Analysis and Visualization of Hierarchical Graphs

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Thesis submitted in partial fulfillment of the requirements for the

Masters' of Science degree in Computer Science and Engineering

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Abstract

In this work, we explore cutting-edge path and channel decomposition algorithms and applications. Our algorithms are linear or almost linear, and our results are very close to the optimum. Additionally, we developed a general-purpose hierarchical graph drawing framework based on path/channel decomposition that helped us reveal critical aspects of graph hierarchies.

More precisely, we will show how to create a sub-optimal channel decomposition of a DAG (directed acyclic graph) in almost linear time. The number of vertex-disjoint channels our algorithm creates is very close to the minimum. The time complexity of our algorithm is $O(|E| + c * l)$, where c is the number of path concatenations and l is the longest path of the graph. We will give a detailed explanation in the following sections. This fundamental concept has a wide area of applications. We will focus on a few of them. We will extensively describe how to solve the transitive closure of graphs and answer queries in constant time by creating a known indexing scheme. Our method needs $O(k_c * |E_{red}|)$ time and $O(k_c * |V|)$ space. The factor k_c is a sub-optimal number of channels, E_{red} is the set of non-transitive edges, and $|V|$ is the number of nodes. Moreover, we show that E_{red} is bounded, $E_{red} <= width * |V|$, and we illustrate how to find a subset of E_{tr} (the set of transitive edges) without calculating transitive closure. We accompany our approach and algorithms with extensive experimental work. Our experiments reveal that our methods are not merely theoretically efficient since the performance is even better in practice.

Furthermore, we developed the Path-Based Framework (PBF). PBF is a recent graph drawing framework that resembles but also differs from the classical Sugiyama technique. PBF is based on the concepts of path and channel decomposition. We extended that idea. We draw all edges, apply edge bundling, minimize the height using a compaction technique, and reduce the width by applying algorithms similar to task scheduling. As a result, we present a generic framework suitable for hierarchical graph drawings. We conducted a user study to evaluate the performance and investigate its usability and readability. We put tasks on graph drawings calculated by our framework, and we compare the users' performance against the graph drawing results as computed by OGDF, which follows the Sugiyama technique. Our findings reveal a clear advantage in using the generic PBF over OGDF based on a set of tasks.

Keywords: Algorithms, graph algorithms, performance, channel decomposition, path decomposition, transitive closure, transitive reduction, hierarchy, query processing, DAG, data structures, network analysis.

Τίτλος

Περίληψη

Σε αυτό το έργο εξερευνήσαμε αλγορίθμους αιχμής για διάσπαση γράφων σε μονοπάτια και κανάλια. Οι αλγόριθμοι μας είναι γραμμικοί ή σχεδόν γραμμικοί, και τα αποτελέσματα τους είναι πολύ κοντά στο βέλτιστο. Επιπρόσθετα, αναπτύξαμε ένα πλαίσιο οπτικοποίησης ιεραρχικών γραφημάτων που βασίζεται στην διάσπαση σε μονοπάτια και κανάλια και μας βοηθάει να αποκαλύψουμε κρίσιμες πτυχές των ιεραρχιών ενός γράφου.

Ακριβέστερα, θα δείξουμε πώς να δημιουργήσουμε μια υποβέλτιστη διάσπαση σε κανάλια ενός άκυκλου κατευθυνόμενου γραφήματος σε σχεδόν γραμμικό χρόνο. Ο αριθμός των καναλιών που δημιουργεί ο αλγόριθμος μας, τα οποία δεν μοιράζονται κοινούς κόμβους, είναι πολύ κοντά στο ελάχιστο. Η χρονική πολυπλοκότητα του αλγορίθμου μας είναι $O(|E| + c * l)$, όπου c είναι ο αριθμός των καναλιών και l ο αριθμός της μεγαλύτερης διαδρομής του γράφου. Θα δώσουμε αναλυτική εξήγηση στα επόμενα κεφάλαια. Αυτή η θεμελιώδης έννοια έχει ένα ευρύ φάσμα εφαρμογών. Θα επικεντρωθούμε σε μερικές από αυτές. Θα περιγράψουμε εκτενώς πώς να λύσουμε το πρόβλημα της μεταβατικής κλειστότητας και πώς να απαντάμε ερωτήματα σε σταθερό χρόνο δημιουργώντας ένα γνωστό σχήμα από δείκτες. Η μέθοδος μας χρειάζεται $O(k_c * |E_{red}|)$ χρόνο και $O(k_c * |V|)$ χώρο. Ο όρος k_c είναι το μέγεθος μιας υποβέλτιστης διάσπασης καναλιών, ο όρος E_{red} είναι το σύνολο των μη-μεταβατικών ακμών του γράφου, και ο όρος $|V|$ υποδηλώνει τον αριθμό των κόμβων. Επιπλέον θα δείξουμε πως το $|E_{red}|$ φράζεται, $|E_{red}| \leq width * |V|$, και θα περιγράψουμε πως μπορούμε να βρούμε ένα υποσύνολο του E_{tr} (σύνολο μεταβατικών ακμών) χωρίς να υπολογίσουμε τη μεταβατική κλειστότητα. Οι μεθοδολογίες μας συνοδεύονται από εκτενής πειράματα. Τα πειράματα μας δείχνουν ότι οι αλγόριθμοι μας δεν είναι απλώς αποδοτικοί στη θεωρία. Στη πράξη η απόδοση είναι ακόμα ποιό μεγάλη.

Ακόμη, έχουμε αναπτύξει το $Path-based-Framework$ (PBF). Το PBF είναι ένα πρόσφατο πλαίσιο οπτικοποίησης ιεραρχικών γραφημάτων που μοιάζει αλλά επίσης διαφέρει από το κλασικό πλαίσιο τεσσάρων φάσεων του $Sugiyama$. Το PBF βασίζεται στη ιδέα της διάσπασης του γράφου σε κανάλια και μονοπάτια. Επεκτείναμε αυτή την ιδέα. Ζωγραφίζουμε όλες τις ακμές, εφαρμόζουμε επικάλυψη ακμών, ελαχιστοποιούμε το ύψος, και μειώνουμε το πλάτος του γραφήματος εφαρμόζοντας τεχνικές όμοιες με αυτές του χρονο-προγραμματισμού εργασιών. Ως εκ τούτου, παρουσιάζουμε ένα γενικό μοντέλο οπτικοποίησης ιεραρχικών γραφημάτων. Τρέξαμε μια έρευνα χρήστη προκειμένου να μετρήσουμε την ευχρηστία και την αναγνωσιμότητα του. Βάλαμε ερωτήσεις σε γράφος που οπτικοποιήθηκαν από το δικό μας πλαίσιο, και συγκρίναμε την επίδοση των χρηστών σε σχέση με την οπτικοποίηση της βιβλιοθήκης $OGDF$ που χρησιμοποιεί το πλαίσιο του $Sygiyama$. Τα αποτελέσματα μας φανέρωσαν ένα ξεκάθαρο πλεονέκτημα του PBF σε σχέση με το $OGDF$ σε αυτά τα ερωτήματα.

Λέξεις κλειδιά: Αλγόριθμοι, αλγόριθμοι γράφων, απόδωση, ιεραρχίες γράφων,

διάσπαση γράφου σε κανάλια, διάσπαση γράφου σε μονοπάτια, συνένωση μονοπατιών, μεταβατική κλειστότητα, συμπιεσμένη μεταβατική κλειστότητα, μεταβατική αφαίρεση, διαχείριση ερωτημάτων, σχήμα δεικτών, ιεραρχικά γραφήματα, πειραματική εργασία, ‘Ακυκλοι γράφοι, δομές δεδομένων, ανάλυση δικτύων.

Ευχαριστίες

Θα ήθελα να ευχαριστήσω θερμά τον κ. Τόλλη Ιωάννη, ο οποίος ήταν ο επόπτης μου κατα τη διάρκεια των μεταπτυχιακών μου σπουδών. Ήταν το άτομο που μοιράζομουν τις σκέψεις μου και τα ερευνιτικά μου ενδιαφέροντα. Επίσης θα ήθελα να ευχαριστήσω κ. Παναγιώτη Λιονάκη για την εξαίρετη συνεργασία μας. Τέλος, νιώθω την ανάγκη να ευχαριστήσω την οικογένειά μου για την αγάπη, κατανόηση, και υποστήριξή τους.

στους γονείς μου

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 On Graph Hierarchies

The arrival of new technologies, advanced sensors, and the increasing tendency of people to interact and use them, passively or actively, has led us to manage, analyze, and interpret an enormous amount of data. To achieve that, we develop more efficient and faster tools and methods. Graph theory is a critical mathematical modeling method employed in several applications of technology. In this work, we explore graph hierarchies.

Hierarchical and often directed graphs are the de facto representation for many applications in various domains including research and business. Such graphs often represent hierarchical relationships between objects in a structure or in a more complex network such as in PERT applications. The analysis and visualization of these directed (often acyclic) graphs has received significant attention recently.

We developed a general-purpose hierarchical graph drawing framework that departs from the classical four-phase framework of Sugiyama and produces readable drawings. We call it Path-Based Framework since it is based on Path Channel Decomposition. In addition to [56], we draw all edges, apply edge bundling, minimize the height using a compaction technique, and reduce the width by applying algorithms similar to task scheduling.

Furthermore, in this work, we developed a cut-edge channel decomposition technique. Channel decomposition has a wide area of applications as in distributed computing [40, 68], in bioinformatics [10, 36], in graph visualization [56], it can facilitate answering reachability queries [41, 64, 45], and many more. We focus on answering reachability queries. We bound the transitive edges and propose linear time preprocessing steps that facilitate every transitive closure algorithm. The experiments show the efficiency of our proposals. Answering efficiently reachability queries is an important research topic mostly driven by various arising real-world applications, such as XML databases, GIS, web mining, social network analysis, ontologies, and bioinformatics.

Definitions and Abbreviations

- **Path/Chain:** In a path every vertex is connected by a direct edge to its successor, while in a chain any vertex is connected to its successor by a directed path. The vertices of a path/chain are in ascending topological order.
- **DAG:** Directed acyclic graph.
- **Paths/Chains decomposition of a DAG:** Let G = (V,E) be a DAG. A path/chain decomposition of G is a set of vertex-disjoint paths/chains. The decomposition includes all vertices of G. There is an example of a path and a chain decomposition in figure 2.1.
  - k_p: We use this abbreviation to refer to the number of paths of a path decomposition of a graph.
  - k_c: We use this abbreviation to refer to the number of chains of a chain decomposition of a graph.
- **Width:** The maximal number of mutually unreachable vertices of the graph [20].
  - The number of chains in a minimal chain decomposition of a graph is equal to its width.
- **Transitive edge:** A edge (v_1, v_2) of a DAG G is transitive if there is a path longer than one that connects v_1 and v_2.
- **DAG G(V,E):** A DAG G. V represent the set of nodes and E the set of edges.
  - E_{tr} : The set of all transitive edges. $E_{tr} \subset E$.
  - E'_{tr} : A subset of E_{tr}.
  - E_{red} : $E_{red} = E - E_{tr}$, $E_{red} \subseteq E$.
  - $G(V, E_{red})$: The transitive reduction [6] of G(V,E). The transitive reduction is unique for DAGs. It contains the minimum number of edges needed to form the same transitive relation with G(V, E).
- **Sink vertex**: A vertex with no outgoing edges.
- **Source vertex**: A vertex with no incoming edges.

Chapter 2

Path/Channel Decomposition

2.1 Introduction

Searching for efficient ways to decompose the graph into chains, we could not find an efficient solution that scales on large graphs. An efficient chain decomposition has many applications and can facilitate many algorithms and systems. In this work, we develop an almost linear chain decomposition algorithm that produces a set of chains with almost minimum cardinality. We use the notion of chain decomposition to offer bounds to the transitive edges and explore how it facilitates in transitive closure problem.

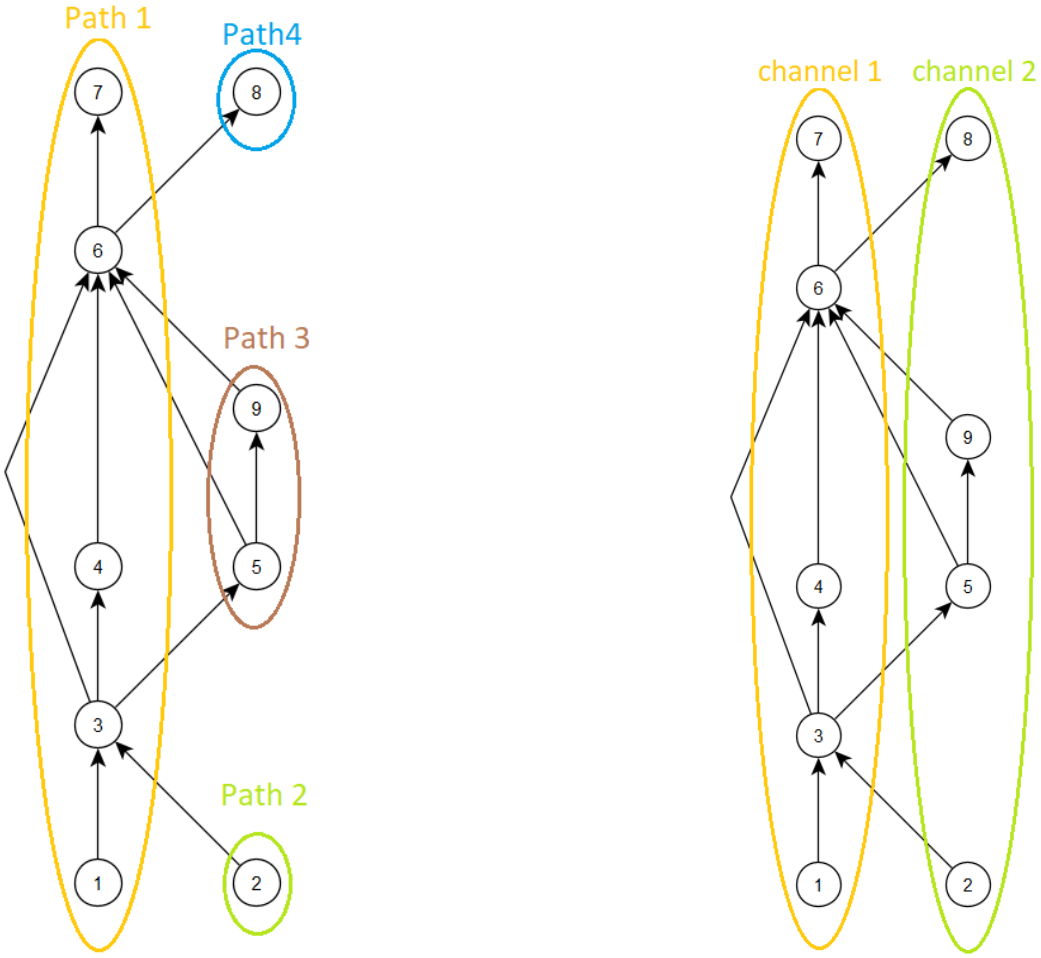
In section 2.2, we present path decomposition approaches, and in section 2.3, chain decomposition and path concatenation. Additionally, we show experiments and evaluate the performance of our heuristic. Next, we examine a few outcomes. In section 2.4, we prove that $E_{red} \leq width * |V|$, and see how we can in linear time, remove a subset of transitive edges and bound $E - E'_{tr}$ by $O(k * V)$ given a path/chain decomposition of size k. Finally, section 2.5 demonstrates how to build a known indexing scheme for transitive closure reporting experimental results.

In our research, we extensively used the Path-Based Framework. A new general-purpose hierarchical graph drawing framework that reveals critical aspects of graph hierarchies, see chapter 3.

We conducted all the experiments using a laptop PC (Intel(R) Core(TM) i5-6200U CPU, 8 GB of main memory).

2.2 Path Decomposition

Jagadish in [42] categorized path decomposition techniques into two categories. Chain Order Heuristics and Node Order Heuristics. The first construct the paths one by one, while the second creates the paths in parallel. He also described the benefits of topological sorting. More precisely, in [1], the author presented chain decomposition heuristics based on Chain Order Heuristic and Node Order Heuristic. He utilized a list of all successors and not only the immediate for



(a) A path decomposition of the graph. It consists of 4 paths.

(b) A chain decomposition of the graph. It consists of 2 chains.

Figure 2.1: On the left, there is a path decomposition of graph G. On the right, a chain decomposition of G.

each vertex. However, his algorithms require $O(n^2)$ time using the precomputed transitive closure. That is inefficient, especially for large graphs, and we will not examine them further. Our heuristic does not need any precomputation and decomposes the graph into a number k_c of chains in $O(|E| + c * l)$ time which in practice is almost linear. Factor c is the number of concatenations, and l is the longest path of the graph. We will describe our technique in detail in the next section.

In this section, we describe the linear time algorithms for path decomposition. We use topological sorting and examine the vertices in ascending order.

Chain Order Heuristic

The chain-order heuristic starts from a vertex and keeps on extending the chain to the extent possible. The chain ends when no more unused immediate successors can be found. As you can see in Algorithm 1, the first for loop finds an unused vertex and creates a chain. The inner while loop extends the chain.

Algorithm 1 Path Decomposition

procedure ChainOrderHeuristic(G, T)
INPUT: A DAG $G = (V, E)$, and a topological sorting $T(v_1, ..., v_i, ..., v_N)$ of G
OUTPUT: A path decomposition of G
  $K \leftarrow \emptyset$ //Set of paths
  Mark all nodes **unused**
  for every **unused** vertex $v_i \in T$ in ascending topological order **do**
    $current \leftarrow v_i$
    $C \leftarrow$ new Chain()
    Add *current* **to** C
    while there is an **unused** immediate successor s of the **current** node **do**
      add s **to** C
      $current \leftarrow s$
    end while
    add C **to** K
  end for
end procedure

Node Order Heuristic

The node-order heuristic examines each node and assigns it to an existing chain. If there is no matching, then a new chain is created for the vertex. Algorithm 2 illustrates the node order heuristic.

Algorithm 2 Path Decomposition

procedure NODEORDERHEURISTIC(G, T)
INPUT: A DAG $G = (V, E)$, and a topological sorting $T(v_1, ..., v_i, ..., v_N)$ of G
OUTPUT: A path decomposition of G
  $K \leftarrow \emptyset$ //Set of paths
  for every vertex $v_i \in T$ **in ascending topological order do**
    if v_i is an immediate successor of the last node of a chain C **then**
      add v_i **to** C
    else
      $C \leftarrow$ new Chain()
      add v_i **to** C
      add C **to** K
    end if
  end for
end procedure

2.3 Chain Decomposition

In this section, we present a path concatenation technique that takes as input a path decomposition and constructs a chain decomposition in $O(|E| + c * l)$ time, where c is the number of path concatenations and l is the longest path of the graph. In order to apply our path concatenation algorithm, we must find a path decomposition of the Graph. We can use an already known linear-time algorithm based on Node-Order Heuristic or Chain Order Heuristic.

2.3.1 Path Concatenation

Our concatenation algorithm can work in combination with every path decomposition algorithm. Given a graph G(V, E) and its path decomposition D_p with k_p paths we build a chain decomposition of k_c chains in $O(|E| + (k_p - k_c) * l)$ time, where l is the longest path of G. Since each concatenation reduces the number of chains by one, factor $(k_p - k_c)$ is the number of path concatenations.

For every path, we start a reverse DFS lookup function from the first vertex of the chain, looking for the last vertex of another chain. DFS lookup function is the well-known depth-first search graph traversal for path finding. If the DFS lookup function detects the last vertex of a chain, then it concatenates the chains. If we do merely that the algorithm will run in $O(k_p * m)$ since we run k_p DFS functions. In our case, every DFS lookup function will take advantage of the previous DFS lookup functions' executions. DFS for path finding returns the path between the source vertex and the target vertex. In our case, the path between the first vertex of a chain and the last vertex of another chain. Hence, every execution goes through a set of vertices V_i that can be split into two vertex disjoint sets, R_i and P_i. In P_i belong the vertices of the path from the source vertex to the destination

vertex. In R_i belong every vertex in $V_i - P_i$. If no path is found then $V_i = R_i$ and $P_i = \emptyset$.

Notice that every vertex in the set R_i is not the last vertex of a chain. If it was then it would belong to P_i and not to R_i. The same way, for every vertex in R_i, all its predecessors are in R_i too. Hence, if a forthcoming reverse_DFS_lookup function meets a vertex of R_i, there is no reason to proceed with its predecessors. All the above are basic DFS theory.

Algorithm 3 Path Concatenation

```
procedure CONCATENATION(G, D)
INPUT: A DAG G = (V, E), and a path decomposition D of G
OUTPUT: A chain decomposition of G
    for each path: p_i ∈ D do
        f_i ← first vertex of p_i
        (R_i, P_i) ← reverse_DFS_lookup(G, f_i)
        if P_i ≠ ∅ then
            l_i ← destination vertex of P_i //Last vertex of a path
            Merge_Paths( l_i, f_i )
        end if
        G ← G \ R_i
    end for
end procedure
```

Algorithm 2 shows our chain concatenation technique. As you see, the DFS lookup function is invoked for every starting vertex of a path. Every reverse DFS lookup function goes through the set R_i and the set P_i, examining the nodes and their incident edges. P_i is the path from the first vertex of a chain to the last vertex of another. The set R_i contains all of the vertices the function went through except the vertices of P_i.

Theorem 2.3.1. *The time complexity of algorithm 3 is* $O(|E| + (k_p - k_c) * l)$.

Proof. Assume that we have k_p paths. We call k_p times the reverse_DFS_lookup function. Hence, we have (R_i, P_i) sets, $0 \leq i < k_p$. In every loop, we delete the vertices of R_i. Hence, $R_i \cap R_j = \emptyset$,$0 \leq i, j < k_p$ and $i \neq j$. R_i and R_j are vertex disjoint sets. We conclude that $\bigcup_{i=0}^{k_p-1} R_i \subseteq N$ and $\sum_{i=0}^{k_p-1} |R_i| \leq |N|$.

Every path on the graph cannot be longer than the longest path. $P_i, 0 \leq i \leq k_p$, is not empty if and only if concatenation has occurred. Hence, $\sum_{i=0}^{k_p-1} |P_i| \leq c * l$ where c is the number of concatenations and l is the longest path of the graph. Since every concatenation reduce the number of chains by one, $c = k_p - k_c$. □

2.3.2 Chain Decomposition Heuristic: A Better Approach

Previously, we described how to produce a chain decomposition applying a post-processing path concatenation step. At this point, we will demonstrate an approach which not only runs in O(m+c*l) time. It also finds a close to optimal chain decomposition.

Algorithm 4 is our Node Order Heuristic variation. It is like the Node Order heuristic but with two additions. The first is that when we visit a vertex with out-degree 1, we add its unique immediate successor to its path. The second is that we do not merely search for the first available immediate predecessor that is the last vertex of a path. Instead of the first available vertex, we choose the vertex with the biggest out-degree. Our aim using this heuristic is to create a chain construction in which more concatenations will occur. Algorithm 4 goes through all vertices. For every vertex, examines all the outgoing (line 8) and all the incoming edges (line 19). Hence, the time complexity is linear.

Algorithm 5 illustrates our path decomposition of Algorithm 4 in combination with chain concatenation. The only addition to algorithm 4 is the if-statement of line 10 and its block. If we do not find an immediate predecessor, we search all predecessors using the reverse_DFS_lookup function. The differentiation of our concatenation is that it does not take part as a post-processing step. It is applied on time when the algorithm does not find an immediate predecessor that is the last vertex of a chain. We do it to avoid transitive edges that could lead to false matches.

2.3.3 Experiments

In this section, we present experiments on graphs created by NetworkX [37]. We used three different random graph generator models. Erdos-Renyi, Barabasi, and Watts-Strogatz model. For every model, we created 12 graphs. Six of 5000 nodes and six graphs of 10000 nodes and average degree 5,10,20,40,80, and 160. We examine the performance of heuristics in terms of the chains' number. We obtain a minimum set of chains by using the Fulkerson's method [21]. Our aim is to reveal the behavior of the width and the behavior of the heuristics in these models.

Fulkerson's method:

1. Construct transitive closure $G^*(V, E')$ of the graph, where $V = \{v_1, ..., v_n\}$.
2. Construct a bipartite graph B with bipartite (V_1, V_2), where $V1 = \{x_1, x_2, ..., x_n\}$, $V2 = \{y_1, y_2, ..., y_n\}$. An edge (x_i, y_j) is formed whenever $(v_i, v_j) \in E'$
3. Find a maximal matching M of B. The width of the graph is $n - |M|$. In order to construct the minimum set of chains, for any two edges $e_1, e_2 \in M$, if $e_1 = (x_i, y_t)$ and $e_2 = (x_t, y_j)$ then connect e_1 to e_2

Random Graph Generators:

Algorithm 4 Path Decomposition

```
1: procedure NODE-ORDER BASED VARIATION(G, T)
   INPUT: A DAG G = (V, E), and a topological sorting T(v_1, ..., v_i, ..., v_N) of
   G
   OUTPUT: A path decomposition of G
2:     K ← ∅ //Set of paths
3:     for every vertex v_i ∈ T in ascending topological order do
4:         Chain C
5:         if u_i is assigned to a chain then
6:             C ← u_i's chain
7:         else if v_i is not assigned to a chain then
8:             l_i ← choose the immediate predecessor with the lowest outdegree
9:                   that is the last vertex of a chain
10:            if l_i ≠ null then
11:                C ← path indicated by l_i
12:                add v_i to C
13:            else
14:                C ← new Chain()
15:                add v_i to C
16:            end if
17:            add C to K
18:        end if
19:        if there is an immediate successor s_i of u_i with in-degree 1 then
20:            add s_i to C
21:        end if
22:    end for
23: end procedure
```

- **Erdős-Rényi model**: The generator returns a binomial graph. The generator's parameters are two, the number of nodes n and a probability p. Every edge in this model has a probability p to be formed.

- **Barabási–Albert**: A graph of n nodes is grown by attaching new nodes each with m edges that are preferentially attached to existing nodes with high degree. The factors n and m are parameters to the algorithm.

- **Watts–Strogatz**: This model returns a Watts–Strogatz small-world graph. It firstly creates a ring over n nodes. Then each node in the ring is joined to its k nearest neighbors. Then shortcuts are created by replacing some edges as follows: for each edge (u,v) in the underlying "n-ring with k nearest neighbors" with probability b replace it with a new edge (u,w) with uniformly random choice of existing node w. The factors n,k, and b are the generator's parameters.

Algorithm 5 Chain Decomposition

```
1: procedure NodeOrder based variation with concatenation(G, T)
   INPUT: A DAG G = (V, E), and a topological sorting T(v_1, ..., v_i, ..., v_N) of
   G
   OUTPUT: A path decomposition of G
2:     K ← ∅ //Set of paths
3:     for every vertex v_i ∈ T in ascending topological order do
4:         Chain C
5:         if u_i is assigned to a chain then
6:             C ← u_i's chain
7:         else if v_i is not assigned to a chain then
8:             l_i ← choose the immediate predecessor with the lowest outdegree
9:                   that is the last vertex of a chain
10:            if l_i = null then
11:                (R_i, P_i) ← reverse_DFS_lookup(G, u_i)
12:                if P_i ≠ ∅ then
13:                    l_i ← destination vertex of P_i
14:                end if
15:                G ← G \ R_i
16:            end if
17:            if l_i ≠ null then
18:                C ← path indicated by l_i
19:                add v_i to C
20:            else
21:                C ← new Chain()
22:                add v_i to C
23:            end if
24:            add C to K
25:        end if
26:        if there is an immediate successor s_i of u_i with in-degree 1 then
27:            add s_i to C
28:        end if
29:    end for
30: end procedure
```

In order to make the directed graphs acyclic, we remove the edges in which the target vertex has a bigger id than the source vertex. For more info about the generators see networkx documentation.

Table 3.1 shows the number of chains created by the heuristics for every graph of 5000 nodes. Table 2.2 does it respectively for the graphs of 10000 nodes. The tables' abbreviations are explained below:

- **CO**: Path decomposition using Chain Order Heuristic.

- **CO conc.**: Chain decomposition using Chain Order Heuristic and our concatenation technique (post-processing step)
- **NO**: Path decomposition using Node Order Heuristic.
- **NO conc.**: Chain decomposition using Node Order Heuristic and our concatenation technique (post-processing step)
- **H3**: Path decomposition using our Node Order Heuristic variation from section 2.3.2.
- **H3 conc.**: Chain decomposition using our technique from section 2.3.2.
- **Width**: The width of the graph (Fulkerson's method).

As you see, in both tables our chain decomposition(H3 conc.) performs better than the others since it produces fewer chains. To visualize how close is the outcome of our heuristic to the width, we made some charts. In Figures 2.3, 2.4, and 2.5, you can see how close is the blue line to the red one for Erdos Renyi, Barabsi Albert, and Watts Strogatz model. The red line indicates the width and the blue the chains produced by our technique.

Furthermore, we explore the behavior of the width on these models. Notice that the Barabasi Albert model produces graphs with a larger width than Erdos-Renyi. Respectively, the Erdos-Renyi model creates graphs with a larger width than Watts-Strogatz. For the Watts Strogatz model, we create two sets of graphs. The first has probability b equals 0.9 and the second 0.3. If the probability b of rewiring an edge is 0, the width would be one. That happens because the generator initially creates a path that goes through all vertices. As probability b grows, the width grows. That's the reason we choose a low and a high probability. Figure 2.2a and 2.2b demonstrates the behavior of the width for each model on the graphs of 5000 and 10000 nodes. Another interesting observation is that the width of the Erdos Renyi model follows the inverse function $width = \frac{\text{nodes}}{\text{average degree}}$.

In this section, we do not present runtime metrics for two reasons. The first is that we do it in the forthcoming section, see tables 2.4,2.5, and 2.3. The second is that all heuristics run in a few milliseconds, and there is no reason for comparison since all scale up on large graphs, much larger than those of 10000 nodes.

2.4 Hierarchies and Transitivity

Proposition 1. *Given a chain decomposition D of a DAG $G(V, E)$, each vertex $v_i \in V$, $0 \leq i < |V|$, can have at most one outgoing non-transitive edge per chain.*

Proof of Proposition 1. Given a graph $G(V, E)$, a decomposition $D(C_1, C_2, ..., C_{k_c})$ of G, and a vertex $v \in V$, assume vertex v has two outgoing edges, (v, t_1) and (v, t_2), and both t_1 and t_2 are in chain C_i. The vertices are in ascending topological order in the chain by definition. Assume t_1 has a lower topological rank than t2. Thus,

|N|=5000

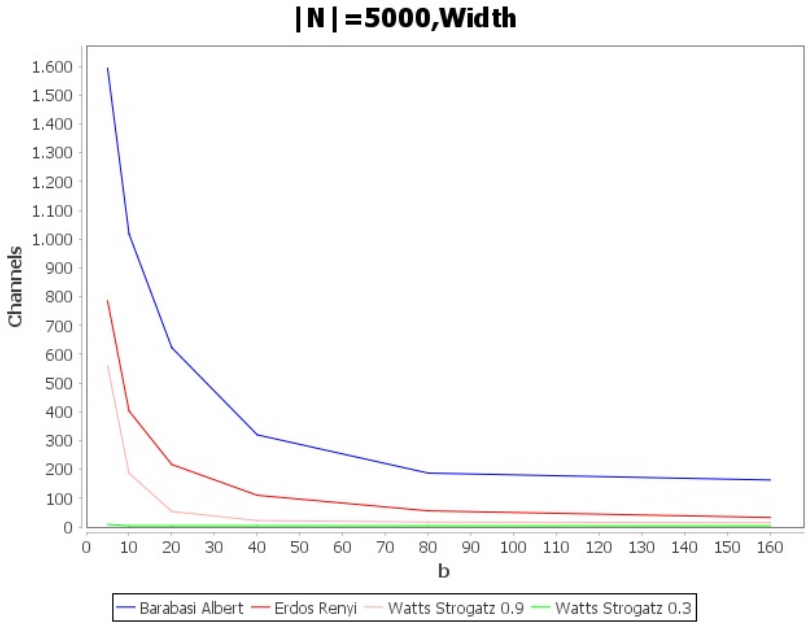
Av. Degree		5	10	20	40	80	160
		Barabasi Albert					
CO		1722	1178	801	471	296	189
CO conc.		1686	1127	747	411	252	164
NO		1792	1250	827	516	306	193
NO conc.		1743	1174	774	445	284	187
H3		1658	1102	720	424	256	165
H3 conc.		1630	1055	664	355	207	163
Width		1593	1018	623	320	187	163
		Erdos Renyi					
CO		1138	710	433	260	148	79
CO conc.		1027	593	356	217	125	69
NO		1184	744	461	263	157	83
NO conc.		1105	686	429	257	153	83
H3		1050	654	401	235	143	80
H3 conc.		923	492	252	139	70	38
Width		785	403	217	110	56	33
		Watts-Strogatz, b=0.9					
CO		948	514	279	161	87	57
CO conc.		794	376	202	107	69	47
NO		995	540	272	126	60	40
NO conc.		865	441	244	119	59	40
H3		891	473	264	145	81	58
H3 conc.		687	212	60	25	20	17
Width		560	187	54	22	17	15
		Watts-Strogatz, b=0.3					
CO		399	240	130	62	39	23
CO conc.		90	57	32	20	12	10
NO		275	88	23	6	7	6
NO conc.		85	40	17	6	7	6
H3		283	162	85	50	28	12
H3 conc.		9	4	4	5	4	5
Width		9	4	4	4	4	4

Table 2.1: Comparing path and chain decomposition algorithms on graphs with 5000 nodes.

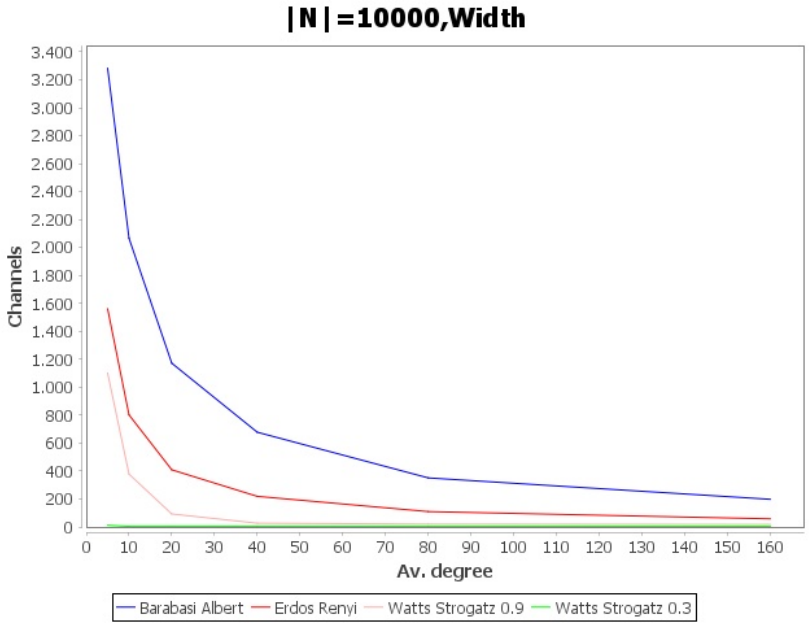
|N|=10000

Av. Degree		5	10	20	40	80	160
		Barabasi Albert					
CO		3501	2401	1537	985	586	357
CO conc.		3441	2301	1415	865	500	294
NO		3635	2519	1645	1033	625	387
NO conc.		3549	2413	1515	959	563	345
H3		3385	2257	1411	911	535	321
H3 conc.		3341	2159	1264	752	400	228
Width		3282	2066	1172	678	351	198
		Erdos Renyi					
CO		2283	1432	871	513	294	165
CO conc.		2015	1213	730	428	251	145
NO		2369	1517	891	531	294	165
NO conc.		2172	1383	833	507	290	163
H3		2135	1325	804	482	272	166
H3 conc.		1837	1003	516	271	139	72
Width		1561	802	409	219	110	58
		Watts-Strogatz, b=0.9					
CO		1869	1064	566	306	170	92
CO conc.		1575	771	381	218	119	72
NO		1975	1083	528	238	101	56
NO conc.		1717	894	455	218	92	56
H3		1748	975	524	269	150	95
H3 conc.		1332	447	100	29	24	22
Width		1101	378	93	27	20	18
		Watts-Strogatz, b=0.3					
CO		816	434	242	133	78	37
CO conc.		184	122	57	38	24	17
NO		565	171	37	10	7	7
NO conc.		165	72	24	9	7	7
H3		534	299	180	96	34	34
H3 conc.		12	4	4	4	4	4
Width		12	4	4	4	4	4

Table 2.2: Comparing path and chain decomposition algorithms on graphs with 10000 nodes.



(a) The width curve on graphs of 5000 nodes.



(b) The width curve on graphs of 10000 nodes.

Figure 2.2: The width curve on graphs of 5000 and 10000 nodes using three different models.

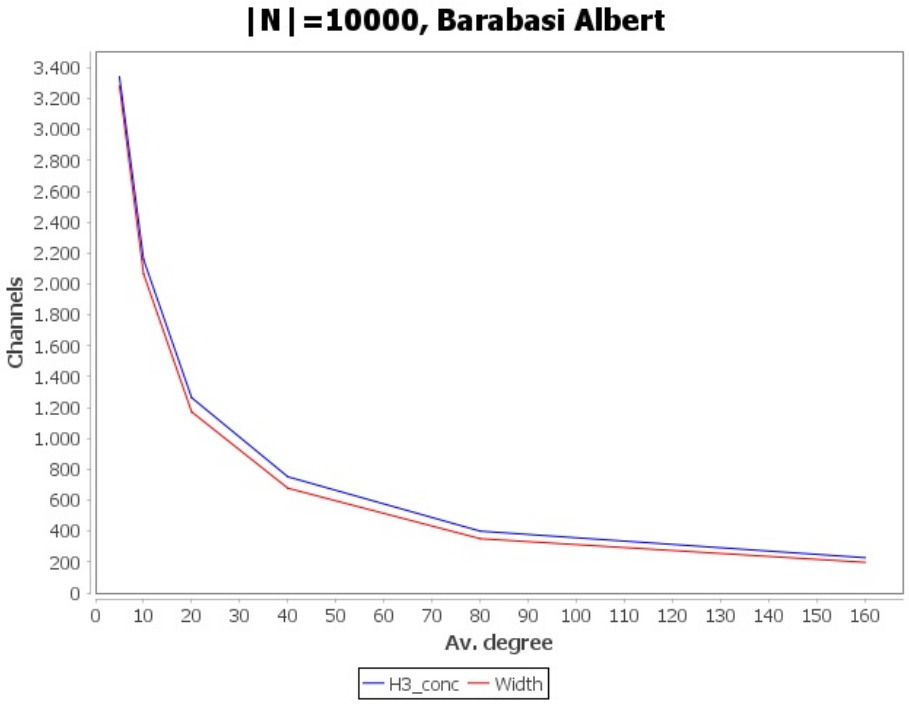


Figure 2.3: The efficiency of our chain decomposition algorithm in Barabasi Albert model.

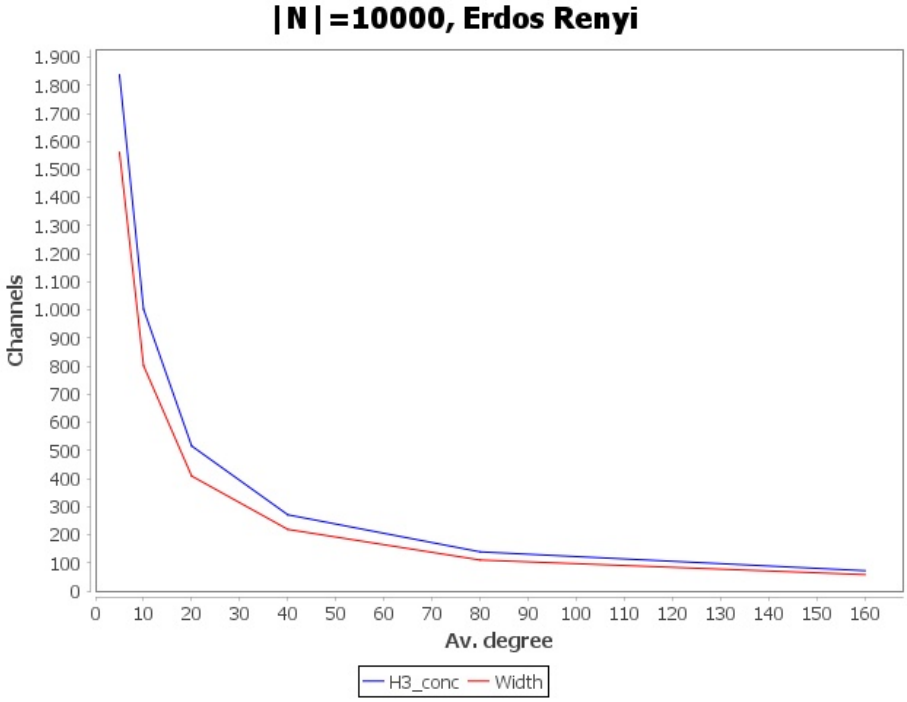


Figure 2.4: The efficiency of our chain decomposition algorithm in Erdos Renyi model.

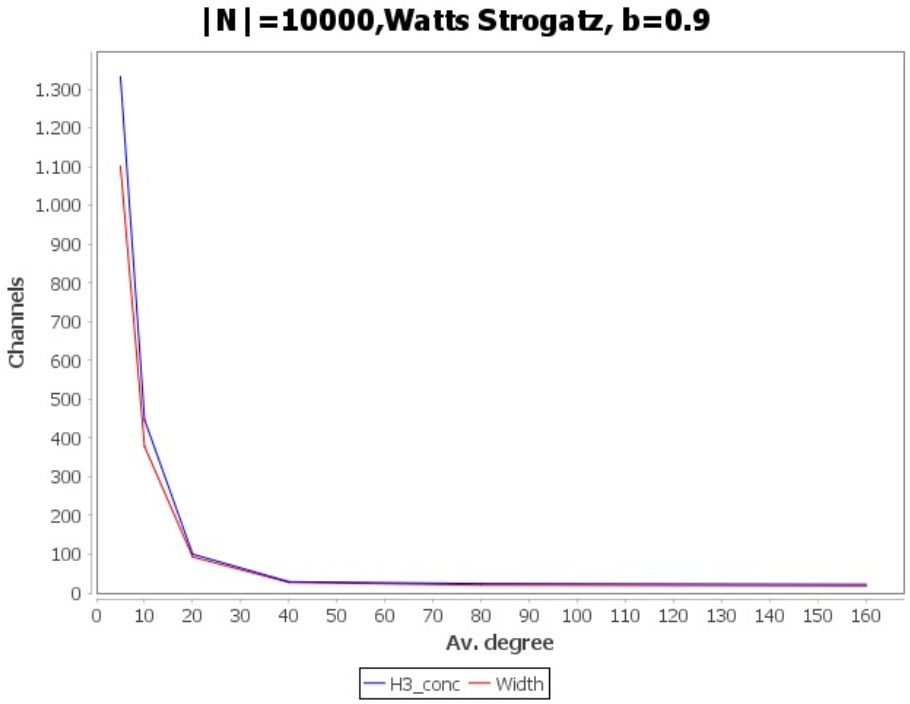
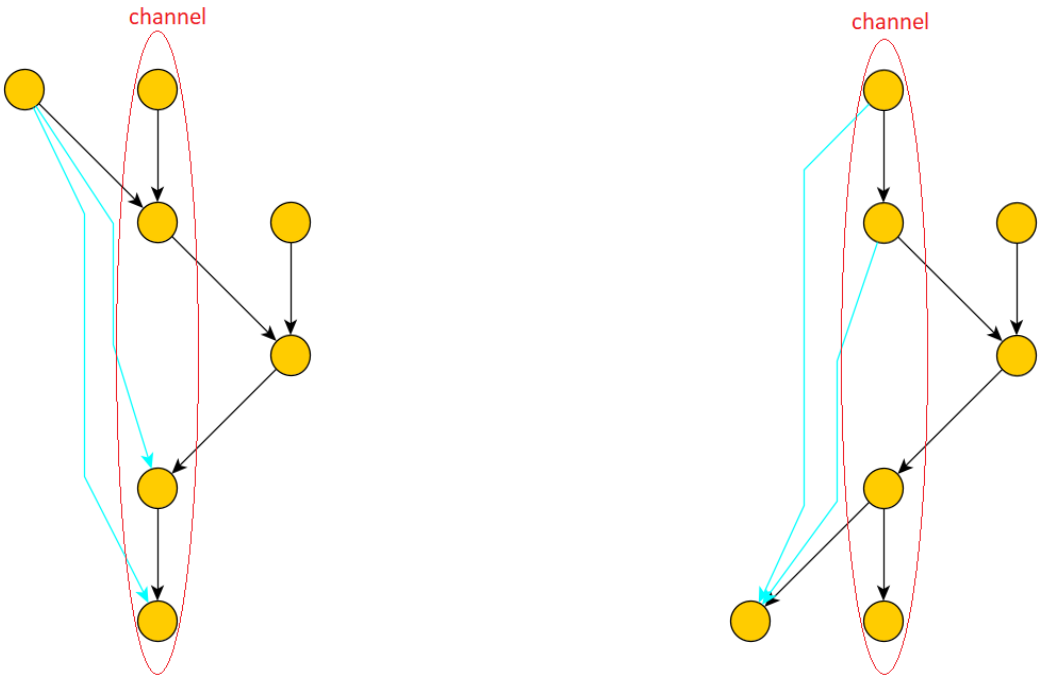


Figure 2.5: The efficiency of our chain decomposition algorithm in Watts-Strogatz model.

there is a path from t_1 to t_2, and accordingly a path from v to t_2 through t_1. Hence, the edge (v, t_2) is transitive. See figure 2.6a. □

Proposition 2. *Given a chain decomposition D of a DAG $G(V, E)$, each vertex $v_i \in V$, $0 \leq i < |V|$, can have at most one incoming non-transitive edge per chain.*

Proof of Proposition 2. Similar to the proof of proposition 1. See figure 2.6b. □



(a) (b)

Figure 2.6: The blue edges are transitive. (a) shows the outgoing transitive edges that end to the same chain. (b) shows the incoming transitive edges that start from the same chain.

Theorem 2.4.1. *The non-transitive edges of a DAG $G(V, E)$ are less or equal to $width * |V|$, $E_{red} = E - E_{tr} \leq width * |V|$.*

Proof of Theorem 2.4.1. Given any DAG G and its width w, there is a chain decomposition of G with w number of chains. From Proposition 1, every vertex of G could have only one outgoing, non-transitive edge per chain, thus its non-transitive outgoing edges can not be more than $w * |V|$. Notice that the same stands for the incoming edges, according to Proposition 2. □

According to theorem 2.4.1, the time complexity of Algorithm 5 can be expressed as $O(k_c * E_{red}) = O(k_c * width * |V|)$ since $E_{red} \leq width * |V|$. Additionally, the chains rarely have the same length. Usually, the decomposition consists of a

few long chains and many shorter chains. Hence, for most of the graphs is not even possible $E_{red} = width * |V|$, E_{red} usually is much less than that. We present experiments in table 2.4 and 2.5.

Also, an essential application of proposition 1 and 2 is that we can find a subset of E_{tr} in linear time. Given a chain decomposition or a path decomposition with k_c chains, we can trace the vertices and their outgoing edges and keep the arcs that point to the lowest point of each chain, rejecting the rest as transitive. We do the same for the incoming edges keeping the edges that come from the highest point (vertex with highest topological rank) of each chain. This way, we find a subset $E'_{tr} \subseteq E_{tr}$. Hence, $E - E'_{tr} \leq k_c * |V|$. This linear time preprocessing can facilitate every transitive closure technique bounding the input edges, and the indegree and outdegree of every vertex by k_c. For example, algorithms based on tree cover, see [5, 13, 69, 73], are practical on sparse graphs and can be enhanced further with a preprocessing step that removes transitive edges.

2.5 Indexing Scheme

In this section, we examine an important application of our chain decomposition technique. We solve the transitive closure problem by creating an indexing scheme.

Jagadish described that indexing scheme in 1990. As we told, Jagadish's heuristic for chain decomposition runs in $O(n^2)$ using the pre-computed transitive closure. Our technique outperforms that. It runs in almost linear time without the precomputed transitive closure, and the outcome is very close to the optimal.

Simon, see [64], built that indexing scheme too. He calculates a path decomposition, boosting the method presented in [35]. The linear time heuristic he presented is Chain Order Heuristic. Our technique follows the same steps for the creation of the indexing scheme. The differentiation is that we do not merely do path decomposition, we do chain decomposition using our algorithm. Simon's algorithm needs $O(k_p * |E_{red}|)$ time and $O(k_p * n)$ space (k_p is the number of paths).

We build our solution in $O(k_c * |E_{red}|)$ time, and we can answer queries in constant time. k_c is the number of chains. $|E_{red}|$ is the number of non-transitive edges. Additionally, we will show that $|E_{red}| \leq width * |V|$. The space complexity of our algorithm is $O(k_c * n)$. Furthermore, we present extensive experimental work, and we show both in theory and practice that our algorithm outperforms Simon's.

By finding the strongly connected components, we can make any graph acyclic. All vertices of a SCC will form a supernode since any vertex is reachable from any other vertex in the same component. That is a well-known step, so we assume that the input of our method is a DAG. The steps given a DAG are:

1. Perform Chain decomposition
2. Sort Adjacency lists
3. Create Indexing Scheme

In step 1, we use our chain decomposition technique that runs in $O(m + c * l)$. Simon performs path decomposition that runs in $O(n + m)$. In step 2, we sort the adjacency lists in O(n+m) time. Lastly, we create the indexing scheme in $O(k_c * |E_{red}|)$ time and $O(k_c * n)$ space. If we had done merely path decomposition, the time complexity would be $O(k_p * |E_{red}|)$ and $O(k_p * n)$ space. Probably, you have already noticed the relation between step 1 and step 3.

2.5.1 The Indexing Scheme

Assume there is a chain decomposition of a DAG G with size k_c. Its indexing scheme includes a pair and an array of indexes with k_c size for every vertex. You can see the example in figure 2.7. The first integer of the pair indicates the node's chain and the second its position in the chain. For example, vertex 1 of figure 2.7 has $(1, 1)$. The node belongs to the $1st$ chain, and it is the $1st$ element in it. Given the chain decomposition, we can easily construct the pairs in O(n) time with a chain decomposition traversal. Every cell of the k_c size array represents a chain. The i-th cell represents the i-th chain. The entry in the i-th cell corresponds to the lowest point of the i-th chain the vertex can reach. For example, the array of vertex 1 is [1,2,3]. The first cell of the array indicates that vertex 1 can reach the $1st$ vertex of the first chain (can reach itself, reflexive property). The second cell of the array indicates that vertex 1 can reach the $2nd$ vertex of the second chain (There is a path from vertex 1 to vertex 7). Finally, the third cell of the array indicates that vertex 1 can reach the $3rd$ vertex of the third chain.

Notice that we do not need the second integer of any pair. If we know the chain a vertex belongs in, we can conclude its position using the array. We present it like that to make it easier to understand.

The process of answering a reachability query is simple. Assume, there is a source vertex S and a target vertex T. To find if the vertex T is reachable from the S, we get T's chain, and we use it as an index in S's array. Hence, we know the lowest point of T's chain vertex S can reach. S can reach T if that point is less or equal to T's position, else it cannot.

2.5.2 Sorting Adjacency lists

Algorithm 6 sorts the adjacency list of every vertex. More precisely, it sorts the adjacency lists of immediate successors in ascending topological order. The variable stack indicates the sorted adjacency list. The algorithm traverses the vertices in reverse topological order $(v_n, ..., v_1)$. For every vertex v_i, $1 \leq i \leq n$, pushes v_i in the stacks of all immediate predecessors. This step could take part even before the chain decomposition as a preprocessing step. We present it in this section to emphasize its crucial role in indexing scheme creation. If the adjacency list is not sorted the time complexity of the algorithm would be $O(k_c * |E|)$ and not $O(k_c * |E_{red}|)$.

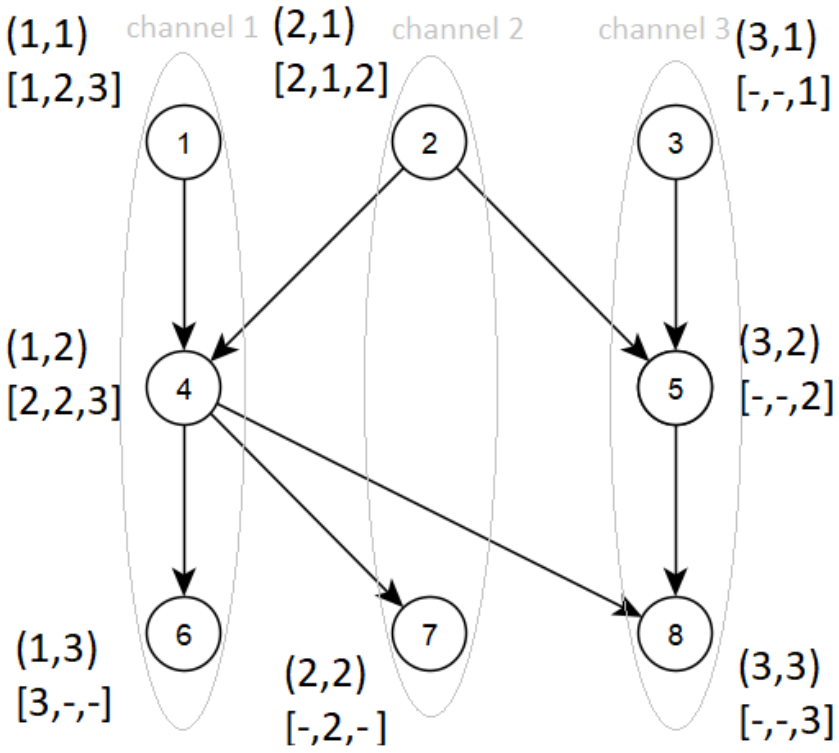


Figure 2.7: An example of an indexing scheme.

Algorithm 6 Sorting Adjacency lists

```
procedure SORT(G, t)
INPUT: A DAG G = (V, E) and a topological sorting t of G
    for each vertex: v_i ∈ G do
        v_i.stack ← new stack()
    end for
    for each vertex v_i in reverse topological order do
        for every incoming edge e(s_j, v_i) do
            s_j.stack.add(v_i)
        end for
    end for
end procedure
```

Proposition 3. *Algorithm 6 sorts the adjacency lists of immediate successors in ascending topological order.*

Proof of Proposition 6. Assume that there is a stack $(u_1, ..., u_n)$, u_1 is the top of the stack. Assume that there is a pair (u_j, u_k) in the stack, where u_j has a bigger topological rank than u_k and u_j precedes u_k. That means the for-loop examined u_j before u_k since it goes through the vertices in reverse topological order. This is a contradiction. The vertex u_j cannot precede u_k if it was examined first by the for-loop. □

2.5.3 Creating the Indexing Scheme.

Algorithm 7 constructs the indexing scheme. The first for-loop initializes the array of indexes. For every vertex, initializes the cell that corresponds to its chain. The rest of the cells initializes with infinite. The indexing scheme initialization illustrated in figure 2.8. The dashes represent the infinite. Notice that after the initialization, the indexes of all sink vertices have been calculated. Since they have no successors, the only vertex they can reach is themselves.

The second for-loop builds the indexing scheme. It goes through vertices in descending topological order. For each vertex, it visits its immediate successors (outgoing edges) in ascending topological order and updates the indexes. Suppose we have the edge (v, s), and we have calculated the indexes of vertex s(s is immediate successor of v). The process of updating the indexes of v with its immediate successor s means that s will pass all its information to the vertex v. Hence, vertex v will be aware that it can reach s and all its successors. Assume the array of indexes of v is $[a_1, a_2, ..., a_{k_c}]$ and the array of s is $[b_1, b_2, ..., b_{k_c}]$. To update the indexes of v using s, we merely trace the arrays and keep the smallest values. For every pair of indexes (a_i, b_i), $0 \leq i \leq kc$, the new value of a_i is $\min\{a_i\ ,\ b_i\}$. This process needs k_c steps.

Algorithm 7 Indexing Scheme

```
1: procedure CREATE INDEXING SCHEME(G, T, D)
   INPUT: A DAG G = (V, E), a topological sorting T of G, and the decomposition D of G.
2:     for each vertex: v_i ∈ G do
3:         v_i.indexes ← new table[size of D]
4:         v_i.indexes.fill(∞)
5:         ch_no ← v_i's chain index
6:         pos ← v_i's chain position
7:         v_i.indexes[ ch_no ] ← pos
8:     end for
9:     for each vertex v_i in reverse topological order do
10:        while v_i.stack ≠ ∅ do
11:            target ← v_j.stack.pop()
12:            t_ch ← target's chain index
13:            t_pos ← target's chain position
14:            if t_pos < v_i.indexes[t_ch] then // (v_i, target) is not transitive
15:                v_i.updateIndexes(target.indexes)
16:            end if
17:        end while
18:    end for
19: end procedure
```

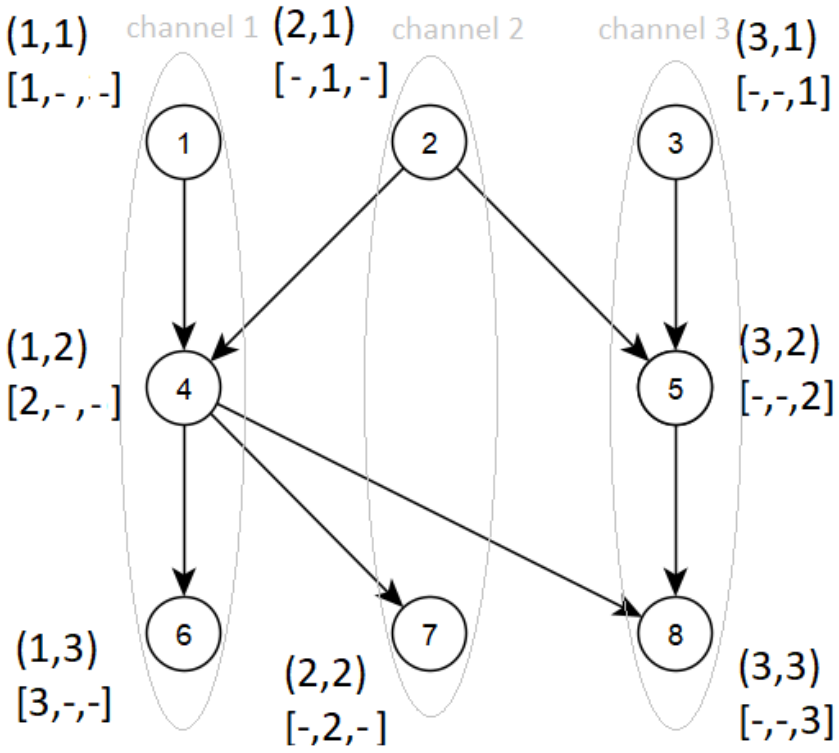


Figure 2.8: Initialization of indexes.

Proposition 4. *Given a vertex v and the calculated indexes of its successors, the while-loop of algorithm 7 (lines 10-17) calculates the indexes of v by updating its array with its non-transitive outgoing edges' successors.*

Proof of Proposition 4. Updating the indexes of vertex v with all its immediate successors will make v aware of all its descendants. The while-loop of Algorithm 7 does not perform the update function for every direct successor. It skips all the transitive edges. Assume there is such a descendant t and the transitive edge (v,t). Since it is transitive, we know by definition that exists a path from v to t with a length of more than 1. Suppose that the path is (v,v1,..,t). The vertex v1 is a predecessor of t and immediate successor of v. Hence it has a lower topological rank than t. The while-loop examines the incident vertices in ascending topological order. Hence, vertex t gets visited after vertex v1. The opposite leads to a contradiction. Consequently, for every incident transitive edge of v, the loop firstly visits a vertex v1 which is a predecessor of t. Thus vertex v will be updated by v1 and aware that the edge (v,t) is transitive. There is no reason to update vertex v indexes since the indexes of t will be greater or equal □

Theorem 2.5.1. *Algorithm 7 computes the indexing scheme in* $O(k_c * |E_{red}|)$ *time.*

Proof of Theorem 2.5.1. In the initialization step, the indexes of all sink vertices have been computed as we described before. Taking vertices in reverse topological order, the first vertex we meet is a sink vertex. When the for-loop of line 9 visits the first non-sink vertex, the indexes of its successors are computed (all its successors are sink vertices). According to proposition 1, we can calculate its indexes, ignoring the transitive edges. Assume the for-loop has reached the vertex v_i in the $i-th$ iteration, and the indexes of its successors are calculated. Similarly, from proposition 1, we can calculate its indexes. By induction, we can calculate the indices of all vertices, ignoring all transitive edges in $O(|E_{red}| * k_c)$ time. □

2.5.4 Experiments

We used the same graph of 5000 and 10000 nodes we described in 2.3.3 produced by three different models of the Networkx. We performed chain decomposition using our approach (H3_conc, Alg. 5), and created the indexing scheme (Alg. 7). Assume the sorting of the adjacency list is a preprocessing step (Alg. 6). We recorded our results in Tables 2.4 and 2.5. Table 2.4 holds the results of graphs with 5000 nodes, and Table 2.5 the results of graphs with 10000 nodes. Next, we explain tables' columns.

- **Av. Degree**: The average degree of the graph
- **Chains**: Number of chains computed by our heuristic (H3_conc).
- $|\mathbf{E_{tr}}|$: Number of transitive edges.
- $|\mathbf{E_{red}}|$: Number of non-transitive edges.

- $|\mathbf{E_{tr}}|/|\mathbf{E}|$: The percentage of transitive edges.

- **H3_conc Time (ms)**: The time, in milliseconds, of the chain decomposition step.

- **Indexing Scheme Time (ms)**: The time, in milliseconds, of the indexing scheme creation step.

- **Total**: The total time(ms) needed to decompose the graph and create the indexing scheme. It is the sum of the two preceding cells.

- **TC**: The time needed by a known algorithm for transitive closure with time complexity $O(|V| * |E|)$. The algorithm performs a DFS function for every vertex to mark reachable vertices. It stores the results in a 2-D adjacency matrix.

The phase of indexing scheme creation needs $k_c * |E_{red}| + E_{tr}$ steps. The numbers on the tables are interesting. As the average degree increases and the graph becomes denser, the cardinality of E_{red} remains almost stable, and the chains decrease. Of course, since the E_{red} does not vary as the average degree increases, the cardinality of E_{tr} increases ($E_{tr} = E - E_{red}$). The algorithm merely traces in linear time the transitive edges. Consequently, the growth of E_{tr} does not affect the run time considerably. As a result, the run time of our technique does not increase as the input graph increases. To demonstrate it clearly, we drew the line chart of figure 2.9 for the graphs of 10000 nodes produced by the Erdos-Renyi model. The blue line represents the run time of the indexing scheme, and the red line the run time of the algorithm based on DFS. The time of the algorithm based on DFS increases as the average degree increases, while the time of the indexing scheme is a straight line parallel to the x-axis. All models follow this pattern. See tables 2.4 and 2.5.

We decompose the graph into chains with our algorithm since it is the most efficient. A chain decomposition is preferable to a path decomposition if we create the indexing scheme. Assume that we have a path decomposition, and we perform chain concatenation. If there is no concatenation between two paths, the concatenation algorithm will run in linear time, which is an acceptable cost. On the other hand, if there are concatenations, for each one of them, the cost is $O(l)$ time units but the gain in the following step of scheme creation is $|V|$ units of space and $|E_{red}|$ units of time. That stands because every concatenation reduces the indexes we need for every vertex by one. Hence, applying path concatenation, we create faster a more compact indexing scheme.

Tables 1 and 2 include metrics of creating the indexing scheme using different decomposition techniques on Erdos Reyni graphs of 10000 nodes. In table 1, we have created the indexing scheme using the chain order heuristic(path decomposition), while in table 2, we use our chain decomposition algorithm.

Av. Degree		Channels	CO Time (ms)	Indexing Scheme Time(ms)	Total
5		2283	8	237	246
10		1432	11	221	231
20		871	10	170	180
40		513	12	152	164
80		294	15	162	177
160		165	21	278	299

(a) Metrics: Creating the indexing scheme in combination with the chain order heuristic.

Av. Degree		Channels	H3_conc Time (ms)	Indexing Scheme Time(ms)	Total
5		1837	9	194	203
10		1003	11	163	174
20		516	16	100	116
40		271	39	108	147
80		139	43	130	173
160		72	75	237	312

(b) Metrics: Creating the indexing scheme in combination with algorithm 5 for chain decomposition.

Table 2.3: The tables present the run time of indexing scheme using path and chain decomposition.

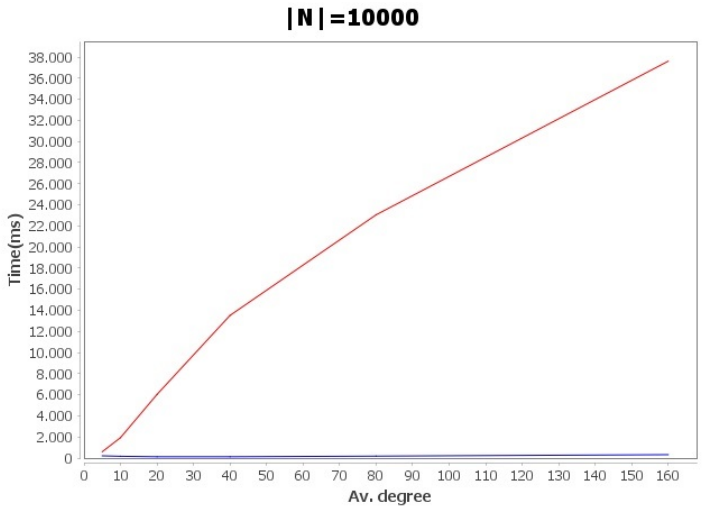


Figure 2.9: Run time comparison between the Indexing Scheme (blue line) and TC (red line) for Erdos-Renyi model on graphs of 10000 nodes. See table 2.5.

Av. Degree		Channels	$\|E_{tr}\|$	$\|E_{red}\|$	$\|E_{tr}\|/\|E\|$	H3_conc Time (ms)	Indexing Scheme Time(ms)	Total	TC
		Barabasi Albert							
5		1630	8054	18921	0.32	3	101	104	137
10		1055	28230	21670	0.57	12	79	91	333
20		664	75801	23799	0.76	6	54	60	638
40		355	180815	22504	0.89	10	48	58	1418
80		207	382422	20854	0.95	122	118	240	3018
160		163	770771	17660	0.98	25	107	132	5464
		Erdos Renyi							
5		923	3440	21466	0.14	6	67	73	172
10		492	24761	25425	0.49	10	51	61	487
20		252	75312	24646	0.75	5	26	31	1079
40		139	175809	22634	0.89	46	51	97	2896
80		70	378015	19435	0.95	16	50	66	5260
160		38	769919	16843	0.98	98	138	236	8609
		Watts-Strogatz, b=0.9							
5		687	7742	17258	0.30	13	71	84	393
10		212	37992	12008	0.76	11	18	29	817
20		60	89272	10728	0.89	23	22	45	1530
40		25	186486	13514	0.93	47	45	92	3704
80		20	386294	13706	0.97	115	103	218	6172
160		17	787066	12934	0.98	253	207	406	9173
		Watts-Strogatz, b=0.3							
5		9	18421	6579	0.74	11	8	19	910
10		4	43505	6495	0.87	8	11	19	1107
20		4	93490	6510	0.93	18	18	36	2176
40		5	193416	6584	0.97	17	18	35	4753
80		4	393348	6652	0.98	98	82	180	7949
160		5	793430	6570	0.99	250	166	416	11757

Table 2.4: Metrics on graphs of 5000 nodes.

Av. Degree		Channels	$\|E_{tr}\|$	$\|E_{red}\|$	$\|E_{tr}\|/\|E\|$	H3_conc Time (ms)	Indexing Scheme Time(ms)	Total	TC
		Barabasi Albert							
5		3341	14544	35431	0.29	7	278	285	441
10		2159	53503	46397	0.54	14	231	245	1379
20		1264	147791	51809	0.74	15	218	233	3347
40		752	355854	52465	0.85	28	188	216	7700
80		400	764926	48350	0.94	271	322	593	14632
160		228	1560464	42967	0.97	81	264	345	24601
		Erdos Renyi							
5		1837	5595	44401	0.11	12	200	212	600
10		1003	44813	55366	0.45	9	161	170	1935
20		516	144276	55310	0.72	16	110	126	6031
40		271	347323	52620	0.87	25	101	126	13522
80		139	749781	46666	0.94	40	145	185	23052
160		72	1548153	39710	0.97	73	249	322	37613
		Watts-Strogatz, b=0.9							
5		1332	13353	36647	0.27	12	175	187	1213
10		447	74782	25218	0.75	9	53	62	3829
20		100	178930	21070	0.89	13	32	45	9279
40		29	373054	26946	0.93	24	60	84	13144
80		24	771374	28626	0.96	266	247	513	25585
160		22	1571957	28043	0.98	80	232	312	36507
		Watts-Strogatz, b=0.3							
5		12	36816	13184	0.73	27	19	46	3468
10		4	86804	13196	0.86	18	45	63	5063
20		4	186756	13244	0.93	10	42	52	12156
40		4	386751	13249	0.97	19	48	67	21055
80		4	786840	13160	0.98	237	187	424	31016
160		4	1586896	13104	0.99	62	167	229	40704

Table 2.5: Metrics on graphs of 10000 nodes.

2.6 Conclusions

In this work, we show not only that finding a chain decomposition can be done in almost linear time but also the number of chains can be very close to the minimum. Our experiments expose the behavior of the width as the density grows, along with the efficiency of our heuristics. We bound the set E_{red} by $width * |V|$ and illustrate how to find a subset of E_{tr} in linear time given a path/chain decomposition. Our approach and theory have a wide area of applications. We applied them to the problem of transitive closure. We built with the most efficient way a known indexing scheme to answer queries in constant time. The time complexity is $O(k_c * |E_{red}|)$, and the space complexity is $O(k_c * |V|)$. Similarly, we ran experimental work revealing the practical efficiency of this approach, especially for dense graphs.

Chapter 3

Path Based Framework

3.1 Introduction

Hierarchical graphs are very important for many applications in several areas of research and business because they often represent hierarchical relationships between objects in a structure. They are directed (often acyclic) graphs and their visualization has received significant attention recently [16, 47, 53]. An experimental study of four algorithms specifically designed for DAGs was presented in [17]. DAGs are usually used to describe processes containing some long paths, such as in PERT applications see for example [18, 25]. The paths can be either application based, e.g. critical paths or user defined. If one desires automatically generated paths, there are several algorithms that compute a path decomposition of minimum cardinality [39, 50, 54, 63]. A new framework to visualize directed graphs and their hierarchies is introduced in [55, 56]. It computes readable hierarchical visualizations in two phases by "hiding" (abstracting) some selected edges while maintaining the complete reachability information of a graph.

In this work we present polynomial time algorithms that follow the main framework of [56]. The produced drawings contain all edges of the input graph and attempt to optimize the height, width and number of bends of the resulting drawing. Hence, we show a general purpose hierarchical graph drawing framework.

The new framework of [56] departs from the typical Sugiyama framework [65] and it consists of two phases: (a) Cycle Removal, (b) the path/channel decomposition and hierarchical drawing step. This framework is based on the idea of partitioning the vertices of a graph into paths/channels, drawing the vertices in each path vertically aligned on some x-coordinate and then drawing the edges between vertices that belong to different paths. The Sugiyama framework has been extensively used in practice, as manifested by the fact that various systems are using it to implement hierarchical drawing techniques. Several systems such as AGD [58], da Vinci [26], GraphViz [31], Graphlet [38], dot [30], OGDF [15], and others implement this framework in order to draw directed graphs. Commercial software such the Tom Sawyer Software TS Perspectives[2] and yWorks [3] essentially

use this framework in order to offer automatic visualizations of directed graphs. The comparative study of [17] concluded that the Sugiyama style algorithms performed better in most of the metrics. For more recent information regarding this framework see [53].

Even though it is very popular, the Sugiyama framework has several limitations: as discussed bellow, most problems and subproblems that are used to optimize the results in various steps of each phase have turned out to be NP-hard. Additionally, several of the heuristics employed to solve these problems give results that are not bound by any approximation. Furthermore, the required manipulations in the graph often increase substantially its complexity, e.g., up to $O(n * m)$ dummy vertices may be inserted in a directed graph $G = (V, E)$ with n vertices and m edges. The overall time complexity of this framework (depending upon implementation) can be as high as $O(n * m)$, or even higher if one chooses algorithms that require exponential time. Finally, another important limitation of this framework is the fact that heuristic solutions and decisions that are made during previous phases (e.g., crossing reduction) will influence severely the results obtained in later phases. Nevertheless, previous decisions cannot be changed in order to obtain better results. By contrast, in the main framework of [56] most problems of the second phase can be solved in polynomial time. If a path decomposition contains k paths, the number of bends introduced is at most $O(k * n)$ and the required area is at most $O(k * n)$. In order to minimize the number of crossings between cross edges and path edges the authors suggest checking all possible $k!$ permutations of the k paths which may be reasonable for small values of k [55]. However, edges between non consecutive vertices in a path, called path transitive edges are not drawn in this framework.

In this work we present algorithms that are based on the framework of [56] and offer experimental results comparing them to the results obtained by running the hierarchical drawing module of OGDF [15], which is based on the Sugiyama framework, and is the most updated research software that implements this framework. Since the "cycle removal phase" is required in both frameworks, we focus our experiments on the case where the input graph G is acyclic(DAG). Our algorithms run in almost linear time, and provide better upper bounds than the ones given in [56]: (a) the height of the resulting drawings is equal to the length of the longest path of G, which is often significantly lower than $n - 1$. (b) The path transitive edges are drawn by our algorithms in such a way that the required extra number of columns is minimized for each path (see Section 3.3.2). The experimental results show that the drawings produced by our algorithms have a significantly lower number of bends and are much smaller in area than the ones produced by OGDF (see section 3.4). On the other hand, the drawings of OGDF have a lower number of crossings when the input graphs are relatively sparse. However, when the graphs are a bit denser (e.g., average degree higher than five) our drawings have less crossings. Of course, it is expected that OGDF would be better than our algorithms in the number of crossings since OGDF places a significant weight in minimizing crossings, whereas we do not explicitly minimize crossings. Thus our

algorithms offer an interesting alternative to visualize hierarchical graphs.

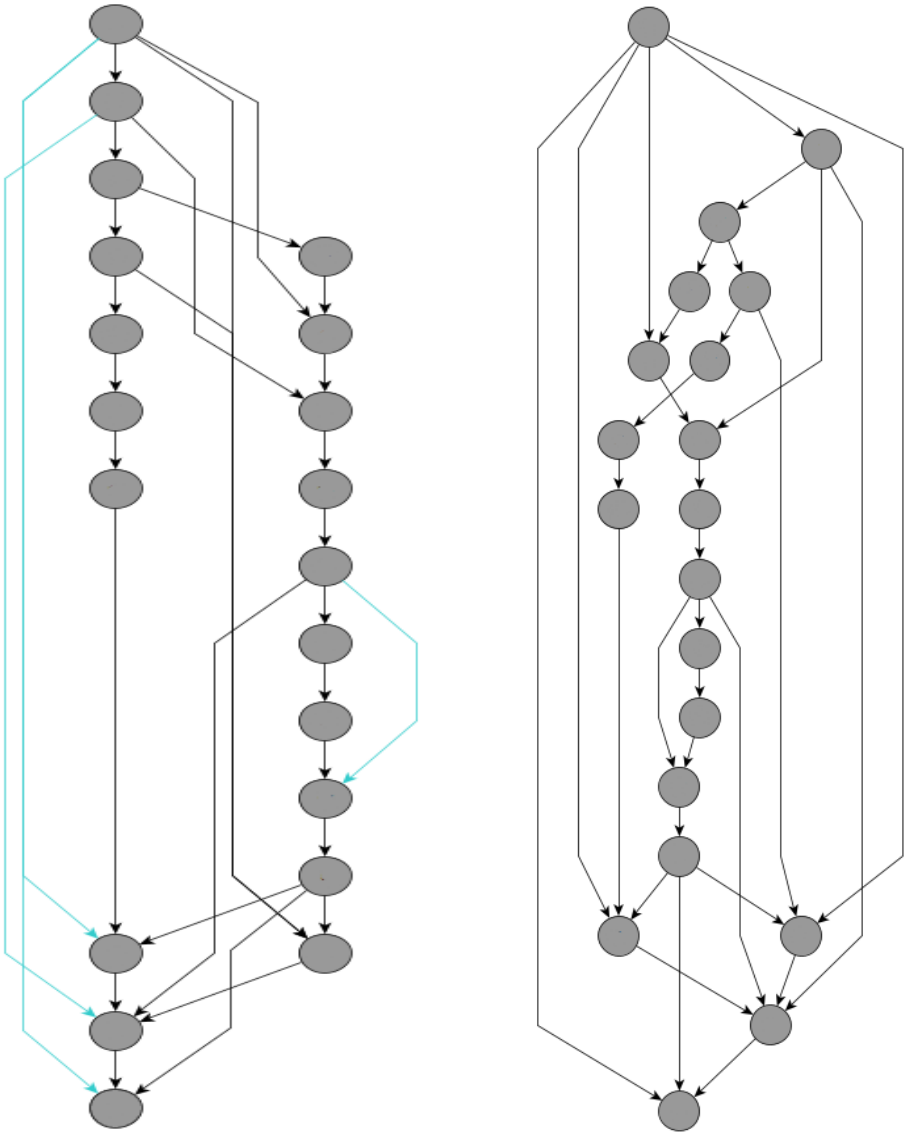
3.2 Overview of the Two Frameworks

To motivate our discussion about the two frameworks considered in this paper we present Figure 3.6 that shows a DAG G drawn by these two frameworks: Part (a) shows a drawing Γ of G computed by our algorithms that customize the path-based framework. Part (b) shows the drawing of G computed by OGDF. The graph consists of 20 nodes and 31 edges. The drawing computed by our algorithms has 12 crossings, 18 bends, width 10, height 15, and area 150. On the other hand, OGDF computes a drawing that has 5 crossings, 22 bends, width 18, height 15 and area 270. We had to normalized these figures in order to have a reasonable comparison, as will be discussed later. As can be observed by these two drawings, the two frameworks produce vastly different drawings with their own advantages and disadvantages.

The Path Based Hierarchical Drawing Framework is based on the idea of partitioning the vertices of the graph G into (a minimum number of) *channels/paths*, that we call *channel/path decomposition* of G, which can be computed in polynomial time. Therefore, it is orthogonal to the Sugiyama framework in the sense that it is a vertical decomposition of G into (vertical) paths/channels. Thus, most resulting problems are *vertically contained*, which makes them simpler, and reduces their time complexity. This framework does not introduce any dummy vertices and keeps the vertices of a path *vertically aligned*. By contrast, the Sugiyama framework performs a horizontal decomposition of a graph, even though the final result is a vertical (hierarchical) visualization.

Let $S_p = \{P_1, ..., P_k\}$ be a path decomposition of G such that every vertex $v \in V$ belongs to exactly one of the paths of S_p. Any path decomposition naturally splits the edges of G into: (a) *path edges* that connect consecutive vertices in the same path, (b) *cross edges* that connect vertices that belong to different paths, and (c) *path transitive edges* that connect non-consecutive vertices in the same path. Given S_p the main algorithm of [57], call it **Algorithm PBH**, draws the vertices of each path P_i *vertically aligned* on some x-coordinate depending on the order of path P_i. The y-coordinate of each vertex is equal to its order in any topological sorting of G. Hence the height of the resulting drawing is $n-1$. In the algorithms of [57] path transitive edges are omitted from the final drawing.

Another advantage of the Path-Based Framework is that it works for any given path decomposition. Therefore, it can be used in order to draw graphs with user-defined or application-defined paths, as is the case in many applications, see chapter 2. If one desires automatically generated paths, there are several algorithms that compute a path decomposition of minimum cardinality [39, 50, 54, 63]. Using a path decomposition with a small cardinality may improve the performance of our algorithm in terms of area, bends, number of crossings and computational time. Since certain critical paths are important for many applications, it is extremely



(a) (b)

Figure 3.1: In (a) we show the drawing Γ based on G as computed by our proposed framework. In (b) we show the drawing of the graph G as computed by OGDF.

important to produce clear drawings where all such paths are vertically aligned. For the rest of this chapter, we will assume that a path decomposition of G is given as part of the input to the algorithm.

OGDF is a self-contained C++ library of graph algorithms, in particular for (but not restricted to) automatic graph drawing. The hierarchical drawing implementation of the Sugiyama framework in OGDF is implemented following [28, 62]. The Sugiyama framework in OGDF according to uses the following default choices: For the first phase of Sugiyama, it uses the *LongestPathRanking* (a ranking module that determines the layering of the graph, i.e., the assignment of vertices into layers) which implements the well-known longest-path ranking algorithm. Next, it performs crossing minimization by using *BarycenterHeuristic*. This module performs two-layer crossing minimization and is applied during the top-down and bottom-up traversals [15]. The crossing minimization is repeated 15 times, and keeps the best. Each repetition (except for the first) starts with randomly permuted nodes on each layer. Finally it computes the final coordinates with *FastHierarchyLayout* which computes the final layout of graph. The two hierarchical drawings shown in Figure 3.6 demonstrate the significant differences in philosophy between the two frameworks.

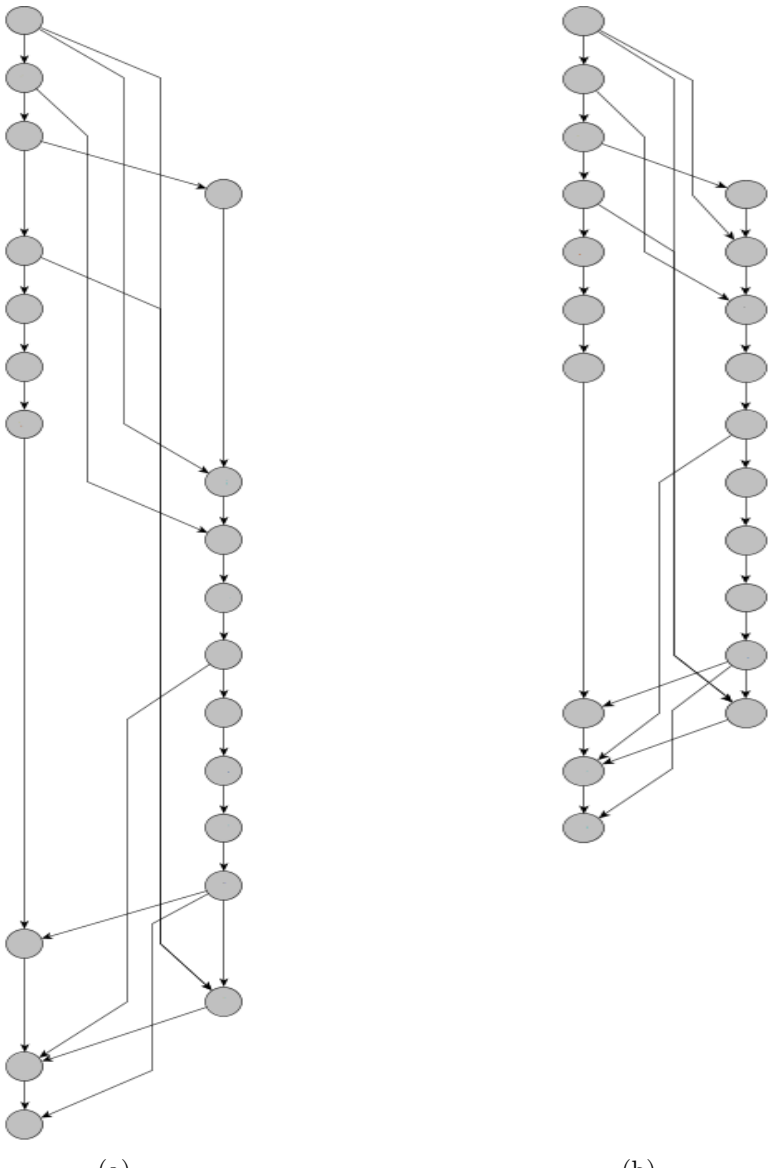
3.3 An Algorithm for Computing Compact Drawings

We present an extension of the framework of [57] by (a) compacting the drawing in the vertical direction, and (b) drawing the path transitive edges that were not drawn in [57]. This approach naturally splits the edges of G into three categories, *path edges*, *cross edges*, and *path transitive edges* that are drawn differently. This clearly adds to the understanding of the user and allows a system to show the different categories separately, without altering the user's mental map.

3.3.1 Compaction

Let $G = (V, E)$ be a DAG with n vertices and m edges. Following the framework of [55, 57] the vertices of V are placed in a unique y-coordinate, which is specified by a topological sorting. Let T be the list of vertices of V in ascending order based on their y-coordinates. We start from the bottom and visit each vertex in T in ascending order. For every vertex v in this order we assign a new y-coordinate, $y(v)$, following a simple rule that compacts the height of the drawing: "If v has no incoming edges then we set its $y(v)$ to 0, else we set $y(v)$ equal to $a + 1$, where a is the *highest* y-coordinate of the vertices that have edges incoming into v."

Algorithm 8 takes as input a DAG G, and a path based hierarchical drawing Γ_1 of G computed by Algorithm PBH and it produces as output a new, compacted, path based hierarchical drawing Γ_2 with height L, where L is the length of a longest path in G. Clearly this simple algorithm can be implemented in $O(n + m)$ time. Figure 3.2 shows an example of two hierarchical drawings of the same graph: Γ_1 is before compaction and Γ_2 is after compaction.



(a) (b)

Figure 3.2: DAG G drawn without its path transitive edges: (a) drawing Γ_1 is computed by Algorithm PBH, and it is the input of Algorithm 8, (b) drawing Γ_2 is the output of Algorithm 8.

Algorithm 8 Compaction(G, Γ_1)
Input: A DAG $G = (V, E)$, and a path based hierarchical drawing Γ_1 of G computed by Algorithm PBH
Output: A compacted path based hierarchical drawing Γ_2 with height L, where L is the length of a longest path in G.

1: **For** each $v \in G$:

- Let E_v be the set of incoming edges, $e = (w, v)$, into v:
  - *a.* **if** $E_v = \emptyset$ **then**:
    - $y(v)$=0
  - *b.* **else**:
    - $y(v)$=max{y-coordinates of vertices w with $(w, v) \in E_v$} + 1

Notice that the first case of the if-statement, is executed only for the first vertex (source) of some paths. Clearly, the rest of the vertices have at least one incoming edge since they belong to some path where every vertex is connected to its predecessor. This is the case for the "else" part. The compacted y-coordinate for the rest of the vertices will always be equal to "max {y coordinates of adjacent vertices to it} +1". Based on these statements and the fact that the drawing after compaction is also a path based hierarchical drawing, we have the next two simple lemmas.

Lemma 3.3.1. *Two vertices of the same path cannot have the same y-coordinate.*

Lemma 3.3.2. *For every vertex v with $y(v) \neq 0$, there is an incoming edge into v that starts from a vertex w such that $y(v) = y(w) + 1$.*

Based on these lemmas the height of the compacted drawing of the graph G is at most L:

Theorem 3.3.3. *Let $G = (V, E)$ be a DAG with n vertices and m edges. Algorithm Compaction computes in $O(n+m)$ time a hierarchical drawing Γ_2 of G with height L, where L is equal to the length of a longest path in G.*

Proof. It is clear that the height of the resulting drawing Γ_2 cannot be lower that L, the length of the longest path, due to Lemma 3.3.1 and the fact that all edges go from a vertex with lower to a vertex with higher y-coordinate. Similarly, the height of the resulting drawing Γ_2 cannot be higher that L since that would imply that there is a y coordinate that does not contain a vertex of a longest path. In this case by the initial assumption and Lemma 3.3.2 there is another path that is longer than L. Hence the height of the resulting drawing Γ_2 is equal to L. The time complexity of Algorithm Compaction is immediate from the fact that we visit each vertex exactly once, in the order specified by T and consider all its incoming edges once. □

3.3.2 Drawing the Path Transitive Edges

An important aspect of our work is the preservation of the mental map of the user that can be expressed by the reachability information of a DAG. At this point, we highlight that for every decomposition path, we have a set of path transitive edges that are not drawn by the framework of [55, 57]. In this subsection we show how to draw these edges while preserving the user's mental map of the previous drawing. Additionally, one may interact with the drawings by hiding the path transitive edges at the click of a button without changing the user's mental map of the complete drawing.

Now we will describe an algorithm that draws the path transitive edges using the minimum extra width (minimum extra number of columns) for each decomposition path. The steps of the algorithm are briefly described as follows:

1. For every vertex of each decomposition path we calculate the indegree and outdegree based only on path transitive edges, i.e., excluding path edges and cross edges.

2. If all indegrees and outdegrees are zero the algorithm is over, if not, we select a vertex v with the highest indegree or outdegree and we bundle all the incoming or outgoing edges of v, respectively. These bundled edges are represented by an *interval* with starting and finishing points, the lowest and highest y-coordinates of the vertices, respectively.

3. Next, we insert each interval on the left side of the path on the first available column such that the interval does not overlap with another interval (see details below).

4. We remove these edges from the set of path transitive edges, update the indegree and outdegree of the vertices and repeat the selection process.

5. The intervals of the rightmost path, are inserted on the right side of the path in order to avoid potential crossing with cross edges.

6. A final, post-processing step can be applied because some crossings between intervals/bundled edges can be removed by changing the order of the columns containing them.

The above algorithm can be implemented to run in time $O(m + n \log n)$ by handling the updates of the indegrees and outdegrees carefully, and placing the appropriate intervals in a (Max Heap) Priority Queue. As expected, the fact that we draw the path transitive edges increases the number of bends, crossings, and area, with respect to not drawing them.

For each decomposition path, suppose we have a set of b of intervals such that each interval I has a start point, s_I, and a finish point f_I. The starting point is the position of the vertex of the interval with the lowest y-coordinate.

Figure 3.3: Bundling of path transitive edges: (a) incoming edges into the last vertex of the path, (b) bundling of incoming edges, (c) outgoing edges from the first vertex of the path, (d) bundling of outgoing edges.

Similarly, the finish point f_I is the position of the node of the interval with the highest y-coordinate. We follow a greedy approach in order to minimize the width (number of columns) for placing the bundled edges. The approach is similar to Task Scheduling [33], for placing the intervals. It uses the optimum number of columns and runs in $O(b \log b)$ time, for each path with b intervals. This is done by considering the intervals of each decomposition path in increasing order of their starting points. We select each interval (resp. task) according to its starting point and place it into the first column that can fit (i.e., does not intersect with another interval). If there are no available columns, we allocate a new column and place the interval there. Since the sum of all b's for all paths in a path decomposition is at most n we conclude that the algorithm runs in $O(n \log n)$ time. The proof of correctness is similar to the one for Task Scheduling in [33] and thus it is omitted here.

Theorem 3.3.4. *Let $G = (V, E)$ be a DAG with n vertices and m edges. There is an algorithm that computes a drawing of G bundling the path transitive edges for each path using the minimum number of columns (width) per path. The algorithm runs in $O(m + n \log n)$ time and computes a compact hierarchical drawing of G.*

3.3.3 Drawing the Cross Edges

Cross edges connect vertices that belong to different paths/channels. When we refer to the vertical distance between two nodes, we mean the absolute value of the subtraction of their levels. The number of bends of every cross edge depends on the vertical distance of its incident nodes. Each cross edge $(u1, u2)$ has:

1. Two bends if the vertical distance between $u1$ and $u2$ is more than two.

2. One bend if the vertical distance between $u1$ and $u2$ is two.

3. The edge is a straight line if the vertical distance between $u1$ and $u2$ is one.

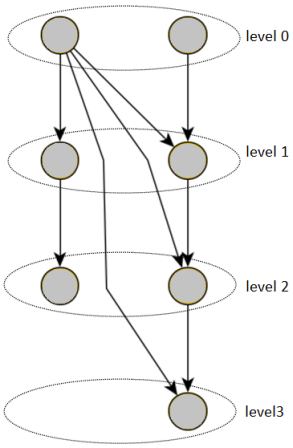


Figure 3.4: An example of cross edges' bends.

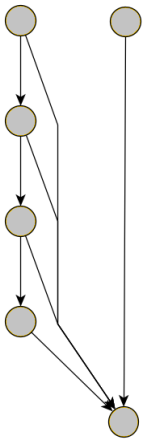


Figure 3.5: An example of cross edges' bundling.

You can see an example in figure 3.4.

We perform bundling for cross edges too. We bundle all incoming cross edges for each vertex(except those with one unit of vertical distance from the target). We can place the bundled cross edges between the paths/channels with the same technique we use for path transitive edges. That technique relies on task scheduling as we have already described. You can see an example of bundled cross edges in figure 3.5.

3.4 Evaluating the two frameworks

Our proposed framework has been evaluated positively with experimental results compared to the results obtained by running the hierarchical drawing module of OGDF. The compared module is based on the Sugiyama framework. In order to evaluate the performance, we used the following standard metrics:

- **Number of crossings.**
- **Number of bends.**
- **Width of the drawing:** The total number of distinct x coordinates that are used by the framework.
- **Height of the drawing:** The total number of distinct y coordinates that are used by the framework.
- **Area of the drawing:** The area of the enclosing rectangle.

Based on these metrics, we conducted a number of experiments to evaluate the performance of the two different hierarchical frameworks. To conduct the experiments, we used a laptop PC (Intel(R) Core(TM) i5-6200U CPU, 8 GB of main memory). Additionally, the metrics of PBF could vary depending on the path/channel decomposition algorithm we use and the ordering of the columns. We used the benchmarks we found at www.graphdrawing.org. The archive consists of graphs with 10 to 100 nodes with average degree about 1.6.

The results showed that our approach differs from the Sugiyama framework completely, since it examines vertically the graph. PBF performs bundling very efficiently and computes the optimal height of the graph in linear time. In most cases, the drawings based on PBF need less area than OGDF. On the other hand, OGDF puts a lot of effort into the crossing minimization step. Hence, OGDF generally has fewer crossings than PBF which does nothing for crossing minimization. All this effort OGDF put in crossing minimization affects overwhelmingly the run time. More specifically, by examining the corresponding run time curves as shown in Figures 3.9a, 3.9b, we observe that PBF grows linearly in contrast to ODFG where it's time complexity is cubic. This indeed is valid, since for example a graph with 80 nodes and an average degree of 5.6, PBF has an execution time

Nodes			Crossings	bends	Width	Height	Area
20	Graph1	PBF	12	14	9	14	126
		OGDF	5	22	18	14	252
	Graph2	PBF	16	17	8	15	120
		OGDF	5	22	18	15	270
	Graph3	PBF	17	14	11	13	143
		OGDF	4	21	19	13	247
	Graph4	PBF	17	14	8	12	96
		OGDF	3	24	16	12	192
	Graph5	PBF	16	17	10	15	150
		OGDF	4	22	14	15	210
50	Graph6	PBF	119	36	24	27	648
		OGDF	39	59	51	27	1377
	Graph7	PBF	103	33	19	25	475
		OGDF	40	55	40	25	1000
	Graph8	PBF	101	38	21	31	651
		OGDF	35	58	45	31	1395
	Graph9	PBF	135	36	21	23	483
		OGDF	39	58	48	23	1104
	Graph10	PBF	136	42	19	31	589
		OGDF	40	64	57	31	1767

Table 3.1: Comparing PBF and OGDF metrics on graphs of average degree 1.6.

of 2 ms while *OGDF* has 148 ms. Similar to that, a bigger graph with 400 nodes and an average degree of 5.6, *PBF* took 7 ms, while *OGDF* took over 16 seconds. Considering all the experimental results we concluded that the two frameworks focus on different aspects and as we initially assumed counting quantitative metrics could not lead to a concrete conclusion. This differentiation led us to conduct a user study in order to evaluate the readability and clarity of PBF comparing it with the Sugiyama framework.

3.4.1 User Study

Users. In total 72 Computer Science related volunteers (students and employees) participated. In order to have more accurate and sophisticated results, we requested only from advanced students to participate i.e., that were familiar with graph theory and graph drawing techniques. The evaluation started on August 6, 2021 and ended on November 6, 2021.

Training. We created a Google form and we invited all the candidate participants to fill it in. Initially, the users were asked to watch a short video used for training: the tutorial described shortly the two hierarchical drawing frameworks (the video is available at `https://youtu.be/BWHc2xO4jmI`).

Datasets. We experimented with a dataset of 3 graph categories with different number of nodes (20 nodes, 50 nodes and 100 nodes i.e. small, medium and

big graphs) with average degree around 1.6. Figure 3.3 shows the number of nodes and edges, respectively for all of the graphs of the 3 different graph categories.

Tasks. We asked the users to answer a set of questions for the PBF and the classical hierarchical drawing framework and carry out a sequence of basic tasks regarding the reachability. Similar to previous experiments (see, e.g., [19],[32],[61]), we decided to choose tasks involving graph reading which are easily understandable also to non-expert users. Thus, we considered the tasks as shown in the Table 3.2.

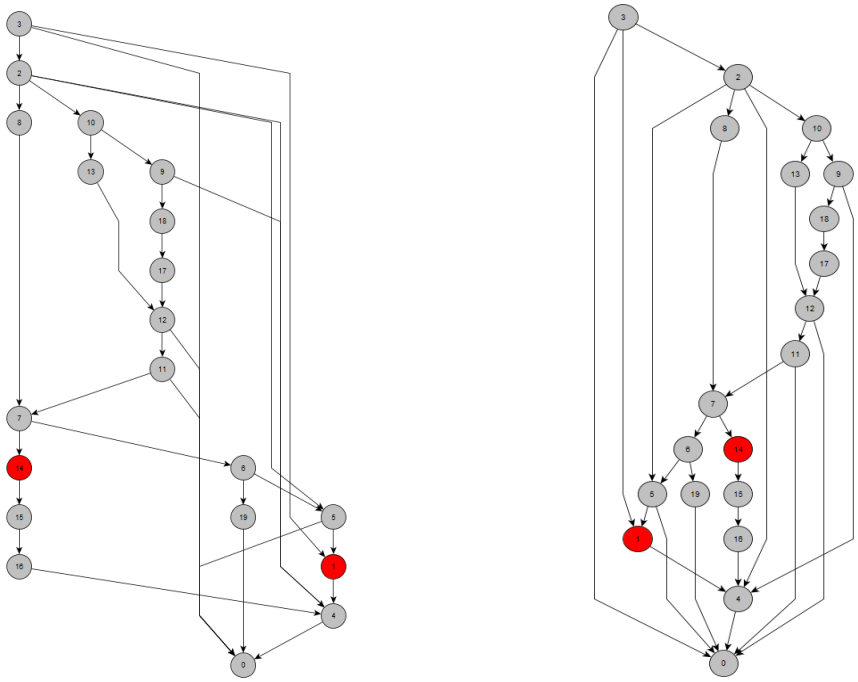
ID	*Task Description*
1	Is there a path between the two highlighted vertices?
2	How long is the shortest path between the two highlighted vertices?
3	Is there a path of length at most 3 that connects the two highlighted nodes?
4	Are all of the green vertices successors of the red vertex?

Table 3.2: The set of tasks participants had to answer for each of the 2 different graph drawing frameworks over various graphs.

For questions on task *"How long is the shortest path between the two highlighted vertices?"*, participants had to choose a number as an answer. We do not require numeric answers for all the tasks. More specific, *"Is there a path between the two highlighted vertices?"*, *"Is there a path of length at most 3 that connects the two highlighted nodes?"* and *"Are all of the green vertices successors of the red vertex?"* participants had to choose an answer among "Yes", "No", or "I do not know". There was no time limit, although participants were expected to answer "I do not know" if a question was too difficult to answer, or extremely time-consuming.

Each of the previous task was repeated for each drawing framework model 4 times: i.e., 2 using small size graphs and 2 using medium size graphs. Note that for each question of the same task we used different highlighted nodes. The only exception was on task 1 where we repeated 3 times (2 using the small graphs and only 1 using the medium size graphs). The reason for that was that the preliminaries results showed that even for a graph with 50 nodes it was difficult to easily answer such question due to the complex readability of the graph. In total, the number of tasks was 30 i.e., 15 questions for PBF and 15 for OGDF. The questions on small graphs (20 nodes) precede those on medium graphs (50 nodes). Finally, to contrast the learning effect, the questions appeared in a randomized order. Figure 3.6 shows a snapshot for the question "Is there a path between the two highlighted vertices?" for both of the 2 drawing frameworks.

Results. Table 3.7 shows the results of the corresponding answers. We recorded the total *number of correct answers* for each question of the 2 different drawing frameworks, for all participants. Also note that the "I do not know" answer was



(a) (b)

Figure 3.6: Snapshots of the drawings used for the user study for the question *"Is there a path between the two highlighted vertices?"*. In (a) we show the drawing computed by PBF and in (b) we show the drawing of the same graph as produced by OGDF.

Graph	**Number of nodes**	**Number of edges**	
Graph1	20	31	*small graphs*
Graph2	20	31	
Graph3	50	82	*medium graphs*
Graph4	50	79	
Graph5	100	163	*large graphs*
Graph6	100	169	

Table 3.3: Graphs dataset.

considered as an error. More specifically, regarding the first graph i.e., *graph*1 for all tasks, we have that both PBF and $OGDF$ have the same performance in terms of correct answers. By examining the rest of the results the average percentage revealed that for all the graphs the performance of both drawing frameworks is not significantly different. The same implies when comparing the average percentage for each task, where again we do not observe any noticeable difference. To further prove this statement, Table 3.8 shows the ratio of participants that answered *"I do not know"* over the number of wrong answers on the various tasks for each of the two drawing frameworks, over the different graphs where we observe that PBF is slightly better than $OGDF$ for all tasks.

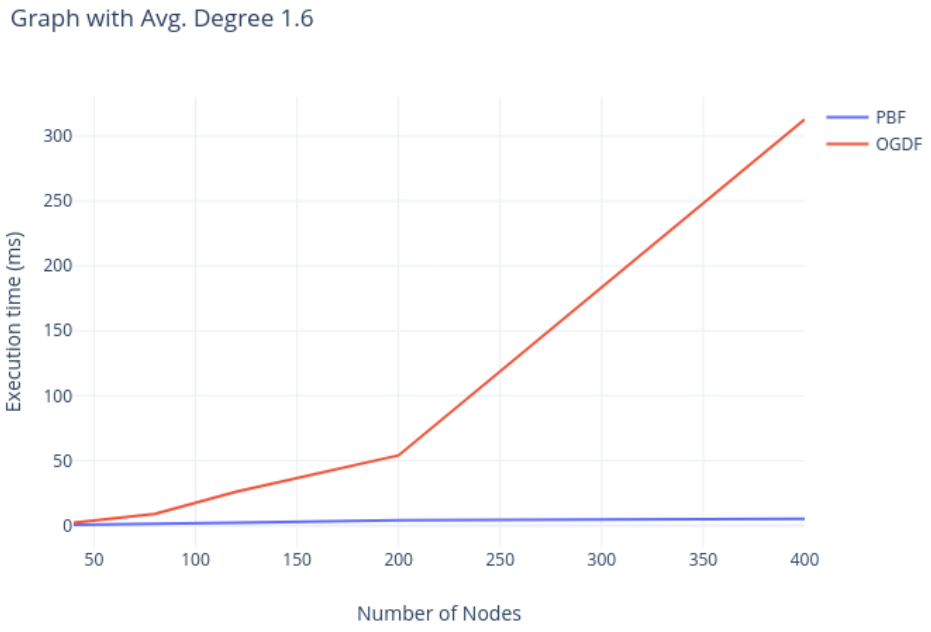
Graph		PBF						OGDF				
		Avg. per graph	Task1	Task2	Task3	Task4		Avg. per graph	Task1	Task2	Task3	Task4
1		95%	71/72	68/72	65/72	69/72		94%	69/72	68/72	67/72	68/72
2		94%	70/72	62/72	69/72	70/72		92%	69/72	70/72	58/72	67/72
3		92%	72/72	68/72	66/72	59/72		90%	69/72	65/72	69/72	55/72
4		92%	-	70/72	63/72	65/72		76%	-	61/72	49/72	55/72
Avg. per task			99%	93%	91%	91%		Avg. per task	96%	91%	84%	85%

Figure 3.7: Results *(number of correct answers)* on the various tasks for each of the drawing framework *(PBF)*, *(OGDF)* over different graphs.

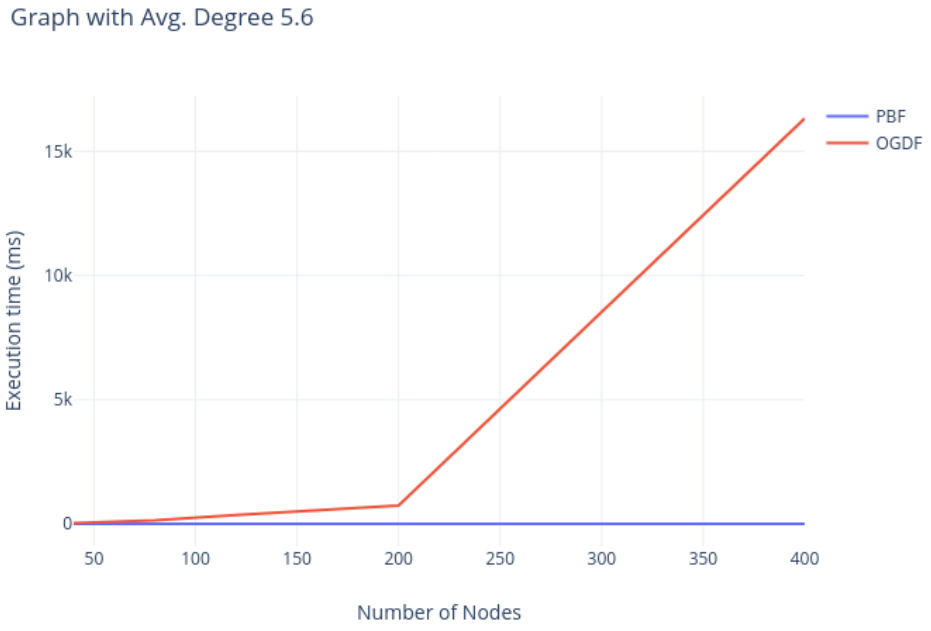
In general, the performance of the participants is slightly better in PBF than in OGDF but since the deviation is small, we cannot state a concrete conclusion.

Graph		PBF						OGDF				
		Avg. per graph	Task1	Task2	Task3	Task4		Avg. per graph	Task1	Task2	Task3	Task4
1		7%	0/1	0/4	0/7	1/3		12%	0/3	1/4	0/5	1/4
2		12%	0/2	1/10	1/3	0/2		8%	0/3	0/2	1/14	1/5
3		9%	-/0	0/4	0/6	2/13		47%	1/3	5/7	1/3	7/17
4		33%	-	2/2	1/9	3/7		24%	-	2/11	3/23	7/17
Avg. per task			0%	15%	8%	24%		Avg. per task	11%	33%	9%	37%

Figure 3.8: Results *(the ratio of participants that answered "I do not know" over the number of wrong answers)* on the various tasks for each of the drawing framework *(PBF)*, *(OGDF)* over different graphs.



(a)



(b)

Figure 3.9: Execution time of *PBF* and *OGDF* on various graphs with average degree of (a) 1.6 and (b) 5.6.

As a second-level of analysis, we perform a comparison of the 2 drawing frameworks. To this respect we asked from the participants to rate each of the 2 models by answering the following tasks:

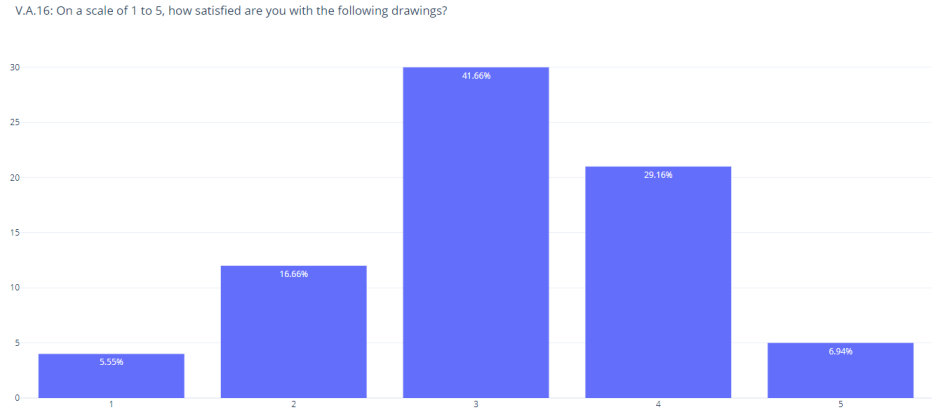
1. On a scale of 1 to 5, how satisfied are you with the following graph drawings?
2. Do you believe would be easy to answer the previous tasks for the following graph?
3. Which of the following drawings of the same graph do you prefer to use in order to answer the previous tasks?

For this scenario, we used the two graphs of 100 nodes (large graphs). Hence, similar to previous experiments, we had two PBF drawings and two OGDF drawings. Since the objective of this section was to evaluate the usability of both drawing frameworks, using the System Usability Score (SUS) [7], we asked the users to answer questions I and II, by giving a rate using the following scale *<Very Unsatisfied, Unsatisfied, Neutral, Satisfied, Very Satisfied>* and *<Strongly Disagree, Disagree, Neutral, Agree, Strongly Agree>* respectively, as shown in Figures 3.10, 3.11. The results show that for the question I, almost 80% of the participants rated PBF from scale 3 to 5, in contrast to 65% of $OGDF$. For the question II, almost 70% of the participants rated PBF from scale 3 to 5, in contrast to 60% of $OGDF$.

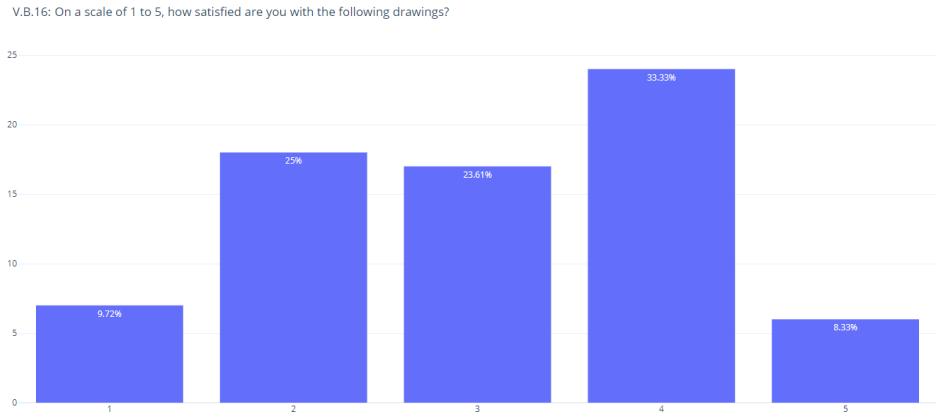
At the end of this evaluation, we perform a direct comparison of the 2 drawing frameworks by asking participants to answer the task c, as shown in Figure 3.12a. It is interesting to note that in overall rating, 58.3% of the participants stated that they prefer the drawing produced by PBF over the $OGDF$ (Figure 3.12b).

Conclusions and Open Problems We present algorithms and experimental results comparing two hierarchical drawing frameworks: (a) the path-based framework and (b) OGDF, which is based on the Sugiyama technique. Our compaction algorithm runs in linear time, and produces drawings with height equal to the length of a longest path of G instead of n−1 which is the height of drawings produced in [21]. In this implementation we present an algorithm to bundle and draw the path transitive edges of G in O(m + n log n) time, which is an extension of the original path based framework [56]. The experimental results show that the drawings produced by our algorithms have significantly lower number of bends and are much smaller in area than the ones produced by OGDF, but they have more crossings for sparse graphs. Thus our algorithms offer an interesting alternative when we visualize hierarchical graphs. They focus on showing important aspects of the graph such as critical paths, path transitive edges, and cross edges. For this reason, this framework is particularly useful in graph visualization systems that encourage user interaction. There are several interesting open problems:

1. Find better algorithms to order the paths.
2. Find techniques to reduce the number of crossings.

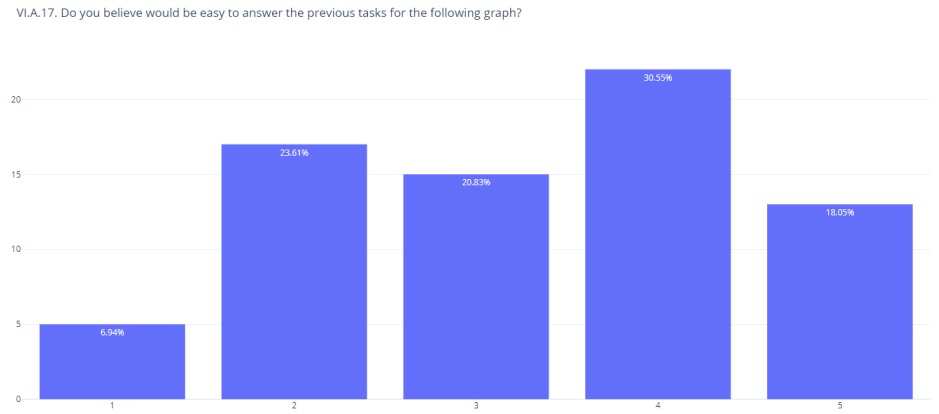


(a)

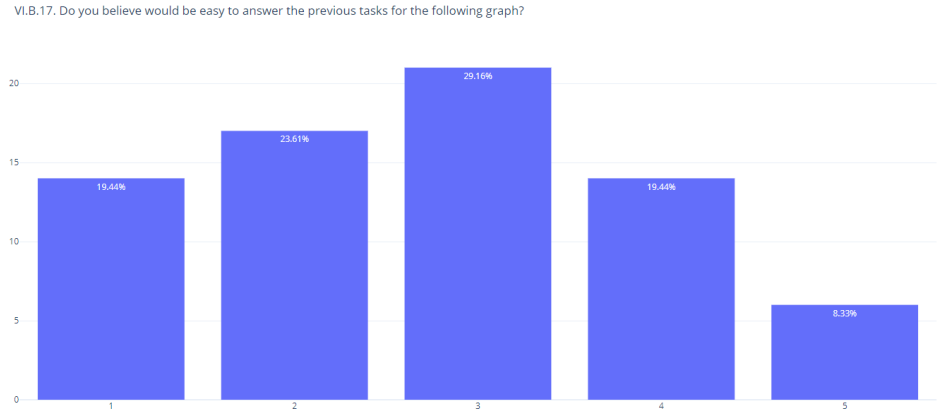


(b)

Figure 3.10: On a scale of 1 to 5, how satisfied are you with the following graph drawings?

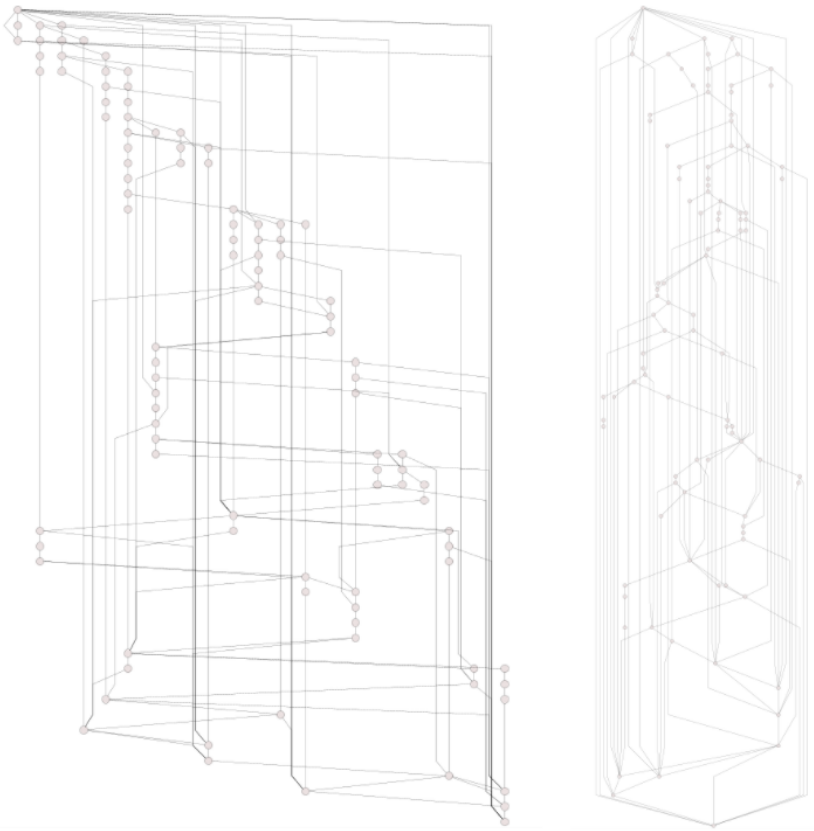


(a)

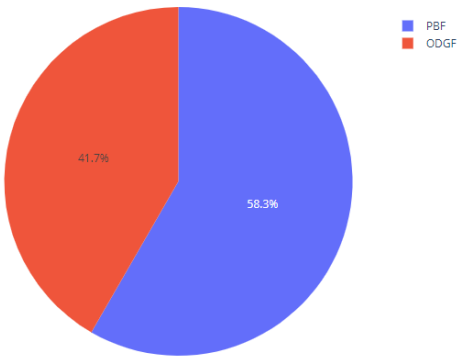


(b)

Figure 3.11: Do you believe would be easy to answer the previous tasks for the following graph?



(a)



(b)

Figure 3.12: In (a) we show snapshots of the same graph as used in our survey. Figure 1 is the drawing as computed by PBF and Figure 2 is the drawing as produced by $OGDF$. In (b) we see the percentage results for the task *"Which of the following drawings of the same graph do you prefer to use in order to answer the previous tasks"*.

3. Allow some extra vertical space between selected vertices in order to make the visualization more visually appealing.

3.5 Conclusions and Open Problems

In this work, we describe a general-purpose hierarchical graph drawing framework based on path/channel decomposition. Our framework naturally splits all the edges into three categories. The path edges, path transitive edges, and cross edges. In addition to [57], we draw all edges of G and apply edge bundling in $O(m + n \log n)$ time. We also minimize the height using a compaction technique and reduce the width by applying algorithms similar to task scheduling. Our compaction algorithm runs in $O(n + m)$ time and produces drawings with height equal to the length of a longest path of G.

We evaluate both the performance and the usability of this new framework compared to $OGDF$ which follows the Sugiyama framework. The experimental performance results show that the two frameworks differ considerably. Generally, the drawings produced by our algorithms have lower number of bends and are much smaller in area than the ones produced by OGDF, but they have more crossings for sparse graphs. Thus our proposed frameworks offer an interesting alternative when we visualize hierarchical graphs. They focus on showing important aspects of the graph such as critical paths, path transitive edges, and cross edges. For this reason, this framework is particularly useful in graph visualization systems that encourage user interaction. Moreover, the task based user evaluation, show that the performance of the participants is slightly better in PBF than in OGDF and the participants prefer PBF in overall rating compared to OGDF.

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