Autonomous and Mobile Robotics

Giorgio Medico

fall semester 2024

Contents

1 Control of Mobile Robots **3**

1.1 Configuration Space . . . 3
1.2 Constraints . . . 3
1.2.1 Fundamental Definitions . . . 3
1.2.2 Non-Holonomic Constraints . . . 3
1.3 Pfaffian Form of Constraints . . . 4
1.3.1 Admissible Speeds . . . 4
1.4 Kinematic Model of a WMR . . . 4
1.4.1 General Formulation . . . 4
1.5 Unicycle Model . . . 4
1.5.1 Kinematic Model . . . 5
1.6 Control Architecture . . . 5
1.6.1 Control Scheme Components . . . 5
1.6.2 Low-Level Control . . . 5
1.6.3 High-Level Control . . . 5
1.7 Motion Control . . . 6
1.7.1 Problem Statement . . . 6
1.7.2 Control Problems . . . 6
1.8 Trajectory Planning . . . 6
1.8.1 Space-time Separation . . . 6
1.8.2 Differential Flatness Planning . . . 7

Chapter 1

Control of Mobile Robots

1.1 Configuration Space

Definition 1.1.1. The configuration space of a mobile robot has dimensions equal to the number of parameters needed to uniquely describe the configuration of the robot. It is heavily dependent on the structure of the considered robot and is equivalent to the Joint Space for manipulators.

For different robot types, we have:

- Unicycle: $q = [x\ y\ \theta]^T \in \mathbb{R}^2 \times S$
- Bicycle: $q = [x\ y\ \theta\ \gamma]^T \in \mathbb{R}^2 \times S^2$

1.2 Constraints

1.2.1 Fundamental Definitions

Definition 1.2.1 (Constraint)**.** A constraint is any condition imposed on a material system that prevents it from assuming a generic position and/or act of motion.

Definition 1.2.2 (Holonomic Constraint)**.** A material system is subject to holonomic constraints if finite relations between the coordinates of the system are present (position constraints) or if differentiable/integrable relations between the coordinates of the system are present.

Definition 1.2.3 (Non-holonomic Constraint)**.** A constraint is said to be non-holonomic if the differential relation between the coordinates is not reducible to finite form.

1.2.2 Non-Holonomic Constraints

Under the simplifying assumption that each wheel rolls without slipping, we can state that:

- Each wheel introduces a non-holonomic constraint since it does not allow normal translations to the rolling direction
- The wheel constraints the instant robot mobility, without typically reducing the configuration space (e.g., parallel parking)

Without constraints, the system is described by:

$$\begin{cases} \dot{x} = v_t \cos\theta + v_n \cos(\theta + \frac{\pi}{2}) \\ \dot{y} = v_t \sin\theta + v_n \sin(\theta + \frac{\pi}{2}) \end{cases} \tag{1.1}$$

Since there is no slipping in normal direction ($v_n = 0$):

$$\begin{cases} \dot{x} = v_t \cos\theta \\ \dot{y} = v_t \sin\theta \end{cases} \Leftrightarrow \tan\theta = \frac{\dot{y}}{\dot{x}} \Leftrightarrow [\dot{x}\sin\theta - \dot{y}\cos\theta = 0] \tag{1.2}$$

This last equation represents the mobility constraint.

1.3 Pfaffian Form of Constraints

The constraints can be expressed in different forms:

- Constraint vector equation: $a(q)\dot{q} = 0$ (1 wheel)
- Constraints matrix equation: $A(q)\dot{q} = 0$ (N wheels)

A constraint that can be written as $A(q)\dot{q} = 0$ is said to be in Pfaffian form.

Definition 1.3.1 (Non-Holonomic Constraint Properties)**.** A non-holonomic constraint:

- Cannot be fully integrated
- Cannot be written in the configuration space
- Does not restrict the space of configurations but the instant robot mobility

1.3.1 Admissible Speeds

Admissible speeds may be generated by a matrix $G(q)$ such that:

$$\mathrm{Im}(G(q)) = \mathrm{Ker}(A(q)), \quad \forall q \in C \tag{1.3}$$

where C is the Configuration Space.

1.4 Kinematic Model of a WMR

1.4.1 General Formulation

The general kinematic model can be written as:

$$\dot{q} = G(q)v \tag{1.4}$$

This formulation:

- Represents the allowable directions of motion in the configuration space
- Binds speeds in the operational space with speeds in the configuration space
- Is required to deal with common problems in mobile robotics:
  - Trajectory planning
  - Control (High level)
  - Robot localization

1.5 Unicycle Model

Definition 1.5.1. A unicycle is a vehicle with a single adjustable wheel with configuration described by $q = [x\ y\ \theta]^T$

The constraint equation is:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \tag{1.5}$$

In Pfaffian Form:

$$A(q)\dot{q} = 0 \text{ with } A(q) = [\sin\theta, -\cos\theta, 0] \tag{1.6}$$

The kernel of $A(q)$ is:

$$\mathrm{Ker}(A(q)) = \mathrm{span}\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{Im}(G(q)) \tag{1.7}$$

1.5.1 Kinematic Model

The unicycle kinematic model is:

$$\dot{q} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{1.8}$$

Where:

- v: linear velocity of the contact point between wheel and ground
- ω: angular velocity of the robot

1.6 Control Architecture

1.6.1 Control Scheme Components

The control architecture consists of:

- **Actuators:** DC motors, stepper motors
- **End-Effector:** General purpose tool, gripper, hand
- **Sensors:**
  - Proprioceptive: encoder, gyro
  - Exteroceptive: bumpers, rangefinders (infrared, ultrasonic), laser, vision (mono, stereo)
- **Control:**
  - Low-level control
  - High-level control

1.6.2 Low-Level Control

Key characteristics:

- Uses high-gain PI controllers to control robot motors
- Deals only with robot actuators control according to high-level control instructions
- Makes the robot a purely kinematic system when gains are high enough

1.6.3 High-Level Control

Main features:

- Processes and computes signals for low-level controller using sensor data
- Views robot as a purely kinematic system
- Uses speed control signals for mobile robots

1.7 Motion Control

1.7.1 Problem Statement

Given a trajectory or desired configuration, design a control law that leads the robot to:

- Reach the desired configuration
- Follow the trajectory (high-level control)

Key aspects:

- Uses kinematic model for motion control
- Assumes kinematic inputs act directly on configuration variables
- For unicycle and bicycle, control inputs are v and ω

1.7.2 Control Problems

Configuration Regulation

The robot must reach a desired configuration:

$$q_d = [x_d, y_d, \theta_d]^T \tag{1.9}$$

starting from an initial configuration:

$$q_0 = [x_0, y_0, \theta_0]^T \tag{1.10}$$

Trajectory Following and Tracking

The robot must asymptotically follow a desired Cartesian trajectory $[x_d(t), y_d(t)]^T$ from initial configuration $q_0 = [x_0, y_0, \theta_0]^T$ with:

- **Trajectory Following:** depends on parameter s (geometric specifications)
- **Trajectory Tracking:** depends on time t (timing specifications)

1.8 Trajectory Planning

1.8.1 Space-time Separation

For a trajectory $q(t), t \in [t_i, t_f]$ from initial configuration $q(t_i) = q_i$ to final configuration $q(t_f) = q_f$, we can decompose into:

- A path $q(s)$, with $\frac{dq(s)}{ds} \neq 0, \forall s$
- A motion law $s = s(t)$, with $s_i \leq s \leq s_f$

Where:

$$\begin{cases} s(t_i) = s_i \\ s(t_f) = s_f \end{cases} \tag{1.11}$$

And s is monotonic: $\dot{s}(t) \geq 0$

1.8.2 Differential Flatness Planning

Definition 1.8.1. A nonlinear dynamic system $\dot{x} = f(x)+g(x)u$ has the property of differential flatness if there exists a set of outputs y (called flat) such that the system's state x and input u can be expressed algebraically as functions of y and its derivatives:

$$\begin{cases} x = x(y, \dot{y}, \ddot{y}, ..., y^{(r)}) \\ u = u(y, \dot{y}, \ddot{y}, ..., y^{(r)}) \end{cases} \tag{1.12}$$

For the unicycle model, the Cartesian coordinates are flat outputs, and:

$$\begin{cases} x' = \tilde{v}\cos\theta \\ y' = \tilde{v}\sin\theta \\ \theta' = \tilde{\omega} \end{cases} \tag{1.13}$$

The orientation is given by:

$$\theta = \theta(x', y') = \arctan(y'/x') + k\pi, \quad k = 0, 1 \tag{1.14}$$