AMORTIZING SWAP PRICING WITH CVA IN A MULTICURVE FRAMEWORK

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1 Introduction

In order to take into account the credit risk associated with lending or borrowing money in the interbank market at different time horizons, we focus our attention on a multicurve framework, which is characterized by a discounting curve (i.e. a curve of discount factors that are used to actualize future cash flows) and several pseudo-discounting curves (one for each Euribor tenor), which lets us deal with floating rates in financial contracts. The underlying idea of this choice is the large basis spreads among different Euribor tenors, which makes problematic the use of a standard bootstrap (with only a discounting curve) and introduces the need of a new bootstrap (section 3).

Since the aim of our analysis is the pricing of a contract in which fixed and floating rates are exchanged with quarterly payments in both legs, we are only interested in Euribor 3m. For this reason, our setting presents a discounting curve and only one pseudo-discounting curve (related to Euribor 3m), leading us to a dual curve framework.

The contract that we aim to price is a so-called "Amortizing Swap" between a bank and a corporate: it is an interest rate derivative in which typically the payments of the fixed leg refer to a decreasing notional, while the ones in the floating leg are all related to the initial value of the notional. However, there exist also amortizing swaps where both the legs are characterized by a decreasing notional. We denote the first case with "single amortizing", while the second with "double amortizing". Moreover, these products are used in hedging instruments with declining principal, such as mortgages.

We first proceed with the risk free pricing of our contract (section 4) and then move on to a valuation which embeds the CVA (section 5), which requires swaption pricing. In order to complete this task, we need to utilize a suitable model for this multicurve framework. Since market data contain swaptions quoted with Bachelier implied volatilities, we apply Bachelier closed formula, properly generalized to this setting.

In section 6, we examine the case of an unwinding of the contract by the corporate: the aim is to find the residual NPV of the instrument in that situation.

In order to employ an alternative method for swaption pricing, we have to consider that this particular context for discounting curve requires the structuring of an adequate model: we need to find a parsimonious multicurve extension of Heath-Jarrow-Morton (HJM) models in order to adapt their properties to this framework. In particular, we select the idea proposed in Baviera (2019), namely a multicurve version of the Hull-White (MHW) model, characterized by only 3 parameters, which grants the possibility to avoid too complex calibrations and obtain simple closed formulas when pricing interest rate derivatives (section 7).

Finally, we apply the closed formula coming from the MHW model calibrated in section 6 (obtained through a generalization of the Jamshidian approach) in order to value the amortizing swap (with CVA), as well as the trinomial tree, namely a numerical technique, based on the calibrated model itself (section 8). Furthermore, in the same section we propose an alternative use of the Jamshidian trick for the pricing formula.

The code for our analysis is realized in Matlab and Python.

2 Swap Termsheet

Principal Amount (N):	20 MLN EUR (with amortizing plan)
Trade Date:	24th June 2022
Settle Date:	28th June 2022
Maturity Date:	28th June 2037 (15y), subject to the Modified Following BD convention.
Bank payment dates:	Quarterly, subject to Following BD convention
Daycount:	Act/360
Bank receives:	Euribor 3M

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Corporate payment dates:	Quarterly, subject to Following BD convention
Corporate receives:	2.21%
Daycount:	Act/360

3 Bootstrap in a dual curve framework

As explained in Baviera (2019), in a multicurve setting the discounting curve is defined as usual, namely:

$$B(t,T) = \mathbb{E}\left[D(t,T)|\mathcal{F}_t\right] \quad \forall T \in [t,T^*] \tag{3.1}$$

where *D(t,T)* is the stochastic discount. This implies that the standard relations, such that the forward discount $B(t,T,T+\triangle)$ is equal to the ratio $\frac{B(t,T+\triangle)}{B(t,T)}$, hold. This curve is the one that will be used in order to discount future cash flows.

The equality that relates the Libor rate $L(T,T+\triangle)$, with fixing in T and payment in $T+\triangle$, and the corresponding forward rate $L(t,T,T+\triangle)$ is the following:

$$B(t,T+\triangle)L(t,T,T+\triangle) = \mathbb{E}\left[D(t,T+\triangle)L(T,T+\triangle)|\mathcal{F}_t\right]. \tag{3.2}$$

The tenor $\triangle$ is the one that characterizes the pseudo-discount curve. The forward pseudo-discounts are indeed defined as:

$$\tilde{B}(t,T,T+\triangle) = \frac{1}{1+\delta(T,T+\triangle)L(t,T,T+\triangle)}. \tag{3.3}$$

This relation holds also for non-forward pseudo-discounts. In particular, pseudo and forward pseudo-discounts are connected by:

$$\tilde{B}(t,T,T+\triangle) = \frac{\tilde{B}(t,T+\triangle)}{\tilde{B}(t,T)}. \tag{3.4}$$

As we can understand from formula (3.3), the pseudo-discounts will be used in order to deal with Euribor rates (in such a way that every pseudo-discounting curve lets us treat Euribor rates with a specific tenor), while they will not be employed to actualize future cash flows.
Furthermore, it is convenient to define the so-called "multiplicative spread" (or spread), which is:

$$\beta(t,T,T+\triangle) = \frac{B(t,T,T+\triangle)}{\tilde{B}(t,T,T+\triangle)}. \tag{3.5}$$

Following the methodology proposed by Baviera&Cassaro (2015), in order to derive the discounting and pseudo-discounting curve from market data in a multicurve framework, we need to apply a particular kind of bootstrap, the so-called "Mr. Crab's bootstrap".
The procedure starts with the construction of the discounting curve, for which we choose the Effective OverNight Index Average (EONIA) curve. In particular, we employ the quoted Overnight Indexed Swap (OIS) rates to obtain the discount factors, splitting the procedure into two cases:

- if the OIS has maturity lower or equal to 1 year, the formula is:

$$B(t_0,t_e) = \frac{1}{1+\delta(t_0,t_e)R^{OIS}(t_0,t_e)} \tag{3.6}$$

- if the OIS has maturity longer than 1 year, the formula becomes:

$$B(t_0,t_e) = \frac{1-R^{OIS}(t_0,t_e)\sum_{k=1}^{i-1}\delta_k B(t_0,t_k)}{1+\delta_i R^{OIS}(t_0,t_i)} \tag{3.7}$$

where $B(t_0,t_e)$ is the discount factor for the time interval (t_0,t_e), $R^{OIS}(t_0,t_e)$ is the OIS rate for the period (t_0,t_e) and δ_k is the year fraction for the calculation period (t_{k-1},t_k).
It can be noticed that the discounts are derived in a standard single curve modeling setting.

Once completed the discounting curve bootstrap, the methodology proceeds with the pseudo-discounting curve (with tenor 3 months in our case).

First, we consider the 3-month Euribor rate for the 3-month depo and we compute the pseudo-discount $\tilde{B}(t_0, t_3)$ using formula (3.3).
Then, we take into account the 3x6 Forward Rate Agreement (FRA) and we calculate the 6-month pseudo-discount $\tilde{B}(t_0, t_6)$ using (3.3) and (3.4):

$$\tilde{B}(t_0, t_6) = \tilde{B}(t_0, t_3)\tilde{B}(t_0, t_3, t_6). \tag{3.8}$$

At this point, exploiting the pseudo-discounts $\tilde{B}(t_0, t_3)$ and $\tilde{B}(t_0, t_6)$, we derive through an interpolation the 4- and 5-month pseudo-discounts, from which, employing the 1x4 and 2x5 FRAs, we obtain the 1- and 2-month pseudo-discounts moving "backwards":

$$\tilde{B}(t_0, t_i) = \frac{\tilde{B}(t_0, t_{i+3})}{\tilde{B}(t_0, t_i, t_{i+3})}. \tag{3.9}$$

In this passage we can understand how this non-standard bootstrap methodology moves forwards and backwards, as the name "Mr Crab's bootstrap" underlines. Then, for expiries up to two years we consider the first 7 STIR futures. From them we derive the forward rates, which are equal to the future rates under the hypothesis of no convexity adjustment, and consequently we calculate all the pseudo-discounts through the expression:

$$\tilde{B}(t_0, t_{e,i}) = \tilde{B}(t_0, t_{s,i})\tilde{B}(t_0, t_{s,i}, t_{e,i}), \tag{3.10}$$

where $B(t_0, t_{s,i}, t_{e,i})$ is computed using formula (3.3) from the forward rates (since the STIR futures have a Euribor 3m as underlying), and $t_{s,i}, t_{e,i}$ are respectively the start date and the expiry date of the i-th future. $B(t_0, t_{s,i})$ is obtained via interpolation.
In the next passage we calculate the last forward rate of the second year, exploiting the form of the swap contract, which is:

$$\sum_{k=1}^{f \times i} w_k F_k(t_0) = \mathcal{I}(i), \tag{3.11}$$

where

$$w_k = \delta_k B(t_0, t_k) \tag{3.12}$$

and

$$\mathcal{I}(i) = S(t_0, t_i) \sum_{k=1}^{i} \delta_k B(t_0, t_k). \tag{3.13}$$

This equality comes from the definition of the swap rate and the condition of NPV of the contract equal to zero. The frequency f is 4 in the case of Euribor 3m, while 2 in the case of Euribor 6m. Letter i represents the year.
Using this relation, we can write the last forward rate of the second year in the following way:

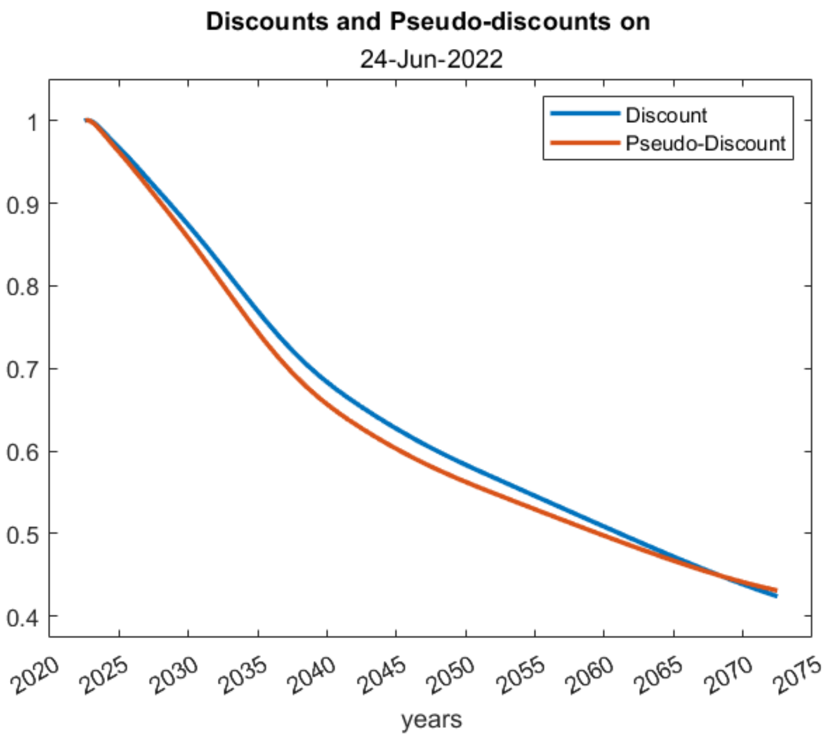
$$F_8(t_0) = \frac{1}{w_8}\left[\mathcal{I}(2) - \sum_{k=1}^{2\times f-1} w_k F_k(t_0)\right], \tag{3.14}$$

where the first seven forward rates are obtained through an interpolation.
At this point, we interpolate all the swaps in order to have a complete set for all the years from 2 to 50 and, starting from the 3-year one, we utilize them in order to exploit equality (3.11) and derive all the forward rates year by year, using interpolation (in particular the first 3 forward rates of each year are expressed as functions of the last forward rate, via interpolation, in such a way that all rates of the each year are found from the same equation (3.11)).
The obtained rates are then used to compute the pseudo-discounts for all the years with this formula:

$$\tilde{B}(t_0, t_{i-1}, t_i) = \prod_{k \in Y(i)} \frac{1}{1 + \delta_k F_k(t_0)} \quad i \geq 3, \tag{3.15}$$

where $k \in Y(i)$ indicates the 3-month forward rates in the i-th year.

We completed all these passages based on market data for trade date 24^{th} *Jun 2022* (hence settlement date 28^{th} *June 2022*) obtaining the following discounting and pseudo-discounting curve.



From the plot above it can be noticed that, due to the presence of negative rates in market data, the discounting and pseudo-discounting curve reach values higher than 1 for short time horizons. In particular, the 3-month pseudo-discount is 1.0006 (hence bigger than 1), which implies that the first Euribor rate is negative, as we can see from the definition of pseudo-discount at formula (3.3), which can be rewritten as:

$$L(t,T,T+\triangle) = \frac{1-\tilde{B}(t,T,T+\triangle)}{\delta(T,T+\triangle)\tilde{B}(t,T,T+\triangle)}. \tag{3.16}$$

This peculiar aspect will have consequences on the comparison between the NPV of the amortizing swap on 24^{th} *Jun 2022* and the one on the unwinding date.

4 Amortizing swap risk free pricing

After having computed both the needed discounting and pseudo-discounting curve, we can proceed with the risk free computation of the NPV of our contract. As reported in the given termsheet, we are dealing with an amortizing swap in which the bank pays quarterly an annual fixed rate R equal to *2.21%*, while receives quarterly floating cash flows referring to *Euribor 3 months*, with a principal amount of *20 Mln €*.

Starting from the pricing in a single amortizing framework, we have to compute and discount back to the settlement date (*28th June 2022*) all futures payments of the considered contract, since

$$NPV_{\text{risk free SA}}(t_0) = \mathbb{E}\left[\sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) D(t_0, T_{i+1}) (L(T_i, T_{i+1}) N_0 - RN_i) \middle| \mathcal{F}_{t_0}\right], \tag{4.1}$$

which can be seen as the difference between the NPV of the floating leg and the NPV of the fixed leg:

$$NPV_{\text{risk free SA}}(t_0) = NPV_{\text{floating leg SA}}(t_0) - NPV_{\text{fixed leg}}(t_0), \tag{4.2}$$

where

$$\begin{aligned} NPV_{\text{floating leg SA}}(t_0) &= \mathbb{E}\left[\sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) D(t_0, T_{i+1}) L(T_i, T_{i+1}) N_0 \middle| \mathcal{F}_{t_0}\right] \\ &= \sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) B(t_0, T_{i+1}) L(t_0, T_i, T_{i+1}) N_0 \\ &= N_0 \sum_{i=0}^{\omega-1} B(t_0, T_{i+1}) \left[\frac{1}{\tilde{B}(t_0, T_i, T_{i+1})} - 1\right], \end{aligned} \tag{4.3}$$

while in the $NPV_{\text{fixed leg}}$ the whole stochasticity is represented just by the stochastic discount $D(t_0, T_{i+1})$, since all other terms are purely deterministic, leading us to a simple formula for this component of the NPV:

$$\begin{aligned} NPV_{\text{fixed leg}}(t_0) &= \mathbb{E}\left[\sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) D(t_0, T_{i+1}) RN_i \middle| \mathcal{F}_{t_0}\right] \\ &= \sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) B(t_0, T_{i+1}) RN_i. \end{aligned} \tag{4.4}$$

It is better to recall that $L(T_i, T_{i+1})$ is a martingale under the T_i-forward measure, R is the fixed rate established at the beginning of the contract and N_i is the notional to which refers the payment at time T_{i+1}. Moreover, it is important to remember that, due to the fact that we are operating in a dual curve framework, we cannot perform a trivial telescopic sum when computing the $NPV_{\text{floating leg}}$, because of the presence of pseudo-discount factors.

The same reasoning can be applied also to the double amortizing case (DA), paying attention to the computation of the $NPV_{\text{floating leg}}$, where we can no longer bring the notional outside the summation, since now in the floating leg we do not have only the initial notional N_0, but a sequence of notionals N_i decreasing in time. The $NPV_{\text{fixed leg}}$, instead, remains unchanged, indeed

$$NPV_{\text{risk free DA}}(t_0) = \mathbb{E}\left[\sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) D(t_0, T_{i+1}) (L(T_i, T_{i+1}) N_i - RN_i) \middle| \mathcal{F}_{t_0}\right], \tag{4.5}$$

where relation (4.2) still holds. The only difference is in the computation of

$$\begin{aligned} NPV_{\text{floating leg DA}}(t_0) &= \mathbb{E}\left[\sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) D(t_0, T_{i+1}) L(T_i, T_{i+1}) N_i \middle| \mathcal{F}_{t_0}\right] \\ &= \sum_{i=0}^{\omega-1} \delta(T_i, T_{i+1}) B(t_0, T_{i+1}) L(t_0, T_i, T_{i+1}) N_i \\ &= \sum_{i=0}^{\omega-1} N_i B(t_0, T_{i+1}) \left[\frac{1}{\tilde{B}(t_0, T_i, T_{i+1})} - 1\right]. \end{aligned} \tag{4.6}$$

Thanks to formula (4.1), we can finally display the results of the $NPV_{\text{risk free}}$ in both cases as follows

	$NPV_{\text{risk free}}$ €
SA	2'942'545.03
DA	-112'976.06

There is an evident difference between the single amortizing case (SA) and the double amortizing one (DA), not only in terms of order of magnitude, but also in sign. The reason why this happens should be researched in the definition of the two different situations. Indeed, considering a decreasing sequence of notionals also on the floating leg (DA case), we have to keep in mind that Euribor payments refer to a smaller notional N_{i-1} at each time T_i if compared to the single amortizing case (SA). This means that at each payment date, floating and fixed leg refer to the same notional over time in DA case (while in SA the notional related to the floating leg (constant) is higher than the one associated to the fixed leg (declining)). Moreover if we take a look at Euribor future values in t_0, computed according to formula (3.16), we notice that they are less than the fixed exchanged rate R for the first payments. If we combine these two aspects, namely smaller notionals and negative difference between floating and fixed rates at the beginning, the result is of course a reduced $NPV_{\text{risk free}}$ in DA case with respect to the SA case (in particular it becomes negative).

5 Amortizing swap pricing with CVA

A more interesting and realistic approach is to take into considerations that the corporate, namely bank's counterparty, could default, clearly having an impact on the pricing of the derivative of interest. This translates into the need of a proper measure of risk for this possible event, introducing the Counterparty Credit Valuation Adjustment (CVA) as a metric embedding the new risk and formally defined as the difference between the value of a position traded with a default-free counterparty and the value of the same position when traded with a defaultable counterparty (as stated in *Counterparty credit risk, collateral and funding* by D. Brigo, M. Morini and A. Pallavicini (2013)).

$$NPV(t_0) = NPV_{\text{risk free}}(t_0) - CVA(t_0) \tag{5.1}$$

For the sake of simplicity, we will treat only a unilateral default risk (assuming that the bank is default free), with no collateral and without any cost of funding. We will also assume a deterministic Loss Given Default (LGD) equal to *40%*. Moreover, we are going to consider two different scenarios with different constant Credit Default Swap (CDS) spread *s* equal to *300 basis points* and *500 basis points* (bps), in order to create various stressed scenarios when considering a possible default of the counterparty involved in the contract. This assumption leads to constant values of intensities λ according to the formula

$$\lambda = \frac{s}{LGD}, \tag{5.2}$$

which will make survival probabilities computation much simpler, compared to a more concrete situation in which we should extract the intensities from a separated market CDS bootstrap. Moreover, two further hypotheses are needed: we have to assume that the default event τ can only happen at future times coinciding with the contract payment dates and, finally, that the probability of a default event to happen is independent from interest rates evolution. The final needed ingredients for a complete CVA computation are of course the survival probabilities corresponding to each of the obtained constant values of λ:

$$\mathbb{P}(\tau > T_i) = e^{-\int_{t_0}^{T_i} \lambda(s)ds} = e^{-\lambda(T_i - t_0)}, \tag{5.3}$$

where $\mathbb{P}(\tau > T_i)$ is the probability that the default time τ is going to be strictly after the payment date T_i.

We have finally reached the real aim of this section, which is to provide a formula for the CVA introduced at the beginning of this chapter under all the assumptions and simplifications pointed out during the last steps:

$$\begin{aligned}
CVA(t_0) &= LGD \cdot \mathbb{E}\left[\mathbb{1}_{\tau \leq T_\omega}(NPV(\tau))^+\right] \\
&= LGD \int_{T_\alpha}^{T_\omega} PS(t_0, T_i, T_\omega, R, S(t_0, s, T_\omega), \sigma_{s,\omega})\mathbb{P}(\tau \leq \mathrm{s})ds \\
&= LGD \sum_{i=\alpha+1}^{\omega-1} \mathbb{P}(\tau \in (T_{i-1}, T_i])PS(t_0, T_i, T_\omega, R, S(t_0, T_i, T_\omega), \sigma_{i,\omega}) \\
&= LGD \sum_{i=\alpha+1}^{\omega-1} (\mathbb{P}(\tau > T_{i-1}) - \mathbb{P}(\tau > T_i))PS(t_0, T_i, T_\omega, R, S(t_0, T_i, T_\omega), \sigma_{i,\omega}),
\end{aligned} \tag{5.4}$$

where *PS* is the current (in t_0) price of a Payer Swaption having expiry T_i, maturity of the underlying swap in T_ω, fixed rate *R*, forward swap rate $S_{i,\omega}(t_0)$ (evaluated in t_0, starting in T_i, until T_ω) and implied volatility $\sigma_{i,\omega}$ (namely the one related to the swaption's expiry and tenor).

The reason why we end up with this formula is pretty straightforward. Indeed, once we separate default probabilities and interest rates evolution, we remain with an option on the residual value of the underlying swap, which is nothing else than a sum of swaptions with different expiries coinciding with the original swap payment dates, each of them weighted by the probability of defaulting between the two time instants T_{i-1} and T_i.

The remaining part of this section will be devoted to the explanation of the derivation of the swaption prices.

Since market implied volatilities are available, we compute swaption market prices by Bachelier closed formula for a payer swaption (keeping in mind that we are dealing with an amortizing swap (SA and DA), hence we have to pay attention to a couple of topics):

$$PS_{\alpha\omega}(t_0) = B(t_0, T_\alpha) BPV_{\alpha\omega}(t_0)[(S_{\alpha\omega}(t_0) - R)N(d) + \hat{\sigma}_{\alpha\omega}\sqrt{T_\alpha - t_0}\phi(d)], \tag{5.5}$$

where R is the fixed rate, $N(\cdot)$ is the standard normal CDF, $\phi(\cdot)$ the standard normal density function and $\hat{\sigma}_{\alpha\omega}$ is the corresponding implied volatility. It is important to spend more time on really understanding this last term $\hat{\sigma}_{\alpha\omega}$. Indeed, we can no longer simply use quoted implied volatilities available on the market, but we have to perform and interpolation in order to move from plain vanilla swaptions to swaptions having an amortizing swap as their underlying.

First of all we have to reconstruct a full matrix of implied volatilities based on plain vanilla swaptions by performing a linear interpolation for every expiry and for every tenor, obtaining a full set of implied volatilities for every swaption of interest. After that, we have to differentiate between the single amortizing and double amortizing case: for the second case we can already use these interpolated volatilities since both the fixed and the floating leg refer to the same notional N_i at each time T_{i+1} (they decrease over time on both legs), while the situation becomes more complicated in the first case, where at each time T_{i+1} the two legs refer to different notionals (always N_0 for the floating leg and N_i instead for the fixed leg).
The reason why in the single amortizing case for all the swaptions (with expiries in all payment dates of the swap except the last one) the floating leg refers always to the principal amount N_0 comes from the definition of the CVA: it is related to the residual NPV of the future positive cash flows of the contract in case of default of the counterparty (as wee can see in *Counterparty credit risk, collateral and funding* by D. Brigo, M. Morini and A. Pallavicini (2013)), which in SA case is the positive part of the difference between the floating rate of the same initial notional and the fixed rate of a decreasing notional. In order to clarify this aspect, a proof on the form of the CVA for both single and double amortizing is provided in the appendix A.
Coming back to the volatilities for the single amortizing case, in order to overcome the illustrated problem, we have to linearly interpolate again in terms of $\widehat{BPV}_{\alpha\omega}$, since we have to remember that

$$BPV_{\alpha\omega \text{ plain vanilla}}(t_0) = \sum_{i=\alpha}^{\omega-1} \delta(T_i, T_{i+1}) B(t_0, T_{i+1}), \tag{5.6}$$

while in this case we have that the $BPV_{\alpha\omega}$ needed for Bachelier formula in (4.5) is equal to

$$BPV_{\alpha\omega}(t_0) = \sum_{i=\alpha}^{\omega-1} N_i \delta(T_i, T_{i+1}) B(t_0, T_{i+1}). \tag{5.7}$$

Our aim is to interpolate all the implied volatilities $\sigma_{\alpha\omega}$ in the corresponding $BPV_{\alpha\omega \text{ plain vanilla}}$, evaluating them in the $\widehat{BPV}_{\alpha\omega}$ defined as

$$\widehat{BPV}_{\alpha\omega}(t_0) = \sum_{i=\alpha}^{\omega-1} \frac{N_i}{N_0} \delta(T_i, T_{i+1}) B(t_0, T_{i+1}), \tag{5.8}$$

in order to obtain the desired $\hat{\sigma}_{\alpha\omega}$ needed for our closed formula (5.5). This $\widehat{BPV}_{\alpha\omega}$ comes from the fact that in the SA case the floating leg always refers to the initial notional N_0, which can be taken out of the $NPV_{\text{floating leg}}$ when computing the expectation of the sum of all future floating payments and it can be put at the denominator of the other leg, namely the $NPV_{\text{fixed leg}}$, leading to this new quantity called $\widehat{BPV}_{\alpha\omega}$.

Moreover, we have to remember that in the SA case the definition of the forward swap rate $S_{\alpha\omega}(t_0)$ is slightly different from the plain vanilla one, while the situation becomes way more delicate when computing this quantity in the DA case. Indeed, even if the general definition still holds as

$$S_{\alpha\omega}(t_0) = \frac{\mathcal{N}_{\alpha\omega}(t_0)}{\widehat{BPV}_{\alpha\omega}(t_0)} \tag{5.9}$$

and formula (5.8) can be used in order to obtain $\widehat{BPV}_{\alpha\omega}$, the computation of $\mathcal{N}_{\alpha\omega}(t_0)$ has to be done separately, since this quantity comes from the $NPV(t_0)$ of all future payments related to the floating leg of the contract. Thus, in the SA case, we simply obtain that

$$\begin{aligned}
\mathcal{N}_{\alpha\omega\,\mathrm{SA}}(t_0) &= \mathbb{E}\left[\sum_{i=\alpha}^{\omega-1}\delta(T_i,T_{i+1})D(t_0,T_{i+1})L(T_i,T_{i+1})N_0\middle|\mathcal{F}_{t_\alpha}\right]/B(t_0,T_\alpha)\\
&= \sum_{i=\alpha}^{\omega-1}\delta(T_i,T_{i+1})B(t_0,T_\alpha,T_{i+1})L(t_0,T_i,T_{i+1})N_0\\
&= \left[1 - B(t_0,T_\alpha,T_\omega) + \sum_{i=\alpha}^{\omega-1}B(t_0,T_\alpha,T_i)[\beta(t_0,T_i,T_{i+1})-1]\right]N_0,
\end{aligned} \tag{5.10}$$

while in the DA case we have a decreasing sequence of notionals affecting also the floating leg, which translates into

$$\begin{aligned}
\mathcal{N}_{\alpha\omega\,\mathrm{DA}}(t_0) &= \mathbb{E}\left[\sum_{i=\alpha}^{\omega-1}\delta(T_i,T_{i+1})D(t_0,T_{i+1})L(T_i,T_{i+1})N_i\middle|\mathcal{F}_{t_\alpha}\right]/B(t_0,T_\alpha)\\
&= \sum_{i=\alpha}^{\omega-1}\delta(T_i,T_{i+1})B(t_0,T_\alpha,T_{i+1})L(t_0,T_i,T_{i+1})N_i\\
&= N_{\alpha-1} - B(t_0,T_\alpha,T_\omega)N_{w-1} + \sum_{i=\alpha}^{\omega-1}B(t_0,T_\alpha,T_i)[\beta(t_0,T_i,T_{i+1})N_i - N_{i-1}].
\end{aligned} \tag{5.11}$$

For the sake of completeness, we provide proofs of formula (5.10) in the appendix C and (5.11) in the appendix B.
Finally, after having computed $S_{\alpha\omega}(t_0)$, we need the definition of the last element in Bachelier closed formula (5.5), namely

$$d = \frac{S_{\alpha\omega}(t_0) - R}{\hat{\sigma}_{\alpha\omega}\sqrt{T_\alpha - t_0}}. \tag{5.12}$$

This procedure allows us to find a value for the CVA, hence to consider also a possible counterparty default risk. Summing up, by using (5.1), we display our results for both single amortizing and double amortizing cases

SA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	2’942’545.03	672’727.31	2’269’817.72
$spread_{500bps}$	2’942’545.03	899’912.04	2’042’632.99

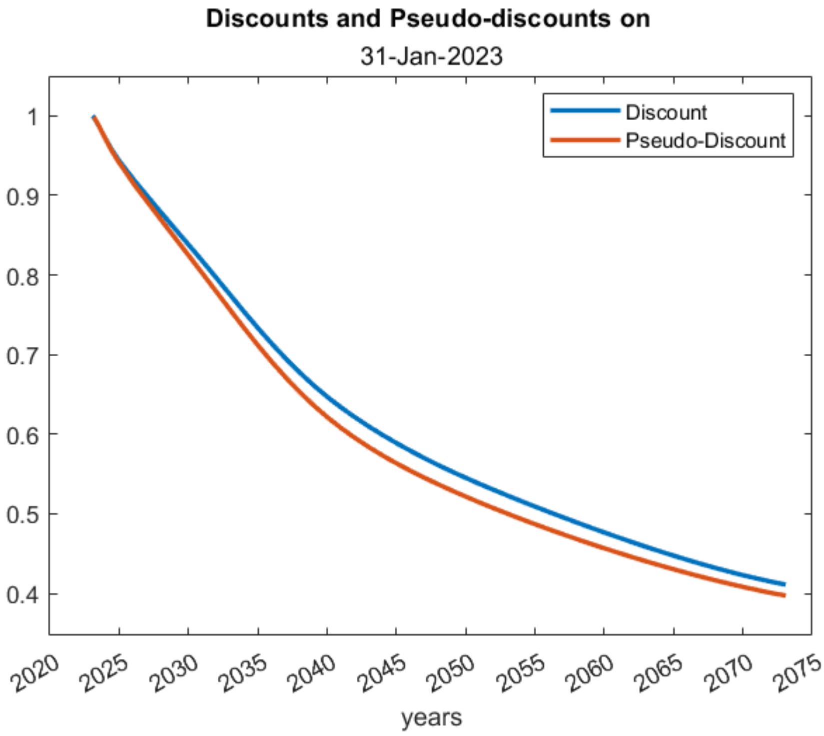
DA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	-112’976.06	123’394.76	-236’370.82
$spread_{500bps}$	-112’976.06	169’273.73	-282’249.79

As expected, a higher value of the CDS spread, corresponding to a higher intensity λ and hence affecting directly the survival probabilities, results into a greater CVA, lowering the real value of the NPV when considering also corporate default risk.

6 Pricing at unwinding

We now want to represent a peculiar situation in which we observe an early redemption from the corporate, namely the contract is closed on purpose at a certain time, without having reached its maturity. This situation in known in the financial industry as *unwinding* and the counterparty leaving the contract has to pay to the other conuterparty the residual NPV of the contract at that particular date.
In our specific case, we assume that this situation happens on the *31st January 2023*, meaning that we have to compute once again the discounting and pseudo-discounting curve from the new market data available.

A relevant issue that must be enlightened is that since the unwinding is on the 31st January and the Euribor is determined every 3 months and paid a quarter later, the contract is closed in the middle of a period between two payment dates. Hence, we can easily compute the expectation of all future floating leg payments as we have already done in the sections above, except for the Euribor paid on the first payment date after the unwinding date (28th March 2023), because it has been already determined on the previous reset date (28th December 2022), reason for which we select it from online market data, where it is available and quoted at *2.202%*.



After the new bootstrap procedure, we can perform the pricing of the amortizing swap by following exactly the same method as in the previous section (hence taking into account also the CVA), obtaining the following results:

SA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	3’931’808.39	681’653.27	3’250’155.12
$spread_{500bps}$	3’931’808.39	927’834.29	3’003’974.09

DA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	806’034.11	129’104.92	676’929.18
$spread_{500bps}$	806’034.11	181’487.50	624’546.61

We can notice that the new NPV is way larger than the one computed when the contract started. Indeed, going back to the final considerations of section 3, we recall that the first Euribor rate was negative. This means that the bank was receiving a negative cash flow in the first time interval, namely it was paying also the floating leg to its counterparty. Moreover, even though we know that all floating payments refer to the same initial notional (in the single amortizing case), we have to keep in mind that the first payment (on the 28^{th} September 2022) is affected by the highest discount factor (equal to 1.0008, namely a negative zero rate) when computing the NPV with respect to the initial settlement date. Thus, this value is the largest among all the ones in the floating leg.
It is immediate to understand that having it with a negative sign will affect significantly the final value of the NPV, reason for which it was negative on the original settlement date. On the other hand, on unwinding date all the forward pseudo-discounts are strictly lower than 1, which leads to the absence of negative Euribor rates. Consequently, on the 31^{st} January 2023 from the bank perspective there are only positive cash flows in the floating leg, which determines a higher NPV of the contract with respect to the original one (in particular in the double amortizing case, where it switches from negative to positive).

In conclusion, analogously to section 5, we can notice that if the CDS spread increases, the NPV with CVA decreases.

7 Multicurve Hull-White model calibration

The next subject of our study is to provide a suitable extension of the well known Hull-White model to the multicurve framework (MHW). Although different models have been proposed in the financial literature, we want to prioritize a parsimonious model, which allows us to avoid calibration issues without taking into account too many parameters. For this reason, we are going to rely on the 3 factors *(a, σ, γ)* model proposed in Baviera (2019), coming up with an explicit formulation for the dynamics of forward discount factors and spreads β, useful when pricing different kinds of swaptions.
We will initially focus our work on the calibration by using plain vanilla physical delivery receiver swaptions and then extend the whole analysis also to physical delivery payer swaptions, thanks to the swaption put-call parity.

First of all we have to recall the payoff of a physical delivery receiver swaption at its expiry T_α:

$$\begin{aligned}\mathcal{R}_{\alpha\omega}(T_\alpha) &= [BPV_{\alpha\omega}(T_\alpha)(R - S_{\alpha\omega}(T_\alpha))^+] \\ &= [(BPV_{\alpha\omega}(T_\alpha)R - \mathcal{N}_{\alpha\omega}(T_\alpha))^+],\end{aligned} \tag{7.1}$$

where *R* is the fixed rate, namely the strike of the swaption, while

$$BPV_{\alpha\omega}(T_\alpha) = \sum_{i=\alpha+1}^{\omega} \delta(T_{i-1}, T_i) B(T_\alpha; T_\alpha, T_i) \tag{7.2}$$

and

$$\mathcal{N}_{\alpha\omega}(T_\alpha) = 1 - B(T_\alpha, T_\alpha, T_\omega) + \sum_{i=\alpha}^{\omega-1} B(T_\alpha, T_\alpha, T_i)[\beta(T_\alpha, T_i, T_{i+1}) - 1]. \tag{7.3}$$

Moreover, introducing $c_i = R\delta(T_{i-1}, T_i)$ for $i < \omega$ and $c_\omega = R\delta(T_{\omega-1}, T_\omega) + 1$, we can develop a new formulation for the payoff initially stated in (7.1):

$$\mathcal{R}_{\alpha\omega}(T_\alpha) = \left[\sum_{i=\alpha+1}^{\omega} c_i B(T_\alpha, T_\alpha, T_i) + \sum_{i=\alpha+1}^{\omega-1} B(T_\alpha, T_\alpha, T_i) - \sum_{i=\alpha}^{\omega-1} \beta(T_\alpha, T_i, T_{i+1}) B(T_\alpha, T_\alpha, T_i)\right]^+. \tag{7.4}$$

From now on, we will assume that the whole process dynamics is driven by a standard normal random variable *x*, allowing us to rewrite each element of the final payoff (7.4) as a simple function of this quantity. Furthermore, as reported in Baviera (2019), there exists a unique value x^* such that $S_{\alpha\omega}(T_\alpha)$ is equal to the strike of interest *R*. This preliminary step is fundamental in order to extend the Jamshidian (1989) approach to our multicurve framework, obtaining at the end a simple closed formula for our swaptions.

Before proceeding, let us introduce such r.v. as

$$x = \frac{1}{\zeta_\alpha} \int_{t_0}^{T_\alpha} \sigma e^{-a(T_\alpha - u)} dW_u^{(\alpha)}, \tag{7.5}$$

with

$$\zeta_\alpha^2 = \begin{cases} \sigma^2 \frac{1 - e^{-2a(T_\alpha - t_0)}}{2a} & \text{if } a \in \mathbb{R}^+ \backslash \{0\} \\ \sigma^2 (T_\alpha - t_0) & \text{if } a = 0 \end{cases} \tag{7.6}$$

where $W_t^{(\alpha)}$ is a Brownian Motion with respect to the the T_α-forward measure, under which $B(t, T_\alpha, T_i)$ is a martingale. We also recall that $\beta(t, T_i, T_{i+1})$ is a martingale under the T_i-forward measure. Thus, we can write the dynamics of forward discount factors and spreads β in a MHW model as

$$B(T_\alpha, T_\alpha, T_i) = B(t_0, T_\alpha, T_i) e^{-\varsigma_{\alpha i} x - \frac{1}{2}\varsigma_{\alpha i}^2} \tag{7.7}$$

and

$$\beta(T_\alpha, T_i, T_{i+1})B(T_\alpha, T_\alpha, T_i) = \beta(t_0, T_i, T_{i+1})B(t_0, T_\alpha, T_i)e^{-\nu_{\alpha i}x - \frac{1}{2}\nu_{\alpha i}^2}, \tag{7.8}$$

where $\varsigma_{\alpha i} = (1-\gamma)\upsilon_{\alpha i}$ represents the volatility for the evolution of the forward discount factors, while $\nu_{\alpha i} = \upsilon_{\alpha i} - \gamma\upsilon_{\alpha i+1}$ embeds the stochasticity of the spread β, with

$$\upsilon_{\alpha i} = \zeta_\alpha \frac{1 - e^{-a(T_i - T_\alpha)}}{a}, \tag{7.9}$$

in which the index *i* varies according to the summations in formula (7.4), while α is related to the time instant T_α, namely the expiry of the considered swaption.
After having introduced the above quantities, it is straightforward to rewrite (7.4) as a function of the only variable *x*, obtaining

$$\mathcal{R}_{\alpha\omega}(T_\alpha) = [f(x)]^+, \tag{7.10}$$

where

$$\begin{aligned} f(x) = &\sum_{i=\alpha+1}^{\omega} c_i B(t_0, T_\alpha, T_i)e^{-\varsigma_{\alpha i}x - \frac{1}{2}\varsigma_{\alpha i}^2} + \sum_{i=\alpha+1}^{\omega-1} B(t_0, T_\alpha, T_i)e^{-\varsigma_{\alpha i}x - \frac{1}{2}\varsigma_{\alpha i}^2} \\ &- \sum_{i=\alpha}^{\omega-1} \beta(t_0, T_i, T_{i+1})B(t_0, T_\alpha, T_i)e^{-\nu_{\alpha i}x - \frac{1}{2}\nu_{\alpha i}^2}, \end{aligned} \tag{7.11}$$

allowing us to state that, according to a MHW mode, $\exists!\ x^*$ such that $f(x^*) = 0$. Moreover the function *f(X)* is grater than zero for $x < x^*$ and, even if it is not a monotone decreasing function of the variable *x*, we are only interested in the fact that there exists a unique value of *x* satisfying the equality $S_{\alpha\omega}(T_\alpha)$ = *R*. We can finally move on in our modelling procedure by computing the expected value of the discounted payoff of the receiver swaption (RS) previously stated in (7.1) and (7.4), but taking into consideration all the results obtained up to this moment, hence

$$\begin{aligned} RS_{\alpha\omega}(t_0) &= \mathbb{E}[D(t_0, T_\alpha)\mathcal{R}_{\alpha\omega}(T_\alpha)|\mathcal{F}_{t_0}] \\ &= \mathbb{E}[D(t_0, T_\alpha)BPV_{\alpha\omega}(T_\alpha)(R - S_{\alpha\omega}(t_\alpha))^+|\mathcal{F}_{t_0}] \\ &= \mathbb{E}[D(t_0, T_\alpha)[f(x)]^+|\mathcal{F}_{t_0}] \\ &= \mathbb{E}[D(t_0, T_\alpha)f(x)\mathbb{I}_{x \le x^*}|\mathcal{F}_{t_0}]. \end{aligned} \tag{7.12}$$

We can eventually price a physical delivery receiver swaption according to the following closed formula:

$$\begin{aligned} RS_{\alpha\omega}(t_0) = B(t_0, T_\alpha)\Bigg\{ &\sum_{i=\alpha+1}^{\omega} c_i B(t_0, T_\alpha, T_i)N(x^* + \varsigma_{\alpha i}) + \sum_{i=\alpha+1}^{\omega-1} B(t_0, T_\alpha, T_i)N(x^* + \varsigma_{\alpha i}) \\ &- \sum_{i=\alpha}^{\omega-1}\beta(t_0, T_i, T_{i+1})B(t_0, T_\alpha, T_i)N(x^* + \nu_{\alpha i})\Bigg\}, \end{aligned} \tag{7.13}$$

where $N(\cdot)$ is the standard normal CDF and x^* is the unique value that nullifies *f(x)* defined in (7.11).

Exploiting this result, we can calibrate the MHW model introduced in this section in a very parsimonious way, preforming a minimization of the $\mathbb{L}^2$ distance between market prices (computed according to Bachelier formula) and model prices (obtained thanks to formula (7.13)) with respect to the only 3 parameters *a*, σ, γ on which the model relies.

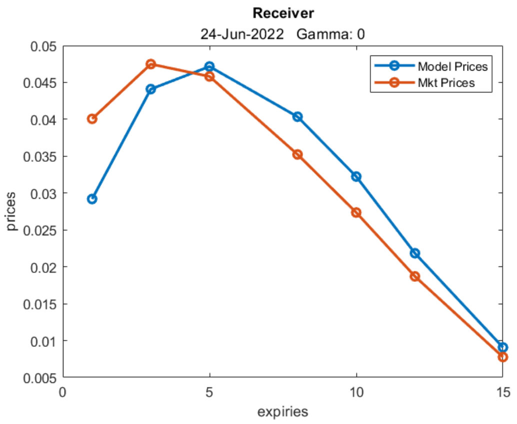
However, since considering also γ as a degree of freedom will lead to a more complex calibration, we will only perform a 2-factors minimization of the $\mathbb{L}^2$ distance, which now becomes a function of only *a* and σ, while we consider three possible fixed values of γ equal to [*0 0.5 1*]. The optimization problem we obtain is the following:

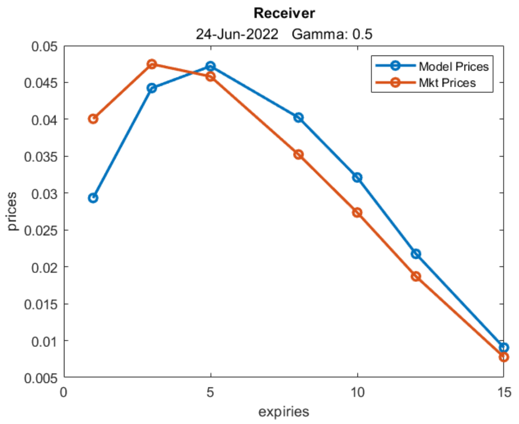
$$\{a, \sigma\} = min_{a,\sigma} \sum_{i=1}^{N} [RS_i^{Market}(t_0) - RS_i^{Model}(a, \sigma; t_0)]^2. \tag{7.14}$$

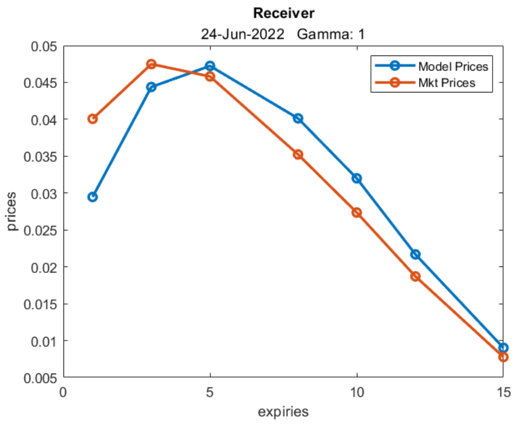
When performing the calibration above, we consider only the "diagonal" swaption values (N=7) for some expiries and tenors, namely *[1y15y, 3y12y, 5y10y, 8y7y, 10y5y, 12y3y, 15y1y]*.

We can finally display the calibrated parameters and the minimized $\mathbb{L}^2$ distance for the original trade date 24^{th} June 2022, together with the plot of the differences between market and model prices for each given value of γ:

24^{th} June 2022	a (%)	σ (%)	$\mathbb{L}^2$ distance (10^{-4})
$\gamma = 0$	0.41	0.91	1.9293
$\gamma = 0.5$	0.53	0.91	1.8629
$\gamma = 1$	0.71	0.92	1.7973



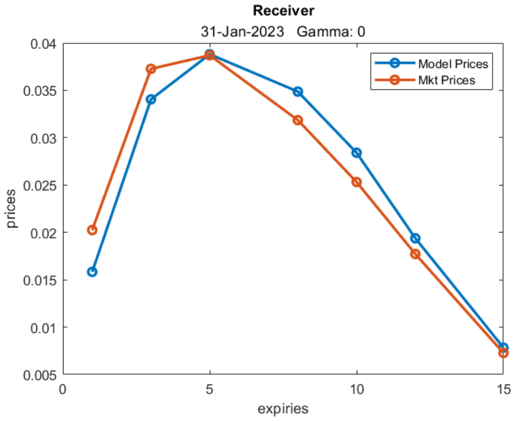


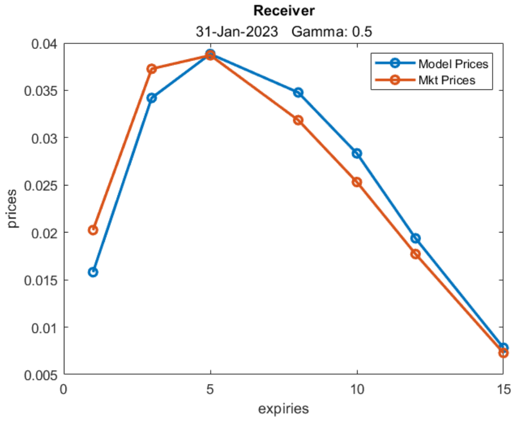


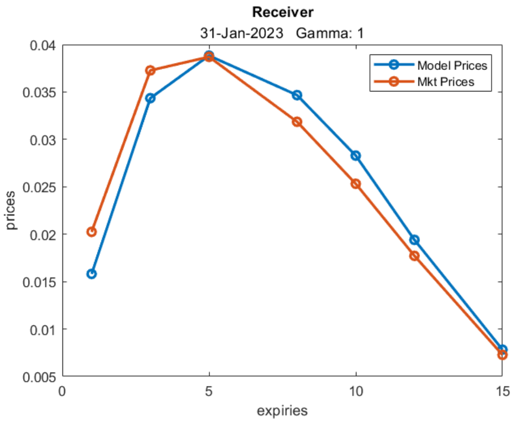
Moreover we provide the calibration also on the unwinding date of 31^{st} January 2023, showing once again all the calibrated parameters and the minimized $\mathbb{L}^2$ distance for each fixed value of γ:

31^{st} January 2023	a (%)	σ (%)	$\mathbb{L}^2$ distance (10^{-4})
$\gamma = 0$	2.47	1.03	0.5161
$\gamma = 0.5$	3.59	1.11	0.4994
$\gamma = 1$	4.79	1.21	0.4811

and, even in this second calibration, we can see the differences between market and model prices for each value of γ in the following plots:







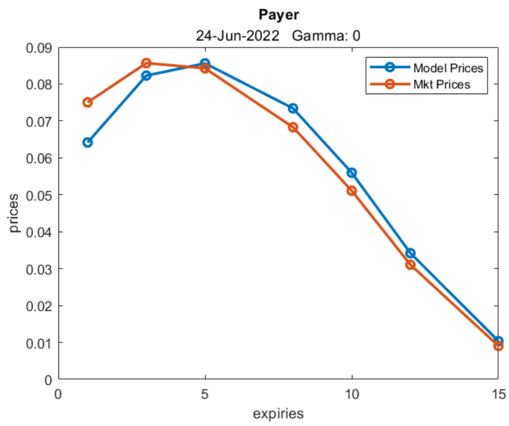
Furthermore, we repeat the exact same procedure in order to calibrate the model considering this time physical delivery Payer Swaptions $PS_{\alpha\omega}(t_0)$, since a suitable put-call parity is available for physical delivery swaptions, defined as

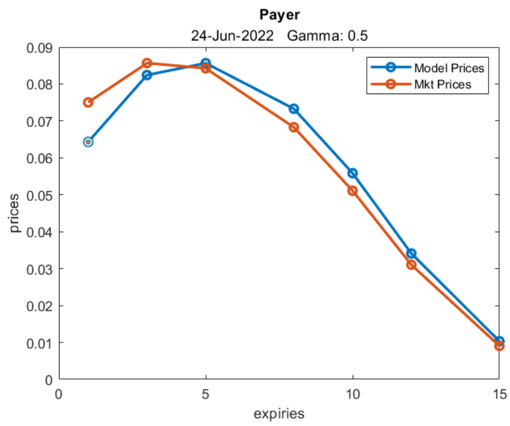
$$PS_{\alpha\omega}(t_0) = RS_{\alpha\omega}(t_0) - B(t_0, T_\alpha)BPV_{\alpha\omega}(t_0)(R - S_{\alpha\omega}(t_0)), \tag{7.15}$$

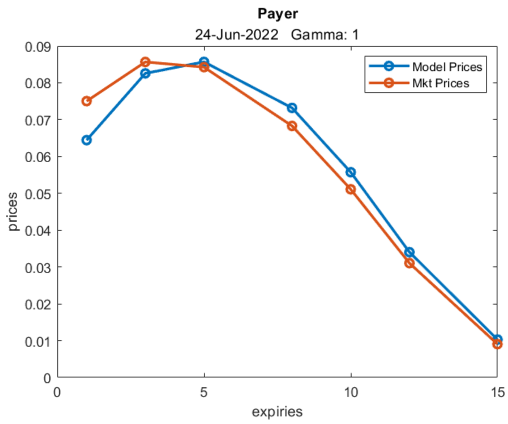
where $BPV_{\alpha\omega}(t_0)$ is reported in formula (7.2) and $S_{\alpha\omega}(t_0)$ is the usual forward swap rate.

Hence, we can show the results obtained also with this new calibration for each fixed value of γ, starting with the initial trade date 24^{th} June 2022

24^{th} June 2022	a (%)	σ (%)	$\mathbb{L}^2$ distance (10^{-4})
$\gamma = 0$	0.41	0.91	1.9293
$\gamma = 0.5$	0.53	0.91	1.8629
$\gamma = 1$	0.71	0.92	1.7973

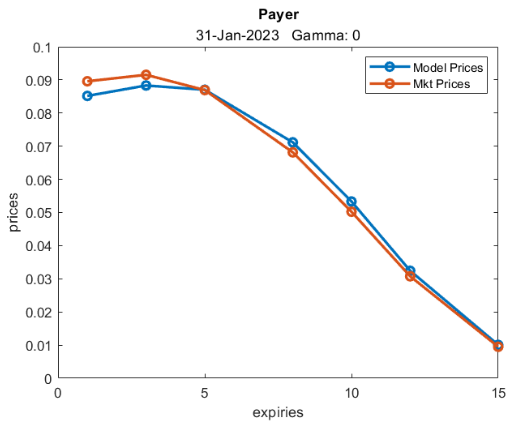


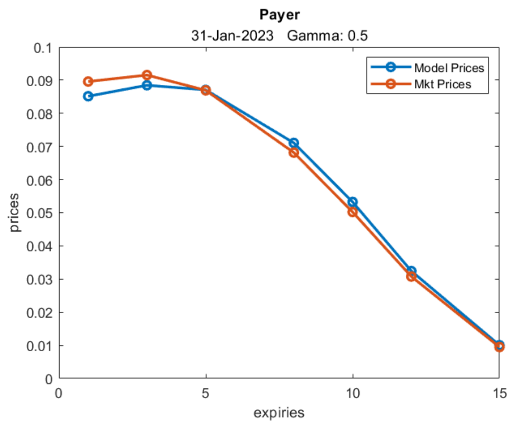


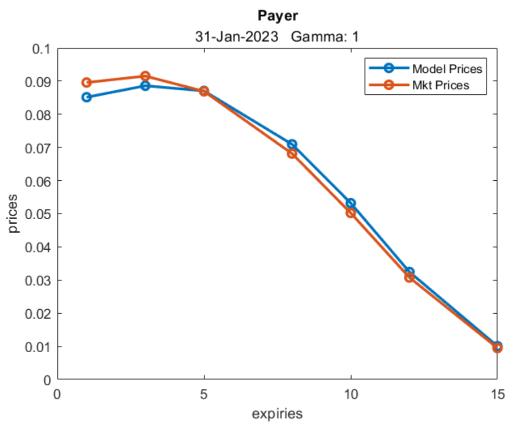


and ending, exactly as in the previous case, with the visualization of the results of our last calibration on 31^{st} January 2023:

31^{st} January 2023	a (%)	σ (%)	$\mathbb{L}^2$ distance (10^{-4})
$\gamma = 0$	2.47	1.03	0.5161
$\gamma = 0.5$	3.58	1.11	0.4994
$\gamma = 1$	4.79	1.21	0.4811

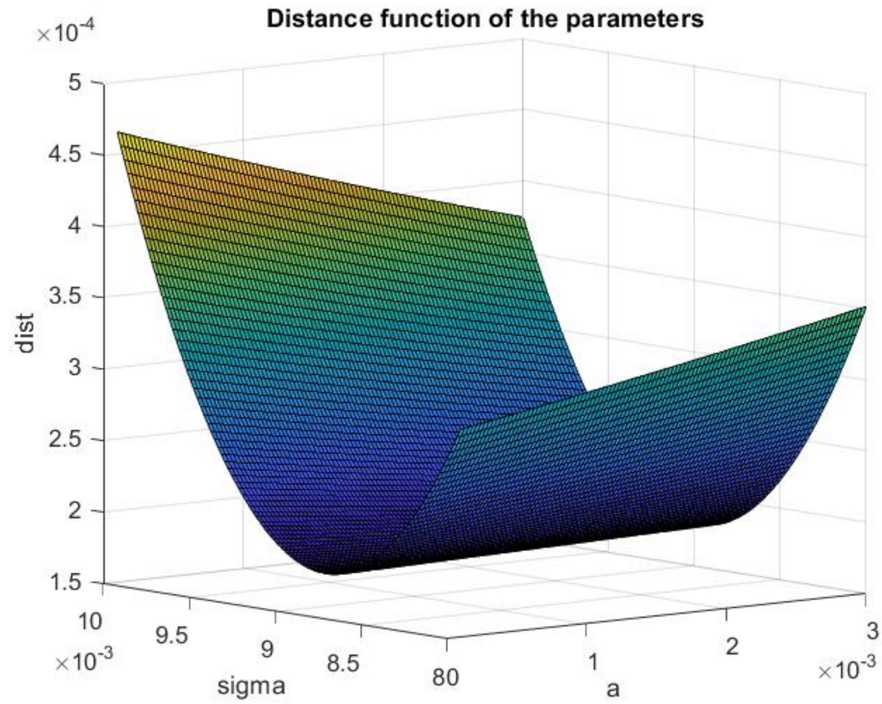






Exactly as expected, there are no relevant differences on the calibrated parameters when considering physical delivery payer swaptions or physical delivery receiver swaptions. Indeed, if we compare the obtained a and σ in the two cases, we can notice some differences only after several decimal digits.

An important remark is that during the code implementation we observed a relevant instability of the minimization algorithm related to the initial conditions (high variations of the output for small variations of initial conditions). For this reason, we plotted the $\mathbb{L}^2$ distance of the model prices from the market prices as function of the parameters a and σ (for all γ values, the two settlement dates and both receiver and payer swaptions). In each situation we obtained a very similar figure, which is reported below:



We can understand from this plot that there are many local minima in the function (in particular there is a set of possible values for the parameter a corresponding to local minima). This could be a problem for the minimization algorithm, since the iterative procedure could stop in several different points (identifying a local minima), based on various initial conditions. This fact justifies that we could end up with very different parameters changing the input of the algorithm and explains why we obtain different results in Python and Matlab.

Hence, we tried to find the optimal parameters minimizing the distance function from the plot and we obtained for the parameter a the value zero. With this strategy we ended up with the same optimal parameters in Python and Matlab.

However, the value zero for the parameter a gives problems to the Matlab function *fzero* as well as the Python function *fsolve*. For this reason, we decided to keep the initial conditions coming from the Matlab optimization and the associated optimal parameters returned by the minimization algorithms (even though they are different in Matlab and Python), since the $\mathbb{L}^2$ distance of the model prices from the market prices does not change significantly modifying the value of the parameter a (as it can be seen in the plot of the distance function, where we have an almost constant function in the parameter a once we have reached a suitable σ).

All the reported results hereinafter are based on the Matlab minimization.

8 Pricing with MHW model

The final section of our work is devoted to the implementation of pricing techniques, focusing especially on numerical techniques such as a suitable modification of the already known trinomial tree to our multicurve framework and an alternative formula coming from a different use of the Jamshidian trick proposed by us, everything referring to the initial trade date of 24^{th} June 2022. We will then compare the resulting prices with the ones coming from model closed formula obtained after the calibration performed in the previous section.

Before starting, we assume for the whole discussion provided in this section a fixed vale of γ equal to *0*, namely we consider deterministic spread β, allowing us to neglect its stochasticity and leaving us with the following simplification

$$\varsigma_{\alpha i} = \nu_{\alpha i} = \upsilon_{\alpha i}, \tag{8.1}$$

which leads us to the conclusion that $\beta(T_\alpha, T_\alpha, T_i) = \beta(t_0, T_\alpha, T_i)$.

Starting from the trinomial tree, it's necessary to point out that the regular procedure can no longer be applied as in the simpler single curve framework.
The reason behind this problem lies in the definition of the dynamics of the forward discounts. In this case the stochastic process "x" is defined as:

$$x = \frac{1}{\zeta_\alpha} \int_{t_0}^{T_\alpha} \sigma e^{-a(T_\alpha - u)} dW_u^{(\alpha)}, \tag{8.2}$$

while in the single curve framework the stochastic process in the discounts' dynamics was an Ornstein-Uhlenbeck (OU) process:

$$y_{T_\alpha} = \sigma \int_{t_0}^{T_\alpha} e^{-a(T_\alpha - u)} dW_u^{(\alpha)}. \tag{8.3}$$

If we compute the variance of the process in (8.3), exploiting Itô isometry, we get:

$$\begin{aligned} Var(y_{T_\alpha}) &= \int_{t_0}^{T_\alpha} \sigma^2 e^{-2a(T_\alpha - u)} du \\ &= \sigma^2 \frac{1 - e^{-2a(T_\alpha - t_0)}}{2a} \end{aligned} \tag{8.4}$$

which is exactly ζ_α^2.
Thus, we can understand that the stochastic process in the multicurve framework is just a normalization of the previous OU process and it can be rewritten as:

$$x = \frac{y_{T_\alpha}}{\zeta_\alpha}. \tag{8.5}$$

For this reason we can extend the trinomial tree to our setting by using the same discretization as in the single curve context, which is now applied to y_{T_α}, namely the OU process, and then normalize for the proper ζ_α in order to obtain the values of x to insert in the discounts' dynamics. With this procedure we can simply exploit all the previous theory on the trinomial tree, paying attention to the discussed normalization when dealing with the discounts.
We underline that for the tree we use a monthly number of steps instead of a yearly one, since it seemed a more suitable choice to treat swaptions with expiries shorter than 1 year. In particular, we select 6 steps for each month, which results as a good compromise between accuracy and computational cost.

We can finally price the amortizing swap for both single amortizing (SA) and double amortizing (DA) cases, taking into account also CVA in order to consider the risk coming from a possible counterparty default, summing up everything in the following tables:

Trinomial tree SA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	2'942'545.02	695'817.13	2'246'727.89
$spread_{500bps}$	2'942'545.02	930'545.48	2'011'999.54

Trinomial tree DA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	-112'976.05	113'395.75	-226'371.81
$spread_{500bps}$	-112'976.05	154'483.31	-267'459.37

Furthermore, we want to provide an alternative consideration on the Jamshidian trick already used in the implementation of the calibrated model above, but with a different reorganization of the terms involved.

Indeed, starting from the usual definition of a physical delivery payer swaption price

$$PS_{\alpha\omega}(t_0) = \mathbb{E}[D(t_0, T_\alpha)BPV_{\alpha\omega}(t_0)(S_{\alpha\omega}(t_0) - R)^+|\mathcal{F}_{t_0}] \tag{8.6}$$

and after a series of computations reported in appendix D, we end up with

$$\begin{aligned} PS_{\alpha\omega}(t_0) = \mathbb{E}\Bigg[D(t_0,T_\alpha)\Bigg\{&\beta(T_\alpha,T_\alpha,T_{\alpha+1})N_\alpha + \sum_{i=\alpha+1}^{\omega-1} B(T_\alpha,T_\omega)[\beta(T_\alpha,T_i,T_{i+1})N_i - N_{i+1}] \\ &- B(T_\alpha,T_\omega)N_{\omega-1} - \sum_{i=\alpha+1}^{\omega}\delta(T_{i-1},T_i)B(T_\alpha,T_i)N_{i-1}R\Bigg\}^+\Bigg|\mathcal{F}_{t_0}\Bigg] \end{aligned} \tag{8.7}$$

which can be finally reshaped into

$$PS_{\alpha\omega}(t_0) = \mathbb{E}\left[D(t_0,T_\alpha)\left\{\bar{K} - \sum_{i=\alpha+1}^{\omega}\bar{c}_i B(T_\alpha,T_i)\right\}^+\middle|\mathcal{F}_{t_0}\right], \tag{8.8}$$

where in the single amortizing case (SA) the coupons $\bar{c}_i$ are defined as follows

$$\begin{aligned} \bar{c}_{i\,\text{SA}} &= -\beta(T_\alpha,T_i,T_{i+1})N_i + N_{i-1} + \delta(T_{i-1},T_i)RN_0 \quad \forall i = \alpha+1,...,\omega-1 \\ \bar{c}_{\omega\,\text{SA}} &= N_{\omega-1} + \delta(T_{\omega-1},T_\omega)RN_0, \end{aligned}$$

while in the double amortizing case (DA) the coupons of the CB are defined as

$$\begin{aligned} \bar{c}_{i\,\text{DA}} &= -\beta(T_\alpha,T_i,T_{i+1})N_i + N_{i-1} + \delta(T_{i-1},T_i)RN_{i-1} \quad \forall i = \alpha+1,...,\omega-1 \\ \bar{c}_{\omega\,\text{DA}} &= N_{\omega-1} + \delta(T_{\omega-1},T_\omega)RN_{\omega-1}. \end{aligned}$$

As reported in the formula above, we can see our payer swaption as a put option on a coupon bond (CB) with strike $\bar{K} = \beta(T_\alpha,T_\alpha,T_{\alpha+1})N_\alpha$ and perform the Jamshidian trick on this derivative, splitting it in the sum of $\omega-\alpha-1$ put options on zero coupon bonds (ZCB) with different strikes $\tilde{k}_i$.

This procedure is now possible since the first forward discount factor multiplying $\beta(T_\alpha,T_\alpha,T_{\alpha+1})N_\alpha$ would be $B(T_\alpha,T_\alpha,T_\alpha) = 1$, namely we have isolated a quantity which does not depend on the evolution of the r.v. *x*, allowing us to obtain a strike independent from such *x*.

We are now ready to perform the Jamshidian trick as already done in section 7, rewriting the payoff described by formula (8.8) as a function $f_J(x)$ depending only on *x*, and relying on the fact that $\exists!\ x^*_J$ such that $f_J(x^*_J) = 0$, where the dynamics of each discount factor $B(T_\alpha,T_i) = B(T_\alpha,T_\alpha,T_i)$ comes directly from formula (7.7), while there is no dynamics for the spreads β.

Thus, we can find x^*_J nullifying

$$f_J(x) = \bar{K} - \sum_{i=\alpha+1}^{\omega}\bar{c}_i B(t_0,T_\alpha,T_i)e^{-\varsigma_{\alpha i}x - \frac{1}{2}\varsigma^2_{\alpha i}}. \tag{8.9}$$

We can now define the new strikes $\tilde{k}_i$ for each one of the puts on the ZCB as

$$\tilde{k}_i = B(t_0,T_\alpha,T_i)e^{-\varsigma_{\alpha i}x^*_J - \frac{1}{2}\varsigma^2_{\alpha i}} \tag{8.10}$$

and finally rewrite formula (8.8) by applying the Jamshidian trick as follows

$$\begin{aligned} PS_{\alpha\omega}(t_0) &= \mathbb{E}[D(t_0,T_\alpha)(\bar{K}-CB)^+|\mathcal{F}_{t_0}] \\ &= \sum_{i=\alpha+1}^{\omega} \bar{c}_i \mathbb{E}[D(t_0,T_\alpha)(\tilde{k}_i - ZCB)^+|\mathcal{F}_{t_0}] \\ &= \sum_{i=\alpha+1}^{\omega} \bar{c}_i \mathcal{P}_i(t_0), \end{aligned} \tag{8.11}$$

where $\mathcal{P}_i(t_0)$ is the price of a put option on a ZCB having maturity T_i, with derivative expiry at T_α and strike $\tilde{k}_i$, computed according to

$$\begin{aligned} \mathcal{P}_i(t_0) &= \mathbb{E}[D(t_0,T_\alpha)(\tilde{k}_i - B(T_\alpha,T_i))^+|\mathcal{F}_{t_0}] \\ &= B(t_0,T_\alpha)\tilde{k}_i(1-N(x_J^*)) - B(t_0,T_i)(1-N(x_J^*+\varsigma_{\alpha i})). \end{aligned} \tag{8.12}$$

In order to clarify this result, the proof for formula (8.12) is available in appendix E.

We can finally price all the payer swaptions needed for the CVA, hence our alternative solution produces the following results:

Alternative Jamshidian SA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	2'942'545.02	717'745.96	2'224'799.06
$spread_{500bps}$	2'942'545.02	955'451.63	1'987'093.39

Alternative Jamshidian DA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	-112'976.05	114'332.04	-227'308.10
$spread_{500bps}$	-112'976.05	155'183.50	-268'159.56

In conclusion, we want to compare all the results obtained in this section to the ones coming from an NPV priced by using closed formulas according to the model calibrated in section 7. More precisely, we price all the physical delivery payer swaptions $PS_{\alpha\omega}(t_0)$ needed for CVA computation starting from the corresponding physical delivery receiver swaptions $RS_{\alpha\omega}(t_0)$ with same expiry T_α, same maturity T_ω and referring to the same amortizing swap thanks to model formula (7.13). The only difference from previous section is that we were working with plain vanilla swaptions, while now we have to take into account the presence of a notional at each time, indeed we have to deal with a function $f_{SA}(x)$ different from the one stated in (7.11). Hence, for the single amortizing case, we have to find the value x^*_{SA} such that $f_{SA}(x^*_{DA})$ = 0, where

$$\begin{aligned} f_{SA}(x) = &\sum_{i=\alpha+1}^{\omega} N_{i-1}c_i B(t_0,T_\alpha,T_i)e^{-\varsigma_{\alpha i}x-\frac{1}{2}\varsigma_{\alpha i}^2} + \sum_{i=\alpha+1}^{\omega-1} N_0 B(t_0,T_\alpha,T_i)e^{-\varsigma_{\alpha i}x-\frac{1}{2}\varsigma_{\alpha i}^2} \\ &-\sum_{i=\alpha}^{\omega-1} N_0\beta(t_0,T_i,T_{i+1})B(t_0,T_\alpha,T_i)e^{-\nu_{\alpha i}x-\frac{1}{2}\nu_{\alpha i}^2}, \end{aligned} \tag{8.13}$$

leading to the following closed formula

$$\begin{aligned} RS_{\alpha\omega\,\text{SA}}(t_0) = B(t_0,T_\alpha)\Bigg\{ &\sum_{i=\alpha+1}^{\omega} \frac{N_{i-1}}{N_0} c_i B(t_0,T_\alpha,T_i)N(x^*_{SA}+\varsigma_{\alpha i}) \\ &+ \sum_{i=\alpha+1}^{\omega-1} B(t_0,T_\alpha,T_i)N(x^*_{SA}+\varsigma_{\alpha i}) \\ &- \sum_{i=\alpha}^{\omega-1} B(t_0,T_\alpha,T_i)\beta(t_0,T_i,T_{i+1})N(x^*_{SA}+\nu_{\alpha i}) \Bigg\} N_0. \end{aligned} \tag{8.14}$$

It's better to recall that $c_i = R\delta(T_{i-1}, T_i)$ for $i < \omega$ and $c_\omega = R\delta(T_{\omega-1}, T_\omega) + 1$.

Instead, for the double amortizing case, we have to consider a slightly modified function $f_{DA}(x)$ since now we have a sequence of decreasing notionals on both swap's legs. Although the procedure is exactly the same as before, looking for the value x^*_{DA} such that $f_{DA}(x^*_{DA}) = 0$, where

$$\begin{aligned} f_{DA}(x) = &\sum_{i=\alpha+1}^{\omega} N_{i-1} c_i B(t_0, T_\alpha, T_i) e^{-\varsigma_{\alpha i} x - \frac{1}{2}\varsigma_{\alpha i}^2} + \sum_{i=\alpha+1}^{\omega-1} N_{i-1} B(t_0, T_\alpha, T_i) e^{-\varsigma_{\alpha i} x - \frac{1}{2}\varsigma_{\alpha i}^2} \\ &- \sum_{i=\alpha}^{\omega-1} N_{i-1} \beta(t_0, T_i, T_{i+1}) B(t_0, T_\alpha, T_i) e^{-\nu_{\alpha i} x - \frac{1}{2}\nu_{\alpha i}^2} \end{aligned} \tag{8.15}$$

and obtaining the double amortizing closed formula defined as

$$\begin{aligned} RS_{\alpha\omega\,\text{DA}}(t_0) = B(t_0, T_\alpha) \Bigg\{ &\sum_{i=\alpha+1}^{\omega} N_{i-1} c_i B(t_0, T_\alpha, T_i) N(x^*_{DA} + \varsigma_{\alpha i}) \\ &+ \sum_{i=\alpha+1}^{\omega-1} N_{i-1} B(t_0, T_\alpha, T_i) N(x^*_{DA} + \varsigma_{\alpha i}) \\ &- \sum_{i=\alpha}^{\omega-1} N_{i-1} \beta(t_0, T_i, T_{i+1}) B(t_0, T_\alpha, T_i) N(x^*_{DA} + \nu_{\alpha i}) \Bigg\}, \end{aligned} \tag{8.16}$$

where each quantity is defined as it was in section 7 and, as always, the notional N_i refers to the payment at time T_{i+1}.

Finally, we are able to price each of the wanted $PS_{\alpha\omega}(t_0)$ thanks to swaption put-call parity previously defined in (7.15). The obtained results are as follows:

Model SA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	2'942'545.02	723'448.05	2'219'096.97
$spread_{500bps}$	2'942'545.02	961'943.47	1'980'601.55

Model DA	$NPV_{\text{risk free}}$ €	CVA €	NPV €
$spread_{300bps}$	-112'976.05	119'261.08	-232'237.14
$spread_{500bps}$	-112'976.05	161'572.46	-274'548.52

As we can notice from the results shown in the tables above, there are (small) differences between prices computed according to the different techniques introduced in this section and the ones obtained in section 5. A possible explanation can be the fact that the parameters derived in the calibration performed in section 7 lead to a greater $\mathbb{L}^2$ distance between model and market prices with respect to the one reported in Baviera (2019). Indeed the prices in this section, depending on the same calibrated parameters a and σ, are closer to each other.

Appendix

A CVA for single and double amortizing

Exploiting the formula in Brigo & Masetti (2005), we can write the expected value with CVA of an interest rate swap as:

$$\mathbb{E}_t\left[\Pi(t)\right] = \mathbb{E}_t\left[\Pi^{\text{risk free}}(t)\right] - LGD \cdot \mathbb{E}_t\left[\mathbb{I}_{\tau \leq T_\omega} D(t,\tau)(NPV(\tau))^+\right] \tag{A.1}$$

where the second term represents the CVA. Hence, using a discretization based on payment dates, we obtain:

$$CVA = LGD \sum_{j=1}^{\omega-1} \mathbb{Q}\{T_{j-1} < \tau \leq T_j\} \mathbb{E}_t\left[D(t,T_j)(\mathbb{E}_{T_j}\Pi(T_j,T_\omega))^+\right] \tag{A.2}$$

where the last expected value is the residual NPV of the positive future cash flows of the contract (in our case an amortizing swap) at time T_j.
We can therefore focus on this value, which assumes different forms in the two cases that we consider:

- Single amortizing

$$\begin{aligned}
&\mathbb{E}_t\left[D(t,T_j)(\mathbb{E}_{T_j}\Pi(T_j,T_\omega))^+\right] \\
&= \mathbb{E}_t\left[D(t,T_j)\mathbb{E}_{T_j}\left[\left(\sum_{i=j+1}^{\omega} N_0\delta_i D(T_j,T_i)L(T_{i-1},T_i) - \sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)R\right)^+\right]\right] \\
&= \mathbb{E}_t\left[D(t,T_j)\mathbb{E}_{T_j}\left[\left(\sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)S_{j\omega}(T_j) - \sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)R\right)^+\right]\right] \\
&= \mathbb{E}_t\left[D(t,T_j)BPV_{j\omega}(T_j)(S_{j\omega}(T_j)-R)^+\right]
\end{aligned} \tag{A.3}$$

where we used the equation on the swap rate:

$$\sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)S_{j\omega}(T_j) = N_0 \sum_{i=j+1}^{\omega} \delta_i D(T_j,T_i)L(T_{i-1},T_i) \tag{A.4}$$

and we have:

$$BPV_{j\omega}(T_j) = \sum_{i=j+1}^{\omega} N_{i-1}\delta_i B(T_j,T_i) \tag{A.5}$$

We therefore obtained the pricing formula of a payer swaption for each possible default time.

It can be noticed from formula (A.4) that, dividing both terms by N_0, the $\widehat{BPV}_{j\omega}(T_j)$ for each swaption has at denominator the same initial notional N_0.

- Double amortizing

$$
\begin{aligned}
&\mathbb{E}_t\left[D(t,T_j)(\mathbb{E}_{T_j}\Pi(T_j,T_\omega))^+\right] \\
&\quad = \mathbb{E}_t\left[D(t,T_j)\mathbb{E}_{T_j}\left[\left(\sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)L(T_{i-1},T_i) - \sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)R\right)^+\right]\right] \\
&\quad = \mathbb{E}_t\left[D(t,T_j)\mathbb{E}_{T_j}\left[\left(\sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)S_{j\omega}(T_j) - \sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)R\right)^+\right]\right] \\
&\quad = \mathbb{E}_t\left[D(t,T_j)BPV_{j\omega}(T_j)(S_{j\omega}(T_j) - R)^+\right]
\end{aligned}
\tag{A.6}
$$

where we used the equation on the swap rate:

$$
\sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)S_{j\omega}(T_j) = \sum_{i=j+1}^{\omega} N_{i-1}\delta_i D(T_j,T_i)L(T_{i-1},T_i) \tag{A.7}
$$

and we have:

$$
BPV_{j\omega}(T_j) = \sum_{i=j+1}^{\omega} N_{i-1}\delta_i B(T_j,T_i) \tag{A.8}
$$

In this case we arrive again to the pricing formula of a payer swaption for each possible time of default, but in this case we cannot define a $\widehat{BPV}_{j\omega}(T_j)$, because also the floating leg refers to the same decreasing notional as the fixed leg (formula (A.7)). We can understand that in this situation we do not have to interpolate the quoted plain vanilla volatilities in terms of BPV.

B Proof of the formula (5.11)

$$\mathbb{N}_{\alpha\omega\,\mathrm{DA}}(t_0) = \mathbb{E}\left[\sum_{i=\alpha}^{\omega-1} D(t_0, T_{i+1})\delta_i L(t_i, T_{i+1}) N_i | \mathcal{F}_{t_0}\right] / B(t_0, T_\alpha) \tag{B.1}$$

$$= \sum_{i=\alpha}^{\omega-1} \delta_i N_i B(t_0, T_{i+1}) L(t_0, T_i, T_{i+1}) / B(t_0, T_\alpha)$$

$$= \sum_{i=\alpha}^{\omega-1} \frac{(1 - \tilde{B}(t_0, T_i, T_{i+1})) B(t_0, T_{i+1}) N_i}{\tilde{B}(t_0, T_i, T_{i+1}) B(t_0, T_\alpha)}$$

$$= \sum_{i=\alpha}^{\omega-1} \frac{B(t_0, T_i) B(t_0, T_i, T_{i+1})(1 - \tilde{B}(t_0, T_i, T_{i+1})) N_i}{\tilde{B}(t_0, T_i, T_{i+1}) B(t_0, T_\alpha)}$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \beta(t_0, T_i, T_{i+1})(1 - \tilde{B}(t_0, T_i, T_{i+1})) N_i$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \beta(t_0, T_i, T_{i+1}) N_i - \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) B(t_0, T_i, T_{i+1}) N_i$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \beta(t_0, T_i, T_{i+1}) N_i - \sum_{i=\alpha}^{\omega-1} \frac{B(t_0, T_i) B(t_0, T_{i+1}) N_i}{B(t_0, T_\alpha) B(t_0, T_i)}$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \beta(t_0, T_i, T_{i+1}) N_i - \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_{i+1}) N_i$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \beta(t_0, T_i, T_{i+1}) N_i - \sum_{i=\alpha+1}^{\omega} B(t_0, T_\alpha, T_i) N_{i-1}$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \beta(t_0, T_i, T_{i+1}) N_i - \left(B(t_0, T_\alpha, T_\omega) N_{\omega-1} + \sum_{i=\alpha+1}^{\omega-1} B(t_0, T_\alpha, T_i) N_{i-1} \right)$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \beta(t_0, T_i, T_{i+1}) N_i - \left(B(t_0, T_\alpha, T_\omega) N_{\omega-1} - B(t_0, T_\alpha, T_\alpha) N_{\alpha-1} + \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) N_{i-1} \right)$$

$$= \sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i) \left[\beta(t_0, T_i, T_{i+1}) N_i - N_{i-1} \right] + N_{\alpha-1} - B(t_0, T_\alpha, T_\omega) N_{\omega-1}$$

where we used:

$$L(t_0, T_i, T_{i+1}) = \frac{1 - \tilde{B}(t_0, T_i, T_{i+1})}{\delta_i \tilde{B}(t_0, T_i, T_{i+1})}$$

$$\beta(t_0, T_i, T_{i+1}) = \frac{B(t_0, T_i, T_{i+1})}{\tilde{B}(t_0, T_i, T_{i+1})}$$

$$B(t_0, T_\alpha, T_\alpha) = 1$$

C Proof of the formula (5.10)

In order to prove formula (5.10) we can simply repeat the passages of the proof of formula (5.11) but replacing the decreasing notional with a constant notional N_0 (due to the single amortizing case). It is immediate to understand that in this situation we can collect the constant notional and arrive, following exactly the same structure as before, to an analogous expression to (5.11). Thus, the final formula (5.10) is just a modification of (5.11) where all the terms are multiplied by N_0:

$$\mathbb{N}_{\alpha\omega\,\mathrm{SA}}(t_0) = \left[\sum_{i=\alpha}^{\omega-1} B(t_0, T_\alpha, T_i)\left[\beta(t_0, T_i, T_{i+1}) - 1\right] + 1 - B(t_0, T_\alpha, T_\omega)\right] N_0 \tag{C.1}$$

D Jamshidian Variation

$$\begin{aligned}
P_{\alpha\omega}(t_0) &= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)BPV_{\alpha\omega}(T_\alpha)\left\{S_{\alpha\omega}(T_\alpha) - R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{BPV_{\alpha\omega}(T_\alpha)S_{\alpha\omega}(T_\alpha) - BPV_{\alpha\omega}(T_\alpha)R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{\mathcal{N}_{\alpha\omega}(T_\alpha) - BPV_{\alpha\omega}(T_\alpha)R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{\mathbb{E}_\alpha\left[\sum_{i=\alpha}^{\omega-1} D(T_\alpha,T_{i+1})\delta_i L(T_i,T_{i+1})N_i\right] - BPV_{\alpha\omega}(T_\alpha)R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{\sum_{i=\alpha}^{\omega-1} B(T_\alpha,T_{i+1})\left(\frac{1}{\tilde{B}(T_\alpha,T_i,T_{i+1})} - 1\right)N_i - \sum_{i=\alpha}^{\omega-1}\delta_i B(T_\alpha,T_{i+1})N_i R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{\sum_{i=\alpha}^{\omega-1} B(T_\alpha,T_{i+1})\left(\frac{\beta(T_\alpha,T_i,T_{i+1})}{B(T_\alpha,T_i,T_{i+1})} - 1\right)N_i - \sum_{i=\alpha}^{\omega-1}\delta_i B(T_\alpha,T_{i+1})N_i R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{\sum_{i=\alpha}^{\omega-1} B(T_\alpha,T_{i+1})\left(\frac{\beta(T_\alpha,T_i,T_{i+1})B(T_\alpha,T_i)}{B(T_\alpha,T_{i+1})} - 1\right)N_i\right.\right. \\
&\qquad\qquad\left.\left. - \sum_{i=\alpha}^{\omega-1}\delta_i B(T_\alpha,T_{i+1})N_i R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{\sum_{i=\alpha}^{\omega-1}\left(\beta(T_\alpha,T_i,T_{i+1})B(T_\alpha,T_i) - B(T_\alpha,T_{i+1})\right)N_i\right.\right. \\
&\qquad\qquad\left.\left. - \sum_{i=\alpha}^{\omega-1}\delta_i B(T_\alpha,T_{i+1})N_i R\right\}^+\right] \\
&= \mathbb{E}_{t_0}\left[D(t_0,T_\alpha)\left\{\sum_{i=\alpha}^{\omega-1}\beta(T_\alpha,T_i,T_{i+1})B(T_\alpha,T_i)N_i - \sum_{i=\alpha}^{\omega-1} B(T_\alpha,T_{i+1})N_i\right.\right. \\
&\qquad\qquad\left.\left. - \sum_{i=\alpha}^{\omega-1}\delta_i B(T_\alpha,T_{i+1})N_i R\right\}^+\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{t_0}\Bigg[D(t_0,T_\alpha)\Bigg\{\beta(T_\alpha,T_{\alpha+1})N_\alpha + \sum_{i=\alpha+1}^{\omega-1} B(T_\alpha,T_i)\Bigg(\beta(T_\alpha,T_i,T_{i+1})N_i - N_{i-1}\Bigg) \\
&\qquad\qquad - B(T_\alpha,T_\omega)N_{\omega-1} - \sum_{i=\alpha+1}^{\omega} \delta_i B(T_\alpha,T_i)N_{i-1}R\Bigg\}^+\Bigg] \\
&= \mathbb{E}_{t_0}\Bigg[D(t_0,T_\alpha)\Bigg\{\beta(T_\alpha,T_{\alpha+1})N_\alpha - \sum_{i=\alpha+1}^{\omega} c_i B(T_\alpha,T_i)\Bigg\}^+\Bigg]
\end{aligned}
\tag{D.1}
$$

where

$$
\begin{aligned}
c_i &= -\beta(T_\alpha,T_i,T_{i+1})N_i + N_{i-1} + R\delta_i N_{i-1} \qquad \forall i = \alpha+1,\dots,\omega-1 \\
c_\omega &= (R\delta_\omega + 1)\, N_{\omega-1}
\end{aligned}
$$

E Put option on a ZCB

$$
\begin{aligned}
\mathcal{P}_i(t_0) &= \mathbb{E}[D(t_0,T_\alpha)(\tilde{k}_i - B(T_\alpha,T_i))^+|\mathcal{F}_{t_0}] \\
&= \mathbb{E}[D(t_0,T_\alpha)(\tilde{k}_i - B(T_\alpha,T_\alpha,T_i))^+|\mathcal{F}_{t_0}] \\
&= \mathbb{E}[D(t_0,T_\alpha)(\tilde{k}_i - B(t_0,T_\alpha,T_i)e^{-\varsigma_{\alpha i}x-\frac{1}{2}\varsigma_{\alpha i}^2})^+|\mathcal{F}_{t_0}] \\
&= B(t_0,T_\alpha)\mathbb{E}[(\tilde{k}_i - B(t_0,T_\alpha,T_i)e^{-\varsigma_{\alpha i}x-\frac{1}{2}\varsigma_{\alpha i}^2})\mathbb{I}_{x\geq x_J^*}|\mathcal{F}_{t_0}] \\
&= B(t_0,T_\alpha)\mathbb{E}_{t_0}[\tilde{k}_i\mathbb{I}_{x\geq x_J^*} - B(t_0,T_\alpha,T_i)e^{-\varsigma_{\alpha i}x-\frac{1}{2}\varsigma_{\alpha i}^2}\mathbb{I}_{x\geq x_J^*}] \\
&= B(t_0,T_\alpha)\tilde{k}_i(1-N(x_J^*)) - B(t_0,T_\alpha)B(t_0,T_\alpha,T_i)\mathbb{E}_{t_0}[e^{-\varsigma_{\alpha i}x-\frac{1}{2}\varsigma_{\alpha i}^2}\mathbb{I}_{x\geq x_J^*}] \\
&= B(t_0,T_\alpha)\tilde{k}_i(1-N(x_J^*)) - B(t_0,T_i)\int_{x_J^*}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\cdot e^{-\varsigma_{\alpha i}x-\frac{1}{2}\varsigma_{\alpha i}^2}dx \\
&= B(t_0,T_\alpha)\tilde{k}_i(1-N(x_J^*)) - B(t_0,T_i)\int_{x_J^*}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x+\varsigma_{\alpha i})^2}dx \\
&= B(t_0,T_\alpha)\tilde{k}_i(1-N(x_J^*)) - B(t_0,T_i)\int_{x_J^*+\varsigma_{\alpha i}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy \\
&= B(t_0,T_\alpha)\tilde{k}_i(1-N(x_J^*)) - B(t_0,T_i)(1-N(x_J^*+\varsigma_{\alpha i}))
\end{aligned}
\tag{E.1}
$$

where we performed a change of variable according to which $y = x + \varsigma_{\alpha i}$.

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