On the Convergence of Three Applied Stochastic Models related to Reflected Jump Diffusions, Fast-Slow Dynamical Systems, and Optimistic Policy Iteration

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Abstract

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This dissertation explores three stochastic models: additive functionals of reflected jump-diffusion processes, two-time scale dynamical systems forced by α-stable Lévy noise, and a variant of the Optimistic Policy Iteration algorithm in Reinforcement Learning. The connecting thread between all three projects is in showing convergence of these objects whose results have direct applied implications.

A large deviation principle is established for general additive processes of reflected jump-diffusions on a bounded domain, both in the normal and oblique setting. A characterization of the large deviation rate function, which quantifies the rate of exponential decay for the rare event probabilities of the additive processes, is provided. This characterization relies on a solution of a partial integro-differential equation with boundary constraints that is numerically solved with its implementation provided. It is then applied to a few practical examples, in particular, a reflected jump-diffusion arising from applications to biochemical reactions.

We derive the weak convergence of the functional central limit theorem for a fast-slow dynamical system driven by two independent, symmetric, and multiplicative $\alpha-$stable noise processes. To do this, a strong averaging principle is established by solving an auxiliary Poisson equation where the regularity properties of the solution are essential to the proof. The latter allow for the order of convergence to the averaged process of $1-\frac{1}{\alpha}$ to be established and subsequently used to show weak convergence of the scaled deviations of the slow process from its average. The theory is then applied to a Monte Carlo simulation of an illustrative example.

In the Optimistic Policy Iteration algorithm, Monte Carlo simulations of trajectories for some known environment are used to evaluate a value function and greedily update the policy which we show converges to its optimal value almost surely. This is done for undiscounted costs and without restricting which states are used for updating. We employ the greedy lookahead policies used in previous results thereby extending current research to discount factor $\alpha = 1$. The first-visit variation of this algorithm follows as a corollary and we further extend previous known results when the first state is picked for updating.

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Introduction

Can we show it converges?

Each of the three stochastic processes studied in this dissertation take on a different form: one is an additive process of a jump-diffusion, another the scaled deviations of a two-time scale dynamical system, and the third the path an algorithm takes within some random environment; yet they all ask the same fundamental question. There is a secondary common element connecting the three projects. They each deal with an applied model. Establishing the limit has direct applications to non purely mathematical questions in a variety of fields. Satisfy the model's assumptions and one can now say something about its long term behavior. The mode of convergence might change, the context might be different, but given some random object: what can we say about its limit?

The remainder of this chapter is dedicated to introducing the three projects and putting them into context.[1]

Large Deviations for Additive Functionals of Reflected Jump-Diffusions

Many applied stochastic systems in finance, economics, queueing theory, and electrical engineering are modeled by jump-diffusions [Kou07; Run03; Tan03; WG03a; WG03b; DM+15]. Some important properties of climate systems were explained by the addition of jumps in modeling [Dit99; DD09]. In many biological models, the inclusion of jumps in diffusion models has been useful: in neuronal systems [Jah+11; SG13], as well as in ecology and evolution [JMW12; CFM06]. Our motivation comes from problems in systems biology, where basic intracellular processes are modeled by chemical reaction networks [Bal+06; Wil09; Wil18; GAK15]. Due to the complexity of multi-scale features in chemical reaction systems, the most appropriate approximation of their inherent stochasticity may require jump-diffusion models.

Although many practical results for Lévy processes are well explored, relatively fewer are available for the more general jump-diffusions. In some applications, modeling by pure Lévy processes is inadequate as both jump and diffusion rates will genuinely depend on the current state of the system. For example, in chemical reaction networks, jump rates and diffusion coefficients are derived from rates of interactions between different molecular species, and these rates inherently depend on the amount of species types presently in the system. Consequently, one needs to consider stochastic differential equations driven by Poisson random measures. Many systems also take values on positive and bounded spaces, because of the natural constraints on the amounts of species in the system. For chemical

[1] The text introducing the first project was taken from the published paper [PZ22]. The subsequent two introductions are proposed for their related papers.

reaction models, the counts of molecules often satisfy some conservation relations in the system which keep these counts bounded above. The same is true for ecological constraints based on carrying capacities, and for some financial and engineering models with restrictions. The reflection of the process when it reaches the boundary of its domain may be built into model dynamics. For example, jump-diffusions modeling chemical reaction systems need to have oblique reflections at the boundaries defined in order to match the same behaviour of jump Markov models [And+19].

Long term behaviour of these models reveals their stability and equilibria, and the portion of time spent in different parts of the state space. Ergodic theory quantifies averages of integrated functions of process paths, martingale methods provide standard deviations from these averages, and large deviation theory provides more detailed results on rare departures from average behaviour. Large deviation rate functions quantify the values of dominant terms in integrated exponential functions of the process. Our study is motivated by the fact that many important biological mechanisms rely on the occurrence of rare events. In some mechanisms they lead to transitions to a new stable state and these transitions typically arise from the intrinsic stochasticity of the system [BVOC11; BQ10]. Due to population proliferation (cell growth+division, or species demography) rare events have many opportunities to occur and, though rare on the level of an individual molecule, occur with reasonable probability on the scale of the whole population. Time additive functionals of the process, or dynamical observables, are of particular interest for experimental studies with limited access to precise values at specific time points, and easier access to empirical distributions. In chemical reaction network models, occupation measures can be used to distinguish the orders of magnitudes of certain subsets of reactions within the system, and thus help determine which approximating model is most appropriate [MP14].

We are interested in computing long time statistics for time additive functionals of reflected jump-diffusions. Exact and explicit expressions can be found only in very special cases, and for most processes of interest one has to rely on numerical methods of evaluation. While long-term averages and standard deviations can be simulated using Monte Carlo methods [AG07], large deviations are non-trivial to assess numerically. For Markov processes and small noise diffusions one can devise simulation methods of rare events using large deviation rate functions and importance sampling techniques [BB04; VEW12], the efficiency of which is dependent on the application in question ([RT+09], e.g. in climate modeling [RWB18])). For chemical reaction dynamics there are several numerical methods for simulating functional large deviations of complete paths in small noise diffusion models (e.g. [WRVE04; VEW12]), or in pure jump Markov processes (e.g. [AWTW05]). However, large deviations of time-integrated additive functionals should be less arduous than functional large deviation paths. The framework for additive functionals relies on taking a limit as the time of integration approaches infinity, rather than a limit in which the noise of the model vanishes.

The first theoretical results for large deviations of occupation times and empirical measures for ergodic Markov processes date back to Donsker-Varadhan [DV76; DV83], with additional approaches established by Gärtner [Gär77] and Stroock [Str12]. In Chapter 1 we use the results of Fleming-Sheu-Soner [FSS87] to get the large deviation principle for additive functionals of reflected jump-diffusion processes assuming they are ergodic. This technique ensures the existence and uniqueness of a solution to a boundary value partial integro-differential equation (PIDE), which identifies the limiting logarithmic moment generating function of the additive process (for fixed parameter value in the generating function) as an eigenvalue for a second order linear operator paired with its eigenfunction. Only in some special cases is the explicit form for this eigenvalue available (c.f. [FKZ15] for reflected one-dimensional Brownian motion and its local time on the boundary).

For one-dimensional reflected diffusions and Lévy processes, a similar boundary value PIDE was obtained to characterize the limiting logarithmic moment generating function for additive functionals in ([GW15], [And+15] Sec 14.4). Our results cover the more general reflected jump-diffusions in multi-dimensional space, and provide sufficient assumptions for the existence and uniqueness of a solution to this PIDE, in order to establish the large deviation principle for the process. We thus achieve more general theoretical conclusions with the potential of a greater range of applicability.

In order to use our result in practice, we additionally provide a numerical technique for calculating the limiting logarithmic moment generating function based on numerically solving the eigenvalue problem associated to the PIDEs. We do this by way of finite-differences to approximate the derivatives and numerical quadrature or weighted sums for the integral term. Similar methods can be implemented for multidimensional problems with a small number of variables, with some care regarding an efficient evaluation of the integral term. We test our results and their numerical implementation on two special cases, a reflected Brownian motion and a reflected birth-death process, for which a comparison with analytic solutions is possible. We then use our results on an application that is analytically intractable: an example of a jump-diffusion model that approximates a system of chemical reactions. We use our large deviation results to calculate the long term mean local time at its two deterministic stable states, and the probability of departures from it. We then use this additive functional to compare the long-term behaviour of two types of approximate models for this system: a reflected diffusion process (based on the Constrained Langevin approximation developed in [LW19]), and a reflected jump-diffusion process that allows a subset of its dynamics to have sizeable noise.

Functional Central Limit Theorem for a Fast-Slow Dynamical System Driven by Symmetric and Multiplicative α-Stable Noise

Multiscale dynamical systems are natural models for systems whose dynamics evolve on different time scales; specifically for Chapter 2, a fast time scale and a slow time scale. Such systems may be found in numerous applications such as atmospheric and oceanic sciences, molecular dynamics, mathematical finance, nonlinear PDEs, material science, chemical engineering, etc (see the introductory applied text [PS08] and references therein). The separation between the two time scales is dictated by a separation of time scales parameter $\epsilon \in (0, 1)$; the smaller ϵ becomes, the faster the fast process evolves relative to the slow process. To study the asymptotic behavior of the slow process as $\epsilon \to 0$, one averages out or eliminates the effects of the fast process thereby exposing an averaged system. Intuitively, one may think of such a limit as a functional law of large numbers where the entire process limits to its average. This makes the original system more amenable to analytic methods.

One such tractable analytic result that motivates the development to come is to establish a functional central limit theorem between the slow process and the averaged process. The objective is to characterize the asymptotic behavior between the scaled deviations of these two processes via weak convergence of processes. Having the order of convergence of the slow process to the averaged process available, and applying it to an appropriate asymptotic expansion, is one way to proceed with the functional central limit theorem and the method used herein. Examples of previous work in establishing functional central limit theorems can be found in [KY05; LX22; PV01; WR12; Yan+22] etc. However, before discussing a central limit theorem, one needs an average to consider deviations from.

In the Gaussian setting, Khasminskii pioneered a time discretization technique to establish the averaging principle, or, the asymptotic behavior of the slow process as ϵ vanishes [Kha68]. Rigorous treatments of various techniques in this setting can be found in the standard reference by [FMJWA12]. Subsequently, many other techniques involving the martingale problem [Pap77], perturbation analysis and the use of Kolmogorov's equation [PS08] were developed. Each technique allows for different conclusions. For example, if one is interested in establishing convergence rates for the averaging principle, time discretization approaches could potentially allow one to discuss strong convergence rates, whereas asymptotic expansions of Kolmogorov's equations are usually tied to weak convergence rates.

It is not always possible to attain sharp enough strong convergence rates by using the methods discussed in the previous paragraph. Therefore, the approach taken here involves a technique pioneered in [PV01; PV03; PV05] which uses the solution to an auxiliary Poisson equation, but developed upon to show the order of convergence. The treatment follows past results which commonly use the same method to establish this type of theorem, e.g. [RSX19; RSX21; SXX22; SX23; Zha+20] just to name a few. One freezes the dependence of the slow process within the fast process and studies the ergodic properties of this frozen equation. These properties allow for a solution to a related Poisson equation, but, more importantly, the study of its regularity properties. The latter are then applied to prove the necessary bounds which provide a strong order of convergence of $1-\frac{1}{\alpha}$ between the slow process and its averaged process. Finally, making use of this order of convergence, some bounds established in the averaging proof, and a martingale convergence argument, one may conclude with the functional central limit theorem result.

Such problems have been studied in many cases including the Brownian case, jump diffusions, fractional Brownian motion, stochastic partial differential equations, and driven by jump processes, e.g. [BYY17; BDS23; Liu12; WR12; YY04]. The focus here will be in adding to the literature with respect to fast-slow systems driven by symmetric α-stable motion, e.g. [SXX22; Yan+22; Zha+20]. These processes have independent increments that are α-stable distributed, i.e. $\mathcal{S}_\alpha((dt)^{\frac{1}{\alpha}}, 0, 0)$. At $\alpha = 2$, one recovers Brownian motion, but for general α, one loses square integrability. They are $\frac{1}{\alpha}$ self-similar, that is, for $c > 0$, the Lévy motion has the same finite dimensional distribution as $\{c^{\frac{1}{\alpha}} X_t\}_{t\geq 0}$ (see [Sam17] for more details).

To conclude the chapter, an example of a fast-slow dynamical system is studied numerically. The chapter's assumptions are checked for the example and a Euler-Maruyama iterative scheme is employed to simulate the paths of the two-time scale process. These are used to show convergence to the averaged system and to approximate the scaled deviations. The latter are shown to converge weakly to a limiting process constructed from the numerical discretization of the Poisson equation and its application within the context of the functional central limit theorem. In essence, all the theory developed within the chapter is made visual for purely illustrative purposes on a toy example, but one can imagine such techniques used within an applied setting.

On Convergence of Undiscounted Optimistic Policy Iteration without State Update Restrictions

In recent years, reinforcement learning has received a great deal of applied interest. With the advent of superhuman capabilities in games of perfect information through neural networks and Monte Carlo Tree Search methods, e.g. [Sil+16; Sil+17a; Sil+17b], in games of imperfect information, e.g. [BS19; Bro+20], and more recently using human feedback to align model outputs, e.g. [Zie+19; Ouy+22; Fer+23], reinforcement learning has been advancing quickly at a practical level. This provides ample motivation to address some of the long standing open theoretical problems, one of which is the main concern of Chapter 3.

Reinforcement learning algorithms generally deal with finding an optimal policy from an agent interacting within some system. Given a model of the environment, one possibility is to employ a Monte Carlo simulation of the environment and observe costs (or rewards) resulting from completed trajectories to update the value function. This is then used in a greedy update of the policy. If the variation of the algorithm is valid, the policy will eventually converge optimally. The exploring starts condition requiring that any state could be picked as the initial state for updating the value function ensures every state is visited infinitely often. Monte Carlo simulations and exploring starts together give us the famous algorithm initially presented in [Sut99] and described fully in the classic textbook where the authors claim that establishing convergence of this algorithm is "one of the most fundamental open theoretical questions in reinforcement learning" (pg 99, [SB18]). The algorithm is closely related to Optimistic Policy Iteration in the Dynammic Programming literature [BT96; Tsi02]. In contrast to its non-optimistic variants, we do not wait for the policy evaluation step to converge before updating the policy, rather in the Monte Carlo setting, we simply update greedily upon termination of each trajectory.

There have been many variations of this algorithm studied in the literature to progress the settlement of this open problem. The first case shown to converge can be found in [Tsi02] under the assumption that every state is updated synchronously and each trajectory has discount factor $\alpha \in (0, 1)$. They used the stochastic iterative techniques of Chapter 4 in [BT96]. This is in contrast to the probabilistic graph arguments of [Wan+20]. In the latter paper, they establish convergence of the original algorithm without the assumptions of [Tsi02] but with the condition of an optimal policy feed-forward environment for the dynamics of the model. In [Che18], they extend [Tsi02] to the stochastic shortest path setting under the assumption that every policy leads to termination, also known as the proper policy assumption. This is further relaxed in [Liu21]. They prove convergence for the same undiscounted case but without the proper policy assumption and for an asynchronous version where a single trajectory is used to update a, possibly, non-uniformly distributed selection for the initial state used in the update procedure. Therefore, the problem has been fully solved for all discount factors $\alpha \in (0, 1]$ when the first state is used for updating. Finally, following the work in [Efr+18a; Efr+18b; Efr+19; EGM20] on lookahead policies, [WS23] established convergence of the first-visit variation. In this case, provided sufficient lookahead, a single trajectory is simulated and the discounted costs following a first-visit to a state is used for updating.

The variation we will study in Chapter 3 is the stochastic shortest path, or undiscounted Monte Carlo case with no restriction on the set of states that are updated within a trajectory. This allows for the first-visit variation to follow as a corollary. We generalize the exploring starts assumption and claim the proper policy assumption can be relaxed as done in [Liu21]. To carry all this out, we use the greedy lookahead policy technique proposed in [WS23] with the modifications [Liu21] used to generalize [Tsi02]. That is, we use the contracting factor β emanating from the dynamic programming operators to be introduced. These serve the same purpose as the discount factor α in the discounted case. Putting all this together, we arrive at a general result for this simulation based reinforcement learning algorithm. The main limitation of the result is that it requires sufficient lookahead to work, but it is not likely to be resolvable otherwise (see Example 5.11 in [BT96] or Example 18 in [Liu21] which suggest an assumption-less algorithm would fail to converge).

Each chapter will start with its outline followed by preliminaries and assumptions, the main result, and end with any related results or applications. The concluding chapter summarizes the thesis and discusses possible future avenues of research.

Chapter 1

Large Deviations for Additive Functionals of Reflected Jump-Diffusions[1]

Outline

§1.1 Introduces the reflected jump-diffusion model, the associated boundary process, and the additive functional.

§1.2 Presents the derivation of the partial-integro differential equation (PIDE) which is followed by the main result for the large deviations of the additive process, and specializes to the one-dimensional case.

§1.3 Two derived analytical examples for the rate function are compared to their numerical approximation. Then we give numerical results for an application to a system of chemical reactions given by a jump Markov model.

§1.4 A full description of the numerical scheme used to solve the PIDE for the examples is presented.

[1]This chapter is the full presentation of the published article [PZ22] with minor aesthetic modifications to create uniformity between chapters.

§1.1 Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and let $m \geq 0$ and $d \geq 1$. Denote the Skorokhod space of right-continuous functions with finite left-hand limits (RCLL) by $\mathcal{D} \equiv \mathcal{D}_{[0,\infty)}(\mathbb{R}^d)$ and the set of RCLL functions in $\overline{S}$ by $\mathcal{D}(S) = \{Z \in \mathcal{D} : Z(t) \in \overline{S}, t \geq 0\}$, where $S \subset \mathbb{R}^d$ will be made precise in the context of normal or oblique reflections later in this section. Consider a d-dimensional jump diffusion $X = \{X(t)\}_{t\geq 0}$ to be a $\mathcal{D}$-measurable process that satisfies and

$$\begin{aligned} X(t) =& X(0) + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dB(s) \\ &+ \int_0^t \int_{\mathcal{M}} \gamma(X(s-), y)N(ds, dy), \quad X(0) = x_0 \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $B = \{B(t)\}_{t\geq 0}$ is an m-dimensional Brownian motion, adapted to, and a martingale with respect to $\{\mathcal{F}(t)\}_{t\geq 0}$; $\mathcal{M}$ is a Borel measurable subset of $\mathbb{R}^n$; and the random counting measure $N(t, \mathcal{M}) = \sum_{0<s\leq t} \mathbf{1}_{\{\gamma(X(s-),y)\neq 0: y\in\mathcal{M}\}}$ is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$, independent of B, and has state-dependent, time-homogenous intensity measure $dt \cdot \nu_{X(t-)}(A), A \subseteq \mathcal{M}$ such that $\int_{|y|\leq 1} |y| \sup_{z\in\mathbb{R}^d} \nu_z(dy) < \infty$ [GM04]. Since the jumps of (1.1) are assumed to be of finite variation on finite time intervals, the sum of all jumps is well defined and we may rewrite the jump integral term in its canonical form $\int_0^t \int_{\mathcal{M}} \gamma(X(s-), y)N(ds, dy) = \sum_{0<s\leq t:\Delta X(s)\neq 0} \Delta X(s)$ where $\Delta X(s) = X(s) - X(s-) = \gamma(X(s-), y)$. Note that, after compensating for the mean, we have that for each $z \in \mathbb{R}^d, A \subseteq \mathcal{M}, \widetilde{N}(t, A) = N(t, A) - t \cdot \nu_z(A)$ is a martingale-valued measure.

We assume that the measurable functions $\mu : \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, and $\gamma : \mathbb{R}^d \times \mathcal{M} \to \mathbb{R}^d$ are Lipschitz in order to ensure the existence of a unique strong solution to the stochastic differential equation (SDE) (1.1) (c.f. Theorem V.7, [PP05]). We additionally impose a linear growth condition on the coefficients, that is, we assume $\exists C_1 > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d$

$$\begin{aligned} &||\mu(x_1) - \mu(x_2)||^2 + ||\sigma(x_1) - \sigma(x_2)||^2 \\ &+ \left|\left| \int_{\mathcal{M}} \gamma(x_1, y)\nu_{x_1}(dy) - \int_{\mathcal{M}} \gamma(x_2, y)\nu_{x_2}(dy) \right|\right|^2 \leq C_1||x_1 - x_2||^2, \end{aligned} \tag{1.2}$$

and $\exists C_2 > 0$ such that for all $x \in \mathbb{R}^d$

$$||\mu(x)||^2 + ||\sigma(x)||^2 + \int_{\mathcal{M}} ||\gamma(x, y)||^2 \nu_x(dy) \leq C_2(1 + |x|^2), \tag{1.3}$$

where $||\cdot||$ denotes the appropriate Euclidean norm.

We next introduce a solution to the stochastic differential equation with reflection (SDER). It is related to the Skorokhod problem (SP), which for a given process ϕ defines an associated boundary process η, with finite variation on finite time intervals, whose variation increases only when ϕ is on the boundary of a given domain, in such a way that ensures the reflected process $\varphi = \phi + \eta$ remains in the domain. The SP has been established for various processes in different domains including: multidimensional diffusions in convex domains [Tan79], in general domains satisfying conditions (A), (B) (defined below) and admissibility conditions [LS84], where the latter conditions were relaxed in [Sai87]. It was also extended to processes with RCLL paths in convex polyhedra [DI91], for general semimartingales in convex regions [AL91], and in non-smooth domains [Cos92].

We consider processes with both normal and oblique reflections, with slightly different sets of assumptions that ensure the respective SDER is well-defined. Let $S \subset \mathbb{R}^d$ be a bounded convex set, and $\mathcal{N}_x$ be the set of all inward unit normal vectors at $x \in \partial S$: $\mathcal{N}_x = \cup_{r>0}\mathcal{N}_{x,r}$, $\mathcal{N}_{x,r} = \{\mathbf{n} \in \mathbb{R}^d : |\mathbf{n}| = 1, B(x - r\mathbf{n}, r) \cap S = \emptyset\}$, where $B(z, r) = \{y \in \mathbb{R}^d : |y - z| < r, z \in \mathbb{R}^d, r > 0\}$ is the ball with radius $r > 0$ around a point $z \in \mathbb{R}^d$. Consider the following assumptions on S from [Slo93; LS84; Sai87]:

(A) There exists a constant $r_0 > 0$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$ for every $x \in \partial S$;

(B) There exist constants $\delta > 0, \beta \geq 1$ such that for every $x \in \partial S$ and for every $\mathbf{n} \in \cup_{y \in B(x,\delta)\cap\partial S}\mathcal{N}_y$, there exists a unit vector $\mathbf{e}_x$ such that $\langle \mathbf{e}_x, n\rangle \geq \frac{1}{\beta}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product.

Condition (A) guarantees the existence of a unit normal vector at each point on the boundary with a uniform sphere about itself. This condition is satisfied by the boundedness assumption on S (c.f. [AL91], [Tan79]) and $r_0 = +\infty$ by the convexity of S (c.f. Remark 1(iii), [Slo93]). Condition (B) is equivalent to defining a uniform cone on the interior of S at each boundary point. Call $V \in \mathcal{D}(S)$ a solution to the stochastic differential equation with normal reflection if there exists an associated boundary process $L \in \mathcal{D}(\mathbb{R})$ such that $L(0) = 0$, and $L(t) = \int_0^t \mathbf{1}_{\{V(s)\in\partial S\}}dL(s)$, that is L is equal to its own total boundary variation on $[0, t]$, and for all $t \geq 0$, $V(t)$ satisfies the equation

$$\begin{aligned} V(t) =& V(0) + \int_0^t \mu(V(s))ds + \int_0^t \sigma(V(s))dB(s) \\ &+ \int_0^t\int_{\mathcal{M}} \gamma(V(s-), y)N(ds, dy) + \int_0^t \mathbf{n}(V(s))dL(s), \quad V(0) \in \overline{S}. \end{aligned} \tag{1.4}$$

Intuitively, L captures the precise amount by which V would escape $\overline{S}$ due to the evolution of X and reflects this excess normally on the boundary of S keeping V within $\overline{S}$. If we assume that μ, σ, and γ are bounded on S, it was shown (Theorem 5, [Slo93]) by a step function approximation that there exists a solution of the SP with normal reflection and a unique strong solution to the SDER (1.4).

We also consider oblique reflections at the boundary as they are used in chemical reaction approximations to match the combined behaviour of all the reactions that are active on the boundaries (c.f. [LW19]). An analogous reflected process exists if we impose some additional assumptions. Let $S \subset \mathbb{R}^d$ be a bounded simply connected region with a smooth, connected and orientable boundary ∂S (the condition on the boundary can be relaxed, c.f. Remark 5 [MR85]). Define a twice continuously differentiable vector field ρ in a neighborhood of $\overline{S}$ such that $-\rho(x) \cdot n(x) \geq \epsilon > 0, \forall x \in \partial S$. Assume that for all $x \in \overline{S}, y \in \mathcal{M}, x + \gamma(x, y) \in \overline{S}$, that is, all jumps from $\overline{S}$ remain in $\overline{S}$ (c.f. Section 2, [MR85]). Call $V \in \mathcal{D}(S)$ a solution to the stochastic differential equation with oblique reflection if there exists a continuous associated boundary process L such that $L(0) = 0$ and $L(t) = \int_0^t \mathbf{1}_{\{V(s)\in\partial S\}} dL(s)$, and for all $t \geq 0$ $V(t)$ satisfies the equation

$$\begin{aligned} V(t) =& V(0) + \int_0^t \mu(V(s))ds + \int_0^t \sigma(V(s))dB(s) \\ &+ \int_0^t \int_{\mathcal{M}} \gamma(V(s-), y)N(ds, dy) + \int_0^t \rho(V(s))dL(s), \quad V(0) \in \overline{S}. \end{aligned} \tag{1.5}$$

The existence and uniqueness of a solution to (1.5) was shown in [MR85] by first establishing a solution for normal reflection by a penalization argument and then constructing a diffeomorphism between $\overline{S}$ and a closed unit ball where ρ is mapped to an outward normal vector to extend existence and uniqueness to appropriate oblique reflections as well. Note that by setting $\rho \equiv \mathbf{n}$, we get the normally reflected process (1.4) as a special case of (1.5), but with more restrictions imposed on S and L for the oblique case.

Solutions to SDERs have been established under different assumptions on its driving processes and domains. The first results [Tan79] are for a diffusion process in a convex domain with normal reflection, extended by [LS84] for a diffusion with normal and arbitrary reflections in their domain. It was further shown in [DI93] that obliquely reflected diffusions also exist in non-smooth domains with corners. For RCLL processes, it was shown that there exists a unique solution in the positive half-space [CMEKM80], and more recently [ŁS03] established existence of arbitrarily reflected semimartingales allowing jumps at the boundary of the domain, using convergence of approximating processes in the S-topology.

We next introduce the additive functional of the reflected jump-diffusion process V. Let f be a bounded continuous function on $\overline{S}$, and define the additive functional Λ as

$$\Lambda(t) = \int_0^t f(V(s))ds + \int_0^t f(V(s))\, dL^c(s), \tag{1.6}$$

where L^c is the continuous part of the associated boundary process L,

$$L^c(t) = L(t) - \sum_{0<s\leq t:\Delta L(s)\neq 0} \Delta L(s). \tag{1.7}$$

Using a sequence of continuous functions approximating the step function $f = \mathbf{1}_A$, we can recover (by convergence of associated additive functionals) the total occupation time of V in an arbitrary $A \subset S$. For $A \subset \partial S$, we can also recover the total variation over A of the continuous part of the associated boundary process L^c. Many other quantities of interest may be studied by an appropriate choice of f.

§1.2 Main Results

Our main result considers the large deviation principle for the additive functional Λ in terms of a PIDE for calculating its logarithmic moment generating (spectral radius) function.

We make the following assumptions on the transition semigroup of the reflected jump-diffusion V (which by uniqueness of solutions to the SDER is a Markov process). These ensure V is a Feller process whose occupation measure converges to the invariant measure exponentially fast (c.f. [DV83] p.187 I-II, [FSS87] (2.1)-(2.3)). Assume there exists a probability measure μ on $\overline{S}$ such that for all $t > 0$:

(i) the semigroup T_t of V has a density $p(t,x,y)$ relative to μ, i.e., $T_t u(x) = \int_{\overline{S}} u(y)p(t,x,y)\mu(dy)$;

(ii) $0 < a(t) \leq p(t,x,y)$ for some strictly positive function $a(t)$ and for all $x, y \in \overline{S}$; and

(iii) $\lim_{x \to x_0} ||p(t,x,\cdot) - p(t,x_0,\cdot)||_{L^1(S,\mu)} = 0$.

Theorem 1.1 PIDE & EXPONENTIAL MARTINGALE

For all $\theta \in \mathbb{R}$ there exists a unique, up to a constant, positive twice continuously differentiable function $u_\theta(\cdot) \in C^2_+(\overline{S})$ on $\overline{S}$, and a scalar $\psi_\theta \in \mathbb{R}$ such that the pair $(u_\theta(\cdot), \psi_\theta)$ satisfies the partial-integro differential equation

$$\sum_{i=1}^{d} \partial_{x_i} u_\theta(x)\mu_i(x) + \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{x_i x_j} u_\theta(x)(\sigma\sigma^T)_{ij}(x) \\ + \int_{\mathcal{M}} [u_\theta(r(x,y)) - u_\theta(x)]\nu_x(dy) + u_\theta(x)(\theta f(x) - \psi_\theta) = 0, \ \forall x \in S \tag{1.8}$$

subject to boundary conditions (if $L^c \not\equiv 0$)

$$\theta f(x) u_\theta(x) + \sum_{i=1}^{d} \partial_{x_i} u_\theta(x)\rho_i(x) = 0, \ \forall x \in \partial S \tag{1.9}$$

and such that

$$M_\theta(t) = e^{\theta\Lambda(t) - \psi_\theta t} u_\theta(V(t)) \tag{1.10}$$

is a martingale.

Proof. We start by identifying the equations that u_θ and ψ_θ would need to satisfy in order for M_θ to be a martingale. We use of the following notation for ease of exposition. For $x \in S$, $y \in \mathcal{M}$, let $[x + \gamma(x,y)]_{\partial S}$ denote the projection onto ∂S resulting from a jump exceeding the region S. Define $r : S \times \mathcal{M} \to \overline{S}$

$$r(x,y) = \begin{cases} x + \gamma(x,y) & \text{if } x + \gamma(x,y) \in S \\ [x + \gamma(x,y)]_{\partial S} & \text{if } x + \gamma(x,y) \notin S \end{cases}.$$

As V is a semi-martingale we can apply Itô's formula (Theorem II.33, [PP05]) to $M_\theta(t)$. Since L has paths of finite variation on finite intervals we have that $[\Lambda, V]^c(t) = [\Lambda, \Lambda]^c(t) = 0$ and $[V,V]^c(t) = \int_0^t \sigma\sigma^T(V(s))ds$ where $[\cdot,\cdot]^c_t$ denotes the continuous part of the quadratic covariation between the two processes. Itô's formula on (1.10) gives

$$M_\theta(t) - M_\theta(0) = \int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-}(-\psi_\theta)u_\theta(V(s-))ds + \int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-}\theta u_\theta(V(s-))d\Lambda^c(s)$$
$$+ \int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-} \sum_{i=1}^d \partial_{x_i} u_\theta(V(s-))dV^c(s)$$
$$+ \frac{1}{2}\int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-} \sum_{i,j=1}^d \partial^2_{x_i x_j} u_\theta(V(s-))(\sigma\sigma^T)_{ij}(V(s-))ds$$
$$+ \sum_{0<s\le t:\Delta V(s)\neq 0} e^{\theta\Lambda(s-)-\psi_\theta s-}\left(u_\theta(V(s)) - u_\theta(V(s-))\right)$$

Replacing Λ^c, V^c with their definitions, compensating the jumps, collecting all like integrators, and replacing the summation by it's Poisson representation, we have

$$M_\theta(t) - M_\theta(0) = \int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-}\Bigg(\left(\theta f(V(s-)) - \psi_\theta\right) u_\theta(V(s-))$$
$$+ \sum_{i=1}^d \partial_{x_i} u_\theta(V(s-))\mu_i(V(s-)) + \frac{1}{2}\sum_{i,j=1}^d \partial^2_{x_i x_j} u_\theta(V(s-))(\sigma\sigma^T)_{ij}(V(s-))$$
$$+ \int_{\mathcal{M}} \left(u_\theta(r(V(s-),y)) - u_\theta(V(s-))\right)\nu_{V(s-)}(dy)\Bigg)ds$$
$$+ \int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-}\left(\theta f(V(s-))u_\theta(V(s-)) + \sum_{i=1}^d \partial_{x_i} u_\theta(V(s-))\rho_i(V(s-))\right)dL^c(s)$$
$$+ \int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-} \sum_{i=1}^d \partial_{x_i} u_\theta(V(s-)) \sum_{j=1}^d \sigma_{ij}(V(s-))dB_j(s)$$
$$+ \int_0^t \int_{\mathcal{M}} e^{\theta\Lambda(s-)-\psi_\theta s-}\left(u_\theta(r(V(s-),y)) - u_\theta(V(s-))\right)\left(N(ds,dy) - \nu_{V(s-)}(dy)ds\right).$$

For any pair $(u_\theta(\cdot), \psi_\theta)$ satisfying the equations (1.8)-(1.9) we then get

$$M_\theta(t) - M_\theta(0) = \int_0^t e^{\theta\Lambda(s-)-\psi_\theta s-} \sum_{i=1}^d \partial_{x_i} u_\theta(V(s-)) \sum_{j=1}^d \sigma_{ij}(V(s-))dB_j(s)$$
$$+ \int_0^t \int_{\mathcal{M}} e^{\theta\Lambda(s-)-\psi_\theta s-}\left(u_\theta(r(V(s-),y)) - u_\theta(V(s-))\right)\left(N(ds,dy) - \nu_{V(s-)}(dy)ds\right).$$

For $u_\theta(\cdot) \in C^2_+(\overline{S})$ the term $\sum_{i=1}^d \partial_{x_i} u_\theta(\cdot)$ is bounded on $\overline{S}$. Likewise σ is assumed Lipschitz continuous and so it is bounded on $\overline{S}$. Since f is continuous, we also have

$$\sup_{0<s\le t}\left|e^{\theta\Lambda(s)}\right| \le e^{\theta t\cdot\|f\|_{\infty,\overline{S}} + Ct\cdot\|f\|_{\infty,\overline{S}}} < \infty$$

for some constant C satisfying $L^c(t) \le Ct$ and where $\|f\|_{\infty,\overline{S}} = \sup\{|f(x)| : \ x \in \overline{S}\}$. Then we have

$$\int_0^t \int_S \mathbb{E}\big[e^{\theta\Lambda(s-)}[u_\theta(r(V(s-),y)) - u_\theta(V(s-))]\big]\nu_{V(s-)}(dy)ds < 2te^{\theta\widetilde{C}t\|f\|_{\infty,\overline{S}}}\|u_\theta\|_{\infty,\overline{S}} < \infty,$$

for some constant $\widetilde{C}$ large enough. We can conclude that both the integral with respect to Brownian motion and the integral with respect to the compensated Poisson random measure are martingales since any uniformly bounded local martingale is a martingale. Hence their sum with $M_\theta(0)$, i.e. $M_\theta(t)$, is a martingale.

For the existence and uniqueness of u_θ and ψ_θ satisfying (1.8)-(1.9) we next summarize the arguments from Theorem 4.1 in [FSS87]. Let $\widetilde{T}_t$ be the strongly continuous semigroup defined by

$$\widetilde{T}_t u(x) = \mathbb{E}[e^{\theta\Lambda(t)}u(V(t))|V(0) = x].$$

Let $\mathcal{L}$ be the second order linear operator on $u \in C^2(\overline{S})$ given by

$$\mathcal{L}u(x) = \sum_{i=1}^d \partial_{x_i} u(x)\mu_i(x) + \frac{1}{2}\sum_{i,j=1}^d \partial^2_{x_i x_j} u(x)(\sigma\sigma^T)_{ij}(x) + \int_{\mathcal{M}} [u(r(x,y)) - u(x)]\nu_x(dy). \quad (1.11)$$

Similarly to our earlier calculation, Itô's formula gives

$$\begin{aligned} e^{\theta\Lambda(t)}u(V(t)) - u(V(0)) = &\int_0^t e^{\theta\Lambda(s-)}\big(\mathcal{L}u(V(s-)) + \theta f(V(s-))u(V(s-))\big)ds \\ &+ \int_0^t e^{\theta\Lambda(s-)}\left(\theta f(V(s-))u(V(s-)) + \sum_{i=1}^d \partial_{x_i} u(V(s-))\rho_i(V(s-))\right)dL^c(s) \\ &+ \int_0^t e^{\theta\Lambda(s-)}\sum_{i=1}^d \partial_{x_i} u(V(s-))\sum_{j=1}^d \sigma_{ij}(V(s-))dB_j(s) \\ &+ \int_0^t \int_{\mathcal{M}} e^{\theta\Lambda(s-)}\left(u(r(V(s-),y)) - u(V(s-))\right)\left(N(ds,dy) - \nu_{V(s-)}(dy)ds\right). \end{aligned}$$

Taking expectations, and as the last two integrals are martingales, we get that $\widetilde{T}_t u - u = \int_0^t \widetilde{T}_s \widetilde{\mathcal{L}} u\, ds$ holds with the infinitesimal generator of $\widetilde{T}_t$ given by the second order operator $\widetilde{\mathcal{L}} = \mathcal{L} + \theta f$, on the set of functions

$$D(\widetilde{\mathcal{L}}) = \{u \in C^2(\overline{S}) : \theta f(x)u(x) + \sum_{i=1}^d \partial_{x_i} u(x)\rho_i(x) = 0, \forall x \in \partial S\}.$$

Assumptions (i)-(iii) on the semigroup of V imply that the semigroup $\widetilde{T}_t$ also has a density $\widetilde{p}(t,x,y)$ with respect to μ for each $t > 0$ and satisfies (ii)-(iii), which can be easily verified. For each $t > 0$ this ensures existence and uniqueness ([Kra12] Theorems 2.8, 2.10) of a pair $(u_{\theta,t}, \psi_{\theta,t})$ of a positive continuous function $u_{\theta,t}$ on $\overline{S}$ and $\psi_{\theta,t} \in \mathbb{R}$, such that

$$\widetilde{T}_t u_{\theta,t} = e^{\psi_{\theta,t}} u_{\theta,t} \quad \text{and} \quad \max_{x\in\overline{S}} u_{\theta,t}(x) = 1.$$

One can further show (c.f. the argument in [FSS87] p.7), that there exist a probability measure η_θ and $\psi_\theta \in \mathbb{R}$ independent of t, such that for all $v \in C(\overline{S})$ and all $t > 0$

$$\int_{x\in\overline{S}} \widetilde{T}v(x)\eta_\theta(dx) = e^{\lambda_\theta \cdot t}\int_{x\in\overline{S}} v(x)\eta_\theta(dx).$$

Integrating $\widetilde{T}_t u_{\theta,t}$ with respect to η_θ, together with uniqueness of $\psi_{\theta,t}$ then imply that $\psi_{\theta,t} = \psi_\theta\, t$. Iterating the semigroup property gives $\widetilde{T}_{nt} u_{\theta,t} = e^{\psi_\theta\, nt} u_{\theta,t}$, and then uniqueness of $u_{\theta,t}$ implies $u_{\theta,t} = u_{\theta,1}$ $\forall t$ rational. Since $\widetilde{T}_t u_{\theta,1} \in D(\widetilde{\mathcal{L}})$ and is positive, we have $u_{\theta,1} \in D_+(\widetilde{\mathcal{L}})$. Furthermore, $\widetilde{\mathcal{L}} u_{\theta,1} = \psi_\theta u_{\theta,1}$, so that uniqueness of $u_{\theta,1}$ implies uniqueness, up to a constant, of this positive eigenfunction for $\widetilde{\mathcal{L}} = \mathcal{L} + \theta f$.

A positive eigenfunction and eigenvalue of the problem $\widetilde{\mathcal{L}} u_\theta(x) = \psi_\theta u_\theta(x)$ with eigenfunction satisfying the boundary constraint of $D(\widetilde{\mathcal{L}})$ are the desired solution (u_θ, ψ_θ) of (1.8)-(1.9). □

Remark 1.1. *The proof above identifies, for each $\theta \in \mathbb{R}$ and $f \in C(\overline{S})$ bounded, the constant ψ_θ as the principal eigenvalue of the operator $\widetilde{\mathcal{L}} = \mathcal{L} + \theta f$, where $\mathcal{L}$ is the generator of the reflected jump-diffusion V as in* (1.11). *The next result will show it is the limit of log-moment generating function of the additive process Λ. Donsker-Vardhan theory [DV76; DV83] then implies it can also be expressed in terms of a variational problem (c.f. Theorem 1.1. in [FSS87]).*

Viewed as a function of $\theta \in \mathbb{R}$ the eigenvalue ψ_θ is used to establish the large deviations for Λ defined as follows. $\{\Lambda(t)\}_{t\geq 0}$ is said to satisfy the large deviation principle (LDP) with rate t and good rate function I if: $I \not\equiv \infty$; I has compact level sets (so is lower-semicontinuous);

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\Lambda(t) \in C) \leq \inf_{x\in C} I(x),\ \forall C \subset \mathbb{R} \text{ closed},$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\Lambda(t) \in O) \leq \inf_{x\in O} I(x),\ \forall O \subset \mathbb{R} \text{ open}.$$

Using the result of Theorem 4.1 in [FSS87] one can prove this LDP using their Theorem 1.1, or one can use the Gärtner-Ellis theorem (c.f. [DZ09] Theorem 2.3.6, [Hol00] Theorem V.6) as below.

Corollary 1.1 (**Log mgf and Gärtner-Ellis**)**.** *For all $\theta \in \mathbb{R}$ the logarithmic moment generating function of Λ satisfies*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[e^{\theta\Lambda(t)}\right] = \psi_\theta, \tag{1.12}$$

and Λ satisfies the LDP with rate t and good rate function $I = \psi^*$ given by the Legendre transform of ψ

$$\psi^*(x) = \sup_\theta [\theta x - \psi_\theta]. \tag{1.13}$$

Proof. Since $M_\theta(t)$ is a martingale $\mathbb{E}\left[e^{\theta\Lambda(t)-\psi_\theta t} u_\theta(V(t))\right] = u_\theta(V(0))$. By the positivity and boundedness of $u_\theta(x)$, it follows that

$$e^{-\psi_\theta t}\mathbb{E}[e^{\theta\Lambda(t)}] \inf_{x\in\overline{S}} u_\theta(x) \leq u_\theta(V(0)) \leq e^{-\psi_\theta t}\mathbb{E}[e^{\theta\Lambda(t)}] \sup_{x\in\overline{S}} u_\theta(x),$$

which implies

$$\frac{1}{t}\log \mathbb{E}\left[e^{\theta\Lambda(t)}\right] + \frac{1}{t}\log \inf_{x\in\overline{S}} u_\theta(x) \le \psi_\theta + \frac{1}{t}\log u_\theta(V(0)) \le \frac{1}{t}\log \mathbb{E}\left[e^{\theta\Lambda(t)}\right] + \frac{1}{t}\log \sup_{x\in\overline{S}} u_\theta(x). \tag{1.14}$$

Therefore, (1.12) follows by taking $t \to \infty$.

The Gärtner-Ellis theorem requires that the following properties of ψ_θ as a function of θ are satisfied:
(a) $0 \in \mathrm{int}(\mathcal{D}_\psi)$ where $D_\psi = \{\theta \in \mathbb{R} : \psi_\theta < \infty\}$; (b) ψ is lower semi-continuous in θ; (c) ψ_θ is differentiable with respect to θ on $\mathrm{int}(\mathcal{D}_\psi)$; and (d) $D_\psi = \mathbb{R}$ or $\lim_{\theta\in\mathcal{D}_\psi\to\partial D_\psi} |\nabla\psi_\theta| = \infty$.

By Theorem 1.1, there exists a finite eigenvalue ψ_θ for each fixed $\theta \in \mathbb{R}$, so $D_\psi = \mathbb{R}$. Since $\psi_0 = 0$, the origin is in the interior of D_ψ.

Since ψ_θ is the eigenvalue associated with a unique positive eigenfunction of $\mathcal{L} + \theta f$, its differentiability with respect to θ follows by use of the implicit function theorem (c.f. [FSS87] p.2, [KM03] Prop 4.8).

Let $D_{\psi,c} = \{\theta \in \mathbb{R} : \psi_\theta \le c\}$ denote sublevel sets of $\psi.$ with respect to θ, and let $\theta \in \overline{D}_{\psi,c}$. There exists a sequence $\{\theta_n\}_{n\ge 0} \subset D_{\psi,c}$ such that $\theta_n \uparrow \theta$. Then,

$$\lim_{n\to\infty} \psi_{\theta_n} = \lim_{n\to\infty}\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}\left[e^{\theta_n\Lambda(t)}\right] = \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}\left[e^{\theta\Lambda(t)}\right] = \psi_\theta \le c,$$

where the interchange of limits and the passage of the limit through the expectation follows from the monotone convergence theorem and monotonicity in n. Then $\theta \in D_{\psi,c}$ which implies $\overline{D}_{\psi,c} \subset D_{\psi,c}$. As all sublevel sets of ψ_θ are closed, we have that ψ_θ is lower semicontinuous in θ. □

Remark 1.2. *Our main goal was characterizing the logarithmic moment generating function ψ_θ by solving a boundary value PIDE, from which we can also obtain the long term mean and variance for Λ. From the martingale M_θ* (1.10) *we have*

$$\mathbb{E}\left[e^{\theta\Lambda(t)} u_\theta(V(t))\right] = e^{\psi_\theta t} u_\theta(V(0)). \tag{1.15}$$

Assuming u_θ is C^2 as a function of θ, taking derivatives with respect to θ, and using dominated convergence gives

$$\frac{1}{te^{\psi_\theta t}u_\theta(V(0))}\mathbb{E}\left[\Lambda(t)e^{\theta\Lambda(t)}u_\theta(V(t)) + e^{\theta\Lambda(t)}\frac{d}{d\theta}u_\theta(V(t))\right] = \frac{d\psi_\theta}{d\theta} + \frac{1}{t}\frac{\frac{d}{d\theta}u_\theta(V(0))}{u_\theta(V(0))}.$$

Evaluating at $\theta = 0$, and using the fact that at $\theta = 0$ $(u_0(\cdot), \psi_0) \equiv (1, 0)$ solves the PIDE (1.8)-(1.9), we get

$$\frac{1}{t}\mathbb{E}\left[\Lambda(t) + \frac{d}{d\theta}u_\theta(V(t))\big|_{\theta=0}\right] = \frac{d\psi_\theta}{d\theta}\big|_{\theta=0} + \frac{1}{t}\frac{d}{d\theta}u_\theta(V(0))\big|_{\theta=0}.$$

Taking the limit as $t \to \infty$, one gets for the long term mean of Λ that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\Lambda(t)\right] = \frac{d\psi_\theta}{d\theta}\big|_{\theta=0}.$$

Taking second derivatives in (1.15) *with respect to θ, one similarly gets for the long term variance of Λ that*

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{V}\left[\Lambda(t)\right] = \frac{d^2\psi_\theta}{d\theta^2}\big|_{\theta=0}.$$

One dimensional reflected jump-diffusion

In the case of a jump-diffusion in one dimension, much of the general theory simplifies to more explicit formulae. In particular, there is only one possible direction of reflection at each boundary, and there is an explicit formula for the SP mapping taking X to (V, L) that provides a direct construction of the reflected process V.

Without loss of generality, let us assume $S = [0, b]$ for some $b < \infty$. Since L is a process on $\partial S = \{0, b\}$, we can decompose $L = L_0 - L_b$ where we define L_0 as the associated boundary process at 0 and L_b as the associated boundary process at b, in the sense that, for all $t > 0$:

$$\int_0^t \mathbf{1}_{\{V(s)>0\}} dL_0(s) = 0, \quad \int_0^t \mathbf{1}_{\{V(s)<b\}} dL_b(s) = 0. \tag{1.16}$$

The explicit formula for (V, L) was constructed in [Kru+07; Kru+08] providing $V(t) = \mathcal{L}_{[0,b]}(X)(t)$ via the following Skorokhod map

$$\mathcal{L}_{[0,b]}(X)(t) := X(t) - \left[(X(0) - b)^+ \wedge \inf_{0\le u\le t} X(u)\right] \vee \sup_{0\le s\le t} \left[(X(s) - b) \wedge \inf_{s\le u\le t} X(u)\right] \tag{1.17}$$

using the notation $(x - b)^+ = (x - b) \vee 0$. It was also shown in [Kru+07] that the associated boundary processes satisfy

$$L_0(t) = \sup_{0\le s\le t} (L_b(s) - X(s))^+, \;\; L_b(t) = \sup_{0\le s\le t} (X(s) + L_0(s) - b)^+, \tag{1.18}$$

and that this formula with $V = X + L_0 - L_b$ is equivalent to the one in (1.16).

With $S = [0, b]$, the additive functional becomes

$$\Lambda(t) = \int_0^t f(V(s))ds + f(0)L_0^c(t) + f(b)L_b^c(t) \tag{1.19}$$

where L_0^c and L_b^c are the continuous parts of the increasing processes L_0 and L_b, respectively. We can recover them by using continuous approximations to the function $f = \mathbf{1}_0$ to get $\Lambda(t) = L_0^c(t)$; and continuous approximations to $f = \mathbf{1}_b$ to get $\Lambda(t) = L_b^c(t)$.

By Theorem 1.1 we have that $M_\theta(t) = e^{\theta\Lambda(t)-\psi_\theta t}$ is a martingale, and for each $\theta \in \mathbb{R}$ there exists a unique positive function $u_\theta(x)$ and constant ψ_θ satisfying the PIDE

$$\partial_x u_\theta(x)\mu(x) + \frac{1}{2}\partial_{xx}^2 u_\theta(x)\sigma^2(x) + \int_{\mathcal{M}} [u_\theta(r(x, y)) - u_\theta(x)]\nu_x(dy) + u_\theta(x)(\theta f(x) - \psi_\theta) = 0 \tag{1.20}$$

subject to the boundary conditions, if $L^c \not\equiv 0$

$$\theta u_\theta(0)f(0) + \rho_0\partial_x u_\theta(0) = 0, \;\; \theta u_\theta(b)f(b) - \rho_b\partial_x u_\theta(b) = 0. \tag{1.21}$$

Without the jump part, the resulting PDE would be more straightforward to solve numerically. We construct a numerical approximation scheme for $u_\theta(x)$ based on the PIDE (1.20)-(1.21) that allows for jumps (c.f. Section 1.4 for full details), and test it on two examples of PIDEs whose solution we can derive analytically.

§1.3 Examples

The partial integro-differential equation (1.8)-(1.9) for the pair $(u_\theta(x), \psi_\theta)$ can be solved analytically only in a few special cases; even then the expression for ψ is implicit. In general, one has to use numerical methods to solve this PIDE. We next provide a numerical scheme to approximate its solution using $d = 1$ and $S = [0, b]$.

Fix $\theta \in \mathbb{R}$, and let $\widetilde{\mathcal{L}}$ be the second order linear operator defined by $\widetilde{\mathcal{L}}u_\theta(x) - \psi_\theta u_\theta(x) = 0$ in (1.20) on the subset of $C^2([0, b])$ functions $D(\widetilde{\mathcal{L}})$ defined by the boundary constraint (1.21). Since the limit of the logarithmic moment generating function is the eigenvalue ψ_θ of the operator $\widetilde{\mathcal{L}}$ with corresponding eigenfunction $u_\theta(\cdot)$, we will numerically solve the eigenvalue problem $\widetilde{\mathcal{L}}u_\theta(x) = \psi_\theta u_\theta(x)$. We subdivide $\overline{S} = [0, b]$ into $N + 1$ equal subintervals and approximate $\widetilde{\mathcal{L}}$ as a matrix. We treat the derivatives by finite differences and the integral by composite trapezoidal quadrature on intervals with continuous support and as a weighted sum on intervals with discrete support. The boundary conditions are approximated by forward and backward finite-difference schemes and substituted into the matrix where appropriate. Full details of the numerical method are contained in Section 1.4.

We now apply this numerical scheme to two special cases for which we can also derive analytical implicit solutions to the limiting cumulant generating function, in order to test our numerical scheme. We then apply the numerical estimation to our application of interest: a biochemical reaction model and its jump-diffusion approximation.

Reflected Brownian motion with Drift

Let X be a standard Brownian motion in $d = 1$ on $[0, b]$ with drift μ, diffusion σ^2 (and $\nu \equiv 0$) and V be this process normally reflected at the boundaries. Let $\Lambda = L_0^c$ be the local time of X at 0 (so $f =$ continuised version of $\mathbf{1}_0$ as described below (1.7)). Then (1.20)-(1.21) becomes the partial differential equation with boundary constraints

$$\frac{1}{2}\sigma^2\partial_{xx}^2 u_\theta(x) + \mu\partial_x u_\theta(x) - \psi_\theta u_\theta(x) = 0$$
$$\theta u_\theta(0) + \partial_x u_\theta(0) = 0,\ u_\theta(b) = 1,\ \partial_x u_\theta(b) = 0$$

(as u_θ is unique only up to a constant, we are allowed to chose its value at $x = b$). When the characteristic polynomial $\beta^2 + \frac{2\mu}{\sigma^2}\beta - \frac{2\psi_\theta}{\sigma^2} = 0$ has repeated roots, $\psi_\theta = -\mu^2/2\sigma^2$, then

$$u_\theta(x) = e^{\frac{\mu(b-x)}{\sigma^2}}\left(\frac{-b\mu + \sigma^2 + \mu x}{\sigma^2}\right),\ \theta = -\frac{\mu^2 b}{\sigma^2(\sigma^2 - b\mu)}.$$

When the polynomial has real roots, $\psi_\theta > -\mu^2/2\sigma^2$, then for $\alpha = \sqrt{\mu^2 + 2\sigma^2\psi_\theta}$

$$u_\theta(x) = e^{\frac{b(\mu-\alpha)-x(\alpha+\mu)}{\sigma^2}}\left(\frac{(\alpha-\mu)e^{\frac{2\alpha b}{\sigma^2}} + (\alpha+\mu)e^{\frac{2\alpha x}{\sigma^2}}}{2\alpha}\right),\ \theta = \frac{2\sigma^2\psi_\theta e^{\frac{2\alpha b}{\sigma^2}}}{\sigma^2((\alpha-\mu)e^{\frac{2\alpha b}{\sigma^2}} + (\alpha+\mu))}.$$

When the polynomial has complex roots, $\psi_\theta < -\mu^2/2\sigma^2$, then for $\alpha = \sqrt{-(\mu^2 + 2\sigma^2\psi_\theta)}$

$$u_\theta(x) = e^{\frac{\mu(b-x)}{\sigma^2}} \left(\frac{\alpha \cos\left(\frac{\alpha(b-x)}{\sigma^2}\right) - \mu \sin\left(\frac{\alpha(b-x)}{\sigma^2}\right)}{\alpha} \right), \ \theta = \frac{2\sigma^2\psi_\theta \sin(\frac{\alpha b}{\sigma^2})}{\sigma^2(\mu \sin(\frac{\alpha b}{\sigma^2}) - \alpha\cos(\frac{\alpha b}{\sigma^2}))}.$$

In the special case of $\mu = 0, \sigma^2 = 1$: ψ_θ is given implicitly by $\theta = \sqrt{2\psi_\theta}\tanh(b\sqrt{2\psi_\theta})$ with $\psi_0 = 0$. A similar result was obtained by other analytic methods in [FKZ15].

For the numerics, we set $b = 1$ and approximate f by using continuous linear piecewise functions $f(x) = (1-(N+1)x)\mathbf{1}_{\{x<\frac{1}{N+1}\}}$ on $[0,1]$, for the choice of mesh steps N to be defined. Figure 1.1 shows the convergence of the proposed numerical scheme to the analytical solution as the number of mesh steps are increased from $N = 10$ to $N = 110$ for various values of θ. Table 1.1 compares the analytical result of ψ_θ against its numerical approximation, $\widehat{\psi_\theta}$ for θ close to 0. In this approximation the interval $S = [0,1]$ is subdivided $N + 1 = 1001$ sub-intervals.

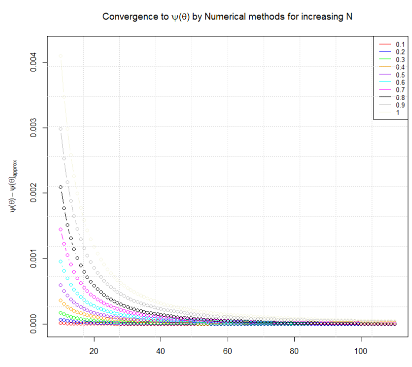


Figure 1.1: Convergence of $\widehat{\psi_\theta}$ for $\theta \in \{\frac{i}{10}\}_1^{10}$ as $\{N\}_{10}^{110}$ increases for a reflected standard Brownian Motion.

θ	ψ_θ	$\widehat{\psi_\theta}$	$\lvert\psi_\theta - \widehat{\psi_\theta}\rvert$
0	0	3.761×10^{-10}	3.761×10^{-10}
0.001	5.003×10^{-4}	5.002×10^{-4}	9.816×10^{-8}
0.002	1.001×10^{-3}	1.001×10^{-3}	3.942×10^{-7}
0.003	1.502×10^{-3}	1.502×10^{-3}	8.859×10^{-7}
0.004	2.004×10^{-3}	2.003×10^{-3}	1.573×10^{-6}
0.005	2.507×10^{-3}	2.504×10^{-3}	2.457×10^{-6}
0.006	3.010×10^{-3}	3.006×10^{-3}	3.536×10^{-6}
0.007	3.513×10^{-3}	3.508×10^{-3}	4.809×10^{-6}
0.008	4.017×10^{-3}	4.011×10^{-3}	6.277×10^{-6}
0.009	4.521×10^{-3}	4.514×10^{-3}	7.937×10^{-6}
0.01	5.027×10^{-3}	5.017×10^{-3}	9.792×10^{-6}

Table 1.1: Comparison between ψ_θ and $\widehat{\psi_\theta}$ when θ is near 0 for a reflected standard Brownian Motion.

Reflected birth-death process

Let X be a pure birth-death process in $d = 1$ on $S = \{0, 1, 2, ..., b\}$ with overall jump rate λ, so $\nu_x(\pm 1) = \frac{\lambda}{2}$ (and $\mu = \sigma \equiv 0$) and V be this process but it reflects on itself at the boundaries: $\nu_x(\pm 1) = \frac{\lambda}{2}$ for $x \in \{1, \dots, b-1\}$ and $\nu_0(+1) = \nu_0(0) = \frac{\lambda}{2}, \nu_b(-1) = \nu_b(0) = \frac{\lambda}{2}$. Let $\Lambda = \int_0^t \mathbf{1}_{V_s \in [0,1)} ds$ and $f(x) = \mathbf{1}_{0 \le x < 1 - \frac{1}{N+1}} + (N+1)(1-x)\mathbf{1}_{\{1-\frac{1}{N+1} \le x < 1\}}$ approximating the function $\mathbf{1}_{[0,1)}$. Then (1.20)-(1.21) becomes the recurrence relation equation

$$-(\lambda + \psi_\theta - \theta f(x))u_\theta(x) + \frac{\lambda}{2}(u_\theta(r(x,1))) + \frac{\lambda}{2}(u_\theta(r(x,-1))) = 0$$

$$\iff \quad \begin{cases} -(\frac{\lambda}{2} + \psi_\theta - \theta)u_\theta(0) + \frac{\lambda}{2}u_\theta(1) = 0, & x = 0 \\ -(\lambda + \psi_\theta)u_\theta(x) + \frac{\lambda}{2}u_\theta(x+1) + \frac{\lambda}{2}u_\theta(x-1) = 0, & x = \{1, 2, ..., b-1\} \\ -(\frac{\lambda}{2} + \psi_\theta)u_\theta(b) + \frac{\lambda}{2}u_\theta(b-1) = 0, & x = b. \end{cases}$$

with sole constraint $u_\theta(b) = 1$ ($L^c \equiv 0$, since X is a pure jump process).

The recurrence equation on $x = \{1, 2, ..., b-1\}$ is a linear difference equation and may be solved by standard methods. Namely, let $u_\theta(x) = \beta^x$ then the characteristic equation is

$$-(\lambda + \psi_\theta)\beta^x + \frac{\lambda}{2}\beta^{x+1} + \frac{\lambda}{2}\beta^{x-1} = 0 \iff \beta^2 - \frac{2}{\lambda}(\lambda + \psi_\theta)\beta + 1 = 0.$$

Solving for β, we have $\beta = \frac{1}{\lambda}\left((\lambda + \psi_\theta) \pm \sqrt{\psi_\theta(2\lambda + \psi_\theta)}\right)$. Consequently,

$$u_\theta(x) = \begin{cases} C_1\left(\frac{1}{\lambda}(\lambda + \psi_\theta)\right)^x + C_2 x\left(\frac{1}{\lambda}(\lambda + \psi_\theta)\right)^x, \\ \quad \text{if } \psi_\theta \in \{0, -2\lambda\} \\ C_1\left(\frac{1}{\lambda}\left((\lambda + \psi_\theta) + \sqrt{\psi_\theta(2\lambda + \psi_\theta)}\right)\right)^x + C_2\left(\frac{1}{\lambda}\left((\lambda + \psi_\theta) - \sqrt{\psi_\theta(2\lambda + \psi_\theta)}\right)\right)^x, \\ \quad \text{otherwise.} \end{cases}$$

Using the boundary condition $u_\theta(b) = 1$, we can derive from the third recurrence relation that $u_\theta(b-1) = 1 + \frac{2}{\lambda}\psi_\theta$, and using this in the second recurrence relation with $x = b-1$, we have $u_\theta(b-2) = 1 + \frac{6}{\lambda}\psi_\theta + \frac{4}{\lambda^2}\psi_\theta^2$. Now we may solve for C_1 and C_2 to find that

$$u_\theta(x) = \begin{cases} \frac{1}{\lambda^3}\left(\frac{\lambda+\psi_\theta}{\lambda}\right)^{1-b+x}\left(\lambda^3 + (5b - 3 - 5x)\lambda^2\psi_\theta + 10(b - 1 - x)\lambda\psi_\theta^2 + 4(b - 1 - x)\psi_\theta^3\right), \\ \quad \psi_\theta \in \{0, -2\lambda\} \\ \frac{1}{2}\left(\left(\frac{\lambda+\psi_\theta-\sqrt{\psi_\theta(2\lambda+\psi_\theta)}}{\lambda}\right)^{x-b} + \frac{\psi_\theta\left(\frac{\lambda+\psi_\theta-\sqrt{\psi_\theta(2\lambda+\psi_\theta)}}{\lambda}\right)^{x-b}}{\sqrt{\psi_\theta(2\lambda+\psi_\theta)}}\right. \\ \qquad \left. + \left(\frac{\lambda+\psi_\theta+\sqrt{\psi_\theta(2\lambda+\psi_\theta)}}{\lambda}\right)^{x-b} + \frac{\psi_\theta\left(\frac{\lambda+\psi_\theta+\sqrt{\psi_\theta(2\lambda+\psi_\theta)}}{\lambda}\right)^{x-b}}{\sqrt{\psi_\theta(2\lambda+\psi_\theta)}}\right), \\ \quad \text{otherwise.} \end{cases}$$

We equate the first recurrence equation to the second recurrence equation when $x = 1$ to derive the identity

$$u_\theta(0) = \frac{\frac{\lambda}{2}}{\frac{\lambda}{2} + \psi_\theta - \theta}u_\theta(1) = \frac{2}{\lambda}(\lambda + \psi_\theta)u_\theta(1) - u_\theta(2),$$

which can now be used to solve for ψ_θ implicitly. Namely for $\psi_\theta \notin \{0, -2\lambda\}$, $\theta = \frac{A}{B}$ where

$$\begin{aligned} A =& \psi_\theta\left(\lambda\left(\left(\frac{-\sqrt{\psi_\theta(2\lambda+\psi_\theta)} + \lambda + \psi_\theta}{\lambda}\right)^b - \left(\frac{\sqrt{\psi_\theta(2\lambda+\psi_\theta)} + \lambda + \psi_\theta}{\lambda}\right)^b\right)\right. \\ &+ \psi_\theta\left(\left(\frac{-\sqrt{\psi_\theta(2\lambda+\psi_\theta)} + \lambda + \psi_\theta}{\lambda}\right)^b - \left(\frac{\sqrt{\psi_\theta(2\lambda+\psi_\theta)} + \lambda + \psi_\theta}{\lambda}\right)^b\right) \\ &\left. - \sqrt{\psi_\theta(2\lambda+\psi_\theta)}\left(\left(\frac{-\sqrt{\psi_\theta(2\lambda+\psi_\theta)} + \lambda + \psi_\theta}{\lambda}\right)^b + \left(\frac{\sqrt{\psi_\theta(2\lambda+\psi_\theta)} + \lambda + \psi_\theta}{\lambda}\right)^b\right)\right), \end{aligned}$$

$$
\begin{aligned}
B =& \psi_\theta \left(\left(\frac{-\sqrt{\psi_\theta(2\lambda+\psi_\theta)}+\lambda+\psi_\theta}{\lambda} \right)^b - \left(\frac{\sqrt{\psi_\theta(2\lambda+\psi_\theta)}+\lambda+\psi_\theta}{\lambda} \right)^b \right) \\
& - \sqrt{\psi_\theta(2\lambda+\psi_\theta)} \left(\left(\frac{-\sqrt{\psi_\theta(2\lambda+\psi_\theta)}+\lambda+\psi_\theta}{\lambda} \right)^b + \left(\frac{\sqrt{\psi_\theta(2\lambda+\psi_\theta)}+\lambda+\psi_\theta}{\lambda} \right)^b \right).
\end{aligned}
$$

For the purposes of the numerical approximations, we will arbitrarily set $\lambda = 50$ and $b = 3$. With this choice, ψ_θ is given implicitly by $\theta = \frac{\psi_\theta(62500+6250\psi_\theta+150\psi_\theta^2+\psi_\theta^3)}{15625+3750\psi_\theta+125\psi_\theta^2+\psi_\theta^3}$. Similarly, using $u_\theta(x)$ when $\psi_\theta \in \{0, -2\lambda\}$, we deduce that with this choice of λ and b we have that $\psi_0 = 0$ and $\psi_{-50} = -100$. Given that S is discrete, numerical approximations of ψ_θ, $\widehat{\psi_\theta}$, can be made by solving for the largest real eigenvalue of the matrix

$$
\begin{bmatrix}
-\frac{50}{2}+\theta & \frac{50}{2} & 0 & 0 \\
\frac{50}{2} & -50 & \frac{50}{2} & 0 \\
0 & \frac{50}{2} & -50 & \frac{50}{2} \\
0 & 0 & \frac{50}{2} & -\frac{50}{2}
\end{bmatrix}
$$

for each θ. Table 1.2 compares the analytical result of ψ_θ against its numerical approximation, $\widehat{\psi_\theta}$, for various values of θ close to 0.

θ	ψ_θ	$\widehat{\psi_\theta}$	$\lvert\psi_\theta - \widehat{\psi_\theta}\rvert$
0	0	-6.476×10^{-317}	6.476×10^{-317}
0.001	2.503×10^{-4}	2.500×10^{-4}	3.410×10^{-7}
0.002	5.007×10^{-4}	5.000×10^{-4}	6.646×10^{-7}
0.003	7.510×10^{-4}	7.501×10^{-4}	9.706×10^{-7}
0.004	1.001×10^{-3}	1.000×10^{-3}	1.259×10^{-6}
0.005	1.252×10^{-3}	1.250×10^{-3}	1.530×10^{-6}
0.006	1.502×10^{-3}	1.500×10^{-3}	1.784×10^{-6}
0.007	1.752×10^{-3}	1.750×10^{-3}	2.020×10^{-6}
0.008	2.003×10^{-3}	2.001×10^{-3}	2.238×10^{-6}
0.009	2.253×10^{-3}	2.251×10^{-3}	2.439×10^{-6}
0.01	2.503×10^{-3}	2.501×10^{-3}	2.623×10^{-6}

Table 1.2: Comparison between ψ_θ and $\widehat{\psi_\theta}$ when θ is near 0 for a reflected birth-death process.

Biochemical reaction model[2]

Our main motivating example comes from jump-diffusion approximation of a biochemical reaction model. The full model is a pure Markov jump process tracking the amount $X_A(t)$ and $X_B(t)$ of molecular species A and B within a cell that also undergoes cellular growth and division. The simplified representation of reactions between A and B is:

[2]The output figures for this model are collected on the last three pages of this subsection.

$$A \overset{\widetilde{\kappa}^{10}_{-1}}{\to} B$$
$$B \overset{\widetilde{\kappa}^{01}_{1}}{\to} A$$
$$A + B \overset{\widetilde{\kappa}^{11}_{-1}}{\to} 2B$$
$$2A + B \overset{\widetilde{\kappa}^{21}_{1}}{\to} 3A$$

where all external factors are captured by reaction constants $\widetilde{\kappa}^{ij}_k$. At the time of cellular division an assignment of one half of doubled molecules results in a Bernoulli($\frac{1}{2}$) random error of ± 1 in the amount of species A compensated by species B, occurring at a rate that is proportional to their product and a division constant $\widetilde{\gamma}$. The jump Markov process for the evolution of this system has the generator:

$$\begin{aligned}\mathcal{G}g(x_A, y_B) &= \widetilde{\kappa}^{10}_{-1} x_A [g(x_A - 1, y_B + 1) - g(x_A, y_B)] + \widetilde{\kappa}^{01}_{1} y_B [g(x_A + 1, y_B - 1) - g(x_A, y_B)] \\ &+ \widetilde{\kappa}^{11}_{-1} x_A y_B [g(x_A - 1, y_B + 1) - g(x_A, y_B)] + \widetilde{\kappa}^{21}_{1} x_A^2 y_B [g(x_A + 1, y_B - 1) - g(x_A, y_B)] \\ &+ \frac{1}{2}\widetilde{\gamma} x_A y_B [g(x_A - 1, y_B + 1) - g(x_A, y_B)] + \frac{1}{2}\widetilde{\gamma} x_A y_B [g(x_A + 1, y_B - 1) - g(x_A, y_B)]\end{aligned}$$

for $g \in \mathcal{C}(\mathbb{N} \times \mathbb{N})$. This reaction and division dynamics has two relevant features:

(1) There is a conservation law in the total sum of species A and B, and letting n denote the initial overall total of both species, $X_A = \frac{x_A}{n}, X_B = \frac{y_B}{n}$ denote the proportions (out of n) of species A, B respectively, all the reactions preserve the initial total $X_A(0) + X_B(0) = 1$ so that $X_B(t) = 1 - X_A(t), \forall t > 0$ reduces the model to $d = 1$. This allows us to express the rates of reactions, which are proportional to the product of source types masses and the chemical reaction constants, where the latter are assumed to scale as $\widetilde{\kappa} = n^{\nu-1}\kappa$ with ν=number of sources in the reaction. The generator of the process $X_A = \frac{x_A}{n}$ for $g \in \mathcal{C}([0,1])$ is:

$$\begin{aligned}\mathcal{G}_n g(x) &= n\big(\kappa^{10}_{-1} x + \kappa^{11}_{-1} x(1-x)\big)\big[g(x - \frac{1}{n}) - g(x)\big] \\ &+ n\big(\kappa^{01}_{1}(1-x) + \kappa^{21}_{1} x^2 (1-x)\big)\big[g(x + \frac{1}{n}) - g(x)\big] \\ &+ \frac{1}{2}\gamma_n x(1-x)\big[g(x - \frac{1}{n}) - g(x)\big] + \frac{1}{2}\gamma_n x(1-x)[g(x + \frac{1}{n}) - g(x)]\end{aligned}$$

showing the overall reaction rates of decreasing and increasing proportions of A as $nr_-(x) = n\kappa^{10}_{-1} x + n\kappa^{11}_{-1} x(1-x)$ and $nr_+(x) = n\kappa^{01}_{1}(1-x) + n\kappa^{21}_{1} x^2(1-x)$, respectively. The division rates for both increasing and decreasing proportions of A are $\xi_n(x) = \frac{1}{2}\gamma_n x(1-x)$, where the relationship of γ_n to n will be explored in the two approximations of the jump Markov chain to follow.

(2) The long term dynamics exhibits a form of noise induced bistability in the proportion of species A, under appropriate assumptions on the constants $\{\kappa^{\cdot\cdot}_{\cdot}\}$ (c.f. ($\star$) below). The rate of change of the mean is:

$$\frac{d}{dt} E\left[X_A(t) | X_A(t) = x\right] = \mu(x) := -\kappa^{10}_{-1} x + \kappa^{01}_{1}(1-x) - \kappa^{11}_{-1} x(1-x) + \kappa^{21}_{1} x^2 (1-x),\ x \in [0,1] \tag{1.22}$$

and since $\mu(x)$ is a cubic, assuming $(\star)$ that it has all real roots in $[0,1]$, then the dynamics of $E[X_A(t)]$ has two stable equilibria and one unstable equilibrium point creating potential barriers on either side of the domain of attraction of the two equilibria. Freidlin-Wentzel theory (Chapter 6, [FW98]) for path properties of Markov processes with $O(n)$ rates and $O(\frac{1}{n})$ jump sizes imply that this process will spend most of its time in the stable equilibria with rare transitions between small neighbourhoods around them created by perturbations due to randomness in the system. The occupation measure process will reflect this and increasingly concentrate at the deterministic stable points. For more details on sample path properties on finite time intervals of biochemical reaction models with division errors (c.f. Section 3.3., [MP14]).

To estimate the mean local time at the two stable equilibria, as well as the large deviations away from this mean, we numerically solve the PIDE for the limiting cumulant generating function ψ_θ, which again reduces to solving for the largest eigenvalue of an $(n+1)\times(n+1)$ matrix. Set $n = 100$, and $\kappa_1^{01} = 1, \kappa_1^{21} = \frac{32}{3}, \kappa_{-1}^{10} = 1, \kappa_{-1}^{11} = \frac{16}{3}$ (for which $(\star)$ is satisfied as the cubic has all real roots), then the two stable equilibria are $x_1 = 0.25$ and $x_2 = 0.75$ on $S = [0,1]$, and we will define $f(x)$ as the continuised version of $\mathbf{1}_{x\in B_{1/(N+1)}(0.25)\cup B_{1/(N+1)}(0.75)}$, so that the additive functional $\Lambda(t)$ measures the time spent around the two stable equilibria.

The jump Markov process parameters are $\nu_x(+\frac{1}{n}) = \xi_n(x) + nr_+(x)$ and $\nu_x(-\frac{1}{n}) = \xi_n(x) + nr_-(x)$ (and $\mu = \sigma^2 \equiv 0$). On the two boundaries the rates of the reaction dynamics and division errors at $x \in \{0,1\}$ have only inward jumps ($r_-(0) = r_+(1) = 0$, $\xi_n(0) = \xi_n(1) = 0$), so no additional reflection is needed to keep the process within $S = [0,1]$, and $V_n = X_A^n$.

When $\gamma_n = n$, the rate of division errors $\xi_n(x) = \frac{1}{2}nx(1-x)$ is of the same order as the rate of reactions. Hence, a rigorous approximation of $X_A^n(t)$ in terms of n can be made in terms of the constrained Langevin process (c.f. [LW19; And+19]) which is a reflected diffusion V_n on $[0,1]$ with small noise

$$\begin{aligned} dV_n(t) =& \mu(V_n(t))dt + \frac{1}{\sqrt{n}}\sqrt{r_+(V_n(t)) + r_-(V_n(t)) + V_n(t)(1-V_n(t))}\,dW(t) \\ &+ \frac{1}{\sqrt{n}}\big(\rho_0 dL_0(t) - \rho_1 dL_1(t)\big). \end{aligned} \tag{1.23}$$

Its drift is equal to $\mu(x) = -\kappa_{-1}^{10}x + \kappa_1^{01}(1-x) - \kappa_{-1}^{11}x(1-x) + \kappa_1^{21}x^2(1-x)$ from (1.22) as the division error is unbiased, the diffusion coefficent $\sigma^2(x) = r_+(x) + r_-(x) + x(1-x)$ is equal to the sum of all rates as the square of all jumps are of size $(\pm\frac{1}{n})^2 = \frac{1}{n^2}$, and the reflection directions are $\rho_0 = \rho_1 = 1$ on associated boundary processes L_0 and L_1 respectively. Large deviation theory for path properties of small noise diffusions (Chapter 5, [DZ09] and Chapter 5, [FW98]) also implies this process spends most of its time in the neighbourhood of stable equilibria with rare excursions transitioning between them. The occupation measure of the process will concentrate near the stable equilibria in the long term limit. To estimate the numerical solution for the limiting cumulant generating function ψ_θ of the same additive functional $\Lambda(t)$ as above, we set the jump-diffusion parameters to $\mu(x), \sigma^2(x)$ as above (and $\nu_x \equiv 0$), and we define a sequence of continuous linear piecewise functions f by:

$$f(x) = \begin{cases} 0 & 0 \le x < 0.25 - \frac{1}{N+1} \\ (N+1)x + 1 - 0.25(N+1) & 0.25 - \frac{1}{N+1} \le x < 0.25 \\ -(N+1)x + 1 + 0.25(N+1) & 0.25 \le x < 0.25 + \frac{1}{N+1} \\ 0 & 0.25 + \frac{1}{N+1} \le x < 0.75 - \frac{1}{N+1} \\ (N+1)x + 1 - 0.75(N+1) & 0.75 - \frac{1}{N+1} \le x < 0.75 \\ -(N+1)x + 1 + 0.75(N+1) & 0.75 \le x < 0.75 + \frac{1}{N+1} \\ 0 & 0.75 + \frac{1}{N+1} \le x \le 1. \end{cases}$$

We use $\psi_{\theta,JMP}$ to denote the numerical solution to the PIDE for the limiting cumulant generating function ψ_θ of the jump Markov process $X_A^n(t)$, and $\psi_{\theta,JDA}^N$ for the reflected diffusion approximation $V_n(t)$ in (1.23), where we use $n = 100$ in the first set of results and $n = 1000$ in the second. We empirically verify the stability of the numerical approximation in mesh size in Figure 1.2: surfaces (a) and (b) approximate $\psi_{\theta,JDA}^N$ for $0 \le \theta \le 1$, $\{N\}_{50}^{150}$ and $\{N\}_{500}^{1500}$, respectively; the bottom plots (c) and (d) compare the approximations of $\psi_{\theta,JMP}$ (in red) and $\psi_{\theta,JDA}^N$ (in blue), for $n = N = 100$ and $n = N = 1000$, respectively.

To estimate the mean $\psi'(0)$ and variance $\psi''(0)$ of local times for fixed $n = N = 1000$, we consider centered finite difference approximations using $\theta = \{-0.01, 0, 0.01\}$ and obtain:

$$\psi'_{0,JMP} \approx 1.2 \times 10^{-3}, \quad \psi''_{0,JMP} \approx 1.6 \times 10^{-5};$$
$$\psi'_{0,JDA} \approx 2.6 \times 10^{-3}, \quad \psi''_{0,JDA} \approx 5.2 \times 10^{-5}.$$

Comparing the results for $n = 100$ with $n = 1000$ shows the numerical approximation is more stable as the rescaling parameter n increases, since then the magnitude of noise decreases in both processes. However, the long-term mean local time differs in the two models regardless of the increase in scaling parameter. This is in contrast with the finite time results which say that the paths of the two processes become closer in n, but can be reasoned by the fact that the sup-norm of the path difference features a multiplying constant that is a function of the length of the time interval [Kur78; KKP14], and that our results are based on taking limits as the time of integration goes to infinity, and not as the noise size goes to zero. Our numerical results indicate that the mean of the local time at equilibria are smaller for the jump Markov model than for the reflected diffusion, indicating a tighter long term concentration of the stationary distribution of the reflected JDA at equilibria compared to that of the JMP. This is in full agreement with the results of [MP14] Theorem 3.1 which say that the functional path large deviation rate of jumps between the two stable equilibria is higher for the jump Markov model than for the reflected diffusion. Since more frequent transitions result in a less concentrated measure our present observation follows.

We next consider the case when division errors occur at higher rate $\gamma_n \gg n$ (e.g. $\gamma_n = 10n^2$) than the rates of reactions. The $\pm\frac{1}{n}$ errors from division now dominate the noise and it is more suitable to approximate the model instead by a process in which only the reaction contributions are modeled by a constrained Langevin equation with small noise while the error division contributions are still modeled by pure jumps. The jump-diffusion model in part reduces computation, but also allows a comparison with the approximate diffusion model in the case $\gamma_n = n$. We now define $\widetilde{V}_n$ to be the following reflected jump-diffusion

$$
\begin{aligned}
d\widetilde{V}_n(t) =& \mu(\widetilde{V}_n(t))dt + \frac{1}{\sqrt{n}}\sqrt{r_+(\widetilde{V}_n(t)) + r_-(\widetilde{V}_n(t))}dW(t) \pm \frac{1}{n}dY_{\pm}^{\xi(\widetilde{V}_n)}(t) \\
&+ \frac{1}{\sqrt{n}}\big(\rho_0 dL_0(t) - \rho_1 dL_1(t)\big),
\end{aligned} \tag{1.24}
$$

where each $Y_-^{\xi(\widetilde{V}_n)}$ and $Y_+^{\xi(\widetilde{V}_n)}$ are counting processes with intensity measure $\xi_n(\widetilde{V}_n) = \frac{1}{2}\gamma_n\widetilde{V}_n(1-\widetilde{V}_n)$. We use $\gamma_n = 10n^2$ and present the results for the local time at the same two equilibria $\{0.25, 0.75\}$ in Figure 1.3.

Derivatives of ψ_θ near $\theta = 0$ inferred from our results for $n = N = 1000$ using centered finite differences at points $\theta = \{-0.01, 0, 0.01\}$:

$$
\begin{aligned}
\psi'_{0,JMP} &\approx 7.1 \times 10^{-3}, \quad \psi''_{0,JMP} \approx 2.9 \times 10^{-5}; \\
\psi'_{0,JDA} &\approx 9.1 \times 10^{-3}, \quad \psi''_{0,JDA} \approx 6.9 \times 10^{-2}
\end{aligned}
$$

indicate that in this scenario the reflected jump-diffusion is a closer approximation of the long term behaviour of the original model. This is a consequence of the fact that the dominant noise comes from jumps that are now present in the same (un-approximated) form in the reflected jump-diffusion as specified in the original model.

Since the magnitude of division error rates is not influenced by the rates of chemical reactions in the system, it is crucial to have a way to distinguish their order of magnitude, which we argue can be done using time-additive functionals (dynamical observables) within experiments. One possible indicator is the local time at the deterministic stable equilibria: when $\gamma_n \gg n$ (e.g. $\gamma_n = 10n^2$) both the jump Markov model and the reflected jump-diffusion from (1.24) spend less time near equilibria, as the stronger noise from division errors counteracts the pull towards the stable equilibria of the drift μ which is defined by the system of reactions [MP14]. Since we have numerically obtained the limiting cumulant generating function for the model with $\gamma_n = n$, we can argue that the comparison of the average value of this local time with that of its average value for the model with $\gamma_n = 10n^2$ provides a distinction between the two cases.

To establish a quantifiable indicator of distinction between the two cases, we also measure the long term local times of the jump Markov model and of the reflected jump-diffusion (1.24) in a neighborhood of the boundary $B_{1/(N+1)}(\{0, 1\})$. All the numerical algorithm parameters are the same as above, except that the piecewise linear continuous functions f approximating $\mathbf{1}_{x\in B_{1/(N+1)}(0)\cup B_{1/(N+1)}(1)}$ are now:

$$
f(x) = \begin{cases}
1 & 0 \le x \le \frac{1}{N+1} \\
2 - (N+1)x & \frac{1}{N+1} \le x < \frac{2}{N+1} \\
0 & \frac{2}{N+1} \le x < 1 - \frac{2}{N+1} \\
2 + (N+1)(x-1) & 1 - \frac{2}{N+1} \le x < 1 - \frac{1}{N+1} \\
1 & 1 - \frac{1}{N+1} \le x \le 1.
\end{cases}
$$

An extra mesh point was used to account for the asymmetry of the neighborhood around the boundary points, which results in more numerically stable outputs.

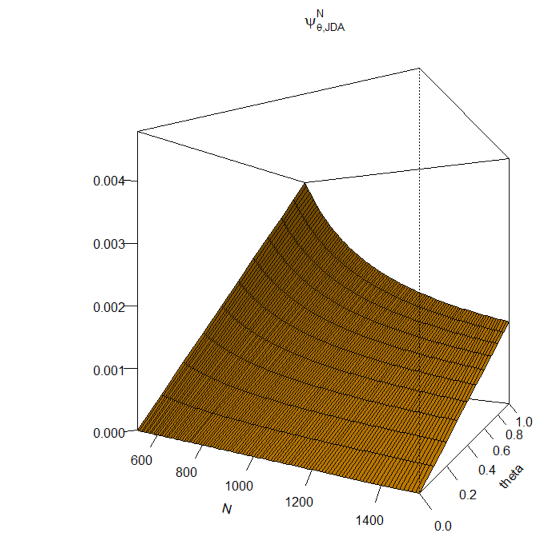
Figure 1.4 displays the results, and the derivatives near $\theta = 0$ for this ψ_θ using $n = N = 1000$ and centered finite differences at $\theta = \{-0.01, 0, 0.01\}$ are approximately:

$$\psi'_{0,JMP} \approx 2.6 \times 10^{-1}, \quad \psi''_{0,JMP} \approx 6.6 \times 10^{-3};$$
$$\psi'_{0,JDA} \approx 1.8 \times 10^{-1}, \quad \psi''_{0,JDA} \approx 5.2 \times 10^{-2}.$$

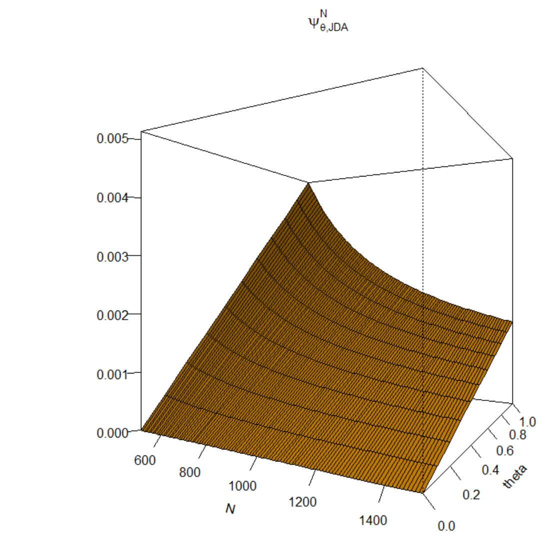
This confirms that the jump-diffusion approximation also accurately reflects the fact that the original process spends substantially more time reflecting at the boundaries than staying near its stable equilibria. In fact, since the noise in the model (after rescaling time) is not small, as the rescaling parameter n increases the paths of the process in finite time are not converging closer to a deterministic path. Instead they exhibit fast passages through the interior of $(0, 1)$ spending most of the time waiting for a reaction on the boundary $\{0, 1\}$ to push it away from the boundary back into the interior. This fully matches the observations of the functional path behaviour of this model in Proposition 4.1 and Figure 3 in Section 4.2 in [MP14]. This type of behaviour has also been called 'discreteness-induced transitions' when analyzed in related models (c.f. [TK01; BDM14; BKW20]).

A comparison of the average long term local time near the boundaries versus the average time at the stable equilibria then presents a quantifiable indicator for distinguishing the order of magnitude of division error intensity, and for retaining jumps in the reflected jump-diffusion approximation of the original model as crucial in the second case. Our example illustrates that numerical estimates of local times constitute valuable dynamic observables for reflected processes with drift of this form. They can be used to assess the closeness of approximating processes to the original model in the long-term (infinite time), as well as to distinguish the order of magnitude of an unbiased source of noise unseen by the drift of the process.

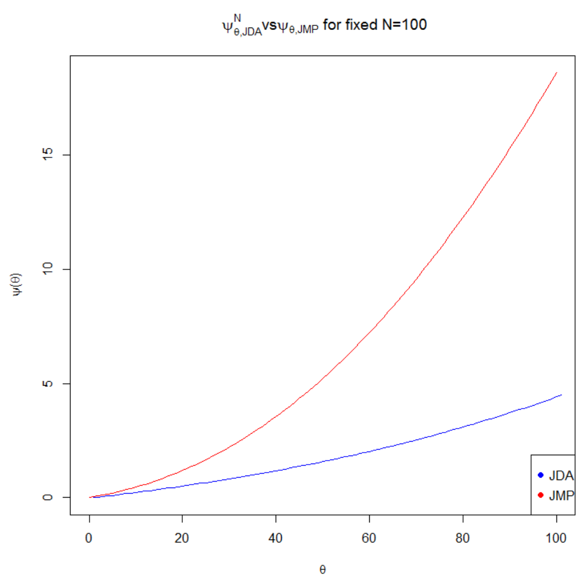
A last comment on the stability of our numerical scheme in this example: we found that mesh sizes comparable to jump sizes in the process work well, while further refinements can lead to numerical instabilities.



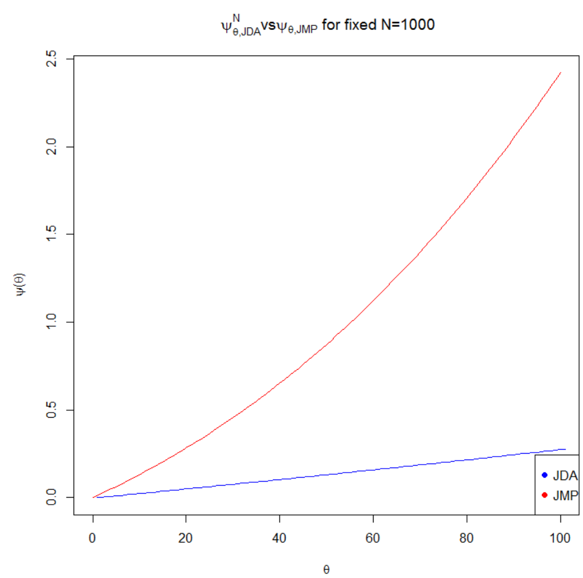
(a) $\psi^N_{\theta,JDA}$ plot for $n = 100$, $0 \leq \theta \leq 1$ as $\{N\}_{50}^{150}$ increases (reflected diffusion)



(b) $\psi^N_{\theta,JDA}$ plot for $n = 1000$, $0 \leq \theta \leq 1$ as $\{N\}_{500}^{1500}$ increases (reflected diffusion)

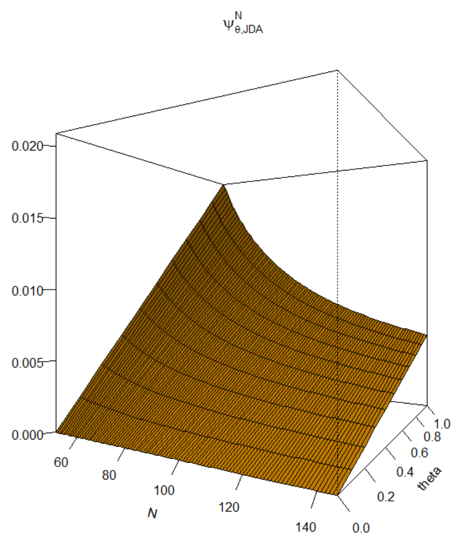


(c) $\psi_{\theta,JMP}$ and $\psi^N_{\theta,JDA}$ plots for $n = N = 100$, $0 \leq \theta \leq 100$.

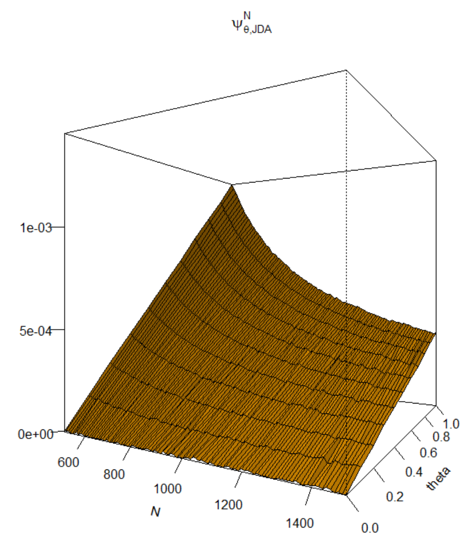


(d) $\psi_{\theta,JMP}$ and $\psi^N_{\theta,JDA}$ plots for $n = N = 1000$, $0 \leq \theta \leq 100$.

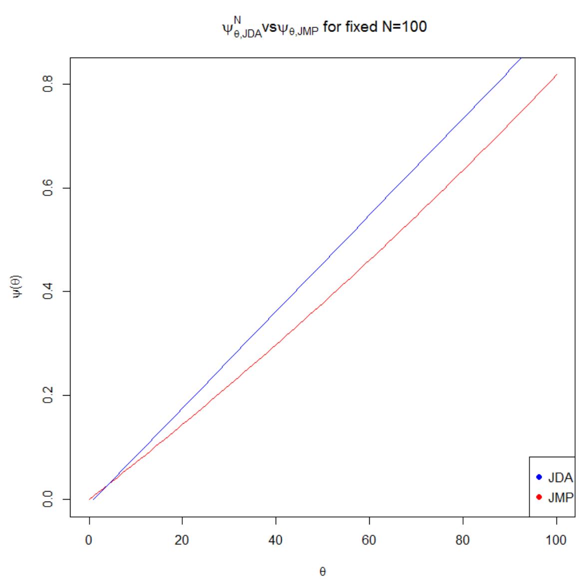
Figure 1.2: Numerical estimate of the limiting cumulant generating function ψ_θ for the long-term local time of $\{0.25, 0.75\}$ of the jump Markov process $\psi_{\theta,JMP}$ and of the reflected diffusion $\psi^N_{\theta,JDA}$ (case $\gamma_n = n$).



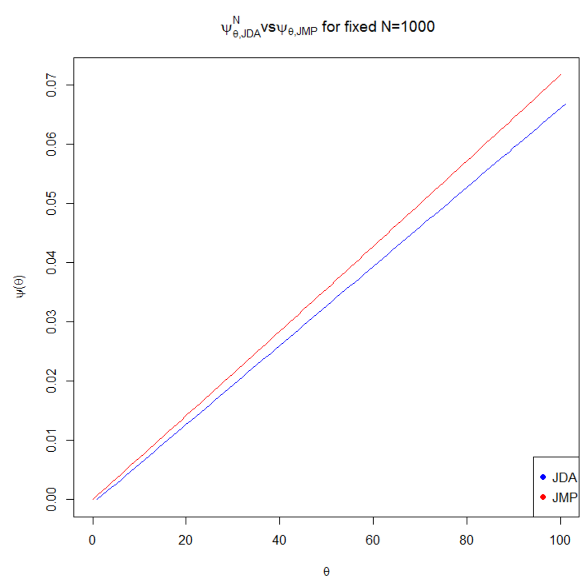
(a) $\psi^N_{\theta,JDA}$ plot for $n = 100$, $0 \leq \theta \leq 1$ as $\{N\}_{50}^{150}$ increases (reflected jump diffusion)



(b) $\psi^N_{\theta,JDA}$ plot for $n = 1000$, $0 \leq \theta \leq 1$ as $\{N\}_{500}^{1500}$ increases (reflected jump diffusion)

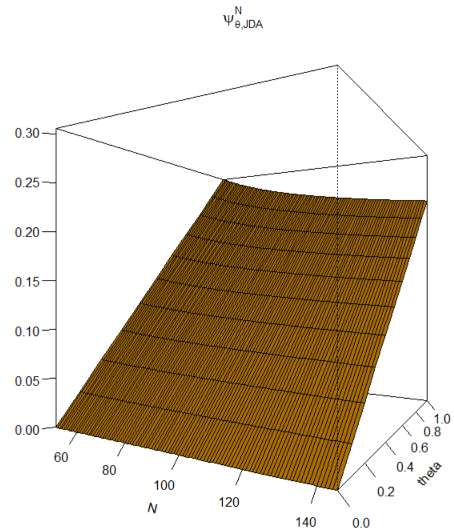


(c) $\psi_{\theta,JMP}$ and $\psi^N_{\theta,JDA}$ plots for $n = N = 100$, $0 \leq \theta \leq 100$.

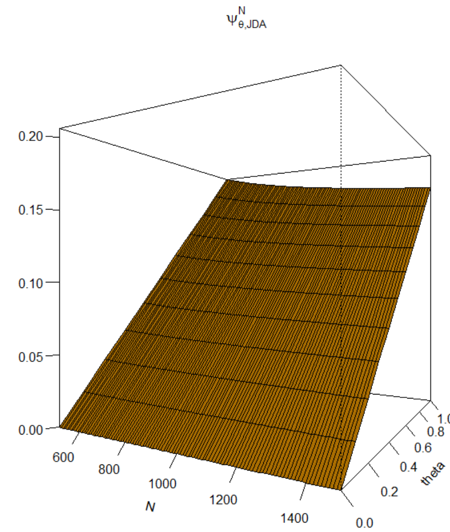


(d) $\psi_{\theta,JMP}$ and $\psi^N_{\theta,JDA}$ plots for $n = N = 1000$, $0 \leq \theta \leq 100$.

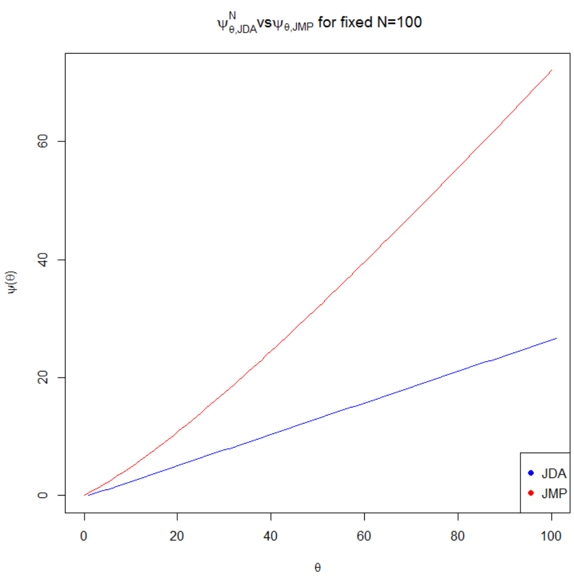
Figure 1.3: Numerical estimates of ψ_θ for the long-term local time of $\{0.25, 0.75\}$ of the jump Markov process $\psi_{\theta,JMP}$ and of the reflected jump-diffusion $\psi^N_{\theta,JDA}$ (case $\gamma_n = 10n^2$).



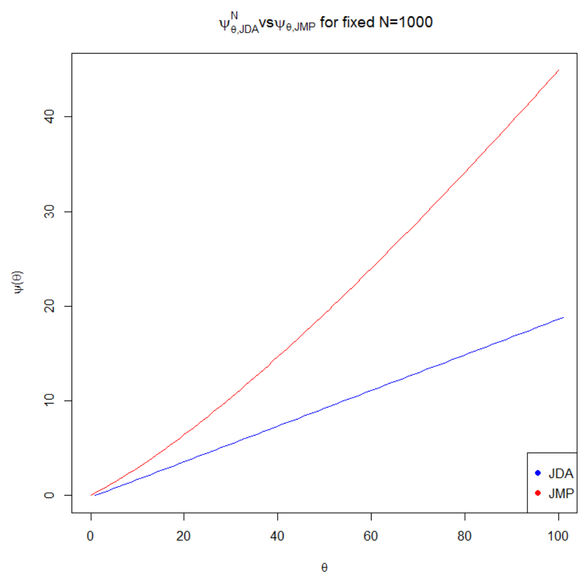
(a) $\psi^N_{\theta,JDA}$ plot for $n=100$, $0 \le \theta \le 1$ as $\{N\}_{50}^{150}$ increases



(b) $\psi^N_{\theta,JDA}$ plot for $n=1000$, $0 \le \theta \le 1$ as $\{N\}_{500}^{1500}$



(c) $\psi_{\theta,JMP}$ and $\psi^N_{\theta,JDA}$ plots for $n=N=100$, $0 \le \theta \le 100$.



(d) $\psi_{\theta,JMP}$ and $\psi^N_{\theta,JDA}$ plots for $n=N=1000$, $0 \le \theta \le 100$.

Figure 1.4: Numerical estimates of ψ_θ for long-term local time of $\{0,1\}$ of the jump Markov process $\psi_{\theta,JMP}$ and of the reflected jump-diffusion $\psi^N_{\theta,JDA}$ ($\gamma_n = 10n^2$).

§1.4 Numerical Solution to PIDE

Fix $\theta \in \mathbb{R}$. Let $\widetilde{\mathcal{L}}$ be the operator

$$\widetilde{\mathcal{L}}u_\theta(x) = \mu(x)\partial_x u_\theta(x) + \frac{\sigma^2(x)}{2}\partial^2_{xx}u_\theta(x) + \theta f(x)u_\theta(x) + \int_{\mathcal{M}}[u_\theta(r(x,y)) - u_\theta(x)]\nu_x(dy);$$
$$\text{with } D(\widetilde{\mathcal{L}}) := \{u_\theta \in C^2([0,b]) : \theta f(0)u_\theta(0) + \rho_0\partial_x u_\theta(0) = 0,\ \theta f(b)u_\theta(b) - \rho_b\partial_x u\theta(b) = 0\}.$$

We will numerically solve for the eigenvalue problem $\widetilde{\mathcal{L}}u_\theta(x) = \psi_\theta u_\theta(x)$, $x \in [0,b]$ subject to the boundary conditions of $D(\widetilde{\mathcal{L}})$, by replacing each derivative by an appropriate finite-difference quotient and the integral term by an appropriate sum.

Select an integer $N > 0$ and divide the length of $[0,b]$ into $(N+1)$ equal subintervals whose endpoints are the mesh points $x_i = ih$, for $i = 0,1,...,N+1$, where $h = \frac{b}{N+1}$. At the interior mesh points, x_i, for $i = 1,2,...,N-1,N$, the PIDE to be approximated is $\widetilde{\mathcal{L}}u_\theta(x_i) = \psi_\theta u_\theta(x_i)$.

Since $\nu_x(dy)$ may be discrete, continuous, or some combination of both, $\mathcal{M}$ as a series of disjoint continuous and discrete intervals. We call the continuous intervals $\mathcal{M}^c_{j_c}$ for $j_c = 1,2,...,n_c$ and the discrete intervals $\mathcal{M}^d_{d_c}$ for $j_d = 1,2,...,n_d$. Then we may rewrite the integral term as

$$\begin{aligned}\int_{\mathcal{M}}[u_\theta(r(x,y)) - u_\theta(x)]\nu_x(dy) = &\sum_{j_c}\int_{\mathcal{M}^c_{j_c}}[u_\theta(r(x,y)) - u_\theta(x)]\nu_x(dy)\\ &+ \sum_{j_d}\int_{\mathcal{M}^d_{j_d}}[u_\theta(r(x,y)) - u_\theta(x)]\nu_x(dy).\end{aligned}$$

The discrete measure may be interpreted as

$$\sum_{j_d}\int_{\mathcal{M}^d_{j_d}}[u_\theta(r(x,y)) - u_\theta(x)]\nu_x(dy) = \sum_{j_d}\sum_{y\in\mathcal{M}^d_{j_d}}[u_\theta(r(x,y)) - u_\theta(x)]\nu_x(y).$$

To approximate the continuous integral, we apply the Composite Trapezoidal rule over each interval $\mathcal{M}^c_{j_c} \equiv [a_{j_c}, b_{j_c}]$. Define the integrand as $g^x(y) \equiv [u_\theta(r(x,y)) - u_\theta(x)]\nu_x(y)$. For each j_c, select an integer $N_{j_c} > 0$ and divide the length of $\mathcal{M}^c_{j_c}$, $(b_{j_c} - a_{j_c})$, into N_{j_c} subintervals. So we have step size $h_{j_c} = \frac{b_{j_c} - a_{j_c}}{N_{j_c}}$ and $y^{j_c}_k = a_{j_c} + kh_{j_c}$ for each $k = 0,1,...,N_{j_c}$. Then the first sum can be written as

$$\sum_{j_c}\int_{\mathcal{M}^c_{j_c}} g^x(y)dy = \sum_{j_c}\left(\frac{h_{j_c}}{2}\left(g^x(a_{j_c}) + 2\sum_{j_c=1}^{N_{j_c}-1} g^x(y^{j_c}_k) + g^x(b_{j_c})\right) - \frac{b_{j_c} - a_{j_c}}{12}h^2_{j_c}(g^x)''(\kappa_{j_c})\right)$$

for some κ_{j_c} in $[a_{j_c}, b_{j_c}]$.

We approximate the derivatives on $(0,b)$ by a centered-difference scheme where η_i and ξ_i are some values in (x_{i-1}, x_{i+1}). Putting all the approximations together results in the finite difference equation

$$\mu(x_i)\left(\frac{u(x_{i+1})-u(x_{i-1})}{2h}-\frac{h^2}{6}u^{(3)}(\eta_i)\right)+\frac{\sigma^2(x_i)}{2}\left(\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2}-\frac{h^2}{12}u^{(4)}(\xi_i)\right)$$
$$+\theta f(x_i)u(x_i)+\sum_{j_c}\left(\frac{h_{j_c}}{2}\left(g^{x_i}(a_{j_c})+2\sum_{j_c=1}^{N_{j_c}-1}g^{x_i}(y_k^{j_c})+g^{x_i}(b_{j_c})\right)-\frac{b_{j_c}-a_{j_c}}{12}h_{j_c}^2(g^{x_i})''(\kappa_{j_c})\right)$$
$$+\sum_{j_d}\sum_{y\in\mathcal{M}_{j_d}^d}[u(r(x_i,y))-u(x_i)]\nu_{x_i}(y)=\psi_\theta u(x_i).$$

We choose forward and backward finite-difference schemes with $O(h^2)$ truncation error to approximate the boundary conditions:

$$\frac{-\frac{3}{2}u_0+2u_1-\frac{1}{2}u_2}{h}=-\frac{f(0)\theta u_0}{\rho_0}\iff u_0=\frac{4\rho_0u_1-\rho_0u_2}{3\rho_0-2\theta f(0)h},$$
$$\frac{\frac{3}{2}u_{N+1}-2u_N+\frac{1}{2}u_{N-1}}{h}=\frac{f(b)\theta u_{N+1}}{\rho_b}\iff u_{N+1}=\frac{\rho_bu_{N-1}-4\rho_bu_N}{2\theta f(b)h-3\rho_b}.$$

After truncating and rearranging together with the boundary conditions, we define the system of linear equations

$$\psi_\theta u_i=\left(-\frac{\mu(x_i)}{2h}+\frac{\sigma^2(x_i)}{2h^2}\right)u_{i-1}+\left(-\frac{\sigma^2(x_i)}{h^2}+\theta f(x_i)\right)u_i+\left(\frac{\mu(x_i)}{2h}+\frac{\sigma^2(x_i)}{2h^2}\right)u_{i+1}$$
$$+\sum_{l=0}^{N+1}\widetilde{g}^{u_l}(x_i)u_l,$$

where each function $\widetilde{g}^{u_l}(x_i)$ represents the sum of all terms in the integral approximations that are factors of u_l, for each $i=1,2,...,N$. Every time u_0 or u_{N+1} is a term in the equation, we replace it with the appropriate boundary condition. This allows us to define the system of equations as an $N\times N$ matrix, with $a_1(x_i)\equiv-\frac{\mu(x_i)}{2h}+\frac{\sigma^2(x_i)}{2h^2}$, $a_2(x_i)\equiv-\frac{\sigma^2(x_i)}{h^2}+\theta f(x_i)$, and $a_3(x_i)\equiv\frac{\mu(x_i)}{2h}+\frac{\sigma^2(x_i)}{2h^2}$, as

$$(\mathbf{A}+\mathbf{G})\mathbf{u}=\psi_\theta\mathbf{u},$$

where

$$\mathbf{A}=\begin{bmatrix}\frac{4\rho_0a_1(x_1)}{3\rho_0-2\theta f(0)h}+a_2(x_1) & -\frac{\rho_0a_1(x_1)}{3\rho_0-2\theta f(0)h}+a_3(x_1) & 0 & \cdots & 0 & 0\\ a_1(x_2) & a_2(x_2) & a_3(x_2) & \cdots & 0 & 0\\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & \cdots & 0 & a_1(x_{N-1}) & a_2(x_{N-1}) & a_3(x_{N-1})\\ 0 & \cdots & 0 & 0 & a_1(x_N)+\frac{\rho_ba_3(x_N)}{2\theta f(b)h-3\rho_b} & a_2(x_N)-\frac{4\rho_ba_3(x_N)}{2\theta f(b)h-3\rho_b}\end{bmatrix},$$

$$\mathbf{G}=\begin{bmatrix}\widetilde{g}^{u_1}(x_1) & \widetilde{g}^{u_2}(x_1) & \widetilde{g}^{u_3}(x_1) & \cdots & \widetilde{g}^{u_N}(x_1)\\ \widetilde{g}^{u_1}(x_2) & \widetilde{g}^{u_2}(x_2) & \widetilde{g}^{u_3}(x_2) & \cdots & \widetilde{g}^{u_N}(x_2)\\ \ddots & \ddots & \ddots & \ddots & \ddots\\ \widetilde{g}^{u_1}(x_N) & \widetilde{g}^{u_2}(x_N) & \widetilde{g}^{u_3}(x_N) & \cdots & \widetilde{g}^{u_N}(x_N)\end{bmatrix},\text{ and }\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_N\end{bmatrix}.$$

Finally, we compute ψ_θ by solving for the eigenvalues of $\mathbf{A}+\mathbf{G}$ and selecting the one with largest value, then its associated eigenvector is the solution $\mathbf{u}$ to the PIDE.

Chapter 2

Functional Central Limit Theorem for a Fast-Slow Dynamical System driven by Symmetric and Multiplicative $\alpha-$Stable Noise

Outline

§2.1 The model with its associated assumptions and definitions are setup. An introduction to the main theorem statements follows.

§2.2 Exponential ergodicity is discussed and the regularity properties of the Poisson equation needed for the main results are studied.

§2.3 The averaging and functional central limit theorem results are proved.

§2.4 Provides an illustrative practical example in a simulation study that illustrates the results of this chapter.

§2.5 Collects some a priori results and repetitive proofs for lemmas that would otherwise detract from the exposition.

Notation

Let k be some positive integer and $\delta \in (0,1)$.

$C_\cdot$	constant dependent on $\cdot$
$\lvert\cdot\rvert$	Euclidean vector norm
$\langle\cdot,\cdot\rangle$	Euclidean inner product
$\lvert\lvert\cdot\rvert\rvert$	Matrix norm
$C^k(\mathbb{R}^d)$	$\{\phi : \mathbb{R}^d \to \mathbb{R} \mid \phi$ and all its partial derivatives up to order k are continuous$\}$
$C_b^k(\mathbb{R}^d)$	$\{\phi \in C^k(\mathbb{R}^d) \mid$ for $1 \leq i \leq k,$ the i-th partial derivatives are bounded$\}$
$C_b^{k+\delta}(\mathbb{R}^d)$	$\{\phi \in C_b^k(\mathbb{R}^d) \mid$ all the k-th partial derivatives are δ-Hölder continuous$\}$

In general, β is reserved for use with multi-index notation, understood as a d-dimensional tuple. So $|\beta| = |(\beta_1, \beta_2, ..., \beta_d)| = \beta_1+\beta_2+...+\beta_k$, $\beta! = \beta_1!\beta_2!...\beta_k!$, and for any d-dimensional vector $x^\beta = x_1^{\beta_1}x_2^{\beta_2}...x_d^{\beta_d}$. Let $\delta_1, \delta_2 \in (0,1)$ and k_1, k_2 be positive integers with the real valued function $\phi : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. The notation $\phi \in C_b^{k_1+\delta_1, k_2+\delta_2}$ reproduced from [SXX22] means that:

(i) let β_1, β_2 be d_1, d_2-tuples, respectively, with $|\beta_1|+|\beta_2| \geq 1$ such that $0 \leq |\beta_1| \leq k_1, 0 \leq |\beta_2| \leq k_2$, then all partial derivatives $\partial_x^{\beta_1}\partial_y^{\beta_2}\phi$ are bounded continuous;

(ii) for any $|\beta_1| = k_1$ and $0 \leq |\beta_2| \leq 1$, $\partial_x^{\beta_1}\partial_y^{\beta_2}\phi$ is δ_1-Hölder continuous with respect to x and index δ_1 uniformly in y;

(iii) and for any $|\beta_2| = k_2$, $\partial_y^{\beta_2}\phi$ is δ_2-Hölder continuous with respect to y with index δ_2 uniformly in x.

§2.1 Preliminaries

Fix $0 < \epsilon < 1$ and consider the following fast-slow dynamical system in $\mathbb{R}^{d_1 \times d_2}$ driven by multiplicative α-stable processes:

$$\begin{cases} dX_t^\epsilon = f(X_t^\epsilon, Y_t^\epsilon)dt + \epsilon^\rho c\left(X_t^\epsilon, Y_t^\epsilon\right) dL_{t,1}, \quad X_0^\epsilon = x \in \mathbb{R}^{d_1}, \\ dY_t^\epsilon = \epsilon^{-1}g(X_t^\epsilon, Y_t^\epsilon)dt + \epsilon^{-\frac{1}{\alpha}}b\left(X_t^\epsilon, Y_t^\epsilon\right) dL_{t,2}, \quad Y_0^\epsilon = y \in \mathbb{R}^{d_2}, \end{cases} \tag{2.1}$$

where $\{L_{t,1}\}_{t\geq 0}$ and $\{L_{t,2}\}_{t\geq 0}$ are independent d_1, d_2-dimensional symmetric α-stable Lévy motions, with stability index $\widetilde{\alpha} \in (1,2]$, and $\alpha \in (1,2)$, respectively, and the parameter $\rho > 1 - \frac{1}{\alpha}$. The coefficient functions are assumed to be Borel measurable and satisfy:

$$\begin{aligned} &f \in C_b^{2+\gamma,2+\delta} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}, \\ &g \in C_b^{1+\gamma,2+\gamma} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}, \\ &c \in C_b^{1,1} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1} \times \mathbb{R}^{d_1}, \\ &b \in C_b^{1+\gamma,2+\gamma} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2} \times \mathbb{R}^{d_2}, \end{aligned} \tag{A1}$$

where $\gamma \in (\alpha - 1, 1)$ and $\delta \in (0,1)$. Call $\{X_t^\epsilon\}_{t\geq 0}$ the slow process and $\{Y_t^\epsilon\}_{t\geq 0}$ the fast process; a simple substitution $t \leftarrow \epsilon t$ shows that Y_t^ϵ evolves at an order $\frac{1}{\epsilon}$ faster than X_t^ϵ by the $\frac{1}{\alpha}$ self-similarity property of α-stable motion.

Remark 2.1. *All first derivatives of the functions f, g, c, b are bounded. Therefore, each of these functions is globally Lipschitz and consequently satisfies the necessary growth condition. It then follows that there exists a unique solution $\{(X_t^\epsilon, Y_t^\epsilon)\}_{t\geq 0}$ to (2.1) (see Theorem 6.2.3, [App09]).*

Due to the lack of square integrability related to the Lévy measure of $L_{t,2}$ and the generator to be introduced, it is not possible to include $\alpha = 2$ in the interval $(1,2)$ or the calculations would not follow. In practice, much has already been studied with respect to dynamical systems forced by Brownian motion and so this does not pose a great limitation in application. By understanding that when $\widetilde{\alpha} = 2$ then $L_{t,1}$ is Brownian motion, no such issue arises in the noise of the slow process.

Furthermore, notice that the function c does not depend on ϵ and so cannot be made to vanish in ϵ (see I_3 in eq 2.60 below). The case $\rho = 1 - \frac{1}{\alpha}$ would lead to a constant which is not sufficient to control the U_t^ϵ term below and the slow process noise could escape to infinity. The strict inequality is necessary.

Define the frozen process as the equation that satisfies

$$dY_t^{x,y} = g(x, Y_t^{x,y}) + b(x, Y_t^{x,y})dL_{t,2}, \quad Y_0^{x,y} = y \in \mathbb{R}^{d_2}. \tag{2.2}$$

Intuitively, this equation describes the evolution of the fast process if the slow process were held fixed on the fast process' time scale. That is,

$$Y_{\epsilon t}^\epsilon = g(X_{\epsilon t}^\epsilon, Y_{\epsilon t}^\epsilon)dt + b(X_{\epsilon t}^\epsilon, Y_{\epsilon t}^\epsilon)dL_{t,2},$$

by the $\frac{1}{\alpha}$ self-similarity property, and then fixing $x = X_{\epsilon t}^\epsilon$. The generator of the frozen process acts as the Poisson equation operator and of great interest to the proof of the averaging principle.

Remark 2.2. *Recall the generator of the α-stable Lévy motion is given by the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ which has the following definition for some function $\phi : \mathbb{R}^d \to \mathbb{R}$*

$$-(-\Delta)^{\frac{\alpha}{2}}\phi(y) := \frac{\alpha 2^{\alpha-1}\Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}}\Gamma(1-\frac{\alpha}{2})} \int_{\mathbb{R}^d\setminus\{0\}} \frac{\phi(y+z) - \phi(y) - \mathbf{1}_{\{|z|<1\}}\langle z, \nabla\phi(y)\rangle}{|z|^{d+\alpha}} dz.$$

Consider the Lévy-Itô decomposition, with small jumps less than 1, for the α-stable Lévy motion

$$dL_{t,2} = \int_{|z|\le 1} z\widetilde{N}_2(dt, dz) + \int_{|z|>1} zN_2(dt, dz).$$

Then one may rewrite $Y_t^{x,y}$ as

$$dY_t^{x,y} = g(x, Y_t^{x,y})dt + \int_{|z|\le 1} b(x, Y_t^{x,y})zd\widetilde{N}_2(dt, dz) + \int_{|z|>1} b(x, Y_t^{x,y})zdN_2(dt, dz),$$

where the Poisson random measure is given by $N_2(t,\zeta) = \sum_{s\le t} \mathbf{1}_\zeta(L_{s,2} - L_{s-,2})$ and the compensated Poisson random measure is given by $\widetilde{N}_2(t,\zeta) = N_2(t,\zeta) - t\nu_2(\zeta)$, with the Lévy measure $\nu_2(dz) = \frac{C_{\alpha,d}dz}{|z|^{d_2+\alpha}}$ where the normalizing constant is given by $C_{\alpha,d} = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}}\Gamma(1-\frac{\alpha}{2})}$.

In this form, it is easy to read the generator (Theorem 6.7.4, [App09]),

$$\begin{aligned}\mathcal{L}\phi^x(y) =& \langle g(x,y), \nabla\phi^x(y)\rangle \\ &+ C_{\alpha,d}\int_{\mathbb{R}^{d_2}\setminus\{0\}} \frac{\phi^x(y + b(x,y)z) - \phi^x(y) - \mathbf{1}_{|z|<1}\langle b(x,y)z, \nabla\phi^x(y)\rangle}{|z|^{d_2+\alpha}} dz.\end{aligned}$$

Now substitute $u := b(x,y)z$ to see an equivalent, more compact form for the generator of $Y_t^{x,y}$ involving the fractional Laplacian,

$$\mathcal{L}\phi^x(y) = \langle b(x,y)^\alpha, -(-\Delta)^{\frac{\alpha}{2}}\phi^x(y)\rangle + \langle g(x,y), \nabla\phi^x(y)\rangle. \tag{2.3}$$

Consider the following definitions which recur throughout the chapter. For any bounded measurable function $\phi : \mathbb{R}^{d_2} \to \mathbb{R}$, define the transition semigroup of $Y_t^{x,y}$ by

$$P_t^x \phi(y) := E\phi(Y_t^{x,y}), \quad y \in \mathbb{R}^{d_2}, t \geq 0. \tag{2.4}$$

Associated to the semigroup is the invariant probability measure which satisfies

$$\mu^x(P_t^x \phi) = \mu^x(\phi). \tag{2.5}$$

$\{P_t^x\}_{t\geq 0}$ is said to be φ-uniformly exponentially ergodic if there are constants $C, \kappa > 0$ and a measurable function $\varphi : \mathbb{R}^{d_2} \to [1, \infty)$ such that

$$\sup_{\phi \leq \varphi} |P_t^x \phi(y) - \mu^x(\phi)| \leq C e^{-\kappa t} \varphi(y), \quad t \geq 0, y \in \mathbb{R}^{d_2}. \tag{2.6}$$

Assume that there exists a $\kappa > 0$ such that for all $x \in \mathbb{R}^{d_1}$ and $y_1, y_2 \in \mathbb{R}^{d_2}$, $\sup_{x \in \mathbb{R}^{d_1}} |f(x, 0)| < \infty$ and the following drift condition holds:

$$\begin{aligned} &\langle g(x, y_1) - g(x, y_2), y_1 - y_2 \rangle + \textstyle\int_{|z|\geq 1} \langle (b(x, y_1) - b(x, y_2))\, z, y_1 - y_2 \rangle \nu_2(dz) \\ &+ \textstyle\int_{\mathbb{R}^{d_2}} |b(x, y_1) - b(x, y_2)|^2 z^2 \nu_2(dz) \leq -\kappa |y_1 - y_2|^2. \end{aligned} \tag{A2}$$

This assumption will be necessary to establish exponential ergodicity in Lemma 2.2 below; but first, one must show that there exists a unique invariant probability measure for $\{P_t^x\}_{t\geq 0}$.

Remark 2.3. *Let*

$$q := \frac{2^{4+2\alpha} \pi^{\frac{d_2}{2}} \Gamma\left(\frac{d_2+\alpha}{2}\right)}{\Gamma\left(\frac{d_2}{2}\right)^2 \Gamma\left(\frac{2-\alpha}{2}\right)} \left(\frac{\alpha}{2-\alpha} + \frac{\alpha}{(\alpha - p)p} \right).$$

Since $g \in C_b^{1+\gamma, 2+\gamma}, b \in C_b^{1+\gamma, 2+\gamma}$, there exists a constant $C > 0$ such that for all $\xi \in \mathbb{R}^{d_2}$ and locally bounded measurable functions ϕ_1, ϕ_2 on $\mathbb{R}^{d_2}$,

$$C^{-1}|\xi| \leq |b(x, y)\xi| \leq C|\xi|, \text{ and} \tag{2.7}$$

$$|g(x, y + h) - g(x, y)| \leq \phi_1 |h|^\gamma, \quad |b(x, y + h) - b(x, y)| \leq \phi_2 |h|^\gamma. \tag{2.8}$$

Furthermore, from condition (A2),

$$\begin{aligned} &\langle g(x, y), y \rangle + \epsilon_0 \phi_1 |y| + q\left(|b(x, y)| + \epsilon_0 \phi_2\right)^\alpha |y|^{2-\alpha} \\ =&\langle g(x, y) - g(x, 0), y \rangle \\ &+ \int_{|z|\geq 1} \langle (b(x, y) - b(x, 0))\, z, y \rangle \nu_2(dz) + \int_{\mathbb{R}^{d_2}} |b(x, y) - b(x, 0)|^2 z^2 \nu_2(dz) \\ &- \int_{\mathbb{R}^{d_2}} |b(x, y) - b(x, 0)|^2 z^2 \nu_2(dz) - \int_{|z|\geq 1} \langle (b(x, y) - b(x, 0))\, z, y \rangle \nu_2(dz) \\ &+ \epsilon_0 \phi_1 |y| + \langle g(x, 0), y \rangle + q\left(|b(x, y)| + \epsilon_0 \phi_2\right)^\alpha |y|^{2-\alpha} \\ \leq& -\kappa |y|^2 + 2C||b||^2 + 2C||b|||y| + \epsilon_0 \phi_1 |y| + |g||y| + q\left(||b|| + \epsilon_0 \phi_2\right)^\alpha (1 + |y|) \\ =& -\kappa |y|^2 + \left(2C||b|| + \epsilon_0 \phi_1 + |g| + q(||b|| + \epsilon_0 \phi_2)^\alpha\right) |y| + \left(2C||b||^2 + q(||b|| + \epsilon_0 \phi_2)^\alpha\right). \end{aligned}$$

Then by Young's inequality,

$$\begin{aligned}
&= -\kappa|y|^2 + \frac{\kappa|y|^2}{2} + \frac{\left(2C||b|| + \epsilon_0\phi_1 + |g| + q(||b|| + \epsilon_0\phi_2)^\alpha\right)^2}{2\kappa} && (2.9)\\
&\quad + \left(2C||b||^2 + q(||b|| + \epsilon_0\phi_2)^\alpha\right)\\
&\leq -\frac{\kappa}{2}|y|^2 + C. && (2.10)
\end{aligned}$$

(2.7) and (2.8) satisfy assumption (H_{loc}) and (2.9) satisfies assumption ($H_{glo}^{r,q}$) with $r = 0$ and $p \in [1, \alpha)$ in Theorem 1.2 of [ZZ23]. Therefore, there exists a unique invariant probability measure μ^x associated to $\{P_t^x\}_{t\geq 0}$.

Lastly, the next remark establishes another quick inequality that is derived directly from the assumption (A2) which will prove useful in the lemmas to come.

Remark 2.4. *Let $\epsilon_0 > 0$ and set $y_1 = y$, $y_2 = y + \epsilon_0$ in (A2) for some fixed $d \in \mathbb{R}^{d_2}$. Then,*

$$\begin{aligned}
&\langle (g(x, y + \epsilon_0 d) - g(x, y)) \frac{\epsilon_0 d}{\epsilon_0 d}, \epsilon_0 d\rangle + \int_{|z|\geq 1} \langle \left((b(x, y + \epsilon_0 d) - b(x, y)) \frac{\epsilon_0 d}{\epsilon_0 d} \right) z, \epsilon_0 d\rangle \nu_2(dz)\\
&+ \int_{\mathbb{R}^{d_2}} \left| (b(x, y + \epsilon_0 d) - b(x, y)) \frac{\epsilon_0 d}{\epsilon_0 d} \right|^2 z^2 \nu_2(dz)\\
&\leq -\kappa|\epsilon_0 d|^2\\
\Longrightarrow \epsilon_0^2 &\left(\langle (g(x, y + \epsilon_0 d) - g(x, y)) \frac{d}{\epsilon_0 d}, d\rangle + \int_{|z|\geq 1} \langle \left((b(x, y + \epsilon_0 d) - b(x, y)) \frac{d}{\epsilon_0 d} \right) z, d\rangle \nu_2(dz) \right.\\
&\left. + \int_{\mathbb{R}^{d_2}} \left| (b(x, y + \epsilon_0 d) - b(x, y)) \frac{d}{\epsilon_0 d} \right|^2 z^2 \nu_2(dz) \right)\\
&\leq -\kappa|\epsilon_0 d|^2.
\end{aligned}$$

Now take $\epsilon_0 \to 0$,

$$\langle \nabla_y g(x, y)d, d\rangle + \int_{|z|\geq 1} \langle \nabla_y b(x, y)zd, d\rangle \nu_2(dz) + \int_{\mathbb{R}^{d_2}} |\nabla_y b(x, y)d|^2 z^2 \nu_2(dz) \leq -\kappa|d|^2. \quad (2.11)$$

The following two results will be proved in Section 2.3, the second of which is the main objective of this chapter. The proposition related to the Poisson equation is interesting in its own right, but the application of the result is the main concern, and therefore will be introduced in Section 2.2.

Theorem 2.1 Averaging Principle

For any initial conditions $(x,y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $T > 0$ and $p \in [1, \alpha)$, there exists a constant $C_{p,T,x,y} > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^\epsilon - \overline{X}_t\right|^p\right] \leq C_{p,T,x,y}\epsilon^{p\left(1-\frac{1}{\alpha}\right)}, \tag{2.12}$$

where $\overline{X}$ is the solution to the averaged equation

$$\overline{X}_t := x + \int_0^t \overline{f}\left(\overline{X}_s\right) ds, \tag{2.13}$$

and $\overline{f}(x) := \int_{\mathbb{R}^{d_2}} f(x,y)\mu^x(dy)$ is the averaged drift where μ^x is the unique invariant measure for frozen equation 2.2.

A Poisson equation involving the generator and the drift terms of the slow and averaged process will be used to establish the rate of convergence in Theorem 2.1. This rate hints at how to appropriately scale the deviations for the next theorem. Furthermore, the solution to the Poisson equation itself aids in fully characterizing the limit of the scaled deviations (eq (2.17) below). In short, this auxiliary Poisson equation provides a technical foundation upon which the main theory rests. It will be introduced and solved shortly (see Section 2.2).

Theorem 2.2 Functional Central Limit

Define the scaled deviations between the slow process and its averaged process by

$$V_t^\epsilon := \epsilon^{\frac{1}{\alpha}-1}\left(X_t^\epsilon - \overline{X}_t\right), \tag{2.14}$$

and let $\mathcal{L}$ be the generator of $Y_t^{x,y}$. Suppose $h(x,y)$ is the unique solution of the Poisson equation

$$-\mathcal{L}h(x,y) = f(x,y) - \overline{f}(x). \tag{2.15}$$

Then for any $T > 0$

$$V_t^\epsilon \Rightarrow V_t,\ \ t \in [0,T], \tag{2.16}$$

where $V_0 = 0$ and V_t solves

$$V_t = \int_0^t \nabla_x \overline{f}\left(\overline{X}_s\right) V_s ds + M_t \tag{2.17}$$

where

$$M_t := \int_0^t \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_2}} h\left(\overline{X}_s, y + b\left(\overline{X}_s, y\right) z\right) - h\left(\overline{X}_s, y\right) \mu^{\overline{X}_s}(dy)\widetilde{N}_2(ds,dz). \tag{2.18}$$

The main difficulties involved relate to the lack of square integrability of α-stable Lévy noise. Finite p-th moments require a little more care in the calculations. Ensuring the solution to the non-local Poisson equation exists and that this solution satisfies all the necessary regularity properties, in the presence of the noise pre-factors, leads to a number of technical lemmas with extra terms to address (see Section 2.5). Before the proofs of the main results, the above mentioned regularity properties for the Poisson equation are addressed.

§2.2 Poisson Equation

This section addresses the non-local Poisson equation in $\mathbb{R}^{d_2}$. The lemmas and corollaries necessary for Proposition 2.1 are stated but proved in Section 2.5 as they only serve a technical function and are often repetitive in nature. Lemmas 2.2, 2.5, and 2.8 are directly referenced in the proposition, but each lemma and corollary is important as they all progressively build on each other.

This first lemma establishes a bound between two frozen processes which is used often within other corollaries and lemmas.

Lemma 2.1. *For any $t \geq 0$, $x_1, x_2 \in \mathbb{R}^{d_1}, y_1, y_2 \in \mathbb{R}^{d_2}$, there exists a constant $C > 0$ such that*

$$E\,|Y_t^{x_1,y_1} - Y_t^{x_2,y_2}| \leq e^{-\frac{\kappa t}{2}}|y_1 - y_2| + C|x_1 - x_2|. \tag{2.19}$$

Proof. See Section 2.5. □

These gradient estimates are a first quick application of Lemma 2.1.

Corollary 2.1. *For any $t \geq 0$, there exists a constant $C > 0$ such that*

$$\sup_{x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} E\,||\nabla_y Y_t^{x,y}|| \leq Ce^{-\frac{\kappa t}{2}}, \qquad \sup_{t\geq 0, x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} E\,||\nabla_x Y_t^{x,y}|| \leq C. \tag{2.20}$$

Proof. See Section 2.5. □

The following lemma says that the semigroup associated to the frozen process is $(1 + |y|)$-uniformly exponentially ergodic. Exponential ergodicity is crucial in establishing the regularity properties of the Poisson equation.

Lemma 2.2. *For any function $\phi \in C_b^1$, there exists a constant $C > 0$ such that for any $t \geq 0$ and $y \in \mathbb{R}^{d_2}$,*

$$\sup_{x\in\mathbb{R}^{d_1}} |P_t^x \phi(y) - \mu^x(\phi)| \leq C||\phi||_1 e^{-\frac{\kappa t}{2}}(1 + |y|), \tag{2.21}$$

where $||\phi||_1 := \sup\limits_{x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|}$.

Proof. From the definition of the transition semigroup and invariant measure for the frozen process

$$\begin{aligned}|P_t^x\phi(y)-\mu^x(\phi)| &= |E\phi(Y_t^{x,y})-\mu^x\left(P_t^x\phi\right)| \\ &= \left|E\phi\left(Y_t^{x,y}\right)-\int_{\mathbb{R}^{d_2}}E\phi\left(Y_t^{x,z}\right)\mu^x(dz)\right| \le ||\phi||_1\int_{\mathbb{R}^{d_2}}E\left|Y_t^{x,y}-Y_t^{x,z}\right|\mu^x(dz),\end{aligned}$$

which by Lemma 2.1 is

$$\le ||\phi||_1\epsilon^{-\frac{\kappa t}{2}}\int_{\mathbb{R}^{d_2}}|y-z|\mu^x(dz) \le C||\phi||_1\epsilon^{-\frac{\kappa t}{2}}(1+|y|), \tag{2.22}$$

since $\sup_{x\in\mathbb{R}^{d_1}}\int_{\mathbb{R}^{d_2}}|z|^p\mu^x(dz)<\infty$ for $p\in[1,\alpha)$ (see eq (2.39) in Section 2.5). □

Similarly to Lemma 2.1 and Corollary 2.1 above, but for the gradient with respect to x.

Lemma 2.3. *For any $t\ge 0$, $x_1,x_2\in\mathbb{R}^{d_1}, y_1,y_2\in\mathbb{R}^{d_2}$, there exists a constant $C>0$ such that*

$$E\left|\left|\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}\right|\right| \le C\left(|x_1-x_2|^\gamma+|x_1-x_2|\right)+Ce^{-\frac{\kappa t}{4}}|y_1-y_2|. \tag{2.23}$$

Proof. See Section 2.5. □

Corollary 2.2. *For any $t\ge 0$, there exists a constant $C>0$ such that*

$$\sup_{x\in\mathbb{R}^{d_1},y\in\mathbb{R}^{d_2}}E\left|\left|\nabla_y\nabla_x Y_t^{x,y}\right|\right| \le Ce^{-\frac{\kappa t}{4}}. \tag{2.24}$$

Proof. See Section 2.5. □

This definition, used to simplify the notation, is frequently encountered throughout the remainder of this section. Define

$$\widehat{f}(x,y,t):=E\left[f\left(x,Y_t^{x,y}\right)\right] \quad \text{and} \quad \widetilde{f}_{t_0}(x,y,t):=\widehat{f}(x,y,t)-\widehat{f}(x,y,t+t_0). \tag{2.25}$$

The following two lemmas establish bounds for these expectations.

Lemma 2.4. *For any $\theta\in(0,1]$ there exists a $C_\theta>0$ such that for any $t\ge 0, x\in\mathbb{R}^{d_1}, y_1,y_2\in\mathbb{R}^{d_2}$,*

$$\left|\left|\nabla_x\widehat{f}(x,y_1,t)-\nabla_x\widehat{f}(x,y_2,t)\right|\right| \le C_\theta e^{-\frac{\kappa\theta t}{4}}|y_1-y_2|^\theta. \tag{2.26}$$

Proof. See Section 2.5. □

Lemma 2.5. *For any $\theta\in(0,1]$, there exists a $C_\theta>0$ and $\eta>0$ such that for any $t_0>0, x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}$,*

$$\left|\left|\nabla_x\widetilde{f}_{t_0}(x,y,t)\right|\right| \le C_\theta e^{-\eta t}\left(1+|y|^\theta\right). \tag{2.27}$$

Proof. See Section 2.5. □

Again, the following lemma-corollary pair bound the difference for the gradient in y.

Lemma 2.6. *For any $t \geq 0$, $x_1, x_2 \in \mathbb{R}^{d_1}, y_1, y_2 \in \mathbb{R}^{d_2}$,*

$$E\,||\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}|| \leq C\left(|x_1 - x_2|^\gamma + |x_1 - x_2|\right) + Ce^{-\frac{\kappa t}{4}}|y_1 - y_2|. \tag{2.28}$$

Proof. See Section 2.5. □

Corollary 2.3. *For any $t \geq 0$, there exists a constant $C > 0$ such that*

$$\sup_{x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} E\left|\left|\nabla_y^2 Y_t^{x,y}\right|\right| \leq Ce^{-\frac{\kappa t}{4}}. \tag{2.29}$$

Proof. See Section 2.5. □

Lastly, one final estimate used in Lemma 2.8 to bound the difference of the gradients with respect to x for the expectations in Lemma 2.8:

Lemma 2.7. *For any $t \geq 0$, $x_1, x_2 \in \mathbb{R}^{d_1}, y_1, y_2 \in \mathbb{R}^{d_2}$,*

$$\sup_{y\in\mathbb{R}^{d_2}} E\,||\nabla_y\nabla_x Y_t^{x_1,y} - \nabla_y\nabla_x Y_t^{x_2,y}|| \leq Ce^{-\frac{\kappa t}{4}}|x_1 - x_2|^\gamma\left(1 + |x_1 - x_2|^{1-\gamma}\right). \tag{2.30}$$

Proof. See Section 2.5. □

Lemma 2.8. *For any $\theta \in (0, 1]$, there exists a $C_\theta > 0$ and $\eta > 0$ such that for any $t_0 > 0, x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}$,*

$$\left|\left|\nabla_x \widetilde{f}_{t_0}(x_1, y, t) - \nabla_x \widetilde{f}_{t_0}(x_2, y, t)\right|\right| \leq Ce^{-\eta t}|x_1 - x_2|^\gamma\left(1 + |x_1 - x_2|^{1-\gamma}\right)(1 + |y|). \tag{2.31}$$

Proof. See Section 2.5. □

The following proposition is the main theorem for this section. The existence of the solution to the Poisson equation and its regularity properties are established. These properties help bound various terms when proving the rate of convergence for the averaging principle and that builds into the functional central limit theorem result.

Proposition 2.1 Poisson Equation

Let $\mathcal{L}$ be the generator of the frozen process $Y_t^{x,y}$ (see eq 2.3 above). Define

$$h(x,y) := \int_0^\infty \left[\mathbb{E}\left[f\left(x, Y_t^{x,y}\right)\right] - \overline{f}(x)\right] dt. \tag{2.32}$$

Then $h(x,y)$ is a solution of the Poisson equation,

$$-\mathcal{L}h(x,y) = f(x,y) - \overline{f}(x). \tag{2.33}$$

Furthermore, there exists a $C > 0$ such that

$$\sup_{x\in\mathbb{R}^{d_1}} |h(x,y)| \leq C(1+|y|), \qquad \sup_{x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} ||\nabla_y h(x,y)|| \leq C, \tag{2.34}$$

and for any $\theta \in (0,1]$, there exists $C_\theta > 0$ such that for any $x_1, x_2 \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}$,

$$\sup_{x\in\mathbb{R}^{d_1}} ||\nabla_x h(x,y)|| \leq C_\theta \left(1 + |y|^\theta\right), \tag{2.35}$$

$$||\nabla_x h(x_1,y) - \nabla_x h(x_2,y)|| \leq C|x_1 - x_2|^\gamma \left(1 + |x_1 - x_2|^{1-\gamma}\right)(1+|y|). \tag{2.36}$$

Proof. Using the exponential ergodicity of Lemma 2.2, $h(x,y)$ is well defined since

$$\begin{aligned}|h(x,y)| &= \left|\int_0^\infty E\left[f\left(x, Y_t^{x,y}\right)\right] - \overline{f}(x) dt\right| = \left|\int_0^\infty P_t^x f(y) - \mu^x(f) dt\right| \\ &\leq C||f||_1(1+|y|)\int_0^\infty e^{-\frac{\kappa t}{2}} dt < \infty.\end{aligned}$$

Notice that the right hand side does not depend on x, therefore, just as easily one may establish the first estimate of (2.34). Namely,

$$\sup_{x\in\mathbb{R}^{d_1}} |h(x,y)| \leq C(1+|y|).$$

By the properties of semigroups, $h(x,y)$ satisfies the Poisson equation (2.33) since

$$\begin{aligned}\mathcal{L}h(x,y) =& \mathcal{L}\left(\lim_{t\to\infty}\int_0^t E\left[f\left(x, Y_s^{x,y}\right)\right] - \overline{f}(x) ds\right) \\ =& \lim_{t\to\infty}\mathcal{L}\left(\int_0^t E\left[f\left(x, Y_s^{x,y}\right)\right] - \overline{f}(x) ds\right),\end{aligned}$$

where the interchanges follows due to dominated convergence,

$$\begin{aligned}&= \lim_{t\to\infty} P_t^x\left[f(x,\cdot) - \overline{f}(x)\right](y) - \left(f(x,y) - \overline{f}(x)\right) \\ &= -\left(f(x,y) - \overline{f}(x)\right).\end{aligned}$$

It follows that

$$\nabla_y h(x,y) = \int_0^\infty E\left[\nabla_y f\left(x, Y_t^{x,y}\right) \cdot \nabla_y Y_t^{x,y}\right] dt$$

where

$$d\nabla_y Y_t^{x,y} = \nabla_y g\left(x, Y_t^{x,y}\right) \cdot \nabla_y Y_t^{x,y} dt + \nabla_y b\left(x, Y_t^{x,y}\right) \cdot \nabla_y Y_t^{x,y} dL_{t,2}, \quad \nabla_y Y_t^{x,y} = I.$$

By Corollary 2.1 and the boundedness of $\nabla_y f$, the second estimate of (2.34) is

$$\nabla_y h(x,y) \leq C \int_0^\infty e^{-\frac{\kappa t}{2}} dt \implies \sup_{x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} ||\nabla_y h(x,y)|| \leq C.$$

We have from Lemma 2.2 and the definition of $\overline{f}$ in terms of the invariant measure,

$$\begin{aligned}
\lim_{t_0\to\infty} \widetilde{f}_{t_0}(x,y,t) &= \lim_{t_0\to\infty} \widehat{f}(x,y,t) - \widehat{f}(x,y,t+t_0) \\
&= \lim_{t_0\to\infty} E\left[f\left(x, Y_t^{x,y}\right)\right] - E\left[f\left(x, Y_{t+t_0}^{x,y}\right)\right] \\
&= E\left[f\left(x, Y_t^{x,y}\right)\right] - \overline{f}(x).
\end{aligned}$$

This further implies,

$$\lim_{t_0\to\infty} \nabla_x \widetilde{f}_{t_0}(x,y,t) = \nabla_x \left[E\left[f\left(x, Y_t^{x,y}\right)\right] - \overline{f}(x)\right].$$

By consequence, Lemma 2.5 gives

$$||\nabla_x h(x,y)|| \leq \int_0^\infty \left|\left|\nabla_x \widetilde{f}_{t_0}(x,y,t)\right|\right| dt \leq C_\theta(1+|y|^\theta) \int_0^\infty e^{-\eta t} dt \leq C_\theta(1+|y|^\theta),$$

which is estimate (2.35).

Finally, for estimate (2.36),

$$\begin{aligned}
||\nabla_x h(x_1,y) - \nabla_x h(x_2,y)|| &= \Bigg|\Bigg| \int_0^\infty \nabla_x \left[E\left[f\left(x_1, Y_t^{x_1,y}\right)\right] - \overline{f}(x_1)\right] dt \\
&\qquad - \int_0^\infty \nabla_x \left[E\left[f\left(x_2, Y_t^{x_2,y}\right)\right] - \overline{f}(x_2)\right] dt \Bigg|\Bigg| \\
&\leq \int_0^\infty \left|\left|\nabla_x \widetilde{f}_{t_0}(x_1,y,t) - \nabla_x \widetilde{f}_{t_0}(x_2,y,t)\right|\right| dt \\
&\leq C|x_1-x_2|^\gamma \left(1+|x_1-x_2|^{1-\gamma}\right)(1+|y|) \int_0^\infty e^{-\eta t} dt \\
&\leq C|x_1-x_2|^\gamma \left(1+|x_1-x_2|^{1-\gamma}\right)(1+|y|),
\end{aligned}$$

which follows from Lemma 2.8. $\square$

§2.3 Proofs of Main Results

The following section details the proofs of the main theorems of this chapter: the averaging principle and the functional central limit theorem. The proof of the averaging principle was inspired by [SXX22]. For the most part, it is identical with the obvious modifications for the additional multiplicative noise term, differing drift condition and keeping the dependence on T explicit. This was done to make it more convenient to show the functional central limit theorem and has the benefit of making explicit why the results hold on the bounded interval $[0,T]$. The proofs of the following moment estimates can be found in Section 2.5.

Lemma 2.9. *For any $p \in [1,\alpha)$ and $T \geq 1$, there exists a constants $C_p > 0$ such that*

$$\sup_{\epsilon\in(0,1)} \mathbb{E}\left[\sup_{t\in[0,T]} |X_t^\epsilon|^p\right] \leq C_p T^p e^{C_p T}(1+|x|^p+|y|^p) \tag{2.37}$$

and

$$\sup_{\epsilon\in(0,1)} \sup_{t\geq 0} \mathbb{E}\,|Y_t^\epsilon|^p \leq C_p(1+|y|^p). \tag{2.38}$$

Proof. See Section 2.5. □

Remark 2.5. *The same steps used to derive (2.38) may be used to derive the estimate*

$$\sup_{t\geq 0} E\,|Y_t^{x,y}|^p \leq C_p(1+|y|^p). \tag{2.39}$$

Lemma 2.10. *For any $p \in [1,\alpha)$ and $T \geq 1$, there exists a constant $C_p > 0$ such that,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^\epsilon|^p\right] \leq C_p T^p \epsilon^{-\frac{p}{\alpha}} + |y|^p\,. \tag{2.40}$$

Proof. See Section 2.5. □

Lemma 2.11. *For any $x \in \mathbb{R}^{d_1}$, (2.13) has a unique solution $\overline{X}_t$. Also, for any $T \geq 1$, there exists a constant $C_p > 0$ such that for any $p \in [1,\alpha)$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\overline{X}_t\right|^p\right] \leq C_{p,T}\left(1+|x|^p\right). \tag{2.41}$$

Proof. See Section 2.5. □

Proof of Theorem 2.1. Begin by writing

$$\begin{aligned}
X_t^\epsilon - \overline{X}_t &= \int_0^t f\left(X_s^\epsilon, Y_s^\epsilon\right) - \overline{f}\left(\overline{X}_s\right) ds + \epsilon^\rho \int_0^t c\left(X_s^\epsilon, Y_s^\epsilon\right) dL_{s,1} \\
&= \int_0^t f\left(X_s^\epsilon, Y_s^\epsilon\right) - \overline{f}\left(X_s^\epsilon\right) ds + \int_0^t \overline{f}\left(X_s^\epsilon\right) - \overline{f}\left(\overline{X}_s\right) ds + \epsilon^\rho \int_0^t c\left(X_s^\epsilon, Y_s^\epsilon\right) dL_{s,1}
\end{aligned}$$

$$= \int_0^t -\mathcal{L}h(X_s^\epsilon, Y_s^\epsilon)ds + \int_0^t \overline{f}\left(X_s^\epsilon\right) - \overline{f}\left(\overline{X}_s\right) ds + \epsilon^\rho \int_0^t c\left(X_s^\epsilon, Y_s^\epsilon\right) dL_{s,1}.$$

The proof of Lemma 2.11 in Section 2.5 establishes the Lipschitz property of $\overline{f}$. Deduce

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^\epsilon - \overline{X}_t\right|^p\right] \leq & C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t -\mathcal{L}h(X_s^\epsilon, Y_s^\epsilon)ds\right|^p\right] \\
& + C_p T^p \int_0^T \mathbb{E}\left[\sup_{s\in[0,T]} \left|X_s^\epsilon - \overline{X}_s\right|^p\right] ds \\
& + \mathbb{E}\left[\sup_{t\in[0,T]} \left|\epsilon^\rho \int_0^t c\left(X_s^\epsilon, Y_s^\epsilon\right) dL_{s,1}\right|^p\right].
\end{aligned}$$

Then by Gronwall's inequality and eq (2.88) in Section 2.5,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^\epsilon - \overline{X}_t\right|^p\right] \leq C_p e^{C_p T^{p+1}} \left(\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t -\mathcal{L}h(X_s^\epsilon, Y_s^\epsilon)ds\right|^p\right] + T^p \epsilon^{\rho p}\right). \tag{2.42}$$

Apply Itô's formula with small jumps of size $\epsilon^{\frac{1}{\alpha}}$ for the fast process,

$$\begin{aligned}
h\left(X_t^\epsilon, Y_t^\epsilon\right) - h(x,y) = & \int_0^t \langle f\left(X_s^\epsilon, Y_s^\epsilon\right), \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle ds \\
& + \int_0^t \langle \epsilon^{-1} g\left(X_s^\epsilon, Y_s^\epsilon\right), \nabla_y h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle ds \\
& + \int_0^t \int_{|z|\geq 1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) N_1(ds, dz) \\
& + \int_0^t \int_{|z|\geq \epsilon^{\frac{1}{\alpha}}} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) N_2(ds, dz) \\
& + \int_0^t \int_{|z|<1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_1(ds, dz) \\
& + \int_0^t \int_{|z|< \epsilon^{\frac{1}{\alpha}}} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_2(ds, dz) \\
& + \int_0^t \int_{|z|<1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \\
& - \langle \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_1(dz) ds \\
& + \int_0^t \int_{|z|< \epsilon^{\frac{1}{\alpha}}} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \\
& - \langle \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_y h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_2(dz) ds.
\end{aligned}$$

Compensate for the small jumps,

$$
\begin{aligned}
&= \int_0^t \langle f\left(X_s^\epsilon, Y_s^\epsilon\right), \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle ds + \int_0^t \langle \epsilon^{-1} g\left(X_s^\epsilon, Y_s^\epsilon\right), \nabla_y h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle ds \\
&\quad + \int_0^t \int_{\mathbb{R}^{d_1}} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_1\left(ds, dz\right) \\
&\quad + \int_0^t \int_{\mathbb{R}^{d_2}} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_2\left(ds, dz\right) \\
&\quad + \int_0^t \int_{|z|\geq 1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \nu_1(dz) ds \\
&\quad + \int_0^t \int_{|z|\geq \epsilon^{\frac{1}{\alpha}}} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \nu_2(dz) ds \\
&\quad + \int_0^t \int_{|z|< 1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \\
&\qquad - \langle \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_1(dz) ds \\
&\quad + \int_0^t \int_{|z|< \epsilon^{\frac{1}{\alpha}}} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \\
&\qquad - \langle \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_y h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_2(dz) ds.
\end{aligned}
$$

Make the substitution $\epsilon^{-\frac{1}{\alpha}} z$ to arrive at

$$
\begin{aligned}
&= \int_0^t \langle f\left(X_s^\epsilon, Y_s^\epsilon\right), \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle ds + \int_0^t \langle \epsilon^{-1} g\left(X_s^\epsilon, Y_s^\epsilon\right), \nabla_y h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle ds \\
&\quad + \int_0^t \int_{\mathbb{R}^{d_1}} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_1\left(ds, dz\right) \\
&\quad + \int_0^t \int_{\mathbb{R}^{d_2}} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_2\left(ds, dz\right) \\
&\quad + \int_0^t \int_{|z|\geq 1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \nu_1(dz) ds \\
&\quad + \epsilon^{-1} \int_0^t \int_{|z|\geq 1} h\left(X_s^\epsilon, Y_s^\epsilon + b\left(X_s^\epsilon, Y_s^\epsilon\right) r\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \nu_2(dr) ds \\
&\quad + \int_0^t \int_{|z|< 1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \\
&\qquad - \langle \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_1(dz) ds \\
&\quad + \epsilon^{-1} \int_0^t \int_{|z|< 1} h\left(X_s^\epsilon, Y_s^\epsilon + b\left(X_s^\epsilon, Y_s^\epsilon\right) r\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \\
&\qquad - \langle b\left(X_s^\epsilon, Y_s^\epsilon\right) r, \nabla_y h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_2(dr) ds.
\end{aligned}
$$

Finally,

$$
= \int_0^t \mathcal{L}^{X^\epsilon} h\left(X_s^\epsilon, Y_s^\epsilon\right) ds + \epsilon^{-1} \int_0^t \mathcal{L} h\left(X_s^\epsilon, Y_s^\epsilon\right) ds + M_{t,1}^\epsilon + M_{t,2}^\epsilon, \tag{2.43}
$$

where

$$\mathcal{L}^{X^\epsilon}h(x,y) := \langle f(x,y), \nabla_x h(x,y)\rangle + \int_{|z|\geq 1} h(x+\epsilon^\rho c(x,y)z, y) - h(x,y)\,\nu_1(dz)$$
$$+ \int_{|z|<1} h(x+\epsilon^\rho c(x,y)z, y) - h(x,y) - \langle \epsilon^\rho c(x,y)z, \nabla_x h(x,y)\rangle \nu_1(dz), \tag{2.44}$$

and

$$M^\epsilon_{t,1} := \int_0^t \int_{\mathbb{R}^{d_1}} h(X^\epsilon_s + \epsilon^\rho c(X^\epsilon_s, Y^\epsilon_s)z, Y^\epsilon_s) - h(X^\epsilon_s, Y^\epsilon_s)\,\widetilde{N}_1(ds, dz), \tag{2.45}$$

$$M^\epsilon_{t,2} := \int_0^t \int_{\mathbb{R}^{d_2}} h\left(X^\epsilon_s, Y^\epsilon_s + \epsilon^{-\frac{1}{\alpha}} b(X^\epsilon_s, Y^\epsilon_s)z\right) - h(X^\epsilon_s, Y^\epsilon_s)\,\widetilde{N}_2(ds, dz), \tag{2.46}$$

are $\mathcal{F}_t-$martingales. Rearrange the terms,

$$\int_0^t -\mathcal{L}h(X^\epsilon_s, Y^\epsilon_s)\,ds = \epsilon\left(h(x,y) - h(X^\epsilon_t, Y^\epsilon_t) + \int_0^t \mathcal{L}^{X^\epsilon} h(X^\epsilon_s, Y^\epsilon_s)\,ds + M^\epsilon_{t,1} + M^\epsilon_{t,2}\right), \tag{2.47}$$

and continue from (2.42),

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X^\epsilon_t - \overline{X}_t\right|^p\right] \leq & C_p e^{C_p T^{p+1}}\epsilon^p \left(\mathbb{E}\left[\sup_{t\in[0,T]}\left|h(x,y) - h(X^\epsilon_t, Y^\epsilon_t)\right|^p\right]\right. \\ & + \mathbb{E}\left[\int_0^T \left|\mathcal{L}^{X^\epsilon}h(X^\epsilon_s, Y^\epsilon_s)\right|^p ds\right] \\ & \left. + \mathbb{E}\left[\sup_{t\in[0,T]}\left|M^\epsilon_{t,1}\right|^p\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\left|M^\epsilon_{t,2}\right|^p\right]\right) + C_p e^{C_p T^{p+1}} T^p \epsilon^{\rho p} \\ =: & C_p e^{C_p T^{p+1}}\epsilon^p \sum_{i=1}^4 J_i + C_p e^{C_p T^{p+1}} T^p \epsilon^{\rho p}.\end{aligned} \tag{2.48}$$

Apply estimates (2.34), (2.40) to J_1,

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left|h(x,y) - h(X^\epsilon_t, Y^\epsilon_t)\right|^p\right] \leq C_p(1+|y|^p) + C_p\left(1 + \mathbb{E}\left[\sup_{t\in[0,T]}|Y^\epsilon_t|^p\right]\right) \\ \leq & C_p(1+|y|^p) + C_p T^p \epsilon^{-\frac{p}{\alpha}} + |y|^p \leq C_p T^p (1+|y|^p)\epsilon^{-\frac{p}{\alpha}}.\end{aligned} \tag{2.49}$$

Notice

$$\begin{aligned}|f(X^\epsilon_s, Y^\epsilon_s)| \leq & |f(X^\epsilon_s, Y^\epsilon_s) - f(X^\epsilon_s, 0)| + |f(X^\epsilon_s, 0) - f(0,0)| + |f(0,0)| \\ \leq & (||\nabla_y f||\,|Y^\epsilon_s| + ||\nabla_x f||\,|X^\epsilon_s| + C) \\ \leq & C(1 + |X^\epsilon_s| + |Y^\epsilon_s|).\end{aligned}$$

By (2.35), (2.36), and Lemma 2.9, then for J_2,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|\mathcal{L}^{X^\epsilon} h\left(X_s^\epsilon, Y_s^\epsilon\right)\right|^p ds\right] \leq C_p T^p \mathbb{E}\left[\int_0^T \left|\left\langle f\left(X_s^\epsilon, Y_s^\epsilon\right), \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\right\rangle\right|^p ds\right] \\
&\quad + C_p T^p \mathbb{E}\left[\int_0^T \left|\int_{|z|\geq 1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \nu_1(dz)\right|^p ds\right] \\
&\quad + C_p T^p \mathbb{E}\left[\int_0^T \left|\int_{|z|<1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right)\right.\right. \\
&\qquad \left.\left. -\langle \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_1(dz)\right|^p ds\right] \\
&\leq C_p T^p \mathbb{E}\left[\int_0^T \left(1 + |X_s^\epsilon|^p + |Y_s^\epsilon|^p\right) C_\theta \left(1 + |Y_s^\epsilon|^\theta\right) ds\right] \\
&\quad + C_p T^p \mathbb{E}\left[\int_0^T \left|\int_{|z|\geq 1} \sup_{x\in\mathbb{R}^{d_1}} ||\nabla_x h\left(x, Y_s^\epsilon\right)|| \left|\epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z\right| \nu_1(dz)\right|^p ds\right] \\
&\quad + C_p T^p \mathbb{E}\left[\int_0^T \left|\int_{|z|<1} \langle \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_x h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right)\rangle\right.\right. \\
&\qquad \left.\left. -\langle \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, \nabla_x h\left(X_s^\epsilon, Y_s^\epsilon\right)\rangle \nu_1(dz)\right|^p ds\right] \\
&\leq C_p T^p \mathbb{E}\left[\int_0^T \left(1 + |X_s^\epsilon|^p + |Y_s^\epsilon|^p\right) \left(1 + |Y_s^\epsilon|^\theta\right) ds\right] \\
&\quad + C_p T^p \mathbb{E}\left[\int_0^T \left|\int_{|z|\geq 1} \left|\epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z\right| \nu_1(dz)\right|^p \left(1 + |Y_s^\epsilon|^p\right) ds\right] \\
&\quad + C_p T^p \mathbb{E}\left[\int_0^T \left|\int_{|z|<1} \left|\epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z\right|^{\gamma+1} \left(1 + \left|\epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z\right|^{1-\gamma}\right) \nu_1(dz)\right|^p \left(1 + |Y_s^\epsilon|^p\right) ds\right] \\
&\leq C_p T^{p+1} \left(1 + |x|^{p'} + |y|^{\frac{\theta p'}{p'-p} \vee (p+\theta)}\right),
\end{aligned} \tag{2.50}
$$

where $p < p' < \alpha$ and θ is small enough such that $\frac{\theta p'}{p'-p} \vee (p+\theta) < \alpha$ (see [SXX22]) and follows by Young's inequality.

For the first martingale term J_3, apply the Burkholder-Davis-Gundy inequality,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left|M_{t,1}^\epsilon\right|^p\right] \leq \\
&\quad C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_{|z|\leq 1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_1\left(ds, dz\right)\right|^p\right] \\
&\quad + C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_{|z|>1} h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_1\left(ds, dz\right)\right|^p\right] \\
&\leq C_p \mathbb{E}\left[\int_0^T \int_{|z|\leq 1} \left|h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right)\right|^2 N_1\left(ds, dz\right)\right]^{\frac{p}{2}} \\
&\quad + C_p \mathbb{E}\left[\int_0^T \int_{|z|>1} \left|h\left(X_s^\epsilon + \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z, Y_s^\epsilon\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right)\right|^p \nu_1\left(dz\right) ds\right]
\end{aligned}
$$

$$
\begin{aligned}
\leq & C_p \mathbb{E}\left[\int_0^T \int_{|z|\leq 1} \sup_{x\in\mathbb{R}^{d_1}} ||\nabla_x h\left(x, Y_s^\epsilon\right)||^2 \left|\epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z\right|^2 N_1\left(ds, dz\right)\right]^{\frac{p}{2}} \\
& + C_p \mathbb{E}\left[\int_0^T \int_{|z|>1} \sup_{x\in\mathbb{R}^{d_1}} ||\nabla_x h\left(x, Y_s^\epsilon\right)||^p \left|\epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) z\right|^p \nu_1\left(dz\right) ds\right].
\end{aligned}
$$

Apply regularity condition (2.35) and (2.38) for any choice of $\theta \in (0, \frac{1}{2}]$

$$
\begin{aligned}
\leq & C_p \epsilon^{2\gamma} ||c||^p \mathbb{E}\left[\int_0^T \int_{|z|\leq 1} |z|^2 \nu_1\left(dz\right) C_\theta \left(1 + \sup_{\epsilon\in(0,1)} \sup_{s\geq 0} |Y_s^\epsilon|^{2\theta}\right) ds\right]^{\frac{p}{2}} \\
& + C_p \epsilon^{2\gamma} ||c||^p \mathbb{E}\left[\int_0^T \int_{|z|>1} |z|^p \nu_1\left(dz\right) C_\theta \left(1 + \sup_{\epsilon\in(0,1)} \sup_{s\geq 0} |Y_s^\epsilon|^{p\theta}\right) ds\right] \\
\leq & C_p T (1 + |y|^p).
\end{aligned}
\tag{2.51}
$$

Similarly for the second martingale term J_4,

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{t\in[0,T]} \left|M_{t,2}^\epsilon\right|^p\right] \leq \\
& \quad C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_{|z|\leq 1} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_2\left(ds, dz\right)\right|^p\right] \\
& \quad + C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_{|z|>1} h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right) \widetilde{N}_2\left(ds, dz\right)\right|^p\right] \\
& \leq C_p \mathbb{E}\left[\int_0^T \int_{|z|\leq 1} \left|h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right)\right|^2 N_2\left(ds, dz\right)\right]^{\frac{p}{2}} \\
& \quad + C_p \mathbb{E}\left[\int_0^T \int_{|z|>1} \left|h\left(X_s^\epsilon, Y_s^\epsilon + \epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right) - h\left(X_s^\epsilon, Y_s^\epsilon\right)\right|^p \nu_2\left(ds\right) dz\right] \\
& \leq C_p \mathbb{E}\left[\int_0^T \int_{|z|\leq 1} ||\nabla_y h||^2 \left|\epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right|^2 \nu_2(dz) ds\right]^{\frac{p}{2}} \\
& \quad + C_p \mathbb{E}\left[\int_0^T \int_{|z|>1} ||\nabla_y h||^2 \left|\epsilon^{-\frac{1}{\alpha}} b\left(X_s^\epsilon, Y_s^\epsilon\right) z\right|^p \nu_2(dz) ds\right].
\end{aligned}
$$

Apply regularity condition (2.34),

$$
\begin{aligned}
\leq & C_p \epsilon^{-\frac{p}{\alpha}} ||b||^p \mathbb{E}\left[\int_0^T \int_{|z|\leq 1} |z|^2 \nu_2(dz) ds\right]^{\frac{p}{2}} + C_p \epsilon^{-\frac{p}{\alpha}} ||b||^p \mathbb{E}\left[\int_0^T \int_{|z|>1} |z|^p \nu_2(dz) ds\right] \\
\leq & C_p T \epsilon^{-\frac{p}{\alpha}}.
\end{aligned}
\tag{2.52}
$$

Putting (2.49),(2.50),(2.51), and (2.52) into (2.48), conclude for some $k > 0$ that

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^\epsilon - \overline{X}_t\right|^p\right] \leq C T^{kp} e^{T^{kp}} (1 + |x|^p + |y|^p) \epsilon^{p(1-\frac{1}{\alpha})} \to 0 \text{ as } \epsilon \to 0.
$$

$\square$

Remark 2.6. *Rather than fix the noise scaling $\epsilon^{-\frac{1}{\alpha}}$ for the fast process, it is possible to introduce a, say, $\widetilde{\rho} \geq -\frac{1}{\alpha}$ and use $\epsilon^{\widetilde{\rho}}$ for the scaling. But when one traces the proof with a variable choice for $\widetilde{\rho}$, it becomes necessary to impose an upper bound $\widetilde{\rho} \leq \frac{1}{\alpha} \log_\epsilon \left(\epsilon^{-1} - \epsilon^{-\frac{1}{\alpha}}\right)$ so that all terms converge with the correct order. If one takes $\epsilon \to 0$, the upper bound converges to $-\frac{1}{\alpha}$. Since $\widetilde{\rho}$ customarily does not depend on ϵ, it seems clear that the choice for scaling should be fixed at $\epsilon^{-\frac{1}{\alpha}}$.*

This concludes the discussion on the averaging principle. In the second half of this section, the functional central limit theorem is proved where some results within the above proof are applied. First, the following lemma establishes tightness since weak convergence follows from tightness of measures and convergence of finite dimensional distributions.

Lemma 2.12 (tightness)**.** *For any $T \geq 0$, there exists some constant $C > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |V_t^\epsilon|\right] \leq C_T. \tag{2.53}$$

Furthermore, $\{V_t^\epsilon\}_{t\geq 0}$ is tight.

Proof. It is easy to see that (2.53) holds by using the averaging rate of convergence (2.12) with $p = 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |V_t^\epsilon|\right] = \epsilon^{\frac{1}{\alpha}-1}\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^\epsilon - \overline{X}_t\right|\right] \leq C_{x,y} T^k e^{T^k},$$

and where the dependence on T is left explicit, as done at the end of the proof for Theorem 2.1.

Let $\tau \in [0, T - \delta_0]$ be a stopping time. By the Markov property, for any $\lambda > 0$ and $\widetilde{\delta} \in (0, \delta_0)$,

$$\begin{aligned}
\mathbb{P}\left(\left|V^\epsilon_{\tau+\widetilde{\delta}} - V^\epsilon_\tau\right| > \lambda\right) =& \mathbb{E}\left[\mathbb{P}\left(\left|V^\epsilon_{s+\widetilde{\delta}} - v\right| > \lambda \middle| (s,v) = (\tau, V^\epsilon_\tau)\right)\right] \\
\leq& \mathbb{P}\left(|V^\epsilon_\tau| > R\right) + \mathbb{E}\left[\mathbb{P}^{(s,v)}\left(\left|V^\epsilon_{s+\widetilde{\delta}} - v\right| > \lambda\right)\right] \\
\leq& \frac{\mathbb{E}\left[|V^\epsilon_\tau|\right]}{R} + \frac{\mathbb{E}\left[\mathbb{E}^{(s,v)}\left[\left|V^\epsilon_{s+\widetilde{\delta}} - v\right|\right]\right]}{\lambda} \\
\leq& \frac{\mathbb{E}\left[\sup_{t\in[0,T]} |V^\epsilon_t|\right]}{R} + \frac{\mathbb{E}\left[\mathbb{E}^{(s,v)}\left[\sup_{s\leq t\leq s+\widetilde{\delta}} |V^\epsilon_t - v|\right]\right]}{\lambda} \\
\leq& \frac{C_{x,y}T^k e^{T^k}}{R} + \frac{\mathbb{E}\left[\sup_{0\leq t\leq \widetilde{\delta}} |V^\epsilon_t|\right]}{\lambda} \\
\leq& \frac{C_{x,y}T^k e^{T^k}}{R} + \frac{C_{x,y}\widetilde{\delta}^k e^{\widetilde{\delta}^k}}{\lambda}.
\end{aligned} \tag{2.54}$$

Let $\widetilde{\delta} \to 0$ and then $R \to 0$,

$$\lim_{\widetilde{\delta}\to 0} \limsup_{\epsilon\to 0} \sup_{\tau\leq\tau+\widetilde{\delta}} \mathbb{P}\left(\left|V^\epsilon_{\tau+\widetilde{\delta}} - V^\epsilon_\tau\right| \geq \lambda\right) = 0.$$

Furthermore, by Markov's inequality,

$$\lim_{R\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |V^\epsilon_t| \geq R\right) \leq \lim_{R\to\infty} \frac{1}{R}\mathbb{E}\left[\sup_{t\in[0,T]} |V^\epsilon_t|\right] = 0. \tag{2.55}$$

Conditions (2.54) and (2.55) imply tightness by Theorem VI-4.5 [JS13]. $\square$

Proof of Theorem 2.2. Decompose V^ϵ_t into

$$\begin{aligned}
V^\epsilon_t &= \epsilon^{\frac{1}{\alpha}-1}\left(\int_0^t f\left(X^\epsilon_s, Y^\epsilon_s\right) - \overline{f}\left(\overline{X}_s\right) ds + \epsilon^\rho \int_0^t c\left(X^\epsilon_s, Y^\epsilon_s\right) dL_{s,1}\right) \\
&= \epsilon^{\frac{1}{\alpha}-1}\left(\int_0^t \mathcal{L}h\left(X^\epsilon_s, Y^\epsilon_s\right) ds + \int_0^t \nabla_x \overline{f}\left(\overline{X}_s\right) \epsilon^{1-\frac{1}{\alpha}} V^\epsilon_s ds\right. \\
&\quad + \int_0^t \overline{f}\left(\overline{X}_s + \epsilon^{1-\frac{1}{\alpha}} V^\epsilon_s\right) - \overline{f}\left(\overline{X}_s\right) - \nabla_x \overline{f}\left(\overline{X}_s\right) \epsilon^{1-\frac{1}{\alpha}} V^\epsilon_s ds \\
&\quad \left. + \int_0^t f\left(X^\epsilon_s, Y^\epsilon_s\right) - \overline{f}\left(X^\epsilon_s\right) - \mathcal{L}h\left(X^\epsilon_s, Y^\epsilon_s\right) ds + \epsilon^\rho \int_0^t c\left(X^\epsilon_s, Y^\epsilon_s\right) dL_{s,1}\right).
\end{aligned} \tag{2.56}$$

For ease of notation, define $\zeta^\epsilon_t := \epsilon^{\frac{1}{\alpha}-1} \int_0^t \mathcal{L}h\left(X^\epsilon_s, Y^\epsilon_s\right) ds$. Consider the simplified equation

$$Z^\epsilon_t = \zeta^\epsilon_t + \int_0^t \nabla_x \overline{f}\left(\overline{X}_s\right) V^\epsilon_s ds, \tag{2.57}$$

and the difference $U^\epsilon_t = V^\epsilon_t - Z^\epsilon_t$. The following shows that as $\epsilon \to 0$, $Z^\epsilon \Rightarrow V$ and that $\mathbb{E}\left[\sup_{t\in[0,T]} |U^\epsilon_t|\right] \to 0$.

Recall from (2.47) that

$$\zeta^\epsilon_t = \epsilon^{\frac{1}{\alpha}}\left(h\left(X^\epsilon_t, Y^\epsilon_t\right) - h\left(x, y\right) - M^\epsilon_{t,1} - M^\epsilon_{t,2} - \int_0^t \mathcal{L}^{X^\epsilon} h(X^\epsilon_s, Y^\epsilon_s) ds\right).$$

It is immediate from estimates (2.51) that $\epsilon^{\frac{1}{\alpha}}\mathbb{E}\left[\sup_{t\in[0,T]} \left|M^\epsilon_{t,1}\right|\right] \leq C\epsilon^{\frac{1}{\alpha}}(1+|y|)$, from (2.50) that $\epsilon^{\frac{1}{\alpha}}\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \mathcal{L}^{X^\epsilon} h(X^\epsilon_s, Y^\epsilon_s) ds\right|\right] \leq C\epsilon^{\frac{1}{\alpha}}\left(1+|x|+|y|\right)$, and from (2.34) that $\epsilon^{\frac{1}{\alpha}} \sup_{x\in\mathbb{R}^{d_1}} |h(x,y)| \leq C\epsilon^{\frac{1}{\alpha}}(1+|y|)$.

Let $\overline{\eta} > 0$. Notice that by equations (2.34) and (2.38), there exists a constant $C_y > 0$ such that for all $K \geq \frac{C_y}{\overline{\eta}}$,

$$\begin{aligned}\mathbb{P}\left(\epsilon^{\frac{1}{\alpha}}\left|h\left(X_t^\epsilon,Y_t^\epsilon\right)\right|>K\right) \leq& \epsilon^{\frac{1}{\alpha}}\frac{\mathbb{E}\left|h\left(X_t^\epsilon,Y_t^\epsilon\right)\right|}{K} \leq \epsilon^{\frac{1}{\alpha}}\frac{C(1+\mathbb{E}|Y_t^\epsilon|)}{K}\\ \leq& \epsilon^{\frac{1}{\alpha}}\frac{C(1+\sup_{\epsilon\in(0,1)}\sup_{t\geq 0}\mathbb{E}|Y_t^\epsilon|)}{K} \leq \epsilon^{\frac{1}{\alpha}}\frac{C(1+C(1+|y|))}{K} \leq \frac{C_y}{K}.\end{aligned}$$

This implies $1-\mathbb{P}(\epsilon^{\frac{1}{\alpha}}|h(X_t^\epsilon,Y_t^\epsilon)|>K)\geq 1-\frac{C_y}{K}$. Consequently,

$$\inf_{\epsilon\in(0,1)}\mathbb{P}(\epsilon^{\frac{1}{\alpha}}|h(X_t^\epsilon,Y_t^\epsilon)|\leq K)\geq 1-\frac{C_y}{K}\geq 1-\overline{\eta}.$$

So for every $\overline{\eta}>0, t\in[0,T]$, there exists a compact set $[0,K]$ satisfying the compact containment condition ([EK09], Remark 7.3). Furthermore, one may identify the limit from the same computation

$$\epsilon^{\frac{1}{\alpha}}\mathbb{E}\left|h\left(X_t^\epsilon,Y_t^\epsilon\right)\right|\leq \epsilon^{\frac{1}{\alpha}}C_y\to 0.$$

Once tightness is established over small intervals of the path, one may conclude that $\epsilon^{\frac{1}{\alpha}}h\left(X_t^\epsilon,Y_t^\epsilon\right)\Rightarrow 0$. To this effect, let $\widetilde{\eta}\in(0,1)$.

$$\begin{aligned}\sup_{0<t\leq\widetilde{\eta}}|h(X_t^\epsilon,Y_t^\epsilon)-h(x,y)| \leq& \sup_{0<t\leq\widetilde{\eta}}|h(X_t^\epsilon,Y_t^\epsilon)-h(x,Y_t^\epsilon)+h(x,Y_t^\epsilon)-h(x,y)|\\ \leq& \sup_{0<t\leq\widetilde{\eta}}\Bigg(\sup_{x\in\mathbb{R}^{d_1}}||\nabla_x h(x,Y_t^\epsilon)|||X_t^\epsilon-x|\\ &+\sup_{x\in\mathbb{R}^{d_1},y\in\mathbb{R}^{d_2}}||\nabla_y h(x,y)|||Y_t^\epsilon-y|\Bigg).\end{aligned}$$

By equations (2.34), (2.35), for any $\theta\in(0,1-\frac{1}{\alpha})$,

$$\begin{aligned}&\leq \sup_{0<t\leq\widetilde{\eta}}\left(C_\theta(1+|Y_t^\epsilon|^\theta)|X_t^\epsilon-x|+C|Y_t^\epsilon-y|\right)\\ &=\sup_{0<t\leq\widetilde{\eta}}\left(C_\theta|X_t^\epsilon-x|+C_\theta|Y_t^\epsilon|^\theta|X_t^\epsilon-x|+C|Y_t^\epsilon-y|\right).\end{aligned}$$

Choose $p=\frac{1}{\theta}>1$ and $q=\frac{1}{1-\theta}>1$, the by Young's inequality

$$\begin{aligned}&\leq \sup_{0<t\leq\widetilde{\eta}}\left(C_\theta|X_t^\epsilon-x|+C_{\theta,p}|Y_t^\epsilon|^{p\theta}+C_{\theta,q}|X_t^\epsilon-x|^q+C|Y_t^\epsilon-y|\right)\\ &\leq C\sup_{0<t\leq\widetilde{\eta}}\left(|X_t^\epsilon|+|X_t^\epsilon|^q+|x|+|x|^q+|Y_t^\epsilon|+|y|\right).\end{aligned}$$

From Lemma 2.9 and Lemma 2.10, it follows that

$$\begin{aligned}\epsilon^{\frac{1}{\alpha}}\mathbb{E}\left[\sup_{0<t\leq\widetilde{\eta}}|h(X_t^\epsilon,Y_t^\epsilon)-h(x,y)|\right] \leq& C\mathbb{E}\left[\sup_{0<t\leq\widetilde{\eta}}\left(|X_t^\epsilon|+|X_t^\epsilon|^q+|x|+|x|^q+|Y_t^\epsilon|+|y|\right)\right]\\ \leq& C\epsilon^{\frac{1}{\alpha}}\Big(C\widetilde{\eta}e^{C\widetilde{\eta}}(1+|x|+|y|)+C\widetilde{\eta}^q e^{C_q\widetilde{\eta}}(1+|x|^q+|y|^q)\\ &+|x|+|x|^q+C\widetilde{\eta}\epsilon^{-\frac{1}{\alpha}}+2|y|\Big).\end{aligned}$$

Take $\epsilon \to 0$ and then $\widetilde{\eta} \to 0$ to conclude that it is tight. Weak convergence to zero has been established as desired.

For the term $\epsilon^{\frac{1}{\alpha}} M^{\epsilon}_{t,2}$, recall (2.46) and notice from the mean value theorem and the boundedness of $\nabla_y h$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\epsilon^{\frac{1}{\alpha}}\int_0^t\int_{\mathbb{R}^{d_2}} h\left(X^\epsilon_s, Y^\epsilon_s + b\left(X^\epsilon_s, Y^\epsilon_s\right)\epsilon^{-\frac{1}{\alpha}} z\right) - h\left(X^\epsilon_s, Y^\epsilon_s\right)\widetilde{N}_2(ds,dz)\right.\right.\\
&\qquad\left.\left. - \int_0^t\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_2}} h\left(\overline{X}_s, y + b\left(\overline{X}_s, y\right) z\right) - h\left(\overline{X}_s, y\right)\mu^{\overline{X}_s}(dy)\widetilde{N}_2(ds,dz)\right|\right]\\
\le& ||\nabla_y h||\,\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\left|b\left(X^\epsilon_s, Y^\epsilon_s\right) - \int_{\mathbb{R}^{d_2}} b\left(\overline{X}_s, y\right)\mu^{\overline{X}_s}(dy)\right|\widetilde{N}_2(ds,dz)\right]\\
\le& ||\nabla_y h||\left(\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\left|b\left(X^\epsilon_s, Y^\epsilon_s\right) - b\left(\overline{X}_s, Y^\epsilon_s\right)\right|\widetilde{N}_2(ds,dz)\right]\right.\\
&\qquad + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\left|b\left(\overline{X}_s, Y^\epsilon_s\right) - E\left[b\left(\overline{X}_s, Y^{\overline{X}_s,y}_{\frac{s}{\epsilon}}\right)\right]\right|\widetilde{N}_2(ds,dz)\right]\\
&\qquad \left. + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\left|E\left[b\left(\overline{X}_s, Y^{\overline{X}_s,y}_{\frac{s}{\epsilon}}\right)\right] - \int_{\mathbb{R}^{d_2}} b\left(\overline{X}_s, y\right)\mu^{\overline{X}_s}(dy)\right|\widetilde{N}_2(ds,dz)\right]\right)\\
\le& ||\nabla_y h||\left(\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\,||\nabla_x b||\left|X^\epsilon_s - \overline{X}_s\right|\widetilde{N}_2(ds,dz)\right]\right.\\
&\qquad + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\,||\nabla_y b||\left|Y^{X^\epsilon_s,y}_{\frac{s}{\epsilon}} - Y^{\overline{X}_s,y}_{\frac{s}{\epsilon}}\right|\widetilde{N}_2(ds,dz)\right]\\
&\qquad \left. + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\sup_{x\in\mathbb{R}^{d_2}}\left|E\left[b\left(x, Y^{x,y}_{\frac{s}{\epsilon}}\right)\right] - \int_{\mathbb{R}^{d_2}} b\left(x,y\right)\mu^{x}(dy)\right|\widetilde{N}_2(ds,dz)\right]\right)\\
\le& ||\nabla_y h||\left(C\,||\nabla_x b||\,\epsilon^{1-\frac{1}{\alpha}}\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|\widetilde{N}_2(ds,dz)\right]\right.\\
&\qquad + ||\nabla_y b||\,\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|C\left|X^\epsilon_s - \overline{X}_s\right|\widetilde{N}_2(ds,dz)\right]\\
&\qquad \left. + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|C\,||b||_1\,e^{-\frac{\kappa s}{\epsilon}}(1+|y|)\widetilde{N}_2(ds,dz)\right]\right)\\
\le& C\epsilon^{1-\frac{1}{\alpha}} + C\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^{d_2}}|z|e^{-\frac{\kappa s}{\epsilon}}\widetilde{N}_2(ds,dz)\right],
\end{aligned}$$

which converges to zero as epsilon becomes small by dominated convergence. Furthermore by estimate (2.52), $\epsilon^{\frac{1}{\alpha}} M^{\epsilon}_{t,2}$ are martingales satisfying

$$\sup_{\epsilon\in(0,1)}\epsilon^{\frac{1}{\alpha}}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\Delta M^\epsilon_{t,2}\right|\right] \le \sup_{\epsilon\in(0,1)} 2\epsilon^{\frac{1}{\alpha}}\mathbb{E}\left[\sup_{t\in[0,T]}\left|M^\epsilon_{t,2}\right|\right] \le C_T < \infty.$$

Then by Corollary 6.30 [JS13], convergence of the stochastic integrals is established. Therefore, as $\epsilon \to 0$ conclude that

$$\zeta^\epsilon_t \Rightarrow \int_0^t\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_2}} h\left(\overline{X}_s, y + b\left(\overline{X}_s, y\right) z\right) - h\left(\overline{X}_s, y\right)\mu^{\overline{X}_s}(dy)\widetilde{N}_2(ds,dz) =: \zeta_t. \tag{2.58}$$

Now consider the continuous mapping $G : \zeta^\epsilon \mapsto Z^\epsilon$. The weak convergence established in (2.58) and the continuous mapping theorem imply

$$Z^\epsilon = G\left(\zeta^\epsilon\right) \Rightarrow G\left(\zeta\right) = V. \tag{2.59}$$

Proceeding to the second part of the proof,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |U_t^\epsilon|\right] \\
&\leq C_{p,T}\epsilon^{\frac{1}{\alpha}-1}\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \overline{f}\left(\overline{X}_s + \epsilon^{1-\frac{1}{\alpha}}V_s^\epsilon\right) - \overline{f}\left(\overline{X}_s\right) - \nabla_x\overline{f}\left(\overline{X}_s\right)\epsilon^{1-\frac{1}{\alpha}}V_s^\epsilon ds\right|\right]\right. \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t f\left(X_s^\epsilon, Y_s^\epsilon\right) - \overline{f}\left(X_s^\epsilon\right) - \mathcal{L}h\left(X_s^\epsilon, Y_s^\epsilon\right) ds\right|\right] \\
&\quad \left. + \mathbb{E}\left[\sup_{t\in[0,T]}\left|\epsilon^\rho\int_0^t c\left(X_s^\epsilon, Y_s^\epsilon\right) dL_{s,1}\right|\right]\right) = C_{p,T}\epsilon^{\frac{1}{\alpha}-1}\sum_{i=1}^3 I_i.
\end{aligned} \tag{2.60}$$

The Poisson equation implies that I_2 is equal to zero and estimate (2.88) in Section 2.5 may be used to show $\epsilon^{\frac{1}{\alpha}-1}I_3$ vanishes in the limit. I_1 is recognized as a first order Taylor expansion. For each $k = 1, ..., d_1$, I_1^k is equal to

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\sum_{i,j=1}^{d_1}\int_0^1 (1-\xi)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}\overline{f}^k\left(\overline{X}_s + \xi\left(X_s^\epsilon - \overline{X}_s\right)\right)d\xi\left(X_s^{\epsilon,i} - \overline{X}_s^i\right)\left(X_s^{\epsilon,j} - \overline{X}_s^j\right)ds\right|\right].$$

From the assumption that $f \in C_b^{2+\gamma,2+\delta}$, all second partial derivatives of $\overline{f}(x) = \int_{\mathbb{R}^{d_2}} f(x,y)\mu^x(dy)$ are bounded. Therefore, by the Cauchy-Schwarz inequality and estimate (2.53)

$$\leq C_{p,T}\mathbb{E}\left[\sup_{t\in[0,T]}\left|V_t^{\epsilon,i}\right|\right]^{\frac{1}{2}}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{\epsilon,j} - \overline{X}_t^j\right|\right]^{\frac{1}{2}} \leq C_{p,T}\epsilon^{\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \to 0.$$

Conclude that (2.60) converges to zero. Combining this with (2.59), then the finite dimensional distributions of V^ϵ converge to V. Having shown tightness in Lemma 2.12, this completes the proof of Theorem 2.2.

□

Remark 2.7. *Notice that this result implies the following trivial case: if the drift of the slow process $f(x,y) \equiv f(x)$ is independent of y then $\overline{f}(x) = f(x)$ and by eq (2.88), $V_t^\epsilon \Rightarrow 0$.*

§2.4 Numerical Study

This section illustrates the above theory through a numerical implementation of the results. The example will depict visually the main components of the theory, that is, the reader will experimentally see that the rate of convergence of X_t^ϵ to $\overline{X}_t$ is bounded by $C\epsilon^{\frac{1}{\alpha}-1}$, the time scaling effect on Y_t^ϵ as ϵ vanishes, the graph of the solution to the Poisson equation with all the the regularity properties, and the weak convergence of V_t^ϵ to V_t. Furthermore, a full explanation of the numerical scheme used is provided and the full code can be found in the following Section 2.4 to allow the reader to verify the implementation's validity.

Before presenting the simulations, the numerical scheme is discussed. There are two types of objects that are numerically approximated. The first are the stochastic processes $\{X_t^\epsilon\}_{t\geq 0}, \{Y_t^\epsilon\}_{t\geq 0}$, and $\{\overline{X}_t\}_{t\geq 0}$ for which a Euler-Maruyama type numerical scheme is employed. Fix N to represent the number of temporal grid points between 0 and T, where $\Delta t = \frac{T}{N}$ represents a single step. Generate N α-stable random variables with the appropriate parameters for dL_t^1 and dL_t^2, or in other words simulate N instances of $\Delta L_n^1 \sim S_{\widetilde{\alpha}}((\Delta t)^{\frac{1}{\widetilde{\alpha}}}, 0, 0), \Delta L_n^2 \sim S_\alpha((\Delta t)^{\frac{1}{\alpha}}, 0, 0)$. Set $X_0^\epsilon = x, Y_0^\epsilon = y, \overline{X}_0 = x$ and for each $n = 1, 2, ..., N$ simulate the evolution with the following iterative scheme

$$
\begin{aligned}
X_{n+1}^\epsilon &= X_n^\epsilon + f\left(X_n^\epsilon, Y_n^\epsilon\right)\Delta t + \epsilon^\rho c\left(X_n^\epsilon, Y_n^\epsilon\right)\Delta L_n^1,\\
Y_{n+1}^\epsilon &= Y_n^\epsilon + \epsilon^{-1} g\left(X_n^\epsilon, Y_n^\epsilon\right)\Delta t + \epsilon^{-\frac{1}{\alpha}} b\left(X_n^\epsilon, Y_n^\epsilon\right)\Delta L_n^2,\\
\overline{X}_{n+1} &= \overline{X}_n + \overline{f}\left(\overline{X}_n\right)\Delta t.
\end{aligned}
$$

And, of course, $V_n^\epsilon := \epsilon^{\frac{1}{\alpha}-1}(X_n^\epsilon - \overline{X}_n)$. That is, once finished simulating the paths for the slow process and the averaged process, use this scaled difference to approximate a path for V_t^ϵ. When simulating, the random noise components are fixed with the same random seed so that all comparisons between different values of ϵ are driven by the same random processes and thus comparable.

The second object to numerically approximate is the solution to the Poisson equation $-\mathcal{L}h(x,y) = f(x,y) - \overline{f}(x)$. This is done on a large enough discretization domain (a, b) such that all values of $y \in (a, b)$ for $a < b \in \mathbb{R}$. Choose K grid points and define the mesh size $h_y = (b-a)/K$. Each grid point is given by $y_i = a + ih_y$ for $i = 0, 1, 2, ..., K$. Let the vector $\mathbf{y} = (y_0, y_1, y_2, ..., y_K)^T$. The following notation is used for the various vectors:

$$
\begin{aligned}
h_x &:= (h(x,y_0), h(x,y_1), h(x,y_2), ..., h(x,y_{K-1}), h(x,y_K))^T\\
f_x &:= (f(x,y_1), f(x,y_2), ..., f(x,y_{K-1}))^T\\
b_x &:= (b(x,y_1), b(x,y_2), ..., b(x,y_{K-1}))^T\\
g_x &:= (g(x,y_1), g(x,y_2), ..., g(x,y_{K-1}))^T\\
e &:= (1, 1, ..., 1)^T.
\end{aligned}
$$

To discretize the fractional Laplacian, fix the splitting parameter $\widetilde{\gamma} = 1 + \frac{\alpha}{2}$, which is experimentally suggested to lead to best convergence (see, [DWZ18]) and controls the accuracy of the discretization scheme by partitioning the problem in an optimal way. This requires the numerical parameter $\kappa_{\widetilde{\gamma}}$ to be set to 1. Furthermore, define the normalizing constant $C^{h}_{\alpha,\widetilde{\gamma}} = \frac{2^{\alpha-1}\alpha\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)\widetilde{\nu}h_y^\alpha}$ with $\widetilde{\nu} := \widetilde{\gamma} - \alpha$. Then by equation (2.10) in [DWZ18], discretize the fractional Laplacian using the following matrix definition reproduced here for convenience,

$$A_{ij} = C^{h_y}_{\alpha,\widetilde{\gamma}} \begin{cases} \sum_{k=2}^{K-1} \frac{(k+1)^{\widetilde{\nu}}-(k-1)^{\widetilde{\nu}}}{k^{\widetilde{\gamma}}} + \frac{K^{\widetilde{\nu}}-(K-1)^{\widetilde{\nu}}}{K^{\widetilde{\gamma}}} + (2^{\widetilde{\nu}} + \kappa_{\widetilde{\gamma}} - 1) + \frac{2\widetilde{\nu}}{\alpha K^\alpha} & j = i, \\ -\frac{(|j-i|+1)^{\widetilde{\nu}}-(|j-i|-1)^{\widetilde{\nu}}}{2|j-i|^{\widetilde{\gamma}}} & j \neq i, i \pm 1, \\ -\frac{1}{2}(2^{\widetilde{\nu}} + \kappa_{\widetilde{\gamma}} - 1) & j = i \pm 1, \end{cases}$$

for $i, j = 1, 2, ..., K-1$. Denote by $\mathbf{A} := [A]_{ij}$ the discretized matrix. Similarly, denote the discretized derivative operator by $\mathbf{D}$, which can be done by standard centered differencing of an appropriate order.

Let $\mathbf{I}$ be the identity matrix. Then

$$\mathbf{h}_x = \begin{cases} (diag(b_x^\alpha)\mathbf{A} + diag(g_x)\mathbf{D})^{-1} \left(\overline{f}(x)\mathbf{I} - diag(f_x)\right) e & \text{i=\{1,2,...,K-1\}} \\ 0 & \text{i=\{0,K\}} \end{cases}$$

is the approximate numerical solution to the Poisson equation.

This numerical solution for h is then used to approximate the integral with respect to the Poisson random measure using the following steps. First apply numerical quadrature to approximate the inner integral with respect to the invariant measure μ^x. Denote by $I(z)$ the integral's approximation as

$$I_x(z) \approx \int_{\mathbb{R}} h\left(x, y + b(x,y)z\right) - h\left(x,y\right)\mu^x(dy).$$

One can search for the closest element y_i in $\mathbf{y}$ and use the index to evaluate $\mathbf{h}_x$ at the specified points. Then interpret the Poisson integral as the sum of the realized jumps less its average; the latter of which is again approximated via quadrature with respect to the Lévy measure over a large enough interval, this time denoted by J_x,

$$J_x \approx \int_{\mathbb{R}} I(z)\nu(dz).$$

Finally, use the following iteration to simulate a path for V_t,

$$V_{n+1} = V_n + \mathbf{D}\overline{f}\left(\overline{X}_n\right)\Delta t + I_{\overline{X}_n}(\Delta L_n) - J_{\overline{X}_n}\Delta t.$$

The chosen application to illustrate the theory is

$$\begin{cases} dX_t^\epsilon = \left(r - \frac{1}{2}Y_t^\epsilon\right)dt + \sqrt{\epsilon}dW_t \\ dY_t^\epsilon = -\epsilon^{-1}Y_t dt + \epsilon^{-\frac{1}{\alpha}}dL_t \end{cases},$$

where $\{W_t\}_{t\geq 0}$ is standard Brownian motion ($\widetilde{\alpha} = 2$), $\{L_t\}_{t\geq 0}$ is the driving symmetric Lévy motion with stability parameter $\alpha = \frac{3}{2}$, $r = 0.03$, $X_0^\epsilon = 0$, and $Y_0^\epsilon = 0$. Numerically, this is simulated with the following iterative scheme for fixed $0 < \epsilon < 1$, and $n \in \{0, 1, 2, ..., N-1\}$:

$$\begin{cases} X_{n+1}^\epsilon = X_n^\epsilon + \left(0.03 - \frac{1}{2}Y_n^\epsilon\right)\Delta t + \sqrt{\epsilon}\Delta W_n, \\ Y_{n+1}^\epsilon = Y_n^\epsilon - \epsilon^{-1}Y_n^\epsilon\Delta t + \epsilon^{-\frac{1}{\alpha}}\Delta L_n \end{cases},$$

where $\{\Delta W_n\}$ are standard normal random variables and $\{\Delta L_n\}$ are $\frac{3}{2}-$stable random variables.

Notice that $\rho = \frac{1}{2} > 1 - \frac{1}{\alpha} = \frac{1}{3}$. As all derivatives are constant, the assumptions (A1) are all satisfied. It is also easy to see that $\sup_{x\in\mathbb{R}} |r| = r < \infty$ and

$$\langle -y_1 + y_2, y_1 - y_2 \rangle \le -|y_1 - y_2|^2;$$

therefore assumption (A2) is also satisfied.

The frozen equation is given by

$$dY_t^{x,y} = -Y_t^{x,y}dt + dL_t,$$

which admits a unique invariant measure $\mu \sim \mathcal{S}_\alpha(\alpha^{-\frac{1}{\alpha}}, 0, 0)$ (see Proposition B.1, [Che+23]). The invariant measure is not analytically expressible, however it is still possible to numerically approximate $\overline{f} \approx \int_{\mathbb{R}} \left(r - \frac{1}{2}y\right) \mu(dy)$ via quadrature. Note that it does not depend on x and so only needs to be approximated once. Furthermore, the approximation to $\overline{X}_n$ will be a linear function in n. Another advantage is that one may solve for the Poisson equation using the simplified form

$$\mathbf{h} = (-\mathbf{A} - \mathbf{I})^{-1} (\overline{f}\mathbf{I} - diag(r\mathbf{e} - \frac{1}{2}\mathbf{y}))\mathbf{e}, \quad \text{for } i = 1, 2, ..., K-1.$$

This is one of a number of small code specializations that become possible with access to the invariant measure. The fact that the system does not depend on x allows for various quantities to be approximated only once (rather than for each x), allowing for a much quicker runtime of the simulations.

For the purpose of the simulation, the model's parameters are fixed at $T = 1, K = 10^2, N = 10^5, a = -0.5$, and $b = 0.5$. Due to the sequential nature of the Euler-Maruyama scheme, it can be slow for large values of N. Therefore, it is important to choose mesh sizes that are fine enough to illustrate the numerical scheme, but not so fine that they would take too much time to run on a personal computer. After running many tests, increasing K did not add much to the approximation because as ϵ becomes small, $Y_{n+1} - Y_n$ takes on increasingly large values. Thus the step size for $\mathbf{y}$ need not be all that small at all. This is not true for N which discretizes time and is very much tied to ϵ. Empirically, N at 10^5 was found to be large enough to show convergence. It is important to make sure $\epsilon \ge \Delta t$ or else numerical instabilities arise because the theoretical scaling in t is of the order ϵ.

In Figure (2.1a), it is visually clear that X_t^ϵ and $\overline{X}_t$ are converging as ϵ tends to 0. Three paths of X_t^ϵ are plotted at $\epsilon = 1.0, \epsilon = 0.00167$, and $\epsilon = 0.00001$. The last of which is very close to $\overline{X}_t$ (the orange dotted line) thus confirming convergence. To make it more explicit, Theorem 2.1 on averaging gives the order of convergence of $\epsilon^{1-\frac{1}{\alpha}}$. In Figure (2.1b) the supremum is approximated by the maximum on $[0, T]$ of $|X_t^\epsilon - \overline{X}_t|$. As can be seen, for all $\epsilon < 10^{-2}$, this maximum stays below the theoretical order of convergence with an approximation of a constant constructed by averaging the implied constant for each ϵ (orange dotted line).

The two plots of Figure (2.1c) and Figure (2.1d) confirm that Y_t^ϵ is behaving as expected. Recall the same random seed is used for each ϵ and so the paths are the same but the scale of Y_t^ϵ is faster relative to X_t^ϵ (Figure (2.1c)), i.e., the random variable $Y_{n+1} - Y_n$ takes increasingly large values. Lemma 2.10 also gives an upper bound as to how fast Y_t^ϵ should grow as $\epsilon \to 0$ and it can be seen in Figure (2.1d) that it does not, in fact, exceed this upper bound (orange dotted line).

Figure (2.1e) plots the numerical solution to the Poisson equation. As expected, it is continuous and satisfies the boundary conditions at $\{a, b\}$ of 0. It is a curve rather than a surface since it does not depend on x, and by consequence it is very easy to see visually that it satisfies all the regularity properties set out in Proposition 2.1.

Figures (2.1f) and (2.1g) are meant to be interpreted together. Figure (2.1f) is an empirical state frequency distribution of ten runs of V_t, but the scale of the states is so small when compared to V_t^ϵ for large ϵ that it appears as a small block in Figure (2.1g) (in brown). Its distribution is hardly discernible from Figure (2.1g) and so Figure (2.1f) was provided for comparison. Having said this, it is evident from Figure (2.1g), that as ϵ becomes smaller, the empirical state frequency distribution of ten path simulations of V_t^ϵ (for each ϵ) are converging to V_t, as the weak convergence result from Theorem 2.2 would imply. One can see the longer tail on the right side with two areas of increased frequency and a slower falloff on the left side being depicted in both Figures (2.1g) and (2.1f).

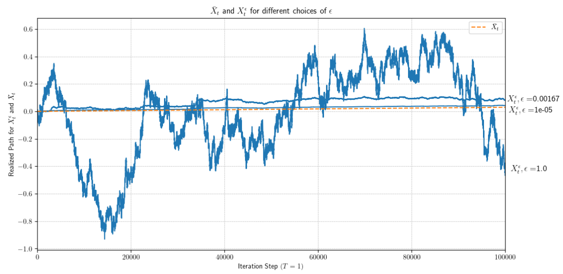
Ten path simulations were generated to collect enough data to build the distribution. When only a single path was generated, there was too little information to see the convergence and simulating more would be beneficial, but quite slow. The empirical transition probability matrix is estimated from these path simulations of V_t^ϵ at various values of ϵ and compared to the empirical transition probability matrix of V_t using Jensen-Shannon divergence in Figure (2.1h). Specifically, each row of the transition probability matrix specifies a probability distribution and the distance between these distributions was compared and then all distances for each row were averaged. As expected, as ϵ becomes small, the average distance between these distributions decreases.

These plots empirically illustrate the validity of this approach to using the theory in practice since the outputs line up nicely with what is expected. Having said this, it is definitely not an efficient approach to simulating the results and it would be interesting to find better ways to derive the same results. In the following section, the reader will find the code used to simulate the various paths and solve the Poisson equation.

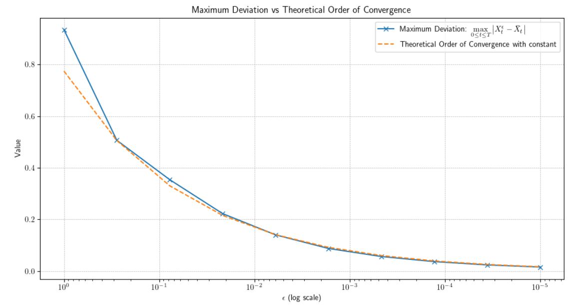
Simulation Code

By specializing the code to this particular example, one can take advantage of certain conveniences that speed up the runtime, which is particularly useful when simulating $N = 10^5$ iterations on a personal computer.

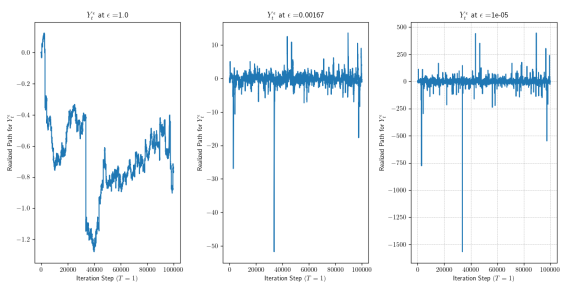
In particular, since there is a known form for the invariant measure, not only is it not necessary to estimate it from runs of the simulation, but one can make use of the numerical approximations of the averaged drift and use that to avoid a potentially computationally costly integral calculation. Also, the fact that the integral does not depend on x when simulating $\overline{X}_t$ allows for a quick cumulative sum calculation rather than an iterative calculation. Indeed, any time either μ^x or ν is required to perform an integral calculation (for example, function `I_mu` and variable `J`), one simply uses pre-built packages to approximate the integrals, which are both convenient and fast.



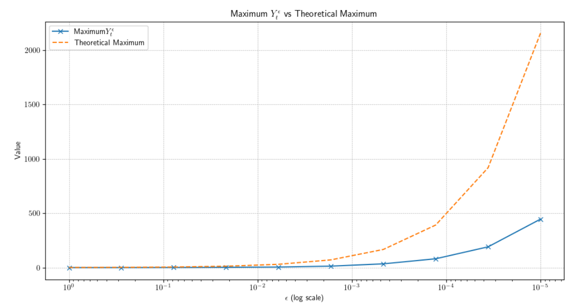
(a) Convergence X_t^ϵ path to $\overline{X}_t$ as $\epsilon \to 0$.



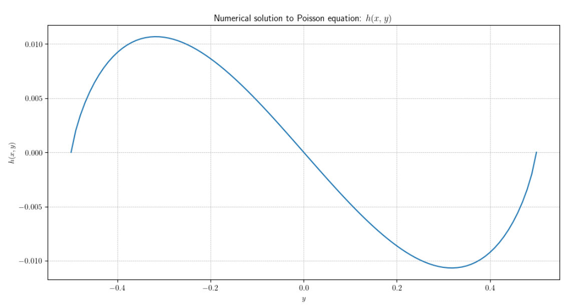
(b) Maximum Deviation between X_t^ϵ and $\overline{X}_t$ as $\epsilon \to 0$.



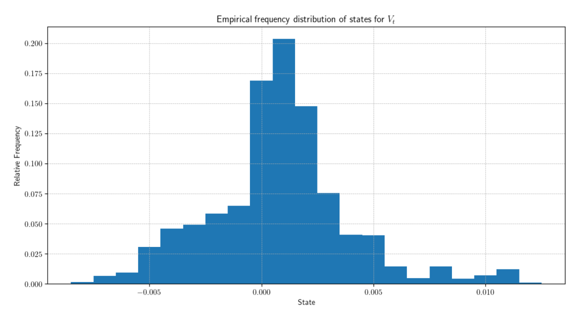
(c) Evolution of Y_t^ϵ as $\epsilon \to 0$.



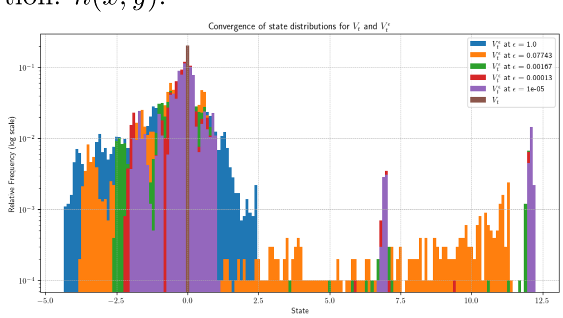
(d) Maximum Y_t^ϵ vs theoretical maximum as $\epsilon \to 0$.



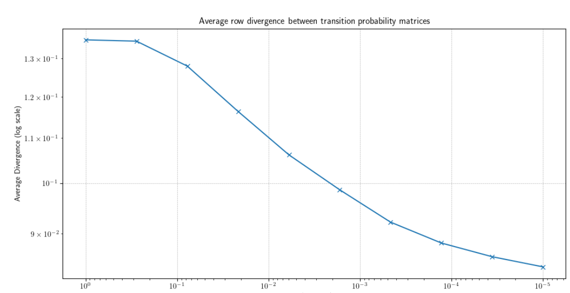
(e) Numerical Solution to the Poisson equation: $h(x, y)$.



(f) Empirical frequency distribution of states for V_t.



(g) Convergence of state distributions for V_t^ϵ to V_t as $\epsilon \to 0$.



(h) Average row divergence between transition probability matrices as $\epsilon \to 0$.

Figure 2.1: Illustrative Plots: The Averaging Principle, Poisson Equation Solution, and the Functional Central Limit Theorem.

Furthermore, the fact that none of the functions depend on x, dramatically speeds up computations since only the solution of a single Poisson equation for an arbitrary value of x is necessary, rather than for each x. That is, a single call to `solve_poisson_equation` rather than potentially $N = 10^5$ (or, more likely, some subsample of it). Likewise, the lack of dependence on x also means that `J` need only be approximated once, rather than for each x and that V_t may be simulated by taking the cumulative sum of the random variables coming from the martingale term rather than having to simulate it iteratively since $\nabla f(x) = 0\ \forall x$.

Python Code

```
def simulate_xy(eps, dL, dW):
        '''simulates a path for X, Y'''
        xt, yt = np.zeros(N+1), np.zeros(N+1)
        xt[0], yt[0] = x0, y0
        for n in range(1, N+1):
                x, y = xt[n-1], yt[n-1]
                xt[n] = x + (r-y/2)*dt + eps**rho*dW[n-1]
                yt[n] = y - eps**(-1)*y*dt + eps**(-1/alpha)*dL[n-1]
        return xt, yt

def mu(y):
        '''returns invariant measure'''
        return levy_stable.pdf(y, alpha=alpha, beta=0, loc=0, scale=alpha**(-1/alpha))
fbar = integrate.quad(lambda y: (r-y/2)*mu(y), -np.inf, np.inf,\
        epsabs=10**(-3), limit=50)[0]
xbar = np.concatenate(([x0], x0 + np.cumsum([fbar*dt]*N)), axis=0)

def discrete_fractional_laplacian():
        '''discretizes the fractional laplacian on the interior of domain'''
        gamma = 1+alpha/2
        kappa_gamma = 2 if gamma == 2 else 1
        normalizing_constant = 2**(alpha-1)*alpha*math.gamma((alpha+1)/2)\
                /(math.sqrt(math.pi**1)*math.gamma(1-alpha/2))
        C = normalizing_constant/((gamma-alpha)*hy**alpha)
        A = np.zeros((K+1,K+1), dtype=float)
        summation = np.sum([((k+1)**(gamma-alpha) - (k-1)**(gamma-alpha))\
                /(k**gamma) for k in range(2,K)])
        for i in tqdm(range(1,K), desc='discretizing fractional laplacian'):
                for j in range(1,K):
                        if i == j:
                                A[i,j] = summation \
                                        + (K**(gamma-alpha) - (K-1)**(gamma-alpha))\
                                        /(K**gamma) + (2**(gamma-alpha)+kappa_gamma-1)\
                                        + (2*(gamma-alpha))/(alpha*K**alpha)
                        elif j == i+1 or j == i-1:
                                A[i,j] = -0.5*(2**(gamma-alpha)+kappa_gamma-1)
                        else:
                                A[i,j] = -((abs(j-i)+1)**(gamma-alpha)-(abs(j-i)-1)\
                                        **(gamma-alpha))/(2*abs(j-i)**gamma)
        return C*A[1:K,1:K]
# solve poisson equation
A = discrete_fractional_laplacian()
h_vec = np.zeros(K+1)
h_vec[1:K] = np.matmul(np.linalg.inv(-A-np.identity(K-1)),\
        np.matmul(fbar*np.identity(K-1)-np.diag([(r-y/2) for y in y_vec[1:K]]),\
        [1]*len(y_vec[1:K])))

def I_mu(z):
        '''approximates the integral with respect to the invariant measure'''
        return integrate.quad(lambda y: (h_vec[np.abs(y_vec - (y + z)).argmin()]\
                - h_vec[np.abs(y_vec - y).argmin()])*mu(y), -np.inf, np.inf,\
```

```
            epsabs=10**(-3), limit=50)[0]

def simulate_v(dL):
        '''simulates a path for V'''
        normalizing_constant = 2**(alpha-1)*alpha*math.gamma((alpha+1)/2)\
                /(math.sqrt(math.pi**1)*math.gamma(1-alpha/2))
        J = integrate.quad(lambda z: normalizing_constant*I_mu(z)\
                /(abs(z)**(1+alpha)), -np.inf, np.inf, epsabs=10**(-3), limit=50)[0]
        results = Parallel(n_jobs=8)(delayed(I_mu)(i) for i in tqdm(dL))
        return np.concatenate(([0],np.cumsum(np.array(results)-J*dt)), axis=0)

def empirical_transition_matrix(lst, all_states):
        trajectories = np.round(np.array(lst), bins)
        possible_states_tm = all_states
        num_states = len(possible_states_tm)
        transition_matrix = np.ones((num_states, num_states), dtype=float)
        for trajectory in trajectories:
                for i in range(len(trajectory)-1):
                        current_state = trajectory[i]
                        next_state = trajectory[i+1]
                        transition_matrix[possible_states_tm == current_state,\
                                possible_states_tm == next_state] += 1.
        row_sums = transition_matrix.sum(axis=1)
        row_sums[row_sums == 0] = 1
        transition_matrix /= row_sums[:, np.newaxis]
        return transition_matrix

def generate_distribution(lst):
        '''estimates a state distribution and a transition probability matrix'''
        trajectories = np.round(np.array(lst),bins)
        possible_states = np.unique(trajectories)
        state_counts = np.zeros(len(possible_states))
        for trajectory in trajectories:
                unique_states, counts = np.unique(trajectory, return_counts=True)
                unique_states = unique_states[~np.isnan(unique_states)]
                for state in unique_states:
                        try:
                                state_counts[possible_states == state] +=\
                                        int(counts[unique_states == state])
                        except:
                                pass
        state_probs = state_counts / (trajectories.shape[0]*trajectories.shape[1])
        return possible_states, state_probs
```

§2.5 Computations

The following section collects the technical proofs of Sections 2.2 and 2.3 that would otherwise take away from the flow of the text.

Poisson Equation Lemmas

Proof of Lemma 2.1. The difference between two frozen processes is given by the process

$$\begin{aligned} Y_t^{x_1,y_1} - Y_t^{x_2,y_2} &= (y_1 - y_2) + \int_0^t g\left(x_1, Y_s^{x_1,y_1}\right) - g\left(x_2, Y_s^{x_2,y_2}\right) ds \\ &\quad + \int_0^t b\left(x_1, Y_s^{x_1,y_1}\right) - b\left(x_2, Y_s^{x_2,y_2}\right) dL_{s,2}. \end{aligned}$$

Apply Itô's formula to the function x^2,

$$
\begin{aligned}
&\left(Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right)^2=\left(y_1-y_2\right)^2+2\int_0^t\langle g\left(x_1,Y_s^{x_1,y_1}\right)-g\left(x_2,Y_s^{x_2,y_2}\right),Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\rangle ds\\
&+\int_0^t\int_{\mathbb{R}^{d_2}}\left(Y_s^{x_1,y_1}-Y_s^{x_2,y_2}+\left(b\left(x_1,Y_s^{x_1,y_1}\right)-b\left(x_2,Y_s^{x_2,y_2}\right)\right)z\right)^2-\left(Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\right)^2\widetilde{N}_2(ds,dz)\\
&+\int_0^t\int_{\mathbb{R}^{d_2}}\left(Y_s^{x_1,y_1}-Y_s^{x_2,y_2}+\left(b\left(x_1,Y_s^{x_1,y_1}\right)-b\left(x_2,Y_s^{x_2,y_2}\right)\right)z\right)^2-\left(Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\right)^2\\
&-\mathbf{1}_{|z|<1}2\langle\left(b\left(x_1,Y_s^{x_1,y_1}\right)-b\left(x_2,Y_s^{x_2,y_2}\right)\right)z,Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\rangle\nu_2(dz)ds.
\end{aligned}
$$

Take expectations,

$$
\begin{aligned}
&E\left[\left(Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right)^2\right]=\left(y_1-y_2\right)^2+\int_0^t E\left[2\langle g\left(x_1,Y_s^{x_1,y_1}\right)-g\left(x_2,Y_s^{x_2,y_2}\right),Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\rangle\right.\\
&+\int_{\mathbb{R}^{d_2}}\left(Y_s^{x_1,y_1}-Y_s^{x_2,y_2}+\left(b\left(x_1,Y_s^{x_1,y_1}\right)-b\left(x_2,Y_s^{x_2,y_2}\right)\right)z\right)^2-\left(Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\right)^2\\
&\left.-\mathbf{1}_{|z|<1}2\langle\left(b\left(x_1,Y_s^{x_1,y_1}\right)-b\left(x_2,Y_s^{x_2,y_2}\right)\right)z,Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\rangle\nu_2(dz)\right]ds,
\end{aligned}
$$

to arrive at

$$
\begin{aligned}
&\frac{d}{dt}E\left[\left(Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right)^2\right]=E\left[2\langle g\left(x_1,Y_t^{x_1,y_1}\right)-g\left(x_2,Y_t^{x_2,y_2}\right),Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\right.\\
&+\int_{\mathbb{R}^{d_2}}\left(Y_t^{x_1,y_1}-Y_t^{x_2,y_2}+\left(b\left(x_1,Y_t^{x_1,y_1}\right)-b\left(x_2,Y_t^{x_2,y_2}\right)\right)z\right)^2-\left(Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right)^2\\
&\left.-\mathbf{1}_{|z|<1}2\langle\left(b\left(x_1,Y_t^{x_1,y_1}\right)-b\left(x_2,Y_t^{x_2,y_2}\right)\right)z,Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\nu_2(dz)\right].
\end{aligned}
$$

By condition (A2),

$$
\begin{aligned}
\frac{d}{dt}E\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right|^2=&E\left[2\langle g\left(x_1,Y_t^{x_1,y_1}\right)-g\left(x_2,Y_t^{x_2,y_2}\right),Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\right.\\
&+2\int_{|z|\geq 1}\langle\left(b\left(x_1,Y_t^{x_1,y_1}\right)-b\left(x_2,Y_t^{x_2,y_2}\right)\right)z,Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\nu_2(dz)\\
&\left.+\int_{\mathbb{R}^{d_2}}\left|b\left(x_1,Y_t^{x_1,y_1}\right)-b\left(x_2,Y_t^{x_2,y_2}\right)\right|^2z^2\nu_2(dz)\right]\\
\leq&E\left[2\langle g\left(x_1,Y_t^{x_1,y_1}\right)-g\left(x_1,Y_t^{x_2,y_2}\right),Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\right.\\
&+2\int_{|z|\geq 1}\langle\left(b\left(x_1,Y_t^{x_1,y_1}\right)-b\left(x_1,Y_t^{x_2,y_2}\right)\right)z,Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\nu_2(dz)\\
&+\int_{\mathbb{R}^{d_2}}\left|b\left(x_1,Y_t^{x_1,y_1}\right)-b\left(x_1,Y_t^{x_2,y_2}\right)\right|^2z^2\nu_2(dz)\\
&+2\langle g\left(x_1,Y_t^{x_2,y_2}\right)-g\left(x_2,Y_t^{x_2,y_2}\right),Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\\
&+2\int_{|z|\geq 1}\langle\left(b\left(x_1,Y_t^{x_2,y_2}\right)-b\left(x_2,Y_t^{x_2,y_2}\right)\right)z,Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\nu_2(dz)\\
&\left.+\int_{\mathbb{R}^{d_2}}\left|b\left(x_1,Y_t^{x_2,y_2}\right)-b\left(x_2,Y_t^{x_2,y_2}\right)\right|^2z^2\nu_2(dz)\right]
\end{aligned}
$$

$$\begin{aligned}
\le & E\Big[-2\kappa\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right|^2 \\
& +C||\nabla_x g|||x_1-x_2|\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right| + C||\nabla_x b|||x_1-x_2|\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right| \\
& +C||\nabla_x b||^2|x_1-x_2|^2\int_{\mathbb{R}^{d_2}} z^2\nu_2(dz)\Big].
\end{aligned}$$

Apply Young's inequality and simplify to get (e.g. letting $C||\nabla_x g|| = \widetilde{C}\sqrt{2\kappa}$ and take $p = q = 2$ in Young's inequality),

$$\frac{d}{dt}E\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right|^2 \le -\kappa E\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right|^2 + C|x_1-x_2|^2.$$

Then by Gronwall's inequality, for any $t\ge 0$,

$$E\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right|^2 \le e^{-\kappa t}|y_1-y_2|^2 + C|x_1-x_2|^2$$

as desired. □

Proof of Corollary 2.1. Lemma 2.1 implies that for any $d>0$,

$$E\left|Y_t^{x,y+d}-Y_t^{x,y}\right| \le e^{-\frac{\kappa t}{2}}\left|d\right|, \quad E\left|Y_t^{x+d,y}-Y_t^{x,y}\right| \le C|d|.$$

Then the results follow immediately by taking d to 0. □

Proof of Lemma 2.3. Consider

$$\begin{aligned}
d\nabla_x Y_t^{x,y} =& \nabla_x g\left(x,Y_t^{x,y}\right)dt + \nabla_y g\left(x,Y_t^{x,y}\right)\nabla_x Y_t^{x,y}dt \\
& + \nabla_x b\left(x,Y_t^{x,y}\right)dL_{t,2} + \nabla_y b\left(x,Y_t^{x,y}\right)\nabla_x Y_t^{x,y}dL_{t,2}, \\
\nabla_x Y_0^{x,y} =& 0.
\end{aligned}$$

Then

$$\begin{aligned}
\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2} =& \int_0^t \nabla_x g\left(x_1,Y_s^{x_1,y_1}\right) - \nabla_x g\left(x_2,Y_s^{x_2,y_2}\right)ds \\
& + \int_0^t \nabla_y g\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1} - \nabla_y g\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}ds \\
& + \int_0^t \nabla_x b\left(x_1,Y_s^{x_1,y_1}\right) - \nabla_x b\left(x_2,Y_s^{x_2,y_2}\right)dL_{s,2} \\
& + \int_0^t \nabla_y b\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1} - \nabla_y b\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}dL_{s,2}.
\end{aligned}$$

Apply Itô's formula to the function x^2,

$$\begin{aligned}
\left(\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\right)^2 =& 2\int_0^t \langle \nabla_x g\left(x_1,Y_s^{x_1,y_1}\right) - \nabla_x g\left(x_2,Y_s^{x_2,y_2}\right) \\
& + \nabla_y g\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1} - \nabla_y g\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}, \nabla_x Y_s^{x_1,y_1} - Y_s^{x_2,y_2}\rangle ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}^{d_2}}\left(\nabla_x Y_s^{x_1,y_1}-\nabla_x Y_s^{x_2,y_2}+\left(\nabla_x b\left(x_1,Y_s^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_s^{x_2,y_2}\right)\right.\right.\\
&\quad\left.\left.+\nabla_y b\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1}-\nabla_y b\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}\right)z\right)^2\\
&\quad-\left(\nabla_x Y_s^{x_1,y_1}-\nabla_x Y_s^{x_2,y_2}\right)^2\widetilde{N}_2(ds,dz)\\
&+\int_0^t\int_{\mathbb{R}^{d_2}}\left(\nabla_x Y_s^{x_1,y_1}-\nabla_x Y_s^{x_2,y_2}+\left(\nabla_x b\left(x_1,Y_s^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_s^{x_2,y_2}\right)\right.\right.\\
&\quad\left.\left.+\nabla_y b\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1}-\nabla_y b\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}\right)z\right)^2-\left(\nabla_x Y_s^{x_1,y_1}-\nabla_x Y_s^{x_2,y_2}\right)^2\\
&\quad-\mathbf{1}_{|z|<1}2\langle\left(\nabla_x b\left(x_1,Y_s^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_s^{x_2,y_2}\right)\right.\\
&\quad\left.+\nabla_y b\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1}-\nabla_y b\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}\right)z,\nabla_x Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\rangle\nu_2(dz)ds.
\end{aligned}
$$

Take expectations,

$$
\begin{aligned}
&E\left|\left|\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}\right|\right|^2=\int_0^t E\left[2\langle\nabla_x g\left(x_1,Y_s^{x_1,y_1}\right)-\nabla_x g\left(x_2,Y_s^{x_2,y_2}\right)\right.\\
&\quad+\nabla_y g\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1}-\nabla_y g\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2},\nabla_x Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\rangle\\
&+\int_{\mathbb{R}^{d_2}}\left(\nabla_x Y_s^{x_1,y_1}-\nabla_x Y_s^{x_2,y_2}+\left(\nabla_x b\left(x_1,Y_s^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_s^{x_2,y_2}\right)\right.\right.\\
&\quad\left.\left.+\nabla_y b\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1}-\nabla_y b\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}\right)z\right)^2-\left(\nabla_x Y_s^{x_1,y_1}-\nabla_x Y_s^{x_2,y_2}\right)^2\\
&\quad-\mathbf{1}_{|z|<1}2\langle\left(\nabla_x b\left(x_1,Y_s^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_s^{x_2,y_2}\right)\right.\\
&\quad\left.\left.+\nabla_y b\left(x_1,Y_s^{x_1,y_1}\right)\nabla_x Y_s^{x_1,y_1}-\nabla_y b\left(x_2,Y_s^{x_2,y_2}\right)\nabla_x Y_s^{x_2,y_2}\right)z,\nabla_x Y_s^{x_1,y_1}-Y_s^{x_2,y_2}\rangle\nu_2(dz)\right]ds,
\end{aligned}
$$

to arrive at

$$
\begin{aligned}
&\frac{d}{dt}E\left|\left|\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}\right|\right|^2=E\left[2\langle\nabla_x g\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x g\left(x_2,Y_t^{x_2,y_2}\right)\right.\\
&\quad+\nabla_y g\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y g\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2},\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}\rangle\\
&\quad+\int_{\mathbb{R}^{d_2}}\left(\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}+\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right.\right.\\
&\quad\left.\left.+\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\right)z\right)^2-\left(\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}\right)^2\\
&\quad-\mathbf{1}_{|z|<1}2\langle\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right.\\
&\quad\left.\left.+\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\right)z,\nabla_x Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\nu_2(dz)\right]\\
&=E\left[2\langle\nabla_x g\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x g\left(x_2,Y_t^{x_2,y_2}\right),\nabla_x Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\right.\\
&\quad+\langle\nabla_y g\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y g\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2},\nabla_x Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\rangle\\
&\quad+\int_{|z|\geq 1}2\langle\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right.\\
&\quad\left.+\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\right)z,\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)\\
&\quad+\int_{\mathbb{R}^{d_2}}\left(\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right.\right.\\
&\quad\left.\left.\left.+\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\right)z\right)^2\nu_2(dz)\right]
\end{aligned}
$$

$$
\begin{aligned}
=&E\Big[2\langle \nabla_x g\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x g\left(x_2, Y_t^{x_2,y_2}\right), \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \\
&\quad + \int_{|z|\geq 1} 2\langle \left(\nabla_x b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x b\left(x_2, Y_t^{x_2,y_2}\right)\right) z, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \nu_2(dz) \\
&\quad + \int_{\mathbb{R}^{d_2}} \left(\nabla_x b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x b\left(x_2, Y_t^{x_2,y_2}\right)\right)^2 z^2 \nu_2(dz) \\
&\quad + \int_{\mathbb{R}^{d_2}} 2\langle \left(\nabla_x b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x b\left(x_2, Y_t^{x_2,y_2}\right)\right) z, \\
&\qquad \nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_1,y_1} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right) \nabla_x Y_t^{x_2,y_2}\rangle \nu_2(dz) \\
&\quad + 2\langle \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_1,y_1} - \nabla_y g\left(x_2, Y_t^{x_2,y_2}\right) \nabla_x Y_t^{x_2,y_2}, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \\
&\quad + \int_{|z|\geq 1} 2\langle \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_1,y_1} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right) \nabla_x Y_t^{x_2,y_2}\right) z, \\
&\qquad \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \nu_2(dz) \\
&\quad + \int_{\mathbb{R}^{d_2}} \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_1,y_1} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right) \nabla_x Y_t^{x_2,y_2}\right)^2 z^2 \nu_2(dz)\Big] \\
=:& \sum_{i=1}^{7} D_i.
\end{aligned}
$$

By the assumption that $g \in C_b^{1+\gamma,2+\gamma}$, mean value theorem, and Young's inequality,

$$
\begin{aligned}
D_1 =& 2\langle \nabla_x g\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x g\left(x_2, Y_t^{x_1,y_1}\right), \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \\
&+ 2\langle g\left(x_2, Y_t^{x_1,y_1}\right) - \nabla_x g\left(x_2, Y_t^{x_2,y_2}\right), \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \\
\leq& 2\left|\nabla_x g\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x g\left(x_2, Y_t^{x_1,y_1}\right)\right| ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}|| \\
&+ \left|g\left(x_2, Y_t^{x_1,y_1}\right) - \nabla_x g\left(x_2, Y_t^{x_2,y_2}\right)\right| ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}|| \\
\leq& 2C\left|x_1 - x_2\right|^\gamma \left|\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\right| \\
&+ 2||\nabla_y \nabla_x g||\, |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}|| \\
\leq& 2C\left(\frac{|x_1 - x_2|^{2\gamma}}{2} + \frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2}\right) \\
&+ 2||\nabla_y \nabla_x g|| \left(e^{-\frac{\kappa t}{2}} \frac{|y_1 - y_2|^2}{2} + C\frac{|x_1 - x_2|^2}{2} + \frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2}\right) \\
\leq& C\left(|x_1 - x_2|^{2\gamma} + |x_1 - x_2|^2 + |e^{-\frac{\kappa t}{2}} y_1 - y_2|^2 + ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2\right).
\end{aligned}
\tag{2.61}
$$

Similarly, noting that $\int_{|z|\geq 1} |z| \nu_2(dz) < \infty$,

$$
\begin{aligned}
D_2 =& \int_{|z|\geq 1} 2\langle \left(\nabla_x b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x b\left(x_2, Y_t^{x_1,y_1}\right)\right) z, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \nu_2(dz) \\
&+ \int_{|z|\geq 1} 2\langle \left(\nabla_x b\left(x_2, Y_t^{x_1,y_1}\right) - \nabla_x b\left(x_2, Y_t^{x_2,y_2}\right)\right) z, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \nu_2(dz)
\end{aligned}
$$

$$\leq 2C\left(\frac{|x_1-x_2|^{2\gamma}}{2}+\frac{||\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}||^2}{2}\right)$$
$$+2C||\nabla_y\nabla_x b||\left(e^{-\frac{\kappa t}{2}}\frac{|y_1-y_2|^2}{2}+C\frac{|x_1-x_2|^2}{2}+\frac{||\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}||^2}{2}\right)$$
$$\leq C\left(|x_1-x_2|^{2\gamma}+|x_1-x_2|^2+e^{-\frac{\kappa t}{2}}|y_1-y_2|^2+||\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_2}||^2\right); \tag{2.62}$$

and noting that $\int_{\mathbb{R}^{d_2}}|z|^2\nu_2(dz)<\infty$,

$$D_3\leq\int_{\mathbb{R}^{d_2}}\left|\left|\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)\right|\right|^2 z^2\nu_2(dz)$$
$$+\int_{\mathbb{R}^{d_2}}\left|\left|\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right|\right|^2 z^2\nu_2(dz)$$
$$\leq C|x_1-x_2|^{2\gamma}+C||\nabla_y\nabla_x b||^2\left|Y_t^{x_1,y_1}-Y_t^{x_2,y_2}\right|^2$$
$$\leq C\left(|x_1-x_2|^{2\gamma}+|x_1-x_2|^2+e^{-\frac{\kappa t}{2}}|y_1-y_2|^2\right). \tag{2.63}$$

By Corollary 2.1, D_4 is equal to

$$\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$
$$+\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$
$$\leq\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$
$$+\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2}-\nabla_y b\left(x_1,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$
$$+\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_1,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$
$$+\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1}-\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$
$$+\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2}-\nabla_y b\left(x_1,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$
$$+\int_{\mathbb{R}^{d_2}}2\langle\left(\nabla_x b\left(x_2,Y_t^{x_1,y_1}\right)-\nabla_x b\left(x_2,Y_t^{x_2,y_2}\right)\right)z,$$
$$\nabla_y b\left(x_1,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}-\nabla_y b\left(x_2,Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\rangle\nu_2(dz)$$

$$
\begin{aligned}
\leq & \int_{\mathbb{R}^{d_2}} 2C|x_1-x_2|^\gamma |z| \, ||\nabla_y b|| \, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}|| \, \nu_2(dz) \\
& + \int_{\mathbb{R}^{d_2}} 2C|x_1-x_2|^\gamma |z| \, \left|\left|\nabla_y^2 b\right|\right| |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}| \, ||\nabla_x Y_t^{x_2,y_2}|| \, \nu_2(dz) \\
& + \int_{\mathbb{R}^{d_2}} 2C|x_1-x_2|^\gamma |z| C|x_1-x_2|^\gamma \, ||\nabla_x Y_t^{x_2,y_2}|| \, \nu_2(dz) \\
& + \int_{\mathbb{R}^{d_2}} 2 \, ||\nabla_y \nabla_x b|| \, |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}| \, |z| \, ||\nabla_y b|| \, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}|| \, \nu_2(dz) \\
& + \int_{\mathbb{R}^{d_2}} 2 \, ||\nabla_y \nabla_x b|| \, |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}| \, |z| \left|\left|\nabla_y^2 b\right|\right| |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}| \, ||\nabla_x Y_t^{x_2,y_2}|| \, \nu_2(dz) \\
& + \int_{\mathbb{R}^{d_2}} 2 \, ||\nabla_y \nabla_x b|| \, |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}| \, |z| C|x_1-x_2|^\gamma \, ||\nabla_x Y_t^{x_2,y_2}|| \, \nu_2(dz) \\
\leq & C|x_1-x_2|^\gamma \, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}|| + C|x_1-x_2|^\gamma \left(e^{-\frac{\kappa t}{2}} |y_1-y_2| + C|x_1-x_2| \right) \\
& + C|x_1-x_2|^{2\gamma} + C\left(e^{-\frac{\kappa t}{2}} |y_1-y_2| + C|x_1-x_2| \right) ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}|| \\
& + C\left(e^{-\frac{\kappa t}{2}} |y_1-y_2| + C|x_1-x_2| \right)^2 + C\left(e^{-\frac{\kappa t}{2}} |y_1-y_2| + C|x_1-x_2| \right) |x_1-x_2|^\gamma .
\end{aligned}
$$

And by Young's inequality

$$
\begin{aligned}
\leq & C\left(\frac{|x_1-x_2|^{2\gamma}}{2} + \frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2} \right) \\
& + C\left(\frac{|x_1-x_2|^{2\gamma}}{2} + e^{-\frac{\kappa t}{2}} \frac{|y_1-y_2|^2}{2} + C\frac{|x_1-x_2|^2}{2} \right) \\
& + C|x_1-x_2|^{2\gamma} + C\left(e^{-\frac{\kappa t}{2}} \frac{|y_1-y_2|^2}{2} + C\frac{|x_1-x_2|^2}{2} + \frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2} \right) \\
& + C\left(e^{-\frac{\kappa t}{2}} \frac{|y_1-y_2|^2}{2} + C\frac{|x_1-x_2|^2}{2} \right) + C\left(e^{-\frac{\kappa t}{2}} \frac{|y_1-y_2|^2}{2} + C\frac{|x_1-x_2|^2}{2} + \frac{|x_1-x_2|^{2\gamma}}{2} \right) \\
\leq & C\left(|x_1-x_2|^{2\gamma} + |x_1-x_2|^2 + e^{-\frac{\kappa t}{2}} |y_1-y_2|^2 + ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2 \right). \qquad (2.64)
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
& D_5 + D_6 + D_7 \leq \\
& \quad 2\langle \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_1,y_1} - \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_2,y_2}, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2} \rangle \\
& \quad + 2\langle \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_2,y_2} - \nabla_y g\left(x_2, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_2,y_2}, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2} \rangle \\
& \quad + 2\langle \nabla_y g\left(x_2, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_2,y_2} - \nabla_y g\left(x_2, Y_t^{x_2,y_2}\right) \nabla_x Y_t^{x_2,y_2}, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2} \rangle \\
& \quad + \int_{|z|\geq 1} 2\langle \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_1,y_1} - \nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \nabla_x Y_t^{x_2,y_2} \right) z, \\
& \quad\quad \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2} \rangle \nu_2(dz)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{|z|\geq 1} 2\langle\left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2} - \nabla_y b\left(x_2, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2}\right) z,\\
&\quad \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \nu_2(dz)\\
&+\int_{|z|\geq 1} 2\langle\left(\nabla_y b\left(x_2, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\right) z,\\
&\quad \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\rangle \nu_2(dz)\\
&+\int_{\mathbb{R}^{d_2}} \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_1,y_1} - \nabla_y b\left(x_1, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2}\right)^2 z^2 \nu_2(dz)\\
&+\int_{\mathbb{R}^{d_2}} \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2} - \nabla_y b\left(x_2, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2}\right)^2 z^2 \nu_2(dz)\\
&+\int_{\mathbb{R}^{d_2}} \left(\nabla_y b\left(x_2, Y_t^{x_1,y_1}\right)\nabla_x Y_t^{x_2,y_2} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right)\nabla_x Y_t^{x_2,y_2}\right)^2 z^2 \nu_2(dz)\Bigg].
\end{aligned}
$$

And by (2.11), $g \in C_b^{1+\gamma,2+\gamma}$, and Young's inequality,

$$
\begin{aligned}
\leq& -2\kappa\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2 + 2\left(|x_1-x_2|^\gamma\, ||\nabla_x Y_t^{x_2,y_2}||\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||\right)\\
&+2\left|\left|\nabla_y^2 g\right|\right|\, |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|\, ||\nabla_x Y_t^{x_1,y_1}||\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||\\
&+2\int_{|z|\geq 1} |z||x_1-x_2|^\gamma\, ||\nabla_x Y_t^{x_2,y_2}||\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||\, \nu_2(dz)\\
&+2\int_{|z|\geq 1} |z|\left|\left|\nabla_y^2 b\right|\right|\, |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|\, ||\nabla_x Y_t^{x_2,y_2}||\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||\, \nu_2(dz)\\
&+\int_{\mathbb{R}^{d_2}} |z|^2|x_1-x_2|^{2\gamma}\, ||\nabla_x Y_t^{x_2,y_2}||^2\, \nu_2(dz)\\
&+\int_{\mathbb{R}^{d_2}} |z|^2\left|\left|\nabla_y^2 b\right|\right|^2 |Y_t^{x_1,y_1} - Y_t^{x_2,y_2}|^2\, ||\nabla_x Y_t^{x_2,y_2}||^2\, \nu_2(dz) \qquad (2.65)\\
\leq& -2\kappa\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2\\
&+2\sup_{t,x,y}||\nabla_x Y_t^{x,y}||\left(\frac{|x_1-x_2|^{2\gamma}}{2} + \frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2}\right)\\
&+2\sup_{t,x,y}||\nabla_x Y_t^{x,y}||\left|\left|\nabla_y^2 g\right|\right|\left(e^{-\frac{\kappa t}{2}}\frac{|y_1-y_2|^2}{2} + C\frac{|x_1-x_2|^2}{2} + \frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2}\right)\\
&+2\int_{|z|\geq 1}|z|\nu_2(dz)\left(\sup_{t,x,y}||\nabla_x Y_t^{x,y}||\left(\frac{|x_1-x_2|^{2\gamma}}{2} + \frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2}\right)\right)\\
&+2\int_{|z|\geq 1}|z|\nu_2(dz)\left(\sup_{t,x,y}||\nabla_x Y_t^{x,y}||\left|\left|\nabla_y^2 b\right|\right|\left(e^{-\frac{\kappa t}{2}}\frac{|y_1-y_2|^2}{2} + C\frac{|x_1-x_2|^2}{2}\right.\right.\\
&\quad\left.\left.+\frac{||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2}{2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^{d_2}} |z|^2\nu_2(dz) \sup_{t,x,y} ||\nabla_x Y_t^{x,y}||^2 \, |x_1-x_2|^{2\gamma} \\
&+\int_{\mathbb{R}^{d_2}} |z|^2\nu_2(dz) \left|\left|\nabla_y^2 b\right|\right|^2 \sup_{t,x,y} ||\nabla_x Y_t^{x,y}||^2 \left(e^{-\frac{\kappa t}{2}}|y_1-y_2|^2 + C|x_1-x_2|^2\right) \\
\leq& C\left(||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}||^2 + |x_1-x_2|^{2\gamma} + |x_1-x_2|^2 + e^{-\frac{\kappa t}{2}}|y_1-y_2|^2\right). \qquad (2.66)
\end{aligned}
$$

The result follows by putting (2.61), (2.62), (2.63), (2.64), and (2.65) together with Gronwall's inequality. □

Proof of Corollary 2.2. Lemma 2.3 implies for any $d>0$,

$$E\left|\left|\nabla_x Y_t^{x,y+d} - \nabla_x Y_t^{x,y}\right|\right| \leq Ce^{-\frac{\kappa t}{4}}|d|.$$

Then the result follows immediately by taking d to 0. □

Proof of Lemma 2.4.

$$
\begin{aligned}
&\left|\left|\nabla_x \widehat{f}(x,y_1,t) - \nabla_x \widehat{f}(x,y_2,t)\right|\right| = ||\nabla_x Ef\left(x,Y_t^{x,y_1}\right) - \nabla_x Ef\left(x,Y_t^{x,y_2}\right)|| \\
\leq& E\,||\nabla_x f\left(x,Y_t^{x,y_1}\right) + \nabla_y f\left(x,Y_t^{x,y_1}\right)\nabla_x Y_t^{x,y_1} - \nabla_x f\left(x,Y_t^{x,y_2}\right) - \nabla_y f\left(x,Y_t^{x,y_2}\right)\nabla_x Y_t^{x,y_2}|| \\
\leq& E\,||\nabla_x f\left(x,Y_t^{x,y_1}\right) - \nabla_x f\left(x,Y_t^{x,y_2}\right)|| \\
&+ E\,||\nabla_y f\left(x,Y_t^{x,y_1}\right)\nabla_x Y_t^{x,y_1} - \nabla_y f\left(x,Y_t^{x,y_2}\right)\nabla_x Y_t^{x,y_2}|| \\
\leq& E\,||\nabla_x f\left(x,Y_t^{x,y_1}\right) - \nabla_x f\left(x,Y_t^{x,y_2}\right)|| \\
&+ E\,||\nabla_y f\left(x,Y_t^{x,y_1}\right)\nabla_x Y_t^{x,y_1} - \nabla_y f\left(x,Y_t^{x,y_2}\right)\nabla_x Y_t^{x,y_1}|| \\
&+ E\,||\nabla_y f\left(x,Y_t^{x,y_2}\right)\nabla_x Y_t^{x,y_1} - \nabla_y f\left(x,Y_t^{x,y_2}\right)\nabla_x Y_t^{x,y_2}|| \\
=:& \sum_{i=1}^{3} E_i.
\end{aligned}
$$

By the generalized mean value theorem, there exists a C_θ for any $\theta \in (0,1]$ such that

$$E_1 \leq C_\theta\,||\nabla_y\nabla_x f||\,E\,||Y_t^{x,y_1} - Y_t^{x,y_2}||^\theta \leq C_\theta e^{-\frac{\kappa\theta t}{2}}|y_1-y_2|^\theta. \qquad (2.67)$$

Likewise,

$$E_2 \leq C_\theta\,||\nabla_x Y_t^{x,y_1}||\,\left|\left|\nabla_y^2 f\right|\right|\,E\,||Y_t^{x,y_1} - Y_t^{x,y_2}||^\theta \leq C_\theta e^{-\frac{\kappa\theta t}{2}}|y_1-y_2|^\theta. \qquad (2.68)$$

Lastly, apply estimate (2.23),

$$E_3 \leq C_\theta\,||\nabla_y f||\,E\,||\nabla_x Y_t^{x,y_1} - \nabla_x Y_t^{x,y_2}||^\theta \leq C_\theta e^{-\frac{\kappa\theta t}{4}}|y_1-y_2|^\theta. \qquad (2.69)$$

Clearly, the result follows from (2.67), (2.68), and (2.69). □

Proof of Lemma 2.5. By the Markov Property

$$\begin{aligned}\widetilde{f}_{t_0}(x,y,t) =&\widehat{f}(x,y,t) - \widehat{f}(x,y,t+t_0)\\ =&\widehat{f}(x,y,t) - Ef\left(x, Y^{x,y}_{t+t_0}\right)\\ =&\widehat{f}(x,y,t) - E\left[E\left[f\left(x, Y^{x,y}_{t+t_0}\right)\middle|\mathcal{F}_{t_0}\right]\right]\\ =&\widehat{f}(x,y,t) - E\widehat{f}\left(x, Y^{x,y}_{t_0}, t\right).\end{aligned}$$

Therefore,

$$\nabla_x \widetilde{f}_{t_0}(x,y,t) = \nabla_x \widehat{f}(x,y,t) - E\left[\nabla_x \widehat{f}\left(x, Y^{x,y}_{t_0}, t\right)\right] - E\left[\nabla_y \widehat{f}\left(x, Y^{x,y}_{t_0}, t\right)\cdot \nabla_x Y^{x,y}_{t_0}\right], \quad (2.70)$$

where $\nabla_x Y^{x,y}_t$ satisfies

$$\begin{aligned}d\nabla_x Y^{x,y}_t =&\nabla_x g\left(x, Y^{x,y}_t\right)dt + \nabla_y g\left(x, Y^{x,y}_t\right)\nabla_x Y^{x,y}_t dt\\ &+ \nabla_x b\left(x, Y^{x,y}_t\right)dL_{t,2} + \nabla_y b\left(x, Y^{x,y}_t\right)\nabla_x Y^{x,y}_t dL_{t,2},\\ \nabla_x Y^{x,y}_0 =&0.\end{aligned}$$

In view of Corollary 2.1,

$$\sup_{x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} \left|\left|\nabla_y \widehat{f}(x,y,t)\right|\right| = \sup_{x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} ||E\left[\nabla_y f\left(x, Y^{x,y}_t\right)\cdot \nabla_y Y^{x,y}_t\right]|| \le Ce^{-\frac{\kappa t}{2}}. \quad (2.71)$$

Then by Lemma 2.4,

$$\begin{aligned}\left|\left|\nabla_x \widetilde{f}_{t_0}(x,y,t)\right|\right| \le& E\left[\nabla_x \widehat{f}(x,y,t) - \nabla_x \widehat{f}\left(x, Y^{x,y}_{t_0}, t\right)\right] + Ce^{-\frac{\kappa t}{2}}\\ \le& C_\theta e^{-\frac{\kappa\theta t}{4}} E|y - Y^{x,y}_{t_0}|^\theta + Ce^{-\frac{\kappa\theta t}{4}}.\end{aligned}$$

By estimate (2.39), conclude

$$\le C_\theta e^{-\frac{\kappa\theta t}{4}}\left(1+|y|^\theta\right).$$

So, for any $\theta\in(0,1]$, choose $\eta = \frac{\kappa\theta t}{4}$. □

Proof of Lemma 2.6. Consider

$$\begin{aligned}&d\nabla_y Y^{x,y}_t = \nabla_y g\left(x, Y^{x,y}_t\right)\cdot\nabla_y Y^{x,y}_t dt + \nabla_y b\left(x, Y^{x,y}_t\right)\cdot \nabla_y Y^{x,y}_t dL_{s,2}\\ &\nabla_y Y^{x,y}_0 = I.\end{aligned}$$

Then

$$\begin{aligned}\nabla_y Y^{x_1,y_1}_t - \nabla_y Y^{x_2,y_2}_t =& \int_0^t \nabla_y g\left(x_1, Y^{x_1,y_1}_s\right)\cdot \nabla_y Y^{x_1,y_1}_s - \nabla_y g\left(x_2, Y^{x_2,y_2}_s\right)\cdot\nabla_y Y^{x_2,y_2}_s ds\\ &+ \int_0^t \nabla_y b\left(x_1, Y^{x_1,y_1}_s\right)\cdot\nabla_y Y^{x_1,y_1}_s - \nabla_y b\left(x_2, Y^{x_2,y_2}_s\right)\cdot\nabla_y Y^{x_2,y_2}_s dL_{s,2}.\end{aligned}$$

Apply Itô's formula to the function x^2,

$$
\begin{aligned}
(\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2})^2 = & \int_0^t 2\langle \nabla_y g\left(x_1, Y_s^{x_1,y_1}\right) \cdot \nabla_y Y_s^{x_1,y_1} - \nabla_y g\left(x_2, Y_s^{x_2,y_2}\right) \cdot \nabla_y Y_s^{x_2,y_2}, \\
& \nabla_y Y_s^{x_1,y_1} - \nabla_y Y_s^{x_2,y_2} \rangle ds \\
& + \int_0^t \int_{\mathbb{R}^{d_2}} (\nabla_y Y_s^{x_1,y_1} - \nabla_y Y_s^{x_2,y_2} \\
& \quad + \left(\nabla_y b\left(x_1, Y_s^{x_1,y_1}\right) \cdot \nabla_y Y_s^{x_1,y_1} - \nabla_y b\left(x_2, Y_s^{x_2,y_2}\right) \cdot \nabla_y Y_s^{x_2,y_2}\right) z)^2 \\
& - (\nabla_y Y_s^{x_1,y_1} - \nabla_y Y_s^{x_2,y_2})^2 \, \widetilde{N}_2(ds, dz) \\
& + \int_0^t \int_{\mathbb{R}^{d_2}} (\nabla_y Y_s^{x_1,y_1} - \nabla_y Y_s^{x_2,y_2} \\
& \quad + \left(\nabla_y b\left(x_1, Y_s^{x_1,y_1}\right) \cdot \nabla_y Y_s^{x_1,y_1} - \nabla_y b\left(x_2, Y_s^{x_2,y_2}\right) \cdot \nabla_y Y_s^{x_2,y_2}\right) z)^2 \\
& - (\nabla_y Y_s^{x_1,y_1} - \nabla_y Y_s^{x_2,y_2})^2 \\
& - \mathbf{1}_{|z|<1} 2\langle \left(\nabla_y b\left(x_1, Y_s^{x_1,y_1}\right) \cdot \nabla_y Y_s^{x_1,y_1} - \nabla_y b\left(x_2, Y_s^{x_2,y_2}\right) \cdot \nabla_y Y_s^{x_2,y_2}\right) z, \\
& \nabla_y Y_s^{x_1,y_1} - \nabla_y Y_s^{x_2,y_2} \rangle \nu_2(dz) ds.
\end{aligned}
$$

Take expectations to arrive at

$$
\begin{aligned}
& \frac{d}{dt} E\left|\left|\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\right|\right|^2 \\
= & E\left[2\langle \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_1,y_1} - \nabla_y g\left(x_2, Y_t^{x_2,y_2}\right) \cdot \nabla_y Y_t^{x_2,y_2}, \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle \right. \\
& + \int_{\mathbb{R}^{d_2}} (\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \\
& \quad + \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_1,y_1} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right) \cdot \nabla_y Y_t^{x_2,y_2}\right) z)^2 \\
& - (\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2})^2 \\
& - \mathbf{1}_{|z|<1} 2\langle \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_1,y_1} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right) \cdot \nabla_y Y_t^{x_2,y_2}\right) z, \\
& \left. \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle \nu_2(dz) \right] \\
= & E\left[2\langle \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_1,y_1} - \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_2,y_2}, \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle \right. \\
& + 2\langle \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_2,y_2} - \nabla_y g\left(x_2, Y_t^{x_2,y_2}\right) \cdot \nabla_y Y_t^{x_2,y_2}, \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle \\
& + \int_{|z|\geq 1} 2\langle \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_1,y_1} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right) \cdot \nabla_y Y_t^{x_2,y_2}\right) z, \\
& \quad \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle \nu_2(dz) \\
& \left. + \int_{\mathbb{R}^{d_2}} \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \cdot \nabla_y Y_t^{x_1,y_1} - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right) \cdot \nabla_y Y_t^{x_2,y_2}\right)^2 z^2 \nu_2(dz) \right] \\
\leq & E\left[2\langle \nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) \cdot \left(\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\right), \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle \right. \\
& + \int_{|z|\geq 1} 2\langle \nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) z \cdot \left(\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\right), \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle \nu_2(dz) \\
& + \int_{\mathbb{R}^{d_2}} \left(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) \cdot \left(\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\right)\right)^2 z^2 \nu_2(dz) \\
& + 2\langle \left(\nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_y g\left(x_2, Y_t^{x_2,y_2}\right)\right) \cdot \nabla_y Y_t^{x_2,y_2}, \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2} \rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{|z|\geq 1} 2\langle(\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right)) \cdot \nabla_y Y_t^{x_2,y_2} z, \nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\rangle \nu_2(dz) \\
&+\int_{\mathbb{R}^{d_2}} \left((\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right)) \cdot \nabla_y Y_t^{x_2,y_2}\right)^2 z^2 \nu_2(dz)\Big] \\
=: &\sum_{i=1}^{6} F_i.
\end{aligned}
$$

From (2.11)

$$
F_1 + F_2 + F_3 \leq -\kappa C E \left|\left|\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\right|\right|^2. \tag{2.72}
$$

$$
\begin{aligned}
F_4 \leq &2E\left[||\nabla_y g\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_y g\left(x_1, Y_t^{x_2,y_2}\right)||\,||\nabla_y Y_t^{x_2,y_2}||\,||\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}||\right] \\
&+ 2E\left[||\nabla_y g\left(x_1, Y_t^{x_2,y_2}\right) - \nabla_y g\left(x_2, Y_t^{x_2,y_2}\right)||\,||\nabla_y Y_t^{x_2,y_2}||\,||\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}||\right] \\
\leq &2\left(\left|\left|\nabla_y^2 g\right|\right| E\left|Y_t^{x_1,y_1} - Y_t^{x_2,y_2}\right| + C|x_1 - x_2|^\gamma\right) e^{-\frac{\kappa t}{2}} E\left|\left|\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\right|\right| \\
\leq &2Ce^{-\frac{\kappa t}{2}}\left(e^{-\frac{\kappa t}{2}}|y_1 - y_2| + C|x_1 - x_2| + |x_1 - x_2|^\gamma\right) E\left|\left|\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}\right|\right| \\
\leq &C\left(E\left|\left|\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\right|\right|^2 + |x_1 - x_2|^{2\gamma} + |x_1 - x_2|^2 + e^{-\frac{\kappa t}{2}}|y_1 - y_2|^2\right),
\end{aligned} \tag{2.73}
$$

where the last line follows by Young's inequality.

In the same way and noting that $\int_{|z|\geq 1} |z|\nu_2(dz) < \infty$ and $\int_{\mathbb{R}^{d_2}} z^2 \nu_2(dz) < \infty$,

$$
\begin{aligned}
F_5 \leq &2CE\left[||\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_y b\left(x_1, Y_t^{x_2,y_2}\right)||\,||\nabla_y Y_t^{x_2,y_2}||\,||\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}||\right] \\
&+ 2CE\left[||\nabla_y b\left(x_1, Y_t^{x_2,y_2}\right) - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right)||\,||\nabla_y Y_t^{x_2,y_2}||\,||\nabla_y Y_t^{x_1,y_1} - \nabla_y Y_t^{x_2,y_2}||\right] \\
\leq &C\left(E\left|\left|\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_2}\right|\right|^2 + |x_1 - x_2|^{2\gamma} + |x_1 - x_2|^2 + e^{-\frac{\kappa t}{2}}|y_1 - y_2|^2\right),
\end{aligned} \tag{2.74}
$$

and

$$
\begin{aligned}
F_6 \leq &C\Big(E\left|\left|\nabla_y b\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_y b\left(x_1, Y_t^{x_2,y_2}\right)\right|\right|^2 \\
&+E\left|\left|\nabla_y b\left(x_1, Y_t^{x_2,y_2}\right) - \nabla_y b\left(x_2, Y_t^{x_2,y_2}\right)\right|\right|^2\Big) E\left|\left|\nabla_y Y_t^{x_2,y_2}\right|\right|^2 \\
\leq &C\left(e^{-\frac{\kappa t}{2}}|y_1 - y_2|^2 + C|x_1 - x_2|^2 + |x_1 - x_2|^{2\gamma}\right).
\end{aligned} \tag{2.75}
$$

The result follows by summing (2.72), (2.73), (2.74), and (2.75) together and using Gronwall's inequality.

□

Proof of Corollary 2.3. Lemma 2.6 implies for any $d > 0$,

$$
E\left|\left|\nabla_y Y_t^{x,y+d} - \nabla_y Y_t^{x,y}\right|\right| \leq Ce^{-\frac{\kappa t}{4}}|d|.
$$

Then the result follows immediately by taking d to 0. □

Proof of Lemma 2.7. Consider

$$\begin{aligned}
d\nabla_y\nabla_x Y_t^{x,y} &= \big(\nabla_y\nabla_x g\left(x, Y_t^{x,y}\right)\cdot\nabla_y Y_t^{x,y} + \nabla_y^2 g\left(x, Y_t^{x,y}\right)\langle\nabla_y Y_t^{x,y}, \nabla_x Y_t^{x,y}\rangle \\
&\quad +\nabla_y g\left(x, Y_t^{x,y}\right)\cdot\nabla_y\nabla_x Y_t^{x,y}\big)\, dt \\
&\quad + \big(\nabla_y\nabla_x b\left(x, Y_t^{x,y}\right)\cdot\nabla_y Y_t^{x,y} + \nabla_y^2 b\left(x, Y_t^{x,y}\right)\langle\nabla_y Y_t^{x,y}, \nabla_x Y_t^{x,y}\rangle \\
&\quad +\nabla_y b\left(x, Y_t^{x,y}\right)\cdot\nabla_y\nabla_x Y_t^{x,y}\big)\, dL_{t,2}, \\
\nabla_y\nabla_x Y_0^{x,y} &= 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\nabla_y\nabla_x Y_t^{x_1,y} - \nabla_y\nabla_x Y_t^{x_2,y} \\
&= \int_0^t \nabla_y\nabla_x g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y Y_s^{x_1,y} - \nabla_y\nabla_x g\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y Y_s^{x_2,y} ds \\
&\quad + \int_0^t \nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\langle\nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\rangle - \nabla_y^2 g\left(x_2, Y_s^{x_2,y}\right)\langle\nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\rangle ds \\
&\quad + \int_0^t \nabla_y g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_1,y} - \nabla_y g\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y} ds \\
&\quad + \int_0^t \nabla_y\nabla_x b\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y Y_s^{x_1,y} - \nabla_y\nabla_x b\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y Y_s^{x_2,y} dL_{s,2} \\
&\quad + \int_0^t \nabla_y^2 b\left(x_1, Y_s^{x_1,y}\right)\langle\nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\rangle - \nabla_y^2 b\left(x_2, Y_s^{x_2,y}\right)\langle\nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\rangle dL_{s,2} \\
&\quad + \int_0^t \nabla_y b\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_1,y} - \nabla_y b\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y} dL_{s,2} \\
&= \int_0^t \big[\nabla_y\nabla_x g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y Y_s^{x_1,y} - \nabla_y\nabla_x g\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y Y_s^{x_2,y} \\
&\qquad + \nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\langle\nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\rangle - \nabla_y^2 g\left(x_2, Y_s^{x_2,y}\right)\langle\nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\rangle \\
&\qquad + \nabla_y g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_1,y} - \nabla_y g\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y}\big]\, ds \\
&\quad + \int_0^t \Big[\int_{|z|\le 1} \big(\nabla_y\nabla_x b\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y Y_s^{x_1,y} - \nabla_y\nabla_x b\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y Y_s^{x_2,y} \\
&\qquad + \nabla_y^2 b\left(x_1, Y_s^{x_1,y}\right)\langle\nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\rangle - \nabla_y^2 b\left(x_2, Y_s^{x_2,y}\right)\langle\nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\rangle \\
&\qquad + \nabla_y b\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_1,y} - \nabla_y b\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y}\big)\, z\Big]\, \widetilde{N}_2(dz, ds) \\
&\quad + \int_0^t \Big[\int_{|z|>1} \big(\nabla_y\nabla_x b\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y Y_s^{x_1,y} - \nabla_y\nabla_x b\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y Y_s^{x_2,y} \\
&\qquad + \nabla_y^2 b\left(x_1, Y_s^{x_1,y}\right)\langle\nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\rangle - \nabla_y^2 b\left(x_2, Y_s^{x_2,y}\right)\langle\nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\rangle \\
&\qquad + \nabla_y b\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_1,y} - \nabla_y b\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y}\big)\, z\Big]\, N_2(dz, ds).
\end{aligned}$$

Take expectations,

$$\begin{aligned}
&E\left|\left|\nabla_y\nabla_x Y_t^{x_1,y} - \nabla_y\nabla_x Y_t^{x_2,y}\right|\right| \le \\
&\quad \int_0^t E\left|\left|\nabla_y\nabla_x g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y Y_s^{x_1,y} - \nabla_y\nabla_x g\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y Y_s^{x_2,y}\right|\right| ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t E\left|\left|\nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\right\rangle - \nabla_y^2 g\left(x_2, Y_s^{x_2,y}\right)\left\langle \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| ds \\
&+\int_0^t E\left|\left|\nabla_y g\left(x_1, Y_s^{x_1,y}\right)\cdot \nabla_y \nabla_x Y_s^{x_1,y} - \nabla_y g\left(x_2, Y_s^{x_2,y}\right)\cdot \nabla_y \nabla_x Y_s^{x_2,y}\right|\right| ds \\
&+\int_0^t \int_{|z|>1} E\left|\left|\nabla_y \nabla_x b\left(x_1, Y_s^{x_1,y}\right)\cdot \nabla_y Y_s^{x_1,y} - \nabla_y \nabla_x b\left(x_2, Y_s^{x_2,y}\right)\cdot \nabla_y Y_s^{x_2,y}\right|\right| |z|\nu_2(dz)ds \\
&+\int_0^t \int_{|z|>1} E\left|\left|\nabla_y^2 b\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\right\rangle\right.\right. \\
&\quad \left.\left. -\nabla_y^2 b\left(x_2, Y_s^{x_2,y}\right)\left\langle \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| |z|\nu_2(dz)ds \\
&+\int_0^t \int_{|z|>1} E\left|\left|\nabla_y b\left(x_1, Y_s^{x_1,y}\right)\cdot \nabla_y \nabla_x Y_s^{x_1,y} - \nabla_y b\left(x_2, Y_s^{x_2,y}\right)\cdot \nabla_y \nabla_x Y_s^{x_2,y}\right|\right| |z|\nu_2(dz)ds \\
=:& \sum_{i=1}^{6}\int_0^t G_i ds.
\end{aligned}
$$

To bound each G_i, make use of the previously established Lemmas 2.1, 2.3, 2.6, and Corollaries 2.1 and 2.2 and the fact that $g, b \in C_b^{1+\gamma,2+\gamma}$.

$$
\begin{aligned}
G_1 \le& E\left|\left|\nabla_y \nabla_x g\left(x_1, Y_s^{x_1,y}\right)\cdot \nabla_y Y_s^{x_1,y} - \nabla_y \nabla_x g\left(x_1, Y_s^{x_1,y}\right)\cdot \nabla_y Y_s^{x_2,y}\right|\right| \\
&+ E\left|\left|\nabla_y \nabla_x g\left(x_1, Y_s^{x_1,y}\right)\cdot \nabla_y Y_s^{x_2,y} - \nabla_y \nabla_x g\left(x_1, Y_s^{x_2,y}\right)\cdot \nabla_y Y_s^{x_2,y}\right|\right| \\
&+ E\left|\left|\nabla_y \nabla_x g\left(x_1, Y_s^{x_2,y}\right)\cdot \nabla_y Y_s^{x_2,y} - \nabla_y \nabla_x g\left(x_2, Y_s^{x_2,y}\right)\cdot \nabla_y Y_s^{x_2,y}\right|\right| \\
\le& CE\left|\left|\nabla_y Y_s^{x_1,y} - \nabla_y Y_s^{x_2,y}\right|\right| + CE\left|\left|\nabla_y \nabla_x g\left(x_1, Y_s^{x_1,y}\right) - \nabla_y \nabla_x g\left(x_1, Y_s^{x_2,y}\right)\right|\right| e^{-\frac{\kappa s}{2}} \\
&+ CE\left|\left|\nabla_y \nabla_x g\left(x_1, Y_s^{x_2,y}\right) - \nabla_y \nabla_x g\left(x_2, Y_s^{x_2,y}\right)\right|\right| e^{-\frac{\kappa s}{2}} \\
\le& C\left(|x_1 - x_2|^\gamma + |x_1 - x_2|\right) + CE\left|Y_s^{x_1,y} - Y_s^{x_2,y}\right|^\gamma e^{-\frac{\kappa s}{2}} + C|x_1 - x_2|^\gamma e^{-\frac{\kappa s}{2}} \\
\le& C\left(|x_1 - x_2|^\gamma + |x_1 - x_2|\right) + C|x_1 - x_2|^\gamma e^{-\frac{\kappa s}{2}}.
\end{aligned} \tag{2.76}
$$

$$
\begin{aligned}
G_2 \le& E\left|\left|\nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y}\right\rangle - \nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| \\
&+ E\left|\left|\nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_2,y}\right\rangle - \nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| \\
&+ E\left|\left|\nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle - \nabla_y^2 g\left(x_1, Y_s^{x_2,y}\right)\left\langle \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| \\
&+ E\left|\left|\nabla_y^2 g\left(x_1, Y_s^{x_2,y}\right)\left\langle \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle - \nabla_y^2 g\left(x_2, Y_s^{x_2,y}\right)\left\langle \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| \\
\le& E\left|\left|\nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_1,y}, \nabla_x Y_s^{x_1,y} - \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| \\
&+ E\left|\left|\nabla_y^2 g\left(x_1, Y_s^{x_1,y}\right)\left\langle \nabla_y Y_s^{x_1,y} - \nabla_y Y_s^{x_2,y}, \nabla_x Y_s^{x_2,y}\right\rangle\right|\right| \\
&+ CE\left[\left|Y_s^{x_1,y} - Y_s^{x_2,y}\right|^\gamma \left|\left|\nabla_y Y_s^{x_2,y}\right|\right| \left|\left|\nabla_x Y_s^{x_2,y}\right|\right|\right] + C|x_1 - x_2|^\gamma E\left[\left|\left|\nabla_y Y_s^{x_2,y}\right|\right| \left|\left|\nabla_x Y_s^{x_2,y}\right|\right|\right] \\
\le& Ce^{-\frac{\kappa s}{2}}\left(|x_1 - x_2|^\gamma + |x_1 - x_2|\right) + Ce^{-\frac{\kappa s}{2}}|x_1 - x_2|^\gamma + Ce^{-\frac{\kappa s}{2}}|x_1 - x_2|^\gamma \\
\le& Ce^{-\frac{\kappa s}{2}}\left(|x_1 - x_2|^\gamma + |x_1 - x_2|\right).
\end{aligned} \tag{2.77}
$$

$$
\begin{aligned}
G_3 \leq & E\left|\left|\nabla_y g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_1,y} - \nabla_y g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y}\right|\right| \\
& + E\left|\left|\nabla_y g\left(x_1, Y_s^{x_1,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y} - \nabla_y g\left(x_1, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y}\right|\right| \\
& + E\left|\left|\nabla_y g\left(x_1, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y} - \nabla_y g\left(x_2, Y_s^{x_2,y}\right)\cdot\nabla_y\nabla_x Y_s^{x_2,y}\right|\right| \\
\leq & 2CE\left|\left|\nabla_y\nabla_x Y_s^{x_2,y}\right|\right| + CE\left[\left|Y_s^{x_1,y} - Y_s^{x_2,y}\right|^\gamma\left|\left|\nabla_y\nabla_x Y_s^{x_2,y}\right|\right|\right] + C|x_1-x_2|^\gamma E\left|\left|\nabla_y\nabla_x Y_s^{x_2,y}\right|\right| \\
\leq & Ce^{-\frac{\kappa s}{4}} + C|x_1-x_2|^\gamma e^{-\frac{\kappa s}{4}}. \qquad (2.78)
\end{aligned}
$$

From (2.76), (2.77), (2.78), and noticing that G_4, G_5, G_6 are identical except they are multiplied by a constant arising from $\int_{|z|>1}|z|\nu_2(dz)$, conclude that

$$
\sup_{y\in\mathbb{R}^{d_2}} E\left|\left|\nabla_y\nabla_x Y_t^{x_1,y} - \nabla_y\nabla_x Y_t^{x_2,y}\right|\right| \leq Ce^{-\frac{\kappa t}{4}}|x_1-x_2|^\gamma\left(1+|x_1-x_2|^{1-\gamma}\right).
$$

□

Proof of Lemma 2.8. From (2.70),

$$
\begin{aligned}
& \left|\left|\nabla_x \widetilde{f}_{t_0}(x_1,y,t) - \nabla_x \widetilde{f}_{t_0}(x_2,y,t)\right|\right| \\
= & \left|\left|\left(\nabla_x\widehat{f}(x_1,y,t) - E\left[\nabla_x\widehat{f}\left(x_1, Y_{t_0}^{x_1,y}, t\right)\right] - E\left[\nabla_y\widehat{f}\left(x_1, Y_{t_0}^{x_1,y}, t\right)\cdot\nabla_x Y_{t_0}^{x_1,y}\right]\right)\right.\right. \\
& \left.\left. - \left(\nabla_x\widehat{f}(x_2,y,t) - E\left[\nabla_x\widehat{f}\left(x_2, Y_{t_0}^{x_2,y}, t\right)\right] - E\left[\nabla_y\widehat{f}\left(x_2, Y_{t_0}^{x_2,y}, t\right)\cdot\nabla_x Y_{t_0}^{x_2,y}\right]\right)\right|\right| \\
\leq & \left|\left|\left(\nabla_x\widehat{f}(x_1,y,t) - E\left[\nabla_x\widehat{f}\left(x_1, Y_{t_0}^{x_1,y}, t\right)\right]\right) - \left(\nabla_x\widehat{f}(x_2,y,t) - E\left[\nabla_x\widehat{f}\left(x_2, Y_{t_0}^{x_1,y}, t\right)\right]\right)\right|\right| \\
& + \left|\left|E\left[\nabla_x\widehat{f}\left(x_2, Y_{t_0}^{x_2,y}, t\right)\right] - E\left[\nabla_x\widehat{f}\left(x_2, Y_{t_0}^{x_1,y}, t\right)\right]\right|\right| \\
& + \left|\left|E\left[\nabla_y\widehat{f}\left(x_1, Y_{t_0}^{x_1,y}, t\right)\cdot\nabla_x Y_{t_0}^{x_1,y}\right] - E\left[\nabla_y\widehat{f}\left(x_2, Y_{t_0}^{x_2,y}, t\right)\cdot\nabla_x Y_{t_0}^{x_2,y}\right]\right|\right| \\
=: & \sum_{i=1}^3 Q_i.
\end{aligned}
$$

Notice $\nabla_x\widehat{f}(x,y,t) = E\left[\nabla_x f\left(x, Y_t^{x,y}\right)\right] + E\left[\nabla_y f\left(x, Y_t^{x,y}\right)\cdot\nabla_x Y_t^{x,y}\right]$. With this, consider

$$
\begin{aligned}
& \left|\left|\nabla_x\widehat{f}\left(x_1,y_1,t\right) - \nabla_x\widehat{f}\left(x_1,y_2,t\right) - \left[\nabla_x\widehat{f}\left(x_2,y_1,t\right) - \nabla_x\widehat{f}\left(x_2,y_2,t\right)\right]\right|\right| \\
= & ||E\left[\nabla_x f\left(x_1, Y_t^{x_1,y_1}\right)\right] + E\left[\nabla_y f\left(x_1, Y_t^{x_1,y_1}\right)\cdot\nabla_x Y_t^{x_1,y_1}\right] \\
& - E\left[\nabla_x f\left(x_1, Y_t^{x_1,y_2}\right)\right] - E\left[\nabla_y f\left(x_1, Y_t^{x_1,y_2}\right)\cdot\nabla_x Y_t^{x_1,y_2}\right] \\
& - E\left[\nabla_x f\left(x_2, Y_t^{x_2,y_1}\right)\right] - E\left[\nabla_y f\left(x_2, Y_t^{x_2,y_1}\right)\cdot\nabla_x Y_t^{x_2,y_1}\right] \\
& + E\left[\nabla_x f\left(x_2, Y_t^{x_2,y_2}\right)\right] + E\left[\nabla_y f\left(x_2, Y_t^{x_2,y_2}\right)\cdot\nabla_x Y_t^{x_2,y_2}\right]|| \\
\leq & ||E\left[\nabla_x f\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_x f\left(x_1, Y_t^{x_1,y_2}\right) - \left(\nabla_x f\left(x_2, Y_t^{x_1,y_1}\right) - \nabla_x f\left(x_2, Y_t^{x_1,y_2}\right)\right)\right]|| \\
& + ||E\left[\nabla_x f\left(x_2, Y_t^{x_1,y_1}\right) - E\left[\nabla_x f\left(x_2, Y_t^{x_2,y_1}\right)\right] - \left(\nabla_x f\left(x_2, Y_t^{x_1,y_2}\right) - E\left[\nabla_x f\left(x_2, Y_t^{x_2,y_2}\right)\right]\right)\right]|| \\
& + ||E\left[\nabla_y f\left(x_1, Y_t^{x_1,y_1}\right)\cdot\nabla_x Y_t^{x_1,y_1} - \nabla_y f\left(x_2, Y_t^{x_2,y_1}\right)\cdot\nabla_x Y_t^{x_2,y_1}\right] \\
& - E\left[\nabla_y f\left(x_1, Y_t^{x_1,y_2}\right)\cdot\nabla_x Y_t^{x_1,y_2} - \nabla_y f\left(x_2, Y_t^{x_2,y_2}\right)\cdot\nabla_x Y_t^{x_2,y_2}\right]|| \\
=: & \sum_{i=1}^3 Q_{1i}.
\end{aligned}
$$

Since $f \in C_b^{2+\gamma,2+\delta}$ and by Lemma 2.1,

$$
\begin{aligned}
Q_{11} \leq & E\left|\left|\int_0^1 \nabla_x \nabla_y f\left(x_1, \xi Y_t^{x_1,y_1} + (1-\xi)\, Y_t^{x_1,y_2}\right) d\xi \cdot \left(Y_t^{x_1,y_1} - Y_t^{x_1,y_2}\right)\right.\right. \\
& \left.\left. - \int_0^1 \nabla_x \nabla_y f\left(x_2, \xi Y_t^{x_1,y_1} + (1-\xi)\, Y_t^{x_1,y_2}\right) d\xi \cdot \left(Y_t^{x_1,y_1} - Y_t^{x_1,y_2}\right)\right|\right| \\
\leq & C|x_1 - x_2|^\gamma E\left|Y_t^{x_1,y_1} - Y_t^{x_1,y_2}\right| \\
\leq & Ce^{-\frac{\kappa t}{2}}|x_1 - x_2|^\gamma |y_1 - y_2|.
\end{aligned} \tag{2.79}
$$

Similarly,

$$
\begin{aligned}
Q_{12} \leq & E\left|\left|\int_0^1 \nabla_x \nabla_y f\left(x_2, \xi Y_t^{x_1,y_1} + (1-\xi)\, Y_t^{x_2,y_1}\right) d\xi \cdot \left(Y_t^{x_1,y_1} - Y_t^{x_2,y_1}\right)\right.\right. \\
& \left.\left. - \int_0^1 \nabla_x \nabla_y f\left(x_2, \xi Y_t^{x_1,y_2} + (1-\xi)\, Y_t^{x_2,y_2}\right) d\xi \cdot \left(Y_t^{x_1,y_2} - Y_t^{x_2,y_2}\right)\right|\right| \\
\leq & E\left|\left|\int_0^1 \nabla_x \nabla_y f\left(x_2, \xi Y_t^{x_1,y_1} + (1-\xi)\, Y_t^{x_2,y_1}\right) d\xi \cdot \left(Y_t^{x_1,y_1} - Y_t^{x_2,y_1}\right)\right.\right. \\
& - \int_0^1 \nabla_x \nabla_y f\left(x_2, \xi Y_t^{x_1,y_2} + (1-\xi)\, Y_t^{x_2,y_2}\right) d\xi \cdot \left(Y_t^{x_1,y_1} - Y_t^{x_2,y_1}\right) \\
& \left.\left. + \int_0^1 \nabla_x \nabla_y f\left(x_2, \xi Y_t^{x_1,y_2} + (1-\xi)\, Y_t^{x_2,y_2}\right) d\xi \cdot \left(Y_t^{x_1,y_1} - Y_t^{x_2,y_1} - Y_t^{x_1,y_2} + Y_t^{x_2,y_2}\right)\right|\right| \\
\leq & \left|\left|\nabla_x \partial_y^2 f\right|\right| E\left[\int_0^1 \left|\xi\left(Y_t^{x_1,y_2} - Y_t^{x_1,y_2}\right) + (1-\xi)\left(Y_t^{x_2,y_1} - Y_t^{x_2,y_2}\right)\right| d\xi \cdot \left|Y_t^{x_1,y_1} - Y_t^{x_2,y_1}\right|\right] \\
& + E\left|Y_t^{x_1,y_1} - Y_t^{x_2,y_1} - Y_t^{x_1,y_2} + Y_t^{x_2,y_2}\right|,
\end{aligned}
$$

and estimating the second term using Taylor's theorem again with Lemma 2.3,

$$
\begin{aligned}
\leq & CE\left[\left(\left|Y_t^{x_1,y_2} - Y_t^{x_1,y_2}\right| + \left|Y_t^{x_2,y_1} - Y_t^{x_2,y_2}\right|\right)\left|Y_t^{x_1,y_1} - Y_t^{x_2,y_1}\right|\right] \\
& + E\left|\int_0^1 \nabla_x Y_t^{\xi x_1 + (1-\xi)x_2, y_1} d\xi \cdot (x_1 - x_2) - \int_0^1 \nabla_x Y_t^{\xi x_1 + (1-\xi)x_2, y_2} d\xi \cdot (x_1 - x_2)\right| \\
\leq & Ce^{-\frac{\kappa t}{2}}|y_1 - y_2||x_1 - x_2| + Ce^{-\frac{\kappa t}{2}}|y_1 - y_2||x_1 - x_2| \\
\leq & Ce^{-\frac{\kappa t}{2}}|y_1 - y_2||x_1 - x_2|.
\end{aligned} \tag{2.80}
$$

It is necessary to further decompose Q_{13},

$$
\begin{aligned}
Q_{13} \leq & E\left|\left|\left(\nabla_y f\left(x_1, Y_t^{x_1,y_1}\right) - \nabla_y f\left(x_2, Y_t^{x_1,y_1}\right)\right) \cdot \nabla_x Y_t^{x_1,y_1}\right.\right. \\
& \left.\left. - \left(\nabla_y f\left(x_1, Y_t^{x_1,y_2}\right) - \nabla_y f\left(x_2, Y_t^{x_1,y_2}\right)\right) \cdot \nabla_x Y_t^{x_1,y_2}\right|\right| \\
& + E\left|\left|\left(\nabla_y f\left(x_2, Y_t^{x_1,y_1}\right) - \nabla_y f\left(x_2, Y_t^{x_2,y_1}\right)\right) \cdot \nabla_x Y_t^{x_1,y_1}\right.\right. \\
& \left.\left. - \left(\nabla_y f\left(x_2, Y_t^{x_1,y_2}\right) - \nabla_y f\left(x_2, Y_t^{x_2,y_2}\right)\right) \cdot \nabla_x Y_t^{x_1,y_2}\right|\right| \\
& + E\left|\left|\nabla_y f\left(x_2, Y_t^{x_2,y_1}\right) \cdot \left(\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_2,y_1}\right)\right.\right. \\
& \left.\left. - \nabla_y f\left(x_2, Y_t^{x_2,y_2}\right) \cdot \left(\nabla_x Y_t^{x_1,y_2} - \nabla_x Y_t^{x_2,y_2}\right)\right|\right|
\end{aligned}
$$

$$=: \sum_{i=1}^{3} Q_{13i}.$$

From Lemma 2.3,

$$\begin{aligned}
Q_{131} \leq & E \Bigg| \Bigg| \int_0^1 \nabla_x \nabla_y f\left(\xi x_1 + (1-\xi)x_2, Y_t^{x_1,y_1}\right) \cdot \langle x_1 - x_2, \nabla_x Y_t^{x_1,y_1} \rangle \\
& \quad - \int_0^1 \nabla_x \nabla_y f\left(\xi x_1 + (1-\xi)x_2, Y_t^{x_1,y_2}\right) \cdot \langle x_1 - x_2, \nabla_x Y_t^{x_1,y_2} \rangle \Bigg| \Bigg| \\
\leq & E \Bigg| \Bigg| \int_0^1 \nabla_x \nabla_y f\left(\xi x_1 + (1-\xi)x_2, Y_t^{x_1,y_1}\right) \cdot \langle x_1 - x_2, \nabla_x Y_t^{x_1,y_1} \rangle \\
& \quad - \int_0^1 \nabla_x \nabla_y f\left(\xi x_1 + (1-\xi)x_2, Y_t^{x_1,y_2}\right) \cdot \langle x_1 - x_2, \nabla_x Y_t^{x_1,y_1} \rangle \\
& \quad + \int_0^1 \nabla_x \nabla_y f\left(\xi x_1 + (1-\xi)x_2, Y_t^{x_1,y_2}\right) \cdot \langle x_1 - x_2, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_1,y_2} \rangle \Bigg| \Bigg| \\
\leq & \left| \left| \nabla_x \nabla_y^2 f \right| \right| |x_1 - x_2| E\left[|Y_t^{x_1,y_1} - Y_t^{x_1,y_2}| \right] ||\nabla_x Y_t^{x_1,y_1}|| \\
& + C|x_1 - x_2| E\, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_1,y_2}|| \\
\leq & C|x_1 - x_2| e^{-\frac{\kappa t}{2}} |y_1 - y_2| + C e^{-\frac{\kappa t}{2}} |y_1 - y_2||x_1 - x_2| \\
\leq & C e^{-\frac{\kappa t}{2}} |x_1 - x_2||y_1 - y_2|.
\end{aligned} \tag{2.81}$$

$$\begin{aligned}
Q_{132} \leq & E \Bigg| \Bigg| \int_0^1 \nabla_y^2 f\left(x_2, \xi Y_t^{x_1,y_1} + (1-\xi) Y_t^{x_2,y_1}\right) d\xi \cdot \langle Y_t^{x_1,y_1} - Y_t^{x_2,y_1}, \nabla_x Y_t^{x_1,y_1} \rangle \\
& - \int_0^1 \nabla_y^2 f\left(x_2, \xi Y_t^{x_1,y_2} + (1-\xi) Y_t^{x_2,y_2}\right) d\xi \cdot \langle Y_t^{x_1,y_2} - Y_t^{x_2,y_2}, \nabla_x Y_t^{x_1,y_2} \rangle \Bigg| \Bigg| \\
\leq & E \Bigg| \Bigg| \int_0^1 \nabla_y^2 f\left(x_2, \xi Y_t^{x_1,y_1} + (1-\xi) Y_t^{x_2,y_1}\right) d\xi \cdot \langle Y_t^{x_1,y_1} - Y_t^{x_2,y_1}, \nabla_x Y_t^{x_1,y_1} \rangle \\
& - \int_0^1 \nabla_y^2 f\left(x_2, \xi Y_t^{x_1,y_2} + (1-\xi) Y_t^{x_2,y_2}\right) d\xi \cdot \langle Y_t^{x_1,y_1} - Y_t^{x_2,y_1}, \nabla_x Y_t^{x_1,y_1} \rangle \\
& + \int_0^1 \nabla_y^2 f\left(x_2, \xi Y_t^{x_1,y_2} + (1-\xi) Y_t^{x_2,y_2}\right) d\xi \cdot \langle Y_t^{x_1,y_1} - Y_t^{x_2,y_1}, \nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_1,y_2} \rangle \\
& + \int_0^1 \nabla_y^2 f\left(x_2, \xi Y_t^{x_1,y_2} + (1-\xi) Y_t^{x_2,y_2}\right) d\xi \cdot \\
& \quad \cdot \langle Y_t^{x_1,y_1} - Y_t^{x_2,y_1} - Y_t^{x_1,y_2} + Y_t^{x_2,y_2}, \nabla_x Y_t^{x_1,y_2} \rangle || \\
\leq & CE \left(|Y_t^{x_1,y_1} - Y_t^{x_1,y_2}|^\delta + |Y_t^{x_2,y_1} - Y_t^{x_2,y_2}|^\delta \right) |Y_t^{x_1,y_1} - Y_t^{x_2,y_1}| \, ||\nabla_x Y_t^{x_1,y_1}|| \\
& + CE\left[|Y_t^{x_1,y_1} - Y_t^{x_2,y_1}| \, ||\nabla_x Y_t^{x_1,y_1} - \nabla_x Y_t^{x_1,y_2}|| \right] \\
& + CE\left[|Y_t^{x_1,y_1} - Y_t^{x_2,y_1} - Y_t^{x_1,y_2} + Y_t^{x_2,y_2}| \, |\nabla_x Y_t^{x_1,y_2}| \right]
\end{aligned}$$

$$
\begin{aligned}
\leq & CE\left(|Y_t^{x_1,y_1}-Y_t^{x_1,y_2}|^\delta+|Y_t^{x_2,y_1}-Y_t^{x_2,y_2}|^\delta\right)|Y_t^{x_1,y_1}-Y_t^{x_2,y_1}|\,||\nabla_x Y_t^{x_1,y_1}|| \\
&+CE\left[|Y_t^{x_1,y_1}-Y_t^{x_2,y_1}|\,||\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_1,y_2}||\right] \\
&+C\sup_{x\in\mathbb{R}^{d_1}}E\left[|\nabla_x Y_t^{x,y_1}-\nabla_x Y_t^{x,y_2}|\,|x_1-x_2|\,|\nabla_x Y_t^{x_1,y_2}|\right] \\
\leq & Ce^{-\frac{\kappa\delta t}{2}}|y_1-y_2|^\delta|x_1-x_2|+C|x_1-x_2|e^{-\frac{\kappa t}{2}}|y_1-y_2|+Ce^{-\frac{\kappa t}{2}}|y_1-y_2||x_1-x_2| \\
\leq & Ce^{-\frac{\kappa\delta t}{2}}|x_1-x_2||y_1-y_2|^\delta\left(1+|y_1-y_2|^{1-\delta}\right). \qquad (2.82)
\end{aligned}
$$

Finally, apply Lemma 2.7 to get

$$
\begin{aligned}
Q_{133}\leq & E\,||(\nabla_y f(x_2,Y_t^{x_2,y_1})-\nabla_y f(x_2,Y_t^{x_2,y_2}))\cdot(\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_1})|| \\
&+E\,||\nabla_y f(x_2,Y_t^{x_2,y_2})\cdot(\nabla_x Y_t^{x_1,y_1}-\nabla_x Y_t^{x_2,y_1}-\nabla_x Y_t^{x_1,y_2}-\nabla_x Y_t^{x_2,y_2})|| \\
\leq & C\left|\left|\nabla_y^2 f\right|\right|E\,||Y_t^{x_2,y_1}-Y_t^{x_2,y_2}||\left(|x_1-x_2|^\gamma+|x_1-x_2|\right) \\
&+C\sup_{y\in\mathbb{R}^{d_2}}E\,||\nabla_y\nabla_x Y_t^{x_1,y}-\nabla_y\nabla_x Y_t^{x_2,y}||\,|y_1-y_2| \\
\leq & Ce^{-\frac{\kappa t}{2}}|y_1-y_2|\left(|x_1-x_2|^\gamma+|x_1-x_2|\right)+Ce^{-\frac{\kappa t}{4}}|x_1-x_2|^\gamma\left(1+|x_1-x_2|^{1-\gamma}\right)|y_1-y_2| \\
\leq & Ce^{-\frac{\kappa t}{4}}|x_1-x_2|^\gamma\left(1+|x_1-x_2|^{1-\gamma}\right)|y_1-y_2|. \qquad (2.83)
\end{aligned}
$$

Add up (2.81), (2.82), and (2.83) to conclude that

$$
Q_{13}\leq Ce^{-\frac{(\kappa\wedge 2\delta)t}{4}}|x_1-x_2|^\gamma|y_1-y_2|^\delta\left(1+|y_1-y_2|^{1-\delta}\right)\left(1+|x_1-x_2|^{1-\gamma}\right). \qquad (2.84)
$$

And (2.79), (2.80), and (2.84) gives

$$
\begin{aligned}
Q_{11}+Q_{12}+Q_{13}\leq & Ce^{-\frac{\kappa t}{2}}|y_1-y_2||x_1-x_2|+Ce^{-\frac{\kappa t}{2}}|y_1-y_2||x_1-x_2| \\
&+Ce^{-\frac{(\kappa\wedge 2\delta)t}{4}}|x_1-x_2|^\gamma|y_1-y_2|^\delta\left(1+|y_1-y_2|^{1-\delta}\right)\left(1+|x_1-x_2|^{1-\gamma}\right) \\
\leq & Ce^{-\frac{(\kappa\wedge 2\delta)t}{4}}|x_1-x_2|^\gamma|y_1-y_2|^\delta\left(1+|y_1-y_2|^{1-\delta}\right)\left(1+|x_1-x_2|^{1-\gamma}\right).
\end{aligned}
$$

Therefore, by equation (2.39)

$$
\begin{aligned}
Q_1\leq & Ce^{-\frac{(\kappa\wedge 2\delta)t}{4}}|x_1-x_2|^\gamma|y-Y_{t_0}^{x_1,y}|^\delta\left(1+|y-Y_{t_0}^{x_1,y}|^{1-\delta}\right)\left(1+|x_1-x_2|^{1-\gamma}\right) \\
\leq & Ce^{-\frac{(\kappa\wedge 2\delta)t}{4}}|x_1-x_2|^\gamma\left(1+|x_1-x_2|^{1-\gamma}\right)(1+|y|). \qquad (2.85)
\end{aligned}
$$

From Corollaries 2.1, 2.2, and $f\in C_b^{2+\gamma,2+\delta}$ then

$$
\begin{aligned}
\nabla_y\nabla_x\widehat{f}(x,y,t)= & \nabla_y\left(E\left[\nabla_x f(x,Y_t^{x,y})\right]+E\left[\nabla_y f(x,Y_t^{x,y})\cdot\nabla_x Y_t^{x,y}\right]\right) \\
= & E\left[\nabla_x\nabla_y f(x,Y_t^{x,y})\cdot\nabla_y Y_t^{x,y}\right]+E\left[\nabla_y^2 f(x,Y_t^{x,y})\langle\nabla_x Y_t^{x,y},\nabla_y Y_t^{x,y}\rangle\right] \\
&+E\left[\nabla_y f(x,Y_t^{x,y})\cdot\nabla_x\nabla_y Y_t^{x,y}\right] \\
\leq & Ce^{-\frac{\kappa t}{4}}.
\end{aligned}
$$

Therefore,

$$Q_2 \leq \left|\left|\nabla_y \nabla_x \widehat{f}\right|\right| E \left|Y_{t_0}^{x_2,y} - Y_{t_0}^{x_1,y}\right| \leq Ce^{-\frac{\kappa t}{4}}|x_1 - x_2|. \tag{2.86}$$

Similarly, but with the addition of Corollary 2.3,

$$\nabla_y \widehat{f}(x,y,t) = E\left[\nabla_y f\left(x, Y_t^{x,y}\right) \cdot \nabla_y Y_t^{x,y}\right] \leq Ce^{-\frac{\kappa t}{2}}$$

and

$$\nabla_y^2 \widehat{f}(x,y,t) = E\left[\nabla_y^2 f\left(x, Y_t^{x,y}\right)\left\langle \nabla_y Y_t^{x,y}, \nabla_y Y_t^{x,y}\right\rangle\right] + E\left[\nabla_y f\left(x, Y_t^{x,y}\right) \cdot \nabla_y^2 Y_t^{x,y}\right] \leq \ Ce^{-\frac{\kappa t}{4}}.$$

Therefore,

$$\begin{aligned}
Q_3 \leq & \left|\left|E\left[\nabla_y \widehat{f}\left(x_1, Y_{t_0}^{x_1,y}, t\right) \cdot \nabla_x Y_{t_0}^{x_1,y}\right] - E\left[\nabla_y \widehat{f}\left(x_1, Y_{t_0}^{x_1,y}, t\right) \cdot \nabla_x Y_{t_0}^{x_2,y}\right]\right|\right| \\
& + \left|\left|E\left[\nabla_y \widehat{f}\left(x_1, Y_{t_0}^{x_1,y}, t\right) \cdot \nabla_x Y_{t_0}^{x_2,y}\right] - E\left[\nabla_y \widehat{f}\left(x_1, Y_{t_0}^{x_2,y}, t\right) \cdot \nabla_x Y_{t_0}^{x_2,y}\right]\right|\right| \\
& + \left|\left|E\left[\nabla_y \widehat{f}\left(x_1, Y_{t_0}^{x_2,y}, t\right) \cdot \nabla_x Y_{t_0}^{x_2,y}\right] - E\left[\nabla_y \widehat{f}\left(x_2, Y_{t_0}^{x_2,y}, t\right) \cdot \nabla_x Y_{t_0}^{x_2,y}\right]\right|\right| \\
\leq & \left|\left|\nabla_y \widehat{f}\right|\right| \left|\left|\nabla_x Y_{t_0}^{x_1,y} - \nabla_x Y_{t_0}^{x_2,y}\right|\right| + \left|\left|\nabla_y^2 \widehat{f}\right|\right| \left|\left|Y_{t_0}^{x_1,y} - Y_{t_0}^{x_2,y}\right|\right| \left|\left|\nabla_x Y_{t_0}^{x_2,y}\right|\right| \\
& + \left|\left|\nabla_x \nabla_y \widehat{f}\right|\right| |x_1 - x_2| \left|\left|\nabla_x Y_{t_0}^{x_2,y}\right|\right| \\
\leq & Ce^{-\frac{\kappa t}{2}}\left(|x_1 - x_2|^\gamma + |x_1 - x_2|\right) + Ce^{-\frac{\kappa t}{4}}|x_1 - x_2| + Ce^{-\frac{\kappa t}{4}}|x_1 - x_2| \\
\leq & Ce^{-\frac{\kappa t}{4}}\left(|x_1 - x_2|^\gamma + |x_1 - x_2|\right).
\end{aligned} \tag{2.87}$$

(2.85), (2.86), and (2.87) together imply the result.

□

Averaging Lemmas

Proof of Lemma 2.9. Notice that the multiplicative noise component of the slow process is bounded:

$$\begin{aligned}
& \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \epsilon^\rho c\left(X_s^\epsilon, Y_s^\epsilon\right) dL_{s,1}\right|^p\right] \\
= & \epsilon^{\rho p}\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_{|z|\leq 1} c\left(X_s^\epsilon, Y_s^\epsilon\right) z\widetilde{N}_1(ds,dz) + \int_0^t \int_{|z|>1} c\left(X_s^\epsilon, Y_s^\epsilon\right) zN_1(ds,dz)\right|^p\right] \\
\leq & \epsilon^{\rho p}\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_{|z|\leq 1} c\left(X_s^\epsilon, Y_s^\epsilon\right) z\widetilde{N}_1(ds,dz)\right|^p\right] + T^p \int_{|z|>1} \mathbb{E}\left[\left|c\left(X_s^\epsilon, Y_s^\epsilon\right)\right|^p\right] |z|^p \nu_1(dz) \\
& + \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_{|z|>1} c\left(X_s^\epsilon, Y_s^\epsilon\right) zN_1(ds,dz)\right|^p\right],
\end{aligned}$$

which by the Burkholder-Davis-Gundy inequality gives,

$$
\begin{aligned}
\leq & \epsilon^{\rho p} C_p ||c||^p \mathbb{E}\left[\left|\int_0^T \int_{|z|\leq 1} |z|^2 N_1(ds,dz)\right|^{\frac{p}{2}}\right] + ||c||^p T^p \int_{|z|>1} |z|^p \nu_1(dz) ds \\
& + \left[\int_0^T \int_{|z|>1} ||c|| |z| \nu_1(dz) ds\right]^p \\
\leq & \epsilon^{\rho p} C_p ||c||^p T^{\frac{p}{2}} \left[\int_{|z|\leq 1} |z|^2 \nu_1(dz)\right]^{\frac{p}{2}} + ||c||^p T^p \int_{|z|>1} |z|^p \nu_1(dz) ds + ||c||^p T^p \left[\int_{|z|>1} |z| \nu_1(dz)\right]^p \\
\leq & C_p T^p \epsilon^{\rho p}. \qquad (2.88)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
|f(X_s^\epsilon, Y_s^\epsilon)| \leq & |f(X_s^\epsilon, Y_s^\epsilon) - f(X_s^\epsilon, 0)| + |f(X_s^\epsilon, 0) - f(0,0)| + |f(0,0)| \\
\leq & ||\nabla_y f|| \, |Y_s^\epsilon| + ||\nabla_x f|| \, |X_s^\epsilon| + |f| \leq C(1 + |X_s^\epsilon| + |Y_s^\epsilon|).
\end{aligned}
$$

Use this and (2.88) to see that

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^\epsilon|^p\right] \leq C_p T^p (1+|x|^p) + C_p \int_0^T \mathbb{E}\left[\sup_{r\in[0,s]} |X_r^\epsilon|^p\right] ds + C_p \int_0^T \mathbb{E}\,|Y_s^\epsilon|^p \, ds. \qquad (2.89)
$$

Then by Gronwall's inequality,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^\epsilon|^p\right] \leq & \left(C_p T^p (1+|x|^p) + C_p \int_0^T \mathbb{E}\,|Y_s^\epsilon|^p \, ds\right) e^{C_p T} \\
\leq & C_p T^p e^{C_p T} \left(1 + |x|^p + \sup_{\epsilon\in(0,1)} \sup_{t\geq 0} \mathbb{E}\,|Y_t^\epsilon|^p\right). \qquad (2.90)
\end{aligned}
$$

An estimate of (2.38) is necessary to complete (2.37). Define the scalar valued function $U : \mathbb{R}^{d_2} \to \mathbb{R}$ as

$$
U(y) := \left(|y|^2 + 1\right)^{\frac{p}{2}}.
$$

Consider the vector of first-order partial derivatives $\nabla = [\partial_{y_1}, .., \partial_{y_{d_2}}]^T$ and to corresponding matrix of second-order partial derivatives $\nabla^2 = \begin{bmatrix} \partial^2_{y_1,y_1} ... \partial^2_{y_1,y_{d_2}} \\ ... \\ \partial^2_{y_{d_2},y_1} ... \partial^2_{y_{d_2},y_{d_2}} \end{bmatrix}$. A quick calculation gives for any $y \in \mathbb{R}^{d_2}$ (see [SXX22]),

$$
|\nabla U(y)| = \left|\frac{py}{\left(|y|^2+1\right)^{1-\frac{p}{2}}}\right| \leq C_p \, |y|^{p-1} \qquad (2.91)
$$

and

$$
||\nabla^2 U(y)|| = \left|\left|\frac{p I_{d_2\times d_2}}{\left(|y|^2+1\right)^{1-\frac{p}{2}}} - \frac{p(p-2)\, y\otimes y}{\left(|y|^2+1\right)^{2-\frac{p}{2}}}\right|\right| \leq \frac{C_p}{(|y|^2+1)^{1-\frac{p}{2}}} \leq C_p. \qquad (2.92)
$$

Apply Itô's formula with small jumps of size $\epsilon^{\frac{1}{\alpha}}$,

$$
\begin{aligned}
&U\left(Y_t^\epsilon\right)-U(y)\\
=&\int_0^t \langle \epsilon^{-1} g(X_s^\epsilon, Y_s^\epsilon), \nabla U(Y_s^\epsilon)\rangle ds+\int_0^t \int_{|z|\geq \epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right) N_2(ds, dz)\\
&+\int_0^t \int_{|z|<\epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right) \widetilde{N}_2(ds, dz)\\
&+\int_0^t \int_{|z|<\epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right)-\langle \epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z, \nabla U\left(Y_s^\epsilon\right)\rangle \nu_2(dz) ds\\
=&\int_0^t \epsilon^{-1}\langle g(X_s^\epsilon, Y_s^\epsilon), \nabla U(Y_s^\epsilon)\rangle ds+\int_0^t \int_{|z|\geq \epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right) \widetilde{N}_2(ds, dz)\\
&+\int_0^t \int_{|z|<\epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right) \widetilde{N}_2(ds, dz)\\
&+\int_0^t \int_{|z|<\epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right)-\langle \epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z, \nabla U\left(Y_s^\epsilon\right)\rangle \nu_2(dz) ds\\
&+\int_0^t \int_{|z|\geq \epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right) \nu_2(dz) ds.
\end{aligned}
$$

Take expectations on both sides and make the substitution $r=\epsilon^{-\frac{1}{\alpha}} z$,

$$
\begin{aligned}
&\mathbb{E}\left[U\left(Y_t^\epsilon\right)-U(y)\right]=\mathbb{E}\left[\int_0^t \epsilon^{-1}\langle g(X_s^\epsilon, Y_s^\epsilon), \nabla U(Y_s^\epsilon)\rangle ds\right]\\
&\quad+\mathbb{E}\left[\int_0^t \int_{|z|<\epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right)-\langle \epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z, \nabla U\left(Y_s^\epsilon\right)\rangle \nu_2(dz) ds\right]\\
&\quad+\mathbb{E}\left[\int_0^t \int_{|z|\geq \epsilon^{\frac{1}{\alpha}}} U\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}} b(X_s^\epsilon, Y_s^\epsilon) z\right)-U\left(Y_s^\epsilon\right) \nu_2(dz) ds\right]\\
=&\epsilon^{-1}\mathbb{E}\left[\int_0^t \langle g(X_s^\epsilon, Y_s^\epsilon), \nabla U(Y_s^\epsilon)\rangle ds\right]\\
&\quad+\epsilon^{-1}\mathbb{E}\left[\int_0^t \int_{|r|<1} U\left(Y_s^\epsilon+b(X_s^\epsilon, Y_s^\epsilon) r\right)-U\left(Y_s^\epsilon\right)-\langle b(X_s^\epsilon, Y_s^\epsilon) r, \nabla U\left(Y_s^\epsilon\right)\rangle \nu_2(dr) ds\right]\\
&\quad+\epsilon^{-1}\mathbb{E}\left[\int_0^t \int_{|r|\geq 1} U\left(Y_s^\epsilon+b(X_s^\epsilon, Y_s^\epsilon) r\right)-U\left(Y_s^\epsilon\right) \nu_2(dr) ds\right].
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{d\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]}{dt}=&\epsilon^{-1}\mathbb{E}\left[\langle g(X_t^\epsilon, Y_t^\epsilon), \nabla U(Y_t^\epsilon)\rangle\right]\\
&+\epsilon^{-1}\mathbb{E}\left[\int_{|r|<1} U\left(Y_s^\epsilon+b(X_s^\epsilon, Y_s^\epsilon) r\right)-U\left(Y_s^\epsilon\right)-\langle b(X_s^\epsilon, Y_s^\epsilon) r, \nabla U\left(Y_s^\epsilon\right)\rangle \nu_2(dr)\right]\\
&+\epsilon^{-1}\mathbb{E}\left[\int_{|r|\geq 1} U\left(Y_s^\epsilon+b(X_s^\epsilon, Y_s^\epsilon) r\right)-U\left(Y_s^\epsilon\right) \nu_2(dr)\right]
\end{aligned}
$$

$$=: \sum_{i=1}^{3} A_i.$$

Condition (A2) and (2.91) imply

$$\begin{aligned}
&\langle g(X_t^\epsilon, Y_t^\epsilon), \nabla U(Y_t^\epsilon)\rangle \\
=&\frac{1}{\left(|Y_t^\epsilon|^2+1\right)^{1-\frac{p}{2}}}\Bigg[\langle g\left(X_t^\epsilon, Y_t^\epsilon\right) - g\left(X_t^\epsilon, 0\right), pY_t^\epsilon\rangle + \int_{|z|\geq 1}\langle (b\left(X_t^\epsilon, Y_t^\epsilon\right) - b\left(X_t^\epsilon, 0\right)) z, pY_t^\epsilon\rangle \nu_2(dz) \\
&+ p\int_{\mathbb{R}^{d_2}} \left|b\left(X_t^\epsilon, Y_t^\epsilon\right) - b\left(X_t^\epsilon, 0\right)\right|^2 z^2 \nu_2(dz) + \langle g\left(X_t^\epsilon, 0\right), pY_t^\epsilon\rangle \\
&- \int_{|z|\geq 1}\langle (b\left(X_t^\epsilon, Y_t^\epsilon\right) - b\left(X_t^\epsilon, 0\right)) z, pY_t^\epsilon\rangle \nu_2(dz) \\
&-p\int_{\mathbb{R}^{d_2}} \left|b\left(X_t^\epsilon, Y_t^\epsilon\right) - b\left(X_t^\epsilon, 0\right)\right|^2 z^2 \nu_2(dz) + \langle g\left(X_t^\epsilon, 0\right), pY_t^\epsilon\rangle\Bigg] \\
\leq&\frac{-\kappa p\left|Y_t^\epsilon\right|^2 + 2||b||p\left|Y_t^\epsilon\right|\int_{|z|\geq 1} z\nu_2(dz) + p||b||^2\int_{\mathbb{R}^{d_2}} z^2\nu_2(dz) + |g|p\left|Y_t^\epsilon\right|}{\left(|Y_t^\epsilon|^2+1\right)^{1-\frac{p}{2}}} \\
\leq&\frac{-\kappa p\left|Y_t^\epsilon\right|^2 + C_p\left(1+\left|Y_t^\epsilon\right|\right)}{\left(|Y_t^\epsilon|^2+1\right)^{1-\frac{p}{2}}} \leq -\eta\left(|Y_t^\epsilon|^2+1\right)^{\frac{p}{2}} + C_p
\end{aligned}$$

for some $\eta > 0$. This implies the estimate

$$A_1 \leq -\frac{\eta\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]}{\epsilon} + \frac{C_p}{\epsilon}. \tag{2.93}$$

By (2.92),

$$\begin{aligned}
A_2 \leq&\epsilon^{-1}\mathbb{E}\left[\int_{|r|<1}\int_0^1 (1-\xi)\nabla^2 U\left(Y_s^\epsilon + \xi b(X_s^\epsilon, Y_s^\epsilon)r\right) d\xi \left|b(X_s^\epsilon, Y_s^\epsilon)r\right|^2 \nu_2(dr)\right] \\
\leq&\epsilon^{-1}C_p||b||^2\mathbb{E}\left[\int_{|r|<1}|r|^2\nu_2(dr)\right] \leq \frac{C_p}{\epsilon}.
\end{aligned} \tag{2.94}$$

Similarly, but using (2.91) instead,

$$\begin{aligned}
A_3 \leq&\epsilon^{-1}\mathbb{E}\left[\int_{|r|\geq 1}\int_0^1 \left|\nabla U\left(Y_s^\epsilon + \xi b(X_s^\epsilon, Y_s^\epsilon)r\right)\right| d\xi |b(X_s^\epsilon, Y_s^\epsilon)r|\nu_2(dr)\right] \\
\leq&\epsilon^{-1}C_p\mathbb{E}\left[\int_{|r|\geq 1}\int_0^1 \left|Y_s^\epsilon + \xi b(X_s^\epsilon, Y_s^\epsilon)r\right|^{p-1} d\xi\nu_2(dr)\right] \\
\leq&\epsilon^{-1}C_p\mathbb{E}\left[\int_{|r|\geq 1}|Y_s^\epsilon|^{p-1} + |r|^{p-1}\nu_2(dr)\right] \leq \frac{\eta\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]}{2\epsilon} + \frac{C_p}{\epsilon}.
\end{aligned} \tag{2.95}$$

Putting (2.93),(2.94), and (2.95) together,

$$\frac{d\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]}{dt} \leq -\frac{\eta\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]}{\epsilon} + \frac{C_p}{\epsilon} + \frac{C_p}{\epsilon} + \frac{\eta\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]}{2\epsilon} + \frac{C_p}{\epsilon}$$

$$\leq -\frac{\eta\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]}{2\epsilon}+\frac{C_p}{\epsilon}.$$

Which implies by Gronwall's inequality,

$$\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]\leq e^{-\frac{\eta t}{2\epsilon}}\left(U(y)+\frac{C_p t}{\epsilon}\right)\leq C_p\left(1+|y|^p\right).$$

Therefore,

$$\mathbb{E}\left[|Y_t^\epsilon|^p\right]\leq\mathbb{E}\left[\left(|Y_t^\epsilon|^2+1\right)^{\frac{p}{2}}\right]=\mathbb{E}\left[U\left(Y_t^\epsilon\right)\right]\leq C_p\left(1+|y|^p\right),$$

which implies (2.38) and in conjunction with (2.90) gives (2.37). □

Proof of Lemma 2.10. Similarly to Lemma 2.9, define the scalar function $U^\epsilon:\mathbb{R}^{d_2}\to\mathbb{R}$ by

$$U^\epsilon(y):=\left(|y|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{\frac{p}{2}}. \tag{2.96}$$

Deduce the following estimates (see [SXX22]),

$$|\nabla U^\epsilon(y)|=\left|\frac{py}{\left(|y|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{1-\frac{p}{2}}}\right|\leq C_p|y|^{p-1}, \tag{2.97}$$

$$\left|\left|\nabla^2U^\epsilon(y)\right|\right|=\left\|\frac{pI_{d_2}}{\left(|y|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{1-\frac{p}{2}}}-\frac{p(p-2)y\otimes y}{\left(|y|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{2-\frac{p}{2}}}\right\|\leq C_p\epsilon^{-\frac{2}{\alpha}\left(\frac{p}{2}-1\right)}. \tag{2.98}$$

Apply Itô's formula,

$$\begin{aligned}
U^\epsilon(Y_t^\epsilon)-U^\epsilon(y)&=\int_0^t\epsilon^{-1}\langle g\left(X_s^\epsilon,Y_s^\epsilon\right),\nabla U^\epsilon(Y_s^\epsilon)\rangle ds\\
&+\int_0^t\int_{|z|<1}U^\epsilon\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right)-U^\epsilon\left(Y_s^\epsilon\right)\widetilde{N}_2(ds,dz)\\
&+\int_0^t\int_{|z|\geq1}U^\epsilon\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right)-U^\epsilon\left(Y_s^\epsilon\right)N_2(ds,dz)\\
&+\int_0^t\int_{|z|<1}U^\epsilon\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right)-U^\epsilon\left(Y_s^\epsilon\right)-\langle\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z,\nabla U^\epsilon(Y_s^\epsilon)\rangle\nu_2(dz)ds\\
&=:\sum_{i=1}^4 B_i.
\end{aligned} \tag{2.99}$$

By (A2) and Young's inequality, for some $C>0$,

$$
\begin{aligned}
\langle g(x,y),y\rangle =& \langle g(x,y)-g(x,0),y-0\rangle + \int_{|z|\geq 1}\langle (b(x,y)-b(x,0))\,z, y-0\rangle \nu_2(dz)\\
&+\int_{\mathbb{R}^{d_2}} |b(x,y)-b(x,0)|^2 z^2\nu_2(dz) + \langle g(x,0),y\rangle - \int_{|z|\geq 1}\langle (b(x,y)-b(x,0))\,z,y\rangle\nu_2(dz)\\
&-\int_{\mathbb{R}^{d_2}} |b(x,y)-b(x,0)|^2 z^2\nu_2(dz)\\
\leq& -\kappa|y|^2+|g||y|+2||b|||y|\int_{|z|\geq 1} z\nu_2(dz)+||b||^2\int_{\mathbb{R}^{d_2}} z^2\nu_2(dz)\\
\leq& -\kappa|y|^2+\left(\sqrt{\kappa}\frac{C}{\sqrt{\kappa}}\right)|y|+C\leq -\kappa|y|^2+\frac{\kappa}{2}|y|^2+\frac{C^2}{\kappa}+C\leq -\frac{\kappa}{2}|y|^2+C.
\end{aligned}
$$

Derive the following,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \langle g\left(X_s^\epsilon,Y_s^\epsilon\right),\nabla U^\epsilon(Y_s^\epsilon)\rangle ds\right|\right] &\leq \mathbb{E}\left[\int_0^T\left|\frac{p\langle g\left(X_s^\epsilon,Y_s^\epsilon\right),Y_s^\epsilon\rangle}{\left(|Y_s^\epsilon|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{1-\frac{p}{2}}}\right|ds\right]\\
\leq \mathbb{E}\left[\int_0^T\left|\frac{p\left(-\frac{\kappa}{2}\left|Y_s^\epsilon\right|^2+C\right)}{\left(|Y_s^\epsilon|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{1-\frac{p}{2}}}\right|ds\right] &\leq \mathbb{E}\left[\int_0^T\left|\frac{C_p}{\epsilon^{-\frac{2}{\alpha}(1-\frac{p}{2})}}\right|ds\right]\leq C_pT\epsilon^{\frac{2}{\alpha}(1-\frac{p}{2})}.
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\left[\sup_{t\in[0,T]}|B_1|\right]\leq C_pT\epsilon^{-1+\frac{2}{\alpha}\left(1-\frac{p}{2}\right)}\leq C_pT\epsilon^{-\frac{p}{\alpha}}. \tag{2.100}
$$

From the remainder term of Taylor's expansion, $\mathbb{E}\left[\sup_{t\in[0,T]}|B_2|\right]$ is

$$
\leq \mathbb{E}\left[\int_0^T\int_{|z|<1}\int_0^1\left|\nabla U^\epsilon\left(Y_s^\epsilon+\xi\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right)\right|d\xi\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|\widetilde{N}_2(ds,dz)\right],
$$

and by the Burkholder-Davis-Gundy inequality with estimate (2.97),

$$
\begin{aligned}
\leq& C_p\mathbb{E}\left[\int_0^T\int_{|z|<1}\int_0^1\left|Y_s^\epsilon+\xi\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|^{2(p-1)}d\xi\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|^2N_2(ds,dz)\right]^{\frac{1}{2}}\\
\leq& C_p\mathbb{E}\left[\int_0^T\int_{|z|<1}|Y_s^\epsilon|^{2(p-1)}\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|^2\right.\\
&\left.+\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|^{2(p-1)}\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|^2N_2(ds,dz)\right]^{\frac{1}{2}}\\
\leq& \frac{1}{4}\mathbb{E}\left[\left(\sup_{t\in[0,T]}|Y_t^\epsilon|^{p-1}\right)\left(16C_p\int_0^T\int_{|z|<1}\left|\epsilon^{-\frac{1}{\alpha}}z\right|^2N_2(ds,dz)\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

$$+ C_p\mathbb{E}\left[\int_0^T \int_{|z|<1} \left|\epsilon^{-\frac{1}{\alpha}}z\right|^{2p} N_2(ds,dz)\right]^{\frac{1}{2}}.$$

Now apply Young's inequality within the left expectation,

$$\begin{aligned}
\leq&\frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^\epsilon|^p\right] + C_p\left(\int_0^T\int_{|z|<1}\left|\epsilon^{-\frac{1}{\alpha}}z\right|^2\nu_2(dz)ds\right)^{\frac{p}{2}} + C_p\left(\int_0^T\int_{|z|<1}\left|\epsilon^{-\frac{1}{\alpha}}z\right|^{2p}\nu_2(dz)ds\right)^{\frac{1}{2}}\\
\leq&\frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^\epsilon|^p\right] + C_pT^{\frac{p}{2}}\epsilon^{-\frac{p}{\alpha}}. \qquad (2.101)
\end{aligned}$$

In the same way,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|B_3|\right] \leq& C\mathbb{E}\left[\int_0^T\int_{|z|\geq 1}\left|U^\epsilon\left(Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right)-U^\epsilon\left(Y_s^\epsilon\right)\right|\nu_2(dz)ds\right]\\
\leq& C\mathbb{E}\left[\int_0^T\int_{|z|\geq 1}\int_0^1\left|\nabla U^\epsilon\left(Y_s^\epsilon+\xi\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right)\right|d\xi\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|\nu_2(dz)ds\right]\\
\leq& C_p\mathbb{E}\left[\int_0^T\int_{|z|\geq 1}\left|Y_s^\epsilon+\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|^{p-1}\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|\nu_2(dz)ds\right]\\
\leq& C_p\mathbb{E}\left[\int_0^T\int_{|z|\geq 1}|Y_s^\epsilon|^{p-1}\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|\nu_2(dz)ds\right]\\
&+ C_p\mathbb{E}\left[\int_0^T\int_{|z|\geq 1}\left|\epsilon^{-\frac{1}{\alpha}}b\left(X_s^\epsilon,Y_s^\epsilon\right)z\right|^p\nu_2(dz)ds\right]\\
\leq& C_p\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^\epsilon|^{p-1}\int_0^T\int_{|z|\geq 1}\left|\epsilon^{-\frac{1}{\alpha}}z\right|\nu_2(dz)ds\right] + C_p\epsilon^{-\frac{p}{\alpha}}\mathbb{E}\left[\int_0^T\int_{|z|\geq 1}|z|^p\nu_2(dz)ds\right]\\
\leq&\frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^\epsilon|^p\right] + C_p\left(\int_0^T\int_{|z|\geq 1}\left|\epsilon^{-\frac{1}{\alpha}}z\right|\nu_2(dz)ds\right)^p + C_p\epsilon^{-\frac{p}{\alpha}}\int_0^T\int_{|z|\geq 1}|z|^p\nu_2(dz)ds\\
\leq&\frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^\epsilon|^p\right] + C_pT^p\epsilon^{-\frac{p}{\alpha}}. \qquad (2.102)
\end{aligned}$$

For the last term, Taylor's expansion again, but with (2.98) gives

$$\mathbb{E}\left[\sup_{t\in[0,T]}|B_4|\right] \leq \mathbb{E}\left[\int_0^T\int_{|z|<1} C_p\epsilon^{-\frac{2}{\alpha}\left(\frac{p}{2}-1\right)}||b||^2\left|\epsilon^{-\frac{1}{\alpha}}z\right|^2\nu_2(dz)ds\right] \leq C_pT\epsilon^{-\frac{p}{\alpha}}. \qquad (2.103)$$

Summing estimates (2.100),(2.101),(2.102), and (2.103) into (2.99), conclude that

$$|Y_t^\epsilon|^p \leq \left(|Y_t^\epsilon|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{\frac{p}{2}} =: U\left(Y_t^\epsilon\right) \leq \left(|y|^2+\epsilon^{-\frac{2}{\alpha}}\right)^{\frac{p}{2}} + C_pT^p\epsilon^{-\frac{p}{\alpha}} + \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^\epsilon|^p\right],$$

which implies the result. □

Proof of Lemma 2.11. Showing $\overline{f}$ is globally Lipschitz implies that (2.13) has a unique solution (see [App09]). By definition of the invariant measure

$$
\begin{aligned}
\left|\overline{f}(x_1)-\overline{f}(x_2)\right| \leq & \left|\overline{f}(x_1)-E\left[f(x_1,Y_t^{x_1,0})\right]\right|+\left|E\left[f(x_2,Y_t^{x_2,0})\right]-\overline{f}(x_2)\right| \\
& +\left|E\left[f(x_1,Y_t^{x_1,0})\right]-E\left[f(x_2,Y_t^{x_2,0})\right]\right| \\
= & \left|P_t^{x_1}f\left(x_1,Y_t^{x_1,0}\right)-\mu^{x_1}(f)\right|+\left|P_t^{x_2}f\left(x_2,Y_t^{x_2,0}\right)-\mu^{x_2}(f)\right| \\
& +E\left|f(x_1,Y_t^{x_1,0})-f(x_1,Y_t^{x_2,0})\right|+E\left|f(x_1,Y_t^{x_2,0})-f(x_2,Y_t^{x_2,0})\right|.
\end{aligned}
$$

Now apply Lemma 2.1 and Lemma 2.2

$$
\begin{aligned}
\leq & C\left|\left|f\right|\right|_1 e^{-\frac{\kappa t}{2}}(1+\sup_{t\geq 0}|Y_t^{x_1,0}|)+C\left|\left|f\right|\right|_1 e^{-\frac{\kappa t}{2}}(1+\sup_{t\geq 0}|Y_t^{x_2,0}|) \\
& +||\nabla_y f||\left|Y_t^{x_1,0}-Y_t^{x_2,0}\right|+||\nabla_x f|||x_1-x_2|,
\end{aligned}
$$

which by (2.39) with $p=1$ gives

$$
\begin{aligned}
\leq & 2C\left|\left|f\right|\right|_1 e^{-\frac{\kappa t}{2}}(1+C_p(1+|y|))+C|x_1-x_2| \\
\leq & C_p e^{-\frac{\kappa t}{2}}+C|x_1-x_2|.
\end{aligned}
$$

Take $t\to\infty$ to see that $\overline{f}$ is globally Lipschitz. Therefore, there exists a unique solution.

By the above Lipschitz condition,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\overline{X}_t\right|^p\right] = & \mathbb{E}\left[\sup_{t\in[0,T]}\left|x+\int_0^t\overline{f}\left(\overline{X}_s\right)ds\right|^p\right] \\
\leq & C_pT^p(1+|x|^p)+\mathbb{E}\left[\int_0^T\left|\overline{f}\left(\overline{X}_s\right)-\overline{f}\left(0\right)\right|^p+\left|\overline{f}\left(0\right)\right|^p ds\right] \\
\leq & C_pT^p(1+|x|^p)+\int_0^T C_p\mathbb{E}\left[\sup_{r\in[0,s]}\left|\overline{X}_r\right|^p\right]ds+T|\overline{f}|^p.
\end{aligned}
$$

The conclusion follows from Gronwall's inequality,

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\overline{X}_t\right|^p\right]\leq C_pT^p\left(1+|x|^p\right)e^{C_pT}.
$$

$\square$

Chapter 3

On Convergence of Undiscounted Optimistic Policy Iteration without State Update Restrictions

Outline

§3.1 Reinforcement Learning preliminaries are introduced with model assumptions. This leads to the algorithm, a stochastic iteration, we will show converges.

§3.2 Main part of the chapter where the convergence result is established following some important lemmas.

§3.3 A discussion on how various assumptions can be relaxed, some quick corollaries, and how to extend previous known results.

As this is only the beginning of the $\alpha = 1$ case and many more variations seem immediately possible, the dissertation's Conclusion lists possibilities for future work.

Notation

Let $x, y \in \mathbb{R}^n$ be n-dimensional vectors and $\xi \in \mathbb{R}^n$ be an n-dimensional vector of strictly positive components. An inequality between vectors is always understood component-wise. Let S be a set and I the identity matrix.

$\lvert\lvert x\rvert\rvert_\infty = \max\limits_{1\leq i\leq n} \lvert x(i)\rvert$	maximum norm
$\lvert\lvert x\rvert\rvert_\xi = \max\limits_{1\leq i\leq n} \frac{\lvert x(i)\rvert}{\xi(i)}$	weighted maximum norm
$e = (1, 1, ..., 1)$	vector of ones
$\lvert S\rvert$	cardinality of S
$diag(x) = x^T I$	matrix whose diagonal is x

§3.1 Preliminaries

Let S and A be the set of states and actions with finite cardinality $|S|$ and $|A|$, respectively. Consider the Markov decision process (MDP) $M = (S, A, P)$ where $P : S \times A \times S \to [0, 1]$ is the transition probability function. Define a policy $\mu : S \mapsto A$ as the function that specifies which action to take from each state. Since the state-action pair is finite, it is evident that there are only finitely many policies. We denote the set of all policies by Π. Furthermore, we denote the deterministic one-stage cost vector by $g_\mu = (g(1, \mu(1))), ..., g(|S|, \mu(|S|)) \in \mathbb{R}^{|S|}$, where each $g : S \times A \to \mathbb{R}$ is assumed bounded. These costs assign a penalty (or reward, depending on the reader's perspective) to each transition.

Remark 3.1. *As mentioned in Section 6 of [Tsi02], one may carry out the same proof with random rewards under the assumption that the conditional expectation of such rewards is equal to g_μ with bounded conditional variance. This amounts to adding zero-mean noise and poses no analytical difficulties, so as is customary for the convenience of exposition, we keep the noise deterministic.*

Fix $i, j \in S$ and $a \in A$ then $p_{ij}(a)$ is the probability of transitioning from state i to state j under action $a = \mu(i)$,

$$p_{ij}(a) := P(i, a, j) = \mathbb{P}(x_{t+1} = j | x_t = i, a_t = a)$$

where $\{(x_t, a_t)\}_{t \geq 0} \subseteq S \times A$ denotes the evolution of the MDP on a fixed trajectory. Notice that by fixing $a \in A$, $P(\cdot, a, \cdot)$ is a matrix with the above transition probabilities as components. To simplify the notation for any fixed policy, consider the matrix

$$P_\mu := (p_{ij}(\mu(i))) \in \mathbb{R}^{|S| \times |S|}.$$

With policy μ, define the associated cost-to-go vector by

$$J^\mu(i) := \mathbb{E}\left[\sum_{t=0}^{\infty} \alpha^t g(x_t, \mu(x_t)) | x_0 = i\right], \quad \forall i \in S,$$

with a fixed discount factor $\alpha \in (0, 1]$. For $\alpha \in (0, 1)$, J^μ is well defined since g is bounded and we have a simple geometric series, but for the $\alpha = 1$ case this chapter is interested in, we must impose the following assumption to ensure it is well behaved (see pg 17, [BT96]).

Assumption 3.1. *Assume there exists a cost-free terminal state, say state 0,*

$$\sum_{j=1}^{n} p_{ij}(a) \leq 1 \quad \text{and} \quad p_{i0}(a) = 1 - \sum_{j=1}^{n} p_{ij}(a)$$

for all $i \in S$ and $a \in A$. Furthermore, assume state 0 is a trap state, in that once the MDP reaches that state, it remains there at no further cost

$$p_{00}(a) = 1 \quad \text{and} \implies p_{0j}(a) = 0 \quad \text{and} \quad g(0, a) = 0 \quad \forall a \in A, j \in S.$$

It is well known that all discounted problems can be seen as a special case of a stochastic shortest path problem (see pg 39-40, [BT96] and Remark 3.3 below). As such, without loss of generality and for the remainder of this chapter, we will assume $\alpha = 1$ and omit it in the notation.

Define the optimal cost-to-go value function as

$$J^*(i) = \min_{\mu \in \Pi} J^\mu(i).$$

There being finitely many policies ensures the minimum is achieved. The objective is to find an optimal policy $\mu^* \in \Pi$ that attains this minimum.

We introduce the standard Bellman equation and associated dynamic programming operators used for this type of analysis. For any vector $J \in \mathbb{R}^{|S|}$, $\mu \in \Pi$, define the operators $T_\mu : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ and $T : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ by

$$T_\mu J(i) = g(i, \mu(i)) + \sum_{j=1}^{n} p_{ij}(\mu(i)) J(j) \quad \text{and} \quad TJ(i) = \min_{a \in \mathcal{A}} \left(g(i,a) + \sum_{j=1}^{n} p_{ij}(a) J(j) \right).$$

In vector form, the notation for these operators simplifies to

$$T_\mu J = g_\mu + P_\mu J \quad \text{and} \quad TJ = \min_{\mu \in \Pi} T_\mu J,$$

respectively. As is known from Proposition 2.1 [BT96], J^μ is the unique fixed point of T_μ,

$$T_\mu J^\mu = J^\mu, \tag{3.1}$$

and similarly, J^* is the unique fixed point of T,

$$TJ^* = J^*. \tag{3.2}$$

In fact, a stationary policy μ is optimal if and only if $T_\mu J^* = TJ^* = J^*$. When μ is such that

$$T_\mu J = TJ \left(= \min_{\mu \in \Pi} T_\mu J \right),$$

we call μ the greedy policy corresponding to the vector J. Such a policy is always attainable. Therefore, convergence has been achieved when $J \equiv J^*$.

When the operator T_μ is composed with copies of itself m times, we call $T_\mu^m J$ the m-step rollout of the policy μ. Similarly, when we apply the Bellman operator k times to a vector J, we call the resulting operator T^k, or the k-times composition of T, the k-step lookahead corresponding to J. The greedy policy corresponding to $T^k J$ is defined as the policy μ that attains

$$T_\mu T^{k-1} J = T^k J \iff \mu \in \arg\min_{\mu \in \Pi} T_\mu T^{k-1} J.$$

Notice that μ need not be unique and so we interpret the $\arg\min$ as the set of minimal policies. In this case, we are considering the greedy policy with respect to some computed k-th future iteration and not the immediate next step.

In the discounted case where $\alpha \in (0,1)$, we can show that T and T_μ are contraction mappings with respect to the maximum norm. This is no longer the case when $\alpha = 1$. Rather, we make the following assumption (see Assumption 2.1 [BT96]).

Assumption 3.2. *All policies in* Π *are proper.*

A policy is said to be proper if the terminal state is reached almost surely regardless of the initial state (see Definition 2.1 [BT96]). It will be possible to relax this assumption (see Remark 3.4 below), but for the optimistic variants of policy iteration, failing to make this assumption could lead to infinite value arising from an improper policy. The standard treatment is to first prove convergence under the assumption that all policies are proper and then to show that an MDP under a relaxed assumption remains close to an MDP using this assumption.

Since all policies are proper, there exists a $\beta \in [0,1)$ and a positive vector ξ such that (see Proposition 2.2 [BT96])

$$\sum_{j=1}^{|S|} p_{ij}(a)\xi(j) \leq \beta\xi(i), \quad \forall i \in S, a \in A, \tag{3.3}$$

which further implies that T and T_μ are contraction mappings with respect to the weighted maximum norm $||J||_\xi := \max_i \frac{|J(i)|}{\xi(i)}$, for all $J \in \mathbb{R}^{|S|}$. In other words,

$$||T_\mu J_1 - T_\mu J_2||_\xi \leq \beta||J_1 - J_2||_\xi \quad \text{and} \quad ||TJ_1 - TJ_2||_\xi \leq \beta||J_1 - J_2||_\xi. \tag{3.4}$$

Define the matrix $\Xi := \text{diag}(\xi(1), ..., \xi(|S|))$. Then (3.3) can be written in its matrix form

$$P_\mu \Xi e \leq \beta\Xi e, \quad \forall \mu \in \Pi, \tag{3.5}$$

where e is the vector of ones and the inequality is interpreted component-wise. Furthermore, Proposition 2.1 (d) [BT96] also gives us the following identity for the limit of the rollout

$$\lim_{m\to\infty} T_\mu^m J = J^\mu, \quad \forall J \in \mathbb{R}^{|S|}, \mu \in \Pi. \tag{3.6}$$

In order to ensure that the lookahead is sufficiently large to guarantee convergence, m and k must satisfy the following assumption.

Assumption 3.3. *Let* $m, k > 0$ *and let* β *be the contracting factor for the operators* T_μ *and* T. *Assume that*

$$\beta^{k-1} + (1+\beta^m)\frac{\beta^{k-1}}{1-\beta}(1+\beta) < 1.$$

We are now ready to introduce the algorithm whose convergence proof is the main result of this chapter. At each iteration index t, we have a vector J_t which, we will show, iteratively converges to J^*. J_0 can be chosen arbitrarily, but all subsequent iterations will be updated as per the algorithm. Given J_t, we compute the k-step lookahead $T^{k-1}J_t$ and then find the greedy policy with respect to this lookahead (policy iteration step), i.e. let μ_t be the policy that solves

$$T_{\mu_t}T^{k-1}J_t = TT^{k-1}J_t. \tag{3.7}$$

It may be computationally difficult to calculate the lookahead $T^{k-1}J_t$. In practice, Monte Carlo Tree Search methods are employed to make its approximation feasible [Sil+17b]. Then use the policy μ_t to simulate a single trajectory.

Define the set of states visited by trajectory t that are chosen for updating as $\mathcal{D}_t$.

Assumption 3.4. *Let $i \in S$ and let $p_t(i)$ be the probability of picking state i, i.e. the probability that $i \in \mathcal{D}_t$. Assume $\inf_{t\geq 0} p_t(i) > 0$ for all $i \in S$.*

This assumption guarantees that every state is picked for updating infinitely often.

Remark 3.2. *The exploring starts assumption that the initial state is drawn from a fixed distribution for each state in S and used for updating, where the probability of picking any state as the initial state is positive, can now be seen as a special case of Assumption 3.4. If each state is guaranteed to be picked infinitely often for updating from the initial state, then it satisfies the assumption.*

We call $\gamma_t(i)$ the state-dependent step-size parameter that satisfies the following standard assumptions.

Assumption 3.5. *The stepsizes $\gamma_t(i)$ are non-negative and satisfy*

$$\sum_{t=0}^{\infty} \gamma_t(i) = \infty \quad \textit{and} \quad \sum_{t=0}^{\infty} \gamma_t(i)^2 < \infty \quad \forall i \in S.$$

Note that when $i \notin \mathcal{D}_t$, we impose the convention that $\gamma_t(i) = 0$. The step size summing to infinity guarantees the algorithm will be able to escape any initial condition and square summability is necessary to ensure updates are not so slow the algorithm converges to the wrong point or possibly even diverges. The step size is state dependent and this generalization is possible due to the use of rollout to achieve the necessary contractive property for convergence.

Having already calculated the lookahead $T^{k-1}J_t$, we use an unbiased estimator of the lookahead's m-step rollout as an approximation for J^{μ_t} by considering the sum of m costs following each $i \in \mathcal{D}_t$ (policy evaluation step), or simply the estimator for trajectory t defined by

$$\sum_{\ell=\tau_i}^{\tau_i+m-1} g(x_\ell, \mu_t(x_\ell)) + T^{k-1}J_t(x_m), \quad \tau_i = \min\{\ell \in \mathbb{N} : x_\ell = i\},\ \forall i \in \mathcal{D}_t. \tag{3.8}$$

To see that this estimator is unbiased for the lookahead's m-step rollout $T_{\mu_t}^m T^{k-1} J_t$, notice that the noise for each state $i \in S$, say $w_t = (w_t(1), ..., w_t(|S|))$, arising from the difference between the estimator and rollout has zero mean:

$$\begin{aligned}
\mathbb{E}\left[w_t(i)\middle|\mathcal{F}_t\right] =& \mathbb{E}\left[\left(\sum_{\ell=\tau_i}^{\tau_i+m-1} g(x_\ell, \mu_t(x_\ell)) + T^{k-1}J_t(x_m)\right) - T_{\mu_t}^m T^{k-1} J_t(i) \middle| \mathcal{F}_t\right] \\
=& g(i, \mu_t(i)) + \sum_{j_1} p_{ij_1}(\mu_t(i)) g(j_1, \mu_t(j_1)) \\
&+ \sum_{j_2}\sum_{j_1} p_{j_2j_1}(\mu_t(j_1)) p_{ij_1}(\mu_t(i)) g(j_2, \mu_t(j_2)) \\
&+ ... \\
&+ \sum_{j_m} ... \sum_{j_1} p_{j_m j_{m-1}}(\mu_t(j_m))...p_{j_1 i}(\mu_t(j_1)) T^{k-1} J_t(j_m) - T_{\mu_t}^m T^{k-1} J_t(i) = 0,
\end{aligned}$$

where $\mathcal{F}_t = \{J_0(i), ..., J_t(i), w_0(i), ..., w_{t-1}(i), \gamma_0(i), ..., \gamma_t(i), i = 1, ..., |S|\}$ denotes the natural filtration or history of the algorithm up to when the step-sizes are determined, but just before the update direction is determined (pg 138, [BT96]). Furthermore, there are only finitely many states and policies, therefore $\mathbb{E}\left[||w_t||_\infty^2 |\mathcal{F}_t\right] \leq C$ for some $C > 0$.

This estimator is then used in the optimistic value iteration update step as follows:

$$J_{t+1}(i) = \begin{cases} (1 - \gamma_t(i)) J_t(i) + \gamma_t(i)(T_{\mu_t}^m T^{k-1} J_t(i) + w_t(i)) & i \in \mathcal{D}_t \\ J_t(i) & i \notin \mathcal{D}_t \end{cases}. \tag{3.9}$$

Let $\chi_t(i)$ be a random variable taking value 1 if state i is selected for updating on trajectory t and 0 otherwise, i.e.

$$\chi_t(i) = \begin{cases} 1 & i \in \mathcal{D}_t \\ 0 & i \notin \mathcal{D}_t, \end{cases}$$

and let $p_t(i)$ denote the probability that state i is ever selected on trajectory t. We do not need to know p_t a priori considering an application of the value iteration update (3.9) does not require it, however knowing it can lead to the desgin of different step sizes (e.g. Section 5 [Liu21]) which may have different desirable convergence properties. Also, notice $p_t(i) = \mathbb{E}[\chi_t(i)]$. Then,

$$\begin{aligned}
J_{t+1}(i) =& \chi_t(i)\left[(1 - \gamma_t(i)) J_t(i) + \gamma_t(i)(T_{\mu_t}^m T^{k-1} J_t(i) + w_t(i))\right] + (1 - \chi_t(i)) J_t(i) \\
=& J_t(i) - \gamma_t(i) p_t(i) J_t(i) + \gamma_t(i) p_t(i) J_t(i) + \chi_t(i) J_t(i) - \chi_t(i) J_t(i) \\
&+ \gamma_t(i) p_t(i) T_{\mu_t}^m T^{k-1} J_t(i) - \gamma_t(i) p_t(i) T_{\mu_t}^m T^{k-1} J_t(i) \\
&- \chi_t(i)\gamma_t(i) J_t(i) + \chi_t(i)\gamma_t(i) T_{\mu_t}^m T^{k-1} J_t(i) + \chi_t(i)\gamma_t(i) w_t(i) \\
=& (1 - \gamma_t(i) p_t(i)) J_t(i) \\
&+ \gamma_t(i) p_t(i) \left(T_{\mu_t}^m T^{k-1} J_t(i) + w_t(i) + \left(\frac{\chi_t(i)}{p_t(i)} - 1\right)\left(T_{\mu_t}^m T^{k-1} J_t(i) - J_t(i) + w_t(i)\right)\right) \\
=& (1 - \gamma_t(i) p_t(i)) J_t(i) + \gamma_t(i) p_t(i) \left(T_{\mu_t}^m T^{k-1} J_t(i) + v_t(i)\right),
\end{aligned}$$

where

$$v_t(i) := w_t(i) + \left(\frac{\chi_t(i)}{p_t(i)} - 1\right)\left(T_{\mu_t}^m T^{k-1} J_t(i) - J_t(i) + w_t(i)\right).$$

For notational ease, this is equivalent to its vector form

$$J_{t+1} = (I - \Gamma_t P_t) J_t + \Gamma_t P_t \left(T_{\mu_t}^m T^{k-1} J_t + v_t\right), \tag{3.10}$$

where I denotes the identity matrix, Γ_t and P_t are matrices with diagonal entries $\gamma_t(i)$ and $p_t(i)$, respectively, v_t is the vector with components $v_t(i)$, and J_t is as above.

§3.2 Convergence Analysis

In this section we prove the main result. There are three lemmas that build on each other to prove the pseudo-contraction property of the operator $H_t^{m,k} := T_{\mu_t}^m T^{k-1}$ which allows one to apply Proposition 4.5 of [BT96] to conclude. This proof follows [WS23] but circumvents the issue of $\alpha = 1$ by using the contracting factor β instead as suggested in [Liu21].

Theorem 3.1 Algorithm Convergence

The sequence J_t generated by the iteration (3.10) converges to J^* almost surely.

The following three lemmas establish some general properties for the operators and will be useful in proving the theorem. The first lemma bounds the difference between the cost-to-go vector (what is being estimated with simulated trajectories) and the lookahead vector. This is then used in the second lemma to bound the rollout vector and the lookahead vector. Finally, the rollout is shown to be a pseudo-contraction with respect to J^* in Lemma 3.3, satisfying a necessary condition to apply Proposition 4.5 in [BT96].

Lemma 3.1. *Let J^μ be the cost-to-go vector, J_t be the iteration generated by* (3.10) *and let β be the contracting factors of the operators T and T_μ with respect to the weighted maximum norm $||\cdot||_\xi$. Then*

$$||J^{\mu_t} - T^{k-1} J_t||_\xi \leq \frac{\beta^{k-1}}{1-\beta} ||T J_t - J_t||_\xi, \quad \forall t \geq 0.$$

Proof. The greedy update (3.7) implies

$$\begin{aligned}
& T_{\mu_t} T^{k-1} J_t = T^k J_t \\
\implies & T_{\mu_t} T^{k-1} J_t = T^{k-1} J_t + (T^k J_t - T^{k-1} J_t) \\
\implies & T_{\mu_t}^2 T^{k-1} J_t = T_{\mu_t}\left(T^{k-1} J_t + (T^k J_t - T^{k-1} J_t)\right) \\
& \qquad = g_{\mu_t} + P_{\mu_t}\left(T^{k-1} J_t + (T^k J_t - T^{k-1} J_t)\right) \\
& \qquad = T_{\mu_t} T^{k-1} J_t + P_{\mu_t}\left(T^k J_t - T^{k-1} J_t\right) \\
& \qquad = T^{k-1} J_t + \left(T^k J_t - T^{k-1} J_t\right) + P_{\mu_t}\left(T^k J_t - T^{k-1} J_t\right)
\end{aligned}$$

$$\implies T_{\mu_t}^3 T^{k-1} J_t = T^{k-1} J_t + \left(T^k J_t - T^{k-1} J_t\right) + P_{\mu_t}\left(T^k J_t - T^{k-1} J_t\right) + P_{\mu_t}^2\left(T^k J_t - T^{k-1} J_t\right).$$

Proceed inductively to derive the following identity

$$T_{\mu_t}^m T^{k-1} J_t = T^{k-1} J_t + \left(I + P_{\mu_t} + \ldots + P_{\mu_t}^{m-1}\right)\left(T^k J_t - T^{k-1} J_t\right). \tag{3.11}$$

Notice that

$$\begin{aligned} P_{\mu_t}^m \left(T^k J_t - T^{k-1} J_t\right) =& P_{\mu_t}^m \Xi \Xi^{-1}\left(T^k J_t - T^{k-1} J_t\right) \\ \leq& P_{\mu_t}^m \Xi ||\Xi^{-1}\left(T^k J_t - T^{k-1} J_t\right)||_\infty e \\ =& P_{\mu_t}^{m-1} P_{\mu_t} \Xi e ||T^k J_t - T^{k-1} J_t||_\xi, \end{aligned}$$

and from (3.5),

$$\begin{aligned} &\leq P_{\mu_t}^{m-1} \beta \Xi e ||T^k J_t - T^{k-1} J_t||_\xi \\ &\leq \beta^m ||T^k J_t - T^{k-1} J_t||_\xi \Xi e. \end{aligned} \tag{3.12}$$

Plug (3.12) into (3.11)

$$\begin{aligned} T_{\mu_t}^m T^{k-1} J_t \leq& T^{k-1} J_t + (I + \beta + \beta^2 + \ldots + \beta^{m-1}) ||T^k J_t - T^{k-1} J_t||_\xi \Xi e \\ \leq& T^{k-1} J_t + \Xi \frac{||T^k J_t - T^{k-1} J_t||_\xi}{1-\beta} e. \end{aligned}$$

Furthermore, since the right hand side no longer depends on m, we have as a consequence of (3.6) that

$$J^{\mu_t} = \lim_{m\to\infty} T_{\mu_t}^m T^{k-1} J_t \leq T^{k-1} J_t + \Xi \frac{||T^k J_t - T^{k-1} J_t||_\xi}{1-\beta} e. \tag{3.13}$$

Repeatedly apply the weighted contraction property established in (3.4),

$$\begin{aligned} J^{\mu_t} - T^{k-1} J_t \leq& \Xi \frac{||TT^{k-1} J_t - TT^{k-2} J_t||_\xi}{1-\beta} e \\ \leq& \Xi \frac{\beta}{1-\beta} ||T^{k-1} J_t - T^{k-2} J_t||_\xi e \\ \leq& \ldots \\ \leq& \Xi \frac{\beta^{k-1} ||T J_t - J_t||_\xi}{1-\beta} e. \end{aligned}$$

By consequence

$$||\Xi^{-1}\left(J^{\mu_t} - T^{k-1} J_t\right)||_\infty \leq \frac{\beta^{k-1}}{1-\beta} ||T J_t - J_t||_\xi,$$

which implies

$$||J^{\mu_t} - T^{k-1} J_t||_\xi \leq \frac{\beta^{k-1}}{1-\beta} ||T J_t - J_t||_\xi$$

as desired. □

Lemma 3.2. *Let J_t be the iteration generated by* (3.10) *and let β be the contracting factors of the operators T and T_μ with respect to the weighted maximum norm $||\cdot||_\xi$. Then*

$$||T_{\mu_t}^m T^{k-1}J_t - T^{k-1}J_t||_\xi \leq \left(\frac{\beta^{m+k-1}}{1-\beta} + \frac{\beta^{k-1}}{1-\beta}\right)||TJ_t - J_t||_\xi, \quad \forall t \geq 0.$$

Proof. By the reverse triangle inequality

$$\begin{aligned}||T_{\mu_t}^m T^{k-1}J_t - T^{k-1}J_t||_\xi - ||T^{k-1}J_t - J^{\mu_t}||_\xi \leq & ||T_{\mu_t}^m T^{k-1}J_t - J^{\mu_t}||_\xi \\ = & ||T_{\mu_t}T_{\mu_t}^{m-1}T^{k-1}J_t - T_{\mu_t}J^{\mu_t}||_\xi \\ \leq & \beta||T_{\mu_t}^{m-1}T^{k-1}J_t - J^{\mu_t}||_\xi,\end{aligned}$$

which follows from the fixed point property of J^{μ_t} (3.1) and an application of the weighted contraction property (3.4). Simply repeat these steps $m-1$ more times to derive

$$\leq \beta^m||T^{k-1}J_t - J^{\mu_t}||_\xi.$$

This implies

$$\begin{aligned}||T_{\mu_t}^m T^{k-1}J_t - T^{k-1}J_t||_\xi \leq & \beta^m||T^{k-1}J_t - J^{\mu_t}||_\xi + ||T^{k-1}J_t - J^{\mu_t}||_\xi \\ \leq & (\beta^m + 1)\frac{\beta^{k-1}}{1-\beta}||TJ_t - J_t||_\xi = \left(\frac{\beta^{m+k-1}}{1-\beta} + \frac{\beta^{k-1}}{1-\beta}\right)||TJ_t - J_t||_\xi,\end{aligned}$$

where the second inequality follows from Lemma 3.1. □

Lemma 3.3. *Let J^* be the optimal cost-to-go vector, J_t be the iteration generated by* (3.10) *and let β be the contracting factors of the operators T and T_μ with respect to the weighted maximum norm $||\cdot||_\xi$. Then*

$$||T_{\mu_t}^m T^{k-1}J_t - J^*||_\xi \leq \left(\beta^{k-1} + (1+\beta^m)\frac{\beta^{k-1}}{1-\beta}(1+\beta)\right)||J_t - J^*||_\xi, \quad \forall t \geq 0.$$

Proof. Lemma 3.2 implies

$$\begin{aligned}||T_{\mu_t}^m T^{k-1}J_t - J^*||_\xi - ||T^{k-1}J_t - J^*||_\xi \leq & ||T_{\mu_t}^m T^{k-1}J_t - T^{k-1}J_t||_\xi \\ \leq & \left(\frac{\beta^{m+k-1}}{1-\beta} + \frac{\beta^{k-1}}{1-\beta}\right)||TJ_t - J_t||_\xi.\end{aligned}$$

Repeatedly apply (3.4) and the fact that $J^* = TJ^*$ to derive

$$\begin{aligned}||T_{\mu_t}^m T^{k-1}J_t - J^*||_\xi \leq & ||T^{k-1}J_t - J^*||_\xi + \left(\frac{\beta^{m+k-1}}{1-\beta} + \frac{\beta^{k-1}}{1-\beta}\right)||TJ_t - J_t||_\xi \\ \leq & \beta^{k-1}||J_t - J^*||_\xi + \left(\frac{\beta^{m+k-1}}{1-\beta} + \frac{\beta^{k-1}}{1-\beta}\right)(||TJ_t - J^*||_\xi + ||J^* - J_t||_\xi) \\ \leq & \beta^{k-1}||J_t - J^*||_\xi + \left(\frac{\beta^{m+k-1}}{1-\beta} + \frac{\beta^{k-1}}{1-\beta}\right)(\beta||J_t - J^*||_\xi + ||J_t - J^*||_\xi) \\ \leq & \left(\beta^{k-1} + (1+\beta^m)\frac{\beta^{k-1}}{1-\beta}(1+\beta)\right)||J_t - J^*||_\xi,\end{aligned}$$

as desired. □

Proof of Theorem 3.1. Fix m and k satisfying Assumption 3.3. Recall (3.10) in component form

$$J_{t+1}(i) = (1 - \gamma_t(i)p_t(i))J_t(i) + \gamma_t(i)p_t(i)\left(T_{\mu_t}^m T^{k-1}J_t(i) + v_t(i)\right).$$

It is evident that $\{J_t\}_{t\geq 0}$ is a sequence of the form (4.23) in [BT96], with $u_t(i) = 0$. In particular by Lemma 3.3, the operator

$$H_t^{m,k} := T_{\mu_t}^m T^{k-1},$$

which is adapted to $\mathcal{F}_t$ by virtue of being an expectation conditional on the filtration, is a pseudo-contraction under the same weighted maximum norm $||\cdot||_\xi$, with the same fixed point J^*, and with the same contraction factor (m and k are fixed for every iteration)

$$\widetilde{\beta} := \left(\beta^{k-1} + (1+\beta^m)\frac{\beta^{k-1}}{1-\beta}(1+\beta)\right) < 1.$$

That is, there exists a vector J^*, a positive vector ξ, and a scalar $\widetilde{\beta} \in [0,1)$, such that

$$||H_t^{m,k}J_t - J^*||_\xi \leq \widetilde{\beta}||J_t - J^*||_\xi, \quad \forall t \geq 0.$$

The step-sizes $\gamma_t(i)$ are non-negative and satisfy the step-size conditions of Assumption 3.5. By Assumption 3.4, we have that $\inf_{t\geq 0} p_t(i) > 0$. Therefore,

$$\sum_{t=0}^{\infty}\gamma_t(i)p_t(i) \geq \inf_{t\geq 0} p_t(i)\sum_{t=0}^{\infty}\gamma_t(i) = \infty \text{ and } \sum_{t=0}^{\infty}\gamma_t(i)^2 p_t^2(i) \leq \sum_{t=0}^{\infty}\gamma_t(i)^2 < \infty.$$

Lastly, we must show that the noise term $v_t(i)$ satisfies Assumption 4.3 in [BT96]. Upon taking expectations, we see that $\mathbb{E}[v_t|\mathcal{F}_t] = 0$. Evidently, by the boundedness of g_μ and there being finitely many policies, we have that $||T_{\mu_t}^m T^{k-1}J_t||_\infty^2$ is bounded by some constant. Therefore,

$$\mathbb{E}[||v_t||_\infty^2|\mathcal{F}_t] \leq A + B||J_t||_\infty^2,$$

where A and B are constant.

We have checked all conditions of Proposition 4.5. Therefore, we may conclude that J_t converges to J^* with probability 1. □

§3.3 Further Discussion

In this section we discuss how some of the assumptions can be relaxed, how some variations either follow as corollaries or can be generalized directly, and we tie the proof back into previous known results.

It is well known that discounted problems can be converted into a stochastic shortest path problems (see pg 37, [BT96]), thus making the former a specialization of the latter.

Remark 3.3. *To convert the discounted problem to a stochastic shortest path problem one may introduce a termination state with probability $1 - \alpha$ of being reached from any state. As such, all policies are proper. Then it can be shown that the two problems are equivalent as the state evolution prior to termination and cost-to-go are the same. Therefore, if one follows this procedure, the MDP in [WS23] is a special case of the MDP in this chapter. By consequence, their results follow.*

Not all policies need to be proper for convergence to be guaranteed. The following remark discusses this.

Remark 3.4 (Relaxing Assumption 3.2)**.** *It was shown in Section 4 of [Liu21] that one can relax the proper policy assumption by replacing it with the following:*

> **Assumption 3.6.** *There exists at least one proper policy and every improper policy yields an infinite cost for at least one initial state.*

In general, the convergence proof would not go through with this modified assumption because there is no guarantee the optimistic value iteration would not explode due to an improper policy. However, modifying an MDP satisfying assumption 3.6 by forcing the trajectory to terminate with small probability and thus guarantee every policy is proper, [Liu21] showed that an optimal value and policy of an MDP satisfying assumption 3.6 will remain close to the original optimal value and policy. This is done by showing that any improper policy cannot be optimal and among proper policies, the value between the modified MDP and the original MDP remain close.

Since the proof in [Liu21] does not rely on the specific variation on the algorithm but rather the uniqueness of J^* for the MDP, it carries over unchanged and so is omitted.

In previous work (e.g. [Tsi02; Liu21]), if one wanted to study the convergence of any asynchronous algorithm, algebraic manipulations had to be employed within very specific contexts to circumvent the issue of non-commuting matrices in the proof of their results. Even so, cases like the first-visit were not covered. The contraction property exhibited by the lookahead policy allows for a different proof technique which avoids the previous pitfall (as used in [WS23]). In conjunction with the techniques employed in [Liu21], this chapter's main theorem extends [WS23] to the stochastic shortest path setting and by not specifying $\mathcal{D}_t$ a priori, we were able to prove a general result about the Optimistic Policy Iteration algorithm. Now we specialize this result further to the first-visit variation using lookahead.

Example (first-visit)**.** *In the classic definition of the first-visit variation [SB18], we are updating based on the average cost of all trajectories following every first visit to a state. Define the set of first-visits on trajectory t by*

$$\mathcal{D}_t := \{i \in S : \exists k \in \mathbb{N} \cup \{0\} \text{ s.t. } x_k = i \text{ and } x_\ell \neq i \ \forall \ell < k\}.$$

The estimator used in the update procedure is (pg 190, [BT96])

$$J_{t+1}(i) = \frac{1}{n_t(i)} \sum_{w=0}^{t} \boldsymbol{1}_{\{i \in \mathcal{D}_w\}} \sum_{\ell=0}^{\infty} g(x_\ell, \mu_w(x_\ell)),$$

where $n_t(i) := \#$ *first visits to state* i *by iteration* t *and* $x_0 = i$. *Upon iterating*

$$= \begin{cases} \frac{n_{t-1}(i)}{n_t(i)} \frac{1}{n_{t-1}(i)} \sum_{w=0}^{t-1} \mathbf{1}_{\{i \in \mathcal{D}_w\}} \sum_{\ell=0}^{\infty} g(x_\ell, \mu_w(x_\ell)) + \frac{1}{n_t(i)} \sum_{\ell=0}^{\infty} g(x_\ell, \mu_t(x_\ell)) & i \in \mathcal{D}_t \\ J_t(i) & i \notin \mathcal{D}_t \end{cases}$$

$$= \begin{cases} \frac{n_t(i)-1}{n_t(i)} J_t(i) + \frac{1}{n_t(i)} \sum_{\ell=0}^{\infty} g(x_\ell, \mu_t(x_\ell)) & i \in \mathcal{D}_t \\ J_t(i) & i \notin \mathcal{D}_t \end{cases},$$

Set $\gamma_t(i) = \frac{1}{n_t(i)}$, $m = \infty$, *and* k *large enough such that*

$$k > 1 + \frac{\log \frac{1}{2}(1-\beta)}{\log \beta}, \tag{3.14}$$

implied by assumption 3.3. Convergence of the first-visit variation is guaranteed by choosing greedy policies based on sufficiently large lookahead since it is a simple matter to check that the step-size satisfies Assumption 3.5. Of course, one may choose to modify the rollout parameter m *to suit any particular application. We simply presented a version that is as close to the original definition as possible.*

For the interested reader, Algorithm 1 of [WS23] presents the pseudo-code for the first-visit variation. It would be identical to this chapter's case with the understanding that α *is allowed to take value one.*

Generalizing past results

It is possible to recover a particular case of the main theorem of [Liu21] from this chapter's Theorem 3.1 and by extension Proposition 1 of [Tsi02]. In the restrictive scenario that every state is guaranteed to be visited on the trajectory before termination, we may set the model parameters to $\mathcal{D}_t = S, \Gamma_t = \gamma_t I$, and $P_t = I$ in iteration (3.10). Furthermore, one would have to allow for a theoretically infinite rollout ($m = \infty$), then for any k large enough, $\lim_{m\to\infty} T_{\mu_t}^m T^{k-1} J_t = J^{\mu_t}$ and we recover

$$J_{t+1} = (1-\gamma_t) J_t + \gamma_t (J^{\mu_t} + w_t),$$

which is iteration (3) studied in [Liu21] under the same assumptions of this chapter. Notice how k is purely formal; it's necessary to satisfy this chapter's requirements for contraction but not necessary to guarantee convergence. Rather one could appeal to the results of [Liu21], consider a 1-step greedy policy, and still be guaranteed convergence.

By making use of lookaheads, we may generalize from this unlikely setup. Say one could not guarantee a visit to every state on each trajectory but still wanted to use the estimator $J^{\mu_t} + w_t$, that is, the sum of all costs to termination. This would lead to a direct generalization of past results: the algorithm can be designed to allow for as many updates as desired for each t. In the process, we provide a sufficient condition for convergence under this generalized iterative scheme. The question of whether this sufficient condition is also necessary remains open, although we do not think it is necessary.

Generate a trajectory with the estimator $T^m_{\mu_t} T^{k-1} J_t(i) + w_t(i)$. Consider the following modified iteration

$$J_{t+1}(i) = \begin{cases} (1-\gamma_t(i))T^{k-1}J_t(i) + \gamma_t(i)(T^m_{\mu_t}T^{k-1}J_t(i) + w_t(i)) & i \in \mathcal{D}_t \\ T^{k-1}J_t(i) & i \notin \mathcal{D}_t \end{cases}. \tag{3.15}$$

Remark 3.5. *If one wished to use the estimator $J^{\mu_t} + w_t$ (i.e. $m = \infty$), we will show that any lookahead satisfying (3.14) will lead to convergence irrespective of how many states are picked for updating, thus allowing for a direct generalization of past results. This is different from iteration (3.10) since we no longer require the restrictive assumptions on the model dynamics discussed above to use the main iterations of [Tsi02; Liu21], but still allows for the flexibility in the choice of state updates.*

Proposition 3.1 Generalized Convergence

Let assumptions 3.1, 3.2, 3.3, 3.4, 3.5 still hold. Further, assume there exists a scalar valued step size $\widehat{\gamma}_t$ that satisfies Assumption 3.5 and

$$\lim_{t\to\infty} P_t = \lim_{t\to\infty} \widehat{\gamma}_t \Gamma_t^{-1}, \quad \forall i \in S \tag{3.16}$$

then iteration (3.15) converges to J^* almost surely.

Remark 3.6. *Assumption (3.16) boils down to asserting the asymptotic equivalence between step size choices emanating from states whose frequency of selection is not restricted and ones that are. It is known that restricting state selection and appropriately selecting step-sizes will lead to convergence through algebraic manipulations (see Section 5, [Liu21]), and therefore this is a natural and possibly obvious sufficient condition. However, it seemed appropriate that in the process of generalizing to iteration (3.15) one would put all cases under a single umbrella. Finding a necessary and sufficient condition is a more interesting question. For example, the following all satisfy assumption (3.16) and use iteration (3.15), thus allowing for a single theorem to capture each of these iterations:*

- *Equation (4) of [Tsi02]: Choose $\mathcal{D}_t = \{i \in S : x_0 = i\}, m = \infty, \Gamma_t = \gamma_t I, P_t = \frac{1}{|S|}I$, and $\widehat{\gamma}_t = \frac{\gamma_t}{|S|}$;*
- *Proposition 16 of [Liu21] with step-size (14): Choose $\mathcal{D}_t = \{i \in S : x_0 = i\}, m = \infty, \Gamma_t = diag(\frac{1}{\#\ times\ i\ selected\ by\ iteration\ t}), P_t = diag(p(i))$, and $\widehat{\gamma}_t = \frac{1}{t+1}$;*
- *Proposition 16 of [Liu21] with step-size (15): Choose $\mathcal{D}_t = \{i \in S : x_0 = i\}, m = \infty, \Gamma_t = diag(\frac{\widehat{\gamma}_t}{p(i)}), P_t = diag(p(i))$, and any $\widehat{\gamma}_t$.*

Example. *As an illustration of sufficiency for Assumption 3.16, consider the example in Section 6 of [Liu21]. It is a discounted problem with $\alpha \in (0,1)$: let $S = \{1,2\}$ and $A = \{l,r\}$ and define the costs as $g(1,r) = g(2,l) = 0$ and $g(1,l) = g(2,r) = 1$. The system is deterministic in that $p_{11}(l) = p_{12}(r) = p_{21}(l) = p_{22}(r) = 1-p_{11}(r) = 1-p_{12}(l) = 1-p_{21}(r) = 1-p_{22}(l) = 1$. The probability of choosing state 1 as the initial state is p. This fully specifies the MDP. For more details see Section 6 in [Liu21] and Example 5.11 in [BT96].*

They show empirically that convergence is achieved for an arbitrary step-size only when states are selected uniformly or if the probability distribution used to select the states is non-uniform, the step-size satisfies Assumption (3.16). All other cases diverge.

It will be easier to work with J_t in vector form. Using similar algebraic manipulations that led to equation (3.10), we may rewrite iteration (3.15) as

$$J_{t+1}(i) = (1 - \widehat{\gamma}_t)T^{k-1}J_t(i) + \widehat{\gamma}_t(T^m_{\mu_t}T^{k-1}J_t(i) + v_t(i) + u_t(i)),$$

where

$$v_t(i) := w_t(i) + \left(\frac{\chi_t(i)}{p_t(i)} - 1\right)(-T^{k-1}J_t(i) + T^m_{\mu_t}T^{k-1}J_t(i)) + \left(\frac{\chi_t(i)\gamma_t(i)}{\widehat{\gamma}_t} - 1\right)w_t(i),$$

and

$$u_t(i) := \left(\frac{\gamma_t(i)}{\widehat{\gamma}_t} - \frac{1}{p_t(i)}\right)\chi_t(i)(-T^{k-1}J_t(i) + T^m_{\mu_t}T^{k-1}J_t(i)).$$

Notice that by the boundedness of g_μ and for some constants $A, B > 0$,

$$\begin{aligned}||T^{k-1}J_t||_\infty &= ||T_{\mu_t}T^{k-2}J_t||_\infty = ||g_{\mu_t} + P_{\mu_t}T^{k-2}J_t||_\infty = ||g_{\mu_t} + P_{\mu_t}g_{\mu_t} + ... + P^m_{\mu_t}J_t||_\infty \\ &\leq A||J_t||_\infty + B.\end{aligned}$$

This, the fact that $T^{k-1}J_t$ is adapted to $\mathcal{F}_t$, and with the boundedness of $T^m_{\mu_t}T^{k-1}J_t$ implies

$$\mathbb{E}\left[v_t(i)|\mathcal{F}_t\right] = 0 \text{ and } \mathbb{E}\left[||v_t(i)||^2_\infty|\mathcal{F}_t\right] \leq \widetilde{A}||J_t||^2_\infty + \widetilde{B}. \tag{3.17}$$

Furthermore, it is evident from (3.16) that there exists a constant $C > 0$ and $\theta_t := C\max_{i\in S}\left|\frac{\gamma_t(i)}{\widehat{\gamma}_t} - \frac{1}{p_t(i)}\right| \to 0$, almost surely, such that

$$|u_t(i)| \leq \theta_t\left(||J_t||_\infty + 1\right), \quad \forall i \in S, t \geq 0. \tag{3.18}$$

We have seen how using β in place of α allows us to generalize to the stochastic shortest path. This applies to this proof as well. As the proof of Proposition 3.1 is virtually identical to the one in [WS23], we outline the major differences allowing for asynchronous behavior and using the contracting factor β in place of the discount factor α, and refer the reader to Appendix A in [WS23] for details.

Proof of Proposition 3.1. We first show that for every $\epsilon > 0$, there exists a sufficiently large t_ϵ such that

$$(1 - \widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_tT^m_{\mu_t}T^{k-1}J_t - \epsilon\Xi e \leq J_{t+1} \leq (1 - \widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_tT^m_{\mu_t}T^{k-1}J_t + \epsilon\Xi e. \tag{3.19}$$

Define the sequence $Y_0 = 0$ and

$$Y_{t+1} = (1 - \widehat{\gamma}_t)Y_t + \widehat{\gamma}_t(v_t + u_t).$$

If we consider the operator that maps Y_t to the zero vector, then such an operator trivially satisfies the pseudo-contraction property. In view of the assumption on $\widehat{\gamma}_t$, (3.17), (3.18), all other assumptions of Proposition 4.5 [BT96] hold and therefore we may conclude that $Y_t \to 0$ almost surely.

Subtract Y_{t+1} from J_{t+1} and rearrange,

$$J_{t+1} = (1-\widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_t T^m_{\mu_t} T^{k-1} J_t + (Y_{t+1} - (1-\widehat{\gamma}_t)Y_t).$$

Since $Y_t \to 0$ a.s. then for all $\delta > 0$ there exists a t_δ, such that for all $t > t_\delta$

$$J_{t+1} \le (1-\widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_t T^m_{\mu_t} T^{k-1} J_t + \delta e.$$

Fix $\epsilon > 0$. Since the above equation holds for all $\delta > 0$, choose $\delta = \epsilon \min_{i \in S} \xi(i)$, then for all $\epsilon > 0$, there exists a t_ϵ such that for all $t > t_\epsilon$,

$$J_{t+1} \le (1-\widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_t T^m_{\mu_t} T^{k-1} J_t + \epsilon \Xi e. \tag{3.20}$$

The left-hand side of (3.19) is established in a symmetrical way.

Recall that $TJ = \min_{\mu \in \Pi} T_\mu J \le T_\mu J \ \forall \mu \in \Pi$ and by Lemma 5 in [Liu21],

$$\begin{aligned}
TJ_{t+1} \le & T_{\mu_t}\left((1-\widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_t T^m_{\mu_t} T^{k-1} J_t + \epsilon \Xi e\right) \\
\le & T_{\mu_t}\left((1-\widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_t T^m_{\mu_t} T^{k-1} J_t\right) + \beta \epsilon \Xi e \\
= & g_{\mu_t} + (1-\widehat{\gamma}_t) g_{\mu_t} - (1-\widehat{\gamma}_t) g_{\mu_t} + (1-\widehat{\gamma}_t) P_{\mu_t} T^{k-1} J_t + \widehat{\gamma}_t P_{\mu_t} T^m_{\mu_t} T^{k-1} J_t + \beta \epsilon \Xi e \\
= & (1-\widehat{\gamma}_t) T^k J_t + \widehat{\gamma}_t T^{m+1}_{\mu_t} T^{k-1} J_t + \beta \epsilon \Xi e.
\end{aligned} \tag{3.21}$$

Now simply subtract (3.20) from (3.21),

$$TJ_{t+1} - J_{t+1} \le (1-\widehat{\gamma}_t)(T^k J_t - T^{k-1} J_t) + \widehat{\gamma}_t (T^{m+1}_{\mu_t} T^{k-1} J_t - T^m_{\mu_t} T^{k-1} J_t) + (1+\beta)\epsilon \Xi e.$$

Remark 3.7. *In contrast to [WS23], we used the matrix Ξ when applying the monotonicity property of T_{μ_t}. In the discounted case, this matrix is not necessary as we do not need to use the weighted maximum norm to establish the result. Beyond this point, the proof is identical to Appendix A in [WS23], with the same Ξ-technique caveat, and so we simply outline the major steps.*

Using an appropriate recursive sequence, namely, set $\delta_{t_\epsilon} := ||TJ_t - J_t||_\xi$, $\widehat{\xi} := \max_{i \in S} \Xi e$, and define $\delta_t = \delta_{t-1}(\beta^{k-1} + \beta^{m+k-1}) + (1+\beta)\widehat{\xi}\epsilon$, then by induction, it can be shown that

$$\limsup_{t\to\infty} TJ_t - J_t \le \lim_{t\to\infty} \delta_t e \downarrow 0 \text{ as } \epsilon \to 0,$$

since Assumption 3.3 guarantees $\beta^{k-1} + \beta^{m+k-1} < 1$. Therefore, for any $\widetilde{\epsilon} > 0$, there exists a $t_{\widetilde{\epsilon}}$ such that for all $t > t_{\widetilde{\epsilon}}$,

$$TJ_t \le J_t + \widetilde{\epsilon}\Xi e \implies T^m_{\mu_t} T^{k-1} J_t \le T^{k-1} J_t + \frac{\beta^{k-1}}{1-\beta}\widetilde{\epsilon}\Xi e.$$

Continuing from (3.19), for $t > t_\epsilon + t_{\widetilde{\epsilon}}$ and without loss of generality $\widehat{\gamma}_t \leq 1$ since $\widehat{\gamma}_t \to 0$,

$$J_{t+1} \leq (1-\widehat{\gamma}_t)T^{k-1}J_t + \widehat{\gamma}_t\left(T^{k-1}J_t + \frac{\beta^{k-1}}{1-\beta}\widetilde{\epsilon}\Xi e\right) + \epsilon\Xi e \leq T^{k-1}J_t + \frac{\beta^{k-1}}{1-\beta}\widetilde{\epsilon}\Xi e + \epsilon\Xi e.$$

And for the reverse inequality,

$$J_{t+1} \geq (1-\widehat{\gamma}_t)(T^m T^{k-1}J_t - \frac{\beta^{k-1}}{1-\beta}\widetilde{\epsilon}\Xi e) + \widehat{\gamma}_t T^m_{\mu_t}T^{k-1}J_t - \epsilon\Xi e \geq T^{m+k-1}J_t - \frac{\beta^{k-1}}{1-\beta}\widetilde{\epsilon}\Xi e - \epsilon\Xi e.$$

Put the two inequalities together,

$$T^{m+k-1}J_t - \frac{\beta^{k-1}}{1-\beta}\widetilde{\epsilon}\Xi e - \epsilon\Xi e \leq J_{t+1} \leq T^{k-1}J_t + \frac{\beta^{k-1}}{1-\beta}\Xi\widetilde{\epsilon}e + \epsilon\Xi e. \tag{3.22}$$

Notice from the weighted maximum norm contraction property applied repeatedly,

$$||T^{m+k-1}J_t - J^*||_\xi \leq \beta^{m+k-1}||J_t - J^*||_\xi, \text{ and } ||T^{k-1}J_t - J^*||_\xi \leq \beta^{k-1}||J_t - J^*||_\xi.$$

Upon subtracting J^* from (3.22),

$$-\beta^{m+k-1}||J_t - J^*||_\xi e - \frac{\beta^{k-1}}{1-\beta}\widetilde{\epsilon}\Xi e - \epsilon\Xi e \leq J_{t+1} - J^* \leq \beta^{k-1}||J_t - J^*||_\xi + \frac{\beta^{k-1}}{1-\beta}\Xi\widetilde{\epsilon}e + \epsilon\Xi e.$$

Therefore $||J_{t+1} - J^*||_\xi \leq (\beta^{k-1} + \beta^{m+k-1})||J_t - J^*||_\xi + \frac{\beta^{k-1}}{1-\beta}\Xi\widetilde{\epsilon}e + \epsilon\Xi e$, which implies

$$\limsup_{t\to\infty} ||J_t - J^*||_\xi \leq \lim_{\epsilon,\widetilde{\epsilon}\to 0} \frac{\frac{\beta^{k-1}}{1-\beta}\Xi\widetilde{\epsilon} + \epsilon\Xi}{\beta^{k-1} + \beta^{m+k-1}} = 0,$$

as desired.

□

Conclusion

Main Convergence Results

Three projects were carried out for this dissertation. We established:

§1 a large deviation principle for additive functionals of reflected jump-diffusions;

§2 a functional central limit theorem for a fast-slow dynamical system driven by symmetric and multiplicative α-stable noise; and

§3 the almost sure convergence of undiscounted optimistic policy iteration without state update restrictions.

For the first project, the existence of a solution to the reflected jump diffusion in both the normal and oblique reflection cases was treated. Then a full characterization of the rate function for the large deviation principle was derived involving the eigen-pair solution of a related partial-integro differential equation using the Gärtner-Ellis theorem by making use of an appropriate exponential martingale. This latter object was shown to be a martingale through the standard Markov process argument of setting the drift of an a-priori conjectured process to zero. In so doing, a partial-integro differential equation was derived where the existence and uniqueness of its solution followed by showing the same for the eigen-pair of a related operator. The results were then specialized to the one dimensional case from which a numerical scheme based on finite differences was devised to solve two standard examples with known analytic solutions and an applied problem in biochemical reactions.

There are a number of places where future work can continue. For one, the additive functional is not as general as it could be; it only considers the continuous part of the associated boundary process. One might ask if this could be extended to consider the entire boundary process. From an application standpoint, the numerical scheme devised is limited to the one-dimensional case. One could extend this to the multi-dimensional setting using the same techniques, although it would probably be more fruitful to consider more advanced techniques in numerical approximation. And, naturally, as this argument has been used in more specialized cases than the one treated in Chapter 2, once can imagine it is likely applicable to processes other than reflected jump-diffusions or additive functionals of the process paths.

The functional central limit theorem result follows the averaging principle, which itself is a consequence of a related Poisson equation involving the generator of the frozen process and the difference between the drift of the slow process and its averaged process. To work out the Poisson equation, one had to first establish the ergodicity for the frozen process and derive a number of technical bounds on gradients and differences between gradients of this process. Then the regularity properties for the solution of the Poisson equation were derived. These regularity properties served as bounds in the proof of the averaging principle along with various moment estimates. From the averaging principle, we were able to derive the scaling for the functional central limit theorem which was established by showing tightness and convergence of the finite dimensional distributions. The chapter was concluded with a numerical study of an illustrative example to put the theory into practice.

There are a few obvious extensions of this work. The Poisson approach has been shown to be useful in a number of previous papers and it is likely to be applicable to other dynamical systems. One could add Brownian motion terms or various other drift terms that would fundamentally change the system but have different applications. The order of scaling can also be manipulated to derive different results. One could also extend this work by establishing a large deviation principle for this system. The numerical study relied on an inefficient Monte Carlo pathwise simulation, and we exploited efficiencies of the practical problem to speed up computation. A possible avenue of research is in how to improve on this simulation or derive better numerical approaches to putting the theory into practice.

Finally, we established the convergence for the optimistic variant of policy iteration in the undiscounted setting and without the need to specify which states are updated a priori. This was done by making use of lookahead policies and the pseudo-contraction property of the operators. We discuss the first-visit, discounted cases, and generalized the variants that use the initial state for updating while providing a sufficient condition for convergence. A seemingly difficult problem would be to find a necessary and sufficient condition for convergence which remains open.

This serves as a first step in the exploration of the $\alpha = 1$ case. It seems likely that these techniques would extend easily to temporal differences: $TD(\lambda), \lambda \in [0, 1)$, where the policy update is carried out within a trajectory, by adapting the techniques in [BT96; Tsi02; Liu21; WS23] to Chapter 3's setting. When a model is available it is often possible to solve for an optimal policy analytically. Therefore, one could look into the model-free case using Q-values and state-action pairs for updating, as done in [Tsi02]. When the problem is too computationally expensive, for example state and action spaces that are too large, we can also consider algorithms that use function approximation, gradient descent, least squares, and feature vectors, as done in the second half of the paper [WS23] among others in this direction: [WS22; Win+21]. Finally, the proof relied on finding a suitable unbiased estimator to update the value function. This precludes the every-visit variation which is known to be consistent but biased. And thus research into the convergence of consistent estimators may allow for a proof of this variation as well.

Bibliography

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[AWTW05] Rosalind J Allen, Patrick B Warren, and Pieter Rein Ten Wolde. "Sampling rare switching events in biochemical networks". In: *Physical review letters* 94.1 (2005), p. 018104.

[And+15] Lars Nørvang Andersen et al. "Lévy Processes with Two-Sided Reflection". In: *Lévy Matters V: Functionals of Lévy Processes*. Cham: Springer International Publishing, 2015, pp. 67–182. ISBN: 978-3-319-23138-9. DOI: 10.1007/978-3-319-23138-9_2. URL: https://doi.org/10.1007/978-3-319-23138-9_2.

[And+19] David F Anderson et al. "On constrained Langevin equations and (bio) chemical reaction networks". In: *Multiscale Modeling & Simulation* 17.1 (2019), pp. 1–30.

[AL91] SV Anulova and B Sh Liptser. "Diffusional approximation for processes with the normal reflection". In: *Theory of Probability & Its Applications* 35.3 (1991), pp. 411–423.

[AG07] Søren Asmussen and Peter W Glynn. *Stochastic simulation: algorithms and analysis*. Vol. 57. Springer, 2007.

[BVOC11] Gábor Balázsi, Alexander Van Oudenaarden, and James J Collins. "Cellular decision making and biological noise: from microbes to mammals". In: *Cell* 144.6 (2011), pp. 910–925.

[Bal+06] Karen Ball et al. "Asymptotic analysis of multiscale approximations to reaction networks". In: *The Annals of Applied Probability* 16.4 (2006), pp. 1925 –1961. DOI: 10.1214/105051606000000420. URL: https://doi.org/10.1214/105051606000000420.

[BDM14] Tommaso Biancalani, Louise Dyson, and Alan J McKane. "Noise-induced bistable states and their mean switching time in foraging colonies". In: *Physical review letters* 112.3 (2014), p. 038101.

[BKW20] Enrico Bibbona, Jinsu Kim, and Carsten Wiuf. "Stationary distributions of systems with discreteness-induced transitions". In: *Journal of The Royal Society Interface* 17.168 (2020), p. 20200243.

[BQ10] Lisa M Bishop and Hong Qian. "Stochastic bistability and bifurcation in a mesoscopic signaling system with autocatalytic kinase". In: *Biophysical journal* 98.1 (2010), pp. 1–11.

[BB04] James Antonio Bucklew and J Bucklew. *Introduction to rare event simulation.* Vol. 5. Springer, 2004.

[CMEKM80] Mireille Chaleyat-Maurel, Nicole El Karoui, and Bernard Marchal. "Réflexion discontinue et systèmes stochastiques". In: *The Annals of Probability* (1980), pp. 1049–1067.

[CFM06] Nicolas Champagnat, Régis Ferrière, and Sylvie Méléard. "Unifying evolutionary dynamics: from individual stochastic processes to macroscopic models". In: *Theoretical population biology* 69.3 (2006), pp. 297–321.

[Cos92] Cristina Costantini. "The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations". In: *Probability theory and related fields* 91 (1992), pp. 43–70.

[DM+15] Krzysztof Dębicki, Michel Mandjes, et al. *Queues and Lévy fluctuation theory.* Springer, 2015.

[DZ09] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications.* Vol. 38. Springer Science & Business Media, 2009.

[Dit99] Peter D Ditlevsen. "Observation of α-stable noise induced millennial climate changes from an ice-core record". In: *Geophysical Research Letters* 26.10 (1999), pp. 1441–1444.

[DD09] Peter D Ditlevsen and Ove D Ditlevsen. "On the stochastic nature of the rapid climate shifts during the last ice age". In: *Journal of Climate* 22.2 (2009), pp. 446–457.

[DV76] M. D. Donsker and S. R. S. Varadhan. "Asymptotic evaluation of certain Markov process expectations for large time. III". In: *Communications on Pure and Applied Mathematics* 29.4 (1976), pp. 389–461. DOI: `https://doi.org/10.1002/cpa.3160290405`.

[DV83] M. D. Donsker and S. R. S. Varadhan. "Asymptotic evaluation of certain Markov process expectations for large time. IV". In: *Communications on Pure and Applied Mathematics* 36.2 (1983), pp. 183–212.

[DI91] Paul Dupuis and Hitoshi Ishii. "On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications". In: *Stochastics: An International Journal of Probability and Stochastic Processes* 35.1 (1991), pp. 31–62.

[DI93] Paul Dupuis and Hitoshi Ishii. "SDEs with oblique reflection on nonsmooth domains". In: *The annals of Probability* (1993), pp. 554–580.

[FSS87] WH Fleming, SJ Sheu, and HM Soner. "A remark on the large deviations of an ergodic Markov process". In: *Stochastics: An International Journal of Probability and Stochastic Processes* 22.3-4 (1987), pp. 187–199.

[FKZ15] Martin Forde, Rohini Kumar, and Hongzhong Zhang. “Large deviations for the boundary local time of doubly reflected Brownian motion”. In: *Statistics & Probability Letters* 96 (2015), pp. 262–268.

[FW98] M. I. Freidlin and A. D. Wentzell. “Random Perturbations”. In: *Random Perturbations of Dynamical Systems*. New York, NY: Springer New York, 1998, pp. 15–43. ISBN: 978-1-4612-0611-8. DOI: 10.1007/978-1-4612-0611-8_2. URL: https://doi.org/10.1007/978-1-4612-0611-8_2.

[GAK15] Arnab Ganguly, Derya Altintan, and Heinz Koeppl. “Jump-diffusion approximation of stochastic reaction dynamics: error bounds and algorithms”. In: *Multiscale Modeling & Simulation* 13.4 (2015), pp. 1390–1419.

[Gär77] Jürgen Gärtner. “On large deviations from the invariant measure”. In: *Theory of Probability & Its Applications* 22.1 (1977), pp. 24–39.

[GM04] Paul Glasserman and Nicolas Merener. “Convergence of a discretization scheme for jump-diffusion processes with state–dependent intensities”. In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 460.2041 (2004), pp. 111–127.

[GW15] Peter W Glynn and Rob J Wang. “Central limit theorems and large deviations for additive functionals of reflecting diffusion processes”. In: *Asymptotic Laws and Methods in Stochastics* 76 (2015), p. 329.

[Hol00] Frank Hollander. *Large deviations*. Vol. 14. American Mathematical Soc., 2000.

[Jah+11] Patrick Jahn et al. “Motoneuron membrane potentials follow a time inhomogeneous jump diffusion process”. In: *Journal of computational neuroscience* 31.3 (2011), pp. 563–579.

[JMW12] Benjamin Jourdain, Sylvie Méléard, and Wojbor A Woyczynski. “Lévy flights in evolutionary ecology”. In: *Journal of mathematical biology* 65 (2012), pp. 677–707.

[KKP14] Hye-Won Kang, Thomas G. Kurtz, and Lea Popovic. “Central limit theorems and diffusion approximations for multiscale Markov chain models”. In: *The Annals of Applied Probability* 24.2 (2014), pp. 721 –759. DOI: 10.1214/13-AAP934. URL: https://doi.org/10.1214/13-AAP934.

[KM03] I. Kontoyiannis and S. P. Meyn. “Spectral theory and limit theorems for geometrically ergodic Markov processes”. In: *The Annals of Applied Probability* 13.1 (2003), pp. 304 –362. DOI: 10.1214/aoap/1042765670. URL: https://doi.org/10.1214/aoap/1042765670.

[Kou07] Steve G Kou. “Jump-diffusion models for asset pricing in financial engineering”. In: *Handbooks in operations research and management science* 15 (2007), pp. 73–116.

[Kra12] MA Krasnoselskii. “Positive solutions of operator equations, Noordhoff, Groningen, 1964”. In: *Douglas R. Anderson Department of Mathematics and Computer Science, Concordia College, Moorhead, MN* 56562 (2012).

[Kru+07] Lukasz Kruk et al. “An explicit formula for the Skorokhod map on $[0, a]$”. In: *The Annals of Probability* 35.5 (2007), pp. 1740 –1768. DOI: 10.1214/009117906000000890. URL: https://doi.org/10.1214/009117906000000890.

[Kru+08] Łukasz Kruk et al. “Double Skorokhod map and reneging real-time queues”. In: *Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz*. Vol. 4. Institute of Mathematical Statistics, 2008, pp. 169–194.

[Kur78] Thomas G Kurtz. “Strong approximation theorems for density dependent Markov chains”. In: *Stochastic Processes and their Applications* 6.3 (1978), pp. 223–240.

[ŁS03] Weronika Łaukajtys and Leszek Słomiński. “Penalization methods for reflecting stochastic differential equations with jumps”. In: *Stochastics and Stochastic Reports* 75.5 (2003), pp. 275–293.

[LW19] Saul C. Leite and Ruth J. Williams. “A constrained Langevin approximation for chemical reaction networks”. In: *The Annals of Applied Probability* 29.3 (2019), pp. 1541 –1608. DOI: 10.1214/18-AAP1421. URL: https://doi.org/10.1214/18-AAP1421.

[LS84] Pierre-Louis Lions and Alain-Sol Sznitman. “Stochastic differential equations with reflecting boundary conditions”. In: *Communications on pure and applied Mathematics* 37.4 (1984), pp. 511–537.

[MP14] John K. McSweeney and Lea Popovic. “Stochastically-induced bistability in chemical reaction systems”. In: *The Annals of Applied Probability* 24.3 (2014), pp. 1226 –1268. DOI: 10.1214/13-AAP946. URL: https://doi.org/10.1214/13-AAP946.

[MR85] José-Luis Menaldi and Maurice Robin. “Reflected diffusion processes with jumps”. In: *The Annals of Probability* (1985), pp. 319–341.

[PP05] Philip E Protter and Philip E Protter. *Stochastic differential equations*. Springer, 2005.

[RWB18] Francesco Ragone, Jeroen Wouters, and Freddy Bouchet. “Computation of extreme heat waves in climate models using a large deviation algorithm”. In: *Proceedings of the National Academy of Sciences* 115.1 (2018), pp. 24–29.

[RT+09] Gerardo Rubino, Bruno Tuffin, et al. *Rare event simulation using Monte Carlo methods*. Vol. 73. Wiley Online Library, 2009.

[Run03] Wolfgang J Runggaldier. “Jump-diffusion models”. In: *Handbook of heavy tailed distributions in finance*. Elsevier, 2003, pp. 169–209.

[SG13] Laura Sacerdote and Maria Teresa Giraudo. “Stochastic integrate and fire models: a review on mathematical methods and their applications”. In: *Stochastic biomathematical models: with applications to neuronal modeling* (2013), pp. 99–148.

[Sai87] Yasumasa Saisho. “Stochastic differential equations for multi-dimensional domain with reflecting boundary”. In: *Probability Theory and Related Fields* 74.3 (1987), pp. 455–477.

[Slo93] Leszek Słomiński. “On existence, uniqueness and stability of solutions of multidimensional SDE’s with reflecting boundary conditions”. In: *Annales de l’IHP Probabilités et statistiques*. Vol. 29. 2. 1993, pp. 163–198.

[Str12] Daniel W Stroock. *An introduction to the theory of large deviations*. Springer Science & Business Media, 2012.

[Tan79] Hiroshi Tanaka. “Stochastic differential equations with reflecting”. In: *Stochastic Processes: Selected Papers of Hiroshi Tanaka* 9 (1979), p. 157.

[Tan03] Peter Tankov. *Financial modelling with jump processes*. CRC press, 2003.

[TK01] Yuichi Togashi and Kunihiko Kaneko. “Transitions induced by the discreteness of molecules in a small autocatalytic system”. In: *Physical review letters* 86.11 (2001), p. 2459.

[VEW12] Eric Vanden-Eijnden and Jonathan Weare. “Rare event simulation of small noise diffusions”. In: *Communications on Pure and Applied Mathematics* 65.12 (2012), pp. 1770–1803.

[WG03a] Amy R Ward and Peter W Glynn. “A diffusion approximation for a Markovian queue with reneging”. In: *Queueing systems* 43 (2003), pp. 103–128.

[WG03b] Amy R Ward and Peter W Glynn. “Properties of the reflected Ornstein–Uhlenbeck process”. In: *Queueing Systems* 44 (2003), pp. 109–123.

[WRVE04] E Weinan, Weiqing Ren, and Eric Vanden-Eijnden. “Minimum action method for the study of rare events”. In: *Communications on pure and applied mathematics* 57.5 (2004), pp. 637–656.

[Wil09] Darren J Wilkinson. “Stochastic modelling for quantitative description of heterogeneous biological systems”. In: *Nature Reviews Genetics* 10.2 (2009), pp. 122–133.

[Wil18] Darren J Wilkinson. *Stochastic modelling for systems biology*. CRC press, 2018.

Functional Central Limit Theorem for a Fast-Slow Dynamical System Driven by Symmetric and Multiplicative $\alpha-$Stable Noise

[App09] David Applebaum. *Lévy processes and stochastic calculus*. Cambridge university press, 2009.

[BYY17] Jianhai Bao, George Yin, and Chenggui Yuan. “Two-time-scale stochastic partial differential equations driven by α-stable noises: Averaging principles”. In: *Bernoulli* 23.1 (2017), pp. 645 –669. DOI: 10.3150/14-BEJ677. URL: https://doi.org/10.3150/14-BEJ677.

[BDS23] Solesne Bourguin, Thanh Dang, and Konstantinos Spiliopoulos. “Moderate deviation principle for multiscale systems driven by fractional Brownian motion”. In: *Journal of Theoretical Probability* 36.1 (2023), p. 1.

[Che+23] Peng Chen et al. "Approximation of the invariant measure of stable SDEs by an Euler–Maruyama scheme". In: *Stochastic Processes and their Applications* (2023).

[DWZ18] Siwei Duo, Hans Werner van Wyk, and Yanzhi Zhang. "A novel and accurate finite difference method for the fractional Laplacian and the fractional Poisson problem". In: *Journal of Computational Physics* 355 (2018), pp. 233–252.

[EK09] Stewart N Ethier and Thomas G Kurtz. *Markov processes: characterization and convergence*. John Wiley & Sons, 2009.

[FMJWA12] I Freidlin Mark, Szücs Joseph, and D Wentzell Alexander. "Random Perturbations of Dynamical Systems". In: *Grundlehren math. Wiss* 260 (2012).

[JS13] Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*. Vol. 288. Springer Science & Business Media, 2013.

[Kha68] RZ Khasminskii. "On an averaging principle for Itô stochastic differential equations". In: *Kibernetica* 4 (1968), pp. 206–279.

[KY05] RZ Khasminskii and G Yin. "Limit behavior of two-time-scale diffusions revisited". In: *Journal of Differential Equations* 212.1 (2005), pp. 85–113.

[LX22] Yun Li and Longjie Xie. "Functional law of large numbers and central limit theorem for slow-fast McKean-Vlasov equations". In: *arXiv preprint arXiv:2207.05949* (2022).

[Liu12] Di Liu. "Strong convergence rate of principle of averaging for jump-diffusion processes". In: *Frontiers of Mathematics in China* 7 (2012), pp. 305–320.

[Pap77] George Papanicolaou. "Martingale approach to some limit theorems". In: *Papers from the Duke Turbulence Conference, Duke Univ., Durham, NC, 1977*. 1977.

[PV01] È. Pardoux and A. Yu. Veretennikov. "On the Poisson Equation and Diffusion Approximation. I". In: *The Annals of Probability* 29.3 (2001), pp. 1061 –1085. DOI: 10.1214/aop/1015345596. URL: https://doi.org/10.1214/aop/1015345596.

[PV03] È. Pardoux and A. Yu. Veretennikov. "On Poisson equation and diffusion approximation 2". In: *The Annals of Probability* 31.3 (2003), pp. 1166 –1192. DOI: 10.1214/aop/1055425774. URL: https://doi.org/10.1214/aop/1055425774.

[PV05] È. Pardoux and A. Yu. Veretennikov. "On the Poisson equation and diffusion approximation 3". In: *The Annals of Probability* 33.3 (2005), pp. 1111 –1133. DOI: 10.1214/009117905000000062. URL: https://doi.org/10.1214/009117905000000062.

[PS08] Grigoris Pavliotis and Andrew Stuart. *Multiscale methods: averaging and homogenization*. Springer Science & Business Media, 2008.

[RSX19] Michael Röckner, Xiaobin Sun, and Longjie Xie. "Strong and weak convergence in the averaging principle for SDEs with Hölder coefficients". In: *arXiv preprint arXiv:1907.09256* (2019).

[RSX21] Michael Röckner, Xiaobin Sun, and Yingchao Xie. “Strong convergence order for slow–fast McKean–Vlasov stochastic differential equations”. In: *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* 57.1 (2021), pp. 547–576. DOI: `10.1214/20-AIHP1087`. URL: `https://doi.org/10.1214/20-AIHP1087`.

[Sam17] Gennady Samoradnitsky. *Stable non-Gaussian random processes: stochastic models with infinite variance*. Routledge, 2017.

[SXX22] Xiaobin Sun, Longjie Xie, and Yingchao Xie. “Strong and weak convergence rates for slow–fast stochastic differential equations driven by α-stable process”. In: *Bernoulli* 28.1 (2022), pp. 343–369.

[SX23] Xiaobin Sun and Yingchao Xie. “The Poisson Equation and Application to Multi-Scale SDEs with State-Dependent Switching”. In: *arXiv preprint arXiv:2304.04969* (2023).

[WR12] W Wang and AJ Roberts. “Average and deviation for slow–fast stochastic partial differential equations”. In: *Journal of Differential Equations* 253.5 (2012), pp. 1265–1286.

[Yan+22] Xiaoyu Yang et al. “The central limit theorem for slow–fast systems with Lévy noise”. In: *Applied Mathematics Letters* 128 (2022), p. 107897.

[YY04] G Yin and H Yang. “Two-time-scale jump-diffusion models with Markovian switching regimes”. In: *Stochastics and Stochastic Reports* 76.2 (2004), pp. 77–99.

[ZZ23] Xiaolong Zhang and Xicheng Zhang. “Ergodicity of supercritical SDEs driven by α-stable processes and heavy-tailed sampling”. In: *Bernoulli* 29.3 (2023), pp. 1933–1958.

[Zha+20] Yanjie Zhang et al. “Weak averaging principle for multiscale stochastic dynamical systems driven by α-stable processes”. In: *arXiv preprint arXiv:2007.08408* (2020).

On Convergence of Undiscounted Optimistic Policy Iteration without State Update Restrictions

[BT96] Dimitri Bertsekas and John N Tsitsiklis. *Neuro-dynamic programming*. Athena Scientific, 1996.

[BS19] Noam Brown and Tuomas Sandholm. “Superhuman AI for multiplayer poker”. In: *Science* 365.6456 (2019), pp. 885–890.

[Bro+20] Noam Brown et al. “Combining deep reinforcement learning and search for imperfect-information games”. In: *Advances in Neural Information Processing Systems* 33 (2020), pp. 17057–17069.

[Che18] Yuanlong Chen. “On the convergence of optimistic policy iteration for stochastic shortest path problem”. In: *arXiv preprint arXiv:1808.08763* (2018).

[EGM20] Yonathan Efroni, Mohammad Ghavamzadeh, and Shie Mannor. "Online planning with lookahead policies". In: *Advances in Neural Information Processing Systems* 33 (2020), pp. 14024–14033.

[Efr+18a] Yonathan Efroni et al. "Beyond the one-step greedy approach in reinforcement learning". In: *International Conference on Machine Learning*. PMLR. 2018, pp. 1387–1396.

[Efr+18b] Yonathan Efroni et al. "Multiple-step greedy policies in online and approximate reinforcement learning". In: *arXiv preprint arXiv:1805.07956* (2018).

[Efr+19] Yonathan Efroni et al. "How to combine tree-search methods in reinforcement learning". In: *Proceedings of the AAAI Conference on Artificial Intelligence*. Vol. 33. 01. 2019, pp. 3494–3501.

[Liu21] Jun Liu. "On the convergence of reinforcement learning with Monte Carlo Exploring Starts". In: *Automatica* 129 (2021), p. 109693.

[Sil+16] David Silver et al. "Mastering the game of Go with deep neural networks and tree search". In: *nature* 529.7587 (2016), pp. 484–489.

[Sil+17a] David Silver et al. "Mastering chess and shogi by self-play with a general reinforcement learning algorithm". In: *arXiv preprint arXiv:1712.01815* (2017).

[Sil+17b] David Silver et al. "Mastering the game of go without human knowledge". In: *nature* 550.7676 (2017), pp. 354–359.

[Sut99] Richard S Sutton. "Open theoretical questions in reinforcement learning". In: *European Conference on Computational Learning Theory*. Springer. 1999, pp. 11–17.

[SB18] Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

[Tsi02] John N Tsitsiklis. "On the convergence of optimistic policy iteration". In: *Journal of Machine Learning Research* 3.Jul (2002), pp. 59–72.

[Wan+20] Che Wang et al. "On the convergence of the monte carlo exploring starts algorithm for reinforcement learning". In: *arXiv preprint arXiv:2002.03585* (2020).

[WS22] Anna Winnicki and R Srikant. "Reinforcement learning with unbiased policy evaluation and linear function approximation". In: *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE. 2022, pp. 801–806.

[WS23] Anna Winnicki and R Srikant. "On The Convergence Of Policy Iteration-Based Reinforcement Learning With Monte Carlo Policy Evaluation". In: *International Conference on Artificial Intelligence and Statistics*. PMLR. 2023, pp. 9852–9878.

[Win+21] Anna Winnicki et al. "The Role of Lookahead and Approximate Policy Evaluation in Reinforcement Learning with Linear Value Function Approximation". In: *arXiv preprint arXiv:2109.13419* (2021).