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A double null analysis of asymptotic symmetries in 3D gravity

Internal Supervisor:
Prof. Silke Klemm

External Supervisors:
Prof. Glenn Barnich
Dott. Luca Ciambelli

Final thesis by:
Giovanni Boldi
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Alla perseveranza di chi fa ciò che ama

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Introduction

In the early sixties, relativists were investigating how to characterize gravitational radiation in the context of General Relativity. The lack of a rigorous derivation of a wave-like solution to Einstein equations was questioning the existence of gravitational waves itself. This was the situation when a series of papers by Trautman [1], Sachs [2],[3],[4] and Bondi [5], [6] with his collaborators van der Burg and Metzner, gave a rigorous approach to the study of gravitational radiation in General Relativity, characterizing the boundary conditions and the typical observables associated to it. Their works also showed that, as one moves infinitely far away from a source of gravitational radiation, the symmetry group that is found at infinity is not the Poincaré group (as one might expect), but instead an infinite-dimensional extension of it. In this so-called "BMS group", the translations are promoted to functions of the angular coordinates, enabling them to act independently on each point of the 2-sphere at infinity (also called "celestial sphere"). Therefore, two independent copies of the BMS group were found, one lying at future null infinity $\mathcal{I}^+$ and one lying at past null infinity $\mathcal{I}^-$. The implications of this result were fully appreciated only much later, with the works of Barnich [7] and Strominger [8]. The renewed interest for the work of Bondi, Sachs et al. came from the attempt to extend the successes of the AdS/CFT correspondence to a $\Lambda = 0$ scenario. A key observation in this direction was that the result obtained by Bondi, Sachs et al., once combined with appropriate matching conditions, relating the action of the BMS group at $\mathcal{I}^+$ and $\mathcal{I}^-$, constrains the gravitational scattering problem [8]. In other terms, one can argue that the BMS group, under some identification of its elements, is a symmetry of the S-matrix of gravity, both for the classical and quantum theory. These asymptotic symmetries revealed to be crucial from a QFT perspective, as their Ward identities were proven to be related to the soft graviton theorem, first formulated by Weinberg in [9]. The connection between asymptotic symmetries and the soft theorems goes beyond gravity itself and expresses, more in general, how the infrared properties of a gauge theory manifest at the symmetry level, and viceversa through the Ward identities. This underlying "infrared connection" enriched both General Relativity and QFT, as from known asymptotic symmetries new soft theorems were discovered and viceversa. Eventually, thanks to this "infrared connection", a soft graviton mode was discovered, whose insertion into the (tree-level) $\mathcal{S}$-matrix was shown to obey the Virasoro-Ward identities of a CFT_2 stress tensor. This fueled the idea that the holographic principle, in asymptotically flat gravity, could be formulated through a certain two-dimensional conformal field theory living on the celestial sphere. This approach then took the name of "celestial holography".

In this work, we revisit the results of Bondi and Sachs, proposing a novel approach to study the asymptotic symmetries of asymptotically flat 3D gravity. The motivation stands in the fact that the Bondi-Sachs formalism, due to its single null foliation, is not able to access both null infinities $\mathcal{I}^+$ and $\mathcal{I}^-$ in the same coordinate chart. This limitation prevents the Bondi-Sachs approach from studying ingoing and outgoing radiation at the same time. Our “double null” approach, on the other hand, exploiting a double null foliation, goes beyond this limitation, offering a new perspective on the two independent copies of the BMS group and how their charges are displayed. Our double null analysis leads to a solution space parametrized by three physical charges: A super-rotation charge $\mathcal{J}$, compatible with the one of the so-called “extended BMS group”, and two independent super-translation charges $\mathcal{M}_u$ and $\mathcal{M}_v$. These two null charges accomplish a chiral decomposition of the typical super-translation charge $\mathcal{M}$, which appears in the BMS group. Besides, it was shown by Rendall in [10], and by Hilditch, Kroon and Zhao in [11], that on a double null foliation the initial value problem is “well-posed”, i.e. given a set of initial data on two intersecting null leaves of the foliation, there is a unique solution to the Cauchy problem which depends continuously on the initial data. This suggests that a double null formulation could be more suitable for a Hamiltonian formulation.

The double null approach is here developed in the simpler setting of three dimensional gravity. Gravity in 4D possesses two kind of degrees of freedom, one related to its dynamical nature, and one related to its topological nature, independently from the number of dimensions considered. The former is what let us discuss gravitational radiation and the scattering problem of gravitons, while the latter is what allows the theory to host black holes and particle defects. 3D gravity, although lacking of any propagating degrees of freedom, still capture the topological degrees of freedom of the theory and it is indeed suitable for describing both black holes [12] and particle defects [13]. This make 3D gravity a simple toy model in which it is easier to develop such a double null formalism. Besides, despite its simplicity, 3D gravity is able to mimic the kinematical features of the Noether charges of 4D gravity.

The work is organized as follows. In the first chapter we introduce Cartan’s formulation of gravity. More specifically, we review the geometric objects on which this construction relies and define the tensors of Riemannian geometry in a torsion-less non-holonomic frame. We then focus on a null tetrad field, introducing the so-called Newman-Penrose formalism. In the second chapter we introduce the theory of surface charges, from a symplectic point of view. First, we review the relevant mathematical structures, namely the variational bicomplex and symplectic manifolds. Several results from these two geometric objects combine to give the “covariant phase space formalism”, which we introduce afterwards. Within this formalism, we review Noether’s theorems and interpret them from a symplectic perspective. This sheds light on what we should identify as a “Noether charge”, leading to the concept of asymptotic symmetries, which we introduce in the end. In the third and last chapter, we finally develop the novel double null approach. A new gauge, compatible with a double null foliation, is proposed. The equations of motion are then solved analytically, retrieving a solution space which includes the well-known phase space of the Bondi-Sachs approach. A set of boundary conditions is given and the residual gauge symmetries of the theory are found. The surface charges associated to them are then computed, revealing the asymptotic symmetry group of the theory. We compute the transformation laws of the fields parametrizing the solution space, under the action of the asymptotic symmetries. Finally, we compute the Lie algebra which the asymptotic symmetries give rise to, and check that the charges form a projective representation of the former, without finding any central extension.

Chapter 1

Tetrad formulation of gravity

The purpose of this first chapter is to review an alternative formulation of general relativity, known as Cartan formulation, which is given in terms of the "tetrad field" or "frame field". This approach casts General Relativity in a form which is similar to gauge theories, highlighting its similarities - and differences - with other fundamental interactions.

Firstly, we introduce Cartan's formalism from a geometric point of view. We review how a metric field can be described through a collection of forms and the geometric structure that this procedure gives rise to. This geometric structure is then compared to the principal bundle of gauge theories, to highlight the differences between gravity and the other fundamental interactions. We then define the tetrad field and recall how to manipulate these new objects. Afterwards, we focus on a null choice of the tetrad field, introducing the so-called Newman-Penrose (NP) formalism. This is a powerful framework, adapted to a null foliation of the spacetime, which will be particularly useful for our double null analysis. For the Cartan formulation of General relativity we follow [14] and [15], while for the discussion of the NP formalism we follow [16].

1.1 Geometric viewpoint

Between 1922 and 1925, in [17], [18], [19], [20], [21], [22] and [23], Cartan firstly introduced and developed the concepts of "torsion" and "non-holonomic spaces", inspired by the theory of elasticity which had been developed by the Cosserat brothers some years before[1]. Cartan formulated these new "espaces généralisées" - as he called them - in terms of 1-forms valued in some Lie algebra $\mathfrak{g}$. These works firstly expressed the idea that a generic geometric structure built on a manifold, such as a tensor field, could be encoded in a collection of differential forms, the "frames". In the context of General Relativity this collection of forms is the so-called tetrad field.

Making use of modern geometric tools, we may intuitively describe this procedure as follows:

- Consider an n-dimensional manifold $\mathcal{M}$ and a geometric structure, built upon it. Assume that this structure is characterized by a suitable space T. In the case of a non-degenerate metric structure for example, the latter would be the space of symmetric and non-degenerate $(0, 2)$ tensors. The first step to encode such a structure in a set of differential forms is defining a

[1] For a historical review see [24].

vector bundle E and build on top of it a "particular configuration" of the geometric structure of interest.

- Consider as fibers V of the vector bundle E copies of $\mathbb{R}^n$. Single out an element $\psi \in T$ and enhance E with it. Notice that, by defining ψ on each V, we are restricting the symmetry group of the bundle, which in general would be $GL(V)$, to the little group of $GL(V)$ preserving ψ, which we call G_ψ.
- The next step consists in "soldering" the fibers V of the vector bundle E to the tangent bundle of the manifold $\mathcal{M}$, i.e. establishing an isomorphism $e : T\mathcal{M} \to V$. This isomorphism is then called "soldering form". Intuitively, the soldering form makes the fibers of the vector bundle tangent to the spacetime, although being - a priori - two completely different spaces. Besides, locally the soldering e is a 1-form on $\mathcal{M}$ with values in V. The space of all possible soldering forms, at each point of $\mathcal{M}$, may be acted upon by the $GL(V)$ group. Defining the right action of $GL(V)$ on the soldering as $e \to g^{-1}e$, the space of soldering forms is endowed with the structure of a $GL(V)$ bundle , which we call "frame bundle" and denote as $F^*\mathcal{M}$.
- Thanks to the to structure ψ defined on V, the notion of "adapted frame" is introduced. An "adapted frame", often referred to also as "frame" or "tetrad field", is a soldering form which maps pointwise the geometric structure of $\mathcal{M}$ to the one given by ψ. In other terms, the soldering now, not only solders the tangent bundle of the manifold to the fibers V, but solders the geometric structures too. Then, the act of *adapting* the soldering to the geometric structure, reduces the structure group of $F^*\mathcal{M}$ from $GL(V)$ to G_ψ, which was the little group preserving the particular structure ψ. The original geometric structure on $\mathcal{M}$ arises now as the pullback of ψ through the frame field e.

Gravity versus gauge theories

All fundamental interactions manifest gauge invariance under some structure group G. This comes from the fact that the Lorentz-covariant fields, that we use to build Lorentz-invariant theories, often describe a redundant amount of degrees of freedom with respect to the physical ones of the system. These redundancies are nicely formulated in geometric terms through the notion of principal bundles. From a geometric point of view, gauge fields are sections of a connection defined on the principal bundle, while matter fields are sections of the vector bundle associated to the principal one. Indeed, given a principal bundle, it is always possible to construct an associated vector bundle, whose fibers are linear representation of the structure group G.

Nevertheless, gravity seems to be fundamentally different from the other interactions for several reasons, such as the non-compactness of its gauge group and the absence of a fixed background geometric structure.

The geometric formulation of a metric field, as the pullback of a vector bundle through tetrad forms, highlights both the similarities and the differences of gravity with other fundamental interactions, from the point of view of its principal bundle structure:

- Gravity, in analogy with gauge theories, has a principle frame bundle $F^*\mathcal{M}$ and a vector bundle E, associated to the metric field. Besides, the fibers V of the vector bundle E carry representations of the frame bundle's structure group G_ψ.

- However, although in an ordinary gauge theory the principal bundle hosts a connection, whose sections we call gauge fields, it is not soldered to the manifold. In general, there is no canonical isomorphism between the principal bundle of a gauge theory, or its associated vector bundle, and the tangent bundle of the manifold. The reason is that, although the fibers of the principal bundle, or its associated vector bundle, are linear representation of the structure group G, the fibers of the tangent bundle are not representations of such a group. Gravity is special because its principal bundle's structure group is Diff($\mathcal{M}$), of which the tangent spaces are linear representations. The frame bundle $F^*\mathcal{M}$ and the vector bundle E are thus soldered to the manifold. In the end, what distinguishes gravity from the other fundamental interactions, is the soldering of its principal bundle structure to the manifold, which is possible only because the structure group of gravity is Diff($\mathcal{M}$).

1.2 Cartan formulation of General Relativity

In the context of General Relativity - with signature (p, q) - the geometric structure of interest is the metric field g, while T is the space of symmetric and non-degenerate $(0, 2)$ tensors. The vector bundle E is defined by its local trivialization $\mathbb{R}^{p,q} \times \mathcal{M}$ and a projection map $\pi : E \to \mathcal{M}$. Moreover, ψ is chosen to be the flat metric η_{ab}. The fibers $V \equiv \pi^{-1}(x) = \mathbb{R}^{p,q}$, with $x \in \mathcal{M}$, have pointwise a scalar product $\langle \cdot, \cdot \rangle$, defined by the flat metric η. The vector bundle E is isomorphic to the tangent bundle of the manifold through the soldering forms. Locally, the soldering can be written as a collection of n linearly-independent 1-forms

$$e^a = e^a_\mu dx^\mu \,, \tag{1.1}$$

where $a = 1, ..., n$. The metric on $\mathcal{M}$ is then defined as the pullback of the metric on E, $\forall \;\; X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ indicates sections of TM,

$$g(X, Y) = \langle e(X), e(Y) \rangle = e^a_\mu e^b_\nu X^\mu Y^\nu \eta_{ab} \Longrightarrow g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{1.2}$$

Orthonormal tetrads can be locally transformed by a Lorentz transformation, being $G_\psi = O(p, q)$, as

$$e^a \to \Lambda^a{}_b e^b \,, \tag{1.3}$$

where $\Lambda^i{}_j \in O(p, q)$. Notice that we have two type of indices available. The latin indices refer to the vector bundle space, where we have the flat metric η; these indices are then raised and lowered using the flat metric. The greek indices, on the other hand, are used as spacetime indices and they are raised and lowered using the metric g. Furthermore, the tetrads, being an isomorphism between $T\mathcal{M}$ and V, let us convert spacetime indices to frame indices and viceversa.

1.2.1 Spin connection and commutation relations

The notion of parallel transport, defined by the affine connection ∇ of $T\mathcal{M}$, is transported to the vector bundle E through the tetrad field e. Although the metric is flat on the vector bundle E, the non-holonomity of the tetrad field produces curvature. The affine connection ∇, on the vector bundle E, accounts for this non-holonomic curvature with the so-called spin coefficients. These coefficients, in analogy to the Levi-Civita symbols of General Relativity, are defined as

$$\nabla e_a = \Gamma^b{}_a e_b \, . \tag{1.4}$$

Notice that the spin coefficients $\Gamma^b{}_a$ are here spacetime 1-forms, since the affine connection ∇ is the one defined on the tangent bundle $T\mathcal{M}$.

Concretely, a vector $v = v^\mu \partial_\mu \in \Gamma(TM)$ may be written as $v = v^a e_a \in \Gamma(TE)$ in the non-holonomic base $\{e_a\}$. Taking its covariant derivative in an arbitrary direction, as an element of the vector bundle E, gives

$$\nabla(v) = \nabla(v^a e_a) = d(v^a) e_a + v^a \nabla(e_a) = d(v^a) e_a + v^a \Gamma^b{}_a e_b = (dv^a + v^b \Gamma^a{}_b) e_a \, , \tag{1.5}$$

from which we read

$$\Rightarrow \nabla(v^a) = dv^a + \Gamma^a{}_b v^b \, . \tag{1.6}$$

Furthermore, if we explicitly write its components, (1.6) becomes

$$\begin{aligned} \nabla_\rho(v^a) &= \partial_\rho v^a + \Gamma^a{}_{b\rho} v^b \\ &= \partial_\rho(v^\mu e^a_\mu) + \Gamma^a{}_{b\rho} v^\mu e^b_\mu \\ &= \partial_\rho v^\mu e^a_\mu + v^\mu \partial_\rho e^a_\mu + \Gamma^a{}_{b\rho} v^\mu e^b_\mu \, , \end{aligned} \tag{1.7}$$

where in the last line we used the fact that, $v = v^\mu \partial_\mu = v^\mu e^a_\mu e_a \Rightarrow v^a = v^\mu e^a_\mu$. If we now apply the same covariant derivative to the vector written in the holonomic coordinate basis $\{\partial_\mu\}$, obtaining

$$\nabla_\rho(v) = (\partial_\rho v^\mu + \Gamma^\mu{}_{\nu\rho} v^\nu) \partial_\mu = (\partial_\rho v^\mu + \Gamma^\mu{}_{\nu\rho} v^\nu) e^a_\mu e_a \, , \tag{1.8}$$

and compare it with (1.5), we find the expression of the Levi-Civita connection, defined on the tangent bundle, in terms of the spin connection $\Gamma^a{}_b$, defined on the vector bundle,

$$\Gamma^\mu{}_{\nu\rho} = e^\mu_a \partial_\rho e^a_\nu + e^\mu_a \Gamma^a{}_{b\rho} e^b_\nu \, . \tag{1.9}$$

Notice that turning the spacetime indices of the Christoffel symbols to frame indices is not sufficient to retrieve the spin connection. On the contrary, an extra term accounting for the non-holonomity of the tetrad field appears. We may restate (1.9) as

$$\nabla_\rho e^a_\nu := -\Gamma^a{}_{b\rho} e^b_\nu = \partial_\rho e^a_\nu - e^a_\mu \Gamma^\mu{}_{\nu\rho}, \tag{1.10}$$

i.e. as the fact that the tetrad's components, when written in coordinate base $\{dx^\mu\}$, transform as a 1-form. It may also be useful to introduce a new "total" covariant derivative ∇^Γ, which parallely transports the tetrad field in both spaces of indices, i.e.

$$0 = \nabla^\Gamma_\mu e^a_\nu := \partial_\mu e^a_\nu + \Gamma^a{}_{b\mu} e^b_\nu - \Gamma^\sigma{}_{\mu\nu} e^a_\sigma \, . \tag{1.11}$$

Notice how, exactly like for the ordinary covariant derivative, every contravariant index is corrected with a plus sign and every covariant index with a minus sign. Moreover, if the index is a spacetime index, the correction is brought by the Christoffel symbols, while if the index is from the vector bundle space, the correction is brought by the spin connection.

We then impose the spin connection to satisfy the metricity condition $\nabla g = 0$,

$$0 = \nabla_\rho(g_{\mu\nu}) = \nabla_\rho(e^a_\mu e^b_\nu \eta_{ab}) = (\Gamma^a{}_{c\rho} e^c_\mu e^b_\nu + \Gamma^b{}_{c\rho} e^c_\nu e^a_\mu)\eta_{ab}. \tag{1.12}$$

Multiplying by $e^\mu_l e^\nu_m$, we get

$$\Rightarrow (\delta^c_l \Gamma^a{}_{c\rho}\delta^b_m + \delta^c_m \Gamma^b{}_{c\rho}\delta^a_l)\eta_{ab} = \Gamma^a{}_{l\rho}\eta_{am} + \Gamma^a{}_{m\rho}\eta_{la} = \Gamma_{ml\rho} + \Gamma_{lm\rho} = 0\,, \tag{1.13}$$

which, omitting the spacetime indices, translates to $\nabla^\Gamma \eta_{ab} = 0$. This expresses the fact that the 1-forms $\Gamma^a{}_b$ are valued in the algebra of $O(p,q)$. In the case of a Riemannian signature, i.e. $\eta_{ab} = \delta_{ab}$, $\Gamma^a{}_b$ would simply be valued in the space of $n \times n$ antisymmetric matrices.

Finally, we introduce the commutation relations $[e_a, e_b]$, which play an important role in the theory of non-holonomic frames. Let us start defining the structure constants as the coefficients $D^c{}_{ab}$ which appear in

$$[e_a, e_b] = D^c{}_{ab} e_c. \tag{1.14}$$

They are anti-symmetric in the indices a and b and they sum up to a total of 24 coefficients (as many as the spin coefficients). The structure constants can be expressed in terms of the spin coefficients as

$$\begin{aligned}
[e_a, e_b]f &= e^\mu_a(e^\nu_b f_{,\nu})_{,\mu} - e^\mu_b(e^\nu_a f_{,\nu})_{,\mu} \\
&= (e^\mu_a e^\nu_{b\,;\mu} - e^\mu_b e^\nu_{a;\mu}) f_{,\nu} \\
&= (-\Gamma_b{}^\nu{}_a + \Gamma_a{}^\nu{}_b) f_{,\nu} \\
&= (-\Gamma_b{}^c{}_a + \Gamma_a{}^c{}_b) e^\nu_c f_{,\nu} \\
&\Rightarrow D^c{}_{ab} = \Gamma^c{}_{ba} - \Gamma^c{}_{ab}.
\end{aligned} \tag{1.15}$$

In the end, by writing explicitly the constant structures in terms of the spin coefficients, we obtain 24 independent relations among the derivatives of the tetrads and the spin coefficients.

1.2.2 Torsion and curvature

Given a spin connection, we can always introduce a $\Lambda^2(\mathcal{M}) \otimes E$ valued object, called "torsion", as

$$T^a := de^a + \Gamma^a{}_b e^b\,, \tag{1.16}$$

First of all, the definition of torsion may be written in the tetrad base as

$$\Rightarrow T^c{}_{ab} = 2\Gamma^c_{[ab]} e_c - D^c{}_{ab}\,. \tag{1.17}$$

We see that torsion can appear from both the antisymmetric component of the spin connection and the constant structures of the non-holonomic base. In a holonomic base, such as a coordinate base $\{\partial_\mu\}$, we have $D^c{}_{ab} = 0$ and we find $T^\rho_{\mu\nu} = 2\Gamma^\rho{}_{[\mu\nu]}$.

Interestingly, there exists a unique non-holonomic torsion-less connection, given by

$$\Gamma^a{}_{b\mu} = e^{\rho a} e^{\sigma}{}_b (-D_{\mu\rho\sigma} + D_{\rho\sigma\mu} + D_{\sigma\mu\rho}), \qquad D_{\mu\rho\sigma} := e_{\mu a} \partial_{[\rho} e^a_{\sigma]} \,. \tag{1.18}$$

We can obtain (1.18), from (1.16),

$$\begin{aligned} T^c &= de^c + \Gamma^c{}_d \wedge e^d = 0 \\ &\Rightarrow \frac{1}{2} e^c_{[\sigma,\rho]} dx^\rho \wedge dx^\sigma + \frac{1}{2} \Gamma^c{}_{d[\rho} e^d_{\sigma]} dx^\rho \wedge dx^\sigma = 0 \\ &\Rightarrow e^c_{[\sigma,\rho]} + \Gamma^c{}_{d[\rho} e^d_{\sigma]} = 0 \\ &\Rightarrow -2 e^{\rho a} e^\sigma_b e_{\mu c} \left(e^c_{[\sigma,\rho]} + \Gamma^c{}_{d[\rho} e^d_{\sigma]} \right) = 0 \,, \end{aligned}$$

and sum the last line above with its cyclic permutations over the indices μ, ρ and σ. This gives

$$\begin{aligned} &\Rightarrow 2 e^{\rho a} e^\sigma_b \left(-e_{\mu c} \partial_{[\rho} e^c_{\sigma]} + e_{\rho c} \partial_{[\sigma} e^c_{\mu]} + e_{\sigma c} \partial_{[\mu} e^c_{\rho]} \right) = \\ &= 2 e^{\rho a} e^\sigma_b \left(-e_{\mu c} \Gamma^c{}_{d[\sigma} e^d_{\rho]} + e_{\rho c} \Gamma^c{}_{d[\mu} e^d_{\sigma]} + e_{\sigma c} \Gamma^c{}_{d[\rho} e^d_{\mu]} \right) \\ &= -e^{\rho a} e^\sigma_b e_{\mu c} (\Gamma^c{}_{d\sigma} e^d_\rho - \Gamma^c{}_{d\rho} e^d_\sigma) + e^{\rho a} e^\sigma_b e_{\rho c} (\Gamma^c{}_{d\mu} e^d_\sigma - \Gamma^c{}_{d\sigma} e^d_\mu) + e^{\rho a} e^\sigma_b e_{\sigma c} (\Gamma^c{}_{d\rho} e^d_\mu - \Gamma^c{}_{d\mu} e^d_\rho) \\ &= -e_{\mu c} e^\sigma_b \Gamma^{ca}{}_\sigma + e_{\mu c} e^{\rho a} \Gamma^c{}_{b\rho} + \Gamma^a{}_{b\mu} - e^d_\mu e^\sigma_b \Gamma^a{}_{d\sigma} + e^{\rho a} e^d_\mu \Gamma_{bd\rho} - \Gamma_b{}^a{}_\mu \\ &= \cancel{-e^d_\mu e^\sigma_b \Gamma_d{}^a{}_\sigma} + \cancel{e^d_\mu e^{\rho a} \Gamma_{db\rho}} + 2\Gamma^a{}_{b\mu} - \cancel{e^d_\mu e^\sigma_b \Gamma^a{}_{d\sigma}} + \cancel{e^{\rho a} e^d_\mu \Gamma_{bd\rho}} - \cancel{\Gamma_b{}^a}{}_\mu \,, \end{aligned} \tag{1.19}$$

from which (1.18) follows.

Remark: The choice of the metric on the vector bundle is completely arbitrary. In general, we could have chosen a metric $\bar{g}_{ab}(x)$, whose components are functions on the spacetime. However, such a choice gives rise to an affine connection which is more involved. Chosen a generic metric, the associated Levi-Civita connection is the unique affine connection on the vector bundle which satisfies the metricity condition $\nabla_a g_{bc} = 0$. Its general form can be shown to be

$$\Gamma_{abc} = \{abc\} + K_{abc} - r_{abc} \,, \tag{1.20}$$

where

$$\{abc\} = \frac{1}{2} (\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}), \tag{1.21}$$

$$K_{abc} = \frac{1}{2} (T_{bac} + T_{cab} - T_{abc}) = -K_{bac}, \tag{1.22}$$

$$r_{abc} = \frac{1}{2} (D_{bac} + D_{cab} - D_{abc}). \tag{1.23}$$

The first term is the ordinary Christoffel symbol of a holonomic frame. The second term is called "contortion" and accounts for the torsion contribution, and lastly, the third one accounts for the non-holonomicity of the tetrad field. Choosing a flat metric on the vector bundle first cancels both the Christoffel and the contortion terms, and secondly, reducing the structure group of the principal frame bundle from $GL(n, \mathbb{R}^{p,q})$ to $O(p,q)$, leads to enhanced symmetries in the Riemann tensor index structure.

Similarly to how we define torsion, we can define the curvature of the spin connection as a 2-form satisfying the Ricci identities, i.e.

$$2\nabla^{\Gamma}_{[\mu}\nabla^{\Gamma}_{\nu]}u^a := R_{\mu\nu}{}^{a}{}_{b}u^b. \tag{1.24}$$

It can be useful, for computational purposes, to convert the curvature tensors to spacetime scalars, by converting all their spacetime indices to frame indices,

$$R_{mn}{}^{a}{}_{b} := R_{\mu\nu}{}^{a}{}_{b}e^{\mu}_{m}e^{\nu}_{n}\,, \tag{1.25}$$

$$R_{ab} := R_{ma}{}^{m}{}_{b}\,, \tag{1.26}$$

$$R := R_{ab}\eta^{ab}\,, \tag{1.27}$$

which are simply the Riemann tensor, the Ricci tensor and the curvature scalar, with all their indices.

1.2.3 Ricci and Bianchi identities in a non-holonomic frame

The "Ricci identities" are a set of 36 linearly-independent equations which give the Riemann tensor's components in terms of the connection's coefficients, which in the metric approach are the Christoffel symbols, while here they are the spin coefficients. They can be derived directly from the definition of curvature given in (1.24), projecting all the indices on a frame base. The Ricci identities then take the form

$$R^{f}{}_{cab} = \partial_a\Gamma^{f}{}_{ca} - \partial_b\Gamma^{f}{}_{ca} + \Gamma^{f}{}_{da}\Gamma^{d}{}_{cb} - \Gamma^{f}{}_{db}\Gamma^{d}{}_{ca} - D^{d}{}_{ab}\Gamma^{f}{}_{cd}\,. \tag{1.28}$$

The Bianchi identities, on the other hand, are a set of 20 linearly-independent equations which relate different derivatives of the Riemann tensor's components. They derive from taking the external covariant derivative of the torsion and curvature forms, namely

$$dT + \Gamma\, T = R\, e \quad \text{and} \quad dR + [\Gamma, R] = 0\,. \tag{1.29}$$

We can explicitly write them as

$$R^{a}{}_{[bcd]} = 0 \quad \text{and} \quad \nabla_{[f}R^{a}{}_{|b|cd]} = 0\,, \tag{1.30}$$

where we have set $T = 0$. The last set of Bianchi identities in (1.30), once implemented with the first set of Bianchi identities, gives us 20 linearly-independent differential equations, where the partial derivatives of the Riemann tensor are expressed in terms of the spin coefficients Γ_{abc}. What we get in the end is

$$0 = R_{ab[cd;f]} = \frac{1}{6}\sum_{[cdf]}\left[R_{abcd,f} - \eta^{nm}\left(\Gamma_{naf}R_{mbcd} + \Gamma_{nbf}R_{amcd} + \Gamma_{ncf}R_{abmd} + \Gamma_{ndf}R_{abcm}\right)\right]\,. \tag{1.31}$$

1.2.4 Einstein-Cartan action

The Einstein-Hilbert action can be reformulated in terms of the tetrad field as the so-called Einstein-Cartan action, which in 4D takes the form

$$
S_{EC}[e,\Gamma] = \frac{1}{32\pi G}\int \mathcal{E}_{abcd}e^a e^b \left(R^{cd} - \frac{\Lambda}{6}e^c e^d\right) \,, \tag{1.32}
$$

where $\mathcal{E}_{abcd}$ is a completely antisymmetric tensor, the orientation is chosen such that $\mathcal{E}_{1234} = +1$, and whenever two forms are written next to each other, as if they were multiplied, a wedge product is implied. Besides, in the action S_{EC} the tetrad e and the spin connection Γ are considered independent variables. Notice that here R^{cd} is the 2-form of curvature defined in (1.24). The integrand then is a top form, integrated over the spacetime manifold $\mathcal{M}$. Notice also that the Einstein-Cartan action is polynomial in the fields and contain up to quartic terms.

Regarding Einstein-Cartan action we shall prove that:

- S_{EC} is equivalent to the Einstein-Hilbert action written in terms of the tetrad field. To see this, substitute inside (1.32) equation (1.18), which was the unique solution for a torsion free spin connection. We obtain

$$
\begin{aligned}
&\Rightarrow \frac{1}{32\pi G}\int \mathcal{E}_{abcd}e^a e^b \left(R^{cd}(\Gamma(e)) - \frac{\Lambda}{6}e^c e^d\right) \\
&= \frac{1}{16\pi G}\int \frac{1}{2}\mathcal{E}_{abcd}e^a_{[\mu}e^b_{\nu}\left(\frac{1}{2}R_{\rho\sigma]}{}^{cd} - \frac{\Lambda}{6}e^c_{\rho}e^d_{\sigma]}\right) dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma \\
&= \frac{1}{16\pi G}\int \frac{1}{4}\mathcal{E}_{abcd}\mathcal{E}^{\mu\nu\rho\sigma}e^a_{[\mu}e^b_{\nu}\left(R_{\rho\sigma]}{}^{cd} - \frac{\Lambda}{3}e^c_{\rho}e^d_{\sigma]}\right) d^4x \\
&= \frac{1}{16\pi G}\int \frac{e}{4}\mathcal{E}_{abcd}\mathcal{E}^{klmn}e^{[\mu}_k e^{\nu}_l e^{\rho}_m e^{\sigma]}_n e^a_{[\mu}e^b_{\nu}\left(R_{\rho\sigma]}{}^{cd} - \frac{\Lambda}{3}e^c_{\rho}e^d_{\sigma]}\right) d^4x \\
&= \frac{1}{16\pi G}\int \frac{e}{4}\mathcal{E}_{abcd}\mathcal{E}^{abmn}e^{[\rho}_m e^{\sigma]}_n\left(R_{[\rho\sigma]}{}^{cd} - \frac{\Lambda}{3}e^c_{[\rho}e^d_{\sigma]}\right) d^4x \\
&= \frac{1}{16\pi G}\int \frac{\sqrt{-g}}{4}\left(2(\delta^m_c\delta^n_d - \delta^n_c\delta^m_d)e^{[\rho}_m e^{\sigma]}_n R_{[\rho\sigma]}{}^{cd} - \frac{24\Lambda}{3}\right) d^4x \\
&= \frac{1}{16\pi G}\int \frac{\sqrt{-g}}{4}\left(4R_{mn}{}^{mn} - 8\Lambda\right) d^4x \\
&= \boxed{\frac{1}{16\pi G}\int \sqrt{-g}\,(R - 2\Lambda)\, d^4x} \,.
\end{aligned} \tag{1.33}
$$

- Varying the action with respect to the spin connection we get the torsion free condition. This means that, when we are on-shell to the equations of motions, the spin connection Γ is an auxiliary field. Computing the variation explicitly we obtain

$$
\begin{aligned}
\frac{\delta S_{EC}[e,\Gamma]}{\delta\Gamma^{kl}} &= \frac{\delta}{\delta\Gamma^{kl}}\int \mathcal{E}_{abcd}e^a e^b\left(R^{cd}(\Gamma) - \frac{\Lambda}{6}e^c e^d\right) \\
&= \frac{\delta}{\delta\Gamma^{kl}}\int \mathcal{E}_{abcd}e^a e^b\left(d\Gamma^{cd} + \Gamma^c{}_n\Gamma^{nd} - \frac{\Lambda}{6}e^c e^d\right) \\
&= \int \mathcal{E}_{abcd}\frac{\delta}{\delta\Gamma^{kl}}e^a e^b\left(d\Gamma^{cd} + \Gamma^{cm}\Gamma^{nd}\eta_{mn} - \frac{\Lambda}{6}e^c e^d\right) \\
&= \int \mathcal{E}_{abcd}\frac{\delta}{\delta\Gamma^{kl}}e^a_{[\mu}e^b_{\nu}\partial_\rho\Gamma^{cd}{}_{\sigma]}\mathcal{E}^{\mu\nu\rho\sigma}d^4x + \int \mathcal{E}_{abcd}e^a e^b\left(\delta^{cm}_{kl}\Gamma^{nd}\eta_{mn} + \delta^{nd}_{kl}\Gamma^{cm}\eta_{mn}\right) \\
&= \int \mathcal{E}_{abcd}\frac{\delta}{\delta\Gamma^{kl}}\partial_{[\rho}(e^a_\mu e^b_\nu)\Gamma^{cd}{}_{\sigma]}\mathcal{E}^{\mu\nu\rho\sigma}d^4x + \cancel{b.t.} - \int \mathcal{E}_{abdk}e^a e^b\left(\Gamma^{nd}\eta_{ln} - \Gamma^{dm}\eta_{ml}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \int \mathcal{E}_{abcd} \frac{\delta}{\delta \Gamma^{kl}} 2d(e^a e^b)\Gamma^{cd} + \int \mathcal{E}_{abdk} 2 e^a e^b \Gamma^d{}_l \\
&= \int \mathcal{E}_{abcd} 2\nabla(e^a e^b) = 0 \\
&\Rightarrow \nabla(e^a e^b) = 0 \\
&\Rightarrow \boxed{\nabla(e^a) = 0}\,.
\end{aligned} \tag{1.34}
$$

- Varying the action with respect to the tetrad field we find Einstein's field equations, written in terms of the tetrad field, i.e.

$$
\begin{aligned}
\frac{\delta S_{EC}[e,\Gamma]}{\delta e^k} &= \frac{\delta}{\delta e^k} \int \mathcal{E}_{abcd} e^a e^b \left[R^{cd}(\Gamma) - \frac{\Lambda}{6} e^c e^d \right] \\
&= \frac{\delta}{\delta e^k_\xi} \int \mathcal{E}_{abcd} e^a_{[\mu} e^b_\nu \left[\frac{1}{2} R^{cd}_{\rho\sigma]}(\Gamma) - \frac{\Lambda}{6} e^c_\rho e^d_{\sigma]} \right] \mathcal{E}^{\mu\nu\rho\sigma} d^4x \\
&= \int \mathcal{E}_{abcd} \left(\delta^a_k \delta^\xi_\mu e^b_\nu [...] + \delta^b_k \delta^\xi_\nu e^a_\mu [...] - \frac{\Lambda}{6} e^a_{[\mu} e^b_\nu (\delta^c_k \delta^\xi_\rho e^d_{\sigma]} + \delta^d_k \delta^\xi_\sigma e^c_{\rho]}) \right) \mathcal{E}^{\mu\nu\rho\sigma} d^4x \\
&= \int \left(2\mathcal{E}_{akcd} e^a_{[\mu} \left[\frac{1}{2} R^{cd}_{\rho\sigma]} - \frac{\Lambda}{6} e^c_\rho e^d_{\sigma]} \right] \mathcal{E}^{\mu\xi\rho\sigma} - \frac{\Lambda}{3} e^a_{[\mu} e^b_\nu e^c_{\rho]} \mathcal{E}_{abck} \mathcal{E}^{\mu\nu\rho\xi} \right) d^4x \\
&= \int 2\mathcal{E}_{akcd} \left(e^a_{[\mu} \left[\frac{1}{2} R^{cd}_{\rho\sigma]} \right] - \frac{\Lambda}{3} e^a_{[\mu} e^c_\rho e^d_{\sigma]} \right) \mathcal{E}^{\mu\xi\rho\sigma} d^4x \\
&= \int 2\mathcal{E}_{akcd} \left(e^a R^{cd} - \frac{\Lambda}{3} e^a e^c e^d \right) = 0 \\
&\Rightarrow \boxed{\mathcal{E}_{kabc} e^a R^{bc} = \mathcal{E}_{kabc} \frac{\Lambda}{3} e^a e^b e^c}\,.
\end{aligned} \tag{1.35}
$$

1.3 The Newman-Penrose formalism

The Newman-Penrose formalism (NP), which was first introduced by Newman and Penrose in [25] (1962), gives us a powerful framework to deal with the null structure of the spacetime. In the case where the spacetime gets foliated by a family of null hypersurfaces, the NP formalism gives great control on how the leaves of the foliation are smoothly deformed along the null directions. Since our final goal is to foliate the spacetime with a double null foliation, the NP formalism represents a powerful tool to develop such a "double null formalism".

It fundamentally consists in taking as tetrad components four null vectors, which do not form an orthonormal tetrad. This may be accomplished by taking complex linear combinations of the timelike and spacelike components of an orthonormal tetrad, such as

$$
\begin{aligned}
l^\mu &= \frac{1}{\sqrt{2}}(e^\mu_1 - e^\mu_2), \qquad n^\mu = \frac{1}{\sqrt{2}}(e^\mu_1 + e^\mu_2), \\
m^\mu &= \frac{1}{\sqrt{2}}(e^\mu_3 + i e^\mu_4), \quad \bar{m}^\mu = \frac{1}{\sqrt{2}}(e^\mu_3 - i e^\mu_4).
\end{aligned} \tag{1.36}
$$

We denote the tetrad frame components and their directional derivatives as

$$
e_a = (l, n, m, \bar{m}) \quad \text{and} \quad \nabla_a = \left(D, \Delta, \delta, \bar{\delta} \right). \tag{1.37}
$$

The flat metric on the vector bundle E is taken off diagonal both in the time-radial sector and in the angular sector:

$$\eta^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \eta_{ab} \,. \tag{1.38}$$

The structure of η_{ab} then implies

$$e^a = \eta^{ab} e_{b\mu} dx^\mu = (n, l, -\bar{m}, -m), \tag{1.39}$$

and that all scalar products between the tetrad components are zero, with the exception of

$$l \cdot n = l^\mu n_\nu = 1, \quad m \cdot \bar{m} = m^\mu \bar{m}_\mu = -1. \tag{1.40}$$

The metric, according to (1.2), is given by

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \,, \tag{1.41}$$

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu \,, \tag{1.42}$$

$$\delta^\mu_\nu = l^\mu n_\nu + n^\mu l_\nu - m^\mu \bar{m}_\nu - \bar{m}^\mu m_\nu \,. \tag{1.43}$$

Similarly, the line element takes the form

$$ds^2 = e^a_\mu e^b_\nu \eta_{ab} \, dx^\mu \wedge dx^\nu = 2l \cdot n - 2m \cdot \bar{m} \,. \tag{1.44}$$

1.3.1 Spin coefficients

The 24 independent spin coefficients are encoded inside 12 complex scalars. To do so, all the indices of the spin coefficients are lowered and turned to frame indices,

$$\begin{aligned}
\Gamma^a{}_{b\mu} &= e^{\rho a} e^\sigma_b \left(-e_{\mu d} \partial_{[\rho} e^d_{\sigma]} + e_{\rho d} \partial_{[\sigma} e^d_{\mu]} + e_{\sigma d} \partial_{[\mu} e^d_{\rho]} \right) \\
\Rightarrow \Gamma_{abc} &= e^\mu_c e^\rho{}_a e^\sigma_b \left(-e_{\mu d} (\partial_\rho e^d_\sigma - \partial_\sigma e^d_\rho) + e_{\rho d} (\partial_\sigma e^d_\mu - \partial_\mu e^d_\sigma) + e_{\sigma d} (\partial_\mu e^d_\rho - \partial_\rho e^d_\mu) \right) \\
&= -e^\sigma_b \partial_a e_{\sigma c} + e^\rho_a \partial_b e_{\rho c} + e^\mu_c \partial_b e_{\mu a} - e^\sigma_b \partial_c e_{\sigma a} + e^\rho_a \partial_c e_{\rho b} - e^\mu_c \partial_a e_{\mu b} \\
&\Rightarrow \boxed{\Gamma_{abc} = D_{[ab]c} - D_{[bc]a} + D_{[ac]b}} \,.
\end{aligned} \tag{1.45}$$

Then, the 12 complex coefficients are defined as

$$\begin{aligned}
\kappa &= \Gamma_{311}, \quad & \varepsilon &= \frac{1}{2}(\Gamma_{211} - \Gamma_{431}), \quad & \pi &= -\Gamma_{421}, \\
\tau &= \Gamma_{312}, \quad & \gamma &= \frac{1}{2}(\Gamma_{212} - \Gamma_{432}), \quad & \nu &= -\Gamma_{422}, \\
\sigma &= \Gamma_{313}, \quad & \beta &= \frac{1}{2}(\Gamma_{213} - \Gamma_{433}), \quad & \mu &= -\Gamma_{423}, \\
\rho &= \Gamma_{314}, \quad & \alpha &= \frac{1}{2}(\Gamma_{214} - \Gamma_{434}), \quad & \lambda &= -\Gamma_{424},
\end{aligned} \tag{1.46}$$

where the middle column may be equivalently written as

$$\begin{aligned}
\Gamma_{211} &= \varepsilon + \bar{\varepsilon}, \quad \Gamma_{431} = \bar{\varepsilon} - \varepsilon, \\
\Gamma_{212} &= \gamma + \bar{\gamma}, \quad \Gamma_{432} = \bar{\gamma} - \gamma, \\
\Gamma_{213} &= \beta + \bar{\alpha}, \quad \Gamma_{433} = \bar{\alpha} - \beta, \\
\Gamma_{214} &= \alpha + \bar{\beta}, \quad \Gamma_{434} = \bar{\beta} - \alpha.
\end{aligned} \tag{1.47}$$

From (1.45), we can write the complex coefficients in terms of the directional derivatives defined from the tetrad components. For example,

$$\kappa = \Gamma_{311} = -D_{[31]1} + D_{[31]1} + \cancel{D_{[11]3}} = 2D_{[31]1} = m^\mu D(l_\mu) - l^\mu \delta(l_\mu). \tag{1.48}$$

Repeating the computation for all the other spin coefficients, we obtain

$$\begin{aligned}
\tau &= \frac{1}{2}\left(m^\mu D(n_\mu) - l^\mu \delta(n_\mu) + m^\mu \Delta(l_\mu) - n^\mu \delta(l_\mu) - l^\mu \Delta(m_\mu) + n^\mu D(m_\mu)\right) \\
\sigma &= m^\mu D(m_\mu) - l^\mu \delta(m_\mu) \\
\rho &= \frac{1}{2}\left(m^\mu D(\bar{m}_\mu) - l^\mu \delta(\bar{m}_\mu) + m^\mu \bar{\Delta}(l_\mu) - \bar{m}^\mu \delta(l_\mu) - l^\mu \bar{\Delta}(m_\mu) + \bar{m}^\mu D(m_\mu)\right) \\
\varepsilon + \bar{\varepsilon} &= n^\mu D(l_\mu) - l^\mu \Delta(l_\mu) \\
\bar{\varepsilon} - \varepsilon &= \frac{1}{2}\left(\bar{m}^\mu \delta(l_\mu) - m^\mu \bar{\delta}(l_\mu) + \bar{m}^\mu D(m_\mu) - l^\mu \bar{\delta}(m_\mu) - m^\mu D(\bar{m}_\mu) + l^\mu \delta(\bar{m}_\mu)\right) \\
\gamma + \bar{\gamma} &= n^\mu D(n_\mu) - l^\mu \Delta(n_\mu) \\
\bar{\gamma} - \gamma &= \frac{1}{2}\left(\bar{m}^\mu \delta(n_\mu) - m^\mu \bar{\delta}(n_\mu) + \bar{m}^\mu \Delta(m_\mu) - n^\mu \bar{\delta}(m_\mu) - m^\mu \Delta(\bar{m}_\mu) + n^\mu \delta(\bar{m}_\mu)\right) \\
\beta + \bar{\alpha} &= \frac{1}{2}\left(n^\mu D(m_\mu) - l^\mu \Delta(m_\mu) + n^\mu \delta(l_\mu) - m^\mu \Delta(l_\mu) - l^\mu \delta(n_\mu) + m^\mu D(n_\mu)\right) \\
\bar{\alpha} - \beta &= \bar{m}^\mu \delta(m_\mu) - m^\mu \bar{\delta}(m_\mu) \\
\alpha + \bar{\beta} &= \frac{1}{2}\left(n^\mu D(\bar{m}_\mu) - l^\mu \Delta(\bar{m}_\mu) + n^\mu \bar{\delta}(l_\mu) - \bar{m}^\mu \Delta(l_\mu) - l^\mu \bar{\delta}(n_\mu) + \bar{m}^\mu D(n_\mu)\right) \\
\bar{\beta} - \alpha &= \bar{m}^\mu \delta(\bar{m}_\mu) - m^\mu \bar{\delta}(\bar{m}_\mu) \\
-\pi &= \frac{1}{2}\left(\bar{m}^\mu \Delta(l_\mu) - n^\mu \bar{\delta}(l_\mu) + \bar{m}^\mu D(n_\mu) - l^\mu \bar{\delta}(n_\mu) - n^\mu D(\bar{m}_\mu) + l^\mu \Delta(\bar{m}_\mu)\right) \\
-\nu &= \bar{m}^\mu \Delta(n_\mu) - n^\mu \bar{\delta}(n_\mu) \\
-\mu &= \frac{1}{2}\left(\bar{m}^\mu \Delta(m_\mu) - n^\mu \bar{\delta}(m_\mu) + \bar{m}^\mu \delta(n_\mu) - m^\mu \bar{\delta}(n_\mu) - n^\mu \delta(\bar{m}_\mu) + m^\mu \Delta(\bar{m}_\mu)\right) \\
-\lambda &= \bar{m}^\mu \Delta(\bar{m}_\mu) - n^\mu \bar{\delta}(\bar{m}_\mu).
\end{aligned} \tag{1.49}$$

1.3.2 Weyl, Ricci and Riemann tensor

In the same way as we encoded the independent components of the spin connection in a set of complex scalars, we define two sets of complex scalars to encode the 20 independent components of the Riemann tensor. The Riemann tensor can indeed be decomposed in its trace, namely the Ricci tensor, and its traceless part, namely the Weyl tensor. Each of these components carries 10 of the 20 independent components of the Riemann tensor. The Weyl tensor encodes the propagating degrees of freedom of gravity. The Ricci tensor, thanks to Einstein equations, encodes the effects of matter on the spacetime curvature. The Riemann then is completely determined by the Weyl, the Ricci and the scalar of curvature and its components satisfy

$$
\begin{aligned}
&R_{1212} = W_{1212} + R_{12} - \frac{1}{6}R, && R_{1324} = W_{1324} + \frac{1}{12}R, && R_{1234} = W_{1234}, \\
&R_{3434} = W_{3434} - R_{34} - \frac{1}{6}R, && R_{1313} = W_{1313}, && R_{2323} = W_{2323}, \\
&R_{1314} = \frac{1}{2}R_{11}, && R_{2324} = \frac{1}{2}R_{22}, && R_{3132} = -\frac{1}{2}R_{33}, \\
&R_{1213} = W_{1213} + \frac{1}{2}R_{13}, && R_{1334} = W_{1334} + \frac{1}{2}R_{13}, && \\
&R_{1223} = W_{1223} - \frac{1}{2}R_{23}, && R_{2334} = W_{2334} + \frac{1}{2}R_{23}, &&
\end{aligned} \tag{1.50}
$$

together with relations obtained by switching $3 \leftrightarrow 4$.

Firstly, let us recall the definition of the Weyl tensor in tetrad components, which is

$$
W_{abcd} = R_{abcd} + \frac{1}{2}(\eta_{ac}R_{bd} + \eta_{bd}R_{ac} - \eta_{bc}R_{ad} - \eta_{ad}R_{bc}) - \frac{1}{6}(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad})R\,. \tag{1.51}
$$

Notice that the Weyl tensor shares the same symmetries in the indices as the Riemann tensor does. The 10 independent components of the Weyl tensor are then written in terms of 5 complex scalars, $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, which are defined as

$$
\begin{aligned}
\Psi_0 &= -W_{1313} = -W_{\mu\nu\rho\sigma}\, l^\mu\, m^\nu\, l^\rho\, m^\sigma\,, \\
\Psi_1 &= -W_{1213} = -W_{\mu\nu\rho\sigma}\, l^\mu\, n^\nu\, l^\rho\, m^\sigma\,, \\
\Psi_2 &= -W_{1342} = -W_{\mu\nu\rho\sigma}\, l^\mu\, m^\nu\, \bar{m}^\rho\, n^\sigma\,, \\
\Psi_3 &= -W_{1242} = -W_{\mu\nu\rho\sigma}\, n^\mu\, l^\nu\, n^\rho\, \bar{m}^\sigma\,, \\
\Psi_4 &= -W_{2424} = -W_{\mu\nu\rho\sigma}\, n^\mu\, \bar{m}^\nu\, n^\rho\, \bar{m}^\sigma\,.
\end{aligned} \tag{1.52}
$$

On the other hand, the Ricci tensor in a tetrad base was defined in (1.26). Its 10 independent components are written in terms of 3 complex scalars $\{\Phi_{01} = \bar{\Phi}_{10}, \Phi_{02} = \bar{\Phi}_{20}, \Phi_{03} = \bar{\Phi}_{30}\}$ and 4 real ones $\{\Phi_{00}, \Phi_{11}, \Phi_{22}, \tilde{\Lambda}\}$, as

$$
\begin{aligned}
&\Phi_{00} := -\frac{1}{2}R_{11} = -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu, && \Phi_{11} := -\frac{1}{4}\left(R_{12} + R_{34}\right) = -\frac{1}{4}R_{\mu\nu}\left(l^\mu n^\nu + m^\mu \bar{m}^\nu\right)\,, \\
&\Phi_{22} := -\frac{1}{2}R_{22} = -\frac{1}{2}R_{\mu\nu}n^\mu n^\nu, && \tilde{\Lambda} := \frac{1}{24}R = \frac{1}{12}\left(R_{12} - R_{34}\right) = -\frac{1}{12}R_{\mu\nu}\left(l^\mu n^\nu - m^\mu \bar{m}^\nu\right)\,, \\
&\Phi_{01} := -\frac{1}{2}R_{13} = -\frac{1}{2}R_{\mu\nu}l^\mu m^\nu, && \Phi_{10} := -\frac{1}{2}R_{14} = -\frac{1}{2}R_{\mu\nu}l^\mu \bar{m}^\nu = \bar{\Phi}_{01}\,, \\
&\Phi_{02} := -\frac{1}{2}R_{33} = -\frac{1}{2}R_{\mu\nu}m^\mu m^\nu, && \Phi_{20} := -\frac{1}{2}R_{44} = -\frac{1}{2}R_{\mu\nu}\bar{m}^\mu \bar{m}^\nu = \bar{\Phi}_{02}\,, \\
&\Phi_{12} := -\frac{1}{2}R_{23} = -\frac{1}{2}R_{\mu\nu}n^\mu m^\nu, && \Phi_{21} := -\frac{1}{2}R_{24} = -\frac{1}{2}R_{\mu\nu}n^\nu \bar{m}^\nu = \bar{\Phi}_{12}\,.
\end{aligned} \tag{1.53}
$$

It is now easy to write down the Riemann components in the terms of the complex NP scalars we just defined, they are

$$
\begin{aligned}
&R_{1212} = -\left(\Psi_2 + \bar{\Psi}_2\right) + 2(4\tilde{\Lambda} - \Phi_{11}), && R_{1324} = \Psi_2 + 2\tilde{\Lambda}, && R_{1234} = \Psi_2 - \bar{\Psi}_2,\\
&R_{3434} = -\left(\Psi_2 + \bar{\Psi}_2\right) - 2(4\tilde{\Lambda} + \Phi_{11}), && R_{1313} = -\Psi_0, && R_{2323} = -\bar{\Psi}_4,\\
&R_{1314} = -\Phi_{00}, && R_{2324} = -\Phi_{22}, && R_{3132} = \Phi_{02},\\
&R_{1213} = -\Psi_1 - \Phi_{01}, && R_{1334} = \Psi_1 - \Phi_{01}, &&\\
&R_{1223} = \bar{\Psi}_3 + \Phi_{12}, && R_{2334} = \bar{\Psi}_3 - \Phi_{12}. &&
\end{aligned}
\tag{1.54}
$$

We continue the review, from now on, focusing on the case of vacuum and asymptotically flat spacetimes, which are going to be the objects of our double analysis later on. We thus set in the following $\Lambda = 0$ and $R_{ab} = 0$.

1.3.3 Commutation relations in the NP formalism

We know from (1.15), that the commutation relations of the tetrad field may be written, in terms of the spin coefficients, as

$$
[e_a, e_b] = D^c{}_{ab} e_c = (\Gamma_{cba} - \Gamma_{cab}) e^c \,. \tag{1.55}
$$

By writing explicitly each component of the commutation relations, as for example

$$
\begin{aligned}
[\Delta, D] = [n, l] = [e_2, e_1] &= (\Gamma_{c12} - \Gamma_{c21}) e^c\\
&= -\Gamma_{121} e^1 + \Gamma_{212} e^2 + (\Gamma_{312} - \Gamma_{321}) e^3 + (\Gamma_{412} - \Gamma_{421}) e^4\\
&= -\Gamma_{121} \Delta + \Gamma D - (\Gamma_{312} - \Gamma_{321}) \bar{\delta} - (\Gamma_{412} - \Gamma_{421}) \delta \,,
\end{aligned}
$$

we obtain the following set of commutation relations,

$$
\begin{aligned}
\Delta D - D\Delta &= (\gamma + \bar{\gamma}) D + (\varepsilon + \bar{\varepsilon}) \Delta - (\bar{\tau} + \pi) \delta - (\tau + \bar{\pi}) \bar{\delta}\\
\delta D - D\delta &= (\bar{\alpha} + \beta - \bar{\pi}) D + \kappa \Delta - (\bar{\rho} + \varepsilon - \bar{\varepsilon}) \delta - \sigma \bar{\delta}\\
\delta \Delta - \Delta \delta &= -\bar{\nu} D + (\tau - \bar{\alpha} - \beta) \Delta + (\mu - \gamma + \bar{\gamma}) \delta + \bar{\lambda} \bar{\delta}\\
\bar{\delta} \delta - \delta \bar{\delta} &= (\bar{\mu} - \mu) D + (\bar{\rho} - \rho) \Delta + (\alpha - \bar{\beta}) \delta + (\beta - \bar{\alpha}) \bar{\delta}
\end{aligned}
\tag{1.56}
$$

This set of equations, together with the Ricci identities, which we review in the following section, are the equations of motion for asymptotically flat General Relativity in the vacuum.

1.3.4 Ricci and Bianchi identities in NP formalism

Starting from (1.50) and exploiting (1.28), the Ricci identities can be written in terms of the spin coefficients. For example,

$$
\begin{aligned}
-\Psi_0 = R_{1313} = \Gamma_{133,1} &- \Gamma_{131,3}\\
&+ \Gamma_{133} \left(\Gamma_{121} + \Gamma_{431} - \Gamma_{413} + \Gamma_{431} + \Gamma_{134}\right)\\
&- \Gamma_{131} \left(\Gamma_{433} + \Gamma_{123} - \Gamma_{213} + \Gamma_{231} + \Gamma_{132}\right)
\end{aligned}
\tag{1.57}
$$

$$
\Rightarrow D\sigma - \delta\kappa = \sigma \left(3\varepsilon - \bar{\varepsilon} + \rho + \bar{\rho}\right) + \kappa \left(\bar{\pi} - \tau - 3\beta - \bar{\alpha}\right) + \Psi_0 \,. \tag{1.58}
$$

Repeating this process for every component of the Riemann tensor, we get a set of 18 complex equations, which are

$$
\begin{aligned}
D\sigma - \delta\kappa &= \sigma\left(3\varepsilon - \bar{\varepsilon} + \rho + \bar{\rho}\right) + \kappa\left(\bar{\pi} - \tau - 3\beta - \bar{\alpha}\right) + \Psi_0 \\
D\rho - \bar{\delta}\kappa &= \left(\rho^2 + \sigma\bar{\sigma}\right) + \rho\left(\varepsilon + \bar{\varepsilon}\right) - \bar{\kappa}\tau - \kappa\left(3\alpha + \bar{\beta} - \pi\right) + \Phi_{00} \\
D\sigma - \delta\kappa &= \sigma\left(\rho + \bar{\rho} + 3\varepsilon - \bar{\varepsilon}\right) - \kappa\left(\tau - \bar{\pi} + \bar{\alpha} + 3\beta\right) + \Psi_0 \\
D\tau - \Delta\kappa &= \rho\left(\tau + \bar{\pi}\right) + \sigma\left(\bar{\tau} + \pi\right) + \tau\left(\varepsilon - \bar{\varepsilon}\right) - \kappa\left(3\gamma + \bar{\gamma}\right) + \Phi_{01} + \Psi_1 \\
D\alpha - \bar{\delta}\varepsilon &= \alpha\left(\rho + \bar{\varepsilon} - 2\varepsilon\right) + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi\left(\varepsilon + \rho\right) + \Phi_{10} \\
D\beta - \delta\varepsilon &= \sigma\left(\alpha + \pi\right) + \beta\left(\bar{\rho} - \bar{\varepsilon}\right) - \kappa\left(\mu + \gamma\right) - \varepsilon\left(\bar{\alpha} - \bar{\pi}\right) + \Psi_1 \\
D\gamma - \Delta\varepsilon &= \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\varepsilon + \bar{\varepsilon}) - \varepsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda \\
D\lambda - \bar{\delta}\pi &= (\rho\lambda + \bar{\sigma}\mu) + \pi(\pi + \alpha - \beta) - \nu\bar{\kappa} - \lambda(3\varepsilon - \bar{\varepsilon}) + \Phi_{20} \\
D\mu - \delta\pi &= (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\varepsilon + \bar{\varepsilon}) - \nu\kappa + \Psi_2 + 2\Lambda \\
D\nu - \Delta\pi &= \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\varepsilon + \bar{\varepsilon}) + \Psi_3 + \Phi_{21} \\
\Delta\lambda - \bar{\delta}\nu &= -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4 \\
\delta\rho - \bar{\delta}\sigma &= \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01} \\
\delta\alpha - \bar{\delta}\beta &= (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \varepsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda \\
\delta\lambda - \bar{\delta}\mu &= \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21} \\
\delta\nu - \Delta\mu &= (\mu^2 + \lambda\bar{\lambda}) + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22} \\
\delta\gamma - \Delta\beta &= \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \varepsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12} \\
\delta\tau - \Delta\sigma &= (\mu\sigma + \bar{\lambda}\rho) + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \kappa\bar{\nu} + \Phi_{02} \\
\Delta\rho - \bar{\delta}\tau &= -(\rho\bar{\mu} + \sigma\lambda) + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2 - 2\Lambda \\
\Delta\alpha - \bar{\delta}\gamma &= \nu(\rho + \varepsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3\,,
\end{aligned} \tag{1.59}
$$

together with their complex conjugate expressions.

The Bianchi identities, on the other hand, can be derived - in terms of the spin coefficients - from (1.31). First, let us write the independent components of the Bianchi identities as a set of complex identities, which is

$$
\begin{aligned}
&R_{13[13;4]} = 0, \quad R_{13[21;4]} = 0, \quad R_{13[13;2]} = 0, \quad R_{13[43;2]} = 0 \\
&R_{42[13;4]} = 0, \quad R_{42[21;4]} = 0, \quad R_{42[13;2]} = 0, \quad R_{42[43;2]} = 0.
\end{aligned}
$$

Consider the first Bianchi identity written above as an example. It may be written, using the symmetries of the Riemann tensor, as

$$R_{1313;4} + R_{1334;1} + R_{1341;3} = 0, \tag{1.60}$$

which can be decomposed, using (1.50), in terms of the Weyl and Ricci tensors, obtaining

$$W_{1313;4} + (W_{1334} + \frac{1}{2}R_{13})_{;1} - \frac{1}{2}R_{11;3} = 0. \tag{1.61}$$

Finally, writing the Weyl in terms of the spin coefficients, as

$$
\begin{aligned}
W_{1313;4} &= W_{1313,4} - \eta^{nm}(\Gamma_{n14}W_{m313} + \Gamma_{n34}W_{1m13} + \Gamma_{n14}W_{13m3} + \Gamma_{n34}W_{131m}) \\
&= W_{1313,4} - 2(\Gamma_{214} + \Gamma_{344})W_{1313} + 2\Gamma_{314}(W_{1213} + W_{4313}) \\
&= -\bar{\delta}\Psi_0 + 4\alpha\Psi_0 - 4\rho\Psi_1
\end{aligned}
$$

and

$$
\begin{aligned}
W_{1334;1} &= W_{1334,1} - \eta^{nm}[\Gamma_{n11}W_{m334} + \Gamma_{n31}(W_{1m34} + W_{13m4}) + \Gamma_{n41}W_{133m}] \\
&= W_{1334,1} - [(\Gamma_{211} + \Gamma_{341})W_{1334} + \Gamma_{131}(W_{1234} - W_{3434}) + \\
&\quad + \Gamma_{231}W_{1314} + \Gamma_{141}W_{1332} + \Gamma_{131}W_{1324} + \Gamma_{241}W_{1331}] \\
&= D\Psi_1 - 2\varepsilon\Psi_1 + 3\kappa\Psi_2 - \pi\Psi_0,
\end{aligned}
$$

and assuming $R_{ab} = 0$, we find the final form

$$
\bar{\delta}\Psi_0 - D\Psi_1 = (4\alpha - \pi)\Psi_0 - 2(2\rho + \varepsilon)\Psi_1 + 3\kappa\Psi_2. \tag{1.62}
$$

Proceeding as described, we can find the complete set of Bianchi identities in the NP formalism:

$$
\begin{aligned}
\bar{\delta}\Psi_0 - D\Psi_1 &= (4\alpha - \pi)\Psi_0 - 2(2\rho + \varepsilon)\Psi_1 + 3\kappa\Psi_2 \\
\bar{\delta}\Psi_1 - D\Psi_2 &= \lambda\Psi_0 + 2(\alpha - \pi)\Psi_1 - 3\rho\Psi_2 + 2\kappa\Psi_3 \\
\bar{\delta}\Psi_2 - D\Psi_3 &= 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\varepsilon - \rho)\Psi_3 + \kappa\Psi_4 \\
\bar{\delta}\Psi_3 - D\Psi_4 &= 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3 + (4\varepsilon - \rho)\Psi_4 \\
\Delta\Psi_0 - \delta\Psi_1 &= (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 \\
\Delta\Psi_1 - \delta\Psi_2 &= \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 \\
\Delta\Psi_2 - \delta\Psi_3 &= 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 \\
\Delta\Psi_3 - \delta\Psi_4 &= 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4
\end{aligned} \tag{1.63}
$$

Therefore, the sets of equations (1.56), (1.59) and (1.63), in the NP formalism, represent the equations of motion of General Relativity in the absence of matter.

1.3.5 Optical scalars

As anticipated before, the NP formalism gives us great control on the null structure of the spacetime. This control over the "geometric degrees of freedom" is encoded in the spin coefficients and we shall now present their physical interpretation.

Let us start recalling the propagation of a tetrad component, under an infinitesimal displacement ξ along an arbitrary direction, which is

$$
\delta_\xi e_{a\mu} = e_{a\mu;\nu}\xi^\nu = \Gamma^b{}_{a\nu}e_{\mu b}\xi^\nu = \Gamma_{bac}e^c_\nu\xi^\nu e^b_\mu = -\Gamma_{abc}e^b_\mu\xi^c\,. \tag{1.64}
$$

Therefore, the displacement $\delta_\xi e_a(c)$ of e_a along the direction c, per unit of displacement, is

$$\delta_\xi e_a(c) = -\Gamma_{abc} e^b \,. \tag{1.65}$$

Now, as an example, choose as e_a in (1.65) the tetrad component $e_1 = l$. If we look at how l changes when infinitesimally displaced in its own direction, we see that

$$\delta_\xi l(1) = Dl = -\Gamma_{1b1} e^b = -\Gamma_{121} e^2 - \Gamma_{131} e^3 - \Gamma_{141} e^4 = (\varepsilon + \bar{\varepsilon}) l - \kappa \bar{m} - \bar{\kappa} m. \tag{1.66}$$

The geometric interpretation of the spin coefficient is then straightforward. The tetrad component l, which is a vector field, defines a null congruence through its integral curves. These curves in general are neither geodesics nor affinely parametrized, but the spin coefficients quantify these obstructions. The control given by the NP formalism then stands, for example, in the fact that we can tune the properties of these congruences of null curves, given by the tetrad components, simply by turning off certain spin coefficients. This is particularly useful when defining gauge fixing conditions from a geometric point of view. For example, from (1.66) we see that if $\kappa = 0$ the integral curves of l become geodesics, which in general are not affinely parametrized. To get an affine parametrization we need the further constraint of $\varepsilon = 0$. Notice also that, the act of constraining spin coefficients to zero is not always a physical assumption on the spacetime. For example, the gauge freedom of the Einstein-Cartan formulation of General Relativity, which is the invariance under diffeomorphisms and local Lorentz transformation, can be always used to fix the affine parametrization of the geodesics and set $\varepsilon = 0$.

The spin coefficients, however, do not encode information regarding only the null congruences defined by the tetrad fields. They also control the obstructions of the tetrad field to admit integral submanifolds of codimension 1, foliating the spacetime. In other terms, they control the "closeness" of the 1-form associated to the each tetrad component. To prove this, let us compute dl. To start, in coordinate components, the covariant derivative of l takes the form

$$\begin{aligned} l_{\mu;\nu} &= \Gamma_{a1b} e^a_\mu e^b_\nu \\ &= (\gamma + \bar{\gamma}) l_\mu l_\nu - (\bar{\alpha} + \beta) l_\mu \bar{m}_\nu - (\alpha + \bar{\beta}) l_\mu m_\nu - \tau \bar{m}_\mu l_\nu \\ &\quad + \sigma \bar{m}_\mu \bar{m}_\nu + \bar{\sigma} m_\mu m_\nu + \rho \bar{m}_\mu m_\nu + \bar{\rho} m_\mu \bar{m}_\nu - \bar{\tau} m_\mu l_\nu \,, \end{aligned} \tag{1.67}$$

where we already assumed that the integral curves of l are affinely parametrized geodesics, namely that $\kappa = \varepsilon = 0$. If we antisymmetrize the indices, then we get

$$l_{[\mu;\nu]} = -(\bar{\alpha} + \beta - \tau) l_{[\mu} \bar{m}_{\nu]} - (\alpha + \bar{\beta} - \bar{\tau}) l_{[\mu} m_{\nu]} + (\rho - \bar{\rho}) \bar{m}_{[\mu} \bar{m}_{\nu]} \,, \tag{1.68}$$

from which derives

$$l_{[\mu;\nu} l_{\xi]} = (\rho - \bar{\rho}) \bar{m}_{[\mu} m_\nu l_{\xi]}. \tag{1.69}$$

This last condition in form notation can be written as $dl \wedge l = 0$ and is a weaker condition with respect to the closure of the 1-form l_μ. If satisfied, in particular, it implies the existence of a foliation, whose leaves are codimension 1 hypersurfaces orthogonal to the integral curves of l^μ; see appendix A for more details.

Recalling Frobenius's theorem, we see from (1.69) that if ρ is real then l is hypersurface orthogonal. We also see from (1.68) that by additionally requiring that $\bar{\alpha} + \beta = \tau$, we get $dl = 0$, i.e. l is the gradient of a scalar field, whose level curves are the null leaves of the foliation. In other terms, the codimension 1 hypersurfaces of the foliation are integrable.

Finally, let us define the following quantities

$$\theta := \frac{1}{2} l^{\mu}{}_{;\mu} = -\frac{1}{2}(\rho + \bar{\rho}), \tag{1.70}$$

$$\omega^2 := \frac{1}{2} l_{[\mu;\nu]} l^{\mu;\nu} = -\frac{1}{4}(\rho - \bar{\rho})^2, \tag{1.71}$$

from which follows

$$\theta^2 + |\sigma|^2 = \frac{1}{2} l_{(\mu;\nu)} l^{\mu;\nu} \,. \tag{1.72}$$

The quantities (1.70) and (1.71) may equivalently be defined as $\theta = -Re(\rho)$ and $\omega = Im(\rho)$.

The scalar quantities θ, ω and σ are called "optical scalars" and control how the congruence of null geodesics[2], given by l, is deformed when it traverse a gravitational field. To see this, first recall that m is a complex vector orthogonal to l. Thus, taken a point $p \in \mathcal{M}$ on $\mathcal{N}$, where $\mathcal{N}$ is a null geodesic of the congruence, the real part of m spans with l a 2-plane. Take then a small circle centered in p, which lies in the 2-plane orthogonal to l. Following the null geodesics of the congruence which intersect the circle centered in p, the circle in general will be expanded, contracted, rotated or sheared into an ellipse. It can be shown that these deformations are quantified by the optical scalars. In particular, θ accounts for the expansion (contraction) of the circle, the "vorticity" ω [3] for its rotation (clockwise or counter-clockwise) and σ for the shear, where $|\sigma|$ tells us the magnitude of the shear, while $\frac{1}{2} arg(\sigma)$ tells us the direction in which it get "squeezed".

1.3.6 Gauge freedom

The Einstein-Cartan formulation enhances the gauge freedom of General Relativity. The theory, beyond its general covariance under diffeomorphisms, gets an additional invariance under local Lorentz transformations, which is brought by the tetrad field. We already mentioned above that the symmetries associated with the vector bundle's indices, are given by the structure group $G_\psi \subset GL(n, \mathbb{R})$ of the frame bundle, i.e. by the little group of $GL(n, \mathbb{R})$ which preserves the metric of the vector bundle. In the NP formalism, the metric chosen is the flat metric η_{ab}, and so the little group G_ψ coincides with the Lorentz group.

Remark: Given a generic metric $g_{ab}(x)$ on the fibers of the vector bundle, there is no little group G_ψ which preserves it, i.e. the tetrad field will have no symmetries. However, the whole theory will be still invariant under diffeomorphisms

In the NP formalism a generic local Lorentz transformation of the tetrad components is decomposed composed in the following three rotations:

- Rotations of class I, which leave l unchanged:

$$l \to l, \quad m \to m + al, \quad \bar{m} \to \bar{m} + \bar{a}l, \quad n \to n + \bar{a}m + a\bar{m} + a\bar{a}l \,. \tag{1.73}$$

[2]Recall that we are **assuming** that l's integral curves form a congruence of null geodesics

[3]Recalling Frobenius theorem, we see that the vorticity of the null geodesics is what spoils the integrability condition of the null leaves

- Rotations of class II, which leave n unchanged:

$$n \to n, \quad m \to m + bn, \quad \bar{m} \to \bar{m} + \bar{b}n, \quad l \to l + \bar{b}m + b\bar{m} + b\bar{b}l\,. \tag{1.74}$$

- Rotations of class III, which leave both l and n unchanged, but rotate m and $\bar{m}$ by an angle θ in the $(m - \bar{m})$-plane:

$$l \to A^{-1}l, \quad n \to An, \quad m \to e^{i\theta}m, \quad \bar{m} \to e^{-i\theta}\bar{m}\,. \tag{1.75}$$

Starting from the I class, we can use (1.49) to deduce the transformation laws of the spin coefficients under a local Lorentz transformation. For example:

$$\begin{aligned}
\sigma = m^\mu D(m_\mu) - l^\mu \delta(m_\mu) &\to (m^\mu + al^\mu)D(m^\mu + al^\mu) - l^\mu\delta(m^\mu + al^\mu) \\
&= m^\mu Dm_\mu - l^\mu\delta m_\mu + a(\cancel{l^\mu Dm_\mu} + m^\mu Dl_\mu - l^\mu\delta l_\mu) + a^2\cancel{l^\mu Dl_\mu} \\
&= \sigma + a\kappa
\end{aligned} \tag{1.76}$$

where the third and the last terms were canceled due to the antisymmetry of spin coefficients in the first two indices, i.e. the torsion-less nature of the spin connection. Similarly, we can derive the transformation laws of the other scalars, obtaining

- Under I class transformations,

  Spin coefficients:

$$\begin{aligned}
&\kappa \to \kappa && \tau \to \tau + a\rho + \bar{a}\sigma + a\bar{a}\kappa \\
&\sigma \to \sigma + a\kappa && \pi \to \pi + 2\bar{a}\varepsilon + (\bar{a})^2\kappa + D\bar{a} \\
&\rho \to \rho + \bar{a}\kappa && \alpha \to \alpha + \bar{a}(\rho + \varepsilon) + (\bar{a})^2\kappa \\
&\varepsilon \to \varepsilon + \bar{a}\kappa && \beta \to \beta + a\varepsilon + \bar{a}\sigma + a\bar{a}\kappa \\
&\gamma \to \gamma + a\alpha + \bar{a}(\beta + \tau) + a\bar{a}(\rho + \varepsilon) + (\bar{a})^2\sigma + a(\bar{a})^2\kappa \\
&\lambda \to \lambda + \bar{a}(2\alpha + \pi) + (\bar{a})^2(\rho + 2\varepsilon) + (\bar{a})^3\kappa + \bar{\delta}\bar{a} + \bar{a}D\bar{a} \\
&\mu \to \mu + a\pi + 2\bar{a}\beta + 2a\bar{a}\varepsilon + (\bar{a})^2\sigma + a(\bar{a})^2\kappa + \delta\bar{a} + aD\bar{a} \\
&\nu \to \nu + a\lambda + \bar{a}(\mu + 2\gamma) + (\bar{a})^2(\tau + 2\beta) + (\bar{a})^3\sigma + a\bar{a}(\pi + 2\alpha) \\
&\quad + a(\bar{a})^2(\rho + 2\varepsilon) + a(\bar{a})^3\kappa + (\Delta + \bar{a}\delta + a\bar{\delta} + a\bar{a}D)\bar{a}\,.
\end{aligned} \tag{1.77}$$

  Weyl scalars:

$$\begin{aligned}
&\Psi_0 \to \Psi_0 \\
&\Psi_1 \to \Psi_1 + \bar{a}\Psi_0 \\
&\Psi_2 \to \Psi_2 + 2\bar{a}\Psi_1 + (\bar{a})^2\Psi_0 \\
&\Psi_3 \to \Psi_3 + 3\bar{a}\Psi_2 + 3(\bar{a})^2\Psi_1 + (\bar{a})^3\Psi_0 \\
&\Psi_4 \to \Psi_4 + 4\bar{a}\Psi_3 + 6(\bar{a})^2\Psi_2 + 4(\bar{a})^3\Psi_1 + (\bar{a})^4\Psi_0\,.
\end{aligned} \tag{1.78}$$

- Under II class transformations the transformations can easily be read from the previous ones, since a II class transformation differs from a I class one just by the exchange of l and n. It is sufficient to exchange the following quantities in the previous transformation laws:

$$\kappa \leftrightarrow -\bar{\nu} \qquad \rho \leftrightarrow -\bar{\mu} \qquad \sigma \leftrightarrow -\bar{\lambda} \tag{1.79}$$

$$\alpha \leftrightarrow -\bar{\beta} \qquad \varepsilon \leftrightarrow -\bar{\gamma} \qquad \pi \leftrightarrow -\bar{\tau} \tag{1.80}$$

$$\Psi_0 \leftrightarrow \bar{\Psi}_4 \quad \Psi_1 \leftrightarrow \bar{\Psi}_3 \quad \Psi_2 \leftrightarrow \bar{\Psi}_2\,. \tag{1.81}$$

- Under III class transformations, on the other hand, we have

  Spin coefficients:

$$\begin{aligned}
&\kappa \to A^{-2}e^{i\theta}\kappa && \sigma \to A^{-1}e^{2i\theta}\sigma && \rho \to A^{-1}\rho && \tau \to e^{i\theta}\tau \\
&\pi \to e^{-i\theta}\pi && \lambda \to Ae^{-2i\theta}\lambda && \mu \to A\mu && \nu \to A^2e^{-i\theta}\nu \\
&\gamma \to A\gamma - \frac{1}{2}\Delta A + \frac{1}{2}i\,A\Delta\theta && && \varepsilon \to A^{-1}\varepsilon - \frac{1}{2}A^{-2}DA + \frac{1}{2}i\,A^{-1}D\theta && \\
&\alpha \to e^{-i\theta}\alpha + \frac{1}{2}ie^{-i\theta}\bar{\delta}\theta - \frac{1}{2}A^{-1}e^{-i\theta}\bar{\delta}A && && \beta \to e^{i\theta}\beta + \frac{1}{2}ie^{i\theta}\delta\theta - \frac{1}{2}A^{-1}e^{i\theta}\delta A\,. &&
\end{aligned}$$

  Weyl scalars:

$$\Psi_0 \to A^{-2}e^{2i\theta}\Psi_0, \quad \Psi_1 \to A^{-1}e^{i\theta}\Psi_1, \quad \Psi_2 \to \Psi_2, \quad \Psi_3 \to Ae^{-i\theta}\Psi_3, \quad \Psi_4 \to A^2e^{-2i\theta}\Psi_4\,.$$

1.4 NP in 3D

The Newman-Penrose formalism in the context of 3D gravity can be derived directly by the 4D case described in the previous section. First of all, the flat metric on the vector bundle becomes

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{1.82}$$

which is not orthogonal anymore. Its inverse, on the other hand, is

$$\eta^{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{1.83}$$

The complex vector field m^μ, which was part of the tetrad's components together with its complex conjugate, now becomes real. As a consequence, all the quantities involved in the NP formalism are real in 3D. In particular, in analogy with (1.37), the tetrad and the covariant derivative frame components become

$$e^\mu_a = (l^\mu\partial_\mu,\ n^\mu\partial_\mu,\ m^\mu\partial_\mu), \quad \nabla_a = (l^\mu D_\mu,\ n^\mu\Delta_\mu,\ m^\mu\delta_\mu), \tag{1.84}$$

where the latter can also be expressed as

$$\nabla_a = n_a D + l_a \Delta - 2m_a \delta. \tag{1.85}$$

The metric, computed from (1.2), is

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - 2m^\mu m^\nu, \tag{1.86}$$

and the set of spin coefficients, from the 12 complex ones given in (1.46), here reduces to 9 real ones given by[4]

$$\begin{aligned}
\kappa &= \Gamma_{311}, \quad \varepsilon = \Gamma_{211}, \quad \pi = -\Gamma_{321}, \\
\tau &= \Gamma_{312}, \quad \gamma = \Gamma_{212}, \quad \nu = -\Gamma_{322}, \\
\sigma &= \Gamma_{313}, \quad \beta = \Gamma_{213}, \quad \mu = -\Gamma_{323}.
\end{aligned} \tag{1.87}$$

Contextually, the Ricci identities given in (1.59), which in the 4D case represented 18 complex equations for the spin coefficients, now reduce to

$$\begin{aligned}
D\sigma - \delta\kappa &= \sigma\left(\varepsilon + 2\sigma\right) - \kappa\left(\tau - \pi + 2\beta\right) \\
D\tau - \Delta\kappa &= 2(\tau + \pi)\sigma - 2\kappa\gamma \\
D\beta - \delta\varepsilon &= 2(\beta + \pi)\sigma - (2\mu + \gamma)\kappa - (\beta - \pi)\varepsilon \\
D\gamma - \Delta\varepsilon &= 2(\tau + \pi)\beta - 2\varepsilon\gamma + 2\tau\pi - 2\kappa\nu \\
D\mu - \delta\pi &= (2\sigma - \varepsilon)\mu + \pi^2 - \kappa\nu \\
D\nu - \Delta\pi &= 2(\pi + \tau)\mu - 2\varepsilon\nu \\
\Delta\mu - \delta\nu &= -(2\mu + \gamma)\mu + (\pi + 2\beta - \tau)\nu \\
\Delta\beta - \delta\gamma &= (\beta - \tau)\gamma - 2(\beta + \tau)\mu + (2\sigma + \varepsilon)\nu \\
\Delta\sigma - \delta\tau &= (\gamma - 2\mu)\sigma - \tau^2 + \nu\kappa
\end{aligned} \tag{1.88}$$

Similarly, also the structure constant equations, expressing the vanishing of torsion, reduce to:

$$\begin{aligned}
Dn^\mu - \Delta l^\mu &= -\gamma l^\mu - \varepsilon n^\mu + 2(\pi + \tau)m^\mu \\
Dm^\mu - \delta l^\mu &= (\pi - \beta)l^\mu - \kappa n^\mu + 2\sigma m^\mu \\
\Delta m^\mu - \delta n^\mu &= \nu l^\mu + (\beta - \tau)n^\mu - 2\mu m^\mu
\end{aligned} \tag{1.89}$$

Since the Weyl tensor vanishes identically in 3 dimension, the Weyl scalars $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4\}$ are zero. Its role is replaced by the Cotton tensor.

[4]The factor $\frac{1}{2}$ present in (1.46) is reabsorbed in the $2+1$ case inside the definition of the spin coefficients

Chapter 2

Surface charges

The theory of surface charges is an interesting construction, involving gauge theories, which starts however by studying some rather different mathematical structures. The part of the construction we all know about, is that associated to each global symmetry, there is always a conserved Noether current, from which a conserved Noether charge may be derived. On the other hand, if we try to do the same with gauge symmetries, the Noether charge associated to it will always be zero on-shell, because of its weakly vanishing Noether current. The part of the construction less known however, is that there are actually some local transformations, which are not mere redundancies of the theory, that carry instead a physical interpretation. In the same way as for the global version of the symmetry, these rather special local transformations may give rise to conserved quantities, what changes is simply the dimension of the surface in which these charges are supported. The implications of this fact are numerous and span from the black hole information paradox to the infrared properties of gravity.

In what follows, we introduce the main steps towards the definition of surface charges. Starting from the relevant mathematical structures, namely the variational bicomplex and symplectic geometry, we arrive to the covariant phase space, in which the discussion of the charges finds its natural setting. Afterwards, we recall Noether's theorems, reformulating them in the language of the covariant phase space, eventually defining the concept of surface charges. As a final step, we define what is an asymptotic symmetry, in preparation for the double null analysis which will take place in the next chapter.

We refer to [26], [27], [15], [28] and [29] for what concerns the mathematical framework and the asymptotic symmetries, while we follow [30] and [31] to discuss surface charges and Noether's identities in the Cartan formalism.

2.1 Mathematical framework

The path to surface charges encompasses some beautiful mathematical objects, which formalizes the geometric and algebraic structures in which familiar operations, such as variational calculus, take place. More specifically, we are going to first introduce the jet bundle, with its variational bicomplex built upon it; the latter will provide us with a series of practical tools, necessary to compute the quantities we are interested in. Secondly, we are going to review the phase space as an example of symplectic manifold and eventually, we are going to bring these two constructions together. The study of the symplectic structure of a jet bundle's variational bicomplex is what it is usually called "covariant phase space formalism" in the literature.

2.1.1 Jet bundles and their variational bicomplex

Jet bundles are the geometrical objects which formalise the variational calculus of relativistic field theories. Introducing jet bundles is analogue - in some sense - to introducing a principle bundle for a gauge theory. We pass from working with sections to working with the algebraic objects from which these sections come from. Although this requires a bit of work and formalism, one is then able to work with actions and functional derivatives in a purely algebraic form. Besides, one is able to unify variational calculus to the exterior algebra[1] of the base manifold in a single object, which will then be introduced as the variational bicomplex.

Firstly, we introduce the jet bundle as a manifold with local coordinates (x^μ, φ^i_μ), where (μ) represents any set of symmetrized indices of cardinality k. These sets of symmetric indices will be formed by the indices brought to the fields by their spacetime derivatives, which are considered as independent variables in the field space. The base space of the jet bundle may be the spacetime manifold, denoted in the following as $\mathcal{M}$, if we are dealing with relativistic field theories, or simply $\mathbb{R}$, if we want to deal with a k^{th} order phase space[2]. The fibers, on the other hand, are copies of an infinite dimensional field space attached at each point of the base manifold. The field space, whose local coordinates are the fields and their derivatives, is often also called k^{th} order "jet space", denoted in the following by F^k. Thus, the jet bundle $J^k(E)$ will locally be

$$J^k(E) = \mathcal{M} \times F^K, \tag{2.1}$$

with E being $E = \mathcal{M} \times F$. Moreover, as for every fiber bundle, taking a section s will account as identifying the coordinates of the jet space F^k with the field "histories" and their derivatives, up to order k:

$$\begin{aligned} s : \mathcal{M} &\longrightarrow J^k(E) \\ x^\mu &\longmapsto (x^\mu, \phi^i(x), \frac{\partial \phi^i(x)}{\partial x^\mu}, ...). \end{aligned}$$

Lastly, a vector field, being a section $\Gamma[T(J^k(E))]$, will have the following form:

$$\mathbf{v} = a^\mu \frac{\partial}{\partial x^\mu} + b^i_{(\mu)} \frac{\partial}{\partial \varphi^i_{(\mu)}}, \tag{2.2}$$

where we used the multi-index notation:

$$(\mu) = \{\emptyset, \mu_1, \mu_1\mu_2, \mu_1...\mu_n\}, \qquad |\mu| = 0, 1, 2, 3, \tag{2.3}$$

[1] with the word "exterior algebra" I am broadly referring to the usual toolkit of operators revolving around a cochain complex, such as the de Rham complex

[2] To be more precise, the jet bundle is formally a construction to be made upon a fiber bundle. The fields are the fibers of the base fiber bundle, the"jet" part is given by identifying the field derivative as independent coordinates of the new bundle.

and the Einstein's convention is such that we are summing on the index's length too.

Now that we have introduced jet bundles, we can build upon them their variational bicomplex. The variation bicomplex is - as its name suggests - a cochain complex, analogue to the de Rham complex of differential forms we are familiar with. What distinguishes it from an ordinary cochain complex, built from a manifold, it's the fibration the jet bundle is equipped with. For this reason our cochain is called "bicomplex", because we can build a complex both thanks to the manifold calculus and the differential properties of the fibers. This allows us to define two directions in which the cochain unfolds. The horizontal direction is the cochain built from the base manifold's calculus, i.e. the de Rham complex, made by all the usual differential forms endowed with a set of completely antisymmetric indices. The vertical direction, on the other hand, consists on the cochain built from the variational calculus of the jet space. So, a form with r dx^μ's and s $\delta\varphi^i_{(\mu)}$'s will be a (r,s)-form, well defined on the variational bicomplex. These two complexes can be proven to be perfectly intertwined in an exact generalized *Mayer-Vietoris* sequence of (r,s) forms on $J^\infty(E)$. Although this result well defines all possible forms of the variational bicomplex, a cochain complex will be particularly relevant to our purposes. This complex $\mathcal{E}(J^\infty(E))$, called the "Euler-Lagrange" hedge complex, is a specific path on the variational bicomplex grid, made of all the form's spaces that naturally appear when studying the dynamics of a system, namely

$$0 \xrightarrow{d} \mathbb{R} \xrightarrow{d} \Omega^{0,0} \xrightarrow{d} \Omega^{1,0} \xrightarrow{d} \dots \xrightarrow{d} \Omega^{top,0} \xrightarrow{\delta} \Omega^{top,1} \xrightarrow{\delta} \Omega^{top,2} \xrightarrow{\delta} \dots \tag{2.4}$$

Now, since we are not going to get into any more detail about neither jet bundles nor the variational bicomplex, we lighten up the notation by making the following assumption. We will always talk about fields as sections of the jet bundle structure. This will make the discussion about charges and Noether's theorems simpler for someone not familiar with the jet bundle formalism. However, everything we said just above can be used to construct the same charges in a more covariant way, without ever referring to the sections, but using instead the field as a natural fiber bundle. Finally, we give the notations and some basic results that will help us when discussing the covariant phase space.

We are going to denote the typical operation connected to the de Rham complex, i.e. the exterior derivative, the interior product and the Lie derivative as $(d, i., \mathcal{L}.)$. Recall also that for these maps Cartan's magic formula holds:

$$\mathcal{L}_\xi \alpha = d\, i_\xi \alpha + i_\xi\, d\alpha, \tag{2.5}$$

with $\alpha \in \Omega^k(\mathcal{M})$, with $k < \dim(\mathcal{M})$ and $\xi \in \Gamma(T\mathcal{M}) \equiv \mathfrak{X}(\mathcal{M})$. Similarly, we denote the operations related to the vertical direction of the bicomplex with $(\delta, I., \mathfrak{L}.)$, for which an analogous version of Cartan's magic formula holds:

$$\mathfrak{L}_\eta \beta = \delta\, I_\eta \beta + I_\eta\, \delta\beta, \tag{2.6}$$

with $\beta \in \Omega^k(\mathcal{F})$ and $\eta \in \mathfrak{X}(\mathcal{F})$, $\mathcal{F} = \Gamma(F)$. Besides, we formalize something we always encounter when varying lagrangians, i.e. the equation of motions and the boundary terms:

Definition 2.1.1 (Source and boundary forms) *Let $\alpha \in \Omega^{top,1}(\mathcal{M} \times \mathcal{F})$ be a mixed form of degree $(top, 1)$. Then:*

- *α is said to be a **source form** if and only if it depends on $\delta\Phi^i(x)$ and none of its derivative. The space of source forms will be denoted by $\Omega^{top,1}_{src}(\mathcal{M}\times\mathcal{F})$.*

- *α is said to be a **boundary form** if and only if there exist a $\beta \in \Omega^{top-1,1}(\mathcal{M}\times\mathcal{F})$ such that $\alpha = d\beta$. The space of boundary forms will be denoted by $\Omega^{top,1}_{bdry}(\mathcal{M}\times\mathcal{F}) = d\Omega^{top-1,1}(\mathcal{M}\times\mathcal{F})$.*

Theorem 2.1.1 (Takens) *The following decomposition holds for the space of mixed form with degree $(top, 1)$:*

$$\Omega^{top,1}(\mathcal{M}\times\mathcal{F}) = \Omega^{top,1}_{src}(\mathcal{M}\times\mathcal{F}) \oplus \Omega^{top,1}_{bdry}(\mathcal{M}\times\mathcal{F}). \tag{2.7}$$

2.1.2 Some elements of symplectic geometry

A symplectic geometry is the structure which arises when we translate, in a purely geometric way, the usual description of a dynamical closed system, i.e. a phase space endowed with Hamilton equations. The idea behind this geometric translation is to express dynamics as "good" morphisms preserving some structure, that we will then call symplectic. But what should this structure be? And what properties it should have?

We can assume it will be some tensorial quantity defined in the exterior algebra of the manifold, but we do not have any clue about which type of tensor it should be nor what properties it should have. To move forward, recall what it meant in Hamiltonian mechanics to specify a dynamics, which in the end is what we are trying to render geometric. Specifying a dynamics meant specifying a smooth function in the phase space, called Hamiltonian, which generats the evolution and is conserved on shell. So, since an Hamiltonian evolution can easily be described as a vector field, whose integral curves are the solutions to the equations of motions, the structure we are looking for is an injective linear map, that will send any smooth function H to a unique vector field X_H. Now, the easiest way to connect a function with a vector field is taking the differential of the function and then take its dual using the canonical connection between vectors and forms. However, since the duality between vectors and forms is a local property, typical of the single tangent space, this map can be accomplished more generally by defining a 2-form, which maps the differential to a vector field [3]. Besides, we request the 2-form to be closed[4] and non-degenerate, to preserve uniqueness. This argument leads to the following definitions of a symplectic form and symplectic manifold:

Definition 2.1.2 (Symplectic form) *Let $\mathcal{P}$ be a $2n$-dimensional manifold and $\omega \in \Omega^2(\mathcal{P})$. The 2-form ω is said to be a **symplectic form** if and only if ω is closed, i.e. $d\omega = 0$, and non degenerate, i.e. $i_\xi\omega = 0$ iff $\xi = 0$, with $\xi \in \mathfrak{X}(\mathcal{P})$ any vector field on $\mathcal{P}$.*

Definition 2.1.3 (Symplectic manifold) *Let $\mathcal{P}$ be a $2n$-dimensional manifold and ω a symplectic form. Then the couple $(\mathcal{P}, \omega)$ is called **symplectic manifold**.*

If we are dealing with a finite dimensional space the non degeneracy condition can be characterized more concretely. Pick coordinates x^i on $\mathcal{P}$, we then have $\omega(x) = \frac{1}{2}\omega_{ij}(x)dx^i \wedge dx^j$, where ω_{ij} is a skew symmetric covariant 2-tensor. The 2-form is non degenerate if and only if $\omega_{ij}(x)$ is invertible

[3] The fact that there is always locally a canonical way of associating a vector field to a 1-form is due to the existence of Darboux coordinates

[4] without asking for the closeness of the 2-form we wouldn't have Darboux theorem and the local existence of canonical coordinates

at every $x \in \mathcal{P}$. Note that an odd dimensional manifold cannot be equipped with a closed and non degenerate 2-form.

Therefore, to encode everything we said above in a sentence, a symplectic structure is what enables a space to develop dynamics from an arbitrary smooth function in a well posed way.

We go on with our review by arguing that one of the most natural examples of a symplectic manifold is the cotangent bundle of a manifold, with the following theorem:

Theorem 2.1.2 *Let $\mathcal{Q}$ be a n-dimensional manifold. Then $\mathcal{P} = T^*\mathcal{Q}$ is a symplectic manifold.*

The reason behind the proof is that every cotangent bundle of a manifold is naturally equipped with a canonical 1-form, also known as "tautological" 1-form, defined as follows:

Definition 2.1.4 (Tautological 1-form) *Let $\mathcal{Q}$ be an n-dimensional manifold and let $\mathcal{P} = T^*\mathcal{Q}$ be its cotangent bundle, which is a manifold itself. Let β be a 1-form defined on $\mathcal{Q}$, a section of $T^*\mathcal{Q}$, i.e. $\beta : \mathcal{Q} \to T^*\mathcal{Q}$. Finally, let σ be a 1-form defined on $\mathcal{P}$. The pullback of σ by β is going to be $\beta^*\sigma = \sigma \circ d\beta$, with $\beta^*\sigma$ being a 1-form on $\mathcal{Q}$ too. The **tautological 1-form** θ is the only 1-form with the property that $\beta^*\theta = \beta$, for any 1-form β on $\mathcal{Q}$.*

Basically, we are saying that there is a canonical isomorphism between the sections of the bundle $\mathcal{P}$ and the space of 1-forms defined on $\mathcal{Q}$, i.e. $\Omega^1(\mathcal{Q})$. Now, when this happens in the general case of a vector bundle, we say that the bundle is soldered to its base, meaning that there exist an isomorphism between the tangent bundle of the base and the elements of the bundle itself. The isomorphism is then called a soldering form. This was exactly the case we encountered in the first chapter when we introduced the tetrad field, which was the solder form of the frame bundle to its base. In the current case however, the soldering between the two is tautological - that's why the name tautological 1-form - since the bundle itself is the cotangent bundle, i.e. $\Omega^1(\mathcal{Q})$.

To give a more concrete description of the tautological 1-form, let q^i be a set of coordinates for $\mathcal{Q}$. Then, $T_q^*\mathcal{Q}$ will be an n-dimensional vector space spanned by dq^i. Coordinates on this vector space are called p^i, so that any $\alpha \in T_q^*\mathcal{Q}$ will look like $p_i(\alpha)dq^i$. Besides, (q^i, p_i) are coordinates on $\mathcal{P} = T^*\mathcal{Q}$. The tautological 1-form on $\mathcal{P} = T^*\mathcal{Q}$ is then given by:

$$\theta(q,p) = p_i dq^i \in \Omega^1(T^*\mathcal{Q}). \tag{2.8}$$

What is so special about this tautological 1-form for our purposes is that it is a symplectic potential, which we define as follows:

Definition 2.1.5 (Symplectic potential) *Let $\theta \in \Omega^1(\mathcal{P})$. θ is said to be a **symplectic potential** if and only if $\omega = d\theta$ is a symplectic form.*

It's indeed easy to verify that the external derivative of the tautological 1-form, i.e. $\omega = d\theta = dp_i \wedge dq^i \in \Omega^2(T^*\mathcal{Q})$ is a symplectic form.

Not only, as we just saw, the cotangent bundle of a manifold is naturally equipped with a symplectic structure, but more in general we can prove that any symplectic manifold always looks locally like a cotangent bundle. This is stated through the following theorem:

Theorem 2.1.3 (Darboux) *Let $\mathcal{P}$ be a symplectic manifold of dimension $2n$ and let ω be its symplectic form. Locally on $\mathcal{P}$ there exist coordinates $z^\mu = (q^1, ..., q^n, p_1, ..., p_n)$ such that*

$$\omega = \frac{1}{2}\omega_{\mu\nu}\, dz^\mu \wedge dz^\nu = dp_i \wedge dq^i. \tag{2.9}$$

Before moving to symmetries and how they are described through symplectic geometry, we mention how symplectic geometry relates to the so-called "Poisson geometries", whose main feature are the well-known Poisson brackets. A Poisson structure is something more general than a symplectic one, characterized by the existence of Poisson brackets between functions on the manifold, which we define as follows:

Definition 2.1.6 (Poisson structure and Poisson manifold) *Let $\mathcal{P}$ be a n-dimensional manifold and let $\mathcal{C}^\infty(\mathcal{P})$ denotes the real algebra of smooth real-valued functions defined on $\mathcal{P}$, with summation and multiplication defined pointwise. A* ***Poisson structure*** *on $\mathcal{P}$ is an $\mathbb{R}$-bilinear map:*

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(\mathcal{P}) \times \mathcal{C}^\infty(\mathcal{P}) \longrightarrow \mathcal{C}^\infty(\mathcal{P}), \tag{2.10}$$

defining a Poisson algebra on $\mathcal{C}^\infty(\mathcal{P})$, i.e. satisfying the following properties:

- *Skew symmetry: $\{f, g\} = -\{g, f\}$.*
- *Jacobi identity: $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$.*
- *Leibniz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$.*

The couple $\big(\{\cdot,\cdot\}, \mathcal{P})\big)$ is then called ***Poisson manifold****.*

Now, since the Poisson brackets of two functions f and g can always be written as the image through a smooth bivector field[5] $\Pi \in \mathfrak{X}^2(\mathcal{P})$ of $df \wedge dg$, i.e.

$$\{f, g\} = \Pi(df \wedge dg)\,, \tag{2.11}$$

whenever we can find a smooth bivector field, we also have a Poisson structure. But this is exactly the case of a symplectic manifold. Indeed, the symplectic form can be used to define Poisson brackets on $\mathcal{C}^\infty(\mathcal{P})$, which we state as follows:

Theorem 2.1.4 (Symplectic $\Rightarrow$ Poisson) *Let $(\mathcal{P}, \omega)$ be a symplectic manifold and let $\Pi := -\frac{1}{2}(\omega^{-1})^{\mu\nu}\partial_\mu \wedge \partial_\nu$ be a bivector field. Let*

$$\{\cdot,\cdot\}_\Pi : \mathcal{C}^\infty(\mathcal{P}) \times \mathcal{C}^\infty(\mathcal{P}) \longrightarrow \mathcal{C}^\infty(\mathcal{P}) \tag{2.12}$$

$$(f, g) \mapsto \{f, g\}_\Pi := \Pi^{\mu\nu}(\partial_\mu f)(\partial_\nu g) \tag{2.13}$$

be the Poisson brackets defined by the bivector field. Then, $\big(\{\cdot,\cdot\}_\Pi, \mathcal{P})\big)$ is a Poisson manifold.

The inverse, on the other hand, does not hold; that is why Poisson geometry is more general than symplectic geometry. There is no degeneracy condition imposed on a Poisson bivector. Because of this, Poisson geometry let us define such a structure on odd-dimensional manifolds, contrary to its symplectic counterpart.

We now move our attention towards how the symplectic structure of our space can induce a dynamics and how this is formalized through the concept of Hamiltonian vector fields. In particular,

[5] A smooth bivector field is a smooth map $\Pi : \mathcal{M} \to \wedge^2 R$, with $\wedge^2 R$ denoting the space of bivectors. A bivector is defined as the wedge product of two vectors.

we saw how a symplectic structure carries a Poisson brackets structure. Now, Poisson brackets naturally induce a mapping between a function on $\mathcal{P}$ and a vector field on $\mathcal{P}$:

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(\mathcal{P}) \longrightarrow \mathfrak{X}(\mathcal{P}) \tag{2.14}$$

$$f \mapsto X_f := \{f,\cdot\}. \tag{2.15}$$

This map is not surjective and its image in $\mathfrak{X}(\mathcal{P})$ is said to be the space of Hamiltonian vector fields. The vector field X_f is then said to be "generated by f" and f is said to be the "Hamiltonian generator of X_f". However, since we want to always refer as much as possible to a naturally symplectic framework, we give now an equivalent definition of hamiltonian vector field, involving directly the symplectic form:

Definition 2.1.7 (Hamiltonian vector field) *A vector field $X \in \mathfrak{X}(\mathcal{P})$ is called **Hamiltonian**, or simply **HVF**, if and only if a function $f_X \in \mathcal{C}^\infty(\mathcal{P})$ exists such that:*

$$i_X\omega = -df_X. \tag{2.16}$$

*Then f_X is said to be the **Hamiltonian generator** of X.*

Firstly, we have the following important result, which will be particularly relevant in the discussion of symmetries:

Proposition 2.1.1 *Let $\{\cdot,\cdot\}$ be the Poisson bracket associated to ω. Then:*

$$\omega\left(X_f, X_g\right) = \{f,g\} \quad \textit{and} \quad [X_f, X_g] = X_{\{f,g\}}, \tag{2.17}$$

which simply states that the Poisson algebra, induced by the symplectic structure, is homomorphic to the Lie algebra of Hamiltonian vector fields.

These vector fields fall in a symplectic description of Hamiltonian dynamics. Take a system described by coordinates $q^i \in \mathcal{Q}$ and canonical conjugate momenta p_i, so that (q^i, p_i) are Darboux coordinates on $\mathcal{P} = T^*\mathcal{Q}$, equipped with its canonical symplectic form ω. In Darboux coordinates take the form $\omega = dp_i \wedge dq^i$ and any vector field, which can be written as $X_f = (X_f)^i_{(q)} \frac{\partial}{\partial q^i} + (X_f)^{(p)}_i \frac{\partial}{\partial p_i}$. If X_f is an HVF, condition (2.16), which is often referred to as "hamiltonian flow equation", holds. Then, by equating the two sides of (2.16), written in Darboux coordinates, i.e.:

$$i_{X_f}\omega = (X_f)^{(p)}_i \, dq^i - (X_f)^i_{(q)} \, dp_i \quad \text{and} \quad -df = -\frac{\partial f}{\partial q^i}dq^i - \frac{\partial f}{\partial p_i}dp_i \, , \tag{2.18}$$

we obtain the following expression for the HVF of f:

$$(X_f)^i_{(q)} = \frac{\partial f}{\partial p_i} \quad \text{and} \quad (X_f)^{(p)}_i = -\frac{\partial f}{\partial q^i} \, . \tag{2.19}$$

This shows an evident resemblance of the Hamiltonian equations of motions and let us make the following statement. Let $\mathcal{H} = \mathcal{H}(q,p)$ be the system's Hamiltonian. Then, the flow of the HVF $X_\mathcal{H}$ reflects the dynamical evolution of the system, in the sense that:

$$X_\mathcal{H} := \{\mathcal{H},\cdot\} = \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial \mathcal{H}}{\partial q^i}\frac{\partial}{\partial p_i} = \dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}_i\frac{\partial}{\partial p_i} \equiv \left.\frac{d}{dt}\right|_{\text{dynamical evolution}} , \tag{2.20}$$

with t being just the parameter of the flow. Hamiltonian vector fields however are not just a way to rephrase the hamiltonian evolution of a system in a more geometrical way, but they formalize any kind of symmetry. Similarly, their Hamiltonian generators formalize the notion of symmetry-generating "charges".

To conclude the introduction of Hamiltonian vector fields, we give some criteria to understand whether a vector field is Hamiltonian or not. The first criteria rely on the first de Rham cohomology of the symplectic manifold. The first de Rham cohomology of $\mathcal{P}$, denoted by $H^1_{dR}(\mathcal{P})$, is the vector space of all closed 1-forms $Z^1(\mathcal{P}) = \{\alpha \in \Omega^1(\mathcal{P}) \mid d\alpha = 0\}$, modulo the exact ones $B^1 = \{\alpha \in \Omega^1(\mathcal{P}) \mid \exists\, \beta \in \Omega^0(\mathcal{P}) \text{ such that } \alpha = d\beta\}$,

$$H^1_{dR} = Z^1(\mathcal{P})/B^1(\mathcal{P}). \tag{2.21}$$

Furthermore, recall that a manifold $\mathcal{P}$ has a trivial first de Rham cohomology if and only if it is simply connected. This is useful for the following:

Proposition 2.1.2 *If $H^1_{dR}(\mathcal{P})$ is trivial, then a vector field $X \in \mathfrak{X}(\mathcal{P})$ is Hamiltonian if and only if $\mathcal{L}_X\omega = 0$,*

which gives us the chance to define the "good" morphisms of a symplectic manifold:

Definition 2.1.8 (Symplectomorphism) *Let $X \in \mathfrak{X}(\mathcal{P})$, then X is called **symplectomorphism** if and only if $\mathcal{L}_X\omega = 0$.*

Alternatively, another criterion which does not (a priori) require any assumption on the cohomology is the following:

Proposition 2.1.3 *Let $X \in \mathfrak{X}(\mathcal{P})$. If ω admits a symplectic potential θ that is X-invariant, i.e. if:*

$$\omega = d\theta \quad \textit{and} \quad \mathcal{L}_X\theta = 0, \tag{2.22}$$

then X is a HVF and the function $f = i_X\theta$ is one of its generators[6].

Symmetries in the symplectic framework

We now want to show how the notion of Hamiltonian vector field can be used to describe the symmetries of a system. We start defining the action of a Lie group on a generic manifold:

Definition 2.1.9 (Lie algebra action and fundamental vector fields) *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and $\mathcal{P}$ a manifold. An action ρ of $\mathfrak{g}$ on $\mathcal{P}$ is a homomorphism of Lie algebras between $\mathfrak{g}$ and $\mathfrak{X}(\mathcal{P})$, i.e. a map:*

$$\rho : \mathfrak{g} \to \mathfrak{X}(\mathcal{P}) \quad \textit{such that} \quad \forall a \in \mathbb{R} \textit{ and } \forall\, \xi, \eta \in \mathfrak{g} \tag{2.23}$$

$$\rho(a\xi + \eta) = a\rho(\xi) + \rho(\eta) \quad \textit{and} \quad \rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)]. \tag{2.24}$$

*Vector fields in the image of ρ are called **fundamental vector fields**.*

[6] Hamiltonian generator functions are always defined up to a constant term.

Basically, a Lie algebra $\mathfrak{g}$ is said to have an action on a manifold $\mathcal{P}$ if there is a set of vector fields on the manifold, in a one-to-one linear correspondence with the algebra elements, representing the algebra. The fundamental vector fields thus are just the representation as vector fields of the symmetry transformations, i.e. of the elements of $\mathfrak{g}$. Furthermore, since our manifold has the additional structure of a symplectic 2-form, we can ask whether the hamiltonian generator of these fundamental vector fields are naturally associated to the algebra elements. Whenever this holds, we talk about the existence of a (co-)momentum map, which we define as follows:

Definition 2.1.10 (Co-momentum map) *Let $(\mathcal{P}, \omega, \mathfrak{g}, \rho)$ be a symplectic manifold acted upon by a Lie algebra $\mathfrak{g}$. The action ρ admits a **co-momentum map** $\check{J} : \mathfrak{g} \to \mathcal{C}^\infty(\mathcal{P})$ if and only if $\forall\, \xi \in \mathfrak{g}$ the function $\check{J}(\xi)$ is the Hamiltonian generator of $\rho(\xi)$, i.e., if and only if:*

$$i_{\rho(\xi)}\omega = -d\check{J}(\xi). \tag{2.25}$$

*The action ρ of $\mathfrak{g}$ on $(\mathcal{P}, \omega)$ is said to be **Hamiltonian** when it admits a co-momentum map.*

Since ρ is linear and so is $\check{J}(\xi)$, the co-momentum map $\check{J}(\xi) : \mathfrak{g} \to \mathcal{C}^\infty(\mathcal{P})$ can equivalently be thought of as a momentum map, i.e., as a map $J : \mathcal{P} \to \mathfrak{g}^*$ such that:

$$\left(\check{J}(\xi)\right)(x) = \langle J(x), \xi\rangle, \tag{2.26}$$

where with $\mathfrak{g}^*$ we denoted the co-algebra of $\mathfrak{g}$. We continue the discussion by giving some sufficient conditions for the action to be Hamiltonian:

Proposition 2.1.4 *Let $(\mathcal{P}, \omega, \mathfrak{g}, \rho)$ be a symplectic manifold acted upon by a Lie algebra $\mathfrak{g}$. Assume that $\mathcal{L}_{\rho(\xi)}\omega = 0$, namely that the dynamics generated by $\rho(\xi)$ preserve the symplectic form, i.e. that the action ρ of $\mathfrak{g}$ on $(\mathcal{P}, \omega)$ is **symplectic**. Then the following 3 statements are equivalent:*

- *$\rho(\xi)$ is Hamiltonian for every $\xi \in \mathfrak{g}$.*
- *$(\mathcal{P}, \omega, \mathfrak{g}, \rho)$ admits a momentum map $J : \mathcal{P} \to \mathfrak{g}^*$.*
- *The linear map $\tilde{\rho} : \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to H^1(\mathcal{P})$, $[\xi] \mapsto [i_{\rho(\xi)}\omega]$ is identically zero.*

From the proposition above, thanks to Whitehead's lemma[7], stating that for every semisimple Lie algebra $\mathfrak{g}$ holds $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, the following corollary follows:

Corollary 2.1.1 *If $\mathfrak{g}$ is semisimple, then $(\mathcal{P}, \omega, \mathfrak{g}, \rho)$ is symplectic if and only if it admits a momentum map J, namely:*

$$\mathcal{L}_{\rho(\xi)}\omega = 0 \;\; \forall\, \xi \in \mathfrak{g} \iff \exists\, J : \mathcal{P} \to \mathfrak{g}^* \;\; \textit{such that} \quad i_{\rho(\xi)}\omega = -d\langle J, \xi\rangle \;\forall \xi \in \mathfrak{g}. \tag{2.27}$$

Now, every momentum map must satisfy an appropriate equivariance property with respect to the action ρ on $\mathcal{P}$. Before showing this, we recall the notions of co-adjoint action and cocycle:

- The coadjoint action $ad^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by the following identity:

$$\forall\, \xi, \eta \in \mathfrak{g} \text{ and } \alpha \in \mathfrak{g}^* : \; \langle ad^*_\xi \alpha, \eta\rangle = -\langle \alpha, ad_\xi \eta\rangle, \text{ where } \quad ad_\xi \eta := [\xi, \eta]. \tag{2.28}$$

[7]Whitehead's lemma states that for any finite-dimensional semisimple Lie algebra $\mathfrak{g}$, the first and second cohomology groups $H^1(\mathfrak{g}, V)$ and $H^2(\mathfrak{g}, V)$ vanish for any finite-dimensional $\mathfrak{g}$-module V. The vanishing of these two cohomology groups implies that any element $g \in \mathfrak{g}$ can be expressed as $g = [a, b]$ for some elements a, $b \in \mathfrak{g}$

- Cocycles are the elements in the kernel of the coboundary map in a cochain complex, where with coboundary map we refer to the usual nilpotent operator which let us move along the cochain complex, e.g. the external derivative for the de Rham complex. In particular, in the following we will mention a particular type of cocycle, i.e. the Chevalley-Eilenberg (CE) 2-cocycle, which is an element $\kappa \in (\mathfrak{g} \wedge \mathfrak{g})^*$, satisfying:

$$\kappa\left([\xi_1, \xi_2], \xi_3\right) + \kappa\left([\xi_3, \xi_1], \xi_2\right) + \kappa\left([\xi_2, \xi_3], \xi_1\right) = 0. \tag{2.29}$$

Proposition 2.1.5 (Equivariance) *Let $(\mathcal{P}, \omega)$ be a symplectic manifold equipped with the Hamiltonian action of a Lie algebra $(\mathfrak{g}, \rho)$ with momentum map $J : \mathcal{P} \to \mathfrak{g}^*$. Then, there exists a CE 2-cocycle κ such that:*

$$\mathcal{L}_\xi J(x) = -ad^*_\xi J(x) + \kappa(\xi, \cdot). \tag{2.30}$$

*If $\kappa = 0$ then the momentum map is said to be **equivariant**.*

This result can equivalently be rephrased, in term of the co-momentum map, as follows:

Corollary 2.1.2 (Charge algebra) *The Poisson algebra between the generators $\check{J}(\xi)$, sometimes called the charge algebra, provides a projective representation of $\mathfrak{g}$, i.e.*

$$\left\{\check{J}(\xi), \check{J}(\eta)\right\} = \check{J}([\xi, \eta]) + \kappa(\xi, \eta). \tag{2.31}$$

If $\kappa = 0$ we can restate the previous results simply by saying that, under the action of a symmetry, an equivariant Hamiltonian generator transform in the co-adjoint representation. Why the name charge algebra? Because $\check{J}(\xi)$ will be, as we will see in the next section, exactly the Noether charges.

We recap the main results 2.1.1 and 2.1.2 given above in the following commutative diagram:

$$\begin{tikzcd}
(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \arrow[rr, "\rho"] \arrow[dr, "\mathrm{J}"'] & & (\mathfrak{X}(\mathcal{P}), [\cdot,\cdot]_{T\mathcal{P}}) \\
& (C^\infty(\mathcal{P}), \{\cdot,\cdot\}) \arrow[ur, "X_\cdot"'] &
\end{tikzcd}$$

This commutative diagram tells us that whenever we are dealing with a manifold which admit the action ρ of a Lie algebra $\mathfrak{g}$, the Lie algebra structure of $\mathfrak{g}$ is represented by the set of fundamental vector fields. Besides, if the action is Hamiltonian then we have something more, namely the Lie algebra structure of $\mathfrak{g}$ gets reflected directly on a set of functions, rather than on vector fields, through the co-momentum map. However, these functions coincide with the Hamiltonian generating functions of the above mentioned fundamental vector fields.

We end recalling the notion of a symmetry orbit, using Frobenius theorem:

Corollary 2.1.3 *Let $(\mathcal{P}, \omega, \mathfrak{g}, \rho)$ be a symplectic space acted upon by a symmetry, and reinterpret the action ρ as a map $\rho : \mathfrak{g} \times \mathcal{P} \to T\mathcal{P}$. Then, $Im(\rho) \subset T\mathcal{P}$ is an integrable distribution according to Frobenius theorem, i.e. the flows generated by ρ are necessarily tangent to a certain foliation of $\mathcal{P}$*

We thus define the symmetry orbits as the leaves of this foliation:

Definition 2.1.11 (Symmetry orbits) *We call* ***symmetry orbits*** *the leaves of the foliation generated by ρ. The symmetry orbit passing through the point z is called O_z and the space of the symmetry orbits is symbolically denoted by:*

$$\mathcal{P}/\mathfrak{g} := \{O_z\}. \tag{2.32}$$

If the action is regular enough, namely if it's proper and has no fixed points, we get the structure of a fiber bundle, whose basis is $\mathcal{P}/\mathfrak{g}$ and whose projection map on the base manifold will be the surjective projection $\pi_g : \mathcal{P} \to \mathcal{P}/\mathfrak{g}$, $z \mapsto O_z$.

2.2 Covariant phase space

The term "covariant phase space formalism" is used in the literature to refer broadly to the mathematical framework in which the symplectic analysis of a Lagrangian field theory can be carried out, and so its dynamics. Its name comes from the fact that, thanks to symplectic geometry and the variational bicomplex, we can deal with the Hamiltonian dynamics of field theories without neither referring to any particular set of canonical coordinates nor to any particular specification of a Cauchy surfaces foliation, maintaining at all times full covariance. But what does it mean to study the Hamiltonian dynamics of a lagrangian field theory? What are the relevant quantities? The covariant phase space mainly helps to shed some light on the Poisson brackets structure of the field space, which derives directly - as we have already seen - from the symplectic 2-form, and the conserved quantities of the system's dynamics, on which we will focus our attention for the rest of the discussion. In what follows we are going to formulate Noether's first theorem, together with the less known second one. Afterwards, we are going to review that conserved quantities are not exclusively associated to global spacetime symmetries or internal symmetries, but they can derive from lower degree conservation laws, associated to "non trivial" gauge symmetries. Nevertheless, the gauge redundancies of a lagrangian theory spoils the non-degeneracy property of the symplectic structure, reducing it to the so called "pre-symplectic" structure. The symplectic structure will then be recovered by performing a so-called "symplectic reduction".

2.2.1 The pre-symplectic structure of a jet bundle

We here assemble the geometric tools gathered in the previous sections and apply them to an infinite dimensional field space. The reason is that, as we saw when we introduced jet bundles, describing the phase space of a field theory. However, we have to take an extra step, which is understanding from where the symplectic structure arises in a Lagrangian field theory. Previously we constructed the symplectic 2-form explicitly just for the case of a finite dimensional cotangent bundle. We want to do the same procedure starting from a Lagrangian density. This is easily done, since variations of the fields may be viewed as mixed forms on the variational bicomplex. Let's start decomposing, using Takens theorem (2.1.1), the Lagrangian density of a typical field theory:

Definition 2.2.1 (Euler-Lagrange form and pre-symplectic potential current)

Let $\mathcal{L} \in \Omega^{top,1}(\mathcal{M} \times \mathcal{F})$ be a lagrangian density. Its field differential is a $(top, 1)$-form, which by Takens theorem, splits uniquely into the sum of a source form E and a boundary form $d\theta$:

$$\delta\mathcal{L} = E + d\theta, \quad E = E_I \wedge \delta\varphi^I. \tag{2.33}$$

*The source form E is then called **Euler-Lagrange** form and the $(top-1, 1)$ form θ is called **pre-symplectic potential current**.*

We denote $(top-1, 1)$-forms with the word "current" because we can always associate to them a $(1, 1)$-form through the Hodge dual

$$\star\,\theta = (-1)^d \theta_\mu dx^\mu, \quad \text{with } d = \dim(\mathcal{M}). \tag{2.34}$$

The components of the $(1, 1)$-form θ_μ can then be read using the following useful formulas:

$$\theta = \theta^\mu \varepsilon_\mu, \tag{2.35}$$

$$d\theta = -\nabla_\mu \theta^\mu \varepsilon, \tag{2.36}$$

where

$$\varepsilon = \star\, 1 = \frac{1}{top!}\sqrt{g}\,\mathcal{E}_{\mu_1 \ldots \mu_{top}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{top}} \equiv \sqrt{g} d^{top}x,$$
$$\varepsilon_\mu = i_{\partial_\mu}\varepsilon = \frac{1}{(top-1)!}\sqrt{g}\,\mathcal{E}_{\mu\nu_1 \ldots \nu_{top}} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{top}}.$$

The relevant quantity for a (pre-)symplectic space however, as we saw, is a $(top-1, 2)$-form that we define as follows:

Definition 2.2.2 (Pre-symplectic current) *The **pre-symplectic current** is a $(top-1, 2)$-form defined as the variation, i.e. the field differential, of the pre-symplectic potential:*

$$\Omega = \delta\theta \in \Omega^{top-1,2}(\mathcal{M} \times \mathcal{F}). \tag{2.37}$$

As before, the following relations hold:

$$\star\,\Omega = (-1)^d \Omega_\mu dx^\mu, \quad \Omega = \Omega^\mu \varepsilon_\mu \quad \text{and} \quad d\Omega = \nabla_\mu \Omega^\mu \varepsilon. \tag{2.38}$$

Theorem 2.2.1 *The pre-symplectic current is conserved on shell:*

$$d\Omega \approx 0. \tag{2.39}$$

Definition 2.2.3 (Symplectic flux) *Consider a bounded spacetime region M, with $\partial M = \bar{\Sigma}_{in} \cup \Sigma_{fin} \cup B$. Σ_{in} and Σ_{fin} being two Cauchy surfaces and B being a timelike boundary. Consider $\int_M d\Omega$:*

$$0 \approx \int_M d\Omega = \int_{\Sigma_{fin}} \Omega - \int_{\Sigma_{in}} \Omega + \int_B \Omega. \tag{2.40}$$

Then,

$$\Omega_{\Sigma_{fin}} - \Omega_{\Sigma_{in}} \approx -\int_B \Omega \tag{2.41}$$

is called ***symplectic flux*** *and corresponds to the leakage of degrees of freedom through the boundaries over the time interval* $[t_{in}, t_{fin}]$.

Thus, if we do not have any symplectic fluxes and if we do not have gauge redundancies, the submanifold of on-shell configuration $\bar{\mathcal{F}}$, together with the pull back on it of the symplectic current integral over Σ, is the symplectic space we introduced above.

To conclude this part and prepare ourselves for the next one, we give the last definitions, which transpose the symmetry arguments - made above for a generic symplectic manifold - to the covariant phase space setting.

Definition 2.2.4 (Lagrangian symmetry) *Let a finite dimensional algebra* $\mathfrak{g}$ *act on* $\mathcal{F}$ *according to the Lie algebra homomorphism* ρ*:*

$$\rho : \mathfrak{g} \to \mathfrak{X}(\mathcal{F}), \quad \xi \mapsto \rho(\xi) := \int_{\mathcal{M}} \delta_\xi \varphi^i \frac{\partial}{\partial \varphi^i}. \tag{2.42}$$

The action $(\rho, \mathfrak{g})$ *is a* ***Lagrangian symmetry*** *of* $\mathcal{L} \in \Omega^{top,0}(\mathcal{M} \times \mathcal{F})$ *if and only if there exists a map* $R : \mathfrak{g} \to \Omega^{top-1,0}(\mathcal{M} \times \mathcal{F})$ *such that:*

$$\mathfrak{L}_{\rho(\xi)} \mathcal{L} = dR(\xi)\,. \tag{2.43}$$

In other words, a symmetry of the lagrangian $\mathcal{L}$ is a transformation that changes it up to boundary terms.

Definition 2.2.5 (Symmetry generator) *Let* $\Omega_\Sigma \in \Omega^2(\mathcal{F})$*. A Lie algebra action* $(\rho, \mathfrak{g})$ *on* $\mathcal{F}$ *admits an* ***off-shell symmetry generator*** *on* Σ, $H_\Sigma : \mathfrak{g} \to \Omega^0(\mathcal{F})$, *if and only if:*

$$I_{\rho(\xi)}\, \Omega_\Sigma = -\delta H_\Sigma(\xi) \quad \forall\, \xi \in \mathfrak{g} \tag{2.44}$$

and it admits an ***on-shell symmetry generator*** *on* Σ *if and only if:*

$$I_{\rho(\xi)}\, \Omega_\Sigma \approx -\delta H_\Sigma(\xi) \quad \forall\, \xi \in \mathfrak{g}. \tag{2.45}$$

The vector field $\rho(\xi)$ *is then said to be respectively an* ***off-shell HVF*** *and an* ***on-shell HVF***.

2.2.2 Noether's theorems

We are now ready to state Noether's theorems. First of all, we note that one should distinguish between two type of symmetries, internal symmetries and spacetime symmetries, that can be both local and global transformations. Depending on the type of symmetry, we need to use different set of differentials and interior products, namely the one related to the de Rham complex for spacetime symmetries and the one related to the field space complex for internal ones. In particular, this leads to two different expressions for the charges. Before starting, keep in mind that all the formalism developed until now is compatible both with internal symmetries and spacetime symmetries. The reason is that, having defined symmetries basically as vector fields on sections, we can always use background independence to pass from the field induced by a diffeomorphism to the diffeomorphism itself; same thing holds for the relative Cartan calculus.

Noether's first theorem

Let's start by the case of internal symmetries:

Theorem 2.2.2 (Noether 1) *If $(\rho, \mathfrak{g})$ is a Lagrangian symmetry of $\mathcal{L} \in \Omega^{top,0}(\mathcal{M} \times \mathcal{F})$, then for all $\xi \in \mathfrak{g}$ the $(top-1, 0)$ form*

$$J(\xi) = I_{\rho(\xi)}\theta - R(\xi), \quad dJ(\xi) = \delta_\xi \varphi^i E_i \approx 0 \tag{2.46}$$

*defines a spacetime **Noether current**, which is conserved on shell.*

Definition 2.2.6 (Noether charge) *To a codim-1 hypersurface Σ we associate the **Noether charge***

$$Q_\Sigma(\xi) := \int_\Sigma J(\xi). \tag{2.47}$$

We can alternatively rephrase Noether's first theorem in the following way:

$$Q_{\Sigma_{fin}}(\xi) - Q_{\Sigma_{in}}(\xi) \approx -\int_B J(\xi), \tag{2.48}$$

where B is the same type of timelike boundary as the one mentioned in (2.2.3), when we talked about symplectic fluxes. Similarly, the left hand side of (2.48) above is called **Noether flux**.

To fully appreciate Noether's first theorem from a symplectic point of view, we give the following theorem, proving that Noether charges correspond exactly to the on shell symmetry generators of the action $(\mathfrak{g}, \rho)$:

Theorem 2.2.3 *Let $\mathcal{L} \in \Omega^{top,0}(\mathcal{M} \times \mathcal{F})$ be a lagrangian density and $\Sigma \hookrightarrow \mathcal{M}$ a boundary-less Cauchy surface, i.e. $\partial\Sigma = \emptyset$. Let $\Omega_\Sigma \in \Omega^2(\mathcal{F})$ be the pre-symplectic 2-form associated to $\mathcal{L}$ at Σ, and let $(\mathfrak{g}, \rho)$ be a lagrangian symmetry of $\mathcal{L}$. Denote J as the Noether current of Noether's first theorem. Then, the Noether charge $Q_\Sigma := \int_\Sigma J$ is an on-shell symmetry generator of $(\mathfrak{g}, \rho)$, namely:*

$$I_{\rho(\xi)}\Omega_\Sigma \approx -\delta Q_\Sigma(\xi) \quad \forall\, \xi \in \mathfrak{g}. \tag{2.49}$$

Moreover, on shell, the charge $Q_\Sigma(\xi)$ is:

- *conserved, i.e. independent of the choice for the Cauchy surface Σ.*
- *free of ambiguities, i.e. it is uniquely determined by the (local) spacetime equivalence class $[\mathcal{L}] = [\mathcal{L} + dl]$.*

In the simplest case, the Noether charge is an off-shell symmetry generator.

Proposition 2.2.1 *If $\mathfrak{L}_{\rho(\xi)}\mathcal{L} = 0$ and $\mathfrak{L}_{\rho(\xi)}\theta = 0 \ \forall \xi \in \mathfrak{g}$, then $R = 0$ and*

$$J(\xi) = I_{\rho(\xi)}\theta \quad \textit{and} \quad I_{\rho(\xi)}\Omega = -\delta J(\xi). \tag{2.50}$$

Integrating on a $\Sigma \hookleftarrow \mathcal{M}$, $Q_\Sigma(\xi) = I_{\rho(\xi)}\theta_\Sigma$ is an off shell symmetry generator of $(\mathfrak{g}, \rho)$.

In the case of spacetime symmetries, we can use background independence to express $R(\xi)$ in a different way:

$$dR(\xi) = \mathfrak{L}_{\rho(\xi)}\mathcal{L} = \mathcal{L}_\xi \mathcal{L} = d\, i_\xi \mathcal{L} \;\Rightarrow\; R(\xi) = i_\xi \mathcal{L}. \tag{2.51}$$

Noether's second theorem

Now, let's imagine we wanted to localize the internal symmetry, meaning that we want to choose pointwise on $\mathcal{M}$ the value of the transformation; the parameter is now a function on the spacetime. This "local" version of the corresponding global symmetry is what we would usually refer to as a "gauge" transformation. What we want to prove in the following is that, although the Noether charge, meant as integral of the Noether current, always vanish on shell, it's still possible to define non trivial conserved quantities associated to them. The reason is that what should be regarded as Noether charge is not merely the integral of the Noether current, but instead the on-shell symmetry generator as defined in (2.49). Recall that Noether currents are always defined up to the divergence of an arbitrary $(top-2,0)$-form, supported on a codim-2 hypersurface called "corner", as mentioned in the second point of theorem (2.2.3). We will then have as Nother charge:

$$Q_\Sigma = \int_\Sigma J + dk \approx \int_{\partial\Sigma} k. \tag{2.52}$$

One of the reason why the corner contribution has been disregarded in the past, is that this $(top-2,0)$-form is completely arbitrary. Because of this, the possibility of defining conserved quantities associated to such local symmetries remained a puzzle for quite some time. Eventually, Barnich, Brandt and Hennaux in [32] and [33], and Wald et al. in [34],[35] and [36], understood that the correct way to define the Noether charges of such local symmetries was not through the Noether currents, but through a $(top-2,1)$-form, which can be single out uniquely from the symplectic structure of the theory. This is exactly the on-shell symmetry generator. In general this quantity will not be neither integrable nor conserved but for our purposes we can consider them both. The reason is that in 3D gravity there are neither symplectic fluxes nor Noether fluxes that can contribute to such obstructions. To argue what we just said, we are going to first introduce Noether's second theorem and afterwards see how this $(top-2,1)$-form can be computed in the case of general relativity.

Let's start recalling two well-known facts about local symmetries. Let us also denote the localized version of the Lie algebra with $\mathfrak{G}$. The first one is the following:

Theorem 2.2.4 *If $(\mathfrak{G}, \rho)$ is a local Lagrangian symmetry, then the equation of motions fails to be linearly independent from each other, since the following expression must identically vanish on shell:*

$$(D^\dagger)^i_\mu E_i = 0, \tag{2.53}$$

where $(D^\dagger)^i_\mu$ is the adjoint operator of D^i_μ, with $D^i_\mu \xi^\mu \equiv \delta_\xi \varphi^i$ appearing in (2.46); the adjoint operator is valid for diffeomorphisms with support in the interior of $\mathcal{M}$, so that we can neglect boundary terms. The relations above are known as ***Noether identities****.*

The theorem above basically asserts that in the phase space of a theory with gauge redundancies the initial value problem is not well posed. Since the equations of motions are not all independent, they are not sufficient to completely determine the dynamics. The other well known fact is that the Noether charge, meant as the integral of the Noether current, vanishes on-shell:

Theorem 2.2.5 (Vanishing charges) *If $(\mathfrak{G}, \rho)$ is a local Lagrangian symmetry and $\partial\Sigma = \emptyset$, then Σ is a Cauchy surface and all the Noether charges of $\mathfrak{G}$ vanish on shell:*

$$Q_\Sigma \approx 0. \tag{2.54}$$

Noether's second theorem then states that the Noether current associated to this local transformation will always be zero on shell, modulo $d-$exact terms:

Theorem 2.2.6 (Noether's second theorem) *Let $(\mathfrak{G}, \rho)$ be a local Lagrangian symmetry, then for all $\xi \in \mathfrak{G}$ the Noether current $J(\xi)$ is exact on shell, i.e. J can be put in the following form:*

$$J(\xi) = \xi^\mu C_\mu + dQ(\xi), \quad \textit{where } C_\mu \approx 0. \tag{2.55}$$

*The term $Q(\xi)$ is called **Noether-Wald surface charge**.*

Theorem 2.2.5 now becomes obvious. In the absence of corners the Noether current always vanishes on shell, being proportional to the equations of motion, and so do the charges as a consequence.

Let's see a simple example to fix the ideas and derive the Noether-Wald surface charge for the Einstein-Hilbert action.

- The starting point is Einstein-Hilbert action, whose Lagrangian density take the following form:

$$\mathcal{L} = \sqrt{g}\left(\frac{1}{2}R - \Lambda\right). \tag{2.56}$$

- We can read the pre-symplectic potential θ as the d-exact term resulting from the variation of the lagrangian:

$$\delta\mathcal{L} = -\frac{1}{2}\sqrt{g}\left(G^{\mu\nu} + \Lambda g^{\mu\nu}\right)\delta g_{\mu\nu} + \frac{1}{2}\nabla_\mu\left[g^{\mu\nu}g^{\rho\sigma}(\nabla_\rho\delta g_{\nu\sigma} - \nabla_\nu\delta g_{\rho\sigma})\right], \tag{2.57}$$

in accordance with (2.2.1).

- With the pre-symplectic potential we can compute the Noether current associated to an infinitesimal diffeomorphism ξ, according to (2.2.2) and (2.51):

$$J^\mu(\xi) = I_{\rho(\xi)}\theta^\mu - R^\mu(\xi), \tag{2.58}$$

with

$$\begin{aligned}
R^\mu(\xi) &= \left(\frac{1}{2}R - \Lambda\right)\xi^\mu, \\
I_{\rho(\xi)}\theta = i_\xi\theta &= \frac{1}{2}\left(\nabla_\rho\nabla^\mu\xi^\rho + \nabla_\rho\nabla^\rho\xi^\mu - 2\nabla^\mu\nabla_\rho\xi^\rho\right) \\
&= \frac{1}{2}g^{\mu\nu}\left(\nabla_\rho\nabla_\nu\xi^\rho + \nabla_\rho\nabla^\rho\xi_\nu - 2\nabla_\nu\nabla_\rho\xi^\rho\right) \\
&= -g^{\mu\nu}[\nabla_\nu, \nabla_\rho]\xi^\rho + \frac{1}{2}g^{\mu\nu}\nabla_\rho\left(\nabla^\rho\xi_\nu - \nabla_\nu\xi^\rho\right) \\
&= g^{\mu\nu}R_{\nu\rho\sigma}{}^\rho\xi^\sigma + \frac{1}{2}\nabla_\rho\left(\nabla^\rho\xi^\mu - \nabla^\mu\xi^\rho\right) \\
&= R^{\mu\sigma}\xi_\sigma + \frac{1}{2}\nabla_\rho\left(\nabla^\rho\xi^\mu - \nabla^\mu\xi^\rho\right),
\end{aligned}$$

where we manipulated the quantities in order to single out in the current two terms, one proportional to the equations of motion and the other one being the divergence of the Noether-Wald charge. In the end we obtain:

$$J^\mu(\xi) = \left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu}\right)\xi_\nu + \frac{1}{2}\nabla_\nu\left(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu\right). \tag{2.59}$$

Here notice that we computed the Hodge dual of the Noether current, whose components may be recovered using (2.35).

Despite the name, the Noether-Wald charge is not the only contribution to the symmetry generator. The pre-symplectic potential contributes as well to the $(top-2,1)$-form, which is the ultimately quantity we will be interested in. Let's see that this is indeed the case:

Proposition 2.2.2 *Let $(\mathfrak{G}, \rho)$ a local Lagrangian symmetry of $\mathcal{L}$. Then there exists a $\mathbb{R}$-linear map r,*

$$r : \mathfrak{G} \to \Omega^{top-2,1}(\mathcal{M} \times \mathcal{F}) \tag{2.60}$$

such that

$$I_{\rho(\xi)}\Omega \approx \delta J(\xi) + dr(\xi). \tag{2.61}$$

We can rephrase this proposition as the following off-shell relation:

$$I_{\rho(\xi)}\Omega = -\delta J(\xi) + (\mathfrak{L}_{\rho(\xi)}\theta - \delta R(\xi)) = -\delta J(\xi) + s, \tag{2.62}$$

with $ds \approx 0$. We can then use the algebraic Poincaré lemma to prove that s is an exact $(top-2,1)$ form. Let us first consider the case of a spacetime symmetry and use background independence to rewrite the last expression above. First we have:

$$\mathfrak{L}_{\rho(\xi)}\theta = \mathcal{L}_\xi\theta = d\, i_\xi\theta + i_\xi\, d\theta. \tag{2.63}$$

Then, recalling the previous result $R(\xi) = i_\xi\mathcal{L}$ we have also:

$$\delta R(\xi) = i_\xi\delta\mathcal{L} = i_\xi E^{\mu\nu}\delta g_{\mu\nu} + i_\xi d\theta. \tag{2.64}$$

Putting everything together, we get:

$$\mathfrak{L}_{\rho(\xi)}\theta - \delta R(\xi) = di_\xi\theta - i_\xi E^{\mu\nu}\delta g_{\mu\nu}, \tag{2.65}$$

which leads to the following Hamiltonian equation flow:

$$I_{\rho(\xi)}\Omega = -\delta J(\xi) + di_\xi\theta - i_\xi E^{\mu\nu}\delta g_{\mu\nu}. \tag{2.66}$$

Take a codimension 1 hypersurface $\Sigma \hookrightarrow \mathcal{M}$ and suppose we can move the operator δ in and out of integrals over Σ[8]. Then we can write:

$$I_{\rho(\xi)}\Omega_\Sigma \approx -\delta Q_\Sigma(\xi) + \int_{\partial\Sigma} i_\xi\theta. \tag{2.67}$$

[8]These manipulations are not harmless and they involve the theory behind embeddings in the covariant phase space. See for example [29] as a reference.

We finally define the infinitesimal surface charge k_ξ:

Definition 2.2.7 *We define as* ***superpotential*** k_ξ*, or equivalently as* ***infinitesimal surface charge****, the unique*[9] $(top-2,1)$*-form satisfying the following identity:*

$$\boxed{I_{\rho(\xi)}\Omega \approx dk_\xi} \tag{2.68}$$

The superpotential can then be expressed in terms of the Noether-Wald surface charge and the pre-symplectic potential as follows:

$$k_\xi \approx -\delta Q_\xi + i_\xi\theta. \tag{2.69}$$

Accordingly, we define the total charge as follows:

Definition 2.2.8 *Let* k_ξ *be the superpotential associated with a diffeomorphism* ξ*. We define the* ***total charge*** *associated to* ξ *as:*

$$\not{\delta} H_\xi = \int_C k_\xi. \tag{2.70}$$

In the case of an internal symmetry the discussion simplifies to the case described in 2.2.1. This implies that the exact $(top-2,1)$ form s, introduced in (2.62), vanishes and the superpotential reduces to

$$\kappa_\xi \approx -\delta Q_\xi \tag{2.71}$$

2.2.3 Noether's identities and surface charges in Cartan formalism

We briefly summarize here the derivation for the Noether identities and the Hodge dual of the superpotential form k, in Cartan formalism. Since our discussion is going to be carried out in tetrad formalism, we derive $k_\xi^{\mu\nu}$ in terms of the tetrad fields and the spin coefficients. As a reference we follow [30], but one could also refer to [31] for a more NP oriented discussion.

We already saw in section 1.2.4 what the variation of Einstein-Cartan action looks like:

$$\delta S^C = \int d^n x \mathbf{e}\left[2(G^a_\mu + \Lambda e^a_\mu)\delta e^\mu_a + e^\mu_a e^\nu_b(\nabla_\mu \delta\Gamma^{ab}_\nu - \nabla_\nu \delta\Gamma^{ab}_\mu)\right]. \tag{2.72}$$

Under an infinitesimal gauge transformation we get

$$\delta_{\xi,\omega} S^C = \int d^n x \left[\frac{\delta\mathcal{L}^C}{\delta e^\mu_a}\delta_{\xi,\omega} e^\mu_a + \frac{\delta\mathcal{L}^C}{\delta\Gamma^{ab}_\mu}\delta_{\xi,\omega}\Gamma^{ab}_\mu\right], \tag{2.73}$$

where

$$\begin{aligned}
\delta_{\xi,\omega} e^a_\mu &= \mathcal{L}_\xi e^a_\mu + \omega^a{}_b e^b_\mu, \\
\delta_{\xi,\omega}\Gamma^{ab}_\mu &= \mathcal{L}_\xi \Gamma^{ab}_\mu - (d\omega^{ab} + \Gamma^a{}_{c\mu}\omega^{cb} + \Gamma^b{}_{c\mu}\omega^{ac}).
\end{aligned}$$

Integrating by parts to isolate the undifferentiated gauge parameters as in 2.2.4, we arrive at the

[9] Up to $(top-3,1)$-forms

following Noether's identities

$$\frac{\delta \mathcal{L}^C}{\delta e^{[a|\mu|}} e^{\mu}_{b]} + \nabla_\mu \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_{\mu}} = 0, \tag{2.74}$$

$$\frac{\delta \mathcal{L}^C}{\delta e^{\mu}_a} \partial_\rho e^{\mu}_a + \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}_{\mu}} \partial_\rho \Gamma^{ab}_{\mu} + \partial_\mu \left(\frac{\delta \mathcal{L}^C}{\delta e^{\rho}_a} e^{\mu}_a - \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_{\mu}} \Gamma^{ab}_{\rho} \right) = 0, \tag{2.75}$$

which are exactly the contracted Bianchi identities.

Turning to the superpotential, the weakly vanishing current associated to the gauge symmetries is

$$J^{\mu}_{\xi,\omega} = \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}_{\mu}} (-\omega^{ab} + \Gamma^{ab}_{\rho} \xi^\rho) - \frac{\delta \mathcal{L}^C}{\delta e^{\rho}_a} e^{\mu}_a \xi^\rho. \tag{2.76}$$

The associated co-dimension 2 form $k_{\xi,\omega} = k^{\mu\nu} (\star\, d^{top-2})_{\mu\nu}$ is then given by

$$k^{\mu\nu}_{\xi,\omega} = \mathbf{e}\left[\left(2\, \delta e^{\mu}_a\, e^{\nu}_b + e^{c}_{\lambda}\, \delta e^{\lambda}_c\, e^{\nu}_a\, e^{\mu}_b \right) \left(-\omega^{ab} + \Gamma^{ab}{}_{\rho}\, \xi^\rho \right) + \delta \Gamma^{ab}{}_{\rho} \left(\xi^\rho e^{\mu}_a\, e^{\nu}_b + 2\, \xi^\mu e^{\nu}_a\, e^{\rho}_b \right) - (\mu \leftrightarrow \nu) \right], \tag{2.77}$$

which can also be written as

$$k_{\xi,\omega} = -\delta K^K_{\xi,\omega} + K^K_{\delta\xi,\delta\omega} - \xi^\nu \frac{\partial}{\partial dx^\nu} \theta_\xi, \tag{2.78}$$

with

$$K^K_{\xi,\omega} = 2\, \mathbf{e}\, e^{\nu}_a e^{\mu}_b (-\omega^{ab} + \Gamma^{ab}{}_{\rho}\, \xi^\rho)(\star\, d^{top-2} x)_{\mu\nu}, \quad \theta_\xi = 2\, \mathbf{e}\, \delta \Gamma^{ab}{}_{\rho} e^{\mu}_a e^{\rho}_b (\star\, d^{top-1} x)_\mu. \tag{2.79}$$

In this expression, $\mathbf{e}$ is the determinant of the tetrad field, which is simply $\sqrt{g}$. Besides, the second term in (2.78) accounts for explicit field dependencies in the gauge symmetry parameters.

2.3 Asymptotic symmetries

We have seen that given a theory invariant under local symmetry transformations, we can associate a unique Noether charge supported on a corner. We wish to apply this machinery to a classical field theory, built upon a manifold $\mathcal{M}$ with a boundary. Here, when we talk about a boundary, we are referring to the boundary in the non-physical compactification of the manifold. However, this fictitious boundary when pushed to infinity in the physical manifold $\mathcal{M}$, does not stop to support the Noether charges. Simply, the value of the fields on the boundary in the compactified manifold translate, in the uncompactified version, to the fall-off behaviours of the fields as they approach infinity. We refer again to [29].

Let's start fixing uniquely a classical field theory on a manifold. Let $\mathcal{M}$ be the manifold with boundary C and let $\mathcal{F}$ be the field space, as before. A classical dynamical theory is then fixed by:

- The dynamics on $\mathcal{M}$, i.e. a well defined action.
- Some set of boundary conditions $\Gamma|_C$ to be specified near the boundary. This fixes uniquely the behaviour of the fields as they approach the boundary.
- Some gauge fixing conditions, in case we are dealing with a gauge theory.

The last point is subtle and we are going to discuss it with more attention later in our analysis. However, given these three points we have a theory. The next step usually is to understand the physical content of this theory, meaning the charges that it hosts. To get them we have to apply the machinary developed above:

- The first step is finding the local symmetry transformations which preserves both the gauge fixing conditions and the fall-off conditions of the theory. We call them **residual gauge symmetries**. If we do not filter this subset, we would be considering transformations between different theories. This could be confusing if one think about these transformations as simply gauge ones. Why should we care about fall-off behaviors of local symmetries? Recall, we saw above that the physical content carried by each one of these transformation lives on the boundary, i.e., near infinity. So, in principle, local symmetries that behave differently at infinity could have different surface charges associated to them, meaning that their action could generate different charges. Without fall-off conditions we would not have control on which surface charges are actually present in the theory.
- The second step is computing the surface charges associated to the residual gauge symmetries. If the charge is zero, then the local transformation is a true gauge transformation, which we call **trivial transformation**, and thus it is a true redundancy of the theory. If the charge, on the other hand, is non-vanishing, then the local symmetry is called **asymptotic symmetry** and it is a true physical transformation, acting non-trivially on the field space. The latter maps the system in a non-physically-equivalent configuration. These are also called **improper** or **large transformations**.
- Finally, the asymptotic symmetry group is defined as the quotient of the residual symmetries with the trivial ones. This is possible, since the trivial symmetry group is an ideal inside the residual symmetry group. This quotienting procedure promotes our presymplectic structure to a proper symplectic one. This procedure is also known by the name of **symplectic reduction**.
- With the symplectic reduction, we got a Poisson structure too. We can then compute the Poisson brackets of the charges, which then gives rise to the algebra organizing the physical observables of the theory.

In general however, these charges are not well defined, i.e. they lack one or more of the following properties:

- **Finiteness**: as we approach the co-dimension 2 corner C considered, the charges remain finite.
- **Integrability**: $i_\xi \Omega = -\int_C Q_\xi$, i.e. the action of the symmetry ξ is Hamiltonian with respect to the symplectic structure. If this holds then the charges are called canonical Noether charges.
- **Conserved**: $Q_\xi|_{C_{fin}} - Q_\xi|_{C_{in}} = \int_{C_{in}}^{C_{fin}} dQ_\xi \approx 0\,.$

Altought we can consider a theory well-defined only when the Noether charges deriving from it are, according to the just mentioned properties, 4D general relativity is the perfect example of a theory giving charges which are not well defined.

Finally, it is important to remark that these surface charges are related to classically observables effects taking place in the asymptotic region of the spacetime, called memory effects. Concretely, a memory effect is a net change in matter distribution of matter due to radiation.

Chapter 3

Double null analysis of 3D asymptotically flat spacetimes

The standard approach to study the asymptotic symmetries and the surface charges of asymptotically flat gravity was developed by Sachs [2][3][4], Bondi, Metzner et al. [5][6] in the sixties. In this so-called "Bondi-Sachs" approach, thanks to a null foliation of the spacetime, the affine parameter u of the null geodesics is taken as null coordinate, substituting the time coordinate t. Such a coordinate system enables us to reach future null infinity $\mathcal{I}^+$ by sending the radial coordinate r to infinity. Alternatively, the Bondi-Sachs approach can as well be adapted to reach the past null infinity $\mathcal{I}^-$, by employing a perpendicular null foliation. Both these choices eventually lead to a physically-relevant solution space, in which the propagating degrees of freedom may be studied. The observable quantities related to them are encoded in the surface charges of the theory. Besides, surprisingly enough, as one approaches the conformal null boundary of the spacetime, the asymptotic symmetries do not reduce to the usual Poincaré group, but to a larger, infinite-dimensional version of it, called "BMS group". This infinite-dimensional-enhancement of the Poincaré group is due to the infrared degrees of freedom of the bulk reaching conformal infinity.

In what follows we develop an alternative approach to study the asymptotic symmetries and the surface charges of an asymptotically flat spacetime in 3D gravity, based on a double null foliation of the spacetime. We formulate such an approach using the NP formalism, being a natural choice for the null framework we are considering. We first propose a "double null" gauge and a set of boundary conditions, leading to a physically-relevant solution space; this is shown to include both Bondi-Sachs solution spaces. Afterwards, we proceed to compute the charges, finding two super-translation charges and a super-rotation charge. The two super-translation charges achieve a chiral decomposition of the ordinary BMS super-traslation charge into two independent null components. Eventually, we compute the asymptotic symmetry algebra and the charge algebra.

In the end we will show that such double null formalism is able, exploiting the double null foliation, to unify both Bondi-Sachs approaches in a single coordinate patch. Our analysis is analytical, in the sense that we do not solve any equation asymptotically. Thus, the discussion will not be limited to the asymptotic boundaries $\mathcal{I}^+$ and $\mathcal{I}^-$, but it will hold in the bulk as well.

3.1 Gauge choice and solution space

A judicious choice of gauge is the starting point of our discussion. This defines, together with some appropriate boundary conditions that we impose later on, the solution space that we intend to study. The gauge fixing conditions in particular are constraints on the tetrad's components, which simplify the equations of motion enough to solve them. However, the choice should not overconstrain the system, in order to admit non-zero surface charges. If we do not have any charge, the spacetime then is simply Minkowski written in some complicated coordinate system. Now, what should drive our choice for the gauge fixing conditions? There is a more practical point of view and a more geometric one. We present in the first section the practical way to proceed and later on, having solved the solution space, we will see how the geometry could have driven our initial choice of the gauge.

3.1.1 Gauge choice

Asymptotically flat metrics in the Bondi-Sachs gauge are written in (u, r, φ) coordinates, which are the ones adapted to asymptote to $\mathcal{I}^+$, taking the form

$$ds^2 = \Theta^+(\varphi)\, du^2 - 2\, du\, dr + \left[\Xi^+(\varphi) + u\, \partial_\varphi \Theta^+(\varphi)\right]\, du\, d\varphi + r^2 d\varphi^2\,, \tag{3.1}$$

with a set of boundary conditions, expressing the asymptotic flatness. The field space is parameterized by two arbitrary functions, representing the generalized mass and angular momentum of the spacetime enclosed by $\mathcal{I}^+$. The same class of spacetimes can as well be described in a (v, r, φ) coordinate system, adapted to asymptote to $\mathcal{I}^-$. As the one above, the metric takes the form

$$ds^2 = \Theta^-(\varphi)\, dv^2 - 2\, dv\, dr + \left[\Xi^-(\varphi) + v\, \partial_\varphi \Theta^-(\varphi)\right]\, dv\, d\varphi + r^2 d\varphi^2\,, \tag{3.2}$$

a priori parameterized by two different functions. Our goal now is to find a setting - some gauge conditions - in (u, v, φ) coordinates, in which the solution space we get combines (3.2) and (3.1) . Employing double null coordinates, the simplest solution compatible with a double null foliation has the du^2 and dv^2 terms set to zero, a $dv\, d\varphi$ term and $du\, d\varphi$ term representing an angular-momentum-like contribution, and a $d\varphi^2$ term. What we expect then is a metric of the form

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & \tilde{J}_u(u, v, \varphi) \\ 1 & 0 & \tilde{J}_v(u, v, \varphi) \\ \tilde{J}_u(u, v, \varphi) & \tilde{J}_v(u, v, \varphi) & \xi(u, v, \varphi) \end{pmatrix}\,, \tag{3.3}$$

that we parametrized using three arbitrary functions of (u, v, φ). It is then straightforward to derive which conditions we have to impose on the most general tetrad field to obtain such a class of metrics, namely

$$l = \partial_v \quad \text{and} \quad n = \partial_u\,. \tag{3.4}$$

These represent our "double null" gauge fixing conditions. No condition has to be imposed on the m-component of the tetrad, except for the orthogonality conditions of the NP formalism. These yield, in terms of the functions parametrizing the metric, the form

$$m = \frac{1}{\sqrt{2\left(2\tilde{J}_u\tilde{J}_v - \xi\right)}}(-\tilde{J}_v\partial_u - \tilde{J}_u\partial_v + \partial_\varphi). \tag{3.5}$$

Furthermore, computing the associated cotetrad by lowering the indices, we find that

$$\begin{aligned} l &= du + \tilde{J}_v d\varphi\,, \\ n &= dv + \tilde{J}_u d\varphi\,, \\ m &= -\sqrt{\frac{2\tilde{J}_u\tilde{J}_v - \xi}{2}}d\varphi\,. \end{aligned} \tag{3.6}$$

In conclusion, (3.4) and (3.5) give us an ansatz for the null tetrad field.

3.1.2 Computing the solution space

As we saw in the previous sections, the equations of motion in the NP formalism consist in two sets of first order differential equations, involving the spin coefficients (1.59) and the tetrad components (1.56), which reduce in 3D gravity respectively to (1.88) and (1.89). To derive the solution space, we compute the action of the tetrad components, that may be used as differential operators, on the coordinates system (u, v, φ):

$$\begin{aligned} &Du = 0, && \Delta u = 1, && \delta u = -\frac{\tilde{J}_v}{\sqrt{2\left(2\tilde{J}_u\tilde{J}_v - \xi\right)}}, \\ &Dv = 1, && \Delta v = 0, && \delta v = -\frac{\tilde{J}_u}{\sqrt{2\left(2\tilde{J}_u\tilde{J}_v - \xi\right)}}, \\ &D\varphi = 0, && \Delta\varphi = 0, && \delta\varphi = -\frac{1}{\sqrt{2\left(2\tilde{J}_u\tilde{J}_v - \xi\right)}}. \end{aligned}$$

Now, applying the commutation relations (1.89) to the coordinate functions, we obtain either differential equations involving the functions that parametrize the null tetrad, or constraints on the spin coefficients:

$$\text{Constraints:} \quad \begin{cases} \varepsilon = 0 \\ \gamma = 0 \\ \tau + \pi = 0 \end{cases} \tag{3.7}$$

$$
\text{Equations:} \quad \begin{cases} P\,\partial_v \tilde{J}_v = \kappa \\ P\,\partial_u \tilde{J}_v = -(\beta+\pi) \\ P\,\partial_u \tilde{J}_u = -\nu \\ P\,\partial_v \tilde{J}_u = -(\pi-\beta) \\ \partial_v P = 2\,\sigma P \\ \partial_u P = -2\,\mu P \end{cases} \tag{3.8}
$$

where we defined

$$
P(u,v,\varphi) := \frac{1}{\sqrt{2\left(2\tilde{J}_u\tilde{J}_v - \xi\right)}} \,. \tag{3.9}
$$

Similarly, the spin coefficients equations, once simplified with (3.7), yield the set of equations

$$
\text{Equations:} \quad \begin{cases} \partial_v\sigma + P\,(\tilde{J}_v\partial_u\kappa + \tilde{J}_u\partial_v\kappa - \partial_\varphi\kappa) = 2\sigma^2 + 2(\pi-\beta)\kappa \\ \partial_v\pi + \partial_u\kappa = 0 \\ \partial_v\beta = 2(\beta+\pi)\sigma - 2\mu\kappa \\ \partial_v\mu + P\,(\tilde{J}_v\partial_u\pi + \tilde{J}_u\partial_v\pi - \partial_\varphi\pi) = 2\sigma\mu + 2\pi^2 - \kappa\nu \\ \partial_v\nu - \partial_u\pi = 0 \\ \partial_u\mu + P\,(\tilde{J}_v\partial_u\nu + \tilde{J}_u\partial_v\nu - \partial_\varphi\nu) = -\mu^2 + 2(\pi+\beta)\nu \\ \partial_u\beta = -2(\beta-\pi)\mu + 2\sigma\nu \\ \partial_u\sigma - P\,(\tilde{J}_v\partial_u\pi + \tilde{J}_u\partial_v\pi + \partial_\varphi\pi) = -2\mu\sigma - 2\pi^2 + \nu\kappa \end{cases} \tag{3.10}
$$

$$
\text{Constraints:} \quad \left\{ 2\pi^2 + 2\kappa\nu = 0 \right. \tag{3.11}
$$

These two set of equations however are still too complicated to solve; we need further gauge fixing conditions. Particularly problematic is how the spin coefficients κ, π and ν enters in (3.10); we also see from (3.8) that these spin coefficients control how $\tilde{J}_v$ and $\tilde{J}_u$ change when moving along the u and v directions. We thus decide to set them to zero, being now able to solve the equations completely. The simplest setting is achieved by setting to zero β as well, so that $\tilde{J}_u$ and $\tilde{J}_v$ depend just on φ. The equations of motion, in our gauge, thus yield

$$
\varepsilon = \gamma = \tau = \pi = \beta = \kappa = \nu = 0\,, \tag{3.12}
$$

$$
\sigma = -\frac{1}{2\left(v - u\,e^{-B(\varphi)} + \tilde{A}(\varphi)\right)}\,, \qquad \mu = -\frac{1}{2\left(v - u\,e^{-B(\varphi)} + \tilde{A}(\varphi)\right)e^{B(\varphi)}}\,, \tag{3.13}
$$

$$
l = \partial_v\,, \qquad n = \partial_u\,, \qquad m = \frac{\left(-\tilde{J}_v(\varphi)\partial_u - \tilde{J}_u(\varphi)\partial_v + \partial_\varphi\right)}{\left(v - u\,e^{-B(\varphi)} + \tilde{A}(\varphi)\right)e^{\psi(\varphi)}}\,, \tag{3.14}
$$

which give rise to the following metric

$$ds^2 = 2\left(du + \tilde{J}_v(\varphi)d\varphi\right)\left(dv + \tilde{J}_u(\varphi)d\varphi\right) - \frac{1}{2}\left(v - u\, e^{-B(\varphi)} + \tilde{A}(\varphi)\right)^2 e^{2\psi(\varphi)}\, d\varphi^2\,. \tag{3.15}$$

3.1.3 A geometric approach to the gauge choice

As anticipated, we here show how the choice of gauge could have been driven by geometric arguments. First of all, recall that we are interested in solutions admitting a double null foliation. The reason is that with a double null foliation we can easily access both conformal null boundaries, being then just leaves of the two foliations. A sufficient condition to have a double null foliation is that the manifold admits two hypersurface orthogonal vector fields (see appendix A for more). But what constraints such an assumption put on the tetrad built on it? We actually anticipated the answer in section 1.3.5: in 3D gravity no one! As we saw, the condition for hypersurface orthogonality set by Frobenius theorem was a reality condition on some spin coefficients, which in 3D gravity are always automatically real (since we do not have a complex tetrad vector). Since asking just for a double null foliation, in 3D, does not enforce additional constraints, we need to design some further condition to regularize the foliation. The typical way to regularize a generic foliation, defined through a vector field, is twofold: we can either regularize its congruence of integral curves or make its associated 1-form, as defined in Frobenius theorem, closed. Let's see what constraints put both request on the fields, namely the tetrad and the spin coefficients:

- The integral curves of the two vector fields are a congruence of null curves, which in general are neither geodesics nor affinely parametrized. Promoting them to be affinely parametrized geodesics is the first step to retrieve the double null gauge. We will then take the two affine parameters as coordinates for our spacetime. What constraints do these two requests put on the ansatz for the fields? Again, referring to section 1.3.5, we can compute the variation of l and n in their directions to find that

$$D\, l = \varepsilon l - \kappa m \quad \text{and} \quad \Delta\, n = -\gamma n + \nu m\,. \tag{3.16}$$

  Therefore, to have two congruences of affinely parametrized geodesics, we need to impose

$$\varepsilon = \gamma = \kappa = \nu = 0. \tag{3.17}$$

- Promoting the 1-form associated to the vector field to a closed form, means promoting the leaves of the foliation to be the level curves of some scalar function. We can compute what obstructions distinguish the property of being hypersurface orthogonal from being the gradient of a scalar function

$$l_{[\mu;\nu]} = \varepsilon l_{[\mu}n_{\nu]} - 2\beta l_{[\mu}m_{\nu]} - 2\kappa m_{[\mu}n_{\nu]} - 2\tau m_{[\mu}l_{\nu]}\,, \tag{3.18}$$
$$n_{[\mu;\nu]} = -\gamma l_{[\mu}n_{\nu]} + 2\beta l_{[\mu}m_{\nu]} + 2\pi m_{[\mu}n_{\nu]} + 2\nu m_{[\mu}l_{\nu]}\,. \tag{3.19}$$

  Suppose we have already imposed that both congruences of integral curves are made of affinely parametrized geodesics. Then we read the constraint

$$\beta = \tau = \pi = 0\,. \tag{3.20}$$

Finally, if we take the two affine parameters as coordinates - and call them u and v - we have derived our gauge fixing conditions. As a final summary, we have obtained that the geometric condition to impose on the spacetime, in order to derive the gauge fixing conditions previously made, is the existence of two vector fields, whose integral curves form a congruence of null geodesics affinely parametrized, and whose associated 1-form is closed.

3.1.4 A physical parametrization of the solution space

We now focus on the solution space that we found. Recall that we started our discussion looking for a double null gauge, stringent enough to let us solve the equations of motion - as we did - but also loose enough to encode all the physical content of the standard BMS approach. We check here that the latter is indeed the case.

The physical content we would like our double null gauge to encode are the four functions appearing inside the BMS metrics in their respective patches; let us call them M_u, M_v, J_u and J_v. The natural question to ask then is:

- Does the gauge we chose encode these degrees of freedom? If so, how are they distributed among the functions $\tilde{J}_u$, $\tilde{J}_v$, B, $\tilde{A}$ and ψ appearing in our solution space?

The answer to the first question is positive, while the second one leads to the reparametrization

$$\tilde{J}_u = J_u e^{-Mv}, \quad \tilde{J}_v = J_v e^{-Mu,}, \quad B = M_v - M_u, \quad \Psi = M_v, \quad \tilde{A} = Ae^{-M_v}. \tag{3.21}$$

The newly parametrized solution space takes then the form

$$\sigma = -\frac{e^{M_v(\varphi)}}{2\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)}\,, \qquad \mu = -\frac{e^{M_u(\varphi)}}{2\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)}\,, \tag{3.22}$$

$$l = \partial_v\,, \qquad n = \partial_u\,, \qquad m = \frac{\left(-J_v(\varphi)e^{-M_u(\varphi)}\,\partial_u - J_u(\varphi)e^{-M_v(\varphi)}\,\partial_v + \partial_\varphi\right)}{\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)}\,, \tag{3.23}$$

which gives rise to the following metric

$$ds^2 = 2\left(du + J_v(\varphi)e^{-M_u(\varphi)}d\varphi\right)\left(dv + J_u(\varphi)e^{-M_v(\varphi)}d\varphi\right) - \frac{1}{2}\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)^2 d\varphi^2\,. \tag{3.24}$$

It turns out to be useful to also rescale all the fields appearing in the solution space by a factor $\frac{1}{\sqrt{2}}$, in order to avoid any factor $\sqrt{2}$ in the charges. Eventually, we obtain the solution space

$$\sigma = -\frac{e^{M_v(\varphi)}}{2\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)}\,, \qquad \mu = -\frac{e^{M_u(\varphi)}}{2\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)}\,, \tag{3.25}$$

$$l = \partial_v\,, \qquad n = \partial_u\,, \qquad m = \frac{\sqrt{2}\,\left(-J_v(\varphi)e^{-M_u(\varphi)}\,\partial_u - J_u(\varphi)e^{-M_v(\varphi)}\,\partial_v + \partial_\varphi\right)}{\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)}\,, \tag{3.26}$$

which leads to

$$ds^2 = 2\left(du + J_v(\varphi)e^{-M_u(\varphi)}d\varphi\right)\left(dv + J_u(\varphi)e^{-M_v(\varphi)}d\varphi\right) - \frac{1}{4}\left(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)\right)^2 d\varphi^2\,. \tag{3.27}$$

This is our starting point for the charge analysis.

It is now straightforward to see how to retrieve the Bondi-Sachs metrics (3.2) and (3.1), in their respective patches, from (3.27). For example, let's say we want to retrieve the one in (u, r, φ) coordinates. This is achieved performing the following steps

- Turn off the parameters M_v, J_v and A.
- Rescale u and e^{M_u} as
$$u \to \frac{u}{2}\,, \quad e^{M_u} \to 2\,e^{M_u}\,. \tag{3.28}$$
- Perform the coordinate transformation
$$r = \frac{v - u\,e^{M_u}}{2} \;\Rightarrow\; v = 2r + u\,e^{M_u}. \tag{3.29}$$

We then obtain the following metric:

$$ds^2 = 2du\,dr + e^{M_u}du^2 + (J_u + u\,e^{M_u})du\,d\varphi - r^2 d\varphi^2, \tag{3.30}$$

which is exactly (3.1), in the opposite signature, under the identifications

$$e^{M_u} = -\,\Theta^+, \quad J_u = -\,\Xi^+. \tag{3.31}$$

The procedure works in a similar way to retrieve the other BMS metric, in the (v, r, φ) patch.

3.1.5 Boundary conditions

In order to specify completely the solution space we need to set some boundary conditions, defining how the fields of the theory fall off at infinity. As for the choice of the gauge, the choice of the boundary conditions has to be meaningful, in the sense that they should be general enough to leave a residual gauge freedom at the boundary, with non zero surface charges associated to them. The way one usually proceed in practice, is finding first the residual gauge symmetries fixed by the gauge fixing conditions only. Then, since this set of symmetries is usually still too large to be manageable or to give finite charges, one looks for a more suitable subset. A good set of boundary conditions does exactly this: It reduces the set of residual gauge symmetries, fixed only by the gauge fixing conditions, to a more suitable subset of symmetries, with finite associated charges. We thus choose the set of "double null" boundary conditions

$$\sigma = -\frac{e^{M_v}}{2\,|e^{M_v}v - e^{M_u}u + A(\varphi))} + \mathcal{O}\left(\frac{1}{(v-u)^2}\right) \,, \tag{3.32}$$

$$\mu = -\frac{e^{M_u}}{2\,|e^{M_v}v - e^{M_u}u + A(\varphi))} + \mathcal{O}\left(\frac{1}{(v-u)^2}\right) \,, \tag{3.33}$$

$$m^\mu = \mathcal{O}\left(\frac{1}{v-u}\right) . \tag{3.34}$$

which are compatible with the definition of asymptotic flatness employed in the Bondi-Sachs approach. The asymptotic region of the spacetime corresponds here to the large $|v-u|$ limit, which corresponds to the large r limit in the Bondi-Sachs approach. In general, being v and u two independent variables, we can distinguish two distinct asymptotic regions through the large u limit and the large v limit, that we may identify respectively with $\mathcal{I}^+$ and $\mathcal{I}^-$. However, for our purposes, this distinction will not be necessary in what follows.

3.2 Residual gauge symmetries

We here wish to compute the diffeomorphisms and local Lorentz transformations preserving both our gauge fixing conditions and the boundary conditions. In other terms, we want to understand how the usual local invariance under diffeomorphisms and local Lorentz transformations has been restricted, under the condition of preserving the double null foliation and the other geometric conditions imposed so far. Let's start recalling how a tetrad vector and a spin coefficient transform under the combined action of a diffeomorphism and a Lorentz local transformation, i.e. under a generic gauge transformation

$$\delta_{\xi,\omega} e^\mu_a = \xi^\nu \partial_\nu e^\mu_a - \partial_\nu \xi^\mu e^\nu_a + \omega_a{}^b e^\mu_b \,, \tag{3.35}$$

$$\delta_{\xi,\omega} \Gamma_{abc} = \xi^\nu \partial_\nu \Gamma_{abc} - D_c \omega_{ab} + \omega_c{}^d \Gamma_{abd} \,, \tag{3.36}$$

where $D_c = (l, n, m)$ is the directional derivative in the tetrad's directions. The gauge transformations we are looking for are the ones whose action has no effect on the tetrad components (3.4) and the spin coefficients (3.12), that we gauge fixed. In other terms, the transformations have to satisfy the following conditions:

$$\delta_{\xi,\omega} l^\mu = \delta_{\xi,\omega} n^\mu = 0, \tag{3.37}$$

$$\delta_{\xi,\omega}\varepsilon = \delta_{\xi,\omega}\gamma = \delta_{\xi,\omega}\tau = \delta_{\xi,\omega}\pi = \delta_{\xi,\omega}\beta = \delta_{\xi,\omega}\kappa = \delta_{\xi,\omega}\nu = 0. \tag{3.38}$$

By using (3.35) and (3.36) to write out explicitly the gauge preserving conditions (3.37) and (3.38), we obtain a set of differential equations for the diffeomorphisms and for the local Lorentz parameters:

$$
\text{Lorentz:}\ \begin{cases} \partial_v \omega^{12} = 0 \\ \partial_u \omega^{12} = 0 \\ \partial_\varphi \omega^{12} = 0 \\ \partial_u \omega^{23} = 2\,\omega^{13}\sigma \\ \partial_v \omega^{13} = -2\,\omega^{23}\mu \\ \partial_v \omega^{23} = 2\,\omega^{23}\sigma \\ \partial_u \omega^{13} = -2\,\omega^{13}\mu \end{cases} \qquad \text{Diffeomorphism:}\ \begin{cases} \partial_v \xi^\varphi = \dfrac{\omega^{23}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \\ \partial_u \xi^\varphi = \dfrac{\omega^{13}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \\ \partial_v \xi^u = -\dfrac{\omega^{23} J_v(\varphi) e^{-M_u(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \\ \partial_u \xi^v = -\dfrac{\omega^{13} J_u(\varphi) e^{-M_v(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \\ \partial_v \xi^v = -\omega^{12} - \dfrac{\omega^{23} J_u(\varphi) e^{-M_v(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \\ \partial_u \xi^u = \omega^{12} - \dfrac{\omega^{13} J_v(\varphi) e^{-M_u(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \end{cases} \tag{3.39}
$$

These equations can be solved, giving us the allowed diffeomorphisms and local Lorentz transformations

$$
\text{Lorentz:}\ \begin{cases} \omega^{12} = cost. \\ \omega^{23} = \dfrac{-e^{C_2(\varphi)+M_u(\varphi)}u + C_1(\varphi)e^{M_v(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \\ \omega^{13} = \dfrac{e^{C_2(\varphi)+M_u(\varphi)}v + A(\varphi)e^{C_2(\varphi)+M_u(\varphi)-M_v(\varphi)} - C_1(\varphi)e^{M_u(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \end{cases}
$$

$$
\text{Diffeomorphism:}\ \begin{cases} \xi^u = \dfrac{[-e^{C_2(\varphi)+M_u(\varphi)-M_v(\varphi)}u + C_1(\varphi)]J_v(\varphi)e^{-M_u(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} + \omega^{12}u + \alpha^u(\varphi) \\ \xi^v = \dfrac{[-e^{C_2(\varphi)}v - A(\varphi)e^{C_2(\varphi)-M_v(\varphi)} + C_1(\varphi)]J_u(\varphi)e^{-M_v(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} - \omega^{12}v + \alpha^v(\varphi) \\ \xi^\varphi = \dfrac{e^{C_2(\varphi)+M_u(\varphi)-M_v(\varphi)}u - C_1(\varphi)}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} + \Upsilon(\varphi) \end{cases} \tag{3.40}
$$

where the functions $C_1(\varphi)$, $C_2(\varphi)$, $\alpha^u(\varphi)$, $\alpha^v(\varphi)$ and $\Upsilon(\varphi)$ are five independent symmetry generators. The function $C_1(\varphi)$ was rescaled by a factor $e^{M_v(\varphi)}$, and $C_2(\varphi)$ by a factor $e^{M_u(\varphi)}$, to obtain a simpler expression. We have thus Now, although these transformations preserve the gauge conditions mentioned above, we still need to verify that they preserve the fall off conditions of the fields. Let us show that, in order to preserve the boundary conditions (3.32), $e^{C_2(\varphi)}$ and ω^{12} have to be set to zero:

- Firstly, compute the variation $\delta_{\xi,\omega} m^u$ where ξ and ω are gauge preserving infinitesimal transformations, i.e. the ones given in (3.40):

$$
\begin{aligned}
\delta_{\xi,\omega} m^u =& \boxed{\frac{e^{C_2+M_u} u}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|}} + \frac{C_1 e^{M_v}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} + \\
&+ \frac{1}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \partial_\varphi \left(\frac{[e^{C_2+M_u-M_v}u - C_1] J_v e^{-M_u}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} - \omega^{12} u - \alpha^u \right) + \\
&+ \left[\frac{\left(e^{C_2+M_u-M_v}u - C_1\right) J_v e^{-M_u}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} - \omega^{12} u - \alpha^u \right] \frac{J_v}{(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi))^2} + \\
&- \left[\frac{\left(e^{C_2}v + A e^{C_2-M_v} - C_1\right) J_u e^{-M_v}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} + \omega^{12} v - \alpha^v \right] \frac{J_v e^{-M_u+M_v}}{(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi))^2} + \\
&- \left(\frac{e^{C_2+M_u-M_v}u - C_1}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} + \Upsilon \right) \partial_\varphi \left(\frac{J_v e^{-M_u}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \right) + \\
&+ \left[\omega^{12} - \frac{e^{C_2-M_v} J_v}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} + \frac{(e^{C_2+M_u-M_v}u - C_1) J_v}{(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi))^2} \right] \cdot \\
&\cdot \frac{J_v e^{-M_u}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|}
\end{aligned}
$$

We see that the first term (here boxed) is the only one which violates the fall off conditions of m^u in the large u limit and so, we conclude that e^{C_2} has to be set to zero. Setting $e^{C_2} = 0$, i.e. $C_2 = i\pi$ is not an issue, since ω^{13} and ω^{23} continue to solve the equations (3.39).

- Secondly, compute the variation $\delta_{\xi,\omega}\sigma$, with $e^{C_2} = 0$. We get

$$
\begin{aligned}
\delta_{\xi,\omega}\sigma =& \boxed{-\frac{e^{M_v}\omega^{12}(v\,e^{M_v} + u\,e^{M_u})}{2(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi))^2}} + \frac{e^{M_v}(\alpha^v\,e^{M_v} + \alpha^u\,e^{M_u})}{2(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi))^2} + \\
&+ \frac{1}{2}\left(\frac{\partial_\varphi(C_1 e^{M_v})}{(e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi))^2} - \frac{C_1 e^{M_v}(\partial_\varphi e^{M_v} v - \partial_\varphi e^{M_u} u)}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|^3} \right) + \\
&- \frac{e^{M_v} C_1 (J_v - J_u)}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|^3} .
\end{aligned}
$$

Again, we see that the first boxed term is the only one violating our boundary conditions (3.32). Because of this, we set $\omega^{12} = 0$ too.

In the end, we have shown that the residual gauge symmetries preserving both the gauge fixing conditions and the fall off conditions of the fields are

$$
\text{Lorentz:} \quad \begin{cases} \omega^{12} = 0 \\ \omega^{23} = \dfrac{C_1(\varphi)\,e^{M_v(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \\ \omega^{13} = -\dfrac{C_1(\varphi) e^{M_u(\varphi)}}{|e^{M_v(\varphi)}v - e^{M_u(\varphi)}u + A(\varphi)|} \end{cases}
$$

$$
\text{Diffeomorphism:} \quad \begin{cases} \xi^u = \dfrac{C_1(\varphi) J_v(\varphi) e^{-M_u(\varphi)}}{|e^{M_v(\varphi)} v - e^{M_u(\varphi)} u + A(\varphi)|} + \alpha^u(\varphi) \\ \xi^v = \dfrac{C_1(\varphi) J_u(\varphi) e^{-M_v(\varphi)}}{|e^{M_v(\varphi)} v - e^{M_u(\varphi)} u + A(\varphi)|} + \alpha^v(\varphi) \\ \xi^\varphi = -\dfrac{C_1(\varphi)}{|e^{M_v(\varphi)} v - e^{M_u(\varphi)} u + A(\varphi)|} + \Upsilon(\varphi) \end{cases} \tag{3.41}
$$

whose charges we proceed now to compute. Notice that the number of symmetry generating functions, satisfying both the gauge fixing conditions and the boundary conditions, matches the number of free functions in our solution space. This is a consistency check, since each charge should have its associated symmetry generating functions.

3.3 Surface charges and asymptotic symmetries

The next step in our analysis is computing the surface charges. We do this before computing the residual gauge symmetries algebra, because some of the solution space parameters could reveal to be just gauge parameters, without any physical relevance. If a parameter does not appear in the charges, as explained in section 2.3, it is just a gauge redundancy. Thus, as seen in section 2.2.3, we need to build the superpotential $k^{\mu\nu}_{\xi,\omega}$ and read out the charges from the exact field derivative term.

3.3.1 Double null 3D charges

To build the superpotential k, recall its form in the tetrad formalism:

$$
k^{\mu\nu}_{\xi,\omega} = \mathbf{e} \left[\left(2\, \delta e^\mu_a\, e^\nu_b + e^c_\lambda\, \delta e^\lambda_c\, e^\nu_a\, e^\mu_b \right) \left(-\omega^{ab} + \Gamma^{ab}{}_\rho\, \xi^\rho \right) + \delta \Gamma^{ab}{}_\rho \left(\xi^\rho e^\mu_a\, e^\nu_b + 2\, \xi^\mu e^\nu_a\, e^\rho_b \right) - (\mu \leftrightarrow \nu) \right], \tag{3.42}
$$

where $\mathbf{e}$ is simply the determinant of the tetrad, i.e. $\sqrt{|g|}$. Inserting our solution space inside (3.42), we obtain

$$
k^{u\varphi}_{\xi,\omega} = 0, \qquad k^{v\varphi}_{\xi,\omega} = 0, \qquad k^{uv}_{\xi,\omega} = \left[\Upsilon \delta \big(J_u + J_v \big) + \alpha^u e^{M_u} \delta M_u + \alpha^v e^{M_v} \delta M_v \right]. \tag{3.43}
$$

The last of the components above, which is also the only non-zero component of the superpotential, can be simplified as follows:

$$
k^{uv}_{\xi,\omega} = \delta \left(\Upsilon J + \alpha^u e^{M_u} + \alpha^v e^{M_v} \right), \tag{3.44}
$$

where we defined the quantity $J := J_u + J_v$. Eventually, we can identify three independent surface charges

$$
\mathcal{M}_u = \int_0^{2\pi} \alpha^u(\varphi) e^{M_u(\varphi)}\, d\varphi\,, \tag{3.45}
$$

$$
\mathcal{M}_v = \int_0^{2\pi} \alpha^v(\varphi) e^{M_v(\varphi)}\, d\varphi\,, \tag{3.46}
$$

$$
\mathcal{J} = \int_0^{2\pi} \Upsilon(\varphi) J(\varphi)\, d\varphi\,. \tag{3.47}
$$

We notice two important features:

- In the superpotential k neither the field A nor the parameter C_1 of the residual symmetries entered. This means that not all the degrees of freedom of the solution space were actually physical. Instead, the parameter A, which would have coupled to C_1, is a purely gauge degree of freedom which we can be set to zero - together with C_1 - in the following. We have thus found out that the asymptotic symmetries are limited to diffeomorphisms of the form

$$\xi = \alpha^u(\varphi)\,\partial_u + \alpha^v(\varphi)\,\partial_v + \Upsilon(\varphi)\,\partial_\varphi\,. \tag{3.48}$$

- Although the super-rotation charge J is compatible with the usual super-rotation charge of bms_3, the super-translation charge M appearing in bms_3 is here decomposed in two independent null components, namely M_u and M_v. This suggests that before imposing any type of antipodal matching conditions, the time super-translation charge admits a chiral decomposition as the sum of two independent null components.

3.3.2 Asymptotic symmetry algebra

Now that we know the asymptotic symmetries of the theory, we can proceed to compute the Lie algebra they give rise to. To compute the structure constants of the Lie algebra, realized by the asymptotic symmetries, we just need to compute their commutators. Doing so, we find that

$$\begin{aligned}[\,\xi,\tilde{\xi}\,] &= [\,\Upsilon\partial_\varphi + \alpha^v\partial_u + \alpha^u\partial_v, \tilde{\Upsilon}\partial_\varphi + \tilde{\alpha}^v\partial_u + \tilde{\alpha}^u\partial_v\,] \\ &= \Big(\Upsilon\partial_\varphi\tilde{\Upsilon} - \tilde{\Upsilon}\partial_\varphi\Upsilon\Big)\,\partial_\varphi + \Big(\Upsilon\partial_\varphi\tilde{\alpha}^v - \tilde{\Upsilon}\partial_\varphi\alpha^v\Big)\,\partial_u + \Big(\Upsilon\partial_\varphi\tilde{\alpha}^u - \tilde{\Upsilon}\partial_\varphi\alpha^u\Big)\,\partial_v\,. \end{aligned} \tag{3.49}$$

The total Lie algebra of the asymptotic symmetries is thus composed by subalgebras in semiproduct between each others. The vectors $\alpha^v(\varphi)\,\partial_u$ and $\alpha^u(\varphi)\,\partial_v$ realize two infinite dimensional abelian subalgebras, which do not act on each other. However, the vector $\Upsilon(\varphi)\,\partial_\varphi$ realizes a $\mathrm{Diff}(S^1)$ subalgebra, which acts on each of the abelian sublagebras. Thus, the whole algebra is

$$\mathrm{Diff}(S^1) \ltimes \left(\mathbb{R}^2\right)^c\,, \tag{3.50}$$

where we denoted with $(\mathbb{R})^c$ the infinite dimensional enhancement of $\mathbb{R}$.

Remark: It can happen that the diffeomorphisms, or the local Lorentz parameters, depend upon the fields. If this is the case, when we perform the usual commutator, i.e. when a diffeomorphism act on another one, its action will change the value of the latter's fields. Since we need to evaluate the diffeomorphisms, at the same point in the solution space, we need to account for this induced variation. This modified version of the commutator is known as the "Barnich-Troessaert" commutator and is defined as

$$[\,\xi(J_u,J_v,...),\tilde{\xi}(J_u,J_v,...)\,]_{BT} := [\xi(J_u,J_v,...),\tilde{\xi}(J_u,J_v,...)\,] - \delta_\xi\tilde{\xi} + \delta_{\tilde{\xi}}\xi\,, \tag{3.51}$$

where the last two terms account for the field dependency. We can write this variation simply as

$$\delta_\xi\tilde{\xi} = \tilde{\xi}(J_u + \delta_{\xi,\omega}J_u,...) - \tilde{\xi}(J_u,...). \tag{3.52}$$

In this case, to get to the asymptotic symmetry algebra we need to know the variation of the solution space parameters under a gauge transformation.

3.3.3 Transformation laws of the fields

While it is not necessary to know the field's variations under the action of the asymptotic symmetries to compute the symmetry algebra, we need them to compute the charge algebra.

These transformation laws can be easily deduced from the variations of the tetrad field and the spin coefficients.

- The first variations we want to compute are $\delta_\xi e^{M_u}$ and $\delta_\xi e^{M_v}$. We may read their values from $\delta_\xi m^\varphi$, which can be written as

$$\delta_\xi m^\varphi = \delta_\xi \left(\frac{1}{|e^{M_v} v - e^{M_u} u|} \right) = - \frac{\delta_\xi e^{M_v}\, v - \delta_\xi e^{M_u}\, u}{(e^{M_v} v - e^{M_u} u)^2} \,. \tag{3.53}$$

On the other hand, if we let the asymptotic symmetries act on m^φ according to (3.35), we obtain

$$\delta_\xi m^\varphi = + \frac{\alpha^u e^{M_u} - \alpha^v e^{M_v}}{(e^{M_v} v - e^{M_u} u)^2} - \frac{\Upsilon(\partial_\varphi e^{M_v}\, v - \partial_\varphi e^{M_u}\, u)}{(e^{M_v} v - e^{M_u} u)^2} - \frac{\partial_\varphi \Upsilon}{|e^{M_v} v - e^{M_u} u|} \,. \tag{3.54}$$

Comparing the two order by order, we see that

$$- \frac{\delta_\xi e^{M_v}\, v - \delta_\xi e^{M_u}\, u}{(e^{M_v} v - e^{M_u} u)^2} = - \frac{\partial_\varphi(\Upsilon e^{M_v}) v - \partial_\varphi(\Upsilon e^{M_u})}{(e^{M_v} v - e^{M_u} u)^2} + \mathcal{O}\left(\frac{1}{(v-u)^2} \right) , \tag{3.55}$$

which leads to

$$\boxed{\delta_\xi e^{M_u} = \partial_\varphi(\Upsilon e^{M_u})} \,, \quad \boxed{\delta_\xi e^{M_v} = \partial_\varphi(\Upsilon e^{M_v})} \,. \tag{3.56}$$

- Similarly, to compute $\delta_\xi J_v$ we can look at $\delta_\xi m^u$. As before, $\delta_\xi m^u$ can be written as

$$\delta_\xi m^u = \delta_\xi \left(\frac{-J_v e^{-M_u}}{|e^{M_v} v - e^{M_u} u|} \right) = - \frac{\delta_\xi J_v\, e^{-M_u} + J_v \delta_\xi e^{-M_u}}{|e^{M_v} v - e^{M_u} u|} + \frac{J_v e^{-M_u} (\delta_\xi e^{M_v}\, v - \delta_\xi e^{M_u}\, u)}{(e^{M_v} v - e^{M_u} u)^2} \,, \tag{3.57}$$

but also as

$$\delta_\xi m^u = - \frac{\partial_\varphi \alpha^u + \Upsilon(\partial_\varphi J_v e^{-M_u} + J_v \partial_\varphi e^{M_u})}{|e^{M_v} v - e^{M_u} u|} + \frac{\Upsilon J_v e^{-M_u} (\partial_\varphi e^{M_v}\, v - \partial_\varphi e^{M_u}\, u)}{(e^{M_v} v - e^{M_u} u)^2} + \mathcal{O}\left(\frac{1}{(v-u)^2} \right) . \tag{3.58}$$

Then, comparing the two expressions order by order, and knowing both $\delta_\xi e^{M_u}$ and $\delta_\xi e^{M_v}$, we can derive

$$\boxed{\delta_\xi J_v = e^{M_u} \partial_\varphi \alpha^u + \partial_\varphi J_v \Upsilon + 2 \partial_\varphi \Upsilon J_v} \,. \tag{3.59}$$

- In a similar way we can compute

$$\boxed{\delta_\xi J_u = e^{M_v} \partial_\varphi \alpha^v + \partial_\varphi J_u \Upsilon + 2 \partial_\varphi \Upsilon J_u} \,. \tag{3.60}$$

Thus, the complete set of transformation laws turns out to be:

$$\delta_\xi e^{M_u} = \partial_\varphi(\Upsilon e^{M_u})\,, \tag{3.61}$$

$$\delta_\xi e^{M_v} = \partial_\varphi(\Upsilon e^{M_v})\,, \tag{3.62}$$

$$\delta_\xi J_v = e^{M_u}\partial_\varphi \alpha^u + \partial_\varphi J_v \Upsilon + 2\partial_\varphi \Upsilon J_v\,, \tag{3.63}$$

$$\delta_\xi J_u = e^{M_v}\partial_\varphi \alpha^v + \partial_\varphi J_u \Upsilon + 2\partial_\varphi \Upsilon J_u\,. \tag{3.64}$$

We conclude that J_u and J_v are currents, while e^{M_u} and e^{M_v} are scalar densities.

3.3.4 Charge algebra

We saw in the previous chapter that the charge algebra is a projective representation of the residual symmetry algebra (3.49). Let's check explicitly that this is indeed the case:

$$\begin{aligned}
\Big\{Q_\xi, Q_{\tilde\xi}\Big\} &:= \delta_{\tilde\xi} Q_\xi \\
&= Q_\xi(J + \delta_{\tilde\xi} J,\, e^{M_u} + \delta_{\tilde\xi} e^{M_u},\, e^{M_v} + \delta_{\tilde\xi} e^{M_v}) - Q_\xi(J,\, e^{M_u},\, e^{M_v}) \\
&= \int \Big(\Upsilon \delta_{\tilde\xi} J + \alpha^u \delta_{\tilde\xi} e^{M_u} + \alpha^v \delta_{\tilde\xi} e^{M_v}\Big)\, d\varphi \\
&= \int \Big[\Upsilon \Big(e^{M_v}\partial_\varphi \tilde\alpha^v + e^{M_u}\partial_\varphi\tilde\alpha^u + \tilde\Upsilon \partial_\varphi J + 2\partial_\varphi J\Big) + \partial_\varphi \Big(\tilde\Upsilon e^{M_u}\Big)\alpha^u + \partial_\varphi\Big(\tilde\Upsilon e^{M_v}\Big)\alpha^v\Big] d\varphi \\
&= \int \Big[\Upsilon \Big(e^{M_v}\partial_\varphi \tilde\alpha^v + e^{M_u}\partial_\varphi\tilde\alpha^u + \underline{\tilde\Upsilon \partial_\varphi J} + 2\partial_\varphi J\Big) + \\
&\qquad\quad + \alpha^u\Big(\partial_\varphi\tilde\Upsilon e^{M_u} + \underline{\tilde\Upsilon\partial_\varphi e^{M_u}}\Big) + \alpha^v\Big(\partial_\varphi\tilde\Upsilon e^{M_v} + \underline{\tilde\Upsilon\partial_\varphi e^{M_v}}\Big) \\
&= \int \Big[\Upsilon\,(e^{M_v}\partial_\varphi\tilde\alpha^v + e^{M_u}\partial_\varphi\tilde\alpha^u + \cancel{2}\partial_\varphi J) + \cancel{\alpha^u\partial_\varphi\tilde\Upsilon e^{M_u}} + \cancel{\alpha^v\partial_\varphi\tilde\Upsilon e^{M_v}}\Big] d\varphi + \\
&\qquad - \int J\Big(\cancel{\partial_\varphi\tilde\Upsilon}\Upsilon + \tilde\Upsilon\partial_\varphi\Upsilon\Big) + e^{M_u}\Big(\partial_\varphi\alpha^u\tilde\Upsilon + \cancel{\alpha^u\partial_\varphi\tilde\Upsilon}\Big) + e^{M_v}\Big(\partial_\varphi\alpha^v\tilde\Upsilon + \cancel{\alpha^v\partial_\varphi\tilde\Upsilon}\Big) + \\
&\qquad\quad + \cancel{\Big[\Upsilon\tilde\Upsilon J\Big]_0^{2\pi}} + \cancel{\Big[\tilde\Upsilon\alpha^u e^{M_u}\Big]_0^{2\pi}} + \cancel{\Big[\tilde\Upsilon\alpha^v e^{M_v}\Big]_0^{2\pi}} \\
&= \int \Big[\Big(\Upsilon\partial_\varphi\tilde\Upsilon - \tilde\Upsilon\partial_\varphi\Upsilon\Big) J + \Big(\Upsilon\partial_\varphi\tilde\alpha^u - \tilde\Upsilon\partial_\varphi\alpha^u\Big) e^{M_u} + \Big(\Upsilon\partial_\varphi\tilde\alpha^v - \tilde\Upsilon\partial_\varphi\alpha^v\Big) e^{M_v}\Big] d\varphi \\
&= Q_{[\,\xi,\tilde\xi\,]}
\end{aligned}$$

Therefore, we conclude that the charges realize a representation of the asymptotic symmetries without a central extension. That is, we have found

$$\boxed{\Big\{Q_\xi, Q_{\tilde\xi}\Big\} = Q_{[\,\xi,\tilde\xi\,]}}\,. \tag{3.65}$$

Conclusions

This work has been devoted to develop an alternative approach to the study of asymptotic symmetries in asymptotically flat 3D gravity. This was accomplished through the construction, using the Newman-Penrose formalism, of a novel "double null" approach. The peculiar feature of our analysis stood in the double null foliation of the spacetime on which it relied. Let us summarize our discussion.

We started this work reviewing Cartan's formulation of General Relativity, in which curvature is expressed as the non-holonomicity of a collection of 1-forms, the tetrads. We reviewed how to manipulate these objects and defined the tensors of Riemannian geometry in a torsion-less non-holonomic frame. The discussion then focused on a certain choice of the tetrad field, namely a null one, and we introduced the Newman-Penrose formalism, which was used to develope our analysis. We then reviewed the theory of surface charges. After having introduced the jet bundle, its variational bicomplex and symplectic manifolds, we combined results from these geometric structures to introduce the covariant phase space formalism. Within this formalism we formulated Noether's theorems and argued that the Noether charges, from a symplectic perspective, are the Hamiltonian generating functions. From this, the existence of gauge symmetries with non-trivial associated Noether charges was derived. These non-trivial gauge transformations were eventually defined as the asymptotic symmetries of the theory considered. Eventually, having gathered the basic tools, we proceeded to develop our "double null" formalism. Firstly we introduced a suitable gauge, adapted to a double null foliation of the spacetime. Our insightful gauge choice led us to a non-trivial solution space, which we proved to include the Bondi-Sachs field spaces. This connection with the solution spaces of the Bondi-Sachs gauge, adapted to approach either $\mathcal{I}^+$ or $\mathcal{I}^-$, gave a clear physical interpretation to the functions M_u, M_v, J_u and J_v, appearing in our field space. A set of boundary conditions was then given (to specify the asymptotics of the theory) and the residual gauge symmetries were computed. The analysis of the charges resulted in two super-translation charges and a super-rotation charge, which displayed interesting features. The super-rotation charge was associated only to the sum $J_u + J_v =: J$, while $J_u - J_v$ revealed to be pure gauge. This did not happen for the super-translation charges, which independently coupled to M_u and M_v. In particular, this displayed a chiral decomposition of the ordinary time super-translation charge as two independent null components. The asymptotic symmetry algebra was then computed, resulting in $\mathrm{Diff}(S^1) \ltimes (\mathbb{R}^2)^C$. Under its action the super-translation charges transformed as scalar densities, while the super-rotation charge transformed as a current. Lastly, the charge algebra was computed and no central extension was found.

This work represents a first step in several directions, one of which points towards the antipodal matching conditions given by Strominger in [8]. In this sense, it could be interesting to investigate if, in a unified approach as the one we developed, these matching conditions arise in a natural way. These could be related, for instance, to the fact that the angular momentum in our analysis enters, at the level of the charges, only as the sum $J_u + J_v$. However, further investigation is required in this direction. Similarly, we plan to investigate possible intersections with the work of Flanagan and Chandrasekaran [37], in which they perform a similar double null analysis for a generic null hypersurface in 4D. Lastly, a natural continuation of this work would be to transpose the whole discussion to four dimension. Doing so, we would be able to give a double null setup, which fully exploits the null structure of the spacetime, to characterize gravitational radiation. Besides, this double null framework in 4D could represent a starting point for a future lightcone approach to the quantization. In this direction, our analysis could intersect some recent works which explore quantization on null hypersurfaces, e.g. see [38], [39].

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Appendix A

Frobenius theorem

When reviewing the BMS formalism and developing the double null approach, we assume the possibility of foliating our spacetime with a family of null-like hypersurfaces. This could lead the reader to wonder under which assumptions this is possible or, alternatively, if there are some sufficient or necessary conditions controlling whether the spacetime admits such a foliation. The answer is given by Frobenius theorem, which is worth to be mentioned in a more extensive way. We follow the discussion about Frobenius theorem given by [40].

At each point $p \in \mathcal{M}$, consider the tangent space $T_p(\mathcal{M})$ and a subspace $W_p \subset T_p(\mathcal{M})$, with dimension $m < n = \dim(\mathcal{M})$, and varying smoothly with p, i.e. $W = \bigcup_{p\in\mathcal{M}} W_p$ is spanned by $\mathcal{C}^\infty$ vector fields. We would like to know if we can find integral submanifolds of W, i.e. whether through each point p we can find an embedded manifold $\mathcal{S}$, such that its tangent space at each point $y \in \mathcal{S}$, coincides with W_y. In other terms, we are asking when a smooth specification of subspaces W is the tangent bundle of some submanifold $S \in \mathcal{M}$. When $\dim(W) = 1$, the problem reduces to finding the integral curves of a smooth vector field, which we know can always be done. However, when $\dim(W) > 1$, it is not always true that we can find such integrable manifolds.

To understand under which conditions this holds, let's assume it to be true. If we could find these integrable manifolds, they would be covered by coordinate patches and their tangent space, which by hypothesis coincide with W at each of their points, would be spanned by a coordinate basis $X_1^\mu, ..., X_m^\mu$, such that $[X_a, X_b] = 0$, $a, b = 1, ..., m$. Furthermore, if we would take any two vectors which lie within W, they could be expressed as linear combinations of these coordinate fields. In other words, it would hold that $\forall\, y, z \in W$:

$$[y, z] = [f^a X_a, g^b X_b] = \left(f^a X_a(g^b) - g^a X_a(f^b)\right) X_b\,, \tag{A.1}$$

which would still be an element of W. Whenever this holds, W is said to be involute, in analogy to the involute of a curve. This leads us to a first formulation of Frobenius theorem:

Theorem A.0.1 (Frobenius theorem, vector form) *A smooth specification W, consisting of m-dim subspaces of the tangent space $T_p(\mathcal{M})$ at each $p \in \mathcal{M}$, possess integral submanifolds if and only if W is involute, i.e. $\forall\, y^\mu, z^\mu \in W$, we have $[y, z]^\mu \in W$.*

Notice now the following. Given $W_p \subset V_p$, we can consider the 1-form $\omega \in V_p^*$ which satisfy $\omega_\mu X^\mu = 0$, for all $X^\mu \in W_p$. Such ω_μ's will span an $(n - m)$ dimensional subspace $T_p^* \subset V_p^*$.

Conversely, an $(n-m)$ dimensional subspace $T_p^* \subset V_x^*$, defines an m dimensional subspace $W_p \subset V_p$ via $\omega_\mu X^\mu = 0$, i.e. $W_p = \langle \ker(\omega_\mu) \rangle_{span}$. According to the first formulation of the theorem, integral manifolds will exist if and only if for all $\omega_a \in T^*$ and all $Y^\mu, Z^\mu \in W$ we will have $\omega_\mu [Y, Z]^\mu = 0$. However, this implies

$$\begin{aligned} 0 &= \omega_\mu \left(Y^\nu \nabla_\nu Z^\mu - Z^\nu \nabla_\nu Y^\mu \right) \\ &= -Z^\mu Y^\nu \nabla_\nu \omega_\mu + Y^\mu Z^\nu \nabla_\nu \omega_\mu \\ &= 2 Y^\mu Z^\nu \nabla_{[\nu} \omega_{\mu]} \,, \end{aligned}$$

which holds if and only if

$$\nabla_{[\nu} \omega_{\mu]} = \sum_{\alpha=1}^{n-m} a^\alpha_{[\mu} v^\alpha_{\nu]} \,, \tag{A.2}$$

where $a^\alpha \in T^*$, so that they annihilates the vector fields contracted with them, while the v^α's are arbitrary 1-forms. This condition on the 1-forms which spans the dual subspace T^*, associated univoquely to the smooth specification W, leads to the second formulation of Frobenius theorem:

Theorem A.0.2 (Frobenius theorem, dual formulation) *Let T^* be a smooth specification of a (n-m) dimensional subspace of 1-forms. Then the associated subspace W of the tangent space admits integral submanifolds if and only if $\forall\, \omega \in T^*$ we have $d\omega = \sum_\alpha a^\alpha \wedge v^\alpha$, where all $\mu^\alpha \in T^*$.*

Let's notice one last thing now. This second formulation of Frobenius theorem gives a useful criterion to understand whether or not a vector field ξ^a is hypersurface orthogonal, i.e. to know whether the $(n-1)$ dimensional subspaces W_p, which are the orthogonal subspaces to the 1-dimensional space spanned by ξ^μ, are integrable. The theorem says that ξ^μ is hypersurface orthogonal if and only if, taken the correspondent dual 1-form ξ_μ, $\nabla_{[\mu} \xi_{\nu]} = \xi_{[\mu} v_{\nu]}$, where $a_\mu = \xi_\mu$ being $\dim(T^*) = 1$. The condition can be rewritten as follows:

$$\boxed{\xi_{[\mu} \nabla_\nu \xi_{\rho]} = 0} \tag{A.3}$$

or, in a more algebraic form:

$$\boxed{\xi \wedge d\xi = 0} \tag{A.4}$$

Under this condition the vector field ξ^μ is hypersurface orthogonal, and its integrable subspaces W_p of co-dimension 1 foliate the spacetime.

Bibliography

[1] Andrzej Trautman. "Radiation and Boundary Conditions in the Theory of Gravitation". In: *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* 6.6 (1958), pp. 407–412. arXiv: 1604.03145 [gr-qc].

[2] R. K. Sachs. "Gravitational waves in general relativity. 6. The outgoing radiation condition". In: *Proc. Roy. Soc. Lond. A* 264 (1961), pp. 309–338. DOI: 10.1098/rspa.1961.0202.

[3] R. K. Sachs. "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times". In: *Proc. Roy. Soc. Lond. A* 270 (1962), pp. 103–126. DOI: 10.1098/rspa.1962.0206.

[4] R. Sachs. "Asymptotic symmetries in gravitational theory". In: *Phys. Rev.* 128 (1962), pp. 2851–2864. DOI: 10.1103/PhysRev.128.2851.

[5] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems". In: *Proc. Roy. Soc. Lond. A* 269 (1962), pp. 21–52. DOI: 10.1098/rspa.1962.0161.

[6] M G.J. van der Burg. "Gravitational waves in general relativity IX. Conserved quantities". In: *Proc. Roy. Soc. (London)* (Sept. 1966). DOI: 10.1098/rspa.1966.0197. URL: https://www.osti.gov/biblio/4526115.

[7] Glenn Barnich and Cedric Troessaert. "Aspects of the BMS/CFT correspondence". In: *JHEP* 05 (2010), p. 062. DOI: 10.1007/JHEP05(2010)062. arXiv: 1001.1541 [hep-th].

[8] Andrew Strominger. "On BMS Invariance of Gravitational Scattering". In: *JHEP* 07 (2014), p. 152. DOI: 10.1007/JHEP07(2014)152. arXiv: 1312.2229 [hep-th].

[9] Steven Weinberg. "Infrared photons and gravitons". In: *Phys. Rev.* 140 (1965), B516–B524. DOI: 10.1103/PhysRev.140.B516.

[10] Alan D. Rendall. "Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations". In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 427 (1990), pp. 221–239. URL: https://api.semanticscholar.org/CorpusID:123093585.

[11] David Hilditch, Juan A. Valiente Kroon, and Peng Zhao. "Revisiting the characteristic initial value problem for the vacuum Einstein field equations". In: *Gen. Rel. Grav.* 52.10 (2020), p. 99. DOI: 10.1007/s10714-020-02747-2. arXiv: 1911.00047 [gr-qc].

[12] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli. "The Black hole in three-dimensional space-time". In: *Phys. Rev. Lett.* 69 (1992), pp. 1849–1851. DOI: 10.1103/PhysRevLett.69.1849. arXiv: hep-th/9204099.

[13] R. Jackiw. *Diverse topics in theoretical and mathematical physics.* WORLD SCIENTIFIC, 1995. DOI: 10.1142/2295. eprint: https://www.worldscientific.com/doi/pdf/10.1142/2295. URL: https://www.worldscientific.com/doi/abs/10.1142/2295.

[14] Kirill Krasnov. *Formulations of General Relativity.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Nov. 2020. ISBN: 978-1-108-67465-2, 978-1-108-48164-9. DOI: 10.1017/9781108674652.

[15] Glenn Barnich. *Advanced general relativity lecture notes.*

[16] Subrahmanyan Chandrasekhar. *The mathematical theory of black holes.* 1985. ISBN: 978-0-19-850370-5.

[17] Élie Cartan. "Sur les équations de structure des espaces généralisés et l'expression analytique du tenseur d'Einstein". In: *Comptes Rendus Académie des Sciences* 174 (1922), 1104ff.

[18] Élie Cartan. "Sur les espaces conformes généralisés et l'Univers optique". In: *CR Acad. Sci. Paris* 174 (1922), pp. 857–859.

[19] Élie Cartan. "Sur les espaces généralisés et la théorie de la relativité". In: *CR Acad. Sci* 174 (1922), pp. 734–737.

[20] Elie Cartan. "Sur les équations de la gravitation d'Einstein". In: *Journal de Mathématiques pures et appliquées* 1 (1922), pp. 141–203.

[21] Élie Cartan. "Sur un théorème fondamental de MH Weyl dans la théorie de l'espace métrique". In: *Comptes Rendus Académie des Sciences* 175 (1922), pp. 82–85.

[22] Élie Cartan. "Sur une définition géométrique du tenseur d'énergie d'Einstein". In: *Comptes Rendus Académie des Sciences* 174 (1922), pp. 437–439.

[23] Élie Cartan. "Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion". In: *Comptes Rendus, Ac. Sc. Paris* 174 (1922), pp. 593–595.

[24] Erhard Scholz. "E. Cartan's attempt at bridge-building between Einstein and the Cosserats – or how translational curvature became to be known as torsion". In: *The European Physical Journal H* 44.1 (Feb. 2019), pp. 47–75. ISSN: 2102-6467. DOI: 10.1140/epjh/e2018-90059-x. URL: http://dx.doi.org/10.1140/epjh/e2018-90059-x.

[25] Ezra Newman and Roger Penrose. "An Approach to gravitational radiation by a method of spin coefficients". In: *J. Math. Phys.* 3 (1962), pp. 566–578. DOI: 10.1063/1.1724257.

[26] Geoffrey Compère. *Advanced Lectures on General Relativity.* Vol. 952. Cham, Switzerland: Springer, Cham, Feb. 2019. ISBN: 978-3-030-04259-2, 978-3-030-04260-8. DOI: 10.1007/978-3-030-04260-8.

[27] Aldo Riello. *Quantum gravity lecture notes.*

[28] Anderson I.M. *The variational bicomplex.* Tech. rep. Formal Geometry and Mathematical Physics, Department of Mathematics, Utah State University, 1989. URL: https://ncatlab.org/nlab/files/AndersonVariationalBicomplex.pdf.

[29] Luca Ciambelli. "From Asymptotic Symmetries to the Corner Proposal". In: *PoS* Modave2022 (2023), p. 002. DOI: 10.22323/1.435.0002. arXiv: 2212.13644 [hep-th].

[30] Glenn Barnich, Pujian Mao, and Romain Ruzziconi. "Conserved currents in the Cartan formulation of general relativity". In: *About Various Kinds of Interactions: Workshop in honour of Professor Philippe Spindel*. Nov. 2016. arXiv: 1611.01777 [gr-qc].

[31] Glenn Barnich, Pujian Mao, and Romain Ruzziconi. "BMS current algebra in the context of the Newman–Penrose formalism". In: *Class. Quant. Grav.* 37.9 (2020), p. 095010. DOI: 10.1088/1361-6382/ab7c01. arXiv: 1910.14588 [gr-qc].

[32] Glenn Barnich, Friedemann Brandt, and Marc Henneaux. "Local BRST cohomology in gauge theories". In: *Phys. Rept.* 338 (2000), pp. 439–569. DOI: 10.1016/S0370-1573(00)00049-1. arXiv: hep-th/0002245.

[33] Glenn Barnich and Friedemann Brandt. "Covariant theory of asymptotic symmetries, conservation laws and central charges". In: *Nucl. Phys. B* 633 (2002), pp. 3–82. DOI: 10.1016/S0550-3213(02)00251-1. arXiv: hep-th/0111246.

[34] J. Lee and Robert M. Wald. "Local symmetries and constraints". In: *J. Math. Phys.* 31 (1990), pp. 725–743. DOI: 10.1063/1.528801.

[35] Vivek Iyer and Robert M. Wald. "Some properties of Noether charge and a proposal for dynamical black hole entropy". In: *Phys. Rev. D* 50 (1994), pp. 846–864. DOI: 10.1103/PhysRevD.50.846. arXiv: gr-qc/9403028.

[36] Robert M. Wald and Andreas Zoupas. "A General definition of 'conserved quantities' in general relativity and other theories of gravity". In: *Phys. Rev. D* 61 (2000), p. 084027. DOI: 10.1103/PhysRevD.61.084027. arXiv: gr-qc/9911095.

[37] Venkatesa Chandrasekaran and Eanna E. Flanagan. "Horizon phase spaces in general relativity". In: *JHEP* 07 (2024), p. 017. DOI: 10.1007/JHEP07(2024)017. arXiv: 2309.03871 [hep-th].

[38] Luca Ciambelli, Laurent Freidel, and Robert G. Leigh. "Quantum Null Geometry and Gravity". In: (July 2024). arXiv: 2407.11132 [hep-th].

[39] Wolfgang Wieland. "Quantum geometry of the null cone". In: (Jan. 2024). arXiv: 2401.17491 [gr-qc].

[40] Robert M. Wald. *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984. DOI: 10.7208/chicago/9780226870373.001.0001.