to the $i-th$ eigenvalue with geometric multiplicity g_i. The Jordan form of the system (with r distinct eigenvalues) is given by a trasformation matrix

$$T^{-1} = \begin{bmatrix} v_{11} & v_{11}^2 & \cdots & v_{1g_1}^{n_1 g_1} & \cdots & v_{rg_r}^{N_{rg_r}} \end{bmatrix}$$

The system resulting from such transformation has block diagonal form:

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_r \end{pmatrix}$$

where the J_i are called Jordan blocks and also have a block diagonal structure

$$J_i = \begin{pmatrix} J_{i1} & 0 & 0 & \cdots & 0 \\ 0 & J_{i2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{ig_i} \end{pmatrix}$$

and the J_{ik} are called mini Jordan blocks

$$J_{ik} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

1.3.5 Schur decomposition

Given $A \in \mathbb{R}^n \times \mathbb{R}^n$ there always exists a change of variables T such that $\tilde{A} = TAT^{-1}$ is upper(lower) triangular. From a system perspective: given a generic linear system it is possible to see it as the cascade of n scalar subsystems
read procedure from notes

1.3.6 Brunowsky canonical form

Definition 1.3.3 (relative degree (SISO case))**.** The triplet (A,B,C) has relative degree r¿0 if the following holds:

- $CA^kB = 0$ for all $k = 0, 1, \dots, r-2$
- $CA^{r-1}B \neq 0$

The relative degree r always exists and $r < n$.
For continuous time systems, r is the number of times the output needs to be derived in order to see a direct effect on it from the output.
we have that

$$rank \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix} = r$$

i.e., the all the rows are linearly independent

Brunowsky change of variables

we choose

$$T = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \\ \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{n-r} \end{pmatrix} := \begin{pmatrix} T_\xi \\ T_\eta \end{pmatrix} \qquad \text{with the rows of } \Phi_i \text{ chosen so that } T \text{ is not singular}$$

This leads to the Brunowsky change of variables:

$$z = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} T_\xi x \\ T_\eta x = Tx \end{pmatrix}$$

This change of variables results in a chain of r integrators, and a "hidden" zero dynamic.

$$\begin{gathered}
\xi_1 = y \\
\dot{\xi}_1 = \xi_2 = \dot{y} \\
\dot{\xi}_2 = \xi_3 = \ddot{y} \\
\vdots \\
\dot{\xi}_{r-1} = \xi_r \\
\dot{\xi}_r = CA^{r-1}(Ax + Bu) = CA^r x + CA^{r-1}Bu \\
\dot{\xi}_r = Q_\xi \xi + Q_\eta \eta + \Delta u \\
(Q_\xi Q_\eta) = CA^r T^{-1} \qquad \Delta = CA^{r-1}B \neq 0 \\
\eta = T\eta x \\
\dot{\eta} = T_\eta (Ax + Bu) \\
\dot{\eta} = \Psi_\xi \xi + \Psi_\eta \eta + T_\eta B u \\
(T_\eta B = 0 \text{ can always be obtained by construction})
\end{gathered}$$

If the input is chosen as the state feedback

$$u(t) = \frac{1}{\Delta}(-Q_\xi \xi(t) - Q_\eta \eta(t) + v(t))$$

with $v(t)$ a new residual input, the I-O relation is equivalent to $\frac{1}{s^r}$

Chapter 2

Stability

Definition 2.0.1 ($\epsilon - \delta_\epsilon$ stability)**.** The equilibrium x=0 is stable if $\forall \epsilon > 0, \quad \exists \delta_\epsilon : \forall ||x(0)|| < \delta_\epsilon \implies ||x(t)|| \leq \epsilon \quad \forall t > 0$

Definition 2.0.2 (asymptotic stability)**.** The equilibrium $x = 0$ is asymptotically stable if it is stable and there exists a set $\mathcal{A} \supset \{0\} : x(0) \in \mathcal{A}$ implies $\lim_{t\to\infty} x(t) = 0$ (attractivity properties). $\mathcal{A}$ is said to be the "Domain of attraction

Definition 2.0.3 (instability)**.** lack of stability

Theorem 2.0.1. A C-T (D-T) linear system is stable iff all the eigenvalues of A have non positive real part (amplitude≤ 1) and possible eigenvalues with zero real part (amplitude=1) have a geometric multiplicity equal to the algebraic one. It is asympototically stable iff the eigenvalues of A all have negative real part (amplitude¡1)

Definition 2.0.4 (positive definite function)**.** A function $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite wrt $\bar{x} \in \mathcal{D}$ if $V(\bar{x}) = 0$ and $V(x) > 0 \forall x \in \mathcal{D} \backslash \{\bar{x}\}$

Definition 2.0.5 (positive semi-definite function)**.** A function $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite wrt $\bar{x} \in \mathcal{D}$ if $V(\bar{x}) = 0$ and $V(x) \geq 0 \forall x \in \mathcal{D} \backslash \{\bar{x}\}$

Definition 2.0.6 (Level set)**.** $\Omega_c := \{x \in \mathbb{R}^n : V(x) \leq c\}$

Theorem 2.0.2 (Direct Lyapunov theorem)**.** Let $\bar{x} \in \mathbb{R}^n$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a positive definite function wrt $\bar{x} \in \mathcal{D}$ and consider the real-valued function $V'(x) : \mathcal{D} \to \mathbb{R}$ defined as $V'(x) = \nabla V(x) f(x)$. The following holds:

- if $V'(x) \leq 0 \quad \forall x \in \mathcal{D}$ then $\bar{x}$ is a stable equilibrium point for the system
- if $V'(x) < 0 \quad \forall x \in \mathcal{D}$ then $\bar{x}$ is an asympotically stable equilibrium point for the system with a certain domain of attraction $\mathcal{A}$

For D-T systems the same can be applied but with $V'(x) := V(f(x)) - V(x)$

Proof. Consider a ball of radius ϵ, $\mathcal{B}_\epsilon$, contained entirely inside $\mathcal{D}$. This ball contains a level set $\Omega_{\bar{c}}$, which contains a ball of radius δ_ϵ, $\mathcal{B}_{delta_\epsilon}$. Since $\forall x \in \mathcal{D}, \quad V(x) > 0$ and $V'(x) \leq 0$, all trajectories starting inside $\mathcal{B}_{delta_\epsilon}$ do not leave $\Omega_{\bar{c}}$, and therefore do not leave $\mathcal{B}_\epsilon$. □

Theorem 2.0.3 (Converse Lyapunov theorem)**.** let $\bar{x}$ be a stable (asymptotically stable) equilibrium point of $\dot{x} = f(x)$ (with domain of attraction $\mathcal{A}$). Then there exists a differentiable $V : \mathcal{D} \to \mathbb{R}$ that is positive definite wrt $\bar{x}$ (with $\mathcal{D} \supset \mathcal{A}$) and fulfilling $V'(x) \leq 0 (V'(x) < 0) \forall x \in \mathcal{D}$.

Theorem 2.0.4 (Direct Lyapunov Theorem (GAS))**.** Let $\bar{x}$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite function wrt $\bar{x}$ which is radially unbounded ($\|x\| \to \infty \implies V(x) \to \infty$) and consider the real valued function $V'(x) : \mathbb{R}^n \to \mathbb{R}$ defined as $V'(x) = \nabla V(x) f(x)$. Then if $V'(x) < 0 \forall x \in \mathbb{R}^n \backslash \{0\}$, then $\bar{x}$ is GAS.

2.1 Quadratic forms

$$V(x) = x^T P x$$

Definition 2.1.1.
- The matrix P is positive semi-definite ($P \geq 0$) if $x^T P x \geq 0 \forall x \in \mathbb{R}^n$
- The matrix P is positive definite ($P > 0$) if $x^T P x > 0 \forall x \in \mathbb{R}^n \backslash \{0\}$
- The matrix P is negative semi-definite ($P \leq 0$) if $x^T P x \leq 0 \forall x \in \mathbb{R}^n$
- The matrix P is negative definite ($P < 0$) if $x^T P x < 0 \forall x \in \mathbb{R}^n \backslash \{0\}$

Divergence of a quadratic form: $\nabla V(x) = 2x^T P$

Observation. In quadratic forms the matrix P can be taken symmetric wlog:

$$P_{notsymm} = P_{symm} + P_{antisymm} \qquad P_{symm} = \frac{P + P^T}{2} \quad P_{antisymm} = \frac{P - P^T}{2}$$
$$\implies x^T P x = x^T (P_{symm} + P_{antisymm}) x = x^T P_{symm} x + x^T P_{antisymm} x$$
$$(x^T P_{symm} x)^T = x^T P_{symm}^T x = x^T P_{symm} x$$
$$(x^T P_{antisymm} x)^T = x^T P_{antisymm}^T x = -x^T P_{antisymm} x \implies$$
$$\implies x^T P_{antisymm} x = 0 \text{ because } x^T P_{antisymm} x \text{ is a scalar}$$

Result. A matrix P is positive definite iff all its leading principal minors are positive.

Properties of symmetric positive definite matrices P:

- P is diagonalizable ($a_i = g_i$)
- The eigenvalues λ_i and eigenvectors v_i are real, $i = 1, \ldots, n$
- $\lambda_i > 0, i = 1, \ldots, n$, and for each pair (v_i, v_j) of eigenvectors $< v_i, v_j >= 0$ (orthogonal)

Geometry of $\Omega_c = \{x \in \mathbb{R}^n \quad : \quad x^T P x \leq c\}$:

Ω_c are ellipsoids with their principal axes direct as the eigenvectors of P and amplitude proportional to the relative eigenvalues.

2.2 Linear systems

$$\left.\begin{array}{r} \dot{x}(t) \\ x(t+1) \end{array}\right\} = Ax(t) \qquad x(0) = x_0 V(x) = x^T P x$$

Theorem 2.2.1.
1. The system is Hurwitz(Schur) iff there exists $P = P^T > 0$ and $Q = Q^T > 0$ solution of the Lyapunov matrix equation:
$$PA + A^T P = -Q \quad \text{(C-T)} \qquad\qquad A^T P A - P = -Q \quad \text{(D-T)}$$
2. If there exists a solution (P, Q) then there exists an infinite number of other solutions, one for each $Q = Q^T > 0$. That is, Q is arbitrary.
3. If the Lyapunov matrix equations are fulfilled with $Q = Q^T \geq 0$ then the system is stable

Proof. (1) $\implies$: pick
$$P = \int_0^\infty e^{A^T s} Q e^{As} \, ds$$
with Q generic. The integrand is a sum of exponential vanishing terms, and therefore the integral exists. We have that

$$PA + A^T P = \int_0^\infty e^{A^T s} Q e^{As} A \, ds \; + \int_0^\infty A^T e^{A^T s} Q e^{As} \, ds \; = \int_0^\infty \frac{d}{ds} e^{A^T s} Q e^{As} \, ds \; = e^{A^T s} Q e^{As} \Big|_0^\infty = -Q$$

If Q is symmetric, i.e. $Q = Q^T$, then

$$P^T = \left[\int_0^\infty e^{A^T s} Q e^{As}\, ds\right]^T = \int_0^\infty \left[e^{A^T s} Q e^{As}\right]^T ds = \int_0^\infty e^{A^T s} Q e^{As}\, ds = P$$

Which proves P is also symmetric. To prove it is positive definite, we can consider that if $Q = Q^T > 0$

$$x^T P x = x^T \int_0^\infty e^{A^T s} Q e^{As}\, ds\ x > 0 \forall s$$

because A is Hurwitz.
$\Longleftarrow$: pick $V(x) = x^T P x$. We have

$$V'(x) =< \nabla V(x), f(x) >= 2x^T P A x = x^T P A x + x^T P A x = x^T P A x + x^T A^T P x$$

therefore

$$V'(x) = x^T (PA + A^T P) x = -x^T Q x$$

if there exists $Q = Q^T > 0$ then due to the Direct Lyapunov Theorem the system is GAS. ☐

Theorem 2.2.2 (indirect Lyapunov theorem)**.** 1. Suppose that there exists a $P = P^T > 0$ and $Q = Q^T > 0$ solution of the Lyapunov matrix equation. Then $\bar{x}$ is LAS for $\dot{x} = f(x)$ with a certain domain of attraction ($V = (x - \bar{x})^T P (x - \bar{x})$ is a possible candidate Lyapunov function)

2. Suppose that A has at least one eigenvalue with positive real part. Then $\bar{x}$ is unstable for $\dot{x} = f(x)$

(1). Let's assume wlog $\bar{x} = 0$.

$$\dot{x} = Ax + g_h(x)$$

where $g_h(x)$ represents the higher order terms of $f(x)$. Picking $V(x) = x^T P x$ we have:

$$V'(x) = 2x^T P (Ax + g_h(x)) = x^T (PA + A^T P) x + 2x^T P g_h(x) = -x^T Q x + 2x^T P g_h(x)$$

The first term on the right-hand side is negative definite, while the second term is in general indefinite. The function $g_h(x)$ satisfies

$$\frac{\|g_h(x)\|}{\|x\|} \to 0 \text{ as } \|x\| \to 0$$

Therefore, for any $\gamma > 0$ there exists $r_\gamma > 0$ such that

$$\|g_h(x)\| < \gamma \|x\|, \quad \forall \|x\| < r_\gamma$$

Hence,

$$V'(x) < -x^T Q x + 2\gamma \|P\| \|x\|^2, \quad \forall \|x\| < r_\gamma$$

but

$$x^T Q x \geq \lambda_{min}(Q) \|x\|^2$$

where $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue of a matrix. Note that $\lambda_{min}(Q)$ is real and positive since Q is symmetric and positive definite. Thus

$$V'(x) < -\left[\lambda_{min}(Q) - 2\gamma \|P\|\right] \|x\|^2, \quad \forall \|x\| < r_\gamma$$

Choosing $\gamma < \lambda_{min}(Q)/2\|P\|$ ensures that $V'(x)$ is negative definite. By the Lyapunov direct theorem, we conclude that the origin is asympotically stable. ☐

Observation. A conservative estimate for the domain of attraction is the largest level set contained in the ball of radius $\lambda_{min}(Q)/4\|P\|$
The theorem is not conclusive when A has eigenvaues on the imaginary axis. In this case the system is said to be non hyperbolic at $\bar{x}$. In the non-hyperbolic case the equilibrium could be either stable, asymptotically stable or unstable for the non linear dynamics. The high-order terms play a role in determining the stability properties of the system.

2.3 Krasovski-La Salle Criterion

Definition 2.3.1 (invariant set)**.** A set M is invariant for $\dot{x} = f(x)$ if, having denoted by $x(t, x_0)$ the trajectory at time t with initial condition x_0 at time $t = 0$, then

$$x_0 \in M \implies x(t, x_0) \in M \forall t \in \mathbb{R}$$

Theorem 2.3.1 (Krasovski-La Salle Criterion)**.** Let $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a positive definite function relative to $\bar{x}$, an equilibrium point of the system $\dot{x} = f(x)$. Suppose that $V'(x) \leq 0 \quad \forall x \in \mathcal{D}' \subseteq \mathcal{D}$. Let $E \subseteq \mathcal{D}'$ be a set where $V'(x) = 0$ and let M be the largest set contained in E which is invariant for the trajectories of the system. Then:

- the equilibrium $\bar{x}$ is stable
- the set M is attractive for the trajectories of the system, namely

$$\lim_{t \to \infty} dist(x(t), M) = 0$$

Corollary. if $M = \bar{x}$ then the equilibrium point is asympototically stable.

2.4 Systms with inputs and outputs

Definition 2.4.1 (Input to State Stability (ISS))**.** A system $\dot{x} = f(x, u)$ fulfilling $f(0, 0) = 0$ is ISS with respect to $\bar{x} = 0$ if there exists a positive definite real valued function $V : \mathcal{D} \to \mathbb{R}$ (ISS Lyapunov function) and a strictly increasing function $\chi : \mathbb{R} \to \mathbb{R}$ fulfilling

$$\|x\| \geq \chi(\|u\|) \quad \implies \quad V'(x, u) = \nabla V(x) f(x, u) < 0$$

Definition 2.4.2 (passivity)**.** A system with an input and output ($\dot{x} = f(x, u)$, $y = h(x)$) is said to be passive if there exists a positive definite real valued function $V : \mathcal{D} \to \mathbb{R}$ (storage function) fulfilling

$$\dot{V}(x(t)) = V'(x, u) = \nabla V(x) f(x, u) \leq u^T y$$

Interpretation: the energy stored in the system in a time interval should be $\leq$ than the energy supplied to it.

Chapter 3

Geometry of trajectories

eigenvalues go **BRRR**

Chapter 4

Reachability and controllability

4.1 Reachability

Definition 4.1.1 (Reachability)**.** Set of reachable states at time t_1:

$$\mathcal{R}^+(t_1) = \{x \in \mathbb{R}^n : x = \Psi(t_1)u([0,t_1)) \text{ for some } u([0,t_1)) \in \mathbb{R}^m\}$$

Reachable set:

$$\mathcal{R}^+ := \mathcal{R}^+(\infty)$$

Theorem 4.1.1. $\mathcal{R}^+ = Im(R) \qquad R = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

D-T case. we assume $x(0) = 0$. We have that the state at time t can be expressed by

$$x(t) = A^t x_0 + \sum_{i=0}^{t-1} A^{t-1-i}Bu(i)$$
$$x(1) = Bu(0)$$
$$x(2) = ABu(0) + Bu(1)$$
$$x(3) = A^2Bu(0) + ABu(1) + Bu(2)$$
$$x(t) = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix} \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix} = R_t u_0^{t-1}$$

we need to prove that $ImR_t = ImR_n \forall t \in \mathbb{N} : t \geq n$: $R_t = \begin{bmatrix} B & AB & \cdots & A^{n-1}|A^nB & A^{n+1} & \cdots \end{bmatrix}$. By the Cayley Hamilton theorem we have that

$$\varphi_A(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I_n = 0 \implies A^nB + \alpha_1 A^{n-1}B + \cdots + \alpha_n B = 0$$
$$\implies A^nB \text{ is L.D from } \{B, AB, \ldots, A^{n-1}B\}$$

This can be extended to A^{n+1} by premultiplying by A.
From this proof we have the added observation that for D-T systems if a state is reachable it is reachable in at most n steps □

Theorem 4.1.2 (PBH Test)**.** The pair (A, B) is completely reachable iff $rank \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \quad \forall \lambda \in \sigma(A)$

Proof. get it done hints in paper notes □

4.2 Computing the input

4.2.1 D-T case

Since $x(t) = R_t u_0^{t-1}$:
case $\mathcal{R}^+ = \mathbb{R}^n, t \geq n$

$$\bar{x} = R_t u_0 t - 1 \qquad u_0 t - 1 = R_t^T (R_t R_t^T) =^{-1} \bar{x}$$

where R_t is a fat full row rank matrix
general case ($t < n, \mathcal{R} \subset \mathbb{R}^n$). The target state, in order to be reached, must fulfil $\bar{x} \in ImR_t$

$$\bar{x} = R_t u_0^{t-1} \qquad u_0^{t-1} = R_t^\dagger \bar{x}$$

The Moore-Penrose pseudoinverse, if the matrix is (right) invertible, boils down to the canonical (right) inverse. Otherwise, it provides a unique solution to $\bar{x} = R_t u_0 t - 1$ if it has one, or if $\bar{x} \notin ImR_t$ it provides the closest solution in the Euclidean sense.

$$||\bar{x} - R_t R_t^\dagger \bar{x}|| \leq ||\bar{x} - R_t u_0^t|| \text{ with } u_0^t \text{ generic input sequence}$$

4.2.2 C-T case

We introduce the reachablity/controllability Gramian at time t:

$$W(t) = \int_0^t e^{A(t-s)} BB^T e^{A^T(t-s)}\, ds$$

Result. The Gramian is non singular for each $t > 0$ iff $\mathcal{R}^+ = \mathbb{R}^n$

let's pick $u(s) = \left[e^{A(t-s)}B\right]^T V(s)$ with $V(s)$ generic for now. From Lagrange we have

$$x(t) = \int_0^t e^{A(t-s)} Bu(s)\, ds \;= \int_0^t e^{A(t-s)} BB^T e^{A^T(t-s)} V(s)\, ds$$

where $e^{A(t-s)} BB^T e^{A^T(t-s)}$ is an nxn supersingular square matrix. If $V(s)$ is constant, we have

$$\bar{x} = x(t) = W(t)V$$

and we can pick $V = W^{-1}(t)\bar{x}$, resulting in:

$$u(s) = B^T e^{A^T(t-s)} W^{-1}(t)\bar{x} \qquad s \in [0, t)$$

if $x_0 \neq 0$, because of linearity the problem can be cast as the problem of steering the state of the system from the origin to the final target in a predetermined interval:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} Bu(s)\, ds \implies u(s) = B^T e^{A^T(t-s)} W^{-1}(t)(\bar{x} - e^{At}x_0) \qquad s \in [0, t)$$

4.3 Controllability

Definition 4.3.1 (Controllability)**.** Set of controllable states at time t_1: $\mathcal{R}^-(t_1) = \{x \in \mathbb{R}^n : x = \Phi(t_1)x + \Psi(t_1)u([0, t_1))$ for some $u([0, t_1)) \in \mathbb{R}^m\}$
Controllable set: $\mathcal{R}^- := \mathcal{R}^-(\infty)$

Theorem 4.3.1. $\mathcal{R}^+ \supseteq \mathcal{R}^-$ in general. $\mathcal{R}^+ = \mathcal{R}^-$ if the system is reversible

4.4 Controllability and state feedback

Controllability properties are invariant to change of coordinates, and are therefore properties of the system:

$$(\tilde{A}, \tilde{B}) = (TAT^{-1}, TB)$$
$$\tilde{R} = \begin{bmatrix}\tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B}\end{bmatrix} = TR \implies ImR = Im\tilde{R}$$

Theorem 4.4.1. (A, B) completely controllable $\implies$ $(A + BK, B)$ completely controllable $\forall K$

Proof. We can apply the PBH test. Assuming

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad K = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix}$$

we have that

$$\begin{bmatrix} \lambda I - (A + BK) & B \end{bmatrix} = \begin{pmatrix} \lambda - a_{11} - b_1 k_1 & -a_{12} - b_1 k_2 & \cdots & -a_{1n} - b_1 k_n & b_1 \\ -a_{21} - b_2 k_1 & \lambda - a_{22} - b_2 k_2 & \cdots & -a_{2n} - b_1 k_n & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n1} - b_n k_1 & -a_{n2} - b_n k_2 & \cdots & \lambda - a_{nn} - b_n k_n & b_n \end{pmatrix}$$

we can apply the following column substitutions without altering the rank of the matrix:

$$c_i \to c_i + c_{n-1} k_i \qquad i = 1, \ldots, n$$

and we obtain the following matrix

$$\begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} & b_1 \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} & b_n \end{pmatrix} = \begin{bmatrix} \lambda I - A & B \end{bmatrix}$$

Therefore, because rank $\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n$ due to it being fully controllable, we conclude that $(A + BK, B)$ is fully controllable $\forall K$ $\square$

4.4.1 controllablilty canonical form

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{pmatrix} \qquad B_c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\varphi_A(\lambda) = det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

Theorem 4.4.2. $m = 1$
A pair (A, B) is similar to (A_c, B_c) $\iff$ the system is completely controllable

$$T = T_c := R_c R^{-1}$$
$$R_c = \begin{bmatrix} B_c & A_c B_c & \cdots & A_c^{n-1} B_c \end{bmatrix}$$

Theorem 4.4.3 (Eigenvalues assignment theorem)**.** A pair (A, B) is completely controllable $\iff$ for all $\{\lambda_1^\star, \ldots, \lambda_n^\star\}$ (set of desired eigenvalues) there exists a K such that $\sigma(A + BK) = \{\lambda_1^\star, \ldots, \lambda_n^\star\}$

($\Longrightarrow$). let $(\alpha_1^\star, \ldots, \alpha_n^\star)$ be s.t. $\{\lambda_1^\star, \ldots, \lambda_n^\star\}$ are roots of $\lambda^n + \alpha_1^\star \lambda^{n-1} + \cdots + \alpha_{n-1}^\star \lambda + \alpha_n^\star = 0$
since (A, B) is completely controllable it can be cast into controllability canonical form through a T_c. Considering a generic K_c we have that

$$A_c + B_c K_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} K_{c_1} & \cdots & K_{c_n} \end{pmatrix}$$

we can pick

$$K_c = (\alpha_n - \alpha_n^\star, \ldots, \alpha_1 - \alpha_1^\star)$$

resulting in the desired eigenvalues. The matrix K is found as follows:

$$z = T_c x$$
$$u = K_c z + v$$
$$\dot{z} = (A_c + B_c K_c)u + B_c v$$
$$u = K_c T_c x$$
$$K = K_c T_c$$

□

4.5 Kalman Decomposition

Definition 4.5.1 (stabilizable system)**.** A system (A, B) is stabilizable if there exists K s.t. $(A + BK)$ is Hurwitz(Schur)

Suppose that (A, B) is not completely controllable, namely rank$R = n_r < n$
Let $\mathcal{R}^+_\perp$ be the orthogonal complement of $\mathcal{R}^+$. It turns out that dim$\mathcal{R}^+_\perp = n - n_r$
Let $\{v_1, \dots, v_{n_r}\}$ be a base of $\mathcal{R}^+$ and let $\{v_{n_r+1}, \dots, v_n\}$ be a base of $\mathcal{R}^+_\perp$. The two sets of vectors are all linearly independent
Consider the change of variables

$$T_K^{-1} = \begin{bmatrix} v_1 & \cdots & v_{n_r} & v_{n_r+1} & \cdots & v_n \end{bmatrix}$$
$$z = T_K x = \begin{pmatrix} z_r \\ z_{n_r} \end{pmatrix} \qquad x = T_K^{-1} z$$

it turns out that $x \in \mathcal{R}^+ \implies z = \begin{pmatrix} \star \\ 0 \end{pmatrix} \qquad x \in \mathcal{R}^+_\perp \implies z = \begin{pmatrix} 0 \\ \star \end{pmatrix}$

Result.

$$\tilde{A}_K = T_K A T_K^{-1} = \begin{pmatrix} \tilde{A}_R & \tilde{A}_J \\ 0 & \tilde{A}_{NR} \end{pmatrix} \qquad \tilde{B}_K = T_K B = \begin{pmatrix} \tilde{B}_R \\ 0 \end{pmatrix} \qquad (\tilde{A}_R, \tilde{B}_R) \text{ completely reachable}$$

with $\tilde{A}_R$ $n_r \times n_r$ and $\tilde{A}_{NR}$ $n - n_r \times n - n_r$.

- The set $\mathcal{R}^+$ is forward invariant for the system dynamics. The "internal dynamics" are completely reachable/controllable
- if $\tilde{A}_{NR}$ is Hurwitz then trajectories starting outside $\mathcal{R}^+$ asymptotically converge to it

Theorem 4.5.1. (A, B) is stabilizable $\iff$ $\tilde{A}_{NR}$ is Hurwitz(Schur)

$\Longleftarrow$. Let (A, B) be generic, with $\mathcal{R}^+ \subset \mathbb{R}^n$. By Kalman,

$$\exists T_K : T_K A T_K^{-1} = \tilde{A}_K = \begin{pmatrix} \tilde{A}_R & \tilde{A}_J \\ 0 & \tilde{A}_{NR} \end{pmatrix} \qquad T_K B = \tilde{B}_K = \begin{pmatrix} \tilde{B}_R \\ 0 \end{pmatrix}$$

In the Kalman Coordinates, we can design a $K_K = \begin{bmatrix} K_{K_R} & K_{K_{NR}} \end{bmatrix}$ such that $\tilde{A}_R + \tilde{B}_R K_{K_R}$ is Hurwitz.:

$$\tilde{A}_K + \tilde{B}_K K_K = \begin{pmatrix} \tilde{A}_R + \tilde{B}_R K_{K_R} & \tilde{A}_J + \tilde{B}_R K_{K_R} \\ 0 & \tilde{A}_{NR} \end{pmatrix} \implies \sigma(\tilde{A}_K + \tilde{B}_K K_K) = \sigma(\tilde{A}_R + \tilde{B}_R K_{K_R}) \cup \sigma(\tilde{A}_{NR})$$

The eigenvalues of $(\tilde{A}_R, \tilde{B}_R$ are completely assignable because it is a completely controllable pair by construction, and the eigenvalues of $\tilde{A}_{NR}$ are stable by definition. We can thus pick

$$K_K = \begin{bmatrix} K - K_R & \star \end{bmatrix}$$

and because

$$z = T_K x \implies u = K_K T_K x + v$$

we have that

$$K = K_K T_K$$

□

4.6 controllability of a cascade

Result. The cascade is controllable $\iff$ the subsystems are controllable and the zeros of the first subsystem do not resonate with the poles of the second subsystem, namely

$$rank \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} = n + 1 \qquad \forall \lambda \in \sigma(F)$$

Chapter 5

Observability and reconstructability

5.1 Observability

Definition 5.1.1 (Set of unobservable states)**.** Set of unobservable states in $[0, t_1)$

$$\mathcal{E}^+_{NO}(t_1) = \{x \in \mathbb{R}^n : C\Phi(t)x = 0 \forall t \in [0, t_1)$$

Unobservable set: $\mathcal{E}^+_{NO} := \mathcal{E}^+_{NO}(\infty)$

Result. the unobservable set $\mathcal{E}^+_{NO}$ is a subspace of $\mathbb{R}^n$

Theorem 5.1.1.

$$\mathcal{E}^+_{NO} = Ker(O) \qquad O := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

O is called the Observability matrix

Proof.

$$\begin{aligned} y(0) &= Cx_0 \\ y(1) &= Cx(1) = CAx_0 \\ y(2) &= Cx(2) = CA^2x_0 \\ y(t) &= CA^tx_0 \\ \implies y_0^{t-1} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t-1) \end{bmatrix} &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix} x_0 = O_t x_0 \end{aligned}$$

The set of x_0 s.t. $y(t) = 0 \forall t < n$ is $KerO_n = \mathcal{E}^+_{NO}(n)$
We need to prove that $KerO_t = KerO_n \forall t \in \mathbb{N} : t > n$
Due to Cayley-Hamilton $\varphi_A(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I_n = 0$, and therefore, premultiplying by C we have

$$\begin{aligned} CA^m = \alpha_1 CA^{n-1} + \cdots + \alpha_n C \quad &\implies \quad CA^n \text{ is L.D. from } \{C, CA, \ldots, CA^{n-1}\} \\ &\implies rank(O_{n+1}) = rank(O) \implies ker(O_{n+1} = ker(O). \\ &\text{This can be extended to } O_{n+i} \text{ by premultiplying by } CA^i \end{aligned}$$

□

Definition 5.1.2 (Observability)**.** observable set: $\mathcal{E}^+ = [\mathcal{E}^+_{NO}$

Observation. $KerM = (ImM^T)_\perp$ with M a generic matrix. Hence:

$$\mathcal{E}^+ = Im[O^T] \qquad O^T := \begin{bmatrix} C^T & A^TC^T & (A^T)^2C^T & \cdots & (A^T)^{n-1}C^T \end{bmatrix}$$

Theorem 5.1.2 (PBH test)**.** the pair (A, C) is completely observable $\iff$

$$rank \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \qquad \forall\lambda(\in \sigma(A))$$

Proof. get it done (lazy cunt) □

5.2 Reconstructing the state from the output

5.2.1 D-T case

since $y_0^{t-1} = O_t x(0)$: case $\mathcal{E}^+ = \mathbb{R}^n, t \geq n$:

$$y_0^{t-1} = O_t x(0) \qquad x(0) = (O_t^T O_t)^{-1} O_t^T y_0^{t-1}$$

where O_t is a tall full column rank matrix

General case ($t < n, \mathcal{E}^+ \subset \mathbb{R}^n$): the initial state $x(0)$ must fulfil $y_0^{t-1} \in ImO_t$. In these cases the equaton $y_0^{t-1} = O_t x(0)$ can be solved for $x(0)$:

$$x(0) = O_t^\dagger y_0^{t-1}$$

If an input is present we can still solve the problem by defining a generalised output:

$$y(t) - \sum_{i=0}^{t-1} CA^{t-i-1}Bu(i) = \bar{y}(t) = CA^t x(0)$$

5.2.2 C-T case

from Lagrange:

$$y(t) = Ce^{At}x_0 \implies \int_0^t e^{A^Ts}C^Ty(s)\,ds \;= \int_0^t e^{A^Ts}C^TCe^{As}x_0\,ds$$

where $e^{A^Tt}C^TCe^{At}$ is an $n \times n$ supersingular matrix. We define the observability Gramian as such:

$$V(t) = \int_0^t e^{A^Ts}C^TCe^{As}\,ds$$

Result. The Gramian is non singular for each $t > 0$ iff $\mathcal{E}^+ = \mathbb{R}^n$

if the Gramian is non singular we can compute the input as follows:

$$\begin{aligned} V(t)x_0 &= \int_0^t e^{A^Ts}C^Ty(s)\,ds \\ \implies x_0 &= V^{-1}(t)\int_0^t e^{A^Ts}C^Ty(s)\,ds \end{aligned}$$

If there is an input present we can define a generalized output:

$$Ce^{At}x_0 = y(t) - \int_0^t e^{A(t-s)}Bu(s)\,ds$$

$$\bar{y}(t) := y(t) - \int_0^t e^{A(t-s)}Bu(s)\,ds$$

$$\implies x(0) = V^{-1}(t)\int_0^t e^{A^Ts}C^T\left[y(s) - \int_0^s e^{A(s-s')}Bu(s')\,ds'\right]ds$$

5.3 Reconstructability

Definition 5.3.1 (Set of non reconstructable states)**.** Set of non reconstructable states in $[0, t_1)$

$$\mathcal{E}_{NO}^{-}(t_1) = \{x \in \mathbb{R}^n : x = \Phi(t_1)z \quad C\Phi(t)z = 0 \forall t \in [0, t_1) \text{ for some } z \in \mathbb{R}^n\}$$

Unobservable set: $\mathcal{E}_{NO}^{-} := \mathcal{E}_{NO}^{-}(\infty)$

Result. the unobservable set $\mathcal{E}_{NO}^{+}$ is a subspace of $\mathbb{R}^n$

Theorem 5.3.1. $\mathcal{E}_{NR}^{-} \supseteq \mathcal{E}_{NO}^{+}$ in general. $\mathcal{E}_{NO}^{+} = \mathcal{E}_{NR}^{-}$ if the system is reversible

5.4 Observability and asymptotic observers

$$(\tilde{A}, \tilde{C}) = (TAT^{-1}, CT^{-1})$$

$$\tilde{O} = \begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{pmatrix} = OT^{-1} \implies ImO = Im\tilde{O} \text{ because T is non singular}$$

5.4.1 Luenberger observer

ouput injection:

Result.

$$(A, C) \text{ observable } \implies (A + LC, C) \text{ completely obeservable } \forall L \in \mathbb{R}^{n \times p}$$

$$\left.\begin{array}{r} \dot{\hat{x}}(t) \\ \hat{x}(t+1) \end{array}\right\} = A\hat{x}(t) + Bu(t) + L(\hat{y}(t) - y(t))$$

where L is called the "output injection matrix" and $\hat{y}(t) - y(t)$ is the "innovation"

Result.

$$(A, C) \text{ observable } \implies (A + LC, C) \text{ completely obeservable } \forall L \in \mathbb{R}^{n \times p}$$

$\hat{x}(0) = x(0) \implies x(t) \equiv \hat{x}(t) \forall t \geq 0$ Let's change variables:

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \to \begin{pmatrix} x \\ e := \hat{x} - x \end{pmatrix} = T \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

we have that

$$\dot{e} = A\hat{x} + Bu + L(C\hat{x} - Cx) - Ax - Bu = Ae + LCe = (A + LC)e$$

The resulting system is no longer a cascade and results completely decoupled

$$\dot{x}(t) = Ax(t) + B(u)$$
$$\dot{e}(t) = (A + LC)e(t)$$

5.4.2 Observability canonical form

$$A_c = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ 0 & 1 & \cdots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{pmatrix} \qquad C_o = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$\varphi_A(\lambda) = det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

Theorem 5.4.1 (p=1)**.** A pair (A, C) is similar to (A_o, C_o) $\iff$ the system is completely observable

$$T = T_o := O_o^{-1} O$$

$$R_o = \begin{bmatrix} C_o & \\ C_o A_o & \\ \vdots & C_o A_o^{n-1} \end{bmatrix}$$

Theorem 5.4.2 (Eigenvalues assignment theorem)**.** A pair (A, C) is completely observable $\iff$ for all $\{\lambda_1^\star, \dots, \lambda_n^\star\}$ (set of desired eigenvalues) there exists an L such that $\sigma(A + LC) = \{\lambda_1^\star, \dots, \lambda_n^\star\}$

($\implies$). let $(\alpha_1^\star, \dots, \alpha_n^\star)$ be s.t. $\{\lambda_1^\star, \dots, \lambda_n^\star\}$ are roots of $\lambda^n + \alpha_1^\star \lambda^{n-1} + \cdots + \alpha_{n-1}^\star \lambda + \alpha_n^\star = 0$
since (A, C) is completely observable it can be cast into observability canonical form through a T_o. Considering a generic L_o we have that

$$A_o = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ 0 & 1 & \cdots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{pmatrix} \qquad C_o = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{o_1} & \cdots & L_{o_n} \end{pmatrix}$$

we can pick

$$L_o = (\alpha_n - \alpha_n^\star, \dots, \alpha_1 - \alpha_1^\star)^T$$

resulting in the desired eigenvalues. The matrix L is found as follows:

$$z = T_o x$$
$$L = T_o^{-1} L_o$$

□

5.5 Kalman Decomposition

Suppose that (A, C) is not completely observalbe, namely rank$O = n_o < n$
Let $\mathcal{E}_\perp^+$ be the orthogonal complement of $\mathcal{E}^+$. It turns out that dim$\mathcal{E}_\perp^+ = n - n_o$
Let $\{v_1, \dots, v_{n_o}\}$ be a base of $\mathcal{E}^+$ and let $\{v_{n_o+1}, \dots, v_n\}$ be a base of $\mathcal{E}_\perp^+$. The two sets of vectors are all linearly independent
Consider the change of variables

$$T_K^{-1} = \begin{bmatrix} v_1 & \cdots & v_{n_o} & v_{n_o+1} & \cdots & v_n \end{bmatrix}$$
$$z = T_K x = \begin{pmatrix} z_o \\ z_{n_o} \end{pmatrix} \qquad x = T_K^{-1} z$$

it turns out that $x \in \mathcal{E}^+ \implies z = \begin{pmatrix} \star \\ 0 \end{pmatrix} \qquad x \in \mathcal{E}_\perp^+ \implies z = \begin{pmatrix} 0 \\ \star \end{pmatrix}$

Result.

$$\tilde{A}_K = T_K A T_K^{-1} = \begin{pmatrix} \tilde{A}_O & 0 \\ \tilde{A}_J & \tilde{A}_{NO} \end{pmatrix} \qquad \tilde{C}_K = C T_K^{-1} = \begin{pmatrix} \tilde{C}_O & 0 \end{pmatrix} \qquad (\tilde{A}_O, \tilde{C}_O) \text{ completely observable}$$

with $\tilde{A}_O$ $n_o r \times n_o$ and $\tilde{A}_{NO}$ $n - n_o \times n - n_o$.

- The set $\mathcal{E}_\perp^+$ is forward invariant for the system dynamics. The "internal dynamics" are not observable
- if $\tilde{A}_{NO}$ is Hurwitz then trajectories starting inside $\mathcal{E}_\perp^+$ asymptotically converge to zero

Definition 5.5.1 (detectability)**.** A system (A, C) is said to be detectable if there exists an L such that $(A+LC)$ is Hurwitz(Schur)

Theorem 5.5.1. (A, C) is detectable $\iff$ $\tilde{A}_{NO}$ is Hurwitz(Schur)

$\Longleftarrow$. Let (A, C) be generic, with $\mathcal{E}^+ \subset \mathbb{R}^n$. By Kalman,

$$\exists T_K : T_K A T_K^{-1} = \tilde{A}_K = \begin{pmatrix} \tilde{A}_O & 0 \\ \tilde{A}_J & \tilde{A}_{NO} \end{pmatrix} \qquad \tilde{C}_K = C T_K^{-1} = \begin{pmatrix} \tilde{C}_O & 0 \end{pmatrix}$$

In the Kalman Coordinates, we can design a $L_K = \begin{bmatrix} L_{K_R} & L_{K_{NO}} \end{bmatrix}$ such that $\tilde{A}_O + L_{K_O}\tilde{C}_O$ is Hurwitz.:

$$\tilde{A}_K + L_{K_O}\tilde{C}_O = \begin{pmatrix} \tilde{A}_O + L_{K_O}\tilde{C}_O & 0 \\ \tilde{A}_J + L_{K_NO}\tilde{C}_O & \tilde{A}_{NO} \end{pmatrix} \implies \sigma(\tilde{A}_K + L_K\tilde{C}_O) = \sigma(\tilde{A}_O + L_{K_O}\tilde{C}_O) \cup \sigma(\tilde{A}_{NO})$$

The eigenvalues of $(\tilde{A}_O, \tilde{C}_O$ are completely assignable because it is a completely controllable pair by construction, and the eigenvalues of $\tilde{A}_{NRO}$ are stable by hypothesis. We can thus pick

$$L_K = \begin{bmatrix} L_{K_O} & \star \end{bmatrix}$$

and because

$$z = T_K x$$

we have that

$$L = T_K^{-1} L_K$$

□

5.6 Observability of a cascade

Result. The cascade is controllable $\iff$ the subsystems are controllable and the poles of the first subsystem do not resonate with the zeros of the second subsystem, namely

$$rank \begin{pmatrix} \lambda I - F & G \\ H & 0 \end{pmatrix} = n + 1 \qquad \forall \lambda \in \sigma(A)$$

5.7 reduced order observer

$$\begin{aligned} \dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n \\ y = Cx \qquad y \in \mathbb{R}^p \end{aligned}$$

we suppose $p = 2$ and $C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$. The observer can therefore be of dimension $n - p$ as p states are already given. We apply a change of coordinates:

$$z = \begin{pmatrix} C \\ \star \end{pmatrix} x \quad \implies \quad z = \begin{pmatrix} y \\ \xi \end{pmatrix} \qquad \begin{cases} \dot{y}(t) = A_y y + A_{y\xi}\xi + B_y u \\ \dot{\xi} = A_{\xi y} y + A_\xi \xi + B_\xi u \end{cases}$$

therefore

$$\begin{pmatrix} A_y & A_{y\xi} \\ A_{\xi y} & A_\xi \end{pmatrix} = TAT^{-1} \qquad \begin{pmatrix} B_y \\ B_\xi \end{pmatrix} = TB$$

we rename

$$\dot{y} - A_y y - B_y u = \bar{y} \text{ and } A_{y\xi} = \bar{C}_\xi$$

we build our reduced order observer as

$$\dot{\hat{\xi}} A_\xi \hat{\xi} + \begin{bmatrix} A_{\xi y} & B_\xi \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} + L(\bar{C}_\xi \hat{\xi} - \bar{y})$$

as with the full Luenberger observer, we define the error:

$$\begin{aligned} e_\xi = \hat{\xi} - \xi \\ \dot{e} = (A_\xi + L\bar{C}_\xi) e_\xi \end{aligned}$$

If $(A_\xi, \bar{C}_\xi)$ is detectable, there exists L such that $(A_\xi + L\bar{C}_\xi)$ is Hurwitz, and the state is recovered as:

$$x = T^{-1}\begin{pmatrix} y \\ \xi \end{pmatrix} \implies \hat{x} = T^{-1}\begin{pmatrix} y \\ \hat{\xi} \end{pmatrix}$$

We know that (A, C) detectable $\implies (A_\xi, \bar{C}_\xi)$ detectable. However, we don't know $\dot{y}$. The solution is another change of variables:

$$\hat{\xi} \to \hat{\chi} = \hat{\xi} + Ly \qquad \implies \qquad \dot{\hat{\chi}} = A_\xi \hat{\xi} + B_\xi \bar{u} + L\bar{C}\xi\hat{\xi}$$

We notice that $\dot{y}$ has been eliminated without any knowledge of it. From the estimate $\hat{\chi}$ we can recover the estimate $\hat{\xi}$ simply by subtracting Ly which is assumed to be available.

Chapter 6

Separation principle

State feedback designed as if x was available + substitute it with the estimate provided by the observer

$$\left.\begin{array}{r}\dot{\hat{x}}(t)\\ \hat{x}(t+1)\end{array}\right\} = A\hat{x}(t) + Bu(t) + L(\hat{y}(t) - y(t))$$
$$u(t) = Kx(t) + v(t)$$

The resulting system is:

$$\left.\begin{array}{r}\dot{\hat{x}}(t)\\ \hat{x}(t+1)\end{array}\right\} = A\hat{x}(t) + B\left[K\hat{x} + v\right] + L(\hat{y}(t) - y(t))$$
$$u = K\hat{x} + v$$

Changing coordinates:

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} x \\ e := \hat{x} - x \end{pmatrix} = T\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

The closed loop is:

$$\left.\begin{array}{r}\dot{e}(t)\\ e(t+1)\end{array}\right\} = (A + LC)e(t) \longrightarrow \quad \left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = (A + BK)x(t) + BKe(t) + Bv$$

This is a cascade, the state matrix for the cascade is

$$\begin{pmatrix} A + BK & BK \\ 0 & A + LC \end{pmatrix}$$

and the eigenvalues of the system are $\sigma(A+BK) \cup \sigma(A+LC)$. Under stabilizability and detectability assumptions it is possible to design K and L so that the closed loop system is Hurwitz/Schur. If the pair (A, C) is completely controllable then the error $e(t)$ can be steered to arbitrarily small values in an arbitrarily small amount of time thus recovering the ideal state feedback dynamics.

Chapter 7

Ultimate Kalman

Suppose that $rankR < n$ and $rankO < n$. We would like to find a change of coordinates that can isolate dynamics that are:

- controllable and not observable
- controllable and observable
- not controllable and not observable
- not controllable and observable

Connections fromt the top to the bottom would all violate non-reachability/non-observability properties of the subsystems, while most connections from the bottom to the top are permissible.

Theorem 7.0.1. Given (A, B, C) The following T:

$$T^{-1} := \begin{bmatrix} \mathcal{R}^+ \cap \mathcal{E}_{NO} & \mathcal{R}^+ \cap \mathcal{E}^+ & \mathcal{R}^+_{NR} \cap \mathcal{E}^+_{NO} & \mathcal{R}^+_{NR} \cap \mathcal{E}^+ \end{bmatrix}$$

makes the system in the following Kalman form:

$$\tilde{A} = TAT^{-1} := \begin{pmatrix} A_{R,NO} & A'_{R,NO} & A''_{R,NO} & A'''_{R,NO} \\ 0 & A_{R,O} & 0 & A'_{R,O} \\ 0 & 0 & A_{NR,NO} & A'_{NR,NO} \\ 0 & 0 & 0 & A_{NR,O} \end{pmatrix} \quad \tilde{B} = TB := \begin{pmatrix} B_{R,NO} \\ B_{R,O} \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{C} = CT^{-1} := \begin{pmatrix} 0 & C_{R,O} & 0 & C_{NR,O} \end{pmatrix}$$

where

$$(A_{R,NO}, B_{R,NO}) \text{ is completely controllable} \qquad (A_{R,O}, C_{R,O}) \text{ is completely observable}$$

$$\left(\begin{pmatrix} A_{R,NO} & A'_{R,NO} \\ 0 & A_{R,O} \end{pmatrix}, \begin{pmatrix} B_{R,NO} \\ B_{R,O} \end{pmatrix} \right) \text{ is completely controllable}$$

$$\left(\begin{pmatrix} A_{R,O} & A'_{R,O} \\ 0 & A_{NR,O} \end{pmatrix}, \begin{pmatrix} C_{R,O} & C_{NR_O} \end{pmatrix} \right) \text{ is completely observable}$$

Chapter 8

NL control via linearization

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = f(x(t),u(t)) \qquad x(0)=x_0$$
$$y(t) = h(x(t))$$

8.1 set point control

Goal: to steer the regulated output y (coincident with the measured output) to a set point $y^\star =$ const. by properly controlling the system via state or output feedback

Let $(x^\star, u^\star)$ be the solution of the system inversion

Observation.

- $(x^\star, u^\star)$ are the desired steady state for the state and the input
- they are uncertain if the system is

8.1.1 Linearization

$$\dot{x} = f(x,u) = f(x^\star,u^\star) + \left.\frac{\partial f(x,u)}{\partial x}\right|_{x^\star,u^\star}(x-x^\star) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{x^\star,u^\star}(u-u^\star) + h.o.t(\tilde{x},\tilde{u})$$
$$y = h(x) = h(x^\star) + \left.\frac{\partial h(x)}{\partial x}\right|_{x^\star}(x-x^\star) + h.o.t(\tilde{x})$$

$$A = \begin{pmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \frac{\partial f_1(x,u)}{\partial x_2} & \dots & \frac{\partial f_1(x,u)}{\partial x_n} \\ \frac{\partial f_2(x,u)}{\partial x_1} & \frac{\partial f_2(x,u)}{\partial x_2} & \dots & \frac{\partial f_2(x,u)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(x,u)}{\partial x_1} & \frac{\partial f_n(x,u)}{\partial x_2} & \dots & \frac{\partial f_n(x,u)}{\partial x_n} \end{pmatrix}_{x=x^\star,u=u^\star} \qquad B = \begin{pmatrix} \frac{\partial f_1(x,u)}{\partial u} \\ \frac{\partial f_2(x,u)}{\partial u} \\ \vdots \\ \frac{\partial f_n(x,u)}{\partial u} \end{pmatrix}_{x=x^\star,u=u^\star} \qquad C^T = \begin{pmatrix} \frac{\partial h(x)}{\partial x} \\ \frac{\partial h(x)}{\partial x} \\ \vdots \\ \frac{\partial h(x)}{\partial x} \end{pmatrix}_{x=x^\star}$$

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = A\tilde{x}(t) + B\tilde{u}(t) + g_f(\tilde{x}(t),\tilde{u}(t)) \qquad \tilde{x}(0)=\tilde{x}_0$$
$$\tilde{y}(t) = C\tilde{x}(t) + g_h(\tilde{x}(t))$$

where

$$\tilde{x} := x - x^\star$$
$$\tilde{u} := u - u^\star$$
$$\tilde{y} := y - y^\star$$

8.1.2 state feedback solution

If (A, B) is stabilizable we could design $u(t)$ so that $\tilde{u}(t) = K\tilde{x}(t)$ with K such that $A + BK$ is Hurwitz/Schur

$$\left.\begin{array}{r}\dot{\tilde{x}}(t)\\ \tilde{x}(t+1)\end{array}\right\} = (A + BK)\tilde{x}(t) + g_f(\tilde{x}(t), K\tilde{x}(t)) \qquad \tilde{x}(0) = \tilde{x}_0$$
$$u(t) = u^\star + K(x(t) - x^\star)$$

By the indirect Lyapunov theorem we know that $\tilde{x} = 0$ is LAS for the nonlinear controlled system.
Robustness issues:

- $u^\star$ is uncertain
- $K : A_\mu B_\mu K$ Hurwitz. If the actual values (A, B) are close to the nominal ones (A_μ, B_μ) it's not a big problem

This solution is only valid in the domain of attraction, which depends on the nonlinearities of the system.

8.1.3 output feedback solution

If (A, B) is stabilizable and (A, C) is detectable we could design $u(t)$ so that $\tilde{u}(t) = K\hat{\tilde{x}}(t)$ with K such that $A + BK$ is Hurwitz/Schur and $\hat{\tilde{x}}$ generated by an asymptotic observer in which the output injection matrix L is chosen so that $A + LC$ is Hurwitz/Schur

$$\left.\begin{array}{r}\dot{\hat{\tilde{x}}}(t)\\ \hat{\tilde{x}}(t+1)\end{array}\right\} = (A + BK)\hat{\tilde{x}}(t) + L(C\hat{\tilde{x}}(t) - \tilde{y}(t))$$
$$u(t) = u^\star + K\hat{\tilde{x}}(t)$$

By letting $e := \hat{\tilde{x}} - \tilde{x}$, it turns out that the closed-loop is (C-T case)

$$\begin{pmatrix}\dot{\tilde{x}}\\ \dot{e}\end{pmatrix} = \begin{pmatrix}A + BK & BK\\ 0 & A + LC\end{pmatrix}\begin{pmatrix}\tilde{x}\\ e\end{pmatrix} + \begin{pmatrix}g_f(\tilde{x}, K\tilde{x} + Ke)\\ -g_f(\tilde{x}, K\tilde{x} + Ke) - Lg_h(\tilde{x})\end{pmatrix}$$

By the indirect Lyapunov theorem we know that $(\tilde{x}, e)$ is LAS for the nonlinear controlled system
Same locality and robustness issues as the previous case

8.1.4 Integral action

To robustify the asymptotic performance of the controller we can add integral action by extending the system:

$$\dot{x}(t) = f(x(t), u(t)) \quad x(0) = x_0$$
$$y(t) = h(x(t))$$
$$\dot{\sigma} = \tilde{y} = h(x) - y^\star \quad \text{bunch of p integrators}$$

We deal with square systems: $u, y \in \mathbb{R}^p$ Target equilibrium:

$$x^\star = f(x^\star, u^\star)$$
$$y^\star = h(x^\star)\sigma^\star = \text{any}$$

Linearization of the system around the target equilibrium:

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} + g_f(\tilde{x}, \tilde{u})$$
$$\dot{\tilde{\sigma}} = C\tilde{x} + g_h(\tilde{x})$$

In compact form:

$$\begin{pmatrix} \dot{\tilde{\sigma}} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\sigma} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \tilde{u} + \begin{pmatrix} g_f(\tilde{x}, \tilde{u}) \\ g_h(\tilde{x}) \end{pmatrix} = \mathcal{A} \begin{pmatrix} \tilde{x} \\ \tilde{\sigma} \end{pmatrix} + \mathcal{B}\tilde{u} + \begin{pmatrix} g_f(\tilde{x}, \tilde{u}) \\ g_h(\tilde{x}) \end{pmatrix}$$

if $(\mathcal{A}, \mathcal{B})$ is stabiilizable then there exists a $\mathcal{K} = [K_x, K_\sigma]$ such that $\mathcal{A} + \mathcal{B}\mathcal{K}$ is Hurwitz. So, let's choose $\tilde{u} = K_x\tilde{x} + K_\sigma\tilde{\sigma}$ so that the origin $(\tilde{x}, \tilde{\sigma}) = (0, 0)$ is LAS (robust). (Note: K_σ is square $(p \times p)$)

Result. if $\mathcal{K}$ is such that $\mathcal{A} + \mathcal{B}\mathcal{K}$ is Hurwitz, then necessarily K_σ is non singular

we can pick $\sigma^\star = \bar{\sigma}^\star := K_\sigma^{-1}(u^\star - K_x x^\star)$

Result. $(\mathcal{A}, \mathcal{B})$ is controllable if (A, B) is controllable and the following non resonance condition is fulifled:

$$rank \begin{pmatrix} -A & B \\ C & 0 \end{pmatrix} = n + p$$

The initial condition must lay within the domain of attraction in order for the LAS property of the system to be of any use. $\sigma(0)$ must therefore be initialized to an appropriate value inside the domain of attraction. If the uncertainties on the system are large, the chosen $\sigma(0)$ might fall outside the actual domain of attraction.

8.1.5 Gain scheduling

The idea is, in order to overcome the locality of the linearization approach, to switch between setpoints, each having their own linearized control, with overlapping domains of attraction.

Let's introduce a gain scheduling variable α and let $(x^\star(\alpha), u^\star(\alpha))$ be solution of

$$\begin{cases} 0 = f(x^\star(\alpha), u^\star(\alpha)) \\ \alpha = h(x^\star(\alpha)) \end{cases}$$

Following the design procedure proposed in the robust integral-based solution let

$$\mathcal{A}(\alpha) = \begin{pmatrix} A(\alpha) & 0 \\ C(\alpha) & 0 \end{pmatrix} \quad \mathcal{B}(\alpha) = \begin{pmatrix} B(\alpha) \\ 0 \end{pmatrix} \text{ with } \mathcal{K}(\alpha) = [K_x(\alpha), K_\sigma(\alpha)] \text{ such that } \mathcal{A}(\alpha) + \mathcal{B}(\alpha)\mathcal{K}(\alpha) \text{ is Hurwitz}$$

This makes a robust controller parametrized by α:

$$\begin{aligned} \dot{\sigma} &= y - \alpha \\ u &= K_x(\alpha)x + K_\sigma(\alpha)\sigma \end{aligned}$$

If $(x(0), \sigma(0))$ are sufficiently close to $(x^\star(\alpha), \bar{\sigma}^\star(\alpha))$ then $x(t) \to x^\star(\alpha)$ and $y(t) \to \alpha$. Idea: to substitute α with a slow varying $y^\star(t)$. We get a linear time-varying controller:

$$\begin{aligned} \dot{\sigma} &= y - y^\star(t) \\ u &= K_x(y^\star(t))x + K_\sigma(y^\star(t))\sigma \end{aligned}$$

Result. $\exists \epsilon_1, \epsilon_2$ (small) positive numbers such that if $\|\dot{y}^\star(t)\| \leq \epsilon_1$ and $\forall (x_0, \sigma_0)$ such that

$$\|(x_0, \sigma_0) - (x^\star(y^\star(0)), \bar{\sigma}^\star(y^\star(0)))\| \leq \epsilon_2$$

then the following holds:

- The trajectories of the closed loop system are bounded
- $\exists c, T > 0$ such that $\|y(t) - y^\star(t)\| \leq c\epsilon_1 \quad \forall t \geq T$
- if $\lim_{t\to\infty} \dot{y}^\star(t) = 0$ then $\lim_{t\to\infty} y(t) - y^\star(t) = 0$

Chapter 9

Feedback linearization

Problem of feedback linearization: Finding a state feedback control law $u = \alpha(x) + v$, whith v an auxiliary input, and a coordinate change $z = \Phi(x)$ such that the controlled system described in the new form $\dot{z} = Az+Bv$ with (A, B) completely controllable.

9.1 relative degree

Definition 9.1.1 (Lie derivative)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$. The Lie derivative of $h(x)$ along $f(x)$ is defined as:

$$L_f h(x) := \frac{dh}{dx} f(x) : \mathbb{R}^n \to \mathbb{R}$$

The 2nd order Lie derivative of $h(x)$ along $f(x)$ is:

$$L_f^2 h(x) := L_f(L_f h(x)) = \frac{d \frac{dh}{dx} f(x)}{dx} f(x) : \mathbb{R}^n \to \mathbb{R}$$

We also define the following:

$$L_f^k h(x) := L_f L_f^{k-1} h(x) \qquad L_f^0 h(x) := h(x)$$
$$L_g L_f h(x) := \frac{d \frac{dh}{dx} f(x)}{dx} g(x) : R^n \to \mathbb{R}$$

Observation. it represents the derivative of $h(x)$ along the direction of $f(x)$
the operator is linear. In particular: $L_f(\alpha_1 h_1(x) + \alpha_2 h_2(x)) = \alpha_1 L_f h_1(x) + \alpha_2 L_f h_2(x)$

Definition 9.1.2 (relative degree)**.** The system $(f(x), g(x), h(x))$ has relative degree r at $\bar{x}$ if

- $L_g L_f^k h(x) = 0$ $\forall x$ in a neighbourhood of $\bar{x}$ and $\forall k = 0, 1, \dots, r-2$
- $L_g L_f^{r-1} h(x) \neq 0$ (and thus by continuity $L_g L_f^{r-1} h(x) \neq 0 \forall x$ in a neighbourhood of $\bar{x}$)

Result. if r exists then $r \leq n$

9.2 Normal form

Result. Suppose that $(f(x), g(x), h(x))$ has relative degree r at $\bar{x}$. Then

$$rank \begin{pmatrix} \frac{dh(x)}{dx} \\ \frac{dL_f h(x)}{dx} \\ \vdots \\ \frac{dL_f^{r-1} h(x)}{dx} \end{pmatrix}_{x=\bar{x}} = r$$

Similarly to what done with the Brunowsky canonical form, we can obtain a diffeomorphism by adding $n-r$ additional functions

$$\Phi(x) = \begin{pmatrix} \Phi_1(x) := \frac{dh(x)}{dx} \\ \vdots \\ \Phi_r(x) := \frac{dL_f^{r-1}h(x)}{dx} \\ \Phi_{r+1}(x) \\ \vdots \\ \Phi_n(x) \end{pmatrix}$$

constructed such that $rank \left.\frac{d\Phi(x)}{dx}\right|_{x=\bar{x}} = n$. Remark: it is always possible to choose $\Phi_i(x), i = r+1, \ldots, n$ such that $L_g\Phi_i(x) = 0$ for all x in a neighbourhood of x
With this change of coordinates

$$z = \Phi(x) \qquad z = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \xi = \begin{pmatrix} \Phi_1(x) \\ \vdots \\ \Phi_r(x) \end{pmatrix} \qquad \eta = \begin{pmatrix} \Phi_{r+1}(x) \\ \vdots \\ \Phi_n(x) \end{pmatrix}$$

we obtain

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} \quad i = 1, \ldots, r-1 \quad y = \xi_1 \\ \dot{\xi}_r &= q(\xi, \eta) + b(\xi, \eta)u \qquad b(\xi, \eta) = L_g L_f^{r-1} h(\Phi^{-1}(z)) \\ \dot{\eta} &= \varphi(\xi, \eta) \end{aligned}$$

9.3 Feedback linearisation

Theorem 9.3.1. The problem of feedback linearisation around a $\bar{x}$ is solvable iff there exists a $h : \mathbb{R}^n \to \mathbb{R}$ such that the system $(f(x), g(x), h(x))$ has a well defined relative degree of n at $\bar{x}$

if part. The normal form of the system is

$$\begin{aligned} \dot{\xi}_{i+1} &= \xi_i \quad i = 1, \ldots, n-1 \\ \dot{\xi}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u \end{aligned}$$

□

The feedback linearising control law is thus

$$u = \frac{1}{L_g L_f^{n-1} h(x)}(-L_f^n h(x) + v)$$

The controllable (A, B) is therefore

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

9.4 Zero dynamics and minimum-phaseness

Let's assume $\bar{x} = 0$ and that $\Phi(\cdot)$ is chosen so that $\Phi(0) = 0$ ($\bar{z} = 0$)
If the system $(f(x), g(x), h(x))$ has a well defined relative degree $r \leq n$ (at $\bar{x} = 0$) then the zero dynamics of the system are $\dot{\eta} = \varphi(\eta, 0)$ (dimension $n-r$)
The terminology "zero dynamics" is justified by the fact that those are dynamics of the system compatible with $y(t) = 0$ $\forall t$ and for the fact that for linear systems the eigenvalues that govern those dynamics coincide with the zeros of the transfer function

Definition 9.4.1 (minimum-phase system)**.** The system $(f(x), g(x), h(x))$ is said to be minimum-phase if $\bar{\eta} = 0$ is LAS for the zero dynamics $\dot{\eta} = \varphi(\eta, 0)$

Definition 9.4.2 (strongly minimum-phase system)**.** The system $(f(x), g(x), h(x))$ is said to be strongly minimum-phase at $\bar{x}$ if the zero dynamics $\dot{\eta} = \varphi(\eta, 0)$ are ISS wrt the input ξ