SET-POINT STABILIZATION PROBLEM

Consider the system

$$\begin{cases} \dot{x} = f(x, u) \\ y_r = h_r(x) \\ y_m = h_m(x) \end{cases}$$

$x(t) \in \mathbb{R}^{n_x}$ (STATE) , $u(t) \in \mathbb{R}^{n_u}$ (CONTROL)

$y_r(t) \in \mathbb{R}^{n_r}$ (REGULATED OUTPUT) , $y_m(t) \in \mathbb{R}^{n_m}$ (MEASURED OUTPUT)

WE ONLY MEASURE $y_m(t)$

Let $y_r^* \in \mathbb{R}^{n_r}$ be a desired value (SET-POINT) for $y_r(t)$

PROBLEM: design a controller of the form

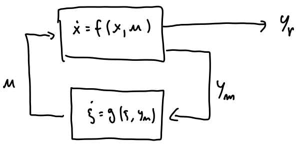
$$\begin{cases} \dot{\xi} = g(\xi, y_m) \\ u = \gamma(\xi, y_m) \end{cases}$$

$\xi(t) \in \mathbb{R}^{n_\xi}$ (CONTROLLER'S STATE)

SUBCASE:
STATIC CONTROLLER:
$u = \gamma(y_m)$

such that the closed-loop system

$$\begin{cases} \dot{x} = f(x, \gamma(\xi, h_m(x))) \\ \dot{\xi} = g(\xi, h_m(x)) \\ y_r = h_r(x) \end{cases} \qquad (*)$$



Satisfies the following:

1. $\exists$ equilibrium point $(x^*, \xi^*) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$ of $(*)$ such that:

$$y_r^* = h_r(x^*)$$

<- The regulated output y_r equals y_r^* at the equilibrium

2. The equilibrium (x^*, ξ^*) is (LOCALLY / GLOBALLY / ...) asymptotically stable for $(*)$

This problem will be our main focus throughout the module

BIBLIOGRAPHY: H. KHALIL, Nonlinear systems (Chap. 12.2)

- NECESSARY CONDITION: If points 1 and 2 are satisfied, then there exist $(x^*, u^*) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ such that:

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

(SOLVABILITY Eq.)

↑

Property of the plant

and the point ξ^* satisfies

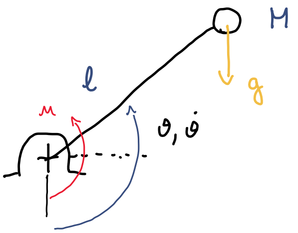
$$\begin{cases} 0 = g(\xi^*, h_m(x^*)) \\ u^* = \gamma(\xi^*, h_m(x^*)) \end{cases}$$

(REGULATOR Eq.)

↑

Prop. of the controller

EXAMPLE: ACTUATED PENDULUM



GOAL: stabilize the pendulum to a given position θ^*

pendulum equations: $(x_1 = \theta, \quad x_2 = \dot{\theta})$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{\beta}{Ml^2} x_2 + \frac{1}{Ml^2} u \\ y_m = x \\ y_r = x_1 \end{cases}$$

SET-POINT: $y_r^* = \theta^*$

Let us solve the SOLVABILITY EQUATIONS : we look for (x^*, u^*) s.t.

$$\begin{cases} 0 = x_2^* \\ 0 = -\frac{g}{\ell}\sin x_1^* - \frac{\beta}{M\ell^2}\overset{0}{\overset{\parallel}{x_2^*}} + \frac{1}{M\ell^2}u^* \\ y_r^* = x_1^* \end{cases} \rightarrow \begin{cases} x^* = (\theta^*, 0) \\ \underline{u^* = Mg\ell \sin\theta^*} \end{cases}$$

the REGULATOR Eqs. tell us that only controller solving the problem must provide this control action at steady-state

the controller does not switch off at steady-state except when $\theta^* = k\pi$ $(k \in \mathbb{N})$

LOCAL STATE-FEEDBACK SOLUTION

- STATE-FEEDBACK : $\boxed{y_m(x) = x}$ (we measure the full state)

- SOLUTION APPROACH BY LINEAR FEEDBACK + LYAPUNOV INDIRECT METHOD

STEP 1) given y_r^*, we solve the SOLVABILITY EQs :

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

STEP 2) we LINEARIZE the system $\dot{x} = f(x,u)$ around (x^*, u^*) obtaining the matrices

$$A = \frac{\partial f}{\partial x}(x^*, u^*) \qquad B = \frac{\partial f}{\partial u}(x^*, u^*)$$

STEP 3) We choose the controller

$$\boxed{u(t) = u^* + K\left(x(t) - x^*\right)} \quad (*) \quad \left(u^* = \text{FEED FORWARD ACTION}\right)$$

where K is chosen so that $\underbrace{A+BK \text{ is Hurwitz}}$

we need (A,B) stabilizable !

RESULT. With the controller $(*)$, the equilibrium x^* is LAS.
Hence, there exists an open set $\mathcal{O} \subset \mathbb{R}^n$ such that $x^* \in \mathcal{O}$ and

$$\forall\, x(0) \in \mathcal{O}, \qquad \lim_{t\to\infty} y_r(t) = y_r^*$$

PROOF.

We can write the plant as

$$\dot{x} = A\tilde{x} + B\hat{u} + f_{HOT}(x,u) \quad (\blacksquare) \qquad \left(\hat{x} = x - x^*,\ \hat{u} = u - u^*\right)$$

where

$$f_{HOT}(x,u) = f(x,u) - A\tilde{x} - B\tilde{u}$$

satisfies

$$\frac{\partial f_{HOT}}{\partial(x,u)}(x^*,u^*) = \left[\frac{\partial f_{HOT}(x^*,u^*)}{\partial x} - A \;\vdots\; \frac{\partial f_{HOT}}{\partial u}(x^*,u^*) - B\right] = 0$$

We can rewrite $(*)$ as

$$\tilde{u} = K\tilde{x}$$

Plugging this into $(\blacksquare)$ leads to the closed-loop system:

$$\dot{x} = (A+BK)\tilde{x} + f_{HOT}\left(x, u^* + K\tilde{x}\right) \doteq F(x)$$

The linearization of $\dot{x} = F(x)$ around x^* is

$$\frac{\partial F}{\partial x}(x^*) = \frac{\partial}{\partial x}\Big[(A+BK)(x-x^*)\Big]\Big|_{x=x^*} + \underbrace{\cancel{\frac{\partial}{\partial x} f_{HOT}(x^*, u^* + K\hat{x})}}_{\text{see above}} \; {=0}$$

$$= A+BK$$

Since $A+BK$ is Hurwitz by design, the result follows from Lyapunov's indirect theorem. $\square$

EXAMPLE:

consider the system

$$\begin{cases} \dot{x} = x^3 + u^2 \\ y_r = y_m = x \end{cases}$$

CASE 1 : set-point control to $y_r^* = 1$

The SOLVABILITY CONDITIONS read

$$\begin{cases} 0 = {x^*}^3 + {u^*}^2 \\ 1 = x^* \end{cases} \Rightarrow {u^*}^2 = -1 \qquad \text{IMPOSSIBLE}$$

we cannot solve case 1.

CASE 2, $y_r^* = 0$.

. the SOLVABILITY CONDITIONS are:

$$\begin{cases} 0 = {x^*}^3 + {u^*}^2 \\ 0 = x^* \end{cases} \Rightarrow \begin{cases} x^* = 0 \\ u^* = 0 \end{cases}$$

• To define the control law we first need to linearize the system around (x^*, u^*)

$$A = \frac{\partial f}{\partial x}(x^*, u^*) = \frac{\partial}{\partial x}\left(x^3 + u^2\right)\Big|_{\substack{x=0\\u=0}} = 3x^2\Big|_{x=0} = 0$$

$$B = \frac{\partial f}{\partial u}(x^*, u^*) = \frac{\partial}{\partial u}\left(x^2 + u^2\right)\Big|_{\substack{x=0\\u=0}} = 2u\Big|_{u=0} = 0$$

$\rightsquigarrow$ $A = 0$, $B = 0$ $\Rightarrow$ (A, B) IS NOT CONTROLLABLE

$\Rightarrow$ we cannot solve case 2

CASE 3. $y_r^* = -1$

the SOLVABILITY CONDITIONS are:

$$\begin{cases} 0 = x^{*3} + u^{*2} \\ -1 = x^* \end{cases} \Rightarrow \begin{cases} x^* = -1 \\ u^* = 1 \end{cases}$$

The linearization matrices are

$$A = 3x^2\Big|_{x=-1} = 3$$

$$B = 2u\Big|_{u=1} = 2$$

$\Rightarrow$ (A, B) is controllable

For every $K < -\frac{3}{2}$, the matrix $A + BK$ is Hurwitz since:

$$A + BK = 3 + 2K < 0$$

Then, $\boxed{\forall K < -\frac{3}{2}}$, the controller

$$u(t) = 1 + K\left(x(t) + 1\right) \qquad (\bullet)$$

locally stabilizes the eq. point where $y^* = -1$.

What about the domain of attraction?

Plugging $(\bullet)$ into the system's equations leads to

$$\dot{x} = x^3 + \left(1 + K(x+1)\right)^2 = x^3 + 1 + K^2(x+1)^2 + 2K(x+1)$$

Changing coordinates from x to $\tilde{x} := x - x^* = x + 1$ leads to

$$\dot{\tilde{x}} = \dot{x} + \dot{1} = \dot{x} = x^3 + 1 + k^2 (x+1)^2 + 2k(x+1)$$

$$= (\tilde{x} - 1)^3 + 1 + k^2 \tilde{x}^2 + 2k\tilde{x}$$

$$= \tilde{x}^3 - 3\tilde{x}^2 + 3\tilde{x} + k^2 \tilde{x}^2 + 2k\tilde{x}$$

$$= \tilde{x}\left(3 + 2k + (k^2 - 3)\tilde{x} + \tilde{x}^2\right)$$

Considering the Lyapunov candidate $V(x) = (x+1)^2 = \tilde{x}^2$, we get

$$\frac{\partial V}{\partial x}(x) \cdot f(x) = 2\tilde{x}\dot{\tilde{x}} = 2\tilde{x}^2 \left(3 + 2k + (k^2 - 3)\tilde{x} + \tilde{x}^2\right)$$

$$= 2V(x) \cdot \underbrace{\left[3 + 2k + (k^2-3)\tilde{x} + \tilde{x}^2\right]}_{\text{we want this to be negative}}$$

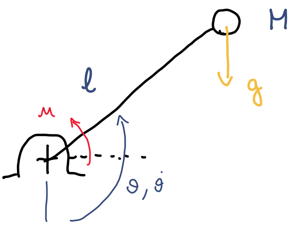
$$\Rightarrow \frac{\partial V}{\partial x}(x) f(x) < 0 \quad \text{as long as} \quad 3 + 2k + (k^2 - 3)\tilde{x} + \tilde{x}^2 < 0$$

so, we need to solve $\tilde{x}^2 + (k^2-3)\tilde{x} + 3 + 2k < 0$

$$\tilde{x} < \frac{3-k^2}{2} + \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)} \quad \vee \quad \tilde{x} > \frac{3-k^2}{2} - \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)}$$

$$D = \left(x^* + \frac{3-k^2}{2} - \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)}\ ,\ x^* + \frac{3-k^2}{2} + \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)}\right) \quad \square$$

EXAMPLE: ACTUATED PENDULUM



GOAL: stabilize the pendulum to a given position θ^*

pendulum equations: $\quad (x_1 = \theta \;,\; x_2 = \dot{\theta})$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{\ell} \sin x_1 - \frac{\beta}{M\ell^2} x_2 + \frac{1}{M\ell^2} u \\ y_m = x \\ y_r = x_1 \end{cases}$$

$$y_r^* = \theta^*$$

we saw before that the SOLVABILITY EQS. have the solution

$$\begin{cases} x^* = (\theta^*, 0) \\ u^* = Mg\ell \sin \theta^* \end{cases}$$

Next, we linearize the system around (x^*, u^*). We have

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \quad \text{where} \quad \begin{cases} f_1(x,u) = x_2 \\ f_2(x,u) = -\frac{g}{\ell} \sin x_1 - \frac{\beta}{M\ell^2} x_2 + \frac{1}{M\ell^2} u \end{cases}$$

- $\frac{\partial f_1}{\partial x_1}(x,u) = 0 \quad , \quad \frac{\partial f_1}{\partial x_2}(x,u) = 1$

- $\frac{\partial f_2}{\partial x_1}(x,u) = -\frac{g}{\ell} \cos x_1 \quad , \quad \frac{\partial f_2}{\partial x_2}(x,u) = -\frac{\beta}{M\ell^2}$

Thus, we obtain:

$$A = \frac{\partial f}{\partial x}(x^*, u^*) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos\theta^* & -\frac{\beta}{M\ell^2} \end{pmatrix}$$

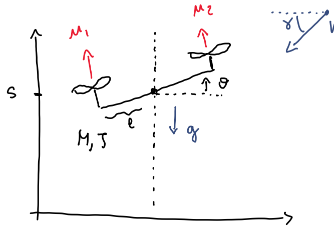
$\leftarrow$ [If $\theta^* > \frac{\pi}{2}$, the forced equilibri[…] are unstable in open loop

$$B = \frac{\partial f}{\partial u}(x^*, u^*) = \begin{pmatrix} \frac{\partial f_1}{\partial u}(x^*, u^*) \\ \frac{\partial f_2}{\partial u}(x^*, u^*) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{M\ell^2} \end{pmatrix}$$

- Since (A, B) is controllable, we can always design K so that $A+BK$ is Hurwitz, for instance:

$$K = \left[\, M\ell g(\cos\theta^* - \varepsilon) \quad 0 \,\right] \qquad \text{for any } \varepsilon > 0$$

EXAMPLE: PLANAR DRONE



$$M\ddot{s} = (u_1 + u_2)\cos\theta - Mg - W\sin\gamma - \mu\dot{s}$$

$$J\ddot{\theta} = -\beta\dot{\theta} + \ell(u_2 - u_1)$$

GOAL: stabilize vertical velocity $\dot{s}$ to 0

We neglect the vertical position and call $x_1 = \dot{s}$, $x_2 = \theta$, $x_3 = \dot{\theta}$

The model reads:

$$\begin{cases} \dot{x}_1 = \frac{1}{M}\cos x_2 \cdot (u_1 + u_2) - g - \frac{W}{M}\sin\gamma - \frac{\mu}{M}x_1 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\frac{\beta}{J}x_3 + \frac{\ell}{J}(u_2 - u_1) \\ y_r = x_1 \\ y_m = x \end{cases}$$

SOLVABILITY CONDITIONS: (GOAL $y_r^* = 0$)

$$\begin{cases} x_1^* = 0 \\ 0 = \cos x_2^* \cdot (u_1^* + u_2^*) - Mg - w\sin\gamma - \mu x_1^* \\ 0 = x_3^* \\ 0 = -\beta x_3^* + \ell(u_2^* - u_1^*) \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1^* = x_3^* = 0 \\ \cos x_2^* (u_1^* + u_2^*) = Mg + w\sin\gamma \\ u_2^* - u_1^* = 0 \end{cases} \Leftrightarrow \begin{cases} x_1^* = 0 \\ x_2^* = \text{FREE} \quad (\neq \frac{k}{2}\pi, k\in\mathbb{N}_{>0}) \\ u_1^* = u_2^* = \dfrac{Mg + w\sin\gamma}{2\cos x_2^*} \end{cases}$$

Next, we linearize:

$$f(x,u) = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \end{pmatrix}$$

$$f_1(x,u) = \frac{1}{M}\cos x_2 (u_1 + u_2) - g - \frac{w}{M}\sin\gamma - \frac{\mu}{M} x_1$$

$$f_2(x,u) = x_3$$

$$f_3(x,u) = -\frac{\beta}{J} x_3 + \frac{\ell}{J}(u_2 - u_1)$$

$$\frac{\partial f}{\partial x}(x,u) = \begin{pmatrix} -\frac{\mu}{M} & -\frac{1}{M}\sin x_2 (u_1+u_2) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{\beta}{J} \end{pmatrix} \Rightarrow A = \begin{pmatrix} -\frac{\mu}{M} & -\frac{Mg + w\sin\gamma}{M}\tan x_2^* & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{\beta}{J} \end{pmatrix}$$

$$\frac{\partial f}{\partial u}(x,u) = \begin{pmatrix} \frac{1}{M}\cos x_2 & \frac{1}{M}\cos x_2 \\ 0 & 0 \\ -\frac{\ell}{J} & \frac{\ell}{J} \end{pmatrix} \to B = \begin{pmatrix} \frac{1}{M}\cos x_2^* & \frac{1}{M}\cos x_2^* \\ 0 & 0 \\ -\frac{\ell}{J} & \frac{\ell}{J} \end{pmatrix}$$

Is (A,B) stabilizable?

$$R = [B \,\vdots\, AB \,\vdots\, A^2B] = \underbrace{\left(\begin{array}{cc|c} \frac{1}{M}\cos x_2^* & \frac{1}{M}\cos x_2^* & -\frac{\mu}{M^2}\cos x_2^* \\ 0 & 0 & -\frac{\ell}{J} \\ -\frac{\ell}{J} & \frac{\ell}{J} & \frac{\beta\ell}{J^2} \end{array}\right.}_{\text{Full-rank (for } x_2^* \neq \frac{k}{2}\pi)} \;\cdots \cancel{*} \cdots \Bigg)$$

yes, (A,B) is controllable

↳ the design is concluded by taking K such that $A+BK$ is Hurwitz and

$$u(t) = u^* + K\left(x(t) - x^*\right)$$

$$= \underbrace{\begin{pmatrix} \dfrac{Mg + w\sin\gamma}{2\cos x_2^*} \\ \dfrac{Mg + w\sin\gamma}{2\cos x_2^*} \end{pmatrix}}_{\text{FEED FORWARD ACTION}} + \underbrace{K \cdot \begin{pmatrix} x_1(t) \\ x_2(t) - x_2^* \\ x_3(t) \end{pmatrix}}_{\text{ERROR FEEDBACK}}$$

LOCAL OUTPUT - FEEDBACK SOLUTION

In this case $\boxed{y_m(x) \neq x}$, typically we only measure a part of the state

We can resort to the linear output - feedback theory, by which we implement

$$u(t) = u^* + K\hat{\tilde{x}}(t)$$

(instead of $u = u^* + K(x - x^*)$)

in which $\hat{\tilde{x}}$ is an estimate of $\tilde{x} \doteq x - x^*$.

we can write the plant as before as

$$\dot{\tilde{x}} = A\tilde{x} + B\hat{u} + f_{HOT}(x, u)$$

where

$$\hat{u} \doteq u - u^* \quad , \quad A = \frac{\partial f}{\partial x}(x^*, u^*) \quad , \quad B = \frac{\partial f}{\partial u}(x^*, u^*)$$

$$f_{HOT}(x, u) = f(x, u) - A\tilde{x} - B\hat{u}$$

) → HIGHER - ORDER TERMS

we add the output equation

$$\tilde{y}_m = h_m(x) - h_m(x^*) = C\tilde{x} + h_{m,HOT}(x)$$

where

$$C \doteq \frac{\partial h_m}{\partial x}(x^*) \qquad \text{and} \qquad \underbrace{h_{m,HOT}(x) = h_m(x) - h_m(x^*) - C\tilde{x}}_{= \text{ HIGHER ORDER TERM}}$$

If (C, A) is detectable, we can find L such that $A + LC$ is Hurwitz, and we can define the Luenberger observer

$$\boxed{\dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + B\tilde{u} + L(C\hat{\tilde{x}} - \tilde{y}_m)}$$

$$= A\hat{\tilde{x}} + B(u^* + K\hat{\tilde{x}}) + L(C\hat{\tilde{x}} - y_m + h_m(x^*))$$

The overall controller is then

$$(\blacktriangle) \quad \begin{cases} \dot{\hat{\tilde{x}}} = (A + BK + LC)\hat{\tilde{x}} - L\tilde{y}_m \\ u = u^* + K\hat{\tilde{x}} \end{cases} \longrightarrow$$

of the form

$$\begin{cases} \dot{\xi} = g(\xi, y_m) \\ u = \gamma(\xi, y_m) \end{cases}$$

with $\xi = \hat{\tilde{x}}$

RESULT. Suppose that (A, B) is stabilizable and (C, A) detectable. The dynamic controller $(\blacktriangle)$ locally stabilizes the equilibrium $(x, \hat{\tilde{x}}) = (x^*, 0)$. Therefore, there exists an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ containing $(x^*, 0)$ such that

$$\forall\, (x(0), \hat{\tilde{x}}(0)) \in \Omega, \qquad \lim_{t\to\infty} y_r(t) = y_r^*$$

PROOF.

Let us analyze the closed-loop system, which has state $(\tilde{x}, \hat{\hat{x}})$

Changing variables from $\hat{\hat{x}}$ to

$$e := \hat{\hat{x}} - \tilde{x}$$

leads to the closed-loop system

$$\dot{\tilde{x}} = A\tilde{x} + B\underbrace{\tilde{u}}_{\tilde{u} = K\hat{\hat{x}} = K(e+\tilde{x})} + f_{HOT}(x,u) = (A+BK)\tilde{x} + BKe + f_{HOT}(x,u)$$

$$\begin{aligned}
\dot{e} = \dot{\hat{\hat{x}}} - \dot{\tilde{x}} &= (A+BK+LC)\hat{\hat{x}} - L\tilde{y}_m - (A+BK)\tilde{x} - BKe - f_{HOT}(x,u) \\
&= (A+BK+LC)(e+\tilde{x}) - (A+BK)\tilde{x} - BKe - L\left(C\tilde{x} + h_{m,HOT}(x)\right) - f_{HOT}(x,u) \\
&= (A+LC)e - \left(f_{HOT}(x,u) + L\,h_{m,HOT}(x)\right)
\end{aligned}$$

hence:

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{e} \end{pmatrix} = \underbrace{\begin{pmatrix} A+BK & BK \\ 0 & A+LC \end{pmatrix}}_{\text{LINEAR PART}} \begin{pmatrix} \tilde{x} \\ e \end{pmatrix} + \underbrace{\begin{pmatrix} f_{HOT}(x,u) \\ -f_{HOT}(x,u) - L h_{m,HOT}(x) \end{pmatrix}}_{\text{HIGHER ORDER TERMS}}$$

The linear part is Hurwitz (recall the SEPARATION PRINCIPLE) → By Lyapunov indirect theorem we conclude that $(\tilde{x}, e) = 0$ is LAS

Thus we conclude that the equilibrium point $(x^*, 0)$ is LAS for $(x, \hat{\hat{x}})$

PROBLEMS OF CONTROL VIA LINEARIZATION :

P1) LOCALITY the guarentees are given by Lyapunov's indirect Theorem that, However, only guerentees the existence of an open non-empty domain of attraction, but it does not say anything about its size

sometimes taking large gains K enlarges the doman of attraction. But this is not always the case.

$\downarrow$

this can be seen in the drone example and in the examples below.

EXAMPLE consider the system

$$\dot{x} = xu + u$$

and let us try to stabilize $x^* = 0$ (requiring $u^* = 0$).
The linear part of the dynamics is defined as

$$A = u^* = 0$$
$$B = x^* + 1 = 1$$

The state-feedback controller is: $u(t) = u^* + K(x(t) - x^*) = K\,x(t)$ for some $K < 0$
This leads to

$$\dot{x} = (x+1)Kx = K(x^2 + x) \qquad (+)$$

No matters how large is $|K|$, for all $x(0) < -1$ the closed-loop trajectories are divergent.

$\Rightarrow$ we cannot enlarge the domain of attraction

in particular the solutions of $(+)$ are (recall, $K < 0$)

$$x(t) = x_0 \frac{e^{Kt}}{x_0 + 1 - x_0 e^{Kt}} \rightarrow \text{ If } x_0 < -1,\ \exists\, \bar{t} = \bar{t}(x_0) > 0 \text{ s.t. } \lim_{t \to \bar{t}} x(t) = \infty$$

P2. ROBUSTNESS ISSUE

The controller depends on the quantities x^* and u^* that are highly sensitive to model uncertainties

To find x^* and u^* we need to solve the SOLVABILITY Eqs.

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

Any slight uncertainty in the knowledge of f, h_m, and h_r may produce some wrong values of x^* and u^*

$\downarrow$

A control law that depends so critically from x^* and u^* is FRAGILE and its implementation in practice may lead to problems

(TYPICAL PROBLEM OF FEEDFORWARD CONTROL)

EXAMPLE

- In the previous example regarding the actuated pendulum we had:

$$\begin{cases} x^* = (\theta^*, 0) \\ u^* = Mg\ell \sin\theta^* \end{cases}$$

$\rightarrow$ u^* depends on $\begin{cases} \bullet \text{ mass } M \\ \bullet \text{ length } \ell \\ \bullet \text{ gravity } g \end{cases}$ ALL UNCERTAIN QUANTITIES!

- In the previous example of the planar drone, we had:

$$\begin{cases} x_1^* = 0 \\ x_2^* = \text{FREE} \quad (\neq k\pi, k \in \mathbb{N}) \\ u_1^* = u_2^* = \dfrac{Mg + w \sin\gamma}{2 \cos x_2^*} \end{cases}$$

u^* depends on M, g, and the vertical wind strength $w \sin\gamma$ that are uncertain

Some of the effects of using wrong x^* and u^* are:

- wrong set-point: the state may not converge to x^*

  $\rightarrow \lim_{t \to \infty} y_p(t) \neq y_p^*$

- destabilization: $x(t)$ may exit the stability domain