Higher Dimensional Topological Order
Higher Category and A Classification in 3+1D

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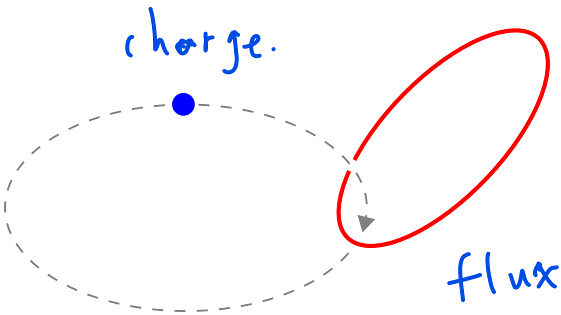
PRX 8, 021074 (2018), arXiv:1704.04221; PRX 9, 021005 (2019), arXiv:1801.08530.

Working Definitions

- Physical definition: topological orders are gapped quantum liquid states without any symmetry.
- In this talk we focus on topological defects and excitations. Properties of excitations determine the phase up to invertible ones.
- Topological defects/excitations: Gapped defects. At fixed-point, physical observables depend on only their topologies (no dependence on metrics, scales, . . . ,) excitations viewed as defects between trivial defects

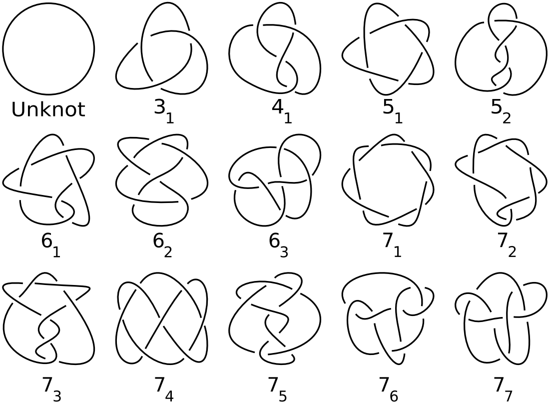
3+1D Topological Order

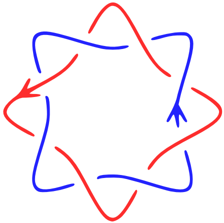
G - gauge.



- String-like excitations in addition to point-like excitations.
- They can braid with each other.
- Particles braid with particles trivially.

Knots and Links?





Difficult!
However, for a classification we do not need to study them!

Motivation

Lesson learned during the study of 2+1D SET (symmetry enriched topological) phases,

- **Tannaka Duality** Reconstruct G from $\mathrm{Rep}(G)$
- $\mathrm{Rep}(G)$: the braided tensor category of group representations.
- Example: $G = SU(2)$, $\mathrm{Rep}(G)$ consists of spins $\{0, 1/2, 1, \dots\}$ plus the following structures:
  - the degeneracy of spins (direct sum): $0 \oplus 0 \oplus 1/2 \oplus 1$.
  - the fusion of spins (tensor product): $1/2 \otimes 1/2 = 0 \oplus 1$.
  - the Clebsch–Gordan coefficients: basis change $\{$tensor product: $|00\rangle, |10\rangle, |01\rangle, |11\rangle\} \Leftrightarrow$ $\{$spin 0 singlet: $|01\rangle - |10\rangle$, spin 1 triplet: $|00\rangle, |01\rangle + |10\rangle, |11\rangle\}$.
  - bosonic exchange, $x \otimes y \to y \otimes x$. can choose fermionic exchange $x \otimes y \to -y \otimes x$ which will reconstruct a super group

Motivation

- **Deligne's Theorem** Symmetric (trivial double exchange) tensor category subject to certain finite condition, must be of the form $\mathrm{Rep}(G, z)$.
- Physically, a finite spectrum of bosons and fermions, must carry the symmetry charge of certain group G.

In 3+1D, particles braids trivially, there is thus a hidden group G.

- Ordinary gauge theory? Almost, but there are examples beyond gauge theory.
- Dijkgraaf-Witten $G, \omega_4 \in H^4[G, U(1)]$ gauge theory?
  Yes if all particles are bosons.
- Gauged SPT (symmetry protect topological) phases? Yes!

Recent Progress

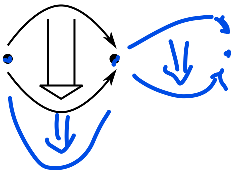
- The mathematical theory of higher (braided) fusion categories was not ready at the time of this work.
- Recent development on higher category theory definition of fusion 2-category by Douglas and Reutter, arXiv:1812.11933, notion of condensation completion by Johnson-Freyd and Gaiotto, arXiv:1905.09566 shed more light on the study of higher dimensional topological orders.
- In particular, Theo Johnson-Freyd arXiv:2003.06663 presented an n-cat-model-independent proof to our classification.
- I will mainly stick to the original simpler ideas and comment on some important modifications.

Outline

- Higher category picture of topological defects/excitations
- Boundary-bulk duality:
  - Boundary: anomalous topological order
  - Bulk: anomaly-free topological order (braiding non-degeneracy)
  - Boundary uniquely determines bulk
- Trivial mutual statistics of low-dimensional excitations
  $\Rightarrow$ point-like excitations determines a hidden "gauge group"
- Condensation of excitations with trivial statistics
  condensing enough excitations can create a boundary
- Applying above ideas in 3+1D leads to a classification:
  3+1D topological orders can all be obtained by gauging SPT.

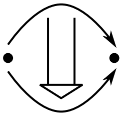
Higher Category

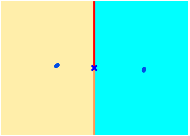
- Category, namely 1-cat, consists of objects (0-morphism), and morphisms (1-morphism) which are arrows between objects.
- 2-cat consists of 0-morphisms, 1-morphisms, and 2-morphisms which are arrows between 1-morphisms.
- ...
- n-cat consists of 0-morphisms, 1-morphisms, ..., n-1-morphisms and n-morphisms which are arrows between n-1-morphisms.
- Globular picture: 0-morphisms are points, 1-morphisms are paths, 2-morphisms are surfaces, ...
- n-morphisms can be composed in n ways.

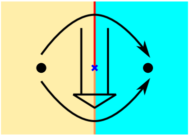


Higher Category of Topological Defects

Dual to the globular picture:

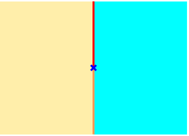






k-morphisms are co-dimension k topological defects.
composition of k-morphisms $=$ fusion of defects

Higher Category of Topological Defects



In n+1 dimensions:

k-morphism	spacial dimension of defects	
0	n	bulk phase
1	n-1	
⋮	⋮	
n-1	1	line defects
n	0	point defects
n+1	“Instanton”	physical operators

They form an (n+1)-category $\mathbf{TO}_{n+1}$.

All n-cat are assumed weak, unitary, and satisfying other necessary physical requirements.

Topological order (potentially anomalous)

Anomaly-free	can be realized by lattice model in the same dimension
Anomalous	must be boundary of lattice model in one higher dimension

Focus on one phase $\mathbf{C} \in \mathbf{TO}_{n+1}$.

- Trivial defects are identity morphisms:

  $\mathrm{id}_{0,\mathbf{C}} \equiv \mathbf{C}$, $\mathrm{id}_{1,\mathbf{C}} \equiv \mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}, \ldots, \mathrm{id}_{k,\mathbf{C}} \equiv \mathrm{id}_{\mathrm{id}_{k-1,\mathbf{C}}} : \mathrm{id}_{k-1,\mathbf{C}} \to \mathrm{id}_{k-1,\mathbf{C}}$, $\mathrm{id}_{n+1,A} \equiv \mathrm{id}_{\mathrm{id}_{n,A}} = 1 \in \mathbb{C}$.

- Excitations are defects between trivial defects.
  Co-dimension k excitations (including defects on them):
  $\mathrm{Hom}(\mathrm{id}_{k-1,\mathbf{C}}, \mathrm{id}_{k-1,\mathbf{C}})$

Excitations in $\mathbf{C}$, $\mathrm{Hom}(\mathbf{C}, \mathbf{C})$
$=$ (n+1)-cat with only one 0-morphism (object) $\mathbf{C}$
$=$ monoidal n-cat $\mathcal{C} := \mathrm{Hom}(\mathbf{C}, \mathbf{C})$
In physical applications require "nice" properties: fusion n-cat

Topological order (anomaly-free)

Braiding is the only physical probe in topological theories. Necessary condition for anomaly-free:

Braiding non-degeneracy

All topological excitations must be detectable via braidings.

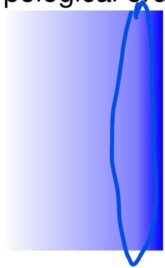
A. Kitaev, Ann. Phys. 321, 2 (2006); M. Levin, PRX 3, 021009 (2013); L. Kong and X.-G. Wen, arXiv:1405.5858

Co-dimension $k \geq 2$ excitations can braid.
(n+1)-cat with only one 0-morphism $\mathbf{C}$ and only 1-morphism $\mathrm{id}_{\mathbf{C}}$
$=$ braided monoidal (n-1)-cat $\mathscr{C} := \mathrm{Hom}(\mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathbf{C}})$
$\mathscr{C}$ should be non-degenerate braided fusion (n-1)-cat

Co-dimension 1 defects can not (full) braid and are determined by co-dimension $k \geq 2$ excitations via codensation completion.

D. Gaiotto, T. Johnson-Freyd, arXiv:1905.09566, T. Johnson-Freyd, arXiv:2003.06663.

Given an n+1D boundary theory, i.e., a (potentially) anomalous topological order in n+1D or a fusion n-cat,



- The boundary theory must involve at least a small neighbourhood in the bulk near the boundary.
- For topological theories there is no scale dependence, a small neighbourhood is the same as the whole bulk.

L. Kong and X.-G. Wen, arXiv:1405.5858; L. Kong, X.-G. Wen, and H. Zheng, Nucl. Phys. B 922, 62 (2017)

Boundary-bulk duality (Holography)

A boundary, a fusion n-cat, uniquely determines the bulk, a non-degenerate braided fusion n-cat,

Higher Drinfeld center (E_1 center)

$\mathcal{Z}_1^{(n)}$: fusion n-cat $\to$ non-degenerate braided fusion n-cat

Concrete constructions: Turaev-Viro TQFT, Levin-Wen model, Walker-Wang model, ...

Anomaly-free condition

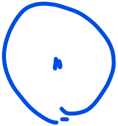
Has a trivial bulk if viewed as a boundary:
A fusion n-cat $\mathcal{C}$ is anomaly-free if $\mathcal{Z}_1^{(n)}(\mathcal{C}) = \boxed{n\mathrm{Vec}.}$ invertible.

L. Kong and X.-G. Wen, arXiv:1405.5858; L. Kong, X.-G. Wen, and H. Zheng, Nucl. Phys. B 922, 62 (2017)

Low-dimensional excitations have symmetric braidings

Full braiding path between low-dimensional excitations is homotopic to trivial path.

- In 3+1D or higher, particle and particle braid symmetrically (boson/fermion).
- In 4+1D or higher, particle-particle and particle-string braidings are symmetric.
- In 5+1D or higher, particle-particle, particle-string and string-string braidings are symmetric.
- ...



Braiding non-degeneracy and even-odd dimensionality

In n+1D, the braiding between p-dimensional excitation and q-dimensional excitation is compare the spacetime dimension n+1 with p+1 (worldsheet) + q+1 (worldsheet) + 1 (braiding path)

$n=3$

$p=1.$

$q=0.$

- Symmetric, if $p+q<n-2$.
- Non-degenerate, if $p+q=n-2$. p-dimensional excitations and $n-2-p$-dimensional excitations detect each other.
- If $p+q>n-2$, can be decomposed to braidings between dimension reduced excitations $p' \leq p$, $q' \leq q$ where $p'+q'=n-2$.

Braiding non-degeneracy and even-odd dimensionality

$z_1(A) = 1$.

Braiding non-degeneracy put strong relations between p-dimensional excitations and $(n-2-p)$-dimensional excitations.
More precisely, according to Johnson-Freyd arXiv:2003.06663

Theorem

If there is a dimension p such that excitations with dimension $\leq p$ are all trivial (i.e. equivalent to $(p+1)\mathrm{Vec}$), then defects with dimension $\geq n-2-p$ are also "trivial" in the sense that higher dimensional defects can all be built from condensations of lower dimensional defects, the topological order is determined by defects with dimension $< n-2-p$.

0 1 2   0 1 2 3

For n odd, low and high dimensional excitations are properly paired.
For n even, in the middle $(n/2-1)$-dimensional excitations pair with themselves.

0 1   0 1 2

They are bosons or fermions with trivial double braidings.
$\Leftrightarrow$ Point-like excitations form a symmetric fusion category
$\Leftrightarrow$ $\mathrm{Rep}(G,z)$, (G,z) is uniquely determined up to isomorphisms.
Here $z \in G$ is involutive $z^2 = 1$ and central $zg = gz, \forall g \in G$.

P. Deligne, Catégories tensorielles, Mosc. Math. J. 2 (2002), no. 2, 227-248

- $z = 1$: usual representation category $\mathrm{Rep}(G)$.
- z is nontrivial: z corresponds to the fermion number parity; the representations where z acts non-trivially are fermions.
  To emphasize the fermionic nature, for non-trivial z, we use the notations $G^f \equiv (G,z), \mathrm{sRep}(G^f) \equiv \mathrm{Rep}(G,z)$, $Z_2^f \equiv \{1,z\}$.

Symmetric braiding is a very strong constraint.

Classification in 3+1D

- In 3+1D, there are only point-like and string-like excitations.
- Point-like excitations must have trivial statistics, fully determined by (G, z).
- Braiding non-degeneracy puts very strong constraints on the string-like excitations.
  Expect: determined by (G, z) plus certain extra data
- Hard to extract due to technical difficulty on braided monoidal 2-cats.
- A "detour": condensation

Conjecture: similar results for odd spacial dimensions:

- Low dimensional excitations have symmetric braidings
  $\Rightarrow$ higher representations of higher (super-)group.
- High dimensional excitations are determined by such higher group to certain extent.

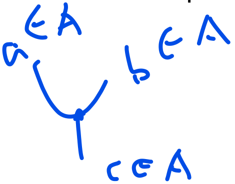
Condensation

Add interactions to make certain subset A of excitations to condense.

- Whether A can be condensed or not depends only on itself: Effectively, the condensate is a “sea” where condensed excitations in A can fluctuate freely.
- Let $|\psi_A\rangle$ be the state of A condensate and $\boxed{W}$ an operator that creates some excitations in A (for example open Wilson loop operators). The above means

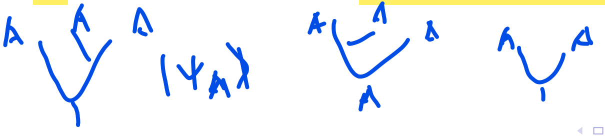
$$W|\psi_A\rangle = |\psi_A\rangle.$$

R



Condensation

- Condensation means making all possible $W = 1$ the most favorable. W have common eigenstates, they should commute (at least in the low energy subspace). Then if there are local projections P_W onto $W = 1$ for all W in a compatible way, it suffices for A to be condensable, by adding interaction of the form $-h\sum P_W$, $h \to +\infty$.
- Such W includes those describing the braidings of the condensed excitations.
  $\Rightarrow$ The mutual statistics of condensed excitations must be trivial.
- P_W corresponds to some algebraic structures on A.



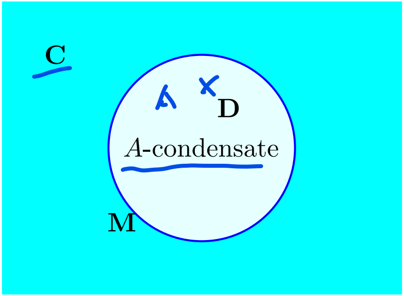
Condensation

- When only point-like excitations are condensed, it is known that A must have an (connected commutative separable) algebra structure. $\Rightarrow A$ consists of bosons.

Review: L. Kong, Anyon condensation and tensor categories, Nuclear Physics B 886 (2014)

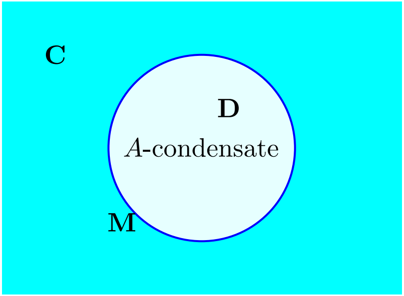
- Whether A can be condensed or not, does not depends on excitations not in A.
- Excitations not in A may be confined or deconfined excitations in the A condensed phase, depending on their mutual statistics with A.

Condensation

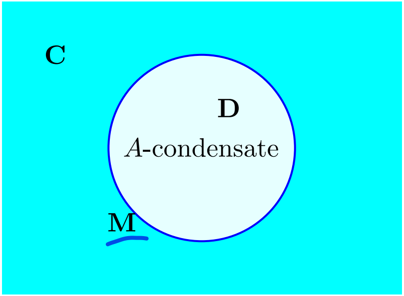


In 2+1D, condensing A in phase $\mathbf{C}$, we obtain a new phase $\mathbf{D}$, together with a gapped defect $\mathbf{M}$ between $\mathbf{C}$ and $\mathbf{D}$.

- A condensate is the new vacuum in $\mathbf{D}$.
- Excitations in the new phase $\mathbf{D}$ and on the interface $\mathbf{M}$ come from the old ones in $\mathbf{C}$ and necessarily carry “representations” of A (A-modules).



- Excitations not condensed are divided into two classes
  - those having trivial mutual statistics with A are deconfined (local A-modules);
  - those having non-trivial mutual statistics with A are confined, and stuck on the interface $\mathbf{M}$.
- Mathematically,
  - A condensed phase $\mathscr{D} = \mathscr{C}_A^{loc}$: local A-modules in $\mathscr{C}$
  - Induced gapped interface $\mathcal{M} = \mathscr{C}_A$: (all) A-modules in $\mathscr{C}$



(Dim A)^2 = Dim C.

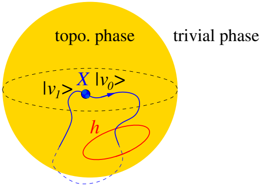
- When A is “large” enough (Langragian algebra) such that $\mathscr{D} = \mathrm{Vec}$ is the trivial phase, $\mathbf{M}$ is a boundary. By boundary-bulk duality we have $\mathscr{C} = \mathcal{Z}_1^{(1)}(\mathcal{M})$. However, in 2+1D not every $\mathscr{C}$ contains a Langragian algebra.
- Fortunately, in 3+1D there is always “large” enough A to create a boundary, which in turn determines the bulk.
  Just need to study such boundary!

All-boson (AB) 3+1D topological orders

PRX 8, 021074 (2018), arXiv:1704.04221

In 3+1D, when all point-like excitations are bosons, they form $\mathrm{Rep}(G)$. Condense them $[A = \mathrm{Fun}(G)]$:

- New phase has no point-like excitations.
- Also no nontrivial string-like excitations due to braiding non-degeneracy. Everything is confined, trivial phase.
- Obtain a boundary (fusion 2-cat) that also has no point-like excitation, only string-like excitations
- Study the braiding between the string on boundary with particles: Strings on boundary given by G (Tannaka Duality).



All-boson (AB) 3+1D topological orders

PRX 8, 021074 (2018), arXiv:1704.04221

- Such fusion 2-cat classified by (G, ω_4), $\omega_4 \in H^4[G, U(1)]$, just G-graded 2-vector-spaces $2\mathrm{Vec}_G^{\omega_4}$.

  Similar as bosonic symmetric protected topological (SPT) phases

  X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G Wen, Phys. Rev. B 87, 155114 (2013), Science 338, 1604 (2012)

- Non-degenerate braided fusion 2-cat whose point-like excitations are $\mathrm{Rep}(G)$, are all of the form $\mathcal{Z}_1^{(2)}(2\mathrm{Vec}_G^{\omega_4})$

  Dijkgraaf-Witten gauge theory in 3+1D

  R. Dijkgraaf and E. Witten, Comm. Math. Phys. 129, 393 (1990)

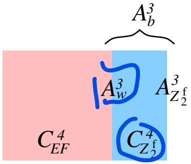
  Gauged bosonic SPT

Emergent-fermion (EF) 3+1D topological orders

PRX 9, 021005 (2019), arXiv:1801.08530

In 3+1D, when some point-like excitations are emergent fermions, they form $\mathrm{sRep}(G^f)$. Condense all bosonic point-like excitations $[A = \mathrm{Fun}(G_b = G^f/Z_2^f)]$:

- In the new phase, point-like excitations form $\mathrm{sRep}(Z_2^f) \simeq \boxed{\mathrm{sVec}}$.
- Such 3+1D topological order $C^4_{Z_2^f}$ is unique. Its string-like excitations can be condensed, after which a boundary $A^3_{Z_2^f}$ with only point-like excitations $\mathrm{sRep}(Z_2^f)$ is obtained. Strictly speaking there are also Majorana chains, as condensation descendent from fermions.
- The gapped interface A^3_w between the original phase C^4_{EF} and $C^4_{Z_2^f}$, the new phase $C^4_{Z_2^f}$ and its boundary $A^3_{Z_2^f}$, form a "sandwich" boundary A^3_b of the original phase.



Emergent-fermion (EF) 3+1D topological orders

PRX 9, 021005 (2019), arXiv:1801.08530

- Alternatively, condensing all bosons together with some strings leads to a boundary of the original phase.
- On this boundary, only non-trivial point-like excitation is the fermion. String-like excitations similarly have group-like fusion rules. Closed strings form G_b. But when considering open strings, there is an extra Z_2^m string corresponding to Majorana chain. There are further two cases:

EF1 String fusion given by $G_b \times Z_2^m$.

Classification similar as group super-cohomology theory for fermionic SPTs.

Z.-C. Gu and X.-G Wen, Phys. Rev. B 90, 115141 (2014)

EF2 String fusion given by a nontrivial Z_2^m extension of G_b. This case must have emergent Majorana zero modes.

This also has counterpart in fermionic SPTs.

A. Kapustin and R. Thorngren, arXiv:1701.08264; Q.-R. Wang and Z.-C. Gu, arXiv:1703.10937

Emergent-fermion (EF) 3+1D topological orders

PRX 9, 021005 (2019), arXiv:1801.08530

- Non-degenerate braided fusion 2-cat whose point-like excitations are $\mathrm{sRep}(G^f)$, are all of the form $\mathcal{Z}_1^{(2)}(\mathcal{A})$, with $\mathcal{A}$ being one of the above two types of fusion 2-cats (called EF 2-cats). They may be realized by higher gauge theories or more complicated tensor network models.

C. Zhu, TL, and X.-G. Wen, PRB 100, 045105 (2019), arXiv:1808.09394.

Gauged fermionic SPT

Main result in short

All 3+1D topological orders correspond to gauged SPTs.

Summary

- Topological defects form n-category
- Anomalous (anomaly-free) topological order and (non-degenerate braided) fusion n-cat
- Boundary-bulk duality and higher Drinfeld center
- Braiding of low-dimensional excitations must be trivial
- Condensation of topological excitations
- Classification in 3+1D   Gauged bosonic/fermionic SPT

Thanks for attention!