$Im(R) = \mathbb{R}^+$ CTcase

- **Result:** $F^T(t)x = 0 \;\; \forall t \in [0,t] \iff x \in Ker(W(t))$

  where $F(t)$ is a $n \times m$ matrix depending on t
  $W(t) \triangleq \int_0^t F(t)F^T(t)\,dt$ is an $n \times n$ matrix depending on t

DEMONSTRATION

$\langle x, W(t)x \rangle = x^T W(t) = \int_0^t x^T F(t) F^T(t) x\,dt = \int_0^t (F^T(x)x)^T F^T(x) x\,dt = \int_0^t \|F^T(x)x\|^2 dt$

so it's true: $\langle x, W(t)x\rangle = 0 \iff F^T(x)x = 0 \;\; \forall x \in [0,t]$

Also $\langle x, W(t)x\rangle = 0 \iff x \perp W(t)x \iff x \perp \alpha$ with $\alpha \in Im(W(t)) \iff x \in (Im(W(t)))^{\perp}$

Having $W(t) = W^T(t)$ then $x \in (Im(W(t)))^{\perp} \iff x \in Ker(W^T(t)) \iff x \in Ker(W(t))$

It's so proven $F^T(t)x = 0 \;\forall t \in [0,t] \iff \langle x, W(t)x\rangle = 0 \iff x \in Ker(W(t))$

- **Result:** $\mathcal{R}^+(t) = Im(W(t))$ where $W(t)$ is the Reachibility Gramian $\left(W(t) = \int_0^t e^{As}BB^Te^{A^Ts}\,ds = \int_0^t e^{A(t-s)}BB^Te^{A^T(t-s)}\,ds\right)$

DEMONSTRATION

$x(t) \in \mathcal{R}^+(t) \iff \exists u([0,t)) : x(t) = \int_0^t e^{A(t-s)}Bu(s)\,ds$ (remember: t is fixed)

Let's consider $x(t) \in Im(W(t))$ $\Rightarrow \exists \alpha \neq 0 : x(t) = W(t)\alpha$ with $\alpha = W^+(t)x(t)$

$\Rightarrow x(t) = W(t)\alpha = \int_0^t e^{A(t-s)}BB^Te^{A^T(t-s)}\alpha\,ds$

$\Rightarrow$ picking $u(s) = B^Te^{A^T(t-s)}\alpha \Rightarrow x(t) = \int_0^t e^{At}Bu(s)\,ds$

$\Rightarrow x(t) \in \mathcal{R}^+$ we have $x^T(t)x(t) = 0$.

Suppose $x(t) \in \mathcal{R}^+(t)$, which means $x^T(t)x(t) = \left(\int_0^t e^{A(t-s)}Bu(s)\,ds\right)^T x(t) = \int_0^t u^T(s)B^Te^{A^T(t-s)}x(t)\,dt$

Suppose also $x(t) \in Ker(W(t)) \Rightarrow B^Te^{A^T(t-s)}x(t) = 0 \;\forall t \in [0,t]$ due to the result demonstrated previosly,

so we have $x^T(t)x(t) = 0$.

That means that all the non zero elements $x(t)$ contained in $Ker(W(t))$ are not contained in $\mathcal{R}^+(t)$ because it would be in contraddiction with beeing non zero.

So we proven: $x(t) \in Im(W(t)) \Rightarrow x(t) \in \mathcal{R}^+(t) \;\wedge\; x(t) \in Ker(W(t)) \setminus \{0\} \Rightarrow x(t) \notin \mathcal{R}^+(t)$

Having that $W(t) = W^T(t) \Rightarrow (Im(W(t)))^{\perp} = Ker(W^T(t)) = Ker(W(t)) \Rightarrow Im(W(t)) \oplus Ker(W(t)) = \mathbb{R}^n$

That means that if $x(t) \notin Im(W(t))$ then $\exists \alpha, \beta \in \mathbb{R}$ with $\beta \neq 0 : x(t) = \alpha\nu_1 + \beta\nu_2$ where $\nu_1 \in Im(W(t)) \wedge \nu_2 \in Ker(W(t))$.

Having that $\nu_2 \notin \mathcal{R}^+(t)$ it's immediatly true that also $x \notin \mathcal{R}^+(t)$ (infact, if that's not the case, then we have $(x(t) - \alpha\nu_1)^1/\beta \in \mathcal{R}^+(t)$ which is in contraddiction with ν_2 not beeing in $\mathcal{R}^+(t)$).

So we also prove $x(t) \notin Im(W(t)) \Rightarrow x(t) \notin \mathcal{R}^+(t)$, equivalent to $x(t) \in \mathcal{R}^+(t) \Rightarrow x(t) \in Im(W(t))$

COROLLARY: $\mathcal{R}^-(t) \triangleq (\mathcal{R}^+(t))^{\perp} = (Im(W(t)))^{\perp} = Ker(W^T(t)) = Ker(W(t))$

- Demonstration of $\mathcal{R}^+ = Im(R)$ for CT systems

Let's consider $x \in \mathcal{R}^- \triangleq \mathcal{R}^-(+\infty)$
$\iff x \in Ker(W(+\infty))$
$\iff B^T e^{A^T s} x = 0 \quad \forall s \in [0, +\infty)$
$\iff \gamma(s) = 0 \; \forall s \in [0, +\infty)$ where $\gamma(s) : [0, +\infty) \to \mathbb{R}^m$

It's immediate $\gamma(s) = 0 \; \forall s \in [0, +\infty) \iff \gamma^{(k)}(s) = 0 \; \forall s \in [0, +\infty) \; \forall k \in \overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$ and also $\gamma^{(k)}(s) = 0 \; \forall s \in [0, +\infty) \; \forall k \in \overline{\mathbb{N}} \Rightarrow \gamma^{(k)}(0) = 0 \; \forall k \in \overline{\mathbb{N}}$. Due to the fact that every function can be expressed as its Taylor expansion which is a linear combination of an infinite set of terms where each term depends on one of the quantities $\gamma^{(k)}(0)$ where $k \in \overline{\mathbb{N}}$, then: $\gamma^{(k)}(0) = 0 \; \forall k \in \overline{\mathbb{N}} \Rightarrow \gamma(s) = 0 \; \forall s \in [0, +\infty)$

So we demonstrate $\gamma(s) = 0 \; \forall s \in [0, +\infty) \iff \gamma^{(k)}(0) = 0 \; \forall k \in \overline{\mathbb{N}}$
Having $\gamma^{(k)}(0) = B^T (A^T)^k x$ we have: $\gamma^{(k)}(0) = 0 \; \forall k \in \overline{\mathbb{N}} \iff B^T (A^T)^k x = 0 \; \forall k \in \overline{\mathbb{N}}$

For the Cayley-Hamilton Theorem, we can write: $A^k = -\alpha_1 A^{k-1} - \alpha_2 A^{k-2} \cdots - \alpha_n I_n \quad \forall k \geq n, k \in \mathbb{Z}$
$\Rightarrow B^T (A^k)^T x = -\alpha_1 B^T (A^{k-1})^T x - \alpha_2 B^T (A^{k-2})^T x \cdots - \alpha_n B^T x \quad \forall k \geq n, k \in \mathbb{Z}$
which means that the following holds:
$B^T (A^T)^k x = 0 \; \forall k \in \overline{\mathbb{N}} \iff B^T (A^T)^k x = 0 \; \forall k \in [0, n-1]$

It's also true:
$B^T (A^T)^k x = 0 \; \forall k \in [0, n-1] \iff [B \;\; AB \;\; A^2 B \; \ldots \; A^{n-1} B]^T x = 0 \iff R^T x = 0 \iff x \in Ker(R^T)$
So we demonstrate $\mathcal{R}^- = Ker(R^T)$

Finally $\mathcal{R}^+ = (\mathcal{R}^-)^\perp = (Ker(R^T))^\perp = Im(R)$