NONLINEAR CHANGES OF COORDINATES

In the first module we considered changes of coordinates of the form

$$x \mapsto z \doteq Tx$$

The the dynamic equations changed as:

$$\dot{x} = Ax + Bu \quad \longmapsto \quad \dot{z} = T(Ax + Bu)\Big|_{x = T^{-1}z} = TAT^{-1}z + TBu$$

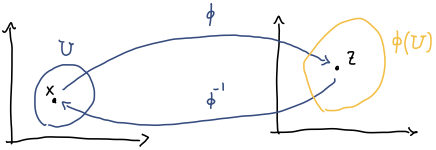
We now develop a similar concept for nonlinear systems

Let $U \subset \mathbb{R}^n$ be open. A function $\phi : U \to \mathbb{R}^n$ is called a DIFFEOMORPHISM if:

- It is INVERTIBLE : $\exists\, \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall x \in U$, $\phi(\phi^{-1}(x)) = x$.
- Both ϕ and ϕ^{-1} are SMOOTH on U

partial derivatives of any order exist and are continuous

[this can be relaxed by asking ϕ and ϕ^{-1} to be just C^1]



If $U = \mathbb{R}^n$, then ϕ is called a GLOBAL DIFFEOMORPHISM

The following result gives sufficient conditions for $\phi : \mathbb{R}^n \to \mathbb{R}^n$ to be a diffeomorphism

RESULT (IMPLICIT FUNCTION THEOREM). Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be smooth, and let $\bar{x} \in \mathbb{R}^n$ be such that the Jacobian of g

$$\frac{dg}{dx}(\bar{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial g_1}{\partial x_n}(\bar{x}) \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial g_n}{\partial x_n}(\bar{x}) \end{pmatrix}$$

is non-singular.

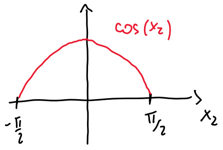
Then, there exists an open set $U \subset \mathbb{R}^n$ containing $\bar{x}$ such that the restriction $\phi : U \to \mathbb{R}^n$, $\phi(x) = g(x)$ of g on U is a diffeomorphism.

EXAMPLE. Let

$$g(x) = \begin{pmatrix} x_1 + x_2 \\ \sin x_2 \end{pmatrix}$$

Its Jacobian is

$$\frac{\partial g}{\partial x}(x) = \begin{pmatrix} 1 & 1 \\ 0 & \cos x_2 \end{pmatrix}$$



For every $\bar{x} \in \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$, $\frac{\partial g}{\partial x}(\bar{x})$ is non-singular

$\Rightarrow \exists\, U \subset \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\phi : U \to \mathbb{R}^2$ (= restriction of g on U) is a diffeomorphism

In this case

$$\phi^{-1}(z) = \begin{pmatrix} z_1 - \arcsin z_2 \\ \arcsin z_2 \end{pmatrix} \qquad \text{and} \qquad U = \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

CHANGE OF VARIABLES

Given a system of the form

$$\dot{x} = f(x, u) \quad , \quad y = h(x)$$

a diffeomorphism $\phi : U \subset \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ induces a CHANGE OF VARIABLES

$$x \mapsto z \doteq \phi(x) \qquad \left(\text{conversely, } x = \phi^{-1}(z)\right)$$

In the new variables we have:

$$\begin{cases} \dot{z} = \tilde{f}(z, u), \\ y = \tilde{h}(z) \end{cases}$$

$$\boxed{\begin{aligned} \tilde{f}(z,u) &\doteq \left.\frac{d\phi}{dx}(x) \cdot f(x,u)\right|_{x = \phi^{-1}(z)} \\ \tilde{h}(z) &\doteq h\left(\phi^{-1}(z)\right) \end{aligned}}$$

Indeed:

$$\dot{z} = \frac{d}{dt}\left(\phi(x)\right) = \frac{d\phi(x)}{dx} \cdot \dot{x} = \frac{d\phi(x)}{dx} \cdot f(x, u)$$

and $x = \phi^{-1}(z)$.

REMARK. If T is invertible then $\phi(x) \doteq Tx$ is a global diffeom.

If $f(x,u) = Ax + Bu$ and $y = Cx$, the previous formula gives

$$\tilde{f}(z,u) = \underbrace{\frac{d\phi(x)}{dx}}_{=T} \cdot f(x,u) \Big|_{\underset{\overset{!}{=} T^{-1}z}{x=\phi^{-1}(z)}} = T \cdot f\left(T^{-1}z, u\right)$$

$$= TAT^{-1}z + TBu$$

$$\tilde{h}(z) = h\left(\phi^{-1}(z)\right) = CT^{-1}z$$

So we recover the linear formulas.

EXAMPLE

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 u \\ \dot{x}_2 = x_1 u \end{cases}$$

change variable through previous $\phi(x) = \begin{pmatrix} x_1 + x_2 \\ \sin x_2 \end{pmatrix}$, $\phi^{-1}(z) = \begin{pmatrix} z_1 - \arcsin z_2 \\ \arcsin z_2 \end{pmatrix}$

$$\tilde{f}(z,u) = \frac{\partial\phi}{\partial x}(x)\, f(x,u) \Big|_{x=\phi^{-1}(z)}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & \cos x_2 \end{pmatrix} \begin{pmatrix} -x_1^3 + x_2 u \\ x_1 u \end{pmatrix} \Bigg|_{x=\phi^{-1}(z)} = \begin{pmatrix} -x_1^3 + (x_1 + x_2) u \\ \cos x_2 \cdot x_1 u \end{pmatrix} \Bigg|_{x=\phi^{-1}(z)}$$

$$= \begin{pmatrix} -(z_1 - \arcsin z_2)^3 + z_1 u \\ (z_1 - \arcsin z_2) \cdot \cos(\arcsin z_2) \cdot u \end{pmatrix}$$

→ In the new coordinates:

$$\begin{cases} \dot{z}_1 = -(z_1 - \arcsin z_2)^3 + z_1 u \\ \dot{z}_2 = (z_1 - \arcsin z_2) \cdot \cos\left(\arcsin z_2\right) \cdot u \end{cases}$$

INPUT - AFFINE SYSTEMS

Systems of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \longleftarrow$$

OUR FOCUS FROM NOW ON

are called "input - affine"

→ given a general system of the form

$$\begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}, \bar{u}) \\ y = \bar{h}(\bar{x}, \bar{u}) \end{cases}$$

I can "extend" the input $\bar{u}$ by defining

$\dot{\bar{u}} = u$ $\quad\quad$ u = new input

$x = (\bar{x}, \bar{u})$ $\quad\quad$ x = new state

The resulting system with input u and state x is input - affine since

$$\begin{aligned} \dot{x} &= f(x) + g(x)\bar{u} \\ y &= h(x) \end{aligned} \quad \text{with} \quad \begin{cases} f(x) = \begin{pmatrix} \bar{f}(x) \\ 0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ h(x) = \bar{h}(x) \end{cases}$$

⇒ Not a big loss of generality to focus on input - affine systems.

RESULT. "Input - affinit - ness" is preserved under changes of variables:

Proof. If

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

and $\phi : U \to \mathbb{R}^{n_x}$ a diffeom.

Then $z = \phi(x)$ satisfies

$$\dot{z} = \frac{\partial \phi}{\partial x}(x) \cdot \dot{x} \Big|_{x = \phi^{-1}(z)} = \underbrace{\frac{\partial \phi}{\partial x}(x) f(x) \Big|_{x=\phi^{-1}(z)}}_{\tilde{f}(z)} + \underbrace{\frac{\partial \phi}{\partial x}(x) g(x) \Big|_{x = \phi^{-1}(z)} \cdot u}_{\tilde{g}(z) \cdot u}$$

$$y = h(x) = \underbrace{h(\phi^{-1}(z))}_{\tilde{h}(z)}$$

∎

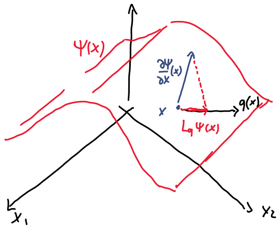
LIE DERIVATIVES

Let $n \in \mathbb{N}$, $q : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}$. The <u>LIE DERIVATIVE of ψ along q</u> is the function

$$L_q \psi : \mathbb{R}^n \to \mathbb{R}$$

defined as

$$L_q \psi(x) = \frac{\partial \psi}{\partial x}(x) \cdot q(x) = \left(\frac{\partial \psi}{\partial x_1}(x) \quad \cdots \quad \frac{\partial \psi}{\partial x_n}(x) \right) \begin{pmatrix} q_1(x) \\ \vdots \\ q_n(x) \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(x)\, q_i(x)$$



$$L_q \psi(x) = \text{projection of } \frac{\partial \psi}{\partial x}(x) \text{ on } q(x)$$

REMARKS

1) $L_q \psi : \mathbb{R}^n \to \mathbb{R}$ $\Rightarrow$ we can define $L_q(L_q \psi)$ as

$$L_q(L_q\psi)(x) = \frac{\partial L_q \psi(x)}{\partial x} \cdot q(x)$$

we use the notation

$$L_q^2 \psi \doteq L_q(L_q \psi)$$

we can as well define

$$L_q^3 \psi \doteq L_q(L_q^2 \psi), \quad L_q^4 \psi \doteq L_q(L_q^3 \psi), \quad \ldots \quad \boxed{L_q^k \psi = L_q L_q^{k-1} \psi}$$

2) If $d : \mathbb{R}^n \to \mathbb{R}^n$ is another function, we can also "mix" Lie derivatives:

$$L_d L_q \psi = \frac{\partial L_q \psi(x)}{\partial x} \cdot d(x)$$

$$L_d^2 L_q^3 \psi, \quad L_q L_d L_q \psi, \quad \ldots$$

3) The operator $L_g : \Psi \mapsto L_g \Psi$ is LINEAR :

$$\forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ and } \Psi_1, \Psi_2 : \mathbb{R}^n \to \mathbb{R}, \quad L_g(\alpha_1 \Psi_1 + \alpha_2 \Psi_2) = \alpha_1 L_g \Psi_1 + \alpha_2 L_g \Psi_2$$

4) Consider an input-affine system "SISO" (single-input-single-output) (namely $n_u = n_y = 1$)

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

Then we have

$$\dot{y} = \frac{d}{dt} h(x) = \frac{\partial h}{\partial x}(x) \cdot \dot{x} = \overbrace{\frac{\partial h}{\partial x}(x)\left(f(x) + g(x)u\right)}^{\text{Lie derivative of the output map along the dynamics}} = L_h f(x) + L_h g(x) \cdot u$$

RELATIVE DEGREE

Consider a SISO system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad n_u = n_y = 1$$

(IN THIS CASE WE LET $\boxed{n = n_x}$ FOR BREVITY)

and a point $\bar{x} \in \mathbb{R}^n$

To see how $u(t)$ affects the output $y(t)$ around $\bar{x}$, we start taking derivatives:

$$\dot{y} = L_f h(x) + L_f g(x) u$$

If $L_f g(\bar{x}) \neq 0$ we stop and say that the system has RELATIVE DEGREE $r = 1$ at $\bar{x}$

$\downarrow$

the input is "ONE DERIVATIVE AWAY"

If, instead, $L_f g(x) = 0$ $\forall x$ in an open set around $\bar{x}$, we keep going: at $x(t) = \bar{x}$ we have

$$\ddot{y} = \frac{d}{dt}\left(L_f h(x) + \underbrace{L_f \cancel{g(x)u}}_{=0}\right) = L_f^2 h(x) + L_g L_f h(x) \cdot u$$

If $L_g L_f h(\bar{x}) \neq 0$ $\to$ we stop and say the RELATIVE DEGREE at $\bar{x}$ is $r = 2$

If $L_g L_f h(x) = 0$ in an open set around $\bar{x}$ we keep going

...

we stop (if possible) when we find $r > 0$ such that $L_g L_f^{r-1} h(\bar{x}) \neq 0$

Intuitively: RELATIVE DEGREE = number of times I need to differentiate the output to have the input appearing

Formally, the system has RELATIVE DEGREE r AT $\bar{x}$ if

1) $L_g L_f^k h(x) = 0$ in an open set around $\bar{x}$ $\forall k = 0, \dots, r-2$

2) $L_g L_f^{r-1} h(\bar{x}) \neq 0$ (by continuity $L_g L_f^{r-1} h(x) \neq 0$ in an open set around $\bar{x}$)

REMARKS.

1) r depends on $\bar{x}$

2) r may not exist

RESULT. If r exists, then $r \leq n$

3) extension to MIMO systems non-trivial

4) for LTI systems: $h(x) = Cx$, $L_f^k h(x) = CA^k x$, $L_g L_f^k h(x) = CA^k B$

↳ the system has relative degree r at any $\bar{x}$ if

1) $CA^k B = 0$ $\forall k = 0, \dots, r-2$

2) $CA^{r-1} B \neq 0$

↳ r equals the difference between the order of the numerator and the denominator of the transfer function $G(s) = C(sI - A)^{-1} B$

↳ r does not depend on $\bar{x}$ (we can speak of "RELATIVE DEGREE of the system")

↳ If (A, B) is controllable then r exists

↳ If $\mathcal{R}^+ \cap (\mathcal{E}^+)^\perp \neq 0$, then r exists

If a system has relative degree r at every $\bar{x} \in \mathbb{R}^n$ we say it has GLOBAL REL. DEGREE r

EXAMPLES (VAN DER POL OSCILLATOR)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + (1-x_1^2)\,x_2 + u \end{cases} \quad \leftarrow \quad \dot{x} = f(x) + g(x)u \quad \text{with}$$

$$f(x) = \begin{pmatrix} x_2 \\ -x_1 + (1-x_1^2)x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we consider two outputs:

CASE I) $y = x_1$

CASE II) $y = \sin x_2$

CASE I $\quad h(x) = x_1$

- $L_g h(x) = \frac{\partial h}{\partial x}(x) \cdot g(x) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

$r \neq 1 \;\; \forall \bar{x} \in \mathbb{R}^n$. Keep going:

- $L_f h(x) = \frac{\partial h}{\partial x}(x) f(x) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ -x_1 + (1-x_1^2)x_2 \end{pmatrix} = x_2$
- $L_g L_f h(x) = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot g(x) = 1$

$\Rightarrow$ the system has GLOBAL REL. DEGREE $r = 2$

CASE II $\quad (h(x) = \sin x_2)$

$$L_g h(x) = \frac{\partial h}{\partial x}(x)\, g(x) = \begin{pmatrix} 0 & \cos x_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos x_2$$

$\Rightarrow r = 1$ at every $\bar{x} \in \left\{ x \in \mathbb{R}^2 : x_2 \neq k\pi - \frac{\pi}{2}, \; k \in \mathbb{Z} \right\}$

If $\bar{x} = \frac{\pi}{2}$ we CANNOT PROCEED since for every open set $U \ni \bar{x}$ there exists $x \in U$ such that $L_g h(x) \neq 0$

$\hookrightarrow$ The system does not have relative degree at $\bar{x} = k\pi - \frac{\pi}{2}$, $k \in \mathbb{Z}$

NORMAL FORM

Consider the system (SISO)

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad x \in \mathbb{R}^n, \qquad u, y \in \mathbb{R}$$

RESULT. If the system has relative degree $r \leq n$ at $\bar{x}$, then

$$\text{rank} \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} h(\bar{x}) \\ \frac{\partial}{\partial x} L_f h(\bar{x}) \\ \vdots \\ \frac{\partial}{\partial x} L_f^{r-1} h(\bar{x}) \end{bmatrix}}_{\in \mathbb{R}^{r \times n}} = r$$

we can define

$$\phi_1(x) \doteq h(x), \qquad \phi_2(x) \doteq L_f h(x), \qquad \dots, \qquad \phi_r(x) = L_f^{r-1} h(x)$$

By the above result, the matrix

$$\frac{\partial}{\partial x} \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \end{bmatrix} \in \mathbb{R}^{r \times n}$$

has full rank r at $x = \bar{x}$.

Let us find $\Phi_{r+1}, \dots, \Phi_n$ such that:

1) The matrix

$$\frac{\partial}{\partial x} \begin{bmatrix} \Phi_1(x) \\ \vdots \\ \Phi_r(x) \\ \Phi_{r+1}(x) \\ \vdots \\ \Phi_n(x) \end{bmatrix} \in \mathbb{R}^{n \times n}$$

has full rank $= n$ at $x = \bar{x}$ ($\Leftrightarrow$ it is nonsingular)

2) $\exists\, \bar U \subset \mathbb{R}^n$ open and containing $\bar x$ such that

$$L_g \Phi_i(x) = 0 \quad , \quad \forall x \in \bar U \quad , \quad \forall i = r+1, \dots, n$$

↑ This is always possible (see ISIDORI, Nonlinear Control Systems)

Then we have constructed a function $\bar\Phi : \bar U \to \mathbb{R}^n$ as

$$\bar\Phi(x) = \left(\phi_1(x), \dots, \phi_n(x) \right)$$

that satisfies

$$\operatorname{rank} \frac{\partial}{\partial x} \Phi(\bar x) = n$$

By the implicit function theorem $\exists\, U \subset \bar U$ open such that $\bar x \in \bar U$ and $\Phi : U \to \mathbb{R}^n$ defined by

$$\Phi(x) = \left(\phi_1(x), \dots, \phi_n(x) \right) \qquad \left(= \text{restriction of } \bar\Phi \text{ on } U \right)$$

is a DIFFEOMORPHISM

↓

We now change coordinates by using this Φ: (only valid for $x \in U$)

$$x \longmapsto z = \Phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} = \xi \in \mathbb{R}^r \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} = \eta \in \mathbb{R}^{n-r} \end{matrix}$$

In the new coordinates we have:

$$y = \xi_1 \quad , \quad \xi = (\xi_1, \dots, \xi_r) \quad , \quad \eta = (\eta_1, \dots, \eta_{n-r})$$

and ξ satisfies the following equations:

$$\dot{\xi}_1 = \dot{h}(x) = \frac{\partial h(x)}{\partial x}\left(f(x)+g(x)u\right) = \underbrace{L_f h(x)}_{=\xi_2} + \underbrace{\cancel{L_g h(x)}}_{=0}\cdot u = \xi_2$$

$$\dot{\xi}_2 = \ddot{h}(x) = \frac{d}{dt}\left(L_f h(x)\right) = \underbrace{L_f^2 h(x)}_{=\xi_3} + \underbrace{\cancel{L_g L_f h(x)}}_{=0}\, u = \xi_3$$

$$\vdots$$

$$\dot{\xi}_{r-1} = \frac{d^{(r-1)}}{dt^{r-1}} h(x) = \underbrace{L_f^{r-1} h(x)}_{=\xi_r} + \underbrace{\cancel{L_g L_f^{r-2}} h(x)}_{=0}\, u = \xi_r$$

$$\dot{\xi}_r = L_f^r h(x) + L_g L_f^{r-1} h(x)\, u$$

$$= \underbrace{L_f^r h\left(\phi^{-1}(\xi,\eta)\right)}_{\doteq\, q(\xi,\eta)} + \underbrace{L_g L_f^{r-1} h\left(\phi^{-1}(\xi,\eta)\right)}_{\doteq\, b(\xi,\eta)}\cdot u$$

$$= q(\xi,\eta) + b(\xi,\eta)\cdot u$$

Instead, η satisfies:

$$\forall i = r+1, \dots, n, \qquad \dot{\eta}_i = \dot{\phi}_i(x) = L_f \phi_i(x) + \underbrace{\cancel{L_g \phi_i(x)}}_{=0 \text{ by construction}}\, u$$

$$= \underbrace{L_f \phi_i\left(\phi^{-1}(\xi,\eta)\right)}_{\doteq\, \psi_i(\xi,\eta)}$$

Therefore we obtain

$$\boxed{\begin{aligned} \dot{\xi}_i &= \xi_{i+1} \qquad i = 1, \dots, r-1 \\ \dot{\xi}_r &= q(\xi,\eta) + b(\xi,\eta)\, u \\ \dot{\eta} &= \psi(\xi,\eta) \\ y &= \xi_1 \end{aligned}}$$

NORMAL FORM

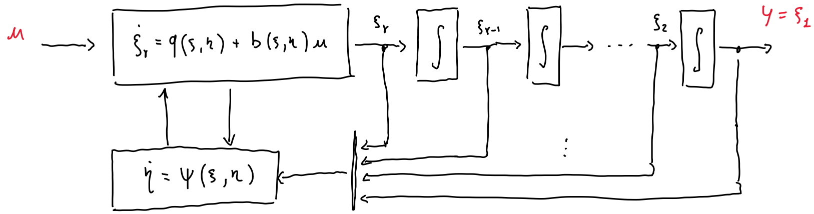
In summary, we have shown that if

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

locally, around $\bar{x}$

has relative degree $= r$ at $x = \bar{x}$, then it is diffeomorphic to a normal form

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} & i = 1, \dots, r-1 \\ \dot{\xi}_r = q(\xi,\eta) + b(\xi,\eta)u \\ \dot{\eta} = \psi(\xi,\eta) \\ y = \xi_1 \end{cases}$$



EXAMPLE

$$\dot{x} = \underbrace{\begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix}}_{g(x)} u$$

$$y = \underbrace{x_3}_{h(x)}$$

Let us compute (if it exists) the diffeomorphism ϕ bringing the system to a normal form

$$\phi_1(x) = h(x)$$

we have:

$$L_g h(x) = \frac{\partial h}{\partial x}(x)\, g(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow$ we keep going:

$$\phi_2(x) = L_f h(x) = \frac{\partial h}{\partial x}(x)\, f(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix} = x_2$$

and

$$L_g L_f h(x) = \frac{\partial L_f h}{\partial x}(x) \cdot g(x) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix} = \underline{\underline{1}} \qquad \forall x$$

$\Rightarrow$ the system has global rel. degree $r = 2$

We miss ϕ_3 $\rightarrow$ we now have to find ϕ_3 such that $\left\{ \frac{\partial}{\partial x}\phi_1(x), \frac{\partial}{\partial x}\phi_2(x), \frac{\partial \phi_3}{\partial x}(x) \right\}$ is linearly independent and $L_g \phi_3(x) = 0$ in some open $U \subset \mathbb{R}^3$

Imposing $L_g \phi_3(x) = 0$ means

$$0 = \frac{\partial}{\partial x}\phi_3(x) \cdot g(x) = \frac{\partial}{\partial x}\phi_3(x) \cdot \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix} = e^{x_2} \frac{\partial}{\partial x_1}\phi_3(x) + \frac{\partial}{\partial x_2}\phi_3(x)$$

A solution is

$$\phi_3(x) = 1 + x_1 - e^{x_2}$$

indeed

$$e^{x_2} \frac{\partial}{\partial x_1}\phi_3(x) + \frac{\partial}{\partial x_2}\phi_3(x) = e^{x_2} \cdot 1 - e^{x_2} = 0$$

Moreover,

$$\frac{\partial}{\partial x}\phi_3(x) = \begin{pmatrix} 1 & -e^{x_2} & 0 \end{pmatrix} \Rightarrow \text{it is linearly independent from } \frac{\partial}{\partial x}\phi_1(x) \text{ and } \frac{\partial}{\partial x}\phi_2(x)$$

The diffeomorphism reads

$$\phi(x) = \begin{pmatrix} x_3 \\ x_2 \\ 1 + x_1 - e^{x_2} \end{pmatrix}$$

It's easy to see that

$$\phi^{-1}(z) = \begin{pmatrix} z_3 - 1 + e^{z_2} \\ z_2 \\ z_1 \end{pmatrix}$$

Since ϕ and ϕ^{-1} are defined and smooth on $U = \mathbb{R}^n$, ϕ is a global diffeomorp.

using

$$f(x) = \begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix} \qquad \text{and} \qquad g(x) = \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix}$$

in the new coordinates $z = \phi(x) = \begin{pmatrix} x_3 \\ x_2 \\ 1 - x_1 + e^{x_2} \end{pmatrix} \begin{matrix} \}\, \xi \\ \\ \}\, \eta \end{matrix}$ we get

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = x_1 x_2 + u = \underbrace{(\eta - 1 + e^{\xi_2})\, \xi_2}_{q(\xi,\eta)} + \underbrace{u}_{b(\xi,\eta) = 1}$$

$$\dot{\eta} = \underbrace{(\eta - 1 + e^{\xi_2})}_{\psi(\xi,\eta)}$$

EXAMPLE

$$\dot{x} = \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} u$$

$$y = x_3$$

Let's build the diffeomorphism :

- $\phi_1(x) = h(x) = x_3$
- $L_g h(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} = 0$

$\rightarrow$ we keep going

- $\phi_2(x) = L_f h(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} = x_1 - x_2$
- $L_g L_f h(x) = \frac{\partial}{\partial x}(L_f h)(x) \cdot g(x) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} = 0$

$\rightarrow$ we keep going :

- $\phi_3(x) = L_f^2 h(x) = \frac{\partial L_f h}{\partial x}(x) \cdot f(x) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} = -x_1 - x_2^2$
- $L_g L_f^2 h(x) = \frac{\partial}{\partial x}(L_f^2 h)(x) \cdot g(x) = \begin{pmatrix} -1 & -2x_2 & 0 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} = -e^{x_2}(1 + 2x_2)$

$\rightarrow\ L_g L_f^2 h(x) \neq 0 \qquad \forall x \in \mathcal{O} \doteq \{ x \in \mathbb{R}^3 : \ x_2 \neq -\frac{1}{2} \}$

$\rightarrow$ Since $\mathcal{O}$ is open, the system has relative degree $r=3$ at every $x \in \mathcal{O}$

Define the function $\tilde{\phi} : \mathcal{O} \to \mathbb{R}^3$ as

$$\tilde{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 - x_2 \\ -x_1 - x_2^2 \end{pmatrix}$$

question: does it exist an open set $U \subset \mathcal{O}$ such that the restriction ϕ of $\tilde{\phi}$ on U is a diffeomorphism ?

- $\tilde{\phi}$ is smooth on $\mathcal{O}$
- Thus we only need an open set $U \subset \mathcal{O}$ such that $\tilde{\phi}$ is invertible on U

$\downarrow$

Let us try to compute ϕ^{-1} : we have to solve in x :

$$\begin{cases} x_3 = z_1 \\ x_1 - x_2 = z_2 \\ -x_1 - x_2^2 = z_3 \end{cases} \rightarrow \begin{cases} x_3 = z_1 \\ x_1 = z_2 + x_2 \\ x_2^2 + x_2 + z_2 + z_3 = 0 \end{cases} \longrightarrow \begin{cases} x_3 = z_1 \\ x_1 = z_2 + x_2 \\ x_2 = \underbrace{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - (z_2+z_3)}} \end{cases}$$

this suggests considering only x such that $|x_2| < \frac{1}{2}$

$\leftarrow$

with $|x_2| < \frac{1}{2}$, we have

$$z_2 + z_3 = \underbrace{-(x_2 + x_2^2) < \frac{1}{4}}$$

Indeed $x_2 \mapsto -(x_2 + x_2^2)$ is decreasing on $(-\frac{1}{2}, \frac{1}{2})$

Therefore we can solve the previous equations as:

$$\begin{cases} x_3 = z_1 \\ x_1 = z_2 + x_2 = z_2 - \frac{1}{2} + \sqrt{\frac{1}{4} - (z_2+z_3)} \\ x_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} - (z_2+z_3)} \end{cases}$$

$\Rightarrow$ For all x in the set

$$U \doteq \{ x \in \mathbb{R}^3 \ | \ |x_2| < \frac{1}{2} \} \subset \mathcal{O}$$

we can invert $\tilde{\phi}(x)$ as

$$\tilde{\phi}^{-1}(z) = \begin{pmatrix} z_2 - \frac{1}{2} + \sqrt{\frac{1}{4} - (z_2+z_3)} \\ -\frac{1}{2} + \sqrt{\frac{1}{4} - (z_2+z_3)} \\ z_1 \end{pmatrix}$$

Moreover, $\tilde{\phi}$ is smooth on U and $\tilde{\phi}^{-1}$ is smooth on $\tilde{\phi}(U)$

$\Rightarrow$ $\phi : U \to \mathbb{R}^3$ defined as $\phi(x) = \tilde{\phi}(x)$ is a diffeomorphism.

In the new variables we have

$$z = \phi(x) = \xi$$

$$\dot{x} = \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} u$$

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \xi_3$$

$$\begin{pmatrix} x_3 \\ x_1 - x_2 \\ -x_1 - x_2^2 \end{pmatrix}$$

$$\dot{\xi}_3 = e^{x_2} u - 2x_2 (\underbrace{x_1}_{-\xi_3} + x_2^2 + e^{x_2} u)$$

$$= q(\xi) + b(\xi) u$$

where

$$q(\xi) = \left[-2x_2 (x_1 + x_2^2) \right] \Big|_{x = \phi^{-1}(\xi)} , \quad b(\xi) = \left[e^{x_2} (1 - 2x_2) \right]_{x = \phi^{-1}(\xi)}$$

FEEDBACK LINEARIZATION

ASSUMPTION. There exists an open set $U \in \mathbb{R}^n$ such that the system has relative degree $r = n$ in U and it has a normal form on U

$\downarrow$

The diffeomorphism $\phi : U \to \mathbb{R}^n$ bringing the system to its normal form reads as

$$\phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}$$

The normal form only has the ξ-variables: $(z = \xi)$

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = q(\xi) + b(\xi)u \end{cases}$$

$$q(\xi) = L_f^n h(x) \Big|_{x = \phi^{-1}(\xi)}$$

$$b(\xi) = L_g L_f^{n-1} h(x) \Big|_{x = \phi^{-1}(\xi)}$$

Consider the controller

$$u = \frac{1}{b(\xi)} \left(-q(\xi) + v \right) \qquad (v = \text{AUXILIARY INPUT})$$

$$= \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + v \right)$$

Then, we have:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = v \end{cases} \rightarrow \text{IT IS } \underline{\text{LINEAR}}:$$

-> The closed-loop system with input $v(t)$ and output $y(t)$ is <u>LINEAR</u> and controllable

$$\begin{cases} \dot{\xi} = A\xi + Bv \\ y = C\xi \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ & & & \ddots & \\ & & & & 1 \\ 0 & 0 & \cdots & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

We can design on the above linear system a control law of the kind

$$v(t) = \gamma(\xi(t), t)$$

to accomplish the control task we desire and the feedback

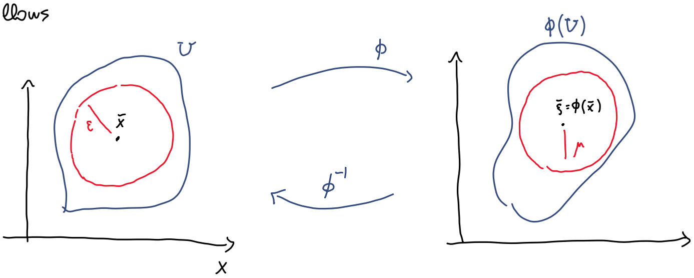
$$u(t) = \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + v(\xi, t)\Big|_{\xi = \phi^{-1}(x)} \right)$$

→ The only thing we should guarantee is that $x(t)$ stays in U

↓

- If $U = \mathbb{R}^n$ → NO PROBLEM → This will be our assumption from now on

- Otherwise, we can map the constraint $x \in U$ to a constraint on ξ as follows



- Since U is open, $\exists\, \bar{x} \in \mathbb{R}^n$, $\exists\, \varepsilon > 0$ such that $\underbrace{B_\varepsilon(\bar{x})}_{\text{Ball of radius } \varepsilon \text{ centered at } \bar{x}} \subset U$

- Since ϕ is a diffeomorphism $\exists\, \mu > 0$ such that $B_\mu(\bar{\xi}) \subset \phi(B_\varepsilon(\bar{x}))$ where $\bar{\xi} := \phi(\bar{x})$

- If v is able to ensure that

$$\|\xi(t) - \bar{\xi}\| < \mu \quad \forall t \geq 0 \qquad \left(\Leftrightarrow \quad \xi(t) \in B_\mu(\bar{\xi}) \right)$$

then for all $t \geq 0$

$$\phi(x(t)) = \xi(t) \in B_\mu(\bar{\xi}) \Rightarrow \quad x(t) = \phi^{-1}(\xi(t)) \subset \phi^{-1}\left(B_\mu(\bar{\xi})\right) \subset \phi^{-1}\left(\phi\left(B_\varepsilon(\bar{x})\right)\right) \subset B_\varepsilon(\bar{x}) \subset U$$

Namely, $x(t) \in U, \forall t \geq 0$.

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