Quasinormal modes of Kerr black holes

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In this review, I describe a simple and highly effective Python code, which calculates the quasinormal modes of Kerr black holes. The code returns both the frequency and the angular separation constant, which are both complex numbers, for every value of the spin parameter α.[a]

I. INTRODUCTION

Black holes that form after the collision of two compact objects are distorted, and the metric that describes them is the sum of a vacuum Kerr solution with a small perturbation. Such black holes vibrate with characteristic oscillation modes and emit gravitational waves, with their frequencies being complex, which justifies the term "quasinormal" (QNM). The real part describes the wave propagation frequency, whereas the imaginary part describes the decay of the emitted gravitational radiation. These frequencies depend solely on the black hole final parameters, which in the case of Kerr black holes are the spin parameter α and total mass M. The gravitational waves of a newly formed black hole are a superposition of QNMs, but since most of them decay very fast, only few of them will leave their mark on the detected signal. Nonetheless, it is actually possible to determine the characteristics of the black hole that produced a specific wave, by analyzing the detected signal. This is the case with the first gravitational wave detection GW 150914 by LIGO on September 14, 2015, where the two colliding black holes eventually formed a new black hole with a spin of $\alpha = 0.68$ and mass of $M = 62.3$.

In the final stage of creation of a black hole, also known as ringdown, one would need a high order perturbation theory in order to describe the phenomena that take place at this stage. However, the linear perturbation theory is the most simple case we can use to describe the basic properties of the final black hole, with its consistency being satisfying. The most elegant way of calculating the quasinormal modes and the angular separation constants of Kerr black holes, is the one proposed by Leaver [1]. These two quantities are the simultaneous roots of two continuous fractions and can be found numerically with high precision.

While there are a few codes available ([5], [7]) that calculate the quasinormal modes and angular separation constants of the Kerr black holes, a new, simple and high precision Python code, along with its results, are presented in this review. Section 2 includes the necessary mathematical background that is needed in order to understand the code flow, and section 3 is devoted in the explanation of its details. In section 4, I present the results of the code and compare them to Berti's. Last but not least, I provide graphs and comments about specific quasinormal modes and the fingerprint that they leave on the gravitational waves observed with the latest detectors.

[a] This code is the main product of my Master's thesis, which is going to be published by Spring 2025, and is available on my GitHub profile upon request.

II. ANALYSIS OF THE KERR SPACETIME PERTURBATIONS

This section provides a brief review of the formalism and calculation method of the quasinormal modes and separation constants that were used to create the code that follows next. For a detailed discussion, one can refer to the original papers [1, 2]. In Boyer-Lindquist coordinates, the metric of the Kerr spacetime can be written as follows:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mr\alpha sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + \alpha^2 + \frac{2Mr\alpha^2 sin^2\theta}{\rho^2}\right)sin^2\theta d\phi^2$$

$$\rho^2 \equiv r^2 + \alpha^2 cos^2\theta \quad \text{and} \quad \Delta \equiv r^2 - 2Mr + \alpha^2$$

Teukolsky was able to prove that the master equation which described the field perturbation, was separable if the field quantities ψ were written as:

$$\psi = \frac{1}{2\pi}\int \sum_{l=|s|}^{\infty}\sum_{m=-l}^{l} S_{lm}(u)R_{lm}(r)e^{im\phi}e^{-i\omega t}d\omega$$

In the above expression, u is defined as $u = cos\theta$ and s corresponds to the spin weight s. Scalar perturbations are described by $s = 0$, electromagnetic ones correspond to $s = -1$ and gravitational perturbations, have a spin of $s = -2$. As for S_{lm}, it denotes the spin-weighted spheroidal harmonics and R_{lm} represent the angular part of the field quantity ψ. Both of these quantities obey a differential equation, with the angular one being:

$$[(1-u^2)S_{lm,u}]_{,u}+\left[\alpha^2\omega^2u^2-2\alpha\omega su+s+A_{lm}-\frac{(m+su)^2}{1-u^2}\right]S_{lm}=0 \tag{1}$$

The equation for the radial part is:

$$\Delta R_{lm,rr}+(s+1)(2r-1)R_{lm,r}+V(r)R_{lm}=0 \tag{2}$$

With $V(r)$ defined as:

$$V(r)=\Big[(r^2+\alpha^2)\omega^2-2\omega r+\alpha^2m^2+is(\alpha m(2r-1)-\omega(r^2-\alpha^2))\Big]\Delta^{-1}+\Big[2is\omega r-\alpha^2\omega^2-A_{lm}\Big]$$

It is worth noting, that the procedure of the presented code, will follow Leaver's convention in which the units used are such that $G=c=2M=1$. This leads to the spin parameter α taking values in the interval $0\leq\alpha<1/2$, with the upper limit corresponding to the extreme Kerr limit. The angular separation constant A_{lm} in the Schwarzschild limit, where $\alpha=0$, is equal to $A_{lm}=l(l+1)-s(s+1)$, and generally depends on the complex frequency ω. In his paper, Leaver used the method of continued fractions in order to calculate the two unknown values, the quasinormal frequency and the angular separation constant. I am not going to dive into the mathematical details of this method, but I provide the two fractions along with the coefficients they include, as they are needed in order to construct the code that will follow in the next section. The angular continued fraction, along with its coefficients will be:

$$k_1=\frac{1}{2}|m-s| \quad \text{and} \quad k_2=\frac{1}{2}|m+s| \tag{3}$$

$$\alpha_n^\theta=-2(n+1)(n+2k_1+1) \tag{4}$$

$$\begin{aligned}\beta_n^\theta&=n(n-1)+2n(k_1+k_2+1-2\alpha\omega)\\&-\Big[2\alpha\omega(2k_1+s+1)-(k_1+k_2)(k_1+k_2+1)\Big]\\&-\Big[\alpha^2\omega^2+s(s+1)+A_{lm}\Big]\end{aligned} \tag{5}$$

$$\gamma_n^\theta=2\alpha\omega(n+k_1+k_2+s) \tag{6}$$

$$0=\beta_0^\theta-\frac{\alpha_0^\theta\gamma_1^\theta}{\beta_1^\theta-}\frac{\alpha_1^\theta\gamma_2^\theta}{\beta_2^\theta-}\frac{\alpha_2^\theta\gamma_3^\theta}{\beta_3^\theta-}... \quad \text{for } n=0 \tag{7}$$

$$\beta_n^\theta-\frac{\alpha_{n-1}^\theta\gamma_n^\theta}{\beta_{n-1}^\theta-}\frac{\alpha_{n-2}^\theta\gamma_{n-1}^\theta}{\beta_{n-2}^\theta-}...\frac{\alpha_0^\theta\gamma_1^\theta}{\beta_0^\theta}=\frac{\alpha_n^\theta\gamma_{n+1}^\theta}{\beta_{n+1}^\theta-}\frac{\alpha_{n+1}^\theta\gamma_{n+2}^\theta}{\beta_{n+2}^\theta-} \quad \text{for } n\geq 1 \tag{8}$$

The fractions given above, can return the value of the angular separation constant A_{lm} if the frequency ω and the parameters m, α and s are known. In other words, the angular separation constant will be a root of the continued fraction. In fact, the continued fraction will have n roots, but one of them will be the most stable. Mathematically, every root of equation (7) is a root of the continued fraction of equation (8) and vice versa, so one could technically use the most simple fraction of the two to find all the roots. In order to calculate numerically the most stable root, one should use the $n-th$ inversion of the continued fraction, which is given by equation (8). The same applies for the corresponding radial continued fraction and the quasinormal frequency ω.

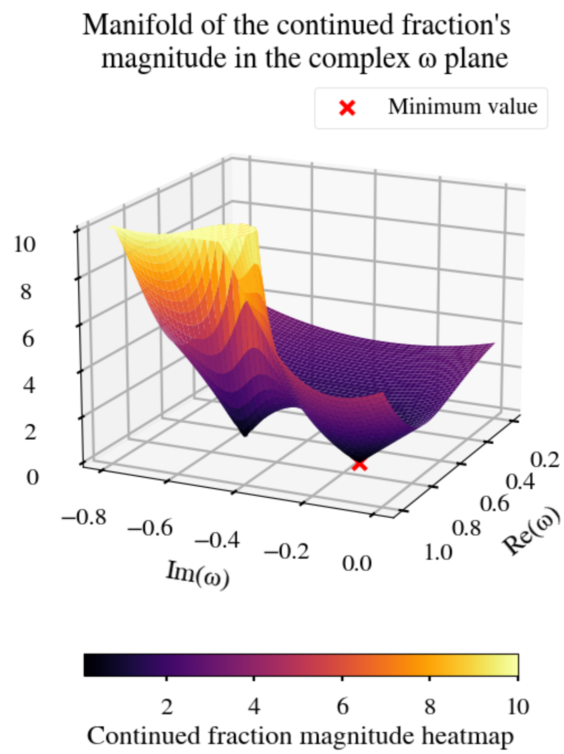

FIG. 1. The manifold of the magnitude of the radial continued fraction, given in equation (18), in the complex ω plane for the case of l=2 of the Schwarzschild black hole. For this case, the separation constant is equal to $A_{lm}=4$.

Visually, it is easier to understand the idea of the stable mode, if we plot the manifold of the continued fraction's magnitude in the complex ω plane. In particular, by using a part of the code discussed in this review, which can calculate the quasinormal modes of the Schwarzschild black hole using Leaver's method, Figure 1 can be easily created. The main idea remains the same for Kerr black holes, but the simple case when $\alpha=0$, which is discussed below, is an effective way of understanding the distinction between stable and unstable roots. The graph presented above, depicts the first two quasinormal modes for the case $l=2$, the fundamental one and the first overtone. They appear as local minimum values, but they have a significant difference. Since in this graph, the continued fraction used, is the one appearing in equation (18), then automatically, the fundamental mode $\omega = 0.747343 - 0.177925i$ is set as the stable one, and the first overtone corresponding to $\omega = 0.693422 - 0.547830i$ as the unstable. This can be seen, as around the fundamental mode, the manifold appears smoother than around the first overtone. This leads to the unstable root of the continued fraction, having a sharp, pointy local minimum, a smaller surface area around it and a magnitude value that is close to zero, but not as close as the magnitude of the stable root. In other words, the stable root, will always correspond to the lowest value of the manifold, which is denoted with a red cross in Figure 1. The numerical method is thus less likely to "find" the first overtone, than the fundamental mode. This applies for every unstable root, so one should use the $n-th$ inversion of the continued fraction to calculate the mode corresponding to index n. It is worth nothing that the manifold of the fraction's magnitude will be reshaped according to which inversion is used each time, but the local minimums will remain in the same place, either as stable or unstable roots.

The radial continued fraction has a similar form, but contains more complicated coefficients than the angular one. All of them are provided below:

$$b = \sqrt{1 - 4\alpha^2} \tag{9}$$

$$\alpha_n^r = n^2 + (c_0 + 1)n + c_0 \tag{10}$$

$$\beta_n^r - 2n^2 + (c_1 + 2)n + c_3 \tag{11}$$

$$\gamma_n^r = n^2 + (c_2 - 3)n + c_4 - c_2 + 2 \tag{12}$$

$$c_0 = 1 - s - i\omega - \frac{2i}{b}\left(\frac{\omega}{2} - \alpha m\right) \tag{13}$$

$$c_1 = -4 + 2i\omega\,(2 + b) + \frac{4i}{b}\left(\frac{\omega}{2} - \alpha m\right) \tag{14}$$

$$c_2 = s + 3 - 3i\omega - \frac{2i}{b}\left(\frac{\omega}{2} - \alpha m\right) \tag{15}$$

$$\begin{aligned} c_3 = \omega^2 \left(4 + 2b - \alpha^2\right) - 2\alpha m\omega - s - 1 + (2 + b)\, i\omega \\ - A_{lm} + \frac{4\omega + 2i}{b}\left(\frac{\omega}{2} - \alpha m\right) \end{aligned} \tag{16}$$

$$\begin{aligned} c_4 = s + 1 - 2\omega^2 - (2s + 3)\, i\omega \\ - \frac{4\omega + 2i}{b}\left(\frac{\omega}{2} - \alpha m\right) \end{aligned} \tag{17}$$

$$0 = \beta_0^r - \frac{\alpha_0^r\gamma_1^r}{\beta_1^r -}\frac{\alpha_1^r\gamma_2^r}{\beta_2^r -}\frac{\alpha_2^r\gamma_3^r}{\beta_3^r -}\ldots \quad \text{for } n = 0 \tag{18}$$

$$\begin{aligned} \beta_n^r - \frac{\alpha_{n-1}^r\gamma_n^r}{\beta_{n-1}^r -}\frac{\alpha_{n-2}^r\gamma_{n-1}^r}{\beta_{n-2}^r -}\ldots\frac{\alpha_0^r\gamma_1^r}{\beta_0^r} \\ = \frac{\alpha_n^r\gamma_{n+1}^r}{\beta_{n+1}^r -}\frac{\alpha_{n+1}^r\gamma_{n+2}^r}{\beta_{n+2}^r -} \quad \text{for } n \geq 1 \end{aligned} \tag{19}$$

Just like the angular continued fraction, the one above will return the most stable root or quasinormal frequency ω, if the angular separation constant A_{lm} and the needed parameters are known. Therefore, the two continuous fractions should be solved simultaneously.

The method described above can generally return a sufficient number of about 60 quasinormal mode frequencies for the Schwarzschild black hole if one uses it for this case, but an way smaller number of modes for the Kerr black hole. The reason behind this, is the fact that the convergence of the continued fractions becomes worse as the imaginary part of the frequency increases, making it impossible to determine the asymptotic behavior of these modes. To address this issue, Nollert ([8]) provided an improvement of this technique and managed to approach the high damping limit. He suggested that the infinite continued fraction appearing on the right side of equations (7) and (18) or (8) and (19), can be calculated with the use of the "rest" of the continued fraction denoted as R_N, starting from a large truncation index N. This fraction is given by the following relation:

$$R_N \equiv \frac{\gamma_{N+1}}{\beta_{N+1} - \alpha_{N+1}R_{N+1}} \tag{20}$$

With R_N being defined as:

$$R_N \equiv \sum_{k=0}^{\infty} C_k N^{-k/2} \tag{21}$$

It should be noted that the definition given in equation (21), is valid for the large, fixed truncation index N and is used only as the starting value. After the first step, which starts from that index N, equation (20) can be summed from bottom to top resulting in the overtone index n. As for the coefficients C_k, which appear in the sum of equation (21), one should use as many as possible to determine the value of R_N for a very large value of the index N. For the calculations presented below, the first three terms are more than enough, with ρ being defined as $\rho = -i\omega$:

$$C_0 = -1, \quad C_1 = \pm\sqrt{2\rho b}, \quad C_2 = \frac{3}{4} - \rho(b + 1) - s \tag{22}$$

III. CODE ANALYSIS

Now that the mathematical background needed for the calculation of the QNMs and separation constants of the

Kerr black hole has been provided, the main idea of the code flow can be explained in detail. The main function of the code needs 12 parameters to be fixed, before it starts calculating the quantities mentioned above. First of all, the interval of the spin parameter α needs to be defined by adjusting its starting and ending values, as well as the step which will be used to increase the value of α in every iteration. It is important to note that since the code uses Leaver's method, the maximum value given for the spin parameter must be $\alpha < 1/2$. The results can be printed or plotted later on using the standard notation $\alpha = 1$ as the maximum spin parameter, if the necessary changes for the QNM frequency are made. Secondly, the user should choose the values for the overtone number n, the harmonic indices (l, m) and the spin weight s, which determine the type of perturbation that is examined. Next, follows the maximum number of terms that is kept in the continuous fractions. In principle, one should keep as many terms as possible to increase accuracy, but this may be costly in terms of performance and speed. In my calculations, I used formally up to 1000 terms for the continuous fractions, with the results being satisfying in most cases. Lastly, the initial values for the real and imaginary parts of the QNM and the separation constant are given, as they will be used later on. The idea behind the initialization of these two quantities, is that the code uses a pair of (ω, A_{lm}) for a certain value of α to calculate the corresponding pair for the next value of α. If the starting value of the spin parameter is equal to zero, then the code will use as initial values for the pair (ω, A_{lm}), the initial value given for the QNM frequency and the value $A_{lm} = l(l+1) - s(s+1)$ for the separation constant, which corresponds to the Schwarzschild limit. Otherwise, if α is positive, it will use the pair of initial values given, so the user should be careful to insert the correct numbers in this area.

The code makes use of three lists in order to save every pair (ω, A_{lm}) that is calculated for every value of α, as they will give the user a complete evolution of the quantities examined for a specific type of perturbation. After the creation of these lists, the main loop of the function starts working. It takes a specific value of the spin parameter α, along with the rest perturbation parameters l, m and s, and defines the functions that return the coefficients provided in equations (3)-(6) and (9)-(17). After that, the angular continued fraction coefficients are calculated with the help of the functions that were defined previously. It should be noted, that the overtone index n, will be used to determine which of the relations (7) and (8) will be used as, like it was stated earlier, the $n-th$ inversion of the continued fraction will make the root corresponding to the overtone index n, the most stable one in terms of numerical calculation. The continued fraction will make use of the maximum number of terms that the user inserted at the start of the code, with this number playing the role of the truncation index N which was mentioned in the previous section. Since this code makes use of Nollert's improvement, the initial value of the infinite continued fraction will be given by equation (21) and the coefficients provided in equation (22). Once this step is done, the only unknown values will be the pair (ω, A_{lm}). The procedure for the radial continued fraction is the same, as the code makes use of the coefficients needed, which results again in a fraction with two unknown parameters, the QNM frequency and the angular separation constant. The only thing remaining up to this point, is to find a solution that satisfies both fractions.

In the last part of the code, a function which takes as equations the two continued fractions and returns the real and imaginary parts of the pair (ω, A_{lm}) is defined. This function will use as variables, the values of the real and imaginary parts of the pair (ω, A_{lm}). These values though, will come from an initial guess for the QNM frequency and angular separation constant, which will be the values of the Schwarzschild limit if $\alpha = 0$, or the ones from the previous iteration if α is positive. The solution of these two continued fractions, can be found using the root option from the SciPy library in Python, along with the hybrid method, denoted as "hybr", which solves the two equations simultaneously. It uses the initial guess values as the starting point in the two equations and tries to find values that will make them almost equal to zero. The tolerance will determine how many iterations will be made until the final set of values for the pair (ω, A_{lm}) is returned. Theoretically, the continued fractions will only be equal to zero if one keeps an infinite number of terms, but since only a certain number can be used, the code will search for values that will approach zero as close as the tolerance level allows it. Finally, the code uses the output values as the starting point for the next value of the spin parameter α which is increased by the step defined from the user at the start, while the results are also stored in the three lists constructed outside the loop. In this way, one can acquire the set of values for the QNM frequency ω and the separation constant A_{lm}, for each value of the parameter α in a certain interval. After the execution of the code, the user can access the three lists and plot the desired graphs. In fact, four of the most common graphs are presented in Figure 2, which depicts the fundamental modes with $l = 2$ and $l = 3$ for every value of the harmonic index m.

IV. RESULTS

In this section, I compare the results of my code, to the ones provided by Berti in his personal webpage for some specific modes of gravitational perturbations. The comparison will be done by finding the relative error between the two codes, in both the real and the imaginary part of the pair (ω, A_{lm}) for various values of the spin parameter α:

$$\text{Relative error} = \frac{\Delta\omega}{\omega_{Berti}} 100\% = \frac{\omega_{Berti} - \omega_{Karikos}}{\omega_{Berti}} 100\%$$

The comparison presented in this section will consider results up to the 10th decimal digit. From the provided results, only the modes with $n = 2$ and $\alpha = 0.998$, denoted with an asterisk (*), needed more terms in the continued fractions in order to be calculated precisely. In particular, these specific values were calculated by keeping 10.000 terms in each fraction. The relative error is expressed in order of $10^{-9} = 10^{-7}\%$ and values that are shown as zeros, correspond to errors smaller than $10^{-12} = 10^{-10}\%$, or total agreement between my values and the ones provided by Berti. The two tables that follow, provide an indicative set of errors, corresponding to the mode with $l = m = 2$ of a Kerr black hole:

α	$n = 0$	$n = 1$	$n = 2$
0	0, 0	0, 0	0, 0
0.4	0, 0.46	0, 0	0.26, 0.12
0.8	0, -0.13	0, 0	0, -0.02
0.9	0, -0.15	0, -0.51	0, -0.17
0.98	0, -1.29	0, 0	-0.12, 0.17
0.998	1.37, -20.0	0, -0.69	0.02, -0.08 *

TABLE I. Relative error of QNM frequencies for gravitational perturbations with $l = m = 2$, for the fundamental mode and the first two overtones.

α	$n = 0$	$n = 1$	$n = 2$
0	0, 0	0, 0	0, 0
0.4	0.09, -1.96	0, 0	0.17, 0
0.8	0, -3.90	0.04, -0.16	0.29, 0.19
0.9	-0.05, -2.84	0.14, 1.58	0.14, 0.28
0.98	-0.15 1.33	0.30, 0.22	0.07, -0.53
0.998	-5.91, -27.4	0.49, -3.85	-0.17, -3.04 *

TABLE II. Relative error of the separation constant for gravitational perturbations with $l = m = 2$, for the fundamental mode and the first two overtones.

In a similar way, the respective errors for the mode with $l = m = 3$ will be:

α	$n = 0$	$n = 1$	$n = 2$
0	0, 0	0, 0	0, 0
0.4	0, 0.03	0, 0.08	0, -0.04
0.8	0, 0.04	0, 0.05	0, -0.02
0.9	0.19, 0.17	-0.19, -0.01	-0.39, 0.03
0.98	-0.23, -0.31	0, 0.03	-0.16, 0.03
0.998	0.28, 1.27	-0.14, -0.34	-0.04, 0.25 *

TABLE III. Relative error of QNM frequencies for gravitational perturbations with $l = m = 3$, for the fundamental mode and the first two overtones.

α	$n = 0$	$n = 1$	$n = 2$
0	0, 0	0, 0	0, 0
0.4	0.01, -1.57	0.05, 0.40	-0.02, -1.43
0.8	0.02, -1.73	-0.05, 0.58	0.05, -0.80
0.9	0, -2.75	-0.01, 1.10	0.14, -0.44
0.98	-0.21, 6.82	0.13, 2.78	0.18, -1.67
0.998	-0.28, 9.79	-0.18, 6.77	-0.09, 5.84 *

TABLE IV. Relative error of the separation constant for gravitational perturbations with $l = m = 3$, for the fundamental mode and the first two overtones.

The relative errors presented above, show an excellent agreement between mine and Berti's codes, as the maximum error is of order $10^{-6}\%$ in only two cases. One may use this code to find the desired quantities for other values of the harmonic indices (l, m), overtone index n or spin weight s, but since my thesis is based on gravitational perturbations, I limited my calculations to the case of $s = -2$. It should be noted, that there are a few ways to get even more accurate results, but this comes to a performance cost. First of all, increasing the number of terms used in the continued fractions could provide slightly higher accuracy of the QNM ω and the separation constant A_{lm}. More precisely, increasing the continued fraction terms from 1000, to 10.000, can generally grant better accuracy usually after the 10th decimal. Perhaps the most effective way of increasing the accuracy of the calculated quantities, is by making the step to which the spin parameter α increases, as small as possible. In my calculations, I used a step of $\delta\alpha = 0.0005$ and got the relative errors presented in the tables above. Making this step even smaller, will grant more accurate results, as the code uses the previously found set of values for the pair (ω, A_{lm}) as an initial guess, in order to calculate the respective pair for the next value of α. This method will surely increase the code execution time, but the impact on the results will be significant. Lastly, making the tolerance of the hybrid root finding method smaller, is also an efficient way of getting more accurate results, but once again the execution time will be a bit longer.

One important comment that should be made about the code presented above, is that all coefficients used, along with the values of the continued fractions, should be as precise as possible. A way to do this is by increasing the number of bits needed to represent a single value in memory. The presented code uses 256 bits for each of these quantities, denoted as numpy.complex256 from the NumPy library in Python, which is the equivalent of long double precision in C language. An equivalent way of increasing the code's accuracy, is to use the Mpmath library to adjust the desired precision, but the difference between the two methods is in most cases unnoticeable. For the same type of gravitational perturbations, I also provide four essential graphs, that explain the behavior of these modes:

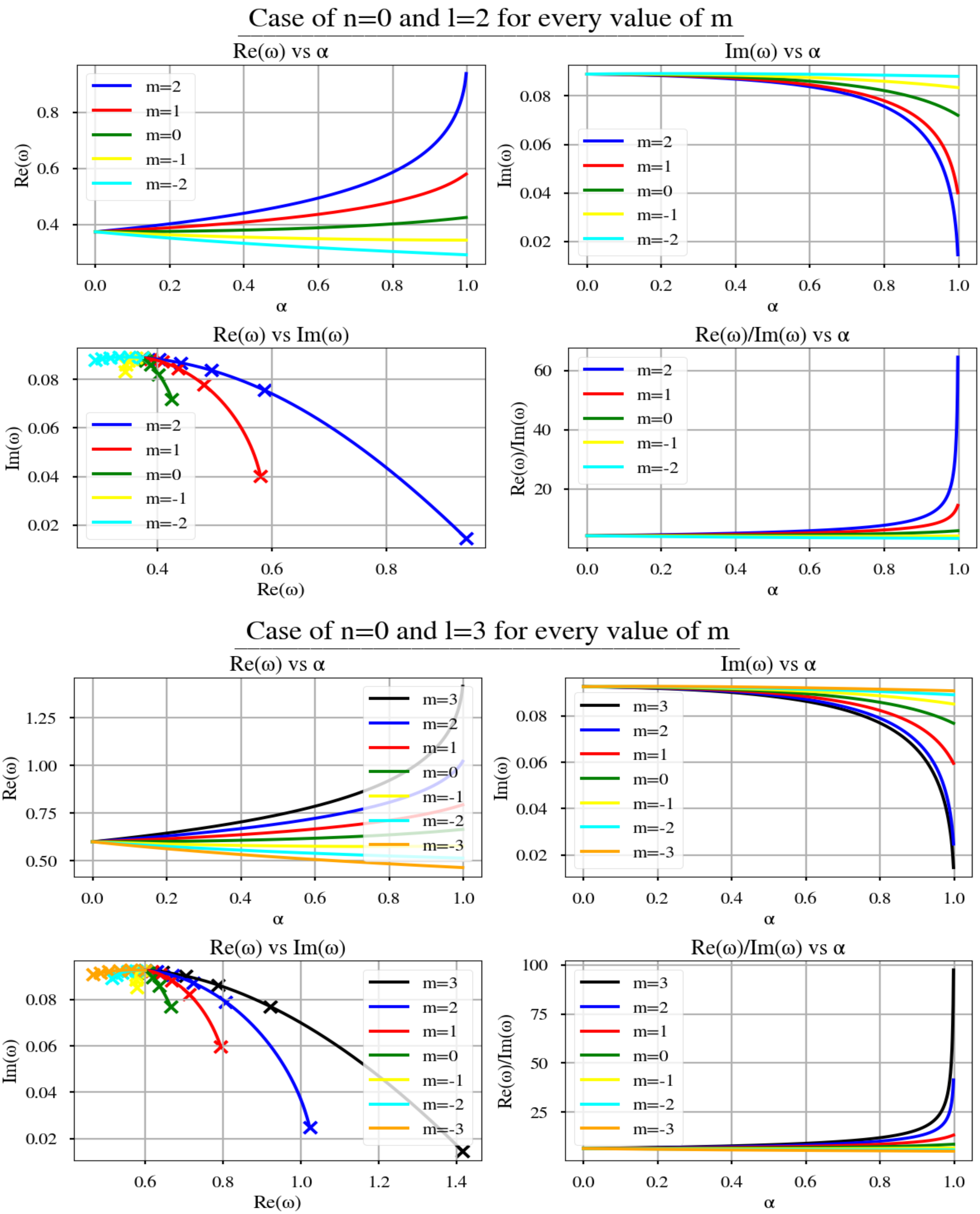

FIG. 2. Four essential graphs that represent gravitational perturbations of spin weight $s = -2$, for the fundamental modes $n = 0$ of $l = 2$ and $l = 3$, for every value of the harmonic index m. For each case, the top two graphs show the real and imaginary part of the QNM frequency as a function of the parameter α. The lower left graph represents the QNM frequency complex plane, in which the crosses are used to symbolize six specific values of α, namely the set $(0, 0.2, 0.4, 0.6, 0.8, 1)$. Lastly, the lower right graph shows the fraction of the real and imaginary parts of the QNM frequency as a function of the parameter α.

The graphs presented above can be easily extracted from the presented code, if one firstly saves the three lists for the spin parameter α, the quasinormal mode frequency ω and the angular separation constant A_{lm} for every possible value of the harmonic index m. Then, it is easy to plot the real and imaginary parts of the QNM, as well as the fraction of these two quantities, as a function of α, or the QNM frequency complex plane. Of course the same procedure can be repeated for the angular separation constant of each case. In both cases of $l = 2$ and $l = 3$, the modes with $l = m$ are the least damped ones, as their imaginary part takes smaller values compared to cases where $l \neq m$. Gravitational waves that were emitted during the ringdown phase, after the collision of two black holes, will contain a superposition of modes. Most of them, will vanish in a short amount of time due to their high damping. For this reason, gravitational wave astronomers often use the least damped mode with $l = m = 2$, in order to extract information from the detected signals by LIGO, VIRGO and other detectors. This is exactly the case with the first ever detected gravitational wave from LIGO on September 14, 2015 ([3],[4]), where the least damped mode was fitted onto the signal data, only to show a superb agreement between theory and observation. More specifically, two black holes with masses of $M = 36.2M_{\odot}$ and $M = 29.1M_{\odot}$ collided and formed a new one, with a mass of $M = 62.3M_{\odot}$, $\alpha = 0.68$ and radiated energy of $M = 3M_{\odot}$. Since the next least damped mode is the one with $l = m = 3$, one could try and fit the superposition of these two modes in the data from the detected signal to try to improve the errors that occur when using only the $l = m = 2$ mode. One could say that these two modes, are basically the fingerprint of the ringdown phase, as they are the most probable ones to be detected.

Lastly, it should be mentioned that the graphs presented above, all correspond to units in which the black hole mass is equal to $M = 1M_{\odot}$ and where the spin parameter α takes values between 0 and 1. This is different from Leaver's convention, but since it is more common to use the units presented in the graphs, I made that shift in the last few lines of the code. In reality, one does not calculate the real and imaginary parts of the quasinormal frequency, but the quantities $MRe(\omega)$ and $MIm(\omega)$ which reduces to the real and imaginary parts when $M = 1M_{\odot}$. If the black hole mass is not equal to $M = 1M_{\odot}$, then one should use the detected frequency components. By evaluating the fraction between the real and imaginary parts of the detected frequency, it is possible to determine the parameter α of the black hole, by knowing the propagation to damping ratio, as this is shown in the lower right graphs of Figure 2. In this graph, the mass does not play any role, as it vanishes both from the numerator and the denominator of the fraction. Then, since the detected frequency components and the parameter α are known, the calculation of the black hole mass is straightforward from the upper two graphs.

V. CONCLUSIONS

In this review, I have mentioned the basic theoretical background needed, in order to construct a code which calculates the quasinormal modes and angular separation constants of a Kerr black hole, for different types of perturbations. The results and graphs were limited to the case of gravitational perturbations, which correspond to a spin weight of $s = -2$, as this is the main topic of my Master's thesis. The presented Python code, can be found on my GitHub page, but the main logic behind it, is well explained in this review. From the numerical results, an excellent agreement with the values given by Berti is observed, with a typical error of order lower than $10^{-6}\%$ in most cases for the fundamental and first two overtones of the modes with $l = m = 2$ and $l = m = 3$. Lastly, four essential graphs of the QNMs corresponding to the two least damped modes were provided, as they play a vital role in modern day gravitational wave astronomy. In the coming weeks, I will attempt to approach the high damping limit of the quasinormal modes of Kerr black holes, analytically, by investigating the asymptotic behavior of modes with $l \neq m$ and $m > 0$, or $m < 0$. Another target is to use this code to perform a numerical analysis of the angular equation and determine the asymptotic behavior of the separation constant, which might give some insights on the radial equation.

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