$$\mathcal{L}(q) = \int Q(\theta) \ln \frac{p(t,\theta)}{Q(\theta)} d\theta \tag{1}$$

$$= \left\langle \ln \frac{p(t,\theta)}{Q(\theta)} \right\rangle_Q \tag{2}$$

$$= \underbrace{\langle \ln p(t,\theta) \rangle_Q}_{(1)} - \underbrace{\langle \ln q(\theta) \rangle_Q}_{(2)} \tag{3}$$

$$\langle \ln p(\boldsymbol{t},\theta) \rangle_Q = \left\langle \ln p(\boldsymbol{t}, \boldsymbol{X}, \boldsymbol{m}, \boldsymbol{W}, z, \sigma^{-2}) \right\rangle_Q \tag{4}$$

$$= \left\langle \ln p(\boldsymbol{t}|\boldsymbol{X}, \boldsymbol{m}, \boldsymbol{W}, z, \sigma^{-2}) p(\boldsymbol{X}) p(\boldsymbol{m}) p(\boldsymbol{W}|z) p(z) p(\sigma^{-2}) \right\rangle_Q \tag{5}$$

$$= \underbrace{\left\langle \ln p(\boldsymbol{t}|\boldsymbol{X}, \boldsymbol{m}, \boldsymbol{W}, \sigma^{-2}, z) \right\rangle_Q}_{(3)} + \underbrace{\langle \ln p(\boldsymbol{X}) \rangle_Q}_{(4)} + \underbrace{\langle \ln p(\boldsymbol{m}) \rangle_Q}_{(5)} + \tag{6}$$

$$\underbrace{\langle \ln p(\boldsymbol{W}|z) \rangle_Q}_{(6)} + \underbrace{\langle \ln p(z) \rangle_Q}_{(8)} + \underbrace{\left\langle \ln p(\sigma^{-2}) \right\rangle_Q}_{(7)}$$

$$\langle \ln p(\boldsymbol{t}|\boldsymbol{\theta}) \rangle_Q = \left\langle \ln \prod_{n=1}^{N} p(t_n|\boldsymbol{X}, \boldsymbol{m}, \boldsymbol{W}, \sigma^{-2}, z) \right\rangle_Q \tag{7}$$

$$= \left\langle \ln \prod_{n=1}^{N} \mathcal{N}\left(t_n \mid \boldsymbol{W}z + \boldsymbol{m}, \sigma^2 \boldsymbol{I}\right) \right\rangle_Q \tag{8}$$

$$= \left\langle \sum_{n=1}^{N} \ln \mathcal{N}\left(t_n \mid \boldsymbol{W}z + \boldsymbol{m}, \sigma^2 \boldsymbol{I}\right) \right\rangle_Q \tag{9}$$

$$= -\frac{Nd}{2} \ln 2\pi + \frac{dN}{2} \left\langle \ln \sigma^{-2} \right\rangle_Q - \left\langle \sigma^{-2} \right\rangle_Q \left(\frac{1}{2} \sum_{n=1}^{N} \gamma_n \right) \tag{10}$$

$$= \frac{dN}{2} (\psi(\tilde{c}_{\sigma^{-2}}) - \ln(\tilde{d}_{\sigma^{-2}})) - \left\langle \sigma^{-2} \right\rangle_Q \left(\tilde{d}_{\sigma^{-2}} - b \right) + const \tag{11}$$

$$= -\frac{dN}{2} \ln(\tilde{d}_{\sigma^{-2}}) - \left\langle \sigma^{-2} \right\rangle_Q \left(\tilde{d}_{\sigma^{-2}} - b \right) + const \tag{12}$$

$$\frac{1}{2} \sum_{n=1}^{N} \gamma_n = \frac{1}{2} \sum_{n=1}^{N} \langle (t_n - \boldsymbol{W}x_n - \boldsymbol{m})^T (t_n - \boldsymbol{W}x_n - \boldsymbol{m}) \rangle_Q \tag{13}$$

$$= \frac{1}{2} \sum_{n=1}^{N} \langle \|t_n\|^2 + \|\boldsymbol{m}\|^2 + x_n^T W^T W x_n \tag{14}$$

$$- 2t_n^T \boldsymbol{W}x_n - 2t_n^T \boldsymbol{m} + 2\boldsymbol{m}^T \boldsymbol{W}x_n \rangle_Q$$

$$= \frac{1}{2} \sum_{n=1}^{N} \|t_n\|^2 + \left\langle \|\boldsymbol{m}\|^2 \right\rangle + Tr\left(\left\langle W^T W \right\rangle \left\langle x_n x_n^T \right\rangle \right) \tag{15}$$

$$- 2t_n^T \left\langle \boldsymbol{W} \right\rangle \left\langle x_n \right\rangle - 2t_n^T \left\langle \boldsymbol{m} \right\rangle + 2 \left\langle \boldsymbol{m}^T \right\rangle \left\langle \boldsymbol{W} \right\rangle \left\langle x_n \right\rangle \tag{16}$$

$$= \tilde{d}_{\sigma^{-2}} - b \tag{17}$$

$$\langle \ln p(\boldsymbol{X}) \rangle_Q = \left\langle \ln \prod_{n=1}^{N} N(x_n|\boldsymbol{0}, \boldsymbol{I}_q) \right\rangle_Q \tag{18}$$

$$= \left\langle \sum_{n=1}^{N} \ln N(x_n|\boldsymbol{0}, \boldsymbol{I}_q) \right\rangle_Q \tag{19}$$

$$= -\frac{Nq}{2} \ln 2\pi - \frac{1}{2} \sum_{n=1}^{N} \left\langle \|x_n\|^2 \right\rangle_Q \tag{20}$$

$$= -\frac{1}{2} \sum_{n=1}^{N} \left\langle \|x_n\|^2 \right\rangle_Q + const \tag{21}$$

$$\langle \ln p(\boldsymbol{m}) \rangle_Q = \left\langle \ln N(\boldsymbol{m}|\boldsymbol{0}, \beta^{-1}\boldsymbol{I}_d) \right\rangle_Q \tag{22}$$

$$= \left\langle -\frac{d}{2} \ln 2\pi - \frac{1}{2} \ln \left|\beta^{-1}\boldsymbol{I}_d\right| - \frac{1}{2}\boldsymbol{m}^T \beta I_d \boldsymbol{\mu} \right\rangle_Q \tag{23}$$

$$= -\frac{\beta}{2} \left\langle \|\boldsymbol{m}\|^2 \right\rangle_Q + const \tag{24}$$

$$\tag{25}$$

$$\langle \ln p(\boldsymbol{W}|z) \rangle_Q = \left\langle \ln \prod_{i=1}^{d} \prod_{j=1}^{q} \frac{1}{\sqrt{2\pi z_{i,j}}} \exp\left(-\frac{w_{i,j}^2}{2 z_{i,j}}\right) \right\rangle_Q \tag{26}$$

$$= -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{q} \langle \ln z_{i,j} \rangle_Q + \frac{\left\langle w_{i,j}^2 \right\rangle_Q}{\langle z_{i,j} \rangle_Q} + const \tag{27}$$

$$= -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{q} \langle \ln z_{i,j} \rangle_Q + \frac{\left\langle w_{i,j}^2 \right\rangle_Q}{\langle z_{i,j} \rangle_Q} + const \tag{28}$$

$$\langle \ln p(z) \rangle_Q = \left\langle \ln \prod_{i=1}^{d} \prod_{j=1}^{q} \frac{1}{z_{i,j}} \right\rangle_Q \tag{29}$$

$$= -\sum_{i=1}^{d} \sum_{j=1}^{q} \langle \ln z_{i,j} \rangle_Q \tag{30}$$

$$\left\langle \ln p(\sigma^{-2}) \right\rangle_Q = \left\langle \ln \Gamma(\sigma^{-2}|c, d) \right\rangle_Q \tag{31}$$

$$= \psi(\tilde{c}_{\sigma^{-2}}) - \ln(\tilde{d}_{\sigma^{-2}}) \tag{32}$$

$$= -\ln(\tilde{d}_{\sigma^{-2}}) + const \tag{33}$$

$$\langle \ln Q(\theta) \rangle_Q = \left\langle \ln \prod_i Q_i(\theta_i) \right\rangle_Q \tag{34}$$

$$= \left\langle \sum_i \ln Q_i(\theta_i) \right\rangle_Q \tag{35}$$

$$= \underbrace{\langle \ln Q(\boldsymbol{X}) \rangle_Q}_{(9)} + \underbrace{\langle \ln Q(\boldsymbol{m}) \rangle_Q}_{(10)} + \underbrace{\langle \ln Q(\boldsymbol{W}) \rangle_Q}_{(11)} + \tag{36}$$

$$\underbrace{\langle \ln Q(z_{i,j}) \rangle_Q}_{(12)} + \underbrace{\langle \ln Q(\sigma^{-2}) \rangle_Q}_{(13)}$$

$$-\langle \ln Q(\boldsymbol{X}) \rangle_Q = -\left\langle \ln \prod_{n=1}^{N} \mathcal{N}(\boldsymbol{x}_n | \mu_x, \Sigma_x) \right\rangle_Q \tag{37}$$

$$= \sum_{n=1}^{N} -\langle \ln \mathcal{N}(\boldsymbol{x}_n | \mu_x, \Sigma_x) \rangle_Q \tag{38}$$

$$= \sum_{n=1}^{N} \frac{q}{2}(1 + \ln(2\pi)) + \frac{1}{2} \ln |\Sigma_x| \tag{39}$$

$$= \frac{Nq}{2}(1 + \ln(2\pi)) + \frac{N}{2} \ln |\Sigma_x| \tag{40}$$

$$= \frac{N}{2} \ln |\Sigma_x| + const \tag{41}$$

$$-\langle \ln Q(\boldsymbol{m}) \rangle_Q = -\langle \ln \mathcal{N}(\mu_m, \Sigma_m) \rangle_Q \tag{42}$$

$$= \frac{q}{2}(1 + \ln(2\pi)) + \frac{1}{2} \ln |\Sigma_m| \tag{43}$$

$$= \frac{1}{2} \ln |\Sigma_m| + const \tag{44}$$

$$-\langle \ln Q(\boldsymbol{W}) \rangle_Q = -\left\langle \ln \prod_{i=1}^{d} \mathcal{N}(\boldsymbol{W}_i | \mu_{\boldsymbol{W}_i}, \Sigma_{\boldsymbol{W}_i}) \right\rangle_Q \tag{45}$$

$$= \sum_{i=1}^{d} -\langle \ln \mathcal{N}(\boldsymbol{W}_i | \mu_{\boldsymbol{W}_i}, \Sigma_{\boldsymbol{W}_i}) \rangle_Q \tag{46}$$

$$= \frac{dq}{2}(1 + \ln(2\pi)) + \sum_{i=1}^{d} \frac{1}{2} \ln |\Sigma_{\boldsymbol{W}_i}| \tag{47}$$

$$= \frac{1}{2} \sum_{i=1}^{d} \ln |\Sigma_{\boldsymbol{W}_i}| + const \tag{48}$$

$$
\begin{aligned}
-\left\langle \ln Q(\sigma^{-2}) \right\rangle_Q =& -\left\langle \ln \Gamma(\sigma^{-2}|c,d) \right\rangle_Q && (49)\\
=& c - \ln d + \ln \Gamma(c) + (1-c)\psi(c) && (50)\\
=& -\ln d + const && (51)
\end{aligned}
$$

$$
\begin{aligned}
\ln Q^*(z_{i,j}) \propto& \mathbb{E}_{\boldsymbol{X},\boldsymbol{W},\boldsymbol{m},\sigma^{-2}}\left[\ln p(\boldsymbol{t},\boldsymbol{X},\boldsymbol{m},\boldsymbol{W},z,\sigma^{-2})\right] && (52)\\
=& \mathbb{E}_W\left[\ln p(W|z)\right] + \ln p(z_{i,j}) + const && (53)\\
=& \mathbb{E}_W\left[-\frac{1}{2}\ln 2\pi z_{i,j} - \frac{w_{i,j}^2}{2z_{i,j}}\right] + \ln\frac{1}{z_{i,j}} + const && (54)\\
=& -\frac{\mathbb{E}_W\left[w_{i,j}^2\right]}{2z_{i,j}} - \frac{1}{2}\ln 2\pi z_{i,j} + \ln\frac{1}{z_{i,j}} + const && (55)\\
& && (56)
\end{aligned}
$$

$$
\begin{aligned}
Q(z_{i,j}) =& \frac{\exp(\ln Q^*(z_{i,j}))}{\int \exp(\ln Q^*(z_{i,j}))\, dz_{i,j}} + const && (57)\\
=& \frac{\frac{1}{z_{i,j}\sqrt{2\pi z_{i,j}}}\exp\left(-\frac{\mathbb{E}_W\left[w_{i,j}^2\right]}{2z_{i,j}}\right)}{\int \frac{1}{z_{i,j}\sqrt{2\pi z_{i,j}}}\exp\left(-\frac{\mathbb{E}_W\left[w_{i,j}^2\right]}{2z_{i,j}}\right)\, dz_{i,j}} + const && (58)\\
=& \frac{\frac{\left(\mathbb{E}_W\left[w_{i,j}^2\right]/2\right)^{1/2}}{\sqrt{\pi}} z_{i,j}^{-1/2-1}\exp\left(-\frac{\mathbb{E}_W\left[w_{i,j}^2\right]/2}{z_{i,j}}\right)}{\int \frac{\left(\mathbb{E}_W\left[w_{i,j}^2\right]/2\right)^{1/2}}{\sqrt{\pi}} z_{i,j}^{-1/2-1}\exp\left(-\frac{\mathbb{E}_W\left[w_{i,j}^2\right]/2}{z_{i,j}}\right)\, dz_{i,j}} + const && (59)\\
=& \frac{\text{Inv-Gamma}(z_{i,j}|1/2, \mathbb{E}_W\left[w_{i,j}^2\right]/2)}{\int \text{Inv-Gamma}(z_{i,j}|1/2, \mathbb{E}_W\left[w_{i,j}^2\right]/2) dz_{i,j}} && (60)\\
=& \text{Inv-Gamma}(z_{i,j}|1/2, \mathbb{E}_W\left[w_{i,j}^2\right]/2) && (61)
\end{aligned}
$$

$$
\begin{aligned}
-\left\langle \ln Q(z_{i,j}) \right\rangle_Q =& -\left\langle \text{Inv-Gamma}(z_{i,j}|1/2, \left\langle w_{i,j}^2 \right\rangle_Q/2) \right\rangle_Q && (62)\\
=& 1/2 + \ln\frac{\left\langle w_{i,j}^2 \right\rangle_Q}{2}\sqrt{\pi} - (1+1/2)\psi(1/2) && (63)\\
=& \ln\left\langle w_{i,j}^2 \right\rangle_Q + const && (64)
\end{aligned}
$$