Inverse Problems
Assignment 1

Georgios Sevastakis

April 2024

The direct problem

1

In this subsection, our aim is to compute the arrival–time anomalies t_γ directly. To calculate the arrival–time anomalies for each of the 12 detectors, for each of the sources, we need to solve the direct problem by making use of the following integral

$$t_\gamma = \int_\gamma s(u)\, du \tag{1}$$

where t_γ denotes the arrival-time anomaly, $s(u)$ the slowness anomaly (the anomaly of the reciprocal velocity) and u is a parameter determining the point we consider on the ray γ.

The integral of equation 1 can be simplified by the following statements that hold true for our problem:

- For each ray γ there are two possible values s for the point we consider; in the absence of the medium $s(u) = 0$, whereas in its presence, $s(u) = 1/v_{object} - 1/v_0$, where v_{object} and v_0 denote the propagation velocity inside and outside the medium respectively.
- $s(u)$ is constant (a step function of two constants to be precise, since the medium is homogeneous) and does not depend on other slowness values of other squares. It can, therefore, be placed outside the integral of equation 1 (linear problem).
- The integral $\int_\gamma du$ will be nothing more that the distance d_γ that the ray γ has traveled inside the medium (since $s(u) = 0$ outside the medium.

Therefore, for each ray γ the arrival–time anomaly can be calculated as follows

$$t_\gamma = \begin{cases} \left(\frac{1}{v_{object}} - \frac{1}{v_0}\right) \cdot d_\gamma & \text{, for rays that cross the medium.} \\ 0 & \text{, for rays that do not cross the medium.} \end{cases} \tag{2}$$

The results are gathered in Table 1.

2

Now, we discretize equation 1 by subdividing the area in the in 11×13 squares of $1\,m$ by $1\,m$.

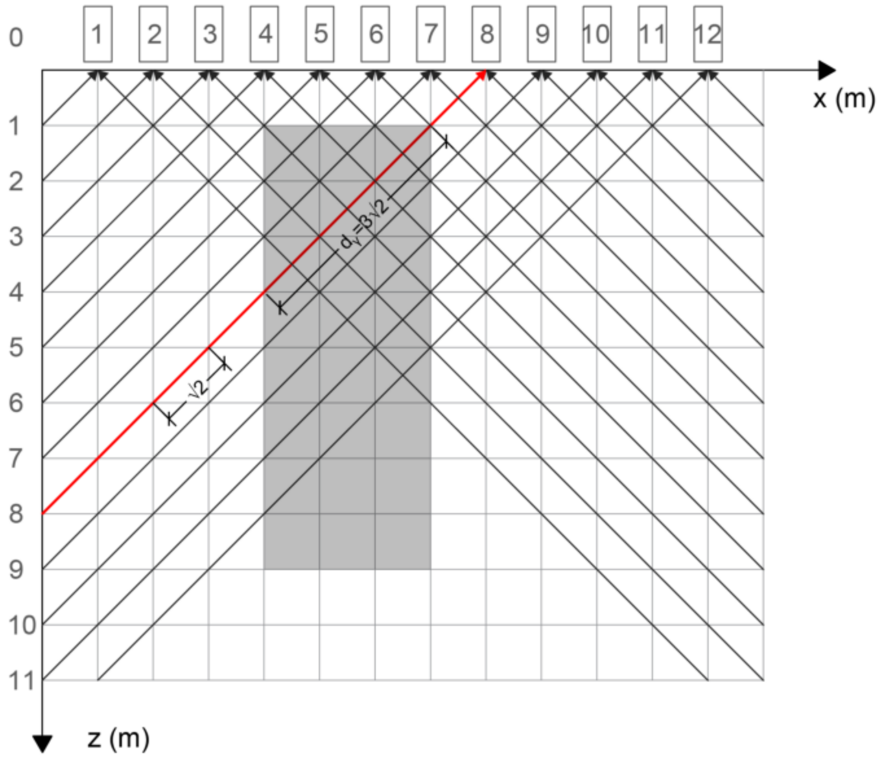

Figure 1: Discretized version of the geometry of the problem; The area is subdivided in squares of $1 \times 1\,m^2$. Notice that since the angle of incidence of the rays is 45^o, each ray covers a distance of $\sqrt{2}\,m$ per square.

By discretizing, we can simplify equation 1 even further

- Since the rays are propagating, forming a 45^o angle (or 135^o depending on the direction) with the x–axis, each ray covers a distance of $\sqrt{2}\,m$ per square. This means that d_γ is always a multiple of $\sqrt{2}$.

Therefore, equation 1 can be expressed as follows

$$t_\gamma = \begin{cases} \left(\frac{1}{v_{object}} - \frac{1}{v_0}\right) \cdot N_\gamma \sqrt{2} & \text{, for rays that cross the medium.} \\ 0 & \text{, for rays that do not cross the medium.} \end{cases} \tag{3}$$

where N_γ denotes the number of squares that a ray γ transverses and that accommodate the medium.

Equation 3 can be written in the following form

$$\underline{d} = \underline{\underline{G}} \cdot \underline{m} \tag{4}$$

where here, G denotes the distance to be covered by a ray in each square and m denotes the slowness (matrix).

By looking at Figure 1, for example, the ray that is emitted from the left and lands on Detector No. 8 experiences a t_γ anomaly with $d_\gamma = 3\sqrt{2}$, since it passes through 3 squares that accommodate the medium.

Since v_0 and v_{object} are known and equal to $5\,m/s$ and $5.2\,m/s$ respectively, we ultimately computed the arrival–time anomalies t_γ by making use of equations 3 and 4. The results are listed in the Table below rounded to the 3rd decimal point.

Detector ID	$t_\gamma\,(s)$	
	Left	Right
1	0.000	-0.033
2	0.000	-0.033
3	0.000	-0.033
4	0.000	-0.022
5	0.000	-0.011
6	-0.011	0.000
7	-0.022	0.000
8	-0.033	0.000
9	-0.033	0.000
10	-0.033	0.000
11	-0.033	0.000
12	-0.033	0.000

Table 1: Computed arrival–time anomalies for each of the 12 detectors and for each of the sources. The minus sign is the byproduct of the 'speed up' of the ray inside the medium (greater propagation velocity inside).

The inverse problem

3

Let's recall Equation 4 and let's formulate the inverse problem

- Having computed the arrival–time anomalies t_γ, we use them now as our data by adding some noise to make them more realistic. The noise added to each t_γ is a number sampled from a Gaussian distribution with $\mu = 0$ and $\sigma = \frac{||t_\gamma||_2}{18\sqrt{N}}$, where N denoted the number of data/measurements, namely the number of t_γ values.
- The G matrix is a matrix that contains the distances that each ray has to cover in each square as mentioned before. To be more specific, G is a 24×143 matrix, where the number of rows represents all rays coming from the left and right and the columns are our 11×13 2D grid vectorized. Therefore, each row contains 0's and $\sqrt{2}$'s depending on if the ray (row) passes from that square (column).
- The slowness matrix m is our unknown and is a 1D vector of size 143 (our 2D grid). It contains all slowness values of the 2D grid.

Therefore, we are after the slowness matrix m which can be found by solving the following equation for m

$$\underline{t_\gamma} = \underline{\underline{G}} \cdot \underline{m} \tag{5}$$

which is of the form

$$\underline{d} = \underline{\underline{G}} \cdot \underline{m} \tag{6}$$

4

The problem is linear, since each square on the grid and its respective slowness value are independent of the other squares and their slowness values. Therefore, the observed data can be expressed as a linear combination of the parameters.

For two sets of slowness values (m_1 and m_2), for which $d_1 = G \cdot m_1$ and $d_2 = G \cdot m_2$ we have the following

$$\underline{d} = \underline{d_1} + \underline{d_2} = \underline{\underline{G}} \cdot \underline{m_1} + \underline{\underline{G}} \cdot \underline{m_2} = \underline{\underline{G}} \cdot (\underline{m_1} + \underline{m_2}) \tag{7}$$

$$\alpha \cdot \underline{d} = \alpha \cdot \underline{\underline{G}} \cdot m = \underline{\underline{G}}(\alpha \cdot \underline{m}) \tag{8}$$

Due to the properties above (Equations 7 and 8), the direct and therefore, the inverse problem are linear.

5

It is evident from the size of the matrix G and the presence of the noise that the solution is not unique. In this case, the system is mixed–determined, since the size of G is 24×143 and noise is present (rendering it ultimately mixed–determined). Therefore, the solution to the problem is not be unique.

6

Since the problem is linear but mixed–determined and gaussian noise is present, Tikhonov regularization is employed in order to mitigate their effects by offering increased stability and reliability to the solution. Tikhonov inversion seeks to minimize the so called (modified) misfit function defined below

$$E(\underline{m}) = ||\underline{t_\gamma} - \underline{\underline{G}} \cdot \underline{m}||^2 + \varepsilon^2 ||\underline{m}||^2 \tag{9}$$

The regularization parameter ε is chosen such that the N data are barely fitted within their uncertainty, so the equation below can be written as follows

$$E(\underline{m}) = ||\underline{t_\gamma} - \underline{\underline{G}} \cdot \underline{m}||^2 - N\sigma^2 \tag{10}$$

where N is the number of data and σ is the standard deviation of the noise.

For suitable small ε determined by the minimization of equation 10 leads to the following approximate formula for the least squares estimate

$$\underline{m} = (\underline{\underline{G}}^T \cdot \underline{\underline{G}} + \varepsilon^2 \cdot \underline{\underline{I}})^{-1} \underline{\underline{G}}^T \cdot \underline{t_\gamma} \tag{11}$$

From equations 10 and 11, we were able to find the optimal parameter ε by computing the modified misfit for different ε values.

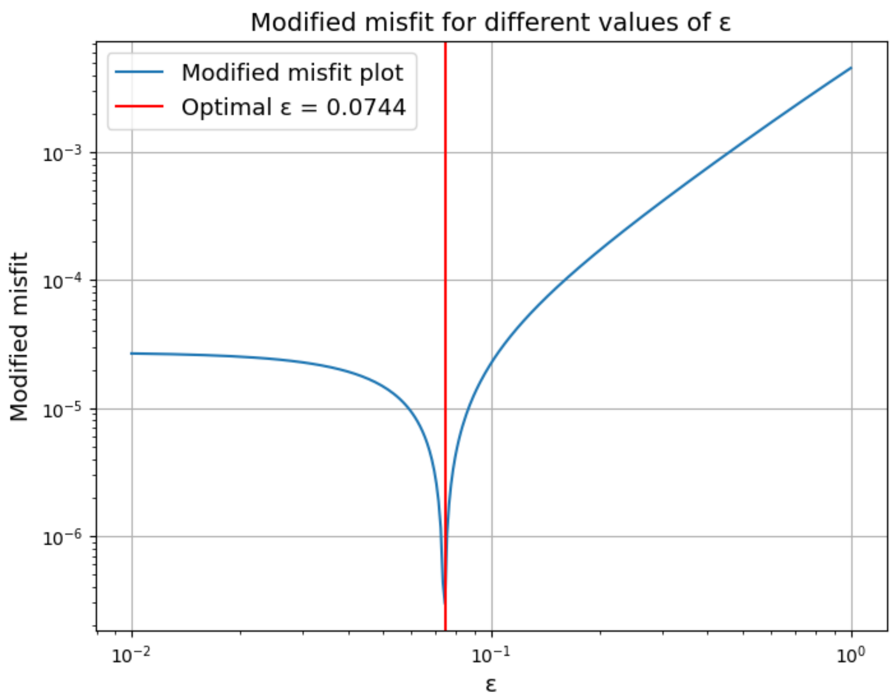

Figure 2: Determining the optimal value for the parameter ε by computing the modified misfit for different values of ε.

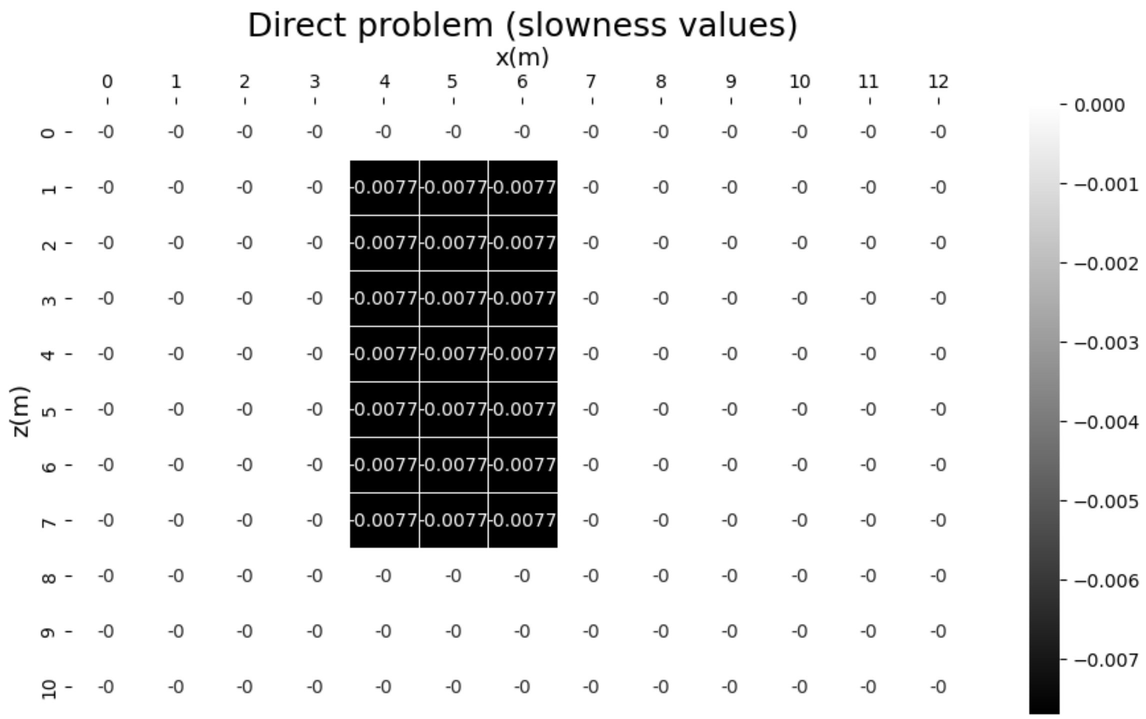

Figure 3: In the direct case, we can compute the slowness values (rounded to the $4th$ decimal point) analytically from equation 3, since the propagation velocities inside and outside the medium as well as the location of the object are known. The geometry of the problem as well as the slowness values are depicted in this Figure.

Having found the optimal ε value, we proceed by plugging it in equation 11 to solve the inverse problem.

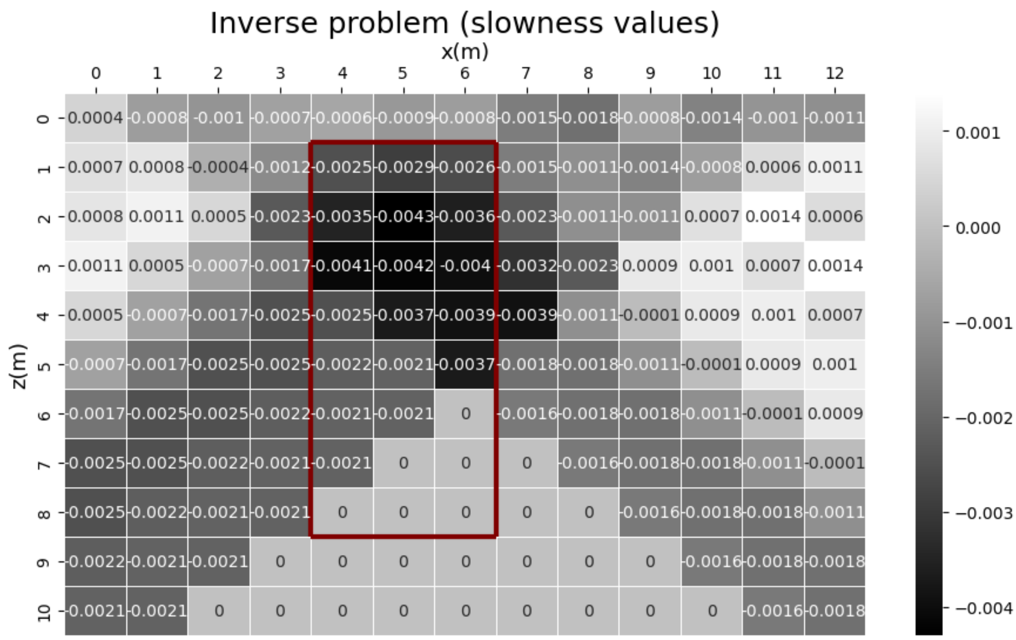

Figure 4: In the inverse case, the slowness values (rounded to the $4th$ decimal point) were computed by employing the Tikhonov regularization, since only the arrival–time anomalies were known (the location of the object as well as the propagation velocities were considered unknown. The geometry of the solution as well as the slowness values computed are depicted in this Figure. The red area represents the actual location of the object we seek to find.

7

Before discussing Figures 3 and 4, we solve the same inverse problem, but for an object that is only one square in size (delta function) and present the results.

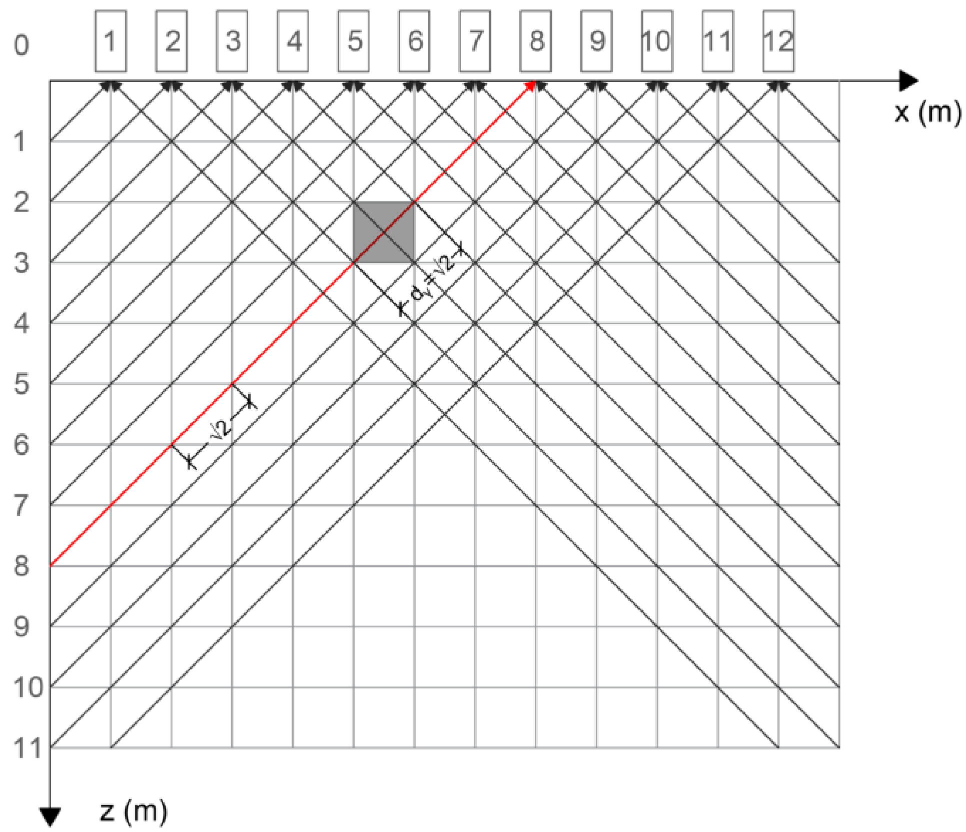

Figure 5: Discretized version of the geometry of the problem (one square); The area is subdivided in squares of $1 \times 1\,m^2$. Each ray covers a distance of $\sqrt{2}\,m$ per square.

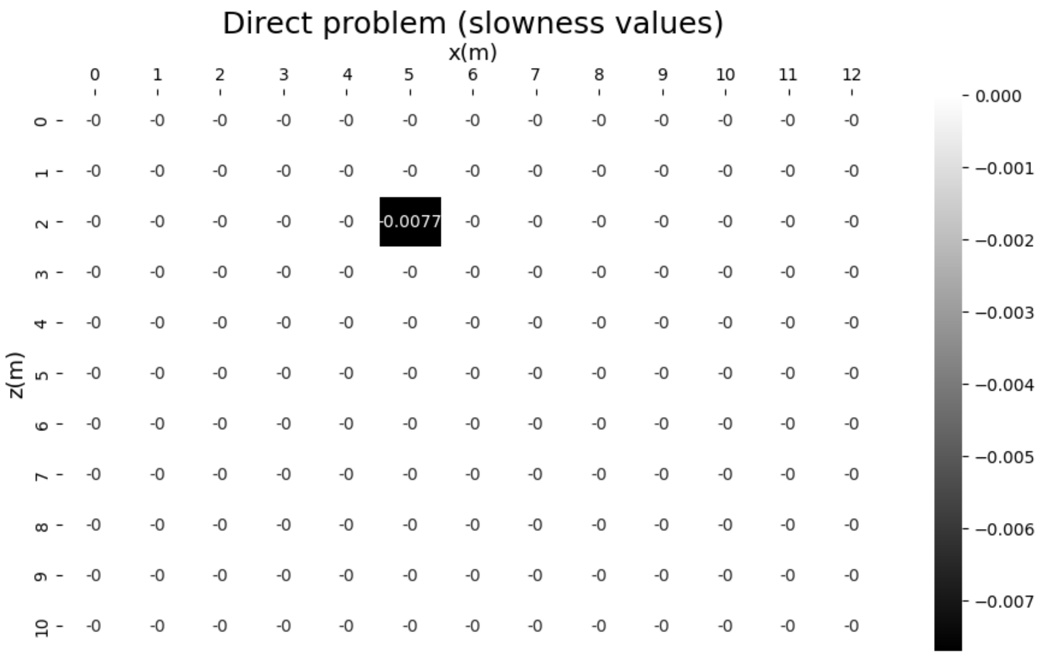

Figure 6: The geometry of the problem as well as the slowness values (rounded to the $4th$ decimal point) for the delta function case are depicted in this Figure.

To solve the inverse problem, we compute the optimal value for the parameter ε as we did in the previous question section

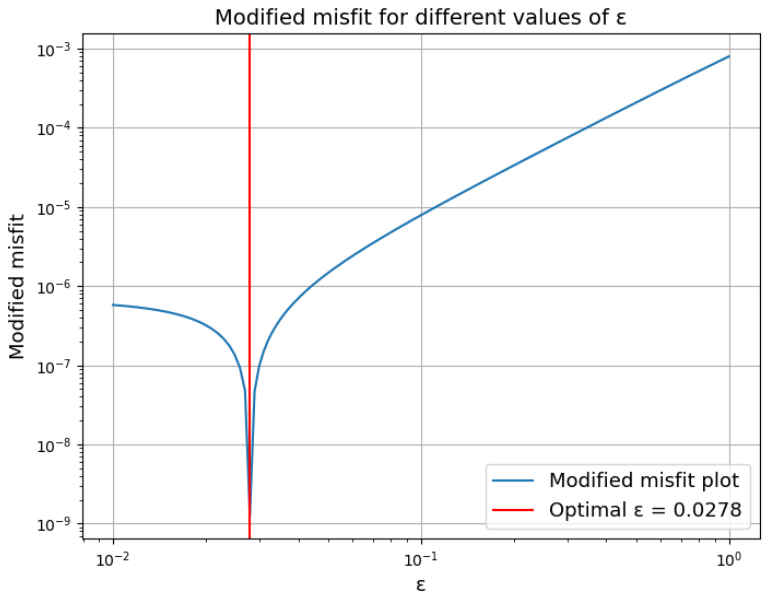

Figure 7: Determining the optimal value for the parameter ε for the delta function case by computing the modified misfit for different values of ε.

Having computed the optimal ε value, we present the resulting image

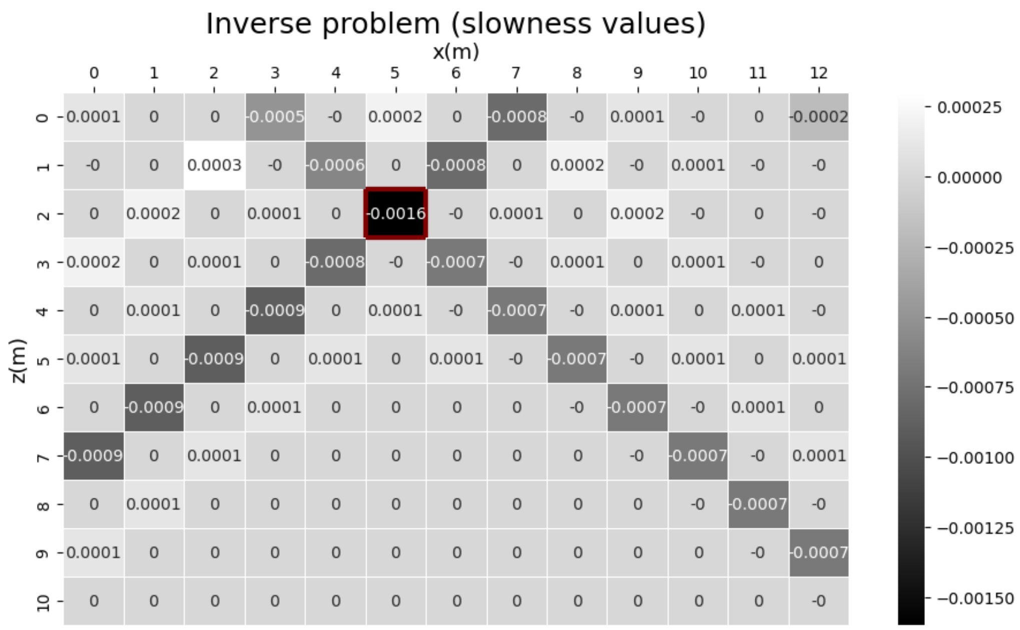

Figure 8: The geometry of the solution as well as the slowness values (rounded to the $4th$ decimal point) computed are depicted in this Figure. The red area represents the actual location of the object we seek to find.

By taking a look at Figure 8, we notice, that the inverse operator is able to detect the object as the square with the highest (in absolute value) slowness value. However, we see that along some of the ray paths we get non zero slowness values, especially along the ray paths that pass through the object, rendering it difficult to accurately determine the position of the object. We notice, also, that the slowness values of each ray (the sum of all values along a ray path) are almost equal to the slowness values of the direct problem. The difference lies in the presence of the noise.

8

Ultimately, the solution to the inverse problem as seen in Figure 4 captures adequately the medium in the true model (Figure 3). Nonetheless, the Tikhonov regularization did not accurately locate the medium by introducing artifacts along the ray paths, especially for those that passed through the medium; instead of accumulating the slowness values in the squares in the medium, it scattered them along the ray paths. This resulted in introducing slowness values on ray paths that do not cross the medium so as to keep the total slowness value zero for the detectors that did not detect any anomaly or equal to 0.0077 for those that did pass through the object. This phenomenon is clear in Figure 8. To mitigate the scattering effect, the spatial resolution needs to be enhanced and/or more detectors need to be added (notice that the bottom right part of the object is not detected due to the lack of detectors). Also, more angles of detection could be added.

Last but not least, the presence of the noise, also resulted in the creation of artifacts. For example, the slowness values at the top right and top left corners of Figure 8 are pure noise.