Chapter 2
Time Series

2.1. Two workhorses

This chapter describes two tractable models of time series: Markov chains and first-order stochastic linear difference equations. These models are organizing devices that put restrictions on a sequence of random vectors. They are useful because they describe a time series with parsimony. In later chapters, we shall make two uses each of Markov chains and stochastic linear difference equations: (1) to represent the exogenous information flows impinging on an agent or an economy, and (2) to represent an optimum or equilibrium outcome of agents' decision making. The Markov chain and the first-order stochastic linear difference both use a sharp notion of a state vector. A state vector summarizes the information about the current position of a system that is relevant for determining its future. The Markov chain and the stochastic linear difference equation will be useful tools for studying dynamic optimization problems.

2.2. Markov chains

A stochastic process is a sequence of random vectors. For us, the sequence will be ordered by a time index, taken to be the integers in this book. So we study discrete time models. We study a discrete-state stochastic process with the following property:

MARKOV PROPERTY: A stochastic process $\{x_t\}$ is said to have the *Markov property* if for all $k \geq 1$ and all t,

$$\text{Prob}\left(x_{t+1}|x_t, x_{t-1}, \ldots, x_{t-k}\right) = \text{Prob}\left(x_{t+1}|x_t\right).$$

We assume the Markov property and characterize the process by a *Markov chain*. A time-invariant Markov chain is defined by a triple of objects, namely, an n-dimensional state space consisting of vectors $e_i, i = 1, \ldots, n$, where e_i is an $n \times 1$ unit vector whose ith entry is 1 and all other entries are zero; an $n \times n$ *transition matrix* P, which records the probabilities of moving from one value of the state to another in one period; and an $(n \times 1)$ vector π_0 whose ith element is the probability of being in state i at time 0: $\pi_{0i} = \text{Prob}(x_0 = e_i)$. The elements of matrix P are

$$P_{ij} = \text{Prob}\left(x_{t+1} = e_j | x_t = e_i\right).$$

For these interpretations to be valid, the matrix P and the vector π_0 must satisfy the following assumption:

Assumption M:

a. For $i = 1, \ldots, n$, the matrix P satisfies

$$\sum_{j=1}^{n} P_{ij} = 1. \tag{2.2.1}$$

b. The vector π_0 satisfies

$$\sum_{i=1}^{n} \pi_{0i} = 1.$$

A matrix P that satisfies property (2.2.1) is called a *stochastic matrix.* A stochastic matrix defines the probabilities of moving from one value of the state to another in one period. The probability of moving from one value of the state to another in *two* periods is determined by P^2 because

$$\begin{aligned} &\text{Prob}\left(x_{t+2} = e_j | x_t = e_i\right) \\ &= \sum_{h=1}^{n} \text{Prob}\left(x_{t+2} = e_j | x_{t+1} = e_h\right) \text{Prob}\left(x_{t+1} = e_h | x_t = e_i\right) \\ &= \sum_{h=1}^{n} P_{ih} P_{hj} = P_{ij}^{(2)}, \end{aligned}$$

where $P_{ij}^{(2)}$ is the i,j element of P^2. Let $P_{i,j}^{(k)}$ denote the i,j element of P^k. By iterating on the preceding equation, we discover that

$$\text{Prob}\left(x_{t+k} = e_j | x_t = e_i\right) = P_{ij}^{(k)}.$$

The unconditional probability distributions of x_t are determined by

$$\begin{aligned}
\pi_1' &= \text{Prob}\,(x_1) = \pi_0' P \\
\pi_2' &= \text{Prob}\,(x_2) = \pi_0' P^2 \\
&\vdots \\
\pi_k' &= \text{Prob}\,(x_k) = \pi_0' P^k,
\end{aligned}$$

where $\pi_t' = \text{Prob}(x_t)$ is the $(1 \times n)$ vector whose ith element is $\text{Prob}(x_t = e_i)$.

2.2.1. Stationary distributions

Unconditional probability distributions evolve according to

$$\pi_{t+1}' = \pi_t' P. \tag{2.2.2}$$

An unconditional distribution is called *stationary* or *invariant* if it satisfies

$$\pi_{t+1} = \pi_t,$$

that is, if the unconditional distribution remains unaltered with the passage of time. From the law of motion $(2.2.2)$ for unconditional distributions, a stationary distribution must satisfy

$$\pi' = \pi' P \tag{2.2.3}$$

or

$$\pi' \, (I - P) = 0.$$

Transposing both sides of this equation gives

$$(I - P')\, \pi = 0, \tag{2.2.4}$$

which determines π as an eigenvector (normalized to satisfy $\sum_{i=1}^n \pi_i = 1$) associated with a unit eigenvalue of P'. We say that P, π is a *stationary Markov chain* if the initial distribution π is such that $(2.2.3)$ holds.

The fact that P is a stochastic matrix (i.e., it has nonnegative elements and satisfies $\sum_j P_{ij} = 1$ for all i) guarantees that P has at least one unit eigenvalue, and that there is at least one eigenvector π that satisfies equation

(2.2.4). This stationary distribution may not be unique because P can have a repeated unit eigenvalue.

Example 1. A Markov chain

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .2 & .5 & .3 \\ 0 & 0 & 1 \end{bmatrix}$$

has two unit eigenvalues with associated stationary distributions $\pi' = [\,1 \quad 0 \quad 0\,]$ and $\pi' = [\,0 \quad 0 \quad 1\,]$. Here states 1 and 3 are both *absorbing* states. Furthermore, any initial distribution that puts zero probability on state 2 is a stationary distribution. See exercises *2.10* and *2.11*.

Example 2. A Markov chain

$$P = \begin{bmatrix} .7 & .3 & 0 \\ 0 & .5 & .5 \\ 0 & .9 & .1 \end{bmatrix}$$

has one unit eigenvalue with associated stationary distribution $\pi' = [\,0 \quad .6429 \quad .3571\,]$. Here states 2 and 3 form an *absorbing subset* of the state space.

2.2.2. Asymptotic stationarity

We often ask the following question about a Markov process: for an arbitrary initial distribution π_0, do the unconditional distributions π_t approach a stationary distribution

$$\lim_{t\to\infty} \pi_t = \pi_\infty ,$$

where π_∞ solves equation (2.2.4)? If the answer is yes, then does the limit distribution π_∞ depend on the initial distribution π_0? If the limit π_∞ is independent of the initial distribution π_0, we say that the process is *asymptotically stationary with a unique invariant distribution*. We call a solution π_∞ a *stationary distribution* or an *invariant distribution* of P.

We state these concepts formally in the following definition:

Definition 2.2.1. Let π_∞ be a unique vector that satisfies $(I - P')\pi_\infty = 0$. If for all initial distributions π_0 it is true that $P^{t\prime}\pi_0$ converges to the same

π_∞, we say that the Markov chain is asymptotically stationary with a unique invariant distribution.

The following theorems can be used to show that a Markov chain is asymptotically stationary.

Theorem 2.2.1. *Let P be a stochastic matrix with $P_{ij} > 0 \; \forall (i,j)$. Then P has a unique stationary distribution, and the process is asymptotically stationary.*

Theorem 2.2.2. *Let P be a stochastic matrix for which $P_{ij}^n > 0 \; \forall (i,j)$ for some value of $n \geq 1$. Then P has a unique stationary distribution, and the process is asymptotically stationary.*

The conditions of Theorem 2.2.1 (and Theorem 2.2.2) state that from any state there is a positive probability of moving to any other state in one (or n) steps. Please note that some of the examples below will violate the conditions of Theorem 2.2.2 for any n.

2.2.3. Forecasting the state

The minimum mean squared error forecast of the state next period is the conditional mathematical expectation:

$$E\left[x_{t+1} | x_t = e_i\right] = \begin{bmatrix} P_{i1} \\ P_{i2} \\ \vdots \\ P_{in} \end{bmatrix} = P' e_i = P'_{i,\cdot} \tag{2.2.5}$$

where $P'_{i,\cdot}$ denotes the transpose of the ith row of the matrix P. In section B.2 of this book's appendix B, we use this equation to motivate the following first-order stochastic difference equation for the state:

$$x_{t+1} = P' x_t + v_{t+1} \tag{2.2.6}$$

where v_{t+1} is a random disturbance that evidently satisfies $E[v_{t+1} | x_t] = 0$.

Now let $\overline{y}$ be an $n \times 1$ vector of real numbers and define $y_t = \overline{y}' x_t$, so that $y_t = \overline{y}_i$ if $x_t = e_i$. Evidently, we can write

$$y_{t+1} = \overline{y}' P' x_t + \overline{y}' v_{t+1}. \tag{2.2.7}$$

The pair of equations (2.2.6), (2.2.7) becomes a simple example of a hidden Markov model when the observation y_t is too coarse to reveal the state. See section B.2 of technical appendix B for a discussion of such models.

2.2.4. Forecasting functions of the state

From the conditional and unconditional probability distributions that we have listed, it follows that the unconditional expectations of y_t for $t \geq 0$ are determined by $Ey_t = (\pi_0' P^t)\overline{y}$. Conditional expectations are determined by

$$E\left(y_{t+1}|x_t = e_i\right) = \sum_j P_{ij}\overline{y}_j = \left(P\overline{y}\right)_i \tag{2.2.8}$$

$$E\left(y_{t+2}|x_t = e_i\right) = \sum_k P_{ik}^{(2)}\overline{y}_k = \left(P^2\overline{y}\right)_i \tag{2.2.9}$$

and so on, where $P_{ik}^{(2)}$ denotes the (i,k) element of P^2 and $(\cdot)_i$ denotes the ith row of the matrix $(\cdot)$. An equivalent formula from (2.2.6), (2.2.7) is $E[y_{t+1}|x_t] = \bar{y}'P'x_t = x_t'P\bar{y}$, which equals $(P\bar{y})_i$ when $x_t = e_i$. Notice that

$$E\left[E\left(y_{t+2}|x_{t+1} = e_j\right)|x_t = e_i\right] = \sum_j P_{ij}\sum_k P_{jk}\overline{y}_k$$
$$= \sum_k\left(\sum_j P_{ij}P_{jk}\right)\overline{y}_k = \sum_k P_{ik}^{(2)}\overline{y}_k = E\left(y_{t+2}|x_t = e_i\right).$$

Connecting the first and last terms in this string of equalities yields $E[E(y_{t+2}|x_{t+1})|x_t] = E[y_{t+2}|x_t]$. This is an example of the "law of iterated expectations." The law of iterated expectations states that for any random variable z and two information sets J, I with $J \subset I$, $E[E(z|I)|J] = E(z|J)$. As another example of the law of iterated expectations, notice that

$$Ey_1 = \sum_j \pi_{1,j}\overline{y}_j = \pi_1'\overline{y} = \left(\pi_0'P\right)\overline{y} = \pi_0'\left(P\overline{y}\right)$$

and that

$$E\left[E\left(y_1|x_0 = e_i\right)\right] = \sum_i \pi_{0,i}\sum_j P_{ij}\overline{y}_j = \sum_j\left(\sum_i \pi_{0,i}P_{ij}\right)\overline{y}_j = \pi_1'\overline{y} = Ey_1.$$

2.2.5. Forecasting functions

There are powerful formulas for forecasting functions of a Markov state. Again, let $\overline{y}$ be an $n \times 1$ vector and consider the random variable $y_t = \overline{y}' x_t$. Then

$$E\left[y_{t+k} | x_t = e_i\right] = \left(P^k \overline{y}\right)_i$$

where $(P^k \overline{y})_i$ denotes the ith row of $P^k \overline{y}$. Stacking all n rows together, we express this as

$$E\left[y_{t+k} | x_t\right] = P^k \overline{y}. \tag{2.2.10}$$

We also have

$$\sum_{k=0}^{\infty} \beta^k E\left[y_{t+k} | x_t = \overline{e}_i\right] = \left[\left(I - \beta P\right)^{-1} \overline{y}\right]_i,$$

where $\beta \in (0,1)$ guarantees existence of $(I - \beta P)^{-1} = (I + \beta P + \beta^2 P^2 + \cdots)$.

2.2.6. Enough one-step-ahead forecasts determine P

One-step-ahead forecasts of a sufficiently rich set of random variables characterize a Markov chain. In particular, one-step-ahead conditional expectations of n independent functions (i.e., n linearly independent vectors $h_1, \ldots, h_n$) uniquely determine the transition matrix P. Thus, let $E[h_{k,t+1} | x_t = e_i] = (P h_k)_i$. We can collect the conditional expectations of h_k for all initial states i in an $n \times 1$ vector $E[h_{k,t+1} | x_t] = P h_k$. We can then collect conditional expectations for the n independent vectors $h_1, \ldots, h_n$ as $Ph = J$ where $h = [\, h_1 \quad h_2 \quad \ldots \quad h_n \,]$ and J is the $n \times n$ matrix consisting of all conditional expectations of all n vectors $h_1, \ldots, h_n$. If we know h and J, we can determine P from $P = J h^{-1}$.

2.2.7. Invariant functions and ergodicity

Let P, π be a stationary n-state Markov chain with the state space $X = [e_i, i = 1, \ldots, n]$. An $n \times 1$ vector $\overline{y}$ defines a random variable $y_t = \overline{y}' x_t$. Let $E[y_\infty | x_0]$ be the expectation of y_s for s very large, conditional on the initial state. The following is a useful precursor to a law of large numbers:

Theorem 2.2.3. *Let $\overline{y}$ define a random variable as a function of an underlying state x, where x is governed by a stationary Markov chain (P, π). Then*

$$\frac{1}{T} \sum_{t=1}^{T} y_t \to E\left[y_\infty | x_0\right] \tag{2.2.11}$$

with probability 1.

To illustrate Theorem 2.2.3, consider the following example:

Example: Consider the Markov chain $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pi_0 = \begin{bmatrix} p \\ (1-p) \end{bmatrix}$ for $p \in (0,1)$. Consider the random variable $y_t = \bar{y}' x_t$ where $\bar{y} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$. The chain has two possible sample paths, $y_t = 10, t \geq 0$, which occurs with probability p and $y_t = 0, t \geq 0$, which occurs with probability $1-p$. Thus, $\frac{1}{T} \sum_{t=1}^{T} y_t \to 10$ with probability p and $\frac{1}{T} \sum_{t=1}^{T} y_t \to 0$ with probability $(1-p)$.

The outcomes in this example indicate why we might want something more than $(2.2.11)$. In particular, we would like to be free to replace $E[y_\infty | x_0]$ with the constant unconditional mean $E[y_t] = E[y_0]$ associated with the stationary distribution π. To get this outcome, we must strengthen what we assume about P by using the following concepts.

Suppose that (P, π) is a stationary Markov chain. Imagine repeatedly drawing x_0 from π and then generating $x_t, t \geq 1$ by successively drawing from transition densities given by the matrix P. We use

Definition 2.2.2. A random variable $y_t = \overline{y}' x_t$ is said to be *invariant* if $y_t = y_0, t \geq 0$, for all realizations of $x_t, t \geq 0$ that occur with positive probability under (P, π).

Thus, a random variable y_t is invariant (or "an invariant function of the state") if it remains constant at y_0 while the underlying state x_t moves through the state space X. Notice how the definition leaves open the possibility that y_0

itself might differ across sample paths indexed by different draws of the initial condition x_0 from the initial (and stationary) density π.

The stationary Markov chain (P, π) induces a joint density $f(x_{t+1}, x_t)$ over (x_{t+1}, x_t) that is independent of calendar time t; P, π and the definition $y_t = \overline{y}' x_t$ also induce a joint density $f_y(y_{t+1}, y_t)$ that is independent of calendar time. In what follows, we compute mathematical expectations with respect to the joint density $f_y(y_{t+1}, y_t)$.

For a finite-state Markov chain, the following theorem gives a convenient way to characterize invariant functions of the state.

Theorem 2.2.4. *Let (P, π) be a stationary Markov chain. If*

$$E\left[y_{t+1} | x_t\right] = y_t \tag{2.2.12}$$

then the random variable $y_t = \overline{y}' x_t$ is invariant.

Proof. By using the law of iterated expectations, notice that

$$\begin{aligned} E\left(y_{t+1} - y_t\right)^2 &= E\left[E\left(y_{t+1}^2 - 2 y_{t+1} y_t + y_t^2\right) | x_t\right] \\ &= E\left[E y_{t+1}^2 | x_t - 2E\left(y_{t+1} | x_t\right) y_t + E y_t^2 | x_t\right] \\ &= E y_{t+1}^2 - 2 E y_t^2 + E y_t^2 \\ &= 0 \end{aligned}$$

where the middle term on the right side of the second line uses that $E[y_t | x_t] = y_t$, the middle term on the right side of the third line uses the hypothesis (2.2.12), and the third line uses the hypothesis that π is a stationary distribution. In a finite Markov chain, if $E(y_{t+1} - y_t)^2 = 0$, then $y_{t+1} = y_t$ for all y_{t+1}, y_t that occur with positive probability under the stationary distribution. ∎

As we shall have reason to study in chapters 17 and 18, *any* (not necessarily stationary) stochastic process y_t that satisfies (2.2.12) is said to be a *martingale*. Theorem 2.2.4 tells us that a martingale that is a function of a finite-state stationary Markov state x_t must be constant over time. This result is a special case of the martingale convergence theorem that underlies some remarkable results about savings to be studied in chapter 17.[1]

[1] Theorem 2.2.4 tells us that a stationary martingale process has so little freedom to move that it has to be constant forever, not just eventually, as asserted by the martingale convergence theorem.

Equation (2.2.12) can be expressed as $P\overline{y} = \overline{y}$ or

$$(P - I)\,\overline{y} = 0, \tag{2.2.13}$$

which states that an invariant function of the state is a (right) eigenvector of P associated with a unit eigenvalue. Thus, associated with unit eigenvalues of P are (1) left eigenvectors that are stationary distributions of the chain (recall equation (2.2.4)), and (2) right eigenvectors that are invariant functions of the chain (from equation (2.2.13)).

Definition 2.2.3. Let (P, π) be a stationary Markov chain. The chain is said to be *ergodic* if the only invariant functions $\overline{y}$ are constant with probability 1 under the stationary unconditional probability distribution π, i.e., $\overline{y}_i = \overline{y}_j$ for all i, j with $\pi_i > 0, \pi_j > 0$.

REMARK: Let $\tilde{\pi}^{(1)}, \tilde{\pi}^{(2)}, \ldots, \tilde{\pi}^{(m)}$ be m distinct 'basis' stationary distributions for an n state Markov chain with transition matrix P. Each $\tilde{\pi}^{(k)}$ is an $(n \times 1)$ left eigenvector of P associated with a distinct unit eigenvalue. Each $\pi^{(j)}$ is scaled to be a probability vector (i.e., its components are nonnegative and sum to unity). The set S of *all* stationary distributions is convex. An element $\pi_b \in S$ can be represented as

$$\pi_b = b_1 \tilde{\pi}^{(1)} + b_2 \tilde{\pi}^{(2)} + \cdots + b_m \tilde{\pi}^{(m)},$$

where $b_j \geq 0, \sum_j b_j = 1$ is a probability vector.

REMARK: A stationary density π_b for which the pair (P, π_b) is an ergodic Markov chain is an extreme point of the convex set S, meaning that it can be represented as $\pi_b = \tilde{\pi}^{(j)}$ for one of the 'basis' stationary densities.

A law of large numbers for Markov chains is:

Theorem 2.2.5. *Let $\overline{y}$ define a random variable on a stationary and ergodic Markov chain (P, π). Then*

$$\frac{1}{T} \sum_{t=1}^{T} y_t \to E\left[y_0\right] \tag{2.2.14}$$

with probability 1.

This theorem tells us that the time series average converges to the population mean of the stationary distribution.

Three examples illustrate these concepts.

Example 1. A chain with transition matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has a unique stationary distribution $\pi = [\,.5 \quad .5\,]'$ and the invariant functions are $[\,\alpha \quad \alpha\,]'$ for any scalar α. Therefore, the process is ergodic and Theorem 2.2.5 applies.

Example 2. A chain with transition matrix $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a continuum of stationary distributions $\gamma \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1-\gamma) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for any $\gamma \in [0,1]$ and invariant functions $\begin{bmatrix} 0 \\ \alpha_1 \end{bmatrix}$ and $\begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}$ for any scalars α_1, α_2. Therefore, the process is not ergodic when $\gamma \in (0,1)$, for note that neither invariant function is constant across states that receive positive probability according to a stationary distribution associated with $\gamma \in (0,1)$. Therefore, the conclusion (2.2.14) of Theorem 2.2.5 does not hold for an initial stationary distribution associated with $\gamma \in (0,1)$, although the weaker result Theorem 2.2.3 does hold. When $\gamma \in (0,1)$, nature chooses state $i = 1$ or $i = 2$ with probabilities $\gamma, 1-\gamma$, respectively, at time 0. Thereafter, the chain remains stuck in the realized time 0 state. Its failure ever to visit the unrealized state prevents the sample average from converging to the population mean of an arbitrary function $\bar{y}$ of the state. Notice that conclusion (2.2.14) of Theorem 2.2.5 does hold for the stationary distributions associated with $\gamma = 0$ and $\gamma = 1$.

Example 3. A chain with transition matrix $P = \begin{bmatrix} .8 & .2 & 0 \\ .1 & .9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has a continuum of stationary distributions $\gamma\,[\,\frac{1}{3} \quad \frac{2}{3} \quad 0\,]' + (1-\gamma)\,[\,0 \quad 0 \quad 1\,]'$ for $\gamma \in [0,1]$ and invariant functions $\alpha_1\,[\,1 \quad 1 \quad 0\,]'$ and $\alpha_2\,[\,0 \quad 0 \quad 1\,]'$ for any scalars α_1, α_2. The conclusion (2.2.14) of Theorem 2.2.5 does not hold for the stationary distributions associated with $\gamma \in (0,1)$, but Theorem 2.2.3 does hold. But again, conclusion (2.2.14) does hold for the stationary distributions associated with $\gamma = 0$ and $\gamma = 1$.

2.2.8. Simulating a Markov chain

It is easy to simulate a Markov chain using a random number generator. The Matlab program `markov.m` does the job. We'll use this program in some later chapters.[2]

2.2.9. The likelihood function

Let P be an $n \times n$ stochastic matrix with states $1, 2, \ldots, n$. Let π_0 be an $n \times 1$ vector with nonnegative elements summing to 1, with $\pi_{0,i}$ being the probability that the state is i at time 0. Let i_t index the state at time t. The Markov property implies that the probability of drawing the path $(x_0, x_1, \ldots, x_{T-1}, x_T) = (\overline{e}_{i_0}, \overline{e}_{i_1}, \ldots, \overline{e}_{i_{T-1}}, \overline{e}_{i_T})$ is

$$\begin{aligned} L &\equiv \text{Prob}\left(\overline{x}_{i_T}, \overline{x}_{i_{T-1}}, \ldots, \overline{x}_{i_1}, \overline{x}_{i_0}\right) \\ &= P_{i_{T-1},i_T} P_{i_{T-2},i_{T-1}} \cdots P_{i_0,i_1} \pi_{0,i_0}. \end{aligned} \tag{2.2.15}$$

The probability L is called the *likelihood*. It is a function of both the sample realization $x_0, \ldots, x_T$ and the parameters of the stochastic matrix P. For a sample $x_0, x_1, \ldots, x_T$, let n_{ij} be the number of times that there occurs a one-period transition from state i to state j. Then the likelihood function can be written

$$L = \pi_{0,i_0} \prod_i \prod_j P_{i,j}^{n_{ij}},$$

a *multinomial* distribution.

Formula (2.2.15) has two uses. A first, which we shall encounter often, is to describe the probability of alternative histories of a Markov chain. In chapter 8, we shall use this formula to study prices and allocations in competitive equilibria.

A second use is for estimating the parameters of a model whose solution is a Markov chain. Maximum likelihood estimation for free parameters θ of a Markov process works as follows. Let the transition matrix P and the initial distribution π_0 be functions $P(\theta), \pi_0(\theta)$ of a vector of free parameters θ. Given a sample $\{x_t\}_{t=0}^T$, regard the likelihood function as a function of the parameters θ. As the estimator of θ, choose the value that maximizes the likelihood function L.

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[2] An index in the back of the book lists Matlab programs.

2.3. Continuous-state Markov chain

In chapter 8, we shall use a somewhat different notation to express the same ideas. This alternative notation can accommodate either discrete- or continuous-state Markov chains. We shall let S denote the state space with typical element $s \in S$. Let state transitions be described by the cumulative distribution function $\Pi(s'|s) = \text{Prob}(s_{t+1} \leq s'|s_t = s)$ and let the initial state s_0 be described by the cumulative distribution function $\Pi_o(s) = \text{Prob}(s_0 \leq s)$. The *transition density* is $\pi(s'|s) = \frac{d}{ds'}\Pi(s'|s)$ and the initial density is $\pi_0(s) = \frac{d}{ds}\Pi_0(s)$. For all $s \in S, \pi(s'|s) \geq 0$ and $\int_{s'} \pi(s'|s)ds' = 1$; also $\int_s \pi_0(s)ds = 1$.[3] Corresponding to (2.2.15), the likelihood function or density over the history $s^t = [s_t, s_{t-1}, \ldots, s_0]$ is

$$\pi\left(s^t\right) = \pi\left(s_t|s_{t-1}\right)\cdots\pi\left(s_1|s_0\right)\pi_0\left(s_0\right). \tag{2.3.1}$$

For $t \geq 1$, the time t unconditional distributions evolve according to

$$\pi_t\left(s_t\right) = \int_{s_{t-1}} \pi\left(s_t|s_{t-1}\right)\pi_{t-1}\left(s_{t-1}\right)d\,s_{t-1}.$$

A stationary or *invariant* distribution satisfies

$$\pi_\infty\left(s'\right) = \int_s \pi\left(s'|s\right)\pi_\infty\left(s\right)d\,s,$$

which is the counterpart to (2.2.3).

Definition: A Markov chain $\big(\pi(s'|s), \pi_0(s)\big)$ is said to be *stationary* if π_0 satisfies

$$\pi_0\left(s'\right) = \int_s \pi\left(s'|s\right)\pi_0\left(s\right)d\,s.$$

Definition: Paralleling our discussion of finite-state Markov chains, we can say that the function $\phi(s)$ is *invariant* if

$$\int \phi\left(s'\right)\pi\left(s'|s\right)ds' = \phi\left(s\right).$$

A stationary continuous-state Markov process is said to be *ergodic* if the only invariant functions $\phi(s')$ are constant with probability 1 under the stationary distribution π_∞.

[3] Thus, when S is discrete, $\pi(s_j|s_i)$ corresponds to $P_{i,j}$ in our earlier notation.

A law of large numbers for Markov processes states:

Theorem 2.3.1. *Let* $y(s)$ *be a random variable, a measurable function of* s*, and let* $\big(\pi(s'|s), \pi_0(s)\big)$ *be a stationary and ergodic continuous-state Markov process. Assume that* $E|y| < +\infty$*. Then*

$$\frac{1}{T}\sum_{t=1}^{T} y_t \to Ey = \int y\,(s)\,\pi_0\,(s)\,ds$$

with probability 1 *with respect to the distribution* π_0*.*

2.4. Stochastic linear difference equations

The first-order linear vector stochastic difference equation is a useful example of a continuous-state Markov process. Here we use $x_t \in I\!R^n$ rather than s_t to denote the time t state and specify that the initial distribution $\pi_0(x_0)$ is Gaussian with mean μ_0 and covariance matrix Σ_0, and that the transition density $\pi(x'|x)$ is Gaussian with mean $A_o x$ and covariance CC'.[4] This specification pins down the joint distribution of the stochastic process $\{x_t\}_{t=0}^{\infty}$ via formula (2.3.1). The joint distribution determines all moments of the process.

This specification can be represented in terms of the first-order stochastic linear difference equation

$$x_{t+1} = A_o x_t + C w_{t+1} \tag{2.4.1}$$

for $t = 0, 1, \ldots$, where x_t is an $n \times 1$ state vector, x_0 is a random initial condition drawn from a probability distribution with mean $Ex_0 = \mu_0$ and covariance matrix $E(x_0 - \mu_0)(x_0 - \mu_0)' = \Sigma_0$, A_o is an $n \times n$ matrix, C is an $n \times m$ matrix, and w_{t+1} is an $m \times 1$ vector satisfying the following:

Assumption A1: w_{t+1} is an i.i.d. process satisfying $w_{t+1} \sim \mathcal{N}(0, I)$.

---

[4] An $n \times 1$ vector z that is multivariate normal has the density function

$$f\,(z) = (2\pi)^{-.5n}\,|\Sigma|^{-.5} \exp\Big(-.5\,(z-\mu)'\,\Sigma^{-1}\,(z-\mu)\Big)$$

where $\mu = Ez$ and $\Sigma = E(z-\mu)(z-\mu)'$.

We can weaken the Gaussian assumption A1. To focus only on first and second moments of the x process, it is sufficient to make the weaker assumption:

ASSUMPTION A2: w_{t+1} is an $m \times 1$ random vector satisfying:

$$E w_{t+1} | J_t = 0 \tag{2.4.2a}$$

$$E w_{t+1} w'_{t+1} | J_t = I, \tag{2.4.2b}$$

where $J_t = [\, w_t \quad \cdots \quad w_1 \quad x_0 \,]$ is the information set at t, and $E[\,\cdot\,|J_t]$ denotes the conditional expectation. We impose no distributional assumptions beyond (2.4.2). A sequence $\{w_{t+1}\}$ satisfying equation (2.4.2a) is said to be a martingale difference sequence adapted to J_t. A sequence $\{z_{t+1}\}$ that satisfies $E[z_{t+1}|J_t] = z_t$ is said to be a martingale adapted to J_t.

An even weaker assumption is

ASSUMPTION A3: w_{t+1} is a process satisfying

$$E w_{t+1} = 0$$

for all t and

$$E w_t w'_{t-j} = \begin{cases} I, & \text{if } j = 0; \\ 0, & \text{if } j \neq 0. \end{cases}$$

A process satisfying assumption A3 is said to be a vector "white noise." [5]

Assumption A1 or A2 implies assumption A3 but not vice versa. Assumption A1 implies assumption A2 but not vice versa. Assumption A3 is sufficient to justify the formulas that we report below for second moments. We shall often append an observation equation $y_t = G x_t$ to equation (2.4.1) and deal with the augmented system

$$x_{t+1} = A_o x_t + C w_{t+1} \tag{2.4.3a}$$

$$y_t = G x_t. \tag{2.4.3b}$$

Here y_t is a vector of variables observed at t, which may include only some linear combinations of x_t. The system (2.4.3) is often called a linear *state-space system.*

[5] Note that (2.4.2a) by itself allows the distribution of w_{t+1} conditional on J_t to be heteroskedastic.

Example 1. Scalar second-order autoregression: Assume that z_t and w_t are scalar processes and that

$$z_{t+1} = \alpha + \rho_1 z_t + \rho_2 z_{t-1} + w_{t+1}.$$

Represent this relationship as the system

$$\begin{bmatrix} z_{t+1} \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 & \alpha \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_t \\ z_{t-1} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w_{t+1}$$

$$z_t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ z_{t-1} \\ 1 \end{bmatrix}$$

which has form (2.4.3).

Example 2. First-order scalar mixed moving average and autoregression: Let

$$z_{t+1} = \rho z_t + w_{t+1} + \gamma w_t.$$

Express this relationship as

$$\begin{bmatrix} z_{t+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & \gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ w_t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_{t+1}$$

$$z_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ w_t \end{bmatrix}.$$

Example 3. Vector autoregression: Let z_t be an $n \times 1$ vector of random variables. We define a vector autoregression by a stochastic difference equation

$$z_{t+1} = \sum_{j=1}^{4} A_j z_{t+1-j} + C_y w_{t+1}, \tag{2.4.4}$$

where w_{t+1} is an $n \times 1$ martingale difference sequence satisfying equation (2.4.2) with $x_0' = \begin{bmatrix} z_0 & z_{-1} & z_{-2} & z_{-3} \end{bmatrix}$ and A_j is an $n \times n$ matrix for each j. We can map equation (2.4.4) into equation (2.4.1) as follows:

$$\begin{bmatrix} z_{t+1} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} z_t \\ z_{t-1} \\ z_{t-2} \\ z_{t-3} \end{bmatrix} + \begin{bmatrix} C_y \\ 0 \\ 0 \\ 0 \end{bmatrix} w_{t+1}. \tag{2.4.5}$$

Define A_o as the state transition matrix in equation (2.4.5). Assume that A_o has all of its eigenvalues bounded in modulus below unity. Then equation (2.4.4) can be initialized so that z_t is *covariance stationary*, a term we now define.

2.4.1. First and second moments

We can use equation (2.4.1) to deduce the first and second moments of the sequence of random vectors $\{x_t\}_{t=0}^{\infty}$. A sequence of random vectors is called a *stochastic process*.

Definition 2.4.1. A stochastic process $\{x_t\}$ is said to be *covariance stationary* if it satisfies the following two properties: (a) the mean is independent of time, $Ex_t = Ex_0$ for all t, and (b) the sequence of autocovariance matrices $E(x_{t+j} - Ex_{t+j})(x_t - Ex_t)'$ depends on the separation between dates $j = 0, \pm 1, \pm 2, \ldots$, but not on t.

We use

Definition 2.4.2. A square real valued matrix A_o is said to be *stable* if all of its eigenvalues modulus are strictly less than unity.

We shall often find it useful to assume that (2.4.3) takes the special form

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{C} \end{bmatrix} w_{t+1} \tag{2.4.6}$$

where $\tilde{A}$ is a stable matrix. That $\tilde{A}$ is a stable matrix implies that the only solution of $(\tilde{A} - I)\mu_2 = 0$ is $\mu_2 = 0$ (i.e., 1 is *not* an eigenvalue of $\tilde{A}$). It follows that the matrix $A_o = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix}$ on the right side of (2.4.6) has one eigenvector associated with a single unit eigenvalue: $(A_o - I) \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = 0$ implies μ_1 is an arbitrary scalar and $\mu_2 = 0$. The first equation of (2.4.6) implies that $x_{1,t+1} = x_{1,0}$ for all $t \geq 0$. Picking the initial condition $x_{1,0}$ pins down a particular eigenvector $\begin{bmatrix} x_{1,0} \\ 0 \end{bmatrix}$ of A_o. As we shall see soon, this eigenvector is our candidate for the unconditional mean of x that makes the process covariance stationary.

We will make an assumption that guarantees that there exists an initial condition $(Ex_0, E(x - Ex_0)(x - Ex_0)')$ that makes the x_t process covariance stationary. Either of the following conditions works:

Condition A1: All of the eigenvalues of A_o in (2.4.3) are strictly less than 1 in modulus.

Condition A2: The state-space representation takes the special form (2.4.6) and all of the eigenvalues of $\tilde{A}$ are strictly less than 1 in modulus.

To discover the first and second moments of the x_t process, we regard the initial condition x_0 as being drawn from a distribution with mean $\mu_0 = Ex_0$ and covariance $\Sigma_0 = E(x - Ex_0)(x - Ex_0)'$. We shall deduce starting values for the mean and covariance that make the process covariance stationary, though our formulas are also useful for describing what happens when we start from other initial conditions that generate transient behavior that stops the process from being covariance stationary.

Taking mathematical expectations on both sides of equation (2.4.1) gives

$$\mu_{t+1} = A_o \mu_t \tag{2.4.7}$$

where $\mu_t = Ex_t$. We will assume that all of the eigenvalues of A_o are strictly less than unity in modulus, except possibly for one that is affiliated with the constant terms in the various equations. Then x_t possesses a stationary mean defined to satisfy $\mu_{t+1} = \mu_t$, which from equation (2.4.7) evidently satisfies

$$(I - A_o)\,\mu = 0, \tag{2.4.8}$$

which characterizes the mean μ as an eigenvector associated with the single unit eigenvalue of A_o. Notice that

$$x_{t+1} - \mu_{t+1} = A_o\,(x_t - \mu_t) + Cw_{t+1}. \tag{2.4.9}$$

Also, the fact that the remaining eigenvalues of A_o are less than unity in modulus implies that starting from any μ_0, $\mu_t \to \mu$.[6]

From equation (2.4.9), we can compute that the law of motion of the unconditional covariance matrices $\Sigma_t \equiv E(x_t - \mu)(x_t - \mu)'$. Thus,

$$E\,(x_{t+1} - \mu)\,(x_{t+1} - \mu)' = A_o E\,(x_t - \mu)\,(x_t - \mu)'\,A_o' + CC'$$

[6] To understand this, assume that the eigenvalues of A_o are distinct, and use the representation $A_o = P\Lambda P^{-1}$ where Λ is a diagonal matrix of the eigenvalues of A_o, arranged in descending order of magnitude, and P is a matrix composed of the corresponding eigenvectors. Then equation (2.4.7) can be represented as $\mu^*_{t+1} = \Lambda\mu^*_t$, where $\mu^*_t \equiv P^{-1}\mu_t$, which implies that $\mu^*_t = \Lambda^t \mu^*_0$. When all eigenvalues but the first are less than unity, Λ^t converges to a matrix of zeros except for the $(1,1)$ element, and μ^*_t converges to a vector of zeros except for the first element, which stays at $\mu^*_{0,1}$, its initial value, which we are free to set equal to 1, to capture the constant. Then $\mu_t = P\mu^*_t$ converges to $P_1\mu^*_{0,1} = P_1$, where P_1 is the eigenvector corresponding to the unit eigenvalue.

or

$$\Sigma_{t+1} = A_o \Sigma_t A_o' + CC'.$$

A fixed point of this matrix difference equation evidently satisfies

$$\Sigma_\infty = A_o \Sigma_\infty A_o' + CC'.$$

We shall use the notation

$$C_x(0) = \Sigma_\infty$$

for the fixed point, which, if it exists, is the covariance matrix $E(x_t - \mu)(x_t - \mu)'$ under the stationary distribution of x.

Thus, to compute $C_x(0)$, we must solve

$$C_x(0) = A_o C_x(0) A_o' + CC', \tag{2.4.10}$$

where $C_x(0) \equiv E(x_t - \mu)(x_t - \mu)'$. Equation (2.4.10) is a *discrete Lyapunov* equation in the $n \times n$ matrix $C_x(0)$. It can be solved with the Matlab program `doublej.m`.

By virtue of (2.4.1) and (2.4.7), note that

$$(x_{t+j} - \mu_{t+j}) = A_o^j (x_t - \mu_t) + Cw_{t+j} + \cdots + A_o^{j-1} Cw_{t+1}.$$

Postmultiplying both sides by $(x_t - \mu_t)'$ and taking expectations shows that the autocovariance sequence satisfies

$$C_x(j) \equiv E(x_{t+j} - \mu)(x_t - \mu)' = A_o^j C_x(0). \tag{2.4.11}$$

The autocovariance matrix sequence $\{C_x(j)\}_{j=-\infty}^{\infty}$ is also called the *autocovariogram*. Once (2.4.10), is solved, the remaining second moments $C_x(j)$ can be deduced from equation (2.4.11).[7]

Suppose that $y_t = Gx_t$. Then $\mu_{yt} = Ey_t = G\mu_t$ and

$$E(y_{t+j} - \mu_{yt+j})(y_t - \mu_{yt})' = GC_x(j) G', \tag{2.4.12}$$

for $j = 0, 1, \ldots$. Equations (2.4.12) show that the autocovariogram for a stochastic process governed by a stochastic linear difference equation obeys the nonstochastic version of that difference equation.

---

[7] Notice that $C_x(-j) = C_x(j)'$.

2.4.2. Summary of moment formulas

Object	Formula
unconditional mean	$\mu_{t+1} = A_o \mu_t$
unconditional covariance	$\Sigma_{t+1} = A_o \Sigma_t A_o' + CC'$
$E[x_t \vert x_0]$	$A_o^t x_0$
$E(x_t - E_0 x_t)(x_t - E_0 x_t)'$	$\sum_{h=0}^{t-1} A_o^h CC'(A_o^h)'$
stationary mean	$(I - A_o)\mu = 0$
stationary variance	$C_x(0) = A_o C_x(0) A_o' + CC'$
stationary autocovariance	$C_x(j) = A_o^j C_x(0)$

The accompanying table summarizes some formulas for various conditional and unconditional first and second moments of the state x_t governed by our linear stochastic state space system A_o, C, G. In section 2.5, we select some moments and use them to form population linear regressions.

2.4.3. Impulse response function

Suppose that the eigenvalues of A_o not associated with the constant are bounded above in modulus by unity. Using the lag operator L defined by $Lx_{t+1} \equiv x_t$, express equation (2.4.1) as

$$(I - A_o L)\, x_{t+1} = C w_{t+1}. \tag{2.4.13}$$

Iterate equation (2.4.1) forward from $t = 0$ to get

$$x_t = A_o^t x_0 + \sum_{j=0}^{t-1} A_o^j C w_{t-j} \tag{2.4.14}$$

Evidently,

$$y_t = G A_o^t x_0 + G \sum_{j=0}^{t-1} A_o^j C w_{t-j} \tag{2.4.15}$$

and $E y_t | x_0 = G A_o^t x_0$. Equations (2.4.14) and (2.4.15) are examples of a moving average representation. Viewed as a function of lag j, $h_j = A_o^j C$ or $\tilde{h}_j = G A_o^j C$ is called the *impulse response function*. The moving average representation and the associated impulse response function show how x_{t+j} or y_{t+j}

is affected by lagged values of the shocks, the w_{t+1}'s. Thus, the contribution of a shock w_{t-j} to x_t is $A_o^j C$.[8]

Equation (2.4.15) implies that the t-step ahead conditional covariance matrices are given by

$$E\left(y_t - E y_t | x_0\right)\left(y_t - E y_t | x_0\right)' = G\left[\sum_{h=0}^{t-1} A_o^h C C' A_o^{h\prime}\right] G'. \tag{2.4.16}$$

2.4.4. Prediction and discounting

From equation (2.4.1) we can compute the useful prediction formulas

$$E_t x_{t+j} = A_o^j x_t \tag{2.4.17}$$

for $j \geq 1$, where $E_t(\cdot)$ denotes the mathematical expectation conditioned on $x^t = (x_t, x_{t-1}, \ldots, x_0)$. Let $y_t = G x_t$, and suppose that we want to compute $E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}$. Evidently,

$$E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} = G\left(I - \beta A_o\right)^{-1} x_t, \tag{2.4.18}$$

provided that the eigenvalues of βA_o are less than unity in modulus. Equation (2.4.18) tells us how to compute an expected discounted sum, where the discount factor β is constant.

[8] The Matlab programs `dimpulse.m` and `impulse.m` compute impulse response functions.

2.4.5. Geometric sums of quadratic forms

In some applications, we want to calculate

$$\alpha_t = E_t \sum_{j=0}^{\infty} \beta^j x'_{t+j} Y x_{t+j}$$

where x_t obeys the stochastic difference equation (2.4.1) and Y is an $n \times n$ matrix. To get a formula for α_t, we use a guess-and-verify method. We guess that α_t can be written in the form

$$\alpha_t = x'_t \nu x_t + \sigma, \tag{2.4.19}$$

where ν is an $(n \times n)$ matrix and σ is a scalar. The definition of α_t and the guess (2.4.19) imply[9]

$$\begin{aligned}\alpha_t &= x'_t Y x_t + \beta E_t \left(x'_{t+1} \nu x_{t+1} + \sigma \right) \\ &= x'_t Y x_t + \beta E_t \left[(A_o x_t + C w_{t+1})' \ \nu \ (A_o x_t + C w_{t+1}) + \sigma \right] \\ &= x'_t \left(Y + \beta A'_o \nu A_o \right) \ x_t + \beta \ \text{trace} \left(\nu C C' \right) + \beta \sigma.\end{aligned}$$

It follows that ν and σ satisfy

$$\begin{aligned}\nu &= Y + \beta A'_o \nu A_o \\ \sigma &= \beta \sigma + \beta \ \text{trace} \ \nu C C'.\end{aligned} \tag{2.4.20}$$

The first equation of (2.4.20) is a *discrete Lyapunov equation* in the square matrix ν and can be solved by using one of several algorithms.[10] After ν has been computed, the second equation can be solved for the scalar σ.

We mention two important applications of formulas (2.4.19) and (2.4.20).

---

[9] Here we are repeatedly using the fact that for two conformable matrices A, B, trace$AB =$ traceBA to conclude that $E(w'_{t+1} C' \nu C w_{t+1}) = E\text{trace}(\nu C w_{t+1} w'_{t+1} C') = \text{trace}(\nu C E w_{t+1} w'_{t+1} C') = \text{trace}(\nu C C')$.

[10] The Matlab control toolkit has a program called `dlyap.m` that works when all of the eigenvalues of A_o are strictly less than unity; the program called `doublej.m` works even when there is a unit eigenvalue associated with the constant.

2.4.5.1. Asset pricing

Let y_t be governed by the state-space system (2.4.3). In addition, assume that there is a scalar random process z_t given by

$$z_t = Hx_t.$$

Regard the process y_t as a payout or dividend from an asset, and regard $\beta^t z_t$ as a stochastic discount factor. The price of a perpetual claim on the stream of payouts is

$$\alpha_t = E_t \sum_{j=0}^{\infty} \left(\beta^j z_{t+j}\right) y_{t+j}. \tag{2.4.21}$$

To compute α_t, we simply set $Y = H'G$ in (2.4.19) and (2.4.20). In this application, the term σ functions as a risk premium; it is zero when $C = 0$.

2.4.5.2. Evaluation of dynamic criterion

Let a state x_t be governed by

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1} \tag{2.4.22}$$

where u_t is a control vector that is set by a decision maker according to a fixed rule

$$u_t = -F_0 x_t. \tag{2.4.23}$$

Substituting (2.4.23) into (2.4.22) gives (2.4.1) where $A_o = A - BF_0$. We want to compute the *value function*

$$v(x_0) = -E_0 \sum_{t=0}^{\infty} \beta^t \left[x_t' R x_t + u_t' Q u_t\right]$$

for fixed positive definite matrices R and Q, fixed decision rule F_0 in (2.4.23), $A_o = A - BF_0$, and arbitrary initial condition x_0. Formulas (2.4.19) and (2.4.20) apply with $Y = R + F_0' Q F_0$ and $A_o = A - BF_0$. Express the solution as

$$v(x_0) = -x_0' P_0 x_0 - \sigma \tag{2.4.24}$$

where by applying formulas (2.4.19) and (2.4.20), P_0 satisfies the following formula:

$$P_0 = R + F_0' Q F_0 + \beta (A - BF_0)' P_0 (A - BF_0). \tag{2.4.25}$$

Given F_0, formula (2.4.25) determines the matrix P_0 in the value function that describes the expected discounted value of the sum of payoffs from sticking forever with this decision rule.

Now consider the following one-period problem. Suppose that we must use decision rule F_0 from time 1 onward, so that the value at time 1 on starting from state x_1 is

$$v(x_1) = -x_1' P_0 x_1 - \sigma. \tag{2.4.26}$$

Taking $u_t = -F_0 x_t$ as given for $t \geq 1$, what is the best choice of u_0? This leads to the optimum problem:

$$\max_{u_0} -\{x_0' R x_0 + u_0' Q u_0 + \beta E (A x_0 + B u_0 + C w_1)' P_0 (A x_0 + B u_0 + C w_1) + \beta \sigma\}. \tag{2.4.27}$$

The first-order conditions for this problem can be rearranged to attain

$$u_0 = -F_1 x_0 \tag{2.4.28}$$

where

$$F_1 = \beta (Q + \beta B' P_0 B)^{-1} B' P_0 A. \tag{2.4.29}$$

Given P_0, formula (2.4.29) gives the best decision rule $u_0 = -F_1 x_0$ if at $t = 0$ you are permitted only a one-period deviation from the rule $u_t = -F_0 x_t$ that has to be used for $t \geq 1$. If $F_1 \neq F_0$, we say that the decision maker would accept the opportunity to deviate from F_0 for one period.

It is tempting to iterate on (2.4.29) and (2.4.25) as follows to seek a decision rule from which a decision maker would not want to deviate for one period: (1) given an F_0, find P_0; (2) reset F equal to the F_1 found in step 1, then to substitute it for F_0 in (2.4.25) to compute a new P, call it P_1; (3) return to step 1 and iterate to convergence. This leads to the two equations

$$\begin{aligned} P_j &= R + F_j' Q F_j + \beta (A - B F_j)' P_j (A - B F_j) \\ F_{j+1} &= \beta (Q + \beta B' P_j B)^{-1} B' P_j A \end{aligned} \tag{2.4.30}$$

which are to be initialized from an arbitrary F_0 that ensures that $\sqrt{\beta}(A - BF_0)$ is a stable matrix. After this process has converged, one cannot find a value-increasing one-period deviation from the limiting decision rule $u_t = -F_\infty x_t$.[11]

[11] It turns out that if you don't want to deviate for one period, then you would never want to deviate, so that the limiting rule is optimal.

As we shall see in chapter 4, this is an excellent algorithm for solving a dynamic programming problem. It is an example of the Howard improvement algorithm. In chapter 5, we describe an alternative algorithm that iterates on the following equations

$$\begin{aligned} P_{j+1} &= R + F_j' Q F_j + \beta \left(A - BF_j\right)' P_j \left(A - BF_j\right) \\ F_j &= \beta \left(Q + \beta B' P_j B\right)^{-1} B' P_j A, \end{aligned} \tag{2.4.31}$$

that is to be initialized from an arbitrary positive semi-definite matrix P_0.[12]

2.5. Population regression

This section explains the notion of a population regression equation. Suppose that we have a state-space system (2.4.3) with initial conditions that make it covariance stationary. We can use the preceding formulas to compute the second moments of any pair of random variables. These moments let us compute a linear regression. Thus, let X be a $p \times 1$ vector of random variables somehow selected from the stochastic process $\{y_t\}$ governed by the system (2.4.3). For example, let $p = 2m$, where y_t is an $m \times 1$ vector, and take $X = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$ for any $t \geq 1$. Let Y be any scalar random variable selected from the $m \times 1$ stochastic process $\{y_t\}$. For example, take $Y = y_{t+1,1}$ for the same t used to define X, where $y_{t+1,1}$ is the first component of y_{t+1}.

We consider the following least-squares approximation problem: find a $1 \times p$ vector of real numbers β that attain

$$\min_\beta E \left(Y - \beta X\right)^2. \tag{2.5.1}$$

Here βX is being used to estimate Y, and we want the value of β that minimizes the expected squared error. The first-order necessary condition for minimizing $E(Y - \beta X)^2$ with respect to β is

$$E \left(Y - \beta X\right) X' = 0, \tag{2.5.2}$$

which can be rearranged as[13]

$$\beta = \left(EYX'\right) \left[E \left(XX'\right)\right]^{-1}. \tag{2.5.3}$$

[12] $P_0 = 0$ is a popular choice.

[13] That $EX'X$ is nonnegative definite implies that the second-order conditions for a minimum of condition (2.5.1) are satisfied.

By using the formulas (2.4.8), (2.4.10), (2.4.11), and (2.4.12), we can compute EXX' and EYX' for whatever selection of X and Y we choose. The condition (2.5.2) is called the least-squares normal equation. It states that the projection error $Y - \beta X$ is orthogonal to X. Therefore, we can represent Y as

$$Y = \beta X + \epsilon \tag{2.5.4}$$

where $E\epsilon X' = 0$. Equation (2.5.4) is called a population regression equation, and βX is called the least-squares projection of Y on X or the least-squares regression of Y on X. The vector β is called the population least-squares regression vector. The law of large numbers for continuous-state Markov processes, Theorem 2.3.1, states conditions that guarantee that sample moments converge to population moments, that is, $\frac{1}{S}\sum_{s=1}^{S} X_s X_s' \to EXX'$ and $\frac{1}{S}\sum_{s=1}^{S} Y_s X_s' \to EYX'$. Under those conditions, sample least-squares estimates converge to β.

There are as many such regressions as there are ways of selecting Y, X. We have shown how a model (e.g., a triple A_o, C, G, together with an initial distribution for x_0) restricts a regression. Going backward, that is, telling what a given regression tells about a model, is more difficult. Many regressions tell little about the model, and what little they have to say can be difficult to decode. As we indicate in sections 2.6 and 2.10, the likelihood function completely describes what a given data set says about a model in a way that is straightforward to decode.

2.5.1. Multiple regressors

Now let Y be an $n \times 1$ vector of random variables and think of regression solving the least squares problem for each of them to attain a representation

$$Y = \beta X + \epsilon \tag{2.5.5}$$

where β is now $n \times p$ and ϵ is now an $n \times 1$ vector of least squares residuals. The population regression coefficients are again given by

$$\beta = E\left(YX'\right)\left[E\left(XX'\right)\right]^{-1}. \tag{2.5.6}$$

We will use this formula repeatedly in section 2.7 to derive the Kalman filter.

2.6. Estimation of model parameters

We have shown how to map the matrices A_o, C into all of the second moments of the stationary distribution of the stochastic process $\{x_t\}$. Linear economic models typically give A_o, C as functions of a set of deeper parameters θ. We shall give examples of some such models in chapters 4 and 5. Those theories and the formulas of this chapter give us a mapping from θ to these theoretical moments of the $\{x_t\}$ process. That mapping is an important ingredient of econometric methods designed to estimate a wide class of linear rational expectations models (see Hansen and Sargent, 1980, 1981). Briefly, these methods use the following procedures to match theory to data. To simplify, we shall assume that at time t, observations are available on the entire state x_t. As discussed in section 2.10, the details are more complicated if only a subset of the state vector or a noisy signal of the state is observed, though the basic principles remain the same.

Given a sample of observations for $\{x_t\}_{t=0}^T \equiv x_t, t = 0, \ldots, T$, the likelihood function is defined as the joint probability distribution $f(x_T, x_{T-1}, \ldots, x_0)$. The likelihood function can be *factored* using

$$\begin{aligned} f\left(x_T, \ldots, x_0\right) = f\left(x_T | x_{T-1}, \ldots, x_0\right) f\left(x_{T-1} | x_{T-2}, \ldots, x_0\right) \cdots \\ f\left(x_1 | x_0\right) f\left(x_0\right), \end{aligned} \tag{2.6.1}$$

where in each case f denotes an appropriate probability distribution. For system (2.4.1), $f(x_{t+1}|x_t, \ldots, x_0) = f(x_{t+1}|x_t)$, which follows from the Markov property possessed by equation (2.4.1). Then the likelihood function has the recursive form

$$f\left(x_T, \ldots, x_0\right) = f\left(x_T | x_{T-1}\right) f\left(x_{T-1} | x_{T-2}\right) \cdots f\left(x_1 | x_0\right) f\left(x_0\right). \tag{2.6.2}$$

If we assume that the w_t's are Gaussian, then the conditional distribution $f(x_{t+1}|x_t)$ is Gaussian with mean $A_o x_t$ and covariance matrix CC'. Thus, under the Gaussian distribution, the log of the conditional density of the n dimensional vector x_{t+1} becomes

$$\begin{aligned} \log f\left(x_{t+1} | x_t\right) = -.5 n \log\left(2\pi\right) - .5 \ \log\det\ \left(CC'\right) \\ - .5 \left(x_{t+1} - A_o x_t\right)' \left(CC'\right)^{-1} \left(x_{t+1} - A_o x_t\right) \end{aligned} \tag{2.6.3}$$

Given an assumption about the distribution of the initial condition x_0, equations (2.6.2) and (2.6.3) can be used to form the likelihood function of a sample of observations on $\{x_t\}_{t=0}^T$. One computes maximum likelihood estimates by using a hill-climbing algorithm to maximize the likelihood function with respect to the free parameters that determine A_o, C.[14]

When the state x_t is not observed, we need to go beyond the likelihood function for $\{x_t\}$. One approach uses filtering methods to build up the likelihood function for the subset of observed variables.[15] In section 2.7, we derive the Kalman filter as an application of the population regression formulas of section 2.5. Then in section 2.10, we use the Kalman filter as a device that tells us how to find state variables that allow us recursively to form a likelihood function for observations of variables that are not themselves Markov.

2.7. The Kalman filter

As a fruitful application of the population regression formula (2.5.6), we derive the celebrated Kalman filter for the state space system for $t \geq 0$:[16]

$$x_{t+1} = A_o x_t + C w_{t+1} \tag{2.7.1}$$

$$y_t = G x_t + v_t \tag{2.7.2}$$

where x_t is an $n \times 1$ state vector and y_t is an $m \times 1$ vector of signals on the hidden state; w_{t+1} is a $p \times 1$ vector iid sequence of normal random variables with mean 0 and identity covariance matrix, and v_t is another iid vector sequence of normal random variables with mean zero and covariance matrix R. We assume that w_{t+1} and v_s are orthogonal (i.e., $E w_{t+1} v_s' = 0$) for all $t+1$ and s greater than or equal to 0. We assume that

$$x_0 \sim \mathcal{N}(\hat{x}_0, \Sigma_0). \tag{2.7.3}$$

This specification implies that

$$y_0 \sim \mathcal{N}(G\hat{x}_0, G\Sigma_0 G' + R). \tag{2.7.4}$$

[14] For example, putting those free parameters into a vector θ, think of A_o, C as being the matrix functions $A_o(\theta), C(\theta)$.

[15] See Hamilton (1994), Canova (2007), DeJong and Dave (2011), and section 2.10 below.

[16] In exercise 2.22, we ask you to derive the Kalman filter for a state space system that uses a different timing convention and that allows the state and measurement noises to be correlated.

The decision maker is assumed to observe $y_t, \dots, y_0$ but not $x_t, \dots, x_0$ at time t. He knows the structure (2.7.1)-(2.7.2) and the first and second moments implied by this structure. We want to find recursive formulas for the population regressions $\hat{x}_t = E[x_t | y_{t-1}, \dots, y_0]$ and the covariance matrices $\Sigma_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$.

We use the insight that the new information in y_0 relative to what is already known ($\hat{x}_0$) is $a_0 \equiv y_0 - G\hat{x}_0$ (and more generally, the new information at t relative to what can be inferred from the past is $a_t = y_t - G\hat{x}_t$). The decision maker regresses what he doesn't know on what he does. Thus, first apply (2.5.6) to compute the population regression

$$x_0 - \hat{x}_0 = L_0 \left(y_0 - G\hat{x}_0\right) + \eta \tag{2.7.5}$$

where η is a matrix of least squares residuals. The least squares orthogonality conditions are

$$E\left(x_0 - \hat{x}_0\right)\left(y_0 - G\hat{x}_0\right)' = L_0 E\left(y_0 - G\hat{x}_0\right)\left(y_0 - G\hat{x}_0\right)'.$$

Evaluating the moment matrices and solving for L_0 gives the formula

$$L_0 = \Sigma_0 G' \left(G\Sigma_0 G' + R\right)^{-1}. \tag{2.7.6}$$

Define $\hat{x}_1 = E[x_1 | y_0]$.[17] Equation (2.7.1) implies that $E[x_1 | \hat{x}_0] = A_o \hat{x}_0$ and that

$$x_1 = A_o \hat{x}_0 + A_o \left(x_0 - \hat{x}_0\right) + C w_1. \tag{2.7.7}$$

Furthermore, applying (2.7.5) shows that $Ex_1 | y_0 = A_o \hat{x}_0 + A_o L_0 (y_0 - G\hat{x}_0)$, which we express as

$$\hat{x}_1 = A_o \hat{x}_0 + K_0 \left(y_0 - G\hat{x}_0\right), \tag{2.7.8}$$

where

$$K_0 = A_o \Sigma_0 G' \left(G\Sigma_0 G' + R\right)^{-1}. \tag{2.7.9}$$

Subtract (2.7.8) from (2.7.7) to get

$$x_1 - \hat{x}_1 = A_o \left(x_0 - \hat{x}_0\right) + C w_1 - K_0 \left(y_0 - G\hat{x}_0\right). \tag{2.7.10}$$

[17] It is understood that the decision maker knows $\hat{x}_0$. Instead of writing $E[x_1 | y_0, \hat{x}_0]$, we choose simply to write $E[x_1 | y_0]$, but we intend the meaning to be the same. More generally, when we write $E[x_t | y^{t-1}]$, it is understood that the mathematical expectation is also conditioned on $\hat{x}_0$.

Use this equation and $y_0 = Gx_0 + v_0$ to compute the following recursion for $E(x_1 - \hat{x}_1)(x_1 - \hat{x}_1)' = \Sigma_1$:

$$\Sigma_1 = (A_o - K_0 G)\,\Sigma_0\,(A_o - K_0 G)' + (CC' + K_0 R K_0'). \tag{2.7.11}$$

Thus, we have that the distribution of $x_1|y_0 \sim \mathcal{N}(\hat{x}_1, \Sigma_1)$.

Iterating the above argument for $t \geq 2$ gives the recursion:[18]

$$a_t = y_t - G\hat{x}_t \tag{2.7.12a}$$

$$K_t = A_o \Sigma_t G' \left(G \Sigma_t G' + R\right)^{-1} \tag{2.7.12b}$$

$$\hat{x}_{t+1} = A_o \hat{x}_t + K_t a_t \tag{2.7.12c}$$

$$\Sigma_{t+1} = CC' + K_t R K_t' + (A_o - K_t G)\,\Sigma_t\,(A_o - K_t G)'. \tag{2.7.12d}$$

System (2.7.12) is the celebrated Kalman filter, and K_t is called the Kalman gain. Equation (2.7.12d) is known as a matrix Ricatti difference equation.

The process $a_t = y_t - E[y_t|y_{t-1}, \ldots, y_0]$ is called the 'innovation' process in y. It is the part of y_t that cannot be predicted from past values of y. Note that $E a_t a_t' = (G\Sigma_t G' + R)$, the moment matrix whose inverse appears on the right side of the least squares regression formula (2.7.12b). A direct calculation that uses the formulas $a_t = G(x_t - \hat{x}_t) + v_t$ and $a_{t-1} = G(x_{t-1} - \hat{x}_{t-1}) + v_{t-1}$ to compute expected values of products shows that $E a_t a_{t-1}' = 0$, and more generally that $E[a_t|a_{t-1}, \ldots, a_0] = 0$. An alternative argument based on first principles proceeds as follows. Let $H(y^t)$ denote the linear space of all linear combinations of y^t. Note that $a_{t+1} = y_{t+1} - E y_{t+1}|y^t$; that $a_t \in H(y^t)$; that by virtue of being a least-squares error, $a_{t+1} \perp H(y^t)$; and that therefore $a_{t+1} \perp a_t$, and more generally, $a_{t+1} \perp a^t$. Thus, $\{a_t\}$ is a 'white noise' process of innovations to the $\{y_t\}$ process.

Sometimes (2.7.12) is called a 'whitening filter' that takes a $\{y_t\}$ process of signals as an input and produces a process $\{a_t\}$ of innovations as an output. The linear space $H(a^t)$ is evidently an orthogonal basis for the linear space $H(y^t)$.

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[18] Substituting for K_t from (2.7.12b) allows us to rewrite (2.7.12d) as

$$\Sigma_{t+1} = A_o \Sigma_t A_o' + CC' - A_o \Sigma_t G' \left(G \Sigma_t G' + R\right)^{-1} G \Sigma_t A_o',$$

a formula that we shall be reminded of when we study dynamic programming for problems with linear constraints and quadratic return functions.

In what will seem to be superficially very different contexts, we shall encounter equations that will remind us of (2.7.12b), (2.7.12d). See chapter 5, page 140.

2.8. Applications of the Kalman filter

2.8.1. Muth's reverse engineering exercise

Phillip Cagan (1956) and Milton Friedman (1956) posited that when people wanted to form expectations of future values of a scalar y_t, they would use the following "adaptive expectations" scheme:

$$y_{t+1}^* = K \sum_{j=0}^{\infty} (1-K)^j \, y_{t-j} \tag{2.8.1a}$$

or

$$y_{t+1}^* = (1-K) \, y_t^* + K y_t, \tag{2.8.1b}$$

where y_{t+1}^* is people's expectation.[19] Friedman used this scheme to describe people's forecasts of future income. Cagan used it to model their forecasts of inflation during hyperinflations. Cagan and Friedman did not assert that the scheme is an optimal one, and so did not fully defend it. Muth (1960) wanted to understand the circumstances under which this forecasting scheme would be optimal. Therefore, he sought a stochastic process for y_t such that equation (2.8.1) would be optimal. In effect, he posed and solved an "inverse optimal prediction" problem of the form "You give me the forecasting scheme; I have to find the stochastic process that makes the scheme optimal." Muth solved the problem using classical (nonrecursive) methods. The Kalman filter was first described in print in the same year as Muth's solution of this problem (Kalman, 1960). The Kalman filter lets us solve Muth's problem quickly.

[19] See Hamilton (1994) and Kim and Nelson (1999) for diverse applications of the Kalman filter. Appendix B (see Technical Appendixes) briefly describes a discrete-state nonlinear filtering problem.

Muth studied the model

$$x_{t+1} = x_t + w_{t+1} \tag{2.8.2a}$$
$$y_t = x_t + v_t, \tag{2.8.2b}$$

where y_t, x_t are scalar random processes, and w_{t+1}, v_t are mutually independent i.i.d. Gaussian random processes with means of zero and variances $Ew_{t+1}^2 = Q, Ev_t^2 = R$, and $Ev_s w_{t+1} = 0$ for all t, s. The initial condition is that x_0 is Gaussian with mean $\hat{x}_0$ and variance Σ_0. Muth sought formulas for $\hat{x}_{t+1} = E[x_{t+1}|y^t]$, where $y^t = [y_t, \ldots, y_0]$.

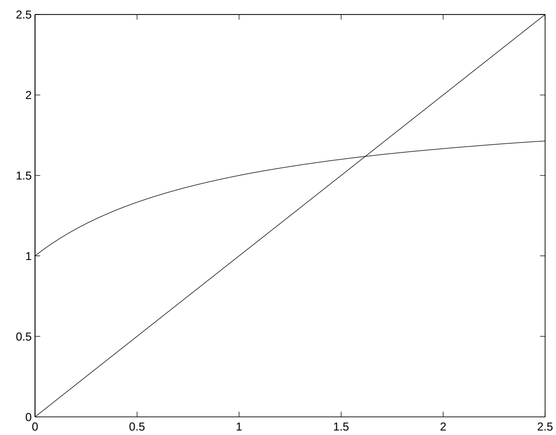

Figure 2.8.1: Graph of $f(\Sigma) = \frac{\Sigma(R+Q)+QR}{\Sigma+R}$, $Q = R = 1$, against the 45-degree line. Iterations on the Riccati equation for Σ_t converge to the fixed point.

For this problem, $A = 1, CC' = Q, G = 1$, making the Kalman filtering equations become

$$K_t = \frac{\Sigma_t}{\Sigma_t + R} \tag{2.8.3a}$$
$$\Sigma_{t+1} = \Sigma_t + Q - \frac{\Sigma_t^2}{\Sigma_t + R}. \tag{2.8.3b}$$

The second equation can be rewritten

$$\Sigma_{t+1} = \frac{\Sigma_t (R+Q) + QR}{\Sigma_t + R}. \tag{2.8.4}$$

For $Q = R = 1$, Figure 2.8.1 plots the function $f(\Sigma) = \frac{\Sigma(R+Q)+QR}{\Sigma+R}$ appearing on the right side of equation (2.8.4) for values $\Sigma \geq 0$ against the 45-degree line. Note that $f(0) = Q$. This graph identifies the fixed point of iterations on $f(\Sigma)$ as the intersection of $f(\cdot)$ and the 45-degree line. That the slope of $f(\cdot)$ is less than unity at the intersection assures us that the iterations on f will converge as $t \to +\infty$ starting from any $\Sigma_0 \geq 0$.

Muth studied the solution of this problem as $t \to \infty$. Evidently, $\Sigma_t \to \Sigma_\infty \equiv \Sigma$ is the fixed point of a graph like Figure 2.8.1. Then $K_t \to K$ and the formula for $\hat{x}_{t+1}$ becomes

$$\hat{x}_{t+1} = (1-K)\,\hat{x}_t + K y_t \tag{2.8.5}$$

where $K = \frac{\Sigma}{\Sigma+R} \in (0,1)$. This is a version of Cagan's adaptive expectations formula. It can be shown that $K \in [0,1]$ is an increasing function of $\frac{Q}{R}$. Thus, K is the fraction of the innovation a_t that should be regarded as 'permanent' and $1-K$ is the fraction that is purely transitory. Iterating backward on equation (2.8.5) gives $\hat{x}_{t+1} = K\sum_{j=0}^{t}(1-K)^j y_{t-j} + (1-K)^{t+1}\hat{x}_0$, which is a version of Cagan and Friedman's geometric distributed lag formula. Using equations (2.8.2), we find that $E[y_{t+j}|y^t] = E[x_{t+j}|y^t] = \hat{x}_{t+1}$ for all $j \geq 1$. This result in conjunction with equation (2.8.5) establishes that the adaptive expectation formula (2.8.5) gives the optimal forecast of y_{t+j} for all horizons $j \geq 1$. This finding is remarkable because for most processes, the optimal forecast will depend on the horizon. That there is a single optimal forecast for all horizons justifies the term *permanent income* that Milton Friedman (1955) chose to describe the forecast of income.

The dependence of the forecast on horizon can be studied using the formulas

$$E\left[x_{t+j}|y^{t-1}\right] = A^j \hat{x}_t \tag{2.8.6a}$$

$$E\left[y_{t+j}|y^{t-1}\right] = GA^j \hat{x}_t \tag{2.8.6b}$$

In the case of Muth's example,

$$E\left[y_{t+j}|y^{t-1}\right] = \hat{y}_t = \hat{x}_t \quad \forall j \geq 0.$$

2.8.2. Jovanovic's application

In chapter 6, we will describe a version of Jovanovic's (1979) matching model, at the core of which is a "signal-extraction" problem that simplifies Muth's problem. Let x_t, y_t be scalars with $A = 1, C = 0, G = 1, R > 0$. Let x_0 be Gaussian with mean μ and variance Σ_0. Interpret x_t (which is evidently constant with this specification) as the hidden value of θ, a "match parameter". Let y^t denote the history of y_s from $s = 0$ to $s = t$. Define $m_t \equiv \hat{x}_{t+1} \equiv E[\theta | y^t]$ and $\Sigma_{t+1} = E(\theta - m_t)^2$. Then the Kalman filter becomes

$$m_t = (1 - K_t)\, m_{t-1} + K_t y_t \tag{2.8.7a}$$

$$K_t = \frac{\Sigma_t}{\Sigma_t + R} \tag{2.8.7b}$$

$$\Sigma_{t+1} = \frac{\Sigma_t R}{\Sigma_t + R}. \tag{2.8.7c}$$

The recursions are to be initiated from (m_{-1}, Σ_0), a pair that embodies all "prior" knowledge about the position of the system. It is easy to see from Figure 2.8.1 that when $Q = 0$, $\Sigma = 0$ is the limit point of iterations on equation $(2.8.7c)$ starting from any $\Sigma_0 \geq 0$. Thus, the value of the match parameter is eventually learned.

It is instructive to write equation $(2.8.7c)$ as

$$\frac{1}{\Sigma_{t+1}} = \frac{1}{\Sigma_t} + \frac{1}{R}. \tag{2.8.8}$$

The reciprocal of the variance is often called the *precision* of the estimate. According to equation (2.8.8) the precision increases without bound as t grows, and $\Sigma_{t+1} \to 0$.[20]

We can represent the Kalman filter in the form

$$m_{t+1} = m_t + K_{t+1} a_{t+1}$$

which implies that

$$E\left(m_{t+1} - m_t\right)^2 = K_{t+1}^2 \sigma_{a,t+1}^2$$

[20] As a further special case, consider when there is zero precision initially ($\Sigma_0 = +\infty$). Then solving the difference equation (2.8.8) gives $\frac{1}{\Sigma_t} = t/R$. Substituting this into equations (2.8.7) gives $K_t = (t+1)^{-1}$, so that the Kalman filter becomes $m_0 = y_0$ and $m_t = [1 - (t+1)^{-1}] m_{t-1} + (t+1)^{-1} y_t$, which implies that $m_t = (t+1)^{-1} \sum_{s=0}^{t} y_t$, the sample mean, and $\Sigma_t = R/t$.

where $a_{t+1} = y_{t+1} - m_t$ and the variance of a_t is equal to $\sigma^2_{a,t+1} = (\Sigma_{t+1} + R)$ from equation (5.6.5). This implies

$$E\left(m_{t+1} - m_t\right)^2 = \frac{\Sigma^2_{t+1}}{\Sigma_{t+1} + R}.$$

For the purposes of our discrete-time counterpart of the Jovanovic model in chapter 6, it will be convenient to represent the motion of m_{t+1} by means of the equation

$$m_{t+1} = m_t + g_{t+1} u_{t+1}$$

where $g_{t+1} \equiv \left(\frac{\Sigma^2_{t+1}}{\Sigma_{t+1}+R}\right)^{.5}$ and u_{t+1} is a standardized i.i.d. normalized and standardized with mean zero and variance 1 constructed to obey $g_{t+1} u_{t+1} \equiv K_{t+1} a_{t+1}$.

2.9. Vector autoregressions and the Kalman filter

2.9.1. Conditioning on the semi-infinite past of y

Under an interesting set of conditions summarized, for example, by Anderson, Hansen, McGrattan, and Sargent (1996), iterations on $(2.7.12b)$, $(2.7.12d)$ converge to time-invariant K, Σ for any positive semi-definite initial covariance matrix Σ_0. A time-invariant matrix $\Sigma_t = \Sigma$ that solves $(2.7.12d)$ is the covariance matrix of x_t around $Ex_t | \{y^{t-1}_{-\infty}\}$, where $\{y^{t-1}_{-\infty}\}$ denotes the semi-infinite history of y_s for all dates on or before $t-1$.[21]

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[21] The Matlab program `kfilter.m` implements the time-invariant Kalman filter, allowing for correlation between the w_{t+1} and v_t Also see exercise 2.22.

2.9.2. A time-invariant VAR

Suppose that the fixed point of (2.7.12d) just described exists. If we initiate (2.7.12d) from this fixed point Σ, then the innovations representation becomes time invariant:

$$\hat{x}_{t+1} = A_o \hat{x}_t + K a_t \tag{2.9.1a}$$

$$y_t = G\hat{x}_t + a_t \tag{2.9.1b}$$

where $E a_t a_t' = G\Sigma G' + R$. Use (2.9.1) to express $\hat{x}_{t+1} = (A - KG)\hat{x}_t + K y_t$. If we assume that the eigenvalues of $A - KG$ are bounded in modulus below unity,[22] we can solve the preceding equation to get

$$\hat{x}_{t+1} = \sum_{j=0}^{\infty} (A - KG)^j \, K y_{t-j}. \tag{2.9.2}$$

Then solving (2.9.1b) for y_t gives the *vector autoregression*

$$y_t = G \sum_{j=0}^{\infty} (A - KG)^j \, K y_{t-j-1} + a_t, \tag{2.9.3}$$

where by construction

$$E\left[a_t y_{t-j-1}'\right] = 0 \quad \forall j \geq 0. \tag{2.9.4}$$

The orthogonality conditions (2.9.4) identify (2.9.3) as a vector autoregression.

[22] Anderson, Hansen, McGrattan, and Sargent (1996) show assumptions that guarantee that the eigenvalue of $A - KG$ are bounded in modulus below unity.

2.9.3. Interpreting VARs

Equilibria of economic models (or linear or log-linear approximations to them – see chapter 11 and the examples in section 2.12 of this chapter and appendix B of chapter 14) typically take the form of the state space system (2.7.1),(2.7.2). This hidden Markov model disturbs the evolution of the state x_t by the $p \times 1$ shock vector w_{t+1} and it perturbs the $m \times 1$ vector of observed variables y_t by the $m \times 1$ vector of measurement errors. Thus, $p+m$ shocks impinge on y_t. An economic theory typically makes w_{t+1}, v_t be directly interpretable as shocks that impinge on preferences, technologies, endowments, information sets, and measurements. The state space system (2.7.1),(2.7.2) is a representation of the stochastic process y_t in terms of these interpretable shocks. But the typical situation is that these shocks can not be recovered directly from the y_ts, even when we know the matrices A_o, G, C, R.

The innovations representation (2.9.1a), (2.9.1b) represents the *same* stochastic process y_t in terms of an $m \times 1$ vector of shocks a_t that would be recovered by running an infinite-order (population) vector autoregression for y_t. Because of its role in constructing the mapping from the original representation (2.7.1),(2.7.2) to the one associated with the vector autoregression (2.9.3), the Kalman filter is a very useful tool for interpreting vector autoregressions.

2.10. Estimation again

The innovations representation that emerges from the Kalman filter is

$$\hat{x}_{t+1} = A_o \hat{x}_t + K_t a_t \tag{2.10.1a}$$

$$y_t = G\hat{x}_t + a_t \tag{2.10.1b}$$

where for $t \geq 1$, $\hat{x}_t = E[x_t | y^{t-1}]$ and $E a_t a_t' = G \Sigma_t G' + R \equiv \Omega_t$. Evidently, for $t \geq 1$, $E[y_t | y^{t-1}] = G\hat{x}_t$ and the distribution of y_t conditional on y^{t-1} is $\mathcal{N}(G\hat{x}_t, \Omega_t)$. The objects $G\hat{x}_t, \Omega_t$ emerging from the Kalman filter are thus sufficient statistics for the distribution of y_t conditioned on y^{t-1} for $t \geq 1$. The sufficient conditions and also the innovation $a_t = y_t - G\hat{x}_t$ can be calculated recursively from (2.7.12). The unconditional distribution of y_0 is evidently $\mathcal{N}(G\hat{x}_0, \Omega_0)$.

As a counterpart to (2.6.2), we can factor the likelihood function for a sample $(y_T, y_{T-1}, \ldots, y_0)$ as

$$f(y_T, \ldots, y_0) = f\left(y_T | y^{T-1}\right) f\left(y_{T-1} | y^{T-2}\right) \cdots f(y_1 | y_0) f(y_0). \qquad (2.10.2)$$

The log of the conditional density of the $m \times 1$ vector y_t is

$$\log f\left(y_t | y^{t-1}\right) = -.5m \log(2\pi) - .5 \log\det\ (\Omega_t) - .5 a_t' \Omega_t^{-1} a_t. \qquad (2.10.3)$$

We can use (2.10.3) and (2.7.12) to evaluate the likelihood function (2.10.2) recursively for a given set of parameter values θ that underlie the matrices A_o, G, C, R. Such calculations are at the heart of efficient strategies for computing maximum-likelihood and Bayesian estimators of free parameters.[23]

The likelihood function is also an essential object for a Bayesian statistician.[24] It completely summarizes how the data influence the posterior via the following application of Bayes' law. Where θ is our parameter vector, y_0^T our data record, and $\tilde{p}(\theta)$ a probability density that summarizes our prior 'views' or 'information' about θ *before* seeing y_0^T, our views about θ *after* seeing y_0^T is summarized by a *posterior probability* $\tilde{p}(\theta | y_0^T)$ that is constructed from Bayes's law via

$$\tilde{p}\left(\theta | y_0^T\right) = \frac{f\left(y_0^T | \theta\right) \tilde{p}(\theta)}{\int f\left(y_0^T | \theta\right) \tilde{p}(\theta)\, d\theta}$$

where the denominator is the marginal joint density $f(y_0^T)$ of y_0^T.

In appendix B, we describe a simulation algorithm for approximating a Bayesian posterior. The algorithm constructs a Markov chain whose invariant distribution *is* the posterior, then iterates the Markov chain to convergence.

---

[23] See Hansen (1982); Eichenbaum (1991); Christiano and Eichenbaum (1992); Burnside, Eichenbaum, and Rebelo (1993); and Burnside and Eichenbaum (1996a, 1996b) for alternative estimation strategies.

[24] See Canova (2007), Christensen and Kiefer (2009), and DeJong and Dave (2011) for extensive descriptions of how Bayesian and maximum likelihood methods can be applied to macroeconomic and other dynamic models.

2.11. The spectrum

For a covariance stationary stochastic process, all second moments can be encoded in a complex-valued matrix called the *spectral density* matrix. The autocovariance sequence for the process determines the spectral density. Conversely, the spectral density can be used to determine the autocovariance sequence.

Under the assumption that A_o is a stable matrix,[25] the state x_t converges to a unique covariance stationary probability distribution as t approaches infinity. The spectral density matrix of this covariance stationary distribution $S_x(\omega)$ is defined to be the Fourier transform of the covariogram of x_t:

$$S_x(\omega) \equiv \sum_{\tau=-\infty}^{\infty} C_x(\tau) e^{-i\omega\tau}. \tag{2.11.1}$$

For the system (2.4.1), the spectral density of the stationary distribution is given by the formula

$$S_x(\omega) = \left[I - A_o e^{-i\omega}\right]^{-1} CC' \left[I - A_o' e^{+i\omega}\right]^{-1}, \quad \forall \omega \in [-\pi, \pi]. \tag{2.11.2}$$

The spectral density summarizes all covariances. They can be recovered from $S_x(\omega)$ by the Fourier inversion formula[26]

$$C_x(\tau) = (1/2\pi) \int_{-\pi}^{\pi} S_x(\omega) e^{+i\omega\tau} d\omega.$$

Setting $\tau = 0$ in the inversion formula gives

$$C_x(0) = (1/2\pi) \int_{-\pi}^{\pi} S_x(\omega) d\omega,$$

which shows that the spectral density decomposes covariance across frequencies.[27] A formula used in the process of generalized method of moments (GMM) estimation emerges by setting $\omega = 0$ in equation (2.11.1), which gives

$$S_x(0) \equiv \sum_{\tau=-\infty}^{\infty} C_x(\tau).$$

---

[25] It is sufficient that the only eigenvalue of A_o not strictly less than unity in modulus is that associated with the constant, which implies that A_o and C fit together in a way that validates (2.11.2).

[26] Spectral densities for continuous-time systems are discussed by Kwakernaak and Sivan (1972). For an elementary discussion of discrete-time systems, see Sargent (1987a). Also see Sargent (1987a, chap. 11) for definitions of the spectral density function and methods of evaluating this integral.

[27] More interestingly, the spectral density achieves a decomposition of covariance into components that are orthogonal across frequencies.

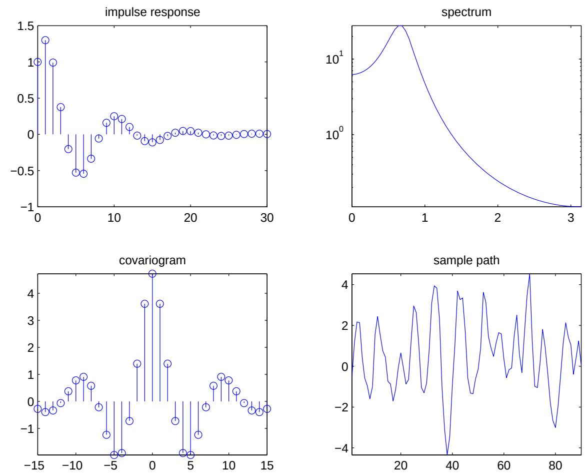

Figure 2.11.1: Impulse response, spectrum, covariogram, and sample path of process $(1 - 1.3L + .7L^2)y_t = w_t$.

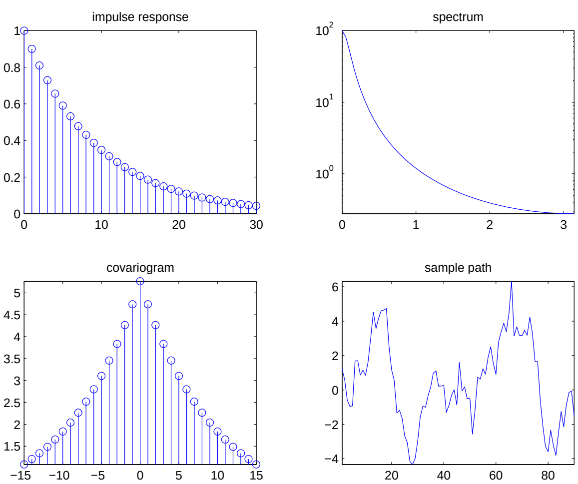

Figure 2.11.2: Impulse response, spectrum, covariogram, and sample path of process $(1 - .9L)y_t = w_t$.

2.11.1. Examples

To give some practice in reading spectral densities, we used the Matlab program `bigshow2.m` to generate Figures 2.11.2, 2.11.3, 2.11.1, and 2.11.4 The program

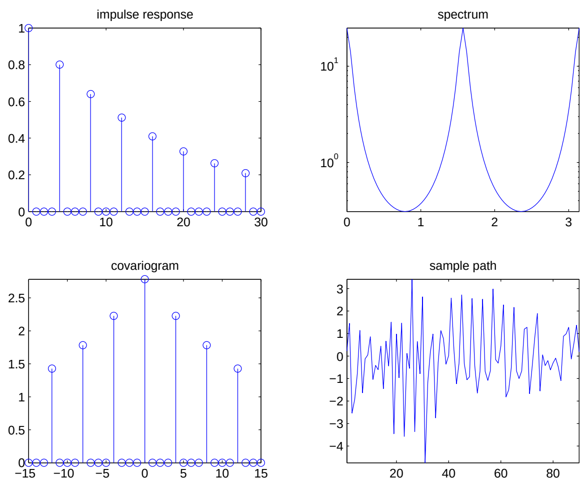

Figure 2.11.3: Impulse response, spectrum, covariogram, and sample path of process $(1 - .8L^4)y_t = w_t$.

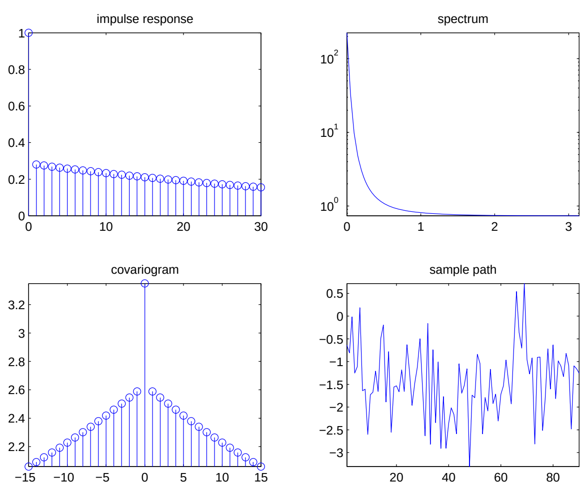

Figure 2.11.4: Impulse response, spectrum, covariogram, and sample path of process $(1 - .98L)y_t = (1 - .7L)w_t$.

takes as an input a univariate process of the form

$$a\,(L)\,y_t = b\,(L)\,w_t,$$

where w_t is a univariate martingale difference sequence with unit variance, where $a(L) = 1 - a_2 L - a_3 L^2 - \cdots - a_n L^{n-1}$ and $b(L) = b_1 + b_2 L + \cdots + b_n L^{n-1}$, and where we require that $a(z) = 0$ imply that $|z| > 1$. The program computes and displays a realization of the process, the impulse response function from w to y, and the spectrum of y. By using this program, a reader can teach himself to read spectra and impulse response functions. Figure 2.11.2 is for the pure autoregressive process with $a(L) = 1 - .9L, b = 1$. The spectrum sweeps downward in what C.W.J. Granger (1966) called the "typical spectral shape" for an economic time series. Figure 2.11.3 sets $a = 1 - .8L^4, b = 1$. This is a process with a strong seasonal component. That the spectrum peaks at π and $\pi/2$ is a telltale sign of a strong seasonal component. Figure 2.11.1 sets $a = 1 - 1.3L + .7L^2, b = 1$. This is a process that has a spectral peak in the interior of $(0, \pi)$ and cycles in its covariogram.[28] Figure 2.11.4 sets a $= 1 - .98L, b = 1 - .7L$. This is a version of a process studied by Muth (1960). After the first lag, the impulse response declines as $.99^j$, where j is the lag length.

2.12. Example: the LQ permanent income model

To illustrate several of the key ideas of this chapter, this section describes the linear quadratic savings problem whose solution is a rational expectations version of the permanent income model of Friedman (1956) and Hall (1978). We use this model as a vehicle for illustrating impulse response functions, alternative notions of the state, the idea of cointegration, and an invariant subspace method.

The LQ permanent income model is a modification (and not quite a special case, for reasons that will be apparent later) of the following "savings problem" to be studied in chapter 17. A consumer has preferences over consumption streams that are ordered by the utility functional

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2.12.1}$$

where E_t is the mathematical expectation conditioned on the consumer's time t information, c_t is time t consumption, $u(c)$ is a strictly concave one-period

[28] See Sargent (1987a) for a more extended discussion.

utility function, and $\beta \in (0,1)$ is a discount factor. The consumer maximizes (2.12.1) by choosing a consumption, borrowing plan $\{c_t, b_{t+1}\}_{t=0}^{\infty}$ subject to the sequence of budget constraints

$$c_t + b_t = R^{-1} b_{t+1} + y_t \tag{2.12.2}$$

where y_t is an exogenous stationary endowment process, R is a constant gross risk-free interest rate, b_t is one-period risk-free debt maturing at t, and b_0 is a given initial condition. We shall assume that $R^{-1} = \beta$. For example, we might assume that the endowment process has the state-space representation

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1} \tag{2.12.3a}$$

$$y_t = U_y z_t \tag{2.12.3b}$$

where w_{t+1} is an i.i.d. process with mean zero and identity contemporaneous covariance matrix, A_{22} is a stable matrix, its eigenvalues being strictly below unity in modulus, and U_y is a selection vector that identifies y with a particular linear combination of the z_t. We impose the following condition on the consumption, borrowing plan:

$$E_0 \sum_{t=0}^{\infty} \beta^t b_t^2 < +\infty. \tag{2.12.4}$$

This condition suffices to rule out Ponzi schemes. The *state* vector confronting the household at t is $[\, b_t \quad z_t \,]'$, where b_t is its one-period debt falling due at the beginning of period t and z_t contains all variables useful for forecasting its future endowment. We impose this condition to rule out an always-borrow scheme that would allow the household to enjoy bliss consumption forever. The rationale for imposing this condition is to make the solution resemble the solution of problems to be studied in chapter 17 that impose nonnegativity on the consumption path. First-order conditions for maximizing (2.12.1) subject to (2.12.2) are[29]

$$E_t u'(c_{t+1}) = u'(c_t), \quad \forall t \geq 0. \tag{2.12.5}$$

For the rest of this section we assume the quadratic utility function $u(c_t) = -.5(c_t - \gamma)^2$, where γ is a bliss level of consumption. Then (2.12.5) implies[30]

$$E_t c_{t+1} = c_t. \tag{2.12.6}$$

[29] We shall study how to derive this first-order condition in detail in later chapters.

[30] A linear marginal utility is essential for deriving (2.12.6) from (2.12.5). Suppose instead that we had imposed the following more standard assumptions on the utility function: $u'(c) >$

Along with the quadratic utility specification, we allow consumption c_t to be negative.[31]

To deduce the optimal decision rule, we have to solve the system of difference equations formed by (2.12.2) and (2.12.6) subject to the boundary condition (2.12.4). To accomplish this, solve (2.12.2) forward and impose $\lim_{T\to+\infty} \beta^T b_{T+1} = 0$ to get

$$b_t = \sum_{j=0}^{\infty} \beta^j \left(y_{t+j} - c_{t+j} \right). \tag{2.12.7}$$

Imposing $\lim_{T\to+\infty} \beta^T b_{T+1} = 0$ suffices to impose (2.12.4) on the debt path. Take conditional expectations on both sides of (2.12.7) and use (2.12.6) and the law of iterated expectations to deduce

$$b_t = \sum_{j=0}^{\infty} \beta^j E_t y_{t+j} - \frac{1}{1-\beta} c_t \tag{2.12.8}$$

or

$$c_t = (1-\beta) \left[\sum_{j=0}^{\infty} \beta^j E_t y_{t+j} - b_t \right]. \tag{2.12.9}$$

If we define the net rate of interest r by $\beta = \frac{1}{1+r}$, we can also express this equation as

$$c_t = \frac{r}{1+r} \left[\sum_{j=0}^{\infty} \beta^j E_t y_{t+j} - b_t \right]. \tag{2.12.10}$$

Equation (2.12.9) or (2.12.10) expresses consumption as equaling economic *income*, namely, a constant marginal propensity to consume or interest factor $\frac{r}{1+r}$ times the sum of nonfinancial wealth $\sum_{j=0}^{\infty} \beta^j E_t y_{t+j}$ and financial wealth $-b_t$. Notice that (2.12.9) or (2.12.10) represents c_t as a function of the *state* $[b_t, z_t]$

$0, u''(c) < 0, u'''(c) > 0$ and required that $c \geq 0$. The Euler equation remains (2.12.5). But the fact that $u''' < 0$ implies via Jensen's inequality that $E_t u'(c_{t+1}) > u'(E_t c_{t+1})$. This inequality together with (2.12.5) implies that $E_t c_{t+1} > c_t$ (consumption is said to be a 'submartingale'), so that consumption stochastically diverges to $+\infty$. The consumer's savings also diverge to $+\infty$. Chapter 17 discusses this 'precautionary savings' divergence result in depth.

[31] That c_t can be negative explains why we impose condition (2.12.4) instead of an upper bound on the level of borrowing, such as the natural borrowing limit of chapters 8, 17, and 18.

confronting the household, where from (2.12.3) z_t contains the information useful for forecasting the endowment process.

2.12.1. Another representation

Pulling together our preceding results, we can regard z_t, b_t as the time t state, where z_t is an *exogenous* component of the state and b_t is an *endogenous* component of the state vector. The system can be represented as

$$\begin{aligned} z_{t+1} &= A_{22} z_t + C_2 w_{t+1} \\ b_{t+1} &= b_t + U_y \left[(I - \beta A_{22})^{-1} (A_{22} - I) \right] z_t \\ y_t &= U_y z_t \\ c_t &= (1 - \beta) \left[U_y (I - \beta A_{22})^{-1} z_t - b_t \right]. \end{aligned}$$

Another way to understand the solution is to show that *after* the optimal decision rule has been obtained, there is a point of view that allows us to regard the state as being c_t together with z_t and to regard b_t as an outcome. Following Hall (1978), this is a sharp way to summarize the implication of the LQ permanent income theory. We now proceed to transform the state vector in this way.

To represent the solution for b_t, substitute (2.12.9) into (2.12.2) and after rearranging obtain

$$b_{t+1} = b_t + \left(\beta^{-1} - 1\right) \sum_{j=0}^{\infty} \beta^j E_t y_{t+j} - \beta^{-1} y_t. \tag{2.12.11}$$

Next, shift (2.12.9) forward one period and eliminate b_{t+1} by using (2.12.2) to obtain

$$c_{t+1} = (1 - \beta) \sum_{j=0}^{\infty} E_{t+1} \beta^j y_{t+j+1} - (1 - \beta) \left[\beta^{-1} (c_t + b_t - y_t) \right].$$

If we add and subtract $\beta^{-1}(1-\beta) \sum_{j=0}^{\infty} \beta^j E_t y_{t+j}$ from the right side of the preceding equation and rearrange, we obtain

$$c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \left(E_{t+1} y_{t+j+1} - E_t y_{t+j+1} \right). \tag{2.12.12}$$

The right side is the time $t+1$ innovation to the expected present value of the endowment process y. It is useful to express this innovation in terms of a moving average representation for income y_t. Suppose that the endowment process has the moving average representation[32]

$$y_{t+1} = d(L)\, w_{t+1} \tag{2.12.13}$$

where w_{t+1} is an i.i.d. vector process with $Ew_{t+1} = 0$ and contemporaneous covariance matrix $Ew_{t+1}w'_{t+1} = I$, $d(L) = \sum_{j=0}^{\infty} d_j L^j$, where L is the lag operator, and the household has an information set[33] $w^t = [w_t, w_{t-1}, \ldots,]$ at time t. Then notice that

$$y_{t+j} - E_t y_{t+j} = d_0 w_{t+j} + d_1 w_{t+j-1} + \cdots + d_{j-1} w_{t+1}.$$

It follows that

$$E_{t+1} y_{t+j} - E_t y_{t+j} = d_{j-1} w_{t+1}. \tag{2.12.14}$$

Using (2.12.14) in (2.12.12) gives

$$c_{t+1} - c_t = (1-\beta)\, d(\beta)\, w_{t+1}. \tag{2.12.15}$$

The object $d(\beta)$ is the present value of the moving average coefficients in the representation for the endowment process y_t.

After all of this work, we can represent the optimal decision rule for c_t, b_{t+1} in the form of the two equations (2.12.12) and (2.12.8), which we repeat here for convenience:

$$c_{t+1} = c_t + (1-\beta) \sum_{j=0}^{\infty} \beta^j \left(E_{t+1} y_{t+j+1} - E_t y_{t+j+1}\right) \tag{2.12.16}$$

$$b_t = \sum_{j=0}^{\infty} \beta^j E_t y_{t+j} - \frac{1}{1-\beta} c_t. \tag{2.12.17}$$

Equation (2.12.17) asserts that the household's debt due at t equals the expected present value of of its endowment minus the expected present value of its

32 Representation (2.12.3) implies that $d(L) = U_y (I - A_{22} L)^{-1} C_2$.

33 A moving average representation for a process y_t is said to be *fundamental* if the linear space spanned by y^t is equal to the linear space spanned by w^t. A time-invariant innovations representation, attained via the Kalman filter as in section 2.7, is by construction fundamental.

consumption stream. A high debt thus indicates a large expected present value of 'surpluses' $y_t - c_t$.

Recalling the form of the endowment process (2.12.3), we can compute

$$
\begin{aligned}
E_t \sum_{j=0}^{\infty} \beta^j z_{t+j} &= (I - \beta A_{22})^{-1} z_t \\
E_{t+1} \sum_{j=0}^{\infty} \beta^j z_{t+j+1} &= (I - \beta A_{22})^{-1} z_{t+1} \\
E_t \sum_{j=0}^{\infty} \beta^j z_{t+j+1} &= (I - \beta A_{22})^{-1} A_{22} z_t.
\end{aligned}
$$

Substituting these formulas into (2.12.16) and (2.12.17) and using (2.12.3a) gives the following representation for the consumer's optimum decision rule:[34]

$$
\begin{aligned}
c_{t+1} &= c_t + (1-\beta)\, U_y\, (I - \beta A_{22})^{-1} C_2 w_{t+1} &\quad (2.12.18a)\\
b_t &= U_y\, (I - \beta A_{22})^{-1} z_t - \frac{1}{1-\beta} c_t &\quad (2.12.18b)\\
y_t &= U_y z_t &\quad (2.12.18c)\\
z_{t+1} &= A_{22} z_t + C_2 w_{t+1} &\quad (2.12.18d)
\end{aligned}
$$

Representation (2.12.18) reveals several things about the optimal decision rule. (1) The *state* consists of the endogenous part c_t and the exogenous part z_t. These contain all of the relevant information for forecasting future c, y, b. Notice that financial assets b_t have disappeared as a component of the state because they are properly encoded in c_t. (2) According to (2.12.18), consumption is a random walk with innovation $(1-\beta)d(\beta)w_{t+1}$ as implied also by (2.12.15). This outcome confirms that the Euler equation (2.12.6) is built into the solution. That consumption is a random walk of course implies that it does not possess an asymptotic stationary distribution, at least so long as z_t exhibits perpetual random fluctuations, as it will generally under (2.12.3).[35] This feature is inherited partly from the assumption that $\beta R = 1$. (3) The impulse

[34] See section B of chapter 8 for a reinterpretation of precisely these outcomes in terms of a competitive equilibrium of a model with a complete set of markets in history- and date-contingent claims to consumption.

[35] The failure of consumption to converge will occur again in chapter 17 when we drop quadratic utility and assume that consumption must be nonnegative.

response function of c_t is a box: for all $j \geq 1$, the response of c_{t+j} to an increase in the innovation w_{t+1} is $(1-\beta)d(\beta) = (1-\beta)U_y(I-\beta A_{22})^{-1}C_2$. (4) Solution (2.12.18) reveals that the joint process c_t, b_t possesses the property that Granger and Engle (1987) called *cointegration*. In particular, *both* c_t and b_t are non-stationary because they have unit roots (see representation (2.12.11) for b_t), but there is a linear combination of c_t, b_t that *is* stationary provided that z_t is stationary. From (2.12.17), the linear combination is $(1-\beta)b_t + c_t$. Accordingly, Granger and Engle would call $[\,(1-\beta) \quad 1\,]$ a cointegrating vector that, when applied to the nonstationary vector process $[\,b_t \quad c_t\,]'$, yields a process that is asymptotically stationary. Equation (2.12.8) can be arranged to take the form

$$(1-\beta)\, b_t + c_t = (1-\beta)\, E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}, \tag{2.12.19}$$

which asserts that the 'cointegrating residual' on the left side equals the conditional expectation of the geometric sum of future incomes on the right.[36]

2.12.2. Debt dynamics

If we subtract equation (2.12.18b) evaluated at time t from equation (2.12.18b) evaluated at time $t+1$ we obtain

$$b_{t+1} - b_t = U_y \left(I - \beta A_{22}\right)^{-1} \left(z_{t+1} - z_t\right) - \frac{1}{1-\beta} \left(c_{t+1} - c_t\right).$$

Substituting $z_{t+1} - z_t = (A_{22} - I) z_t + C_2 w_{t+1}$ and equation (2.12.18a) into the above equation and rearranging gives

$$b_{t+1} - b_t = U_y \left(I - \beta A_{22}\right)^{-1} \left(A_{22} - I\right) z_t. \tag{2.12.20}$$

[36] See Campbell and Shiller (1988) and Lettau and Ludvigson (2001, 2004) for interesting applications of related ideas.

2.12.3. Two classic examples

We illustrate formulas (2.12.18) with the following two examples. In both examples, the endowment follows the process $y_t = z_{1t} + z_{2t}$ where

$$\begin{bmatrix} z_{1t+1} \\ z_{2t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} w_{1t+1} \\ w_{2t+1} \end{bmatrix}$$

where w_{t+1} is an i.i.d. 2×1 process distributed as $\mathcal{N}(0, I)$. Here z_{1t} is a permanent component of y_t while z_{2t} is a purely transitory component.

Example 1. Assume that the consumer observes the state z_t at time t. This implies that the consumer can construct w_{t+1} from observations of z_{t+1} and z_t. Application of formulas (2.12.18) implies that

$$c_{t+1} - c_t = \sigma_1 w_{1t+1} + (1 - \beta)\, \sigma_2 w_{2t+1}. \tag{2.12.21}$$

Since $1-\beta = \frac{r}{1+r}$ where $R = (1+r)$, formula (2.12.21) shows how an increment $\sigma_1 w_{1t+1}$ to the permanent component of income z_{1t+1} leads to a permanent one-for-one increase in consumption and no increase in savings $-b_{t+1}$; but how the purely transitory component of income $\sigma_2 w_{2t+1}$ leads to a permanent increment in consumption by a fraction $(1-\beta)$ of transitory income, while the remaining fraction β is saved, leading to a permanent increment in $-b$. Application of formula (2.12.20) to this example shows that

$$b_{t+1} - b_t = -z_{2t} = -\sigma_2 w_{2t}, \tag{2.12.22}$$

which confirms that none of $\sigma_1 w_{1t}$ is saved, while all of $\sigma_2 w_{2t}$ is saved.

Example 2. Assume that the consumer observes y_t, and its history up to t, but not z_t at time t. Under this assumption, it is appropriate to use an *innovation representation* to form A_{22}, C_2, U_y in formulas (2.12.18). In particular, using our results from section 2.8.1, the pertinent state space representation for y_t is

$$\begin{bmatrix} y_{t+1} \\ a_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -(1-K) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ a_t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_{t+1}$$
$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ a_t \end{bmatrix}$$

where K is the Kalman gain and $a_t = y_t - E[y_t | y^{t-1}]$. From subsection 2.8.1, we know that $K \in [0,1]$ and that K increases as $\frac{\sigma_1^2}{\sigma_2^2}$ increases, i.e., as the ratio of the variance of the permanent shock to the variance of the transitory shock to income increases. Applying formulas (2.12.18) implies

$$c_{t+1} - c_t = [1 - \beta (1 - K)] a_{t+1} \tag{2.12.23}$$

where the endowment process can now be represented in terms of the univariate innovation to y_t as

$$y_{t+1} - y_t = a_{t+1} - (1 - K) a_t. \tag{2.12.24}$$

Equation (2.12.24) indicates that the consumer regards a fraction K of an innovation a_{t+1} to y_{t+1} as *permanent* and a fraction $1-K$ as purely transitory. He permanently increases his consumption by the full amount of his estimate of the permanent part of a_{t+1}, but by only $(1-\beta)$ times his estimate of the purely transitory part of a_{t+1}. Therefore, in total he permanently increments his consumption by a fraction $K + (1-\beta)(1-K) = 1 - \beta(1-K)$ of a_{t+1} and saves the remaining fraction $\beta(1-K)$ of a_{t+1}. According to equation (2.12.24), the first difference of income is a first-order moving average, while (2.12.23) asserts that the first difference of consumption is i.i.d. Application of formula (2.12.20) to this example shows that

$$b_{t+1} - b_t = (K - 1) a_t, \tag{2.12.25}$$

which indicates how the fraction K of the innovation to y_t that is regarded as permanent influences the fraction of the innovation that is saved.

2.12.4. Spreading consumption cross section

Starting from an arbitrary initial distribution for c_0 and say the asymptotic stationary distribution for z_0, if we were to apply formulas (2.4.11) and (2.4.12) to the state space system (2.12.18), the common unit root affecting c_t, b_t would cause the time t variance of c_t to grow linearly with t. If we think of the initial distribution as describing the joint distribution of c_0, b_0 for a cross section of ex ante identical households 'born at time 0, then these formulas would describe the evolution of the cross-section for b_t, c_t as the population of households ages. The distribution would spread out.[37]

[37] See Deaton and Paxton (1994) and Storesletten, Telmer, and Yaron (2004) for evidence that cross section distributions of consumption spread out with age.

2.12.5. Invariant subspace approach

We can glean additional insights about the structure of the optimal decision rule by solving the decision problem in a mechanical but quite revealing way that easily generalizes to a host of problems, as we shall see later in chapter 5. We can represent the system consisting of the Euler equation (2.12.6), the budget constraint (2.12.2), and the description of the endowment process (2.12.3) as

$$\begin{bmatrix} \beta & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{t+1} \\ z_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -U_y & 1 \\ 0 & A_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_t \\ z_t \\ c_t \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \\ C_1 \end{bmatrix} w_{t+1} \tag{2.12.26}$$

where C_1 is an undetermined coefficient. Premultiply both sides by the inverse of the matrix on the left and write

$$\begin{bmatrix} b_{t+1} \\ z_{t+1} \\ c_{t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} b_t \\ z_t \\ c_t \end{bmatrix} + \tilde{C} w_{t+1}. \tag{2.12.27}$$

We want to find solutions of (2.12.27) that satisfy the no-explosion condition (2.12.4). We can do this by using machinery to be introduced in chapter 5. The key idea is to discover what part of the vector $[\, b_t \quad z_t \quad c_t \,]'$ is truly a *state* from the view of the decision maker, being inherited from the past, and what part is a *costate* or *jump* variable that can adjust at t. For our problem b_t, z_t are truly components of the state, but c_t is free to adjust. The theory determines c_t at t as a function of the true state variables $[b_t, z_t]$. A powerful approach to determining this function is the following so-called invariant subspace method of chapter 5. Obtain the eigenvector decomposition of $\tilde{A}$:

$$\tilde{A} = V \Lambda V^{-1}$$

where Λ is a diagonal matrix consisting of the eigenvalues of $\tilde{A}$ and V is a matrix of the associated eigenvectors. Let $V^{-1} \equiv \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix}$. Then applying formula (5.5.11) of chapter 5 implies that if (2.12.4) is to hold, the jump variable c_t must satisfy

$$c_t = -\left(V^{22}\right)^{-1} V^{21} \begin{bmatrix} b_t \\ z_t \end{bmatrix}. \tag{2.12.28}$$

Formula (2.12.28) gives the unique value of c_t that ensures that (2.12.4) is satisfied, or in other words, that the state remains in the "stabilizing subspace."

Notice that the variables on the right side of (2.12.28) conform with those called for by (2.12.10): $-b_t$ is there as a measure of financial wealth, and z_t is there because it includes all variables that are useful for forecasting the future endowments that appear in (2.12.10).

2.13. Concluding remarks

In addition to giving us tools for thinking about time series, the Markov chain and the stochastic linear difference equation have each introduced us to the notion of the state vector as a description of the present position of a system.[38] Subsequent chapters use both Markov chains and stochastic linear difference equations. In the next chapter we study decision problems in which the goal is optimally to manage the evolution of a state vector that can be partially controlled.

A. Linear difference equations

2.A.1. A first-order difference equation

This section describes the solution of a linear first-order scalar difference equation. First, let $|\lambda| < 1$, and let $\{u_t\}_{t=-\infty}^{\infty}$ be a bounded sequence of scalar real numbers. Let L be the lag operator defined by $Lx_t \equiv x_{t-1}$ and let L^{-1} be the forward shift operator defined by $L^{-1}x_t \equiv x_{t+1}$. Then

$$(1 - \lambda L)\, y_t = u_t, \forall t \tag{2.A.1}$$

has the solution

$$y_t = (1 - \lambda L)^{-1}\, u_t + k\lambda^t \tag{2.A.2}$$

[38] See Quah (1990) and Blundell and Preston (1998) for applications of some of the tools of this chapter and of chapter 5 to studying some puzzles associated with a permanent income model.

for any real number k. You can verify this fact by applying $(1-\lambda L)$ to both sides of equation $(2.A.2)$ and noting that $(1-\lambda L)\lambda^t = 0$. To pin down k we need one condition imposed from outside (e.g., an initial or terminal condition) on the path of y.

Now let $|\lambda| > 1$. Rewrite equation $(2.A.1)$ as

$$y_{t-1} = \lambda^{-1} y_t - \lambda^{-1} u_t, \forall t \tag{2.A.3}$$

or

$$\left(1 - \lambda^{-1} L^{-1}\right) y_t = -\lambda^{-1} u_{t+1}. \tag{2.A.4}$$

A solution is

$$y_t = -\lambda^{-1} \left(\frac{1}{1 - \lambda^{-1} L^{-1}} \right) u_{t+1} + k \lambda^t \tag{2.A.5}$$

for any k. To verify that this is a solution, check the consequences of operating on both sides of equation $(2.A.5)$ by $(1-\lambda L)$ and compare to $(2.A.1)$.

Solution $(2.A.2)$ exists for $|\lambda| < 1$ because the distributed lag in u converges. Solution $(2.A.5)$ exists when $|\lambda| > 1$ because the distributed lead in u converges. When $|\lambda| > 1$, the distributed lag in u in $(2.A.2)$ may diverge, so that a solution of this form does not exist. The distributed lead in u in $(2.A.5)$ need not converge when $|\lambda| < 1$.

2.A.2. A second-order difference equation

Now consider the second order difference equation

$$(1 - \lambda_1 L)(1 - \lambda_2 L) y_{t+1} = u_t \tag{2.A.6}$$

where $\{u_t\}$ is a bounded sequence, y_0 is an initial condition, $|\lambda_1| < 1$ and $|\lambda_2| > 1$. We seek a bounded sequence $\{y_t\}_{t=0}^{\infty}$ that satisfies $(2.A.6)$. Using insights from the previous subsection, operate on both sides of $(2.A.6)$ by the forward inverse of $(1-\lambda_2 L)$ to rewrite equation $(2.A.6)$ as

$$(1 - \lambda_1 L) y_{t+1} = -\frac{\lambda_2^{-1}}{1 - \lambda_2^{-1} L^{-1}} u_{t+1}$$

or

$$y_{t+1} = \lambda_1 y_t - \lambda_2^{-1} \sum_{j=0}^{\infty} \lambda_2^{-j} u_{t+j+1}. \tag{2.A.7}$$

Thus, we obtained equation $(2.A.7)$ by "solving stable roots (in this case λ_1 backward, and unstable roots (in this case λ_2) forward". Equation $(2.A.7)$ has a form that we shall encounter often. $\lambda_1 y_t$ is called the 'feedback part' and $-\frac{\lambda_2^{-1}}{1-\lambda_2^{-1}L^{-1}}u_{t+1}$ is called the "feed-forward part' of the solution. We have already encountered solutions of this form. Thus, notice that equation $(2.12.20)$ from subsection 2.12.2 is almost of this form, 'almost' because in equation $(2.12.20)$, $\lambda_1 = 1$. In section 5.5 of chapter 5 we return to these ideas in a more general setting.

B. MCMC approximation of Bayesian posterior

The last twenty years witnessed impressive advances in numerical methods for computing Bayesian and maximum likelihood estimators. In this appendix, we briefly describe a Markov Chain Monte Carlo method that constructs a Bayesian posterior distribution by forming a Markov chain whose invariant distribution equals that posterior distribution.

In the Bayesian method, the following objects are in play:

1. A sample of data y_0^T.
2. A vector $\theta \in \Theta$ of free parameters describing the preferences, technology, and information sets of an economic model.
3. A prior probability distribution $\tilde{p}(\theta)$ over the parameters.
4. A mapping from θ to a state-space representation of an equilibrium of a economic dynamic model. We present examples of this mapping in sections 2.12 and appendix B of chapter 14 and chapters 5 and 7.
5. As described in section 2.7, a mapping from a state space representation of an equilibrium of an economic model to an innovations representation via the Kalman filter and thereby to a recursive representation of a Gaussian log likelihood function

$$\log p\left(y_0^T|\theta\right) = -.5\,(T+1)\,k\log(2\pi) - .5\sum_{t=0}^{T}\log|\Omega_t| - .5\sum_{t=0}^{T}a_t'\Omega_t^{-1}a_t,$$

where $\Omega_t = Ea_ta_t'$.

6. A posterior probability

$$\tilde{p}\left(\theta|y_0^T\right) = \frac{p\left(y_0^T|\theta\right)\tilde{p}\left(\theta\right)}{\int p\left(y_0^T|\theta\right)\tilde{p}\left(\theta\right)d\,\theta}$$

where the denominator is the marginal density of y_0^T.

Our goal is to compute the posterior $\tilde{p}(\theta|y_0^T)$. The Markov Chain Monte Carlo (MCMC) method constructs a Markov chain on a state space Θ for which $\theta \in \Theta$ and such that

a. The chain is easy to sample from.

b. The chain has a unique invariant distribution $\pi(\theta)$.

c. The invariant distribution equals the posterior: $\pi(\theta) = \tilde{p}(\theta|y_0^T)$.

Two key ingredients of the *Metropolis-Hastings algorithm* are

i. The *target* density $\tilde{p}(\theta|y_0^T)$.

ii. A *proposal* or *jumping* density $q(z|\theta; y_0^T)$.

The proposal density should be a good guess at $\tilde{p}(\theta|y_0^T)$. For our applications, a standard choice of a proposal density comes from adjusting the asymptotic distribution associated with the maximum likelihood estimator, $\theta \sim \mathcal{N}(\hat{\theta}_{ML}, \Sigma_\theta)$ where $\Sigma_\theta = V^{-1}$ and $V = -\frac{\partial^2 \log L(\theta|y_0^T)}{\partial\theta\partial\theta'}\Big|_{\theta_{ML}}$. A common choice of a proposal density is:

$$q\left(\theta^*|\theta_j, y_0^T\right) = \mathcal{N}\left(\theta_j, c\Sigma_\theta\right), \tag{2.B.1}$$

where c is a scale parameter. Define the *kernel* $\kappa(\theta|y_0^T)$ by

$$\log\kappa\left(\theta|y_0^T\right) = \log p\left(y_0^T|\theta\right) + \log\tilde{p}\left(\theta\right).$$

Note that

$$\tilde{p}\left(\theta|y_0^T\right) \propto \kappa\left(\theta|y_0^T\right)$$

where the factor of proportionality is the integrating constant $\int p(y_0^T|\theta)\tilde{p}(\theta)d\,\theta$.

The Metropolis-Hastings algorithm defines a Markov chain on Θ by these steps:[39]

[39] See Robert and Casella (2004, ch. 7) for discussions of this algorithm and conditions for convergence.

1. Draw θ_0, $j = 0$.
2. For $j \geq 0$, draw θ^* from $q(\theta^*|\theta_j, y_0^T)$; θ^* is a "candidate" for the next draw of θ_j.
3. Randomly decide whether to accept this candidate by first computing the probability of acceptance

$$r = \frac{\tilde{p}\left(\theta^*|y_0^T\right)}{\tilde{p}\left(\theta_j|y_0^T\right)} = \frac{\kappa\left(\theta^*|y_0^T\right)}{\kappa\left(\theta_j|y_0^T\right)}.$$

(Note that in this step, we only have to compute the kernels, not the integrating constant $\int p(y_0^T|\theta)\tilde{p}(\theta)d\theta$.) Then set

$$\theta_{j+1} = \begin{cases} \theta^* & \text{with probability } \min(r, 1); \\ \theta_j & \text{otherwise.} \end{cases}$$

This algorithm defines the transition density of a Markov chain mapping θ_j into θ_{j+1}. Let the transition density be $\text{Prob}(\theta_{j+1} = \theta^*|\theta_j = \theta) = \Pi(\theta, \theta^*)$. Then we have the following

Proposition: The invariant distribution of the chain is the posterior:

$$\tilde{p}\left(\theta^*|y_0^T\right) = \int \Pi\left(\theta, \theta^*\right) \tilde{p}\left(\theta|y_0^T\right) d\theta.$$

Two practical concerns associated with the Metropolis-Hastings algorithm are, first, whether the chain converges, and, second, the rate of convergence. The literature on MCMC has developed practical diagnostics for checking convergence, and it is important to use these thoughtfully. The rate of convergence is influenced by the acceptance rate, which can be influenced by choice of the scale parameter c when the proposal density is chosen as recommended above. Common piece chooses c to give an acceptance rate between .2 and .4.

Dynare computes maximum likelihood and Bayesian estimates using the above algorithm. See Barillas, Bhandari, Bigio, Colacito, Juillard, Kitao, Matthes, Sargent, and Shin (2012) for some examples.[40]

[40] See <http://www.dynare.org>.

Exercises

Exercise 2.1 Consider the Markov chain $(P, \pi_0) = \left(\begin{bmatrix} .9 & .1 \\ .3 & .7 \end{bmatrix}, \begin{bmatrix} .5 \\ .5 \end{bmatrix} \right)$, and a random variable $y_t = \overline{y} x_t$ where $\overline{y} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Compute the likelihood of the following three histories for y_t for $t = 0, 1, \ldots, 4$:

a. $1, 5, 1, 5, 1$.

b. $1, 1, 1, 1, 1$.

c. $5, 5, 5, 5, 5$.

Exercise 2.2 Consider a two-state Markov chain. Consider a random variable $y_t = \overline{y} x_t$ where $\overline{y} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. It is known that $E(y_{t+1} | x_t) = \begin{bmatrix} 1.8 \\ 3.4 \end{bmatrix}$ and that $E(y_{t+1}^2 | x_t) = \begin{bmatrix} 5.8 \\ 15.4 \end{bmatrix}$. Find a transition matrix consistent with these conditional expectations. Is this transition matrix unique (i.e., can you find another one that is consistent with these conditional expectations)?

Exercise 2.3 Consumption is governed by an n-state Markov chain P, π_0 where P is a stochastic matrix and π_0 is an initial probability distribution. Consumption takes one of the values in the $n \times 1$ vector $\overline{c}$. A consumer ranks stochastic processes of consumption $t = 0, 1 \ldots$ according to

$$E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where E is the mathematical expectation and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some parameter $\gamma \geq 1$. Let $u_i = u(\overline{c}_i)$. Let $v_i = E[\sum_{t=0}^{\infty} \beta^t u(c_t) | x_0 = \overline{e}_i]$ and $V = Ev$, where $\beta \in (0, 1)$ is a discount factor.

a. Let u and v be the $n \times 1$ vectors whose ith components are u_i and v_i, respectively. Verify the following formulas for v and V: $v = (I - \beta P)^{-1} u$, and $V = \sum_i \pi_{0,i} v_i$.

b. Consider the following two Markov processes:
Process 1: $\pi_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Process 2: $\pi_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$, $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$.

For both Markov processes, $\overline{c} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Assume that $\gamma = 2.5, \beta = .95$. Compute the unconditional discounted expected utility V for each of these processes. Which of the two processes does the consumer prefer? Redo the calculations for $\gamma = 4$. Now which process does the consumer prefer?

c. An econometrician observes a sample of 10 observations of consumption rates for our consumer. He knows that one of the two preceding Markov processes generates the data, but he does not know which one. He assigns equal "prior probability" to the two chains. Suppose that the 10 successive observations on consumption are as follows: $1,1,1,1,1,1,1,1,1,1$. Compute the likelihood of this sample under process 1 and under process 2. Denote the likelihood function $\text{Prob}(\text{data}|\text{Model}_i), i = 1,2$.

d. Suppose that the econometrician uses Bayes' law to revise his initial probability estimates for the two models, where in this context Bayes' law states:

$$\text{Prob}\,(M_i)\,|\text{data} = \frac{(\text{Prob}\,(\text{data})|M_i) \cdot \text{Prob}\,(M_i)}{\sum_j \text{Prob}\,(\text{data})|M_j \cdot \text{Prob}\,(M_j)}$$

where M_i denotes model i. The denominator of this expression is the unconditional probability of the data. After observing the data sample, what probabilities does the econometrician place on the two possible models?

e. Repeat the calculation in part d, but now assume that the data sample is $1,5,5,1,5,5,1,5,1,5$.

Exercise 2.4 Consider the univariate stochastic process

$$y_{t+1} = \alpha + \sum_{j=1}^{4} \rho_j y_{t+1-j} + c w_{t+1}$$

where w_{t+1} is a scalar martingale difference sequence adapted to $J_t = [w_t, \ldots, w_1, y_0, y_{-1}, y_{-2}, y_{-3}]$, $\alpha = \mu(1 - \sum_j \rho_j)$ and the ρ_j's are such that the matrix

$$A = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \alpha \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has all of its eigenvalues in modulus bounded below unity.

a. Show how to map this process into a first-order linear stochastic difference equation.

b. For each of the following examples, if possible, assume that the initial conditions are such that y_t is covariance stationary. For each case, state the appropriate initial conditions. Then compute the covariance stationary mean and variance of y_t assuming the following parameter sets of parameter values:

i. $\rho = [\,1.2 \quad -.3 \quad 0 \quad 0\,]$, $\mu = 10, c = 1$.

ii. $\rho = [\,1.2 \quad -.3 \quad 0 \quad 0\,]$, $\mu = 10, c = 2$.

iii. $\rho = [\,.9 \quad 0 \quad 0 \quad 0\,]$, $\mu = 5, c = 1$.

iv. $\rho = [\,.2 \quad 0 \quad 0 \quad .5\,]$, $\mu = 5, c = 1$.

v. $\rho = [\,.8 \quad .3 \quad 0 \quad 0\,]$, $\mu = 5, c = 1$.

Hint 1: The Matlab program `doublej.m`, in particular, the command `X=doublej(A,C*C')` computes the solution of the matrix equation $AXA' + CC' = X$. This program can be downloaded from <https://files.nyu.edu/ts43/public/books.html>.

Hint 2: The mean vector is the eigenvector of A associated with a unit eigenvalue, scaled so that the mean of unity in the state vector is unity.

c. For each case in part b, compute the h_j's in $E_t y_{t+5} = \gamma_0 + \sum_{j=0}^{3} h_j y_{t-j}$.

d. For each case in part b, compute the $\tilde{h}_j$'s in $E_t \sum_{k=0}^{\infty} .95^k y_{t+k} = \sum_{j=0}^{3} \tilde{h}_j y_{t-j}$.

e. For each case in part b, compute the autocovariance $E(y_t - \mu_y)(y_{t-k} - \mu_y)$ for the three values $k = 1, 5, 10$.

Exercise 2.5 A consumer's rate of consumption follows the stochastic process

$$
\begin{aligned}
c_{t+1} &= \alpha_c + \sum_{j=1}^{2} \rho_j c_{t-j+1} + \sum_{j=1}^{2} \delta_j z_{t+1-j} + \psi_1 w_{1,t+1} \\
z_{t+1} &= \sum_{j=1}^{2} \gamma_j c_{t-j+1} + \sum_{j=1}^{2} \phi_j z_{t-j+1} + \psi_2 w_{2,t+1}
\end{aligned} \tag{1}
$$

where w_{t+1} is a 2×1 martingale difference sequence, adapted to $J_t = [\, w_t \quad \ldots w_1 \quad c_0 \quad c_{-1} \quad z_0 \quad z_{-1}\,]$, with contemporaneous covariance matrix $E w_{t+1} w_{t+1}' | J_t = I$, and the coefficients $\rho_j, \delta_j, \gamma_j, \phi_j$ are such that the matrix

$$A = \begin{bmatrix} \rho_1 & \rho_2 & \delta_1 & \delta_2 & \alpha_c \\ 1 & 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \phi_1 & \phi_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has eigenvalues bounded strictly below unity in modulus.

The consumer evaluates consumption streams according to

$$V_0 = E_0 \sum_{t=0}^{\infty} .95^t u\left(c_t\right), \tag{2}$$

where the one-period utility function is

$$u\left(c_t\right) = -.5\left(c_t - 60\right)^2. \tag{3}$$

a. Find a formula for V_0 in terms of the parameters of the one-period utility function (3) and the stochastic process for consumption.

b. Compute V_0 for the following two sets of parameter values:

i. $\rho = [\, .8 \quad -.3\,], \alpha_c = 1, \delta = [\, .2 \quad 0\,], \gamma = [\, 0 \quad 0\,], \phi = [\, .7 \quad -.2\,],\ \psi_1 = \psi_2 = 1$.

ii. Same as for part i except now $\psi_1 = 2, \psi_2 = 1$.

Hint: Remember `doublej.m`.

Exercise 2.6 Consider the stochastic process $\{c_t, z_t\}$ defined by equations (1) in exercise *2.5*. Assume the parameter values described in part b, item i. If possible, assume the initial conditions are such that $\{c_t, z_t\}$ is covariance stationary.

a. Compute the initial mean and covariance matrix that make the process covariance stationary.

b. For the initial conditions in part a, compute numerical values of the following population linear regression:

$$c_{t+2} = \alpha_0 + \alpha_1 z_t + \alpha_2 z_{t-4} + w_t$$

where $Ew_t\,[1 \quad z_t \quad z_{t-4}] = [0 \quad 0 \quad 0]$.

Exercise 2.7 Get the Matlab programs `bigshow2.m` and `freq.m` from <https://files.nyu.edu/ts43/public/books.html>. Use `bigshow2` to compute and display a simulation of length 80, an impulse response function, and a spectrum for each of the following scalar stochastic processes y_t. In each of the following, w_t is a scalar martingale difference sequence adapted to its own history and the initial values of lagged y's.

a. $y_t = w_t$.

b. $y_t = (1 + .5L)w_t$.

c. $y_t = (1 + .5L + .4L^2)w_t$.

d. $(1 - .999L)y_t = (1 - .4L)w_t$.

e. $(1 - .8L)y_t = (1 + .5L + .4L^2)w_t$.

f. $(1 + .8L)y_t = w_t$.

g. $y_t = (1 - .6L)w_t$.

Study the output and look for patterns. When you are done, you will be well on your way to knowing how to read spectral densities.

Exercise 2.8 This exercise deals with Cagan's money demand under rational expectations. A version of Cagan's (1956) demand function for money is

$$m_t - p_t = -\alpha\left(p_{t+1} - p_t\right), \alpha > 0,\ t \geq 0, \tag{1}$$

where m_t is the log of the nominal money supply and p_t is the price level at t. Equation (1) states that the demand for real balances varies inversely with the expected rate of inflation, $(p_{t+1} - p_t)$. There is no uncertainty, so the expected inflation rate equals the actual one. The money supply obeys the difference equation

$$(1 - L)\,(1 - \rho L)\, m_t^s = 0 \tag{2}$$

subject to initial condition for m_{-1}^s, m_{-2}^s. In equilibrium,

$$m_t \equiv m_t^s \quad \forall t \geq 0 \tag{3}$$

(i.e., the demand for money equals the supply). For now assume that

$$|\rho\alpha/(1+\alpha)| < 1. \tag{4}$$

An *equilibrium* is a $\{p_t\}_{t=0}^{\infty}$ that satisfies equations (1), (2), and (3) for all t.

a. Find an expression for an equilibrium p_t of the form

$$p_t = \sum_{j=0}^{n} w_j m_{t-j} + f_t. \tag{5}$$

Please tell how to get formulas for the w_j for all j and the f_t for all t.

b. How many equilibria are there?

c. Is there an equilibrium with $f_t = 0$ for all t?

d. Briefly tell where, if anywhere, condition (4) plays a role in your answer to part a.

e. For the parameter values $\alpha = 1, \rho = 1$, compute and display all the equilibria.

Exercise 2.9 The $n \times 1$ state vector of an economy is governed by the linear stochastic difference equation

$$x_{t+1} = Ax_t + C_t w_{t+1} \tag{1}$$

where C_t is a possibly time-varying matrix (known at t) and w_{t+1} is an $m \times 1$ martingale difference sequence adapted to its own history with $E w_{t+1} w_{t+1}' | J_t = I$, where $J_t = [\, w_t \quad \ldots \quad w_1 \quad x_0 \,]$. A scalar one-period payoff p_{t+1} is given by

$$p_{t+1} = P x_{t+1} \tag{2}$$

The stochastic discount factor for this economy is a scalar m_{t+1} that obeys

$$m_{t+1} = \frac{M x_{t+1}}{M x_t}. \tag{3}$$

Finally, the price at time t of the one-period payoff is given by $q_t = f_t(x_t)$, where f_t is some possibly time-varying function of the state. That m_{t+1} is a stochastic discount factor means that

$$E\,(m_{t+1} p_{t+1} | J_t) = q_t. \tag{4}$$

a. Compute $f_t(x_t)$, describing in detail how it depends on A and C_t.

b. Suppose that an econometrician has a time series data set $X_t = [\, z_t \quad m_{t+1} \quad p_{t+1} \quad q_t \,]$, for $t = 1, \ldots, T$, where z_t is a strict subset of the variables in the state x_t. Assume that investors in the economy see x_t even though the econometrician sees only a subset z_t of x_t. Briefly describe a way to use these data to test implication (4). (Possibly but perhaps not useful hint: recall the law of iterated expectations.)

Exercise 2.10 Let P be a transition matrix for a Markov chain. Suppose that P' has two distinct eigenvectors π_1, π_2 corresponding to unit eigenvalues of P'. Scale π_1 and π_2 so that they are vectors of probabilities (i.e., elements are nonnegative and sum to unity). Prove for any $\alpha \in [0,1]$ that $\alpha\pi_1 + (1-\alpha)\pi_2$ is an invariant distribution of P.

Exercise 2.11 Consider a Markov chain with transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .2 & .5 & .3 \\ 0 & 0 & 1 \end{bmatrix}$$

with initial distribution $\pi_0 = [\, \pi_{1,0} \quad \pi_{2,0} \quad \pi_{3,0} \,]'$. Let $\pi_t = [\, \pi_{1t} \quad \pi_{2t} \quad \pi_{3t} \,]'$ be the distribution over states at time t. Prove that for $t > 0$

$$\begin{aligned} \pi_{1t} &= \pi_{1,0} + .2 \left(\frac{1 - .5^t}{1 - .5} \right) \pi_{2,0} \\ \pi_{2t} &= .5^t \pi_{2,0} \\ \pi_{3t} &= \pi_{3,0} + .3 \left(\frac{1 - .5^t}{1 - .5} \right) \pi_{2,0}. \end{aligned}$$

Exercise 2.12 Let P be a transition matrix for a Markov chain. For $t = 1, 2, \ldots$, prove that the jth column of $(P')^t$ is the distribution across states at t when the initial distribution is $\pi_{j,0} = 1, \pi_{i,0} = 0 \forall i \neq j$.

Exercise 2.13 A household has preferences over consumption processes $\{c_t\}_{t=0}^{\infty}$ that are ordered by

$$-.5 \sum_{t=0}^{\infty} \beta^t \left[(c_t - 30)^2 + .000001 b_t^2 \right] \tag{1}$$

where $\beta = .95$. The household chooses a consumption, borrowing plan to maximize (1) subject to the sequence of budget constraints

$$c_t + b_t = \beta b_{t+1} + y_t$$

for $t \geq 0$, where b_0 is an initial condition, β^{-1} is the one-period gross risk-free interest rate, b_t is the household's one-period debt that is due in period t, and y_t is its labor income, which obeys the second-order autoregressive process

$$\left(1 - \rho_1 L - \rho_2 L^2\right) y_{t+1} = (1 - \rho_1 - \rho_2)\, 5 + .05 w_{t+1}$$

where $\rho_1 = 1.3, \rho_2 = -.4$.

a. Define the *state* of the household at t as $x_t = [\,1 \quad b_t \quad y_t \quad y_{t-1}\,]'$ and the *control* as $u_t = (c_t - 30)$. Then express the transition law facing the household in the form (2.4.22). Compute the eigenvalues of A. Compute the zeros of the characteristic polynomial $(1 - \rho_1 z - \rho_2 z^2)$ and compare them with the eigenvalues of A. (*Hint:* To compute the zeros in Matlab, set $a = [\,.4 \quad -1.3 \quad 1\,]$ and call `roots(a)`. The zeros of $(1 - \rho_1 z - \rho_2 z^2)$ equal the *reciprocals* of the eigenvalues of the associated A.)

b. Write a Matlab program that uses the Howard improvement algorithm (2.4.30) to compute the household's optimal decision rule for $u_t = c_t - 30$. Tell how many iterations it takes for this to converge (also tell your convergence criterion).

c. Use the household's optimal decision rule to compute the law of motion for x_t under the optimal decision rule in the form

$$x_{t+1} = (A - BF^*)\, x_t + C w_{t+1},$$

where $u_t = -F^* x_t$ is the optimal decision rule. Using Matlab, compute the impulse response function of $[\,c_t \quad b_t\,]'$ to w_{t+1}. Compare these with the theoretical expressions (2.12.18).

Exercise 2.14 Consider a Markov chain with transition matrix

$$P = \begin{bmatrix} .5 & .5 & 0 & 0 \\ .1 & .9 & 0 & 0 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with state space $X = \{e_i, i = 1, \ldots, 4\}$ where e_i is the ith unit vector. A random variable y_t is a function $y_t = [1 \quad 2 \quad 3 \quad 4]\, x_t$ of the underlying state.

a. Find all stationary distributions of the Markov chain.

b. Can you find a stationary distribution for which the Markov chain ergodic?

c. Compute all possible limiting values of the sample mean $\frac{1}{T}\sum_{t=0}^{T-1} y_t$ as $T \to \infty$.

Exercise 2.15 Suppose that a scalar is related to a scalar white noise w_t with variance 1 by $y_t = h(L) w_t$ where $h(L) = \sum_{j=0}^{\infty} L^j h_j$ and $\sum_{j=0}^{\infty} h_j^2 < +\infty$. Then a special case of formula (2.11.2) coupled with the observer equation $y_t = G x_t$ implies that the spectrum of y is given by

$$S_y(\omega) = h(\exp(-i\omega))\, h(\exp(i\omega)) = |h(\exp(-i\omega))|^2$$

where $h(\exp(-i\omega)) = \sum_{j=0}^{\infty} h_j \exp(-i\omega j)$.

In a famous paper, Slutsky investigated the consequences of applying the following filter to white noise: $h(L) = (1+L)^n (1-L)^m$ (i.e., the convolution of n two-period moving averages with m difference operators). Compute and plot the spectrum of y for $\omega \in [-\pi, \pi]$ for the following choices of m, n:

a. $m = 10, n = 10$.

b. $m = 10, n = 40$.

c. $m = 40, n = 10$.

d. $m = 120, n = 30$.

e. Comment on these results.

Hint: Notice that $h(\exp(-i\omega)) = (1 + \exp(-i\omega))^n (1 - \exp(-i\omega))^m$.

Exercise 2.16 Consider an n-state Markov chain with state space $X = \{e_i, i = 1, \ldots, n\}$ where e_i is the ith unit vector. Consider the indicator variable $I_{it} = e_i x_t$ which equals 1 if $x_t = e_i$ and 0 otherwise. Suppose that the chain has a unique stationary distribution and that it is ergodic. Let π be the stationary distribution.

a. Verify that $E I_{it} = \pi_i$.

b. Prove that

$$\frac{1}{T}\sum_{t=0}^{T-1} I_{it} = \pi_i$$

as $T \to \infty$ with probability one with respect to the stationary distribution π.

Exercise 2.17 **Lake model**

A worker can be in one of two states, state 1 (unemployed) or state 2 (employed). At the beginning of each period, a previously unemployed worker has probability $\lambda = \int_{\bar{w}}^{B} dF(w)$ of becoming employed. Here $\bar{w}$ is his reservation wage and $F(w)$ is the c.d.f. of a wage offer distribution. We assume that $F(0) = 0, F(B) = 1$. At the beginning of each period an unemployed worker draws one and only one wage offer from F. Successive draws from F are i.i.d. The worker's decision rule is to accept the job if $w \geq \bar{w}$, and otherwise to reject it and remain unemployed one more period. Assume that $\overline{w}$ is such that $\lambda \in (0,1)$. At the beginning of each period, a previously employed worker is fired with probability $\delta \in (0,1)$. Newly fired workers must remain unemployed for one period before drawing a new wage offer.

a. Let the state space be $X = \{e_i, i = 1,2\}$ where e_i is the ith unit vector. Describe the Markov chain on X that is induced by the description above. Compute all stationary distributions of the chain. Under what stationary distributions, if any, is the chain ergodic?

b. Suppose that $\lambda = .05, \delta = .25$. Compute a stationary distribution. Compute the fraction of his life that an infinitely lived worker would spend unemployed.

c. Drawing the initial state from the stationary distribution, compute the joint distribution $g_{ij} = \text{Prob}(x_t = e_i, x_{t-1} = e_j)$ for $i = 1,2, j = 1,2$.

d. Define an indicator function by letting $I_{ij,t} = 1$ if $x_t = e_i, x_{t-1} = e_j$ at time t, and 0 otherwise. Compute

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} I_{ij,t}$$

for all four i,j combinations.

e. Building on your results in part d, construct method of moments estimators of λ and δ. Assuming that you know the wage offer distribution F, construct a method of moments estimator of the reservation wage $\bar{w}$.

f. Compute maximum likelihood estimators of λ and δ.

g. Compare the estimators you derived in parts e and f.

h. *Extra credit.* Compute the asymptotic covariance matrix of the maximum likelihood estimators of λ and δ.

Exercise 2.18 **Random walk**

A Markov chain has state space $X = \{e_i, i = 1, \ldots, 4\}$ where e_i is the unit vector and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A random variable $y_t = \overline{y} x_t$ is defined by $\overline{y} = [\,1 \quad 2 \quad 3 \quad 4\,]$.

a. Find all stationary distributions of this Markov chain.

b. Under what stationary distributions, if any, is this chain ergodic? Compute invariant functions of P.

c. Compute $E[y_{t+1}|x_t]$ for $x_t = e_i, i = 1, \ldots, 4$.

d. Compare your answer to part (c) with (2.2.12). Is $y_t = \overline{y}' x_t$ invariant? If not, what hypothesis of Theorem 2.2.4 is violated?

e. The stochastic process $y_t = \overline{y}' x_t$ is evidently a bounded martingale. Verify that y_t converges almost surely to a constant. To what constant(s) does it converge?

Exercise 2.19 **IQ**

An infinitely lived person's 'true intelligence' $\theta \sim \mathcal{N}(100, 10)$. For each date $t \geq 0$, the person takes a 'test' with the outcome being a univariate random variable $y_t = \theta + v_t$, where v_t is an iid process with distribution $\mathcal{N}(0, 100)$. The person's initial IQ is $IQ_0 = 100$ and at date $t \geq 1$ before the date t test is taken it is $\mathrm{IQ}_t = E\theta | y^{t-1}$, where y^{t-1} is the history of test scores from date 0 until date $t-1$.

a. Give a recursive formula for IQ_t and for $E(\mathrm{IQ}_t - \theta)^2$.

b. Use Matlab to simulate 10 draws of θ and associated paths of y_t, IQ_t for $t = 0, \ldots, 50$.

c. Prove that $\lim_{t\to\infty} E(\mathrm{IQ}_t - \theta)^2 = 0$.

Exercise 2.20 **Random walk**

A scalar process x_t follows the process

$$x_{t+1} = x_t + w_{t+1}$$

where w is an iid $\mathcal{N}(0,1)$ scalar process and $x_0 \sim \mathcal{N}(\hat{x}_0, \Sigma_0)$. Each period, an observer receives two signals in the form of a 2×1 vector y_t that obeys

$$y_t = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_t + v_t$$

where the 2×1 process v_t is iid with distribution $v_t \sim \mathcal{N}(0, R)$ where $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a. Suppose that $\Sigma_0 = 1.36602540378444$. For $t \geq 0$, find formulas for $E[x_t | y^{t-1}]$, where y^{t-1} is the history of y_s for s from 0 to $t-1$.

b. Verify numerically that the matrix $A - KG$ in formula (2.9.3) is stable.

c. Find an infinite-order vector autoregression for y_t.

Exercise 2.21 **Impulse response for VAR**

Find the impulse response function for the state space representation (2.9.1) associated with a vector autoregression.

Exercise 2.22 **Kalman filter with cross-products**

Consider the state space system

$$\begin{aligned} x_{t+1} &= A_o x_t + C w_{t+1} \\ y_{t+1} &= G x_t + D w_{t+1} \end{aligned}$$

where x_t is an $n \times 1$ state vector w_{t+1} is an $m \times 1$ iid process with distribution $\mathcal{N}(0, I)$, y_t is an $m \times 1$ vector of observed variables, and $x_0 \sim \mathcal{N}(\hat{x}_0, \Sigma_0)$. For $t \geq 1$, $\hat{x}_t = E[x_t | y^t]$ where $y^t = [y_t, \ldots, y_1]$ and $\Sigma_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$.

a. Show how to select w_{t+1}, C, and D so that Cw_{t+1} and Dw_{t+1} are mutually uncorrelated processes. Also give an example in which Cw_{t+1} and Dw_{t+1} are correlated.

b. Construct a recursive representation for $\hat{x}_t$ of the form:

$$\hat{x}_{t+1} = A_o \hat{x}_t + K_t a_{t+1}$$
$$y_{t+1} = G\hat{x}_t + a_{t+1}$$

where $a_{t+1} = y_{t+1} - E[y_{t+1}|y^t]$ for $t \geq 0$ and verify that

$$K_t = (CD' + A\Sigma_t G')(DD' + G\Sigma_t G')^{-1}$$
$$\Sigma_{t+1} = (A - K_t G)\Sigma_t (A - K_t G)' + (C - K_t D)(C - K_t D)'$$

and $E a_{t+1} a_{t+1}' = G\Sigma_t G' + DD'$. *Hint:* apply the population regression formula.

Exercise 2.23 **A monopolist, learning, and ergodicity**

A monopolist produces a quantity Q_t of a single good in every period $t \geq 0$ at zero cost. At the beginning of each period $t \geq 0$, before output price p_t is observed, the monopolist sets quantity Q_t to maximize

$$E_{t-1} p_t Q_t \tag{1}$$

where p_t satisfies the linear inverse demand curve

$$p_t = a - bQ_t + \sigma_p \epsilon_t \tag{2}$$

where $b > 0$ is a constant known to the firm, ϵ_t is an i.i.d. scalar with distribution $\epsilon_t \sim \mathcal{N}(0,1)$, and the constant in the inverse demand curve a is a scalar random variable unknown to the firm and whose unconditional distribution is $a \sim \mathcal{N}(\mu_a, \sigma_a^2)$, where $\mu_a > 0$ is large relative to $\sigma_a > 0$. Assume that the random variable a is independent of ϵ_t for all t. Before the firm chooses Q_0, it knows the unconditional distribution of a, but not the realized value of a. For each $t \geq 0$, the firm wants to estimate a because it wants to make a good decision about output Q_t. At the end of each period t, when it must set Q_{t+1}, the firm observes p_t and also of course knows the value of Q_t that it had set. In (1), for $t \geq 1$, $E_{t-1}(\cdot)$ denotes the mathematical expectation conditional on the history of signals $p_s, q_s, s = 0, \ldots, t-1$; for $t = 0$, $E_{t-1}(\cdot)$ denotes the expectation conditioned on no previous observations of p_t, Q_t.

a. What is the optimal setting for Q_0? For each date $t \geq 0$, determine the firm's optimal setting for Q_t as a function of the information p^{t-1}, Q^{t-1} that the firm has when it sets Q_t.

b. Under the firm's optimal policy, is the pair (p_t, Q_t) Markov?

c. 'Finding the state is an art.' Find a recursive representation of the firm's optimal policy for setting Q_t for $t \geq 0$. Interpret the state variables that you propose.

d. Under the firm's optimal rule for setting Q_t, does the random variable $E_{t-1}p_t$ converge to a constant as $t \to +\infty$? If so, prove that it does and find the limiting value. If not, tell why it does not converge.

e. Now suppose that instead of maximizing (1) each period, there is a single infinitely lived monopolist who once and for all before time 0 chooses a plan for an entire sequence $\{Q_t\}_{t=0}^{\infty}$, where the Q_t component has to be a measurable function of (p^{t-1}, q^{t-1}), and where the monopolist's objective is to maximize

$$E_{-1} \sum_{t=0}^{\infty} \beta^t p_t Q_t \tag{3}$$

where $\beta \in (0,1)$ and E_{-1} denotes the mathematical expectation conditioned on the null history. Get as far as you can in deriving the monopolist's optimal sequence of decision rules.

Exercise 2.24 **Stationarity**

A pair of scalar stochastic processes (z_t, y_t) evolves according to the state system for $t \geq 0$:

$$z_{t+1} = .9 z_t + w_{t+1}$$
$$y_t = z_t + v_t$$

where w_{t+1} and v_t are mutually uncorrelated scalar Gaussian random variables with means of 0 and variances of 1. Furthermore, $E w_{t+1} v_s = 0$ for all t, s pairs. In addition, $z_0 \sim \mathcal{N}(\hat{z}_0, \Sigma_0)$.

a. Is $\{z_t\}$ Markov? Explain.

b. Is $\{y_t\}$ Markov? Explain.

c. Define what it would mean for the scalar process $\{z_t\}$ to be *covariance stationary*.

d. Find values of $(\hat{z}_0, \Sigma_0)$ that make the process for $\{z_t\}$ covariance stationary.

e. Assume that y_t is observable, but that z_t is not. Define what it would mean for the scalar process y_t to be *covariance stationary*.

f. Describe in as much detail as you can how to represent the distribution of y_t conditional on the infinite history y^{t-1} in the form $y_t \sim \mathcal{N}(E[y_t|y^{t-1}], \Omega_t)$.

Exercise 2.25 **Consumption**

a. Please use formulas (2.12.18) to verify formulas (2.12.21) and (2.12.23)-(2.12.24) of subsection 2.12.3.

b. Please use formulas (2.8.3) to compute the decision rules in formulas (2.12.21) and (2.12.23) for the following parameter values: $\beta = .95, \sigma_1 = \sigma_2 = 1$.

c. Please use formulas (2.8.3) to compute the decision rules in formulas (2.12.21) and (2.12.23) for the following parameter values: $\beta = .95, \sigma_1 = 2, \sigma_2 = 1$.

d. Please use formula (2.12.20) to confirm formulas (2.12.22) and (2.12.25).

Exercise 2.26 **Math and verbal IQ's**

An infinitely lived person's 'true intelligence' θ has two components, math ability θ_1 and verbal ability θ_2, where $\theta \sim \mathcal{N}\left(\begin{bmatrix} 100 \\ 100 \end{bmatrix}, \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\right)$. For each date $t \geq 0$, the person takes a single 'test' with the outcome being a univariate random variable $y_t = G_t\theta + v_t$, where v_t is an iid process with distribution $\mathcal{N}(0, 50)$ and $G_t = [\,.9 \quad .1\,]$ for $t = 0, 2, 4, \ldots$ and $G_t = [\,.01 \quad .99\,]$ for $t = 1, 3, 5, \ldots$. Here the person takes a math test at t even and a verbal test at t odd (but you have to know how to read English to survive the math test, and you have to know how to tell time in order to plan your time allocation well for the verbal test). The person's initial IQ vector is $IQ_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ and at date $t \geq 1$ before the date t test is taken it is $\mathrm{IQ}_t = E\theta|y^{t-1}$, where y^{t-1} is the history of test scores from date 0 until date $t-1$.

a. Give a recursive formula for IQ_t and for $E(\mathrm{IQ}_t - \theta)(\mathrm{IQ}_t - \theta)'$.

b. Use Matlab to simulate 10 draws of θ and associated paths of y_t, IQ_t for $t = 0, \ldots, 50$.

c. Show computationally or analytically that $\lim_{t\to+\infty} E(\mathrm{IQ}_t-\theta)(\mathrm{IQ}_t-\theta)' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Exercise 2.27 **Permanent income model again**

Each of two consumers named $i=1,2$ has preferences over consumption streams that are ordered by the utility functional

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right) \tag{1}$$

where E_t is the mathematical expectation conditioned on the consumer's time t information, c_t^i is time t consumption of consumer i at time t, $u(c)$ is a strictly concave one-period utility function, and $\beta \in (0,1)$ is a discount factor. The consumer maximizes (1) by choosing a consumption, borrowing plan $\{c_t^i, b_{t+1}^i\}_{t=0}^{\infty}$ subject to the sequence of budget constraints

$$c_t^i + b_t^i = R^{-1} b_{t+1}^i + y_t^i$$

where y_t is an exogenous stationary endowment process, R is a constant gross risk-free interest rate, b_t^i is one-period risk-free debt maturing at t, and $b_0^i = 0$ is a given initial condition. Assume that $R^{-1}=\beta$. We impose the following condition on the consumption, borrowing plan of consumer i:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(b_t^i\right)^2 < +\infty.$$

Assume the quadratic utility function $u(c_t) = -.5(c_t-\gamma)^2$, where $\gamma>0$ is a bliss level of consumption. Negative consumption rates are allowed.

Let $s_t \in \{0,1\}$ be an i.i.d. process with $\mathrm{Prob}(s_t=1)=\mathrm{Prob}(s_1=0)=.5$. The endowment process of consumer 1 is $y_t^1 = 1-.5s_t$ and the endowment process of person 2 is $y_t^2 = .5+.5s_t$. Thus, the two consumers' endowment processes are perfectly negatively correlated i.i.d. processes with means of .75.

a. Find optimal decision rules for consumption for both consumers. Prove that the consumers' optimal decisions imply the following laws of motion for b_t^1, b_t^2:

$$\begin{aligned}
b_{t+1}^1\left(s_t=0\right) &= b_t^1 - .25 \\
b_{t+1}^1\left(s_t=1\right) &= b_t^1 + .25 \\
b_{t+1}^2\left(s_t=0\right) &= b_t^2 + .25 \\
b_{t+1}^2\left(s_t=1\right) &= b_t^2 - .25
\end{aligned}$$

b. Show that for each consumer, c_t^i, b_t^i are co-integrated.

c. Verify that b_{t+1}^i is risk-free in the sense that conditional on information available at time t, it is independent of news arriving at time $t+1$.

d. Verify that with the initial conditions $b_0^1 = b_0^2 = 0$, the following two equalities obtain:

$$\begin{aligned} b_t^1 + b_t^2 &= 0 \quad \forall t \geq 1 \\ c_t^1 + c_t^2 &= 1.5 \quad \forall t \geq 1 \end{aligned}$$

Use these conditions to interpret the decision rules that you have computed as describing a closed pure consumption loans economy in which consumers 1 and 2 borrow and lend with each other and in which the risk-free asset is a one-period IOU from one of the consumers to the other.

e. Define the 'stochastic discount factor of consumer i' as $m_{t+1}^i = \frac{\beta u'(c_{t+1}^i)}{u'(c_t^i)}$. Show that the stochastic discount factors of consumer 1 and 2 are

$$m_{t+1}^1 = \begin{cases} \beta + .25\frac{\beta(1-\beta)}{(\gamma - c_t^1)}, & \text{if } s_{t+1} = 0; \\ \beta - .25\frac{\beta(1-\beta)}{(\gamma - c_t^1)}, & \text{if } s_{t+1} = 1; \end{cases} .$$

$$m_{t+1}^2 = \begin{cases} \beta - .25\frac{\beta(1-\beta)}{(\gamma - c_t^2)}, & \text{if } s_{t+1} = 0; \\ \beta + .25\frac{\beta(1-\beta)}{(\gamma - c_t^2)}, & \text{if } s_{t+1} = 1; \end{cases} .$$

Are the stochastic discount factors of the two consumers equal?

f. Verify that $E_t m_{t+1}^1 = E_t m_{t+1}^2 = \beta$.

Exercise 2.28 **Invertibility**

A univariate stochastic process y_t has a first-order moving average representation

$$y_t = \epsilon_t - 2\epsilon_{t-1} \tag{1}$$

where $\{\epsilon_t\}$ is an i.i.d. process distributed $\mathcal{N}(0,1)$.

a. Argue that ϵ_t cannot be expressed as as linear combination of $y_{t-j}, j \geq 0$ where the sum of the squares of the weights is finite. This means that ϵ_t is not in the space spanned by square summable linear combinations of the infinite history y^t.

b. Write equation (1) as a state space system, indicating the matrices A, C, G.

c. Using the matlab program `kfilter.m` to compute an innovations representation for $\{y_t\}$. Verify that the innovations representation for y_t can be represented as

$$y_t = a_t - .5a_{t-1} \tag{2}$$

where $a_t = y_t - E[y_t | y^{t-1}]$ is a serially uncorrelated process. Compute the variance of a_t. Is it larger or smaller than the variance of ϵ_t?

d. Find an autoregressive representation for y_t of the form

$$y_t = \sum_{j=1}^{\infty} A_j y_{t-j} + a_t \tag{3}$$

where $E a_t y_{t-j} = 0$ for $j \geq 1$. (*Hint:* either use formula (2) or else remember formula (2.9.3).)

e. Is y_t Markov? Is $[\,y_t \quad y_{t-1}\,]'$ Markov? Is $[\,y_t \quad y_{t-1} \quad \cdots \quad y_{t-10}\,]'$ Markov?

f. Extra credit. Verify that ϵ_t *can* be expressed as a square summable linear combination of $y_{t+j}, j \geq 1$.

Chapter 3
Dynamic Programming

This chapter introduces basic ideas and methods of dynamic programming.[1] It sets out the basic elements of a recursive optimization problem, describes a key functional equation called the Bellman equation, presents three methods for solving the Bellman equation, and gives the Benveniste-Scheinkman formula for the derivative of the optimal value function. Let's dive in.

3.1. Sequential problems

Let $\beta \in (0,1)$ be a discount factor. We want to choose an infinite sequence of "controls" $\{u_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t r\left(x_t, u_t\right), \tag{3.1.1}$$

subject to $x_{t+1} = g(x_t, u_t)$, with $x_0 \in I\!R^n$ given. We assume that $r(x_t, u_t)$ is a concave function and that the set $\{(x_{t+1}, x_t) : x_{t+1} \leq g(x_t, u_t), u_t \in I\!R^k\}$ is convex and compact. Dynamic programming seeks a time-invariant *policy function* h mapping the *state* x_t into the control u_t, such that the sequence $\{u_s\}_{s=0}^{\infty}$ generated by iterating the two functions

$$\begin{aligned} u_t &= h\left(x_t\right) \\ x_{t+1} &= g\left(x_t, u_t\right), \end{aligned} \tag{3.1.2}$$

starting from initial condition x_0 at $t = 0$, solves the original problem. A solution in the form of equations (3.1.2) is said to be *recursive*. To find the policy function h we need to know another function $V(x)$ that expresses the optimal value of the original problem, starting from an arbitrary initial condition $x \in X$. This is called the *value function*. In particular, define

$$V\left(x_0\right) = \max_{\{u_s\}_{s=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r\left(x_t, u_t\right), \tag{3.1.3}$$

[1] This chapter aims to the reader to start using the methods quickly. We hope to promote demand for further and more rigorous study of the subject. In particular see Bertsekas (1976), Bertsekas and Shreve (1978), Stokey and Lucas (with Prescott) (1989), Bellman (1957), and Chow (1981). This chapter covers much of the same material as Sargent (1987b, chapter 1).

where again the maximization is subject to $x_{t+1} = g(x_t, u_t)$, with x_0 given. Of course, we cannot possibly expect to know $V(x_0)$ until after we have solved the problem, but let's proceed on faith. If we knew $V(x_0)$, then the policy function h could be computed by solving for each $x \in X$ the problem

$$\max_u \{r(x, u) + \beta V(\tilde{x})\}, \tag{3.1.4}$$

where the maximization is subject to $\tilde{x} = g(x, u)$ with x given, and $\tilde{x}$ denotes the state next period. Thus, we have exchanged the original problem of finding an infinite *sequence* of controls that maximizes expression (3.1.1) for the problem of finding the optimal value function $V(x)$ and a function h that solves the continuum of maximum problems (3.1.4)—one maximum problem for each value of x. This exchange doesn't look like progress, but we shall see that it often is.

Our task has become jointly to solve for $V(x), h(x)$, which are linked by the *Bellman equation*

$$V(x) = \max_u \{r(x, u) + \beta V[g(x, u)]\}. \tag{3.1.5}$$

The maximizer of the right side of equation (3.1.5) is a *policy function* $h(x)$ that satisfies

$$V(x) = r[x, h(x)] + \beta V\{g[x, h(x)]\}. \tag{3.1.6}$$

Equation (3.1.5) or (3.1.6) is a *functional equation* to be solved for the pair of unknown functions $V(x), h(x)$.

Methods for solving the Bellman equation are based on mathematical structures that vary in their details depending on the precise nature of the functions r and g.[2] All of these structures contain versions of the following four findings. Under various particular assumptions about r and g, it turns out that

[2] There are alternative sets of conditions that make the maximization (3.1.4) well behaved. One set of conditions is as follows: (1) r is concave and bounded, and (2) the constraint set generated by g is convex and compact, that is, the set of $\{(x_{t+1}, x_t) : x_{t+1} \leq g(x_t, u_t)\}$ for admissible u_t is convex and compact. See Stokey, Lucas, and Prescott (1989) and Bertsekas (1976) for further details of convergence results. See Benveniste and Scheinkman (1979) and Stokey, Lucas, and Prescott (1989) for the results on differentiability of the value function. In Appendix A (see Technical Appendixes), we describe the mathematics for one standard set of assumptions about (r, g). In chapter 5, we describe it for another set of assumptions about (r, g).

1. The functional equation (3.1.5) has a unique strictly concave solution.
2. This solution is approached in the limit as $j \to \infty$ by iterations on
$$V_{j+1}(x) = \max_u \{r(x,u) + \beta V_j(\tilde{x})\},$$
subject to $\tilde{x} = g(x,u), x$ given, starting from any bounded and continuous initial V_0.
3. There is a unique and time-invariant optimal policy of the form $u_t = h(x_t)$, where h is chosen to maximize the right side of (3.1.5).
4. Off corners, the limiting value function V is differentiable.

Since the value function is differentiable, the first-order necessary condition for problem (3.1.4) becomes[3]

$$r_2(x,u) + \beta V'\{g(x,u)\}\, g_2(x,u) = 0. \tag{3.1.7}$$

If we also assume that the policy function $h(x)$ is differentiable, differentiation of expression (3.1.6) yields[4]

$$\begin{aligned} V'(x) &= r_1[x,h(x)] + r_2[x,h(x)]\ h'(x) \\ &\quad + \beta V'\{g[x,h(x)]\} \Big\{g_1[x,h(x)] + g_2[x,h(x)]\ h'(x)\Big\}. \end{aligned} \tag{3.1.8}$$

When the states and controls can be defined in such a way that only u appears in the transition equation, i.e., $\tilde{x} = g(u)$: the derivative of the value function becomes, after substituting expression (3.1.7) with $u = h(x)$ into (3.1.8),

$$V'(x) = r_1[x,h(x)]\,. \tag{3.1.9}$$

This is a version of a formula of Benveniste and Scheinkman (1979).

At this point, we describe three broad computational strategies that apply in various contexts.

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[3] Here and below, subscript 1 denotes the vector of derivatives with respect to the x components and subscript 2 denotes the derivatives with respect to the u components.

[4] Benveniste and Scheinkman (1979) proved differentiability of $V(x)$ under broad conditions that do not require that $h(x)$ be differentiable. For conditions under which $h(x)$ is differentiable, see Santos (1991,1993).

3.1.1. Three computational methods

There are three main types of computational methods for solving dynamic programs. All aim to solve the functional equation (3.1.4).

Value function iteration. The first method proceeds by constructing a sequence of value functions and associated policy functions. The sequence is created by iterating on the following equation, starting from $V_0 = 0$, and continuing until V_j has converged:

$$V_{j+1}(x) = \max_u \{r(x,u) + \beta V_j(\tilde{x})\}, \tag{3.1.10}$$

subject to $\tilde{x} = g(x,u), x$ given.[5] This method is called *value function iteration* or *iterating on the Bellman equation*.

Guess and verify. A second method involves guessing and verifying a solution V to equation (3.1.5). This method relies on the uniqueness of the solution to the equation, but because it relies on luck in making a good guess, it is not generally available.

Howard's improvement algorithm. A third method, known as *policy function iteration* or *Howard's improvement algorithm*, consists of the following steps:

1. Pick a feasible policy, $u = h_0(x)$, and compute the value associated with operating forever with that policy:

$$V_{h_j}(x) = \sum_{t=0}^{\infty} \beta^t r[x_t, h_j(x_t)],$$

   where $x_{t+1} = g[x_t, h_j(x_t)]$, with $j = 0$.
2. Generate a new policy $u = h_{j+1}(x)$ that solves the two-period problem

$$\max_u \{r(x,u) + \beta V_{h_j}[g(x,u)]\},$$

   for each x.

[5] See Appendix A on functional analysis (see Technical Appendixes) for what it means for a sequence of functions to converge. A proof of the uniform convergence of iterations on equation (3.1.10) is contained in that appendix.

3. Iterate over j to convergence on steps 1 and 2.

In Appendix A (see Technical Appendixes), we describe some conditions under which the policy improvement algorithm converges to the solution of the Bellman equation. The policy improvement algorithm often converges faster than does value function iteration (e.g., see exercise *3.1* at the end of this chapter).[6] The policy improvement algorithm is also a building block for methods used to study government policy in chapter 23.

Each of our three methods for solving dynamic programming problems has its uses. Each is easier said than done, because it is typically impossible analytically to compute even *one* iteration on equation (3.1.10). This fact thrusts us into the domain of computational methods for approximating solutions: pencil and paper are insufficient. Chapter 4 describes computational methods that can applied to problems that cannot be solved by hand. Here we shall describe the first of two special types of problems for which analytical solutions *can* be obtained. It involves Cobb-Douglas constraints and logarithmic preferences. Later, in chapter 5, we shall describe a specification with linear constraints and quadratic preferences. For that special case, many analytic results are available. These two classes have been important in economics as sources of examples and as inspirations for approximations.

3.1.2. Cobb-Douglas transition, logarithmic preferences

Brock and Mirman (1972) used the following optimal growth example.[7] A planner chooses sequences $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln (c_t)$$

subject to a given value for k_0 and a transition law

$$k_{t+1} + c_t = A k_t^{\alpha}, \tag{3.1.11}$$

where $A > 0, \alpha \in (0,1), \beta \in (0,1)$.

[6] The speed of the policy improvement algorithm comes from its implementing Newton's method, which converges quadratically while iteration on the Bellman equation converges at a linear rate. See chapter 4 and Appendix A (see Technical Appendixes).

[7] See also Levhari and Srinivasan (1969).

This problem can be solved "by hand," using any of our three methods. We begin with iteration on the Bellman equation. Start with $v_0(k) = 0$, and solve the one-period problem: choose c to maximize $\ln(c)$ subject to $c + \tilde{k} = Ak^\alpha$. The solution is evidently to set $c = Ak^\alpha, \tilde{k} = 0$, which produces an optimized value $v_1(k) = \ln A + \alpha \ln k$. At the second step, we find $c = \frac{1}{1+\beta\alpha} Ak^\alpha, \tilde{k} = \frac{\beta\alpha}{1+\beta\alpha} Ak^\alpha, v_2(k) = \ln \frac{A}{1+\alpha\beta} + \beta \ln A + \alpha\beta \ln \frac{\alpha\beta A}{1+\alpha\beta} + \alpha(1+\alpha\beta)\ln k$. Continuing, and using the algebra of geometric series, gives the limiting policy functions $c = (1-\beta\alpha)Ak^\alpha, \tilde{k} = \beta\alpha Ak^\alpha$, and the value function $v(k) = (1-\beta)^{-1}\{\ln[A(1-\beta\alpha)] + \frac{\beta\alpha}{1-\beta\alpha}\ln(A\beta\alpha)\} + \frac{\alpha}{1-\beta\alpha}\ln k$.

Here is how the guess-and-verify method applies to this problem. Since we already know the answer, we'll guess a function of the correct form, but leave its coefficients undetermined.[8] Thus, we make the guess

$$v(k) = E + F \ln k, \tag{3.1.12}$$

where E and F are undetermined constants. The left and right sides of equation (3.1.12) must agree for all values of k. For this guess, the first-order necessary condition for the maximum problem on the right side of equation (3.1.10) implies the following formula for the optimal policy $\tilde{k} = h(k)$, where $\tilde{k}$ is next period's value and k is this period's value of the capital stock:

$$\tilde{k} = \frac{\beta F}{1+\beta F} Ak^\alpha. \tag{3.1.13}$$

Substitute equation (3.1.13) into the Bellman equation and equate the result to the right side of equation (3.1.12). Solving the resulting equation for E and F gives $F = \alpha/(1-\alpha\beta)$ and $E = (1-\beta)^{-1}[\ln A(1-\alpha\beta) + \frac{\beta\alpha}{1-\alpha\beta} \ln A\beta\alpha]$. It follows that

$$\tilde{k} = \beta\alpha Ak^\alpha. \tag{3.1.14}$$

Note that the term $F = \alpha/(1-\alpha\beta)$ can be interpreted as a geometric sum $\alpha[1 + \alpha\beta + (\alpha\beta)^2 + \ldots]$.

Equation (3.1.14) shows that the optimal policy is to have capital move according to the difference equation $k_{t+1} = A\beta\alpha k_t^\alpha$, or $\ln k_{t+1} = \ln A\beta\alpha + \alpha \ln k_t$. That α is less than 1 implies that k_t converges as t approaches infinity for any positive initial value k_0. The stationary point is given by the solution of $k_\infty = A\beta\alpha k_\infty^\alpha$, or $k_\infty^{\alpha-1} = (A\beta\alpha)^{-1}$.

[8] This is called the *method of undetermined coefficients*.

3.1.3. Euler equations

In many problems, there is no unique way of defining states and controls, and several alternative definitions lead to the same solution of the problem. When the states and controls can be defined in such a way that only u appears in the transition equation, i.e., $\tilde{x} = g(u)$: the first-order condition for the problem on the right side of the Bellman equation (expression (3.1.7)) in conjunction with the Benveniste-Scheinkman formula (expression (3.1.9)) implies

$$r_2(x_t, u_t) + \beta\, r_1(x_{t+1}, u_{t+1})\; g'(u_t) = 0, \qquad x_{t+1} = g\left(u_t\right).$$

The first equation is called an *Euler equation.* Under circumstances in which the second equation can be inverted to yield u_t as a function of x_{t+1}, using the second equation to eliminate u_t from the first equation produces a second-order difference equation in x_t, since eliminating u_{t+1} brings in x_{t+2}.

3.1.4. A sample Euler equation

As an example of an Euler equation, consider the Ramsey problem of choosing $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize $\sum_{t=0}^{\infty}\beta^t u(c_t)$ subject to $c_t + k_{t+1} = f(k_t)$, where k_0 is given and the one-period utility function satisfies $u'(c) > 0, u''(c) < 0, \lim_{c_t \searrow 0} u'(c_t) = \infty$, and where $f'(k) > 0, f''(k) < 0$. Let the state be k and the control be $\tilde{k}$, where $\tilde{k}$ denotes next period's value of k. Substitute $c = f(k) - \tilde{k}$ into the utility function and express the Bellman equation as

$$v\left(k\right) = \max_{\tilde{k}}\left\{u\left[f\left(k\right) - \tilde{k}\right] + \beta v\left(\tilde{k}\right)\right\}. \tag{3.1.15}$$

Application of the Benveniste-Scheinkman formula gives

$$v'\left(k\right) = u'\left[f\left(k\right) - \tilde{k}\right] f'\left(k\right). \tag{3.1.16}$$

Notice that the first-order condition for the maximum problem on the right side of equation (3.1.15) is $-u'[f(k) - \tilde{k}] + \beta v'(\tilde{k}) = 0$, which, using equation (3.1.16), gives

$$u'\left[f\left(k\right) - \tilde{k}\right] = \beta u'\left[f\left(\tilde{k}\right) - \hat{k}\right] f'\left(\tilde{k}\right), \tag{3.1.17}$$

where $\hat{k}$ denotes the two-period-ahead value of k. Equation (3.1.17) can be expressed as

$$1 = \beta \frac{u'(c_{t+1})}{u'(c_t)} f'(k_{t+1}),$$

an Euler equation that is exploited extensively in the theories of finance, growth, and real business cycles.

3.2. Stochastic control problems

We now consider a modification of problem (3.1.1) to permit uncertainty. Essentially, we add some well-placed shocks to the previous nonstochastic problem. So long as the shocks are either independently and identically distributed or Markov, straightforward modifications of the method for handling the nonstochastic problem will work.

Thus, we modify the transition equation and consider the problem of maximizing

$$E_0 \sum_{t=0}^{\infty} \beta^t r(x_t, u_t), \qquad 0 < \beta < 1, \tag{3.2.1}$$

subject to

$$x_{t+1} = g(x_t, u_t, \epsilon_{t+1}), \tag{3.2.2}$$

with x_0 known and given at $t = 0$, where ϵ_t is a sequence of independently and identically distributed random variables with cumulative probability distribution function $\text{prob}\{\epsilon_t \le e\} = F(e)$ for all t; $E_t(y)$ denotes the mathematical expectation of a random variable y, given information known at t. At time t, x_t is assumed to be known, but $x_{t+j}, j \ge 1$ is not known at t. That is, ϵ_{t+1} is realized at $(t+1)$, after u_t has been chosen at t. In problem (3.2.1)–(3.2.2), uncertainty is injected by assuming that x_t follows a random difference equation.

Problem (3.2.1)–(3.2.2) continues to have a recursive structure, stemming jointly from the additive separability of the objective function (3.2.1) in pairs (x_t, u_t) and from the difference equation characterization of the transition law (3.2.2). In particular, controls dated t affect returns $r(x_s, u_s)$ for $s \ge t$ but not earlier. This feature implies that dynamic programming methods remain appropriate.

The problem is to maximize expression (3.2.1) subject to equation (3.2.2) by choice of a "policy" or "contingency plan" $u_t = h(x_t)$. The Bellman equation (3.1.5) becomes

$$V(x) = \max_u \{r(x,u) + \beta E[V[g(x,u,\epsilon)]|x]\}, \tag{3.2.3}$$

where $E\{V[g(x,u,\epsilon)]|x\} = \int V[g(x,u,\epsilon)]dF(\epsilon)$ and where $V(x)$ is the optimal value of the problem starting from x at $t=0$. The solution $V(x)$ of equation (3.2.3) can be computed by iterating on

$$V_{j+1}(x) = \max_u \{r(x,u) + \beta E[V_j[g(x,u,\epsilon)]|x]\}, \tag{3.2.4}$$

starting from any bounded continuous initial V_0. Under various particular regularity conditions, there obtain versions of the same four properties listed earlier.[9]

The first-order necessary condition for the problem on the right side of equation (3.2.3) is

$$r_2(x,u) + \beta E\Big\{V'[g(x,u,\epsilon)]\; g_2(x,u,\epsilon)\Big|x\Big\} = 0,$$

which we obtained simply by differentiating the right side of equation (3.2.3), passing the differentiation operation under the E (an integration) operator. Off corners, the value function satisfies

$$\begin{aligned} V'(x) &= r_1[x,h(x)] + r_2[x,h(x)]\; h'(x) \\ &\quad + \beta E\Big\{V'\{g[x,h(x),\epsilon]\}\{g_1[x,h(x),\epsilon] + g_2[x,h(x),\epsilon]\; h'(x)\}\Big|x\Big\}. \end{aligned}$$

When the states and controls can be defined in such a way that x does not appear in the transition equation, the formula for $V'(x)$ becomes

$$V'(x) = r_1[x,h(x)].$$

Substituting this formula into the first-order necessary condition for the problem gives the stochastic Euler equation

$$r_2(x,u) + \beta E\Big[r_1(\tilde{x},\tilde{u})\; g_2(x,u,\epsilon)\Big|x\Big] = 0,$$

where tildes over x and u denote next-period values.

[9] See Stokey and Lucas (with Prescott) (1989), or the framework presented in Appendix A (see Technical Appendixes).

3.3. Concluding remarks

This chapter has put forward basic tools and findings: the Bellman equation and several approaches to solving it; the Euler equation; and the Benveniste-Scheinkman formula. To appreciate and believe in the power of these tools requires more words and more practice than we have yet supplied. In the next several chapters, we put the basic tools to work in different contexts with particular specification of return and transition equations designed to render the Bellman equation susceptible to further analysis and computation.

Exercise

Exercise 3.1 **Howard's policy iteration algorithm**

Consider the Brock-Mirman problem: to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t,$$

subject to $c_t + k_{t+1} \leq A k_t^\alpha \theta_t$, k_0 given, $A > 0$, $1 > \alpha > 0$, where $\{\theta_t\}$ is an i.i.d. sequence with $\ln \theta_t$ distributed according to a normal distribution with mean zero and variance σ^2.

Consider the following algorithm. Guess at a policy of the form $k_{t+1} = h_0 (A k_t^\alpha \theta_t)$ for any constant $h_0 \in (0,1)$. Then form

$$J_0 \left(k_0, \theta_0\right) = E_0 \sum_{t=0}^{\infty} \beta^t \ln \left(A k_t^\alpha \theta_t - h_0 A k_t^\alpha \theta_t\right).$$

Next choose a new policy h_1 by maximizing

$$\ln \left(A k^\alpha \theta - k'\right) + \beta E J_0 \left(k', \theta'\right),$$

where $k' = h_1 A k^\alpha \theta$. Then form

$$J_1 \left(k_0, \theta_0\right) = E_0 \sum_{t=0}^{\infty} \beta^t \ln \left(A k_t^\alpha \theta_t - h_1 A k_t^\alpha \theta_t\right).$$

Continue iterating on this scheme until successive h_j have converged.

Show that, for the present example, this algorithm converges to the optimal policy function in one step.

Chapter 4
Practical Dynamic Programming

4.1. The curse of dimensionality

We often encounter problems where it is impossible to attain closed forms for iterating on the Bellman equation. Then we have to adopt numerical approximations. This chapter describes two popular methods for obtaining numerical approximations. The first method replaces the original problem with another problem that forces the state vector to live on a finite and discrete grid of points, then applies discrete-state dynamic programming to this problem. The "curse of dimensionality" impels us to keep the number of points in the discrete state space small. The second approach uses polynomials to approximate the value function. Judd (1998) is a comprehensive reference about numerical analysis of dynamic economic models and contains many insights about ways to compute dynamic models.

4.2. Discrete-state dynamic programming

We introduce the method of discretization of the state space in the context of a particular discrete-state version of an optimal savings problem. An infinitely lived household likes to consume one good that it can acquire by spending labor income or accumulated savings. The household has an endowment of labor at time t, s_t, that evolves according to an m-state Markov chain with transition matrix $\mathcal{P}$ and state space $[\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_m]$. If the realization of the process at t is $\bar{s}_i$, then at time t the household receives labor income of amount $w\bar{s}_i$. The wage w is fixed over time. We shall sometimes assume that m is 2, and that s_t takes on value 0 in an unemployed state and 1 in an employed state. In this case, w has the interpretation of being the wage of employed workers.

The household can choose to hold a single asset in discrete amounts $a_t \in \mathcal{A}$ where $\mathcal{A}$ is a grid $[a_1 < a_2 < \cdots < a_n]$. How the model builder chooses the end points of the grid $\mathcal{A}$ is important, as we describe in detail in chapter 18 on incomplete market models. The asset bears a gross rate of return r that is fixed over time.

The household's maximum problem, for given values of (w, r) and given initial values (a_0, s_0), is to choose a policy for $\{a_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{4.2.1}$$

subject to

$$\begin{aligned} c_t + a_{t+1} &= (r+1) a_t + w s_t \\ c_t &\geq 0 \\ a_{t+1} &\in \mathcal{A} \end{aligned} \tag{4.2.2}$$

where $\beta \in (0,1)$ is a discount factor and r is fixed rate of return on the assets. We assume that $\beta(1+r) < 1$. Here $u(c)$ is a strictly increasing, concave one-period utility function. Associated with this problem is the Bellman equation

$$v(a,s) = \max_{a' \in \mathcal{A}} \left\{ u\left[(r+1) a + ws - a'\right] + \beta E v(a', s') \,|s \right\},$$

where a is next period's value of asset holdings, and s' is next period's value of the shock; here $v(a,s)$ is the optimal value of the objective function, starting from asset, employment state (a,s). We seek a value function $v(a,s)$ that satisfies equation (18.2.3) and an associated policy function $a' = g(a,s)$ mapping this period's (a,s) pair into an optimal choice of assets to carry into next period. Let assets live on the grid $\mathcal{A} = [a_1, a_2, \ldots, a_n]$. Then we can express the Bellman equation as

$$v(a_i, \bar{s}_j) = \max_{a_h \in \mathcal{A}} \left\{ u\left[(r+1) a_i + w\bar{s}_j - a_h\right] + \beta \sum_{l=1}^{m} \mathcal{P}_{jl} v(a_h, \bar{s}_l) \right\}, \tag{4.2.3}$$

for each $i \in [1, \ldots, n]$ and each $j \in [1, \ldots, m]$.

4.3. Bookkeeping

For a discrete state space of small size, it is easy to solve the Bellman equation numerically by manipulating matrices. Here is how to write a computer program to iterate on the Bellman equation in the context of the preceding model of asset accumulation.[1] Let there be n states $[a_1, a_2, \ldots, a_n]$ for assets and two states $[s_1, s_2]$ for employment status. For $j = 1, 2$, define $n \times 1$ vectors $v_j, j = 1, 2$, whose ith rows are determined by $v_j(i) = v(a_i, s_j), i = 1, \ldots, n$. Let $\mathbf{1}$ be the $n \times 1$ vector consisting entirely of ones. For $j = 1, 2$, define two $n \times n$ matrices R_j whose (i, h) elements are

$$R_j\,(i,h) = u\left[(r+1)\,a_i + ws_j - a_h\right], \quad i = 1, \ldots, n, h = 1, \ldots, n.$$

Define an operator $T([v_1, v_2])$ that maps a pair of $n \times 1$ vectors $[v_1, v_2]$ into a pair of $n \times 1$ vectors $[tv_1, tv_2]$:[2]

$$tv_j\,(i) = \max_h \left\{ R_j\,(i,h) + \beta \mathcal{P}_{j1} v_1\,(h) + \beta \mathcal{P}_{j2} v_2\,(h) \right\}$$

for $j = 1, 2$, or

$$\begin{aligned} tv_1 &= \max\{R_1 + \beta \mathcal{P}_{11} \mathbf{1} v_1' + \beta \mathcal{P}_{12} \mathbf{1} v_2'\} \\ tv_2 &= \max\{R_2 + \beta \mathcal{P}_{21} \mathbf{1} v_1' + \beta \mathcal{P}_{22} \mathbf{1} v_2'\}. \end{aligned} \tag{4.3.1}$$

Here it is understood that the "max" operator applied to an $(n \times m)$ matrix M returns an $(n \times 1)$ vector whose ith element is the maximum of the ith row of the matrix M. These two equations can be written compactly as

$$\begin{bmatrix} tv_1 \\ tv_2 \end{bmatrix} = \max \left\{ \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \beta \left(\mathcal{P} \otimes \mathbf{1}\right) \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \right\}, \tag{4.3.2}$$

where $\otimes$ is the Kronecker product.[3]

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[1] Matlab versions of the program have been written by Gary Hansen, Selahattin İmrohoroğlu, George Hall, and Chao Wei.

[2] Programming languages like Gauss and Matlab execute maximum operations over vectors very efficiently. For example, for an $n \times m$ matrix A, the Matlab command `[r,index] =max(A)` returns the two $(1 \times m)$ row vectors `r,index`, where $r_j = \max_i A(i,j)$ and index_j is the row i that attains $\max_i A(i,j)$ for column j [i.e., $\text{index}_j = \text{argmax}_i A(i,j)$]. This command performs m maximizations simultaneously.

[3] If A is an m-by-n matrix and B is a p-by-q matrix, then the Kronecker product $A \otimes B$ is the mp-by-nq block matrix $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$.

The Bellman equation $[v_1 v_2] = T([v_1, v_2])$ can be solved by iterating to convergence on $[v_1, v_2]_{m+1} = T([v_1, v_2]_m)$.

4.4. Application of Howard improvement algorithm

Often computation speed is important. Exercise 3.1 showed that the policy improvement algorithm can be much faster than iterating on the Bellman equation. It is also easy to implement the Howard improvement algorithm in the present setting. At time t, the system resides in one of N predetermined positions, denoted x_i for $i = 1, 2, \ldots, N$. There exists a predetermined set $\mathcal{M}$ of $(N \times N)$ stochastic matrices P that are the objects of choice. Here $P_{ij} = \text{Prob}\,[x_{t+1} = x_j \mid x_t = x_i]$, $i = 1, \ldots, N;\ j = 1, \ldots, N$.

The matrices P satisfy $P_{ij} \geq 0$, $\sum_{j=1}^{N} P_{ij} = 1$, and additional restrictions dictated by the problem at hand that determine the set $\mathcal{M}$. The one-period return function is represented as c_P, a vector of length N, and is a function of P. The ith entry of c_P denotes the one-period return when the state of the system is x_i and the transition matrix is P. The Bellman equation is

$$v_P(x_i) = \max_{P \in \mathcal{M}} \{c_P(x_i) + \beta \sum_{j=1}^{N} P_{ij}\, v_P(x_j)\}$$

or

$$v_P = \max_{P \in \mathcal{M}} \{c_P + \beta P v_P\}\ . \tag{4.4.1}$$

We can express this as

$$v_P = T v_P\ ,$$

where T is the operator defined by the right side of (4.4.1). Following Putterman and Brumelle (1979) and Putterman and Shin (1978), define the operator

$$B = T - I,$$

so that

$$Bv = \max_{P \in \mathcal{M}} \{c_P + \beta P v\} - v.$$

In terms of the operator B, the Bellman equation is

$$Bv = 0. \tag{4.4.2}$$

The policy improvement algorithm consists of iterations on the following two steps.

1. For fixed P_n, solve
$$(I - \beta\, P_n)\; v_{P_n} = c_{P_n} \tag{4.4.3}$$
for v_{P_n}.

2. Find P_{n+1} such that
$$c_{P_{n+1}} + (\beta P_{n+1} - I)\; v_{P_n} = Bv_{P_n} \tag{4.4.4}$$

Step 1 is accomplished by setting
$$v_{P_n} = (I - \beta P_n)^{-1}\, c_{P_n}. \tag{4.4.5}$$

Step 2 amounts to finding a policy function (i.e., a stochastic matrix $P_{n+1} \in \mathcal{M}$) that solves a two-period problem with v_{P_n} as the terminal value function.

Following Putterman and Brumelle, the policy improvement algorithm can be interpreted as a version of Newton's method for finding the zero of $Bv = v$. Using equation (4.4.3) for $n+1$ to eliminate $c_{P_{n+1}}$ from equation (4.4.4) gives
$$(I - \beta P_{n+1})\; v_{P_{n+1}} + (\beta P_{n+1} - I)\; v_{P_n} = Bv_{P_n}$$
which implies
$$v_{P_{n+1}} = v_{P_n} + (I - \beta P_{n+1})^{-1}\, Bv_{P_n}. \tag{4.4.6}$$
From equation (4.4.4), $(\beta P_{n+1} - I)$ can be regarded as the gradient of Bv_{P_n}, which supports the interpretation of equation (4.4.6) as implementing Newton's method.[4]

[4] Newton's method for finding the solution of $G(z) = 0$ is to iterate on $z_{n+1} = z_n - G'(z_n)^{-1}\, G(z_n)$.

4.5. Numerical implementation

We shall illustrate Howard's policy improvement algorithm by applying it to our savings example. Consider a feasible policy function $a' = g(k,s)$. For each j, define the $n \times n$ matrices J_j by

$$J_j\left(a,a'\right) = \begin{cases} 1 & \text{if } \; g\left(a,s_j\right) = a' \\ 0 & \text{otherwise} . \end{cases}$$

Here $j = 1, 2, \ldots, m$ where m is the number of possible values for s_t, and $J_j(a,a')$ is the element of J_j with rows corresponding to initial assets a and columns to terminal assets a'. For a given policy function $a' = g(a,s)$ define the $n \times 1$ vectors r_j with rows corresponding to

$$r_j\left(a\right) = u\left[\left(r+1\right)a + ws_j - g\left(a,s_j\right)\right], \tag{4.5.1}$$

for $j = 1, \ldots, m$.

Suppose the policy function $a' = g(a,s)$ is used forever. Let the value associated with using $g(a,s)$ forever be represented by the m $(n \times 1)$ vectors $[v_1, \ldots, v_m]$, where $v_j(a_i)$ is the value starting from state (a_i, s_j). Suppose that $m = 2$. The vectors $[v_1, v_2]$ obey

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \begin{bmatrix} \beta \mathcal{P}_{11} J_1 & \beta \mathcal{P}_{12} J_1 \\ \beta \mathcal{P}_{21} J_2 & \beta \mathcal{P}_{22} J_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left[I - \beta \begin{pmatrix} \mathcal{P}_{11} J_1 & \mathcal{P}_{12} J_1 \\ \mathcal{P}_{21} J_2 & \mathcal{P}_{22} J_2 \end{pmatrix} \right]^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{4.5.2}$$

Here is how to implement the Howard policy improvement algorithm.

Step 1. For an initial feasible policy function $g_\tau(a,j)$ for $\tau = 1$, form the r_j matrices using equation $(4.5.1)$, then use equation $(4.5.2)$ to evaluate the vectors of values $[v_1^\tau, v_2^\tau]$ implied by using that policy forever.

Step 2. Use $[v_1^\tau, v_2^\tau]$ as the terminal value vectors in equation $(4.3.2)$, and perform one step on the Bellman equation to find a new policy function $g_{\tau+1}(a,s)$ for $\tau + 1 = 2$. Use this policy function, increment τ by 1, and repeat step 1.

Step 3. Iterate to convergence on steps 1 and 2.

4.5.1. Modified policy iteration

Researchers have had success using the following modification of policy iteration: for $k \geq 2$, iterate k times on the Bellman equation. Take the resulting policy function and use equation (4.5.2) to produce a new candidate value function. Then starting from this terminal value function, perform another k iterations on the Bellman equation. Continue in this fashion until the decision rule converges.

4.6. Sample Bellman equations

This section presents some examples. The first two examples involve no optimization, just computing discounted expected utility. Appendix A of chapter 6 describes some related examples based on search theory.

4.6.1. Example 1: calculating expected utility

Suppose that the one-period utility function is the constant relative risk aversion form $u(c) = c^{1-\gamma}/(1-\gamma)$. Suppose that $c_{t+1} = \lambda_{t+1} c_t$ and that $\{\lambda_t\}$ is an n-state Markov process with transition matrix $P_{ij} = \text{Prob}(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i)$. Suppose that we want to evaluate discounted expected utility

$$V\left(c_0, \lambda_0\right) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \tag{4.6.1}$$

where $\beta \in (0,1)$. We can express this equation recursively:

$$V\left(c_t, \lambda_t\right) = u\left(c_t\right) + \beta E_t V\left(c_{t+1}, \lambda_{t+1}\right) \tag{4.6.2}$$

We use a guess-and-verify technique to solve equation (4.6.2) for $V(c_t, \lambda_t)$. Guess that $V(c_t, \lambda_t) = u(c_t) w(\lambda_t)$ for some function $w(\lambda_t)$. Substitute the guess into equation (4.6.2), divide both sides by $u(c_t)$, and rearrange to get

$$w\left(\lambda_t\right) = 1 + \beta E_t \left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} w\left(\lambda_{t+1}\right)$$

or

$$w_i = 1 + \beta \sum_j P_{ij} \left(\lambda_j\right)^{1-\gamma} w_j. \tag{4.6.3}$$

Equation (4.6.3) is a system of linear equations in $w_i, i = 1, \ldots, n$ whose solution can be expressed as

$$w = \left[1 - \beta P \text{ diag}\left(\lambda_1^{1-\gamma}, \ldots, \lambda_n^{1-\gamma}\right)\right]^{-1} \mathbf{1}$$

where $\mathbf{1}$ is an $n \times 1$ vector of ones.

4.6.2. Example 2: risk-sensitive preferences

Suppose we modify the preferences of the previous example to be of the recursive form

$$V(c_t, \lambda_t) = u(c_t) + \beta \mathcal{R}_t V(c_{t+1}, \lambda_{t+1}), \tag{4.6.4}$$

where

$$\mathcal{R}_t(V) = \left(\frac{2}{\sigma}\right) \log E_t \left[\exp\left(\frac{\sigma V_{t+1}}{2}\right)\right] \tag{4.6.5}$$

is an operator used by Jacobson (1973), Whittle (1990), and Hansen and Sargent (1995) to induce a preference for robustness to model misspecification.[5] Here $\sigma \leq 0$; when $\sigma < 0$, it represents a concern for model misspecification, or an extra sensitivity to risk.

We leave it to the reader to propose a method for computing an approximation to a value function that solves the functional equation (4.6.4). (Hint: the method used in example 1 will not apply directly because the homogeneity property exploited there fails to prevail now.)

[5] Also see Epstein and Zin (1989) and Weil (1989) for a version of the $\mathcal{R}_t$ operator.

4.6.3. Example 3: costs of business cycles

Robert E. Lucas, Jr., (1987) proposed that the cost of business cycles be measured in terms of a proportional upward shift in the consumption process that would be required to make a representative consumer indifferent between its random consumption allocation and a nonrandom consumption allocation with the same mean. This measure of business cycles is the fraction Ω that satisfies

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left[(1+\Omega)\, c_t\right] = \sum_{t=0}^{\infty} \beta^t u\left[E_0\left(c_t\right)\right]. \tag{4.6.6}$$

Suppose that the utility function and the consumption process are as in example 1. Then for given Ω, the calculations in example 1 can be used to calculate the left side of equation (4.6.6). In particular, the left side just equals $u[(1+\Omega)c_0]w(\lambda)$, where $w(\lambda)$ is calculated from equation (4.6.3). To calculate the right side, we have to evaluate

$$E_0 c_t = c_0 \sum_{\lambda_t,\ldots,\lambda_1} \lambda_t \lambda_{t-1} \cdots \lambda_1 \pi\left(\lambda_t | \lambda_{t-1}\right) \pi\left(\lambda_{t-1} | \lambda_{t-2}\right) \cdots \pi\left(\lambda_1 | \lambda_0\right), \tag{4.6.7}$$

where the summation is over all possible *paths* of growth rates between 0 and t. In the case of i.i.d. λ_t, this expression simplifies to

$$E_0 c_t = c_0 \left(E\lambda\right)^t, \tag{4.6.8}$$

where $E\lambda_t$ is the unconditional mean of λ. Under equation (4.6.8), the right side of equation (4.6.6) is easy to evaluate.

Given γ, π, a procedure for constructing the cost of cycles—more precisely, the costs of deviations from mean trend—to the representative consumer is first to compute the right side of equation (4.6.6). Then we solve the following equation for Ω:

$$u\left[(1+\Omega)\, c_0\right] w\left(\lambda_0\right) = \sum_{t=0}^{\infty} \beta^t u\left[E_0\left(c_t\right)\right].$$

Using a closely related but somewhat different stochastic specification, Lucas (1987) calculated Ω. He assumed that the endowment is a geometric trend with growth rate μ plus an i.i.d. shock with mean zero and variance σ_z^2. Starting from a base $\mu = \mu_0$, he found μ, σ_z pairs to which the household is indifferent, assuming various values of γ that he judged to be within a reasonable range.[6] Lucas found that for reasonable values of γ, it takes a very small adjustment in the trend rate of growth μ to compensate for even a substantial increase in the "cyclical noise" σ_z, which meant to him that the costs of business cycle fluctuations are small.

Subsequent researchers have studied how other preference specifications would affect the calculated costs. Tallarini (1996, 2000) used a version of the preferences described in example 2 and found larger costs of business cycles when parameters are calibrated to match data on asset prices. Hansen, Sargent, and Tallarini (1999) and Alvarez and Jermann (1999) considered local measures of the cost of business cycles and provided ways to link them to the equity premium puzzle, to be studied in chapter 14.

4.7. Polynomial approximations

Judd (1998) describes a method for iterating on the Bellman equation using a polynomial to approximate the value function and a numerical optimizer to perform the optimization at each iteration. We describe this method in the context of the Bellman equation for a particular problem that we shall encounter later.

In chapter 20, we shall study Hopenhayn and Nicolini's (1997) model of optimal unemployment insurance. A planner wants to provide incentives to an unemployed worker to search for a new job while also partially insuring the worker against bad luck in the search process. The planner seeks to deliver discounted expected utility V to an unemployed worker at minimum cost while providing proper incentives to search for work. Hopenhayn and Nicolini show that the minimum cost $C(V)$ satisfies the Bellman equation

$$C(V) = \min_{V^u} \{c + \beta [1 - p(a)] C(V^u)\} \tag{4.7.1}$$

where c, a are given by

$$c = u^{-1} [\max (0, V + a - \beta\{p(a) V^e + [1 - p(a)] V^u\})] . \tag{4.7.2}$$

[6] See chapter 14 for a discussion of reasonable values of γ. See Table 1 of Manuelli and Sargent (1988) for a correction to Lucas's calculations.

and

$$a = \max \left\{ 0, \frac{\log \left[r\beta \left(V^e - V^u \right) \right]}{r} \right\}. \tag{4.7.3}$$

Here V is a discounted present value that an insurer has promised to an unemployed worker, V_u is a value for next period that the insurer promises the worker if he remains unemployed, $1 - p(a)$ is the probability of remaining unemployed if the worker exerts search effort a, and c is the worker's consumption level. Hopenhayn and Nicolini assume that $p(a) = 1 - \exp(ra)$, $r > 0$.

4.7.1. Recommended computational strategy

To approximate the solution of the Bellman equation (4.7.1), we apply a computational procedure described by Judd (1996, 1998). The method uses a polynomial to approximate the ith iterate $C_i(V)$ of $C(V)$. This polynomial is stored on the computer in terms of $n + 1$ coefficients. Then at each iteration, the Bellman equation is to be solved at a small number $m \geq n + 1$ values of V. This procedure gives values of the ith iterate of the value function $C_i(V)$ at those particular V's. Then we interpolate (or "connect the dots") to fill in the continuous function $C_i(V)$. Substituting this approximation $C_i(V)$ for $C(V)$ in equation (4.7.1), we pass the minimum problem on the right side of equation (4.7.1) to a numerical minimizer. Programming languages like Matlab and Gauss have easy-to-use algorithms for minimizing continuous functions of several variables. We solve one such numerical problem minimization for each node value for V. Doing so yields optimized value $C_{i+1}(V)$ at those node points. We then interpolate to build up $C_{i+1}(V)$. We iterate on this scheme to convergence. Before summarizing the algorithm, we provide a brief description of Chebyshev polynomials.

4.7.2. Chebyshev polynomials

Where n is a nonnegative integer and $x \in I\!R$, the nth Chebyshev polynomial, is

$$T_n(x) = \cos\left(n \cos^{-1} x\right). \tag{4.7.4}$$

Given coefficients $c_j, j = 0, \ldots, n$, the nth-order Chebyshev polynomial approximator is

$$C_n(x) = c_0 + \sum_{j=1}^{n} c_j T_j(x). \tag{4.7.5}$$

We are given a real-valued function f of a single variable $x \in [-1, 1]$. For computational purposes, we want to form an approximator to f of the form (4.7.5). Note that we can store this approximator simply as the $n+1$ coefficients $c_j, j = 0, \ldots, n$. To form the approximator, we evaluate $f(x)$ at $n+1$ carefully chosen points, then use a least-squares formula to form the c_j's in equation (4.7.5). Thus, to interpolate a function of a single variable x with domain $x \in [-1, 1]$, Judd (1996, 1998) recommends evaluating the function at the $m \geq n+1$ points $x_k, k = 1, \ldots, m$, where

$$x_k = \cos\left(\frac{2k-1}{2m}\pi\right), k = 1, \ldots, m. \tag{4.7.6}$$

Here x_k is the zero of the kth Chebyshev polynomial on $[-1, 1]$. Given the $m \geq n+1$ values of $f(x_k)$ for $k = 1, \ldots, m$, choose the least-squares values of c_j

$$c_j = \frac{\sum_{k=1}^{m} f(x_k) T_j(x_k)}{\sum_{k=1}^{m} T_j(x_k)^2}, \ j = 0, \ldots, n \tag{4.7.7}$$

4.7.3. Algorithm: summary

In summary, applied to the Hopenhayn-Nicolini model, the numerical procedure consists of the following steps:

1. Choose upper and lower bounds for V^u, so that V and V^u will be understood to reside in the interval $[\underline{V}^u, \overline{V}^u]$. In particular, set $\overline{V}^u = V^e - \frac{1}{\beta p'(0)}$, the bound required to assure positive search effort, computed in chapter 20. Set $\underline{V}^u = V_{rmaut}$.
2. Choose a degree n for the approximator, a Chebyshev polynomial, and a number $m \geq n+1$ of nodes or grid points.
3. Generate the m zeros of the Chebyshev polynomial on the set $[1, -1]$, given by $(4.7.6)$.
4. By a change of scale, transform the z_i's to corresponding points V_ℓ^u in $[\underline{V}^u, \overline{V}^u]$.
5. Choose initial values of the $n+1$ coefficients in the Chebyshev polynomial, for example, $c_j = 0, \ldots, n$. Use these coefficients to define the function $C_i(V^u)$ for iteration number $i = 0$.
6. Compute the function $\tilde{C}_i(V) \equiv c + \beta[1 - p(a)]C_i(V^u)$, where c, a are determined as functions of (V, V^u) from equations $(4.7.2)$ and $(4.7.3)$. This computation builds in the functional forms and parameters of $u(c)$ and $p(a)$, as well as β.
7. For each point V_ℓ^u, use a numerical minimization program to find $C_{i+1}(V_\ell^u) = \min_{V^u} \tilde{C}_i(V_u)$.
8. Using these m values of $C_{j+1}(V_\ell^u)$, compute new values of the coefficients in the Chebyshev polynomials by using "least squares" [formula $(4.7.7)$]. Return to step 5 and iterate to convergence.

4.7.4. Shape-preserving splines

Judd (1998) points out that because they do not preserve concavity, using Chebyshev polynomials to approximate value functions can cause problems. He recommends the Schumaker quadratic shape-preserving spline. It ensures that the objective in the maximization step of iterating on a Bellman equation will be concave and differentiable (Judd, 1998, p. 441). Using Schumaker splines avoids the type of internodal oscillations associated with other polynomial approximation methods. The exact interpolation procedure is described in Judd (1998, p. 233). A relatively small number of nodes usually is sufficient. Judd and Solnick (1994) find that this approach outperforms linear interpolation and discrete-state approximation methods in a deterministic optimal growth problem.[7]

4.8. Concluding remarks

This chapter has described two of three standard methods for approximating solutions of dynamic programs numerically: discretizing the state space and using polynomials to approximate the value function. The next chapter describes the third method: making the problem have a quadratic return function and linear transition law. A benefit of making the restrictive linear-quadratic assumptions is that they make solving a dynamic program easy by exploiting the ease with which stochastic linear difference equations can be manipulated.

[7] The Matlab program `schumaker.m` (written by Leonardo Rezende of the University of Illinois) can be used to compute the spline. Use the Matlab command `ppval` to evaluate the spline.

Chapter 5
Linear Quadratic Dynamic Programming

5.1. Introduction

This chapter describes the class of dynamic programming problems in which the return function is quadratic and the transition function is linear. This specification leads to the widely used optimal linear regulator problem, for which the Bellman equation can be solved quickly using linear algebra. We consider the special case in which the return function and transition function are both time invariant, though the mathematics is almost identical when they are permitted to be deterministic functions of time.

Linear quadratic dynamic programming has two uses for us. A first is to study optimum and equilibrium problems arising for linear rational expectations models. Here the dynamic decision problems naturally take the form of an optimal linear regulator. A second is to use a linear quadratic dynamic program to approximate one that is not linear quadratic.

Later in the chapter, we tell how the Kalman filtering problem from chapter 2 relates to the linear-quadratic dynamic programming problem. Suitably reinterpreted, formulas that solve the optimal linear regulator are the Kalman filter.

5.2. The optimal linear regulator problem

The undiscounted optimal linear regulator problem is to maximize over choice of $\{u_t\}_{t=0}^{\infty}$ the criterion

$$-\sum_{t=0}^{\infty}\{x_t' R x_t + u_t' Q u_t\}, \tag{5.2.1}$$

subject to $x_{t+1} = Ax_t + Bu_t$, x_0 given. Here x_t is an $(n \times 1)$ vector of state variables, u_t is a $(k \times 1)$ vector of controls, R is a positive semidefinite symmetric matrix, Q is a positive definite symmetric matrix, A is an $(n \times n)$ matrix, and B is an $(n \times k)$ matrix. We guess that the value function is quadratic, $V(x) = -x'Px$, where P is a positive semidefinite symmetric matrix.

Using the transition law to eliminate next period's state, the Bellman equation becomes

$$-x'Px = \max_u \{-x'Rx - u'Qu - (Ax + Bu)' P (Ax + Bu)\}. \tag{5.2.2}$$

The first-order necessary condition for the maximum problem on the right side of equation (5.2.2) is[1]

$$(Q + B'PB)\, u = -B'PAx, \tag{5.2.3}$$

which implies the feedback rule for u:

$$u = -\left(Q + B'PB\right)^{-1} B'PAx \tag{5.2.4}$$

or $u = -Fx$, where

$$F = \left(Q + B'PB\right)^{-1} B'PA. \tag{5.2.5}$$

Substituting the optimizer (5.2.4) into the right side of equation (5.2.2) and rearranging gives

$$P = R + A'PA - A'PB\left(Q + B'PB\right)^{-1} B'PA. \tag{5.2.6}$$

Equation (5.2.6) is called the *algebraic matrix Riccati* equation. It expresses the matrix P as an implicit function of the matrices R, Q, A, B. Solving this equation for P requires a computer whenever P is larger than a 2×2 matrix.

[1] We use the following rules for differentiating quadratic and bilinear matrix forms: $\frac{\partial x'Ax}{\partial x} = (A + A')x$; $\frac{\partial y'Bz}{\partial y} = Bz$, $\frac{\partial y'Bz}{\partial z} = B'y$.

In exercise *5.1*, you are asked to derive the Riccati equation for the case where the return function is modified to

$$-\left(x_t' R x_t + u_t' Q u_t + 2 u_t' H x_t\right).$$

5.2.1. Value function iteration

Under particular conditions to be discussed in the section on stability, equation (5.2.6) has a unique positive semidefinite solution that is approached in the limit as $j \to \infty$ by iterations on the matrix Riccati difference equation[2]

$$P_{j+1} = R + A' P_j A - A' P_j B \left(Q + B' P_j B\right)^{-1} B' P_j A, \tag{5.2.7a}$$

starting from $P_0 = 0$. The policy function associated with P_j is

$$F_{j+1} = \left(Q + B' P_j B\right)^{-1} B' P_j A. \tag{5.2.7b}$$

Equation (5.2.7) is derived much like equation (5.2.6) except that one starts from the iterative version of the Bellman equation rather than from the asymptotic version.

5.2.2. Discounted linear regulator problem

The discounted optimal linear regulator problem is to maximize

$$-\sum_{t=0}^{\infty} \beta^t \{x_t' R x_t + u_t' Q u_t\}, \qquad 0 < \beta < 1, \tag{5.2.8}$$

subject to $x_{t+1} = A x_t + B u_t, x_0$ given. This problem leads to the following matrix Riccati difference equation modified for discounting:

$$P_{j+1} = R + \beta A' P_j A - \beta^2 A' P_j B \left(Q + \beta B' P_j B\right)^{-1} B' P_j A. \tag{5.2.9}$$

The algebraic matrix Riccati equation is modified correspondingly. The value function for the infinite horizon problem is $V(x_0) = -x_0' P x_0$, where P is the

[2] If the eigenvalues of A are bounded in modulus below unity, this result obtains, but much weaker conditions suffice. See Bertsekas (1976, chap. 4) and Sargent (1980).

limiting value of P_j resulting from iterations on equation (5.2.9) starting from $P_0 = 0$. The optimal policy is $u_t = -Fx_t$, where $F = \beta(Q + \beta B'PB)^{-1}B'PA$.

The Matlab program `olrp.m` solves the discounted optimal linear regulator problem. Matlab has a variety of other programs that solve both discrete- and continuous-time versions of undiscounted optimal linear regulator problems.

5.2.3. Policy improvement algorithm

The policy improvement algorithm can be applied to solve the discounted optimal linear regulator problem. We discussed aspects of this algorithm earlier in section 2.4.5.2. Starting from an initial F_0 for which the eigenvalues of $A - BF_0$ are less than $1/\sqrt{\beta}$ in modulus, the algorithm iterates on the two equations

$$P_j = R + F_j'QF_j + \beta\left(A - BF_j\right)' P_j \left(A - BF_j\right) \tag{5.2.10}$$

$$F_{j+1} = \beta\left(Q + \beta B'P_jB\right)^{-1} B'P_jA. \tag{5.2.11}$$

The first equation pins down the matrix for the quadratic form in the value function associated with using a fixed rule F_j forever. The second equation gives the matrix for the optimal first-period decision rule for a two-period problem with second-period value function $-x^{*\prime}P_jx^*$ where x^* is the second-period state. The first equation is an example of a *discrete Lyapunov* or *Sylvester* equation, which is to be solved for the matrix P_j that determines the value $-x_t'P_jx_t$ that is associated with following policy F_j forever. The solution of this equation can be represented in the form

$$P_j = \sum_{k=0}^{\infty} \beta^k \left(A - BF_j\right)'^k \left(R + F_j'QF_j\right)\left(A - BF_j\right)^k .$$

If the eigenvalues of the matrix $A - BF_j$ are bounded in modulus by $1/\sqrt{\beta}$, then a solution of this equation exists. There are several methods available for solving this equation.[3] The Matlab program `policyi.m` solves the undiscounted optimal linear regulator problem using policy iteration. This algorithm is typically much faster than the algorithm that iterates on the matrix Riccati equation. Later we shall present a third method for solving for P that rests on the link between P and shadow prices for the state vector.

[3] The Matlab programs `dlyap.m` and `doublej.m` solve discrete Lyapunov equations. See Anderson, Hansen, McGrattan, and Sargent (1996).

5.3. The stochastic optimal linear regulator problem

The stochastic discounted linear optimal regulator problem is to choose a decision rule for u_t to maximize

$$-E_0 \sum_{t=0}^{\infty} \beta^t \{x_t' R x_t + u_t' Q u_t\}, \qquad 0 < \beta < 1, \tag{5.3.1}$$

subject to x_0 given, and the law of motion

$$x_{t+1} = A x_t + B u_t + C \epsilon_{t+1}, \qquad t \geq 0, \tag{5.3.2}$$

where ϵ_{t+1} is an $(n \times 1)$ vector of random variables that is independently and identically distributed according to the normal distribution with mean vector zero and covariance matrix

$$E \epsilon_t \epsilon_t' = I. \tag{5.3.3}$$

(See Kwakernaak and Sivan, 1972, for an extensive study of the continuous-time version of this problem; also see Chow, 1981.)

The value function for this problem is

$$v(x) = -x' P x - d, \tag{5.3.4}$$

where P is the unique positive semidefinite solution of the discounted algebraic matrix Riccati equation corresponding to equation (5.2.9). As before, it is the limit of iterations on equation (5.2.9) starting from $P_0 = 0$. The scalar d is given by

$$d = \beta (1 - \beta)^{-1} \operatorname{trace} (PCC)'. \tag{5.3.5}$$

Furthermore, the optimal policy continues to be given by $u_t = -F x_t$, where

$$F = \beta \left(Q + \beta B' P' B\right)^{-1} B' P A. \tag{5.3.6}$$

A notable feature of this solution is:

Certainty Equivalence Principle: The decision rule (5.3.6) that solves the stochastic optimal linear regulator problem is identical with the decision rule for the corresponding nonstochastic linear optimal regulator problem.

Proof: Substitute guess (5.3.4) into the Bellman equation to obtain

$$v(x) = \max_u \left\{ -x' R x - u' Q u - \beta E \left[(Ax + Bu + C\epsilon)' P (Ax + Bu + C\epsilon) \right] - \beta d \right\},$$

where ϵ is the realization of ϵ_{t+1} when $x_t = x$ and where $E\epsilon|x = 0$. The preceding equation implies

$$\begin{aligned} v(x) = \max_u \{&-x'Rx - u'Qu - \beta E\{x'A'PAx + x'A'PBu \\ &+ x'A'PC\epsilon + u'B'PAx + u'B'PBu + u'B'PC\epsilon \\ &+ \epsilon'C'PAx + \epsilon'C'PBu + \epsilon'C'PC\epsilon\} - \beta d\}. \end{aligned}$$

Evaluating the expectations inside the braces and using $E\epsilon|x = 0$ gives

$$\begin{aligned} v(x) = \max_u - \{&x'Rx + u'Qu + \beta x'A'PAx + \beta 2x'A'PBu \\ &+ \beta u'B'PBu + \beta E\epsilon'C'PC\epsilon\} - \beta d. \end{aligned}$$

The first-order condition for u is

$$(Q + \beta B'PB)\, u = -\beta B'PAx,$$

which implies equation (5.3.6). Using $E\epsilon'C'PC\epsilon = \mathrm{trace}(PCC)'$, substituting equation (5.3.6) into the preceding expression for $v(x)$, and using equation (5.3.4) gives

$$P = R + \beta A'PA - \beta^2 A'PB\,(Q + \beta B'PB)^{-1}\, B'PA,$$

and

$$d = \beta\,(1 - \beta)^{-1}\,\mathrm{trace}\,(PCC')\,. \qquad \blacksquare$$

5.3.1. Discussion of certainty equivalence

The remarkable thing is that, although through d the objective function (5.3.3) depends on CC', the optimal decision rule $u_t = -Fx_t$ is independent of CC'. This is the message of equation (5.3.6) and the discounted algebraic Riccati equation for P, which are identical with the formulas derived earlier under certainty. In other words, the optimal decision rule $u_t = h(x_t)$ is independent of the problem's noise statistics.[4] The certainty equivalence principle is

[4] Therefore, in linear quadratic versions of the optimum savings problem, there are no precautionary savings. Compare outcomes from section 2.12 of chapter 2 and chapters 17 and 18.

a special property of the optimal linear regulator problem and comes from the quadratic objective function, the linear transition equation, and the property $E(\epsilon_{t+1}|x_t) = 0$. Certainty equivalence does not characterize stochastic control problems generally.

5.4. Shadow prices in the linear regulator

For several purposes,[5] it is helpful to interpret the gradient $-2Px_t$ of the value function $-x_t'Px_t$ as a shadow price or Lagrange multiplier. Thus, associate with the Bellman equation the Lagrangian

$$\begin{aligned} -x_t'Px_t = V(x_t) = \min_{\mu_{t+1}} \max_{u_t, x_{t+1}} -\Big\{ & x_t'Rx_t + u_t'Qu_t + x_{t+1}'Px_{t+1} \\ & + 2\mu_{t+1}' \left[Ax_t + Bu_t - x_{t+1}\right] \Big\}, \end{aligned}$$

where $2\mu_{t+1}$ is a vector of Lagrange multipliers. The first-order necessary conditions for an optimum with respect to u_t and x_{t+1} are

$$\begin{aligned} 2Qu_t + 2B'\mu_{t+1} &= 0 \\ 2Px_{t+1} - 2\mu_{t+1} &= 0. \end{aligned} \tag{5.4.1}$$

Using the transition law and rearranging gives the usual formula for the optimal decision rule, namely, $u_t = -(Q + B'PB)^{-1}B'PAx_t$. Notice that by (5.4.1), the shadow price vector satisfies $\mu_{t+1} = Px_{t+1}$.

In section 5.5, we shall describe a computational strategy that solves for P by directly finding the optimal multiplier process $\{\mu_t\}$ and representing it as $\mu_t = Px_t$. This strategy exploits the *stability* properties of optimal solutions of the linear regulator problem, which we now briefly take up.

[5] In a planning problem in a linear quadratic economy, the gradient of the value function has information from which competitive equilibrium prices can be coaxed. See Hansen and Sargent (2000).

5.4.1. Stability

Upon substituting the optimal control $u_t = -Fx_t$ into the law of motion $x_{t+1} = Ax_t + Bu_t$, we obtain the optimal "closed-loop system" $x_{t+1} = (A - BF)x_t$. This difference equation governs the evolution of x_t under the optimal control. The system is said to be *stable* if $\lim_{t\to\infty} x_t = 0$ starting from any initial $x_0 \in R^n$. Assume that the eigenvalues of $(A - BF)$ are distinct, and use the eigenvalue decomposition $(A - BF) = D\Lambda D^{-1}$ where the columns of D are the eigenvectors of $(A - BF)$ and Λ is a diagonal matrix of eigenvalues of $(A-BF)$. Write the "closed-loop" equation as $x_{t+1} = D\Lambda D^{-1}x_t$. The solution of this difference equation for $t > 0$ is readily verified by repeated substitution to be $x_t = D\Lambda^t D^{-1}x_0$. Evidently, the system is stable for all $x_0 \in R^n$ if and only if the eigenvalues of $(A - BF)$ are all strictly less than unity in absolute value. When this condition is met, $(A - BF)$ is said to be a "stable matrix."[6]

A vast literature is devoted to characterizing the conditions on A, B, R, and Q that imply that F is such that the optimal closed-loop system matrix $(A - BF)$ is stable. These conditions are surveyed by Anderson, Hansen, McGrattan, and Sargent (1996) and can be briefly described here for the undiscounted case $\beta = 1$. Roughly speaking, the conditions on A, B, R, and Q are as follows: First, A and B must be such that it is *possible* to pick a control law $u_t = -Fx_t$ that drives x_t to zero eventually, starting from any $x_0 \in R^n$ ["the pair (A, B) must be stabilizable"]. Second, the matrix R must be such that it is *desirable* to drive x_t to zero as $t \to \infty$.

It would take us too far afield to go deeply into this body of theory, but we can give a flavor of the results by considering the following special assumptions and their implications. Similar results can obtain under weaker conditions relevant for economic problems.[7]

Assumption A.1: The matrix R is positive definite.

There immediately follows:

Proposition 1: Under assumption A.1, if a solution to the undiscounted regulator exists, it satisfies $\lim_{t\to\infty} x_t = 0$.

---

[6] It is possible to amend the statements about stability in this section to permit $A - BF$ to have a single unit eigenvalue associated with a constant in the state vector. See chapter 2 for examples.

[7] See Kwakernaak and Sivan (1972) and Anderson, Hansen, McGrattan, and Sargent (1996) for much weaker conditions.

PROOF: If $x_t \not\to 0$, then $\sum_{t=0}^{\infty} x_t' R x_t \to \infty$. ∎

ASSUMPTION A.2: The matrix R is positive semidefinite.

Under assumption A.2, R is similar to a triangular matrix R^*:

$$R = T' \begin{pmatrix} R_{11}^* & 0 \\ 0 & 0 \end{pmatrix} T$$

where R_{11}^* is positive definite and T is nonsingular. Notice that $x_t' R x_t = x_{1t}^* R_{11}^* x_{1t}^*$ where $x_t^* = T x_t = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} x_t = \begin{pmatrix} x_{1t}^* \\ x_{2t}^* \end{pmatrix}$. Let $x_{1t}^* \equiv T_1 x_t$. These calculations support:

PROPOSITION 2: Suppose that a solution to the optimal linear regulator exists under assumption A.2. Then $\lim_{t\to\infty} x_{1t}^* = 0$.

The following definition is used in control theory:

DEFINITION: The pair (A, B) is said to be *stabilizable* if there exists a matrix F for which $(A - BF)$ is a stable matrix.

The following indicates the flavor of a variety of stability theorems from control theory:[8, 9]

THEOREM: If (A, B) is stabilizable and R is positive definite, then under the optimal rule F, $(A - BF)$ is a stable matrix.

In the next section, we assume that A, B, Q, R satisfy conditions sufficient to invoke such a stability proposition, and we use that assumption to justify a solution method that solves the undiscounted linear regulator by searching among the many solutions of the *Euler equations* for a stable solution.

---

[8] These conditions are discussed under the subjects of controllability, stabilizability, reconstructability, and detectability in the literature on linear optimal control. (For continuous-time linear system, these concepts are described by Kwakernaak and Sivan, 1972; for discrete-time systems, see Sargent, 1980.) These conditions subsume and generalize the transversality conditions used in the discrete-time calculus of variations (see Sargent, 1987a). That is, the case when $(A - BF)$ is stable corresponds to the situation in which it is optimal to solve "stable roots backward and unstable roots forward." See Sargent (1987a, chap. 9). Hansen and Sargent (1981) describe the relationship between Euler equation methods and dynamic programming for a class of linear optimal control systems. Also see Chow (1981).

[9] The conditions under which $(A - BF)$ is stable are also the conditions under which x_t converges to a unique stationary distribution in the stochastic version of the linear regulator problem.

5.5. A Lagrangian formulation

This section describes a Lagrangian formulation of the optimal linear regulator.[10] Besides being useful computationally, this formulation carries insights about the connections between stability and optimality and also opens the way to constructing solutions of dynamic systems not coming directly from an intertemporal optimization problem.[11]

For the undiscounted optimal linear regulator problem, form the Lagrangian

$$\begin{aligned}\mathcal{L} = -\sum_{t=0}^{\infty}\Big\{ & x_t' R x_t + u_t' Q u_t \\ & + 2\mu_{t+1}' \left[A x_t + B u_t - x_{t+1}\right]\Big\}.\end{aligned}$$

First-order conditions for maximization with respect to $\{u_t, x_{t+1}\}$ are

$$\begin{aligned} 2Qu_t + 2B'\mu_{t+1} &= 0 \\ \mu_t &= Rx_t + A'\mu_{t+1}\ ,\ t \geq 0. \end{aligned} \tag{5.5.1}$$

The Lagrange multiplier vector μ_{t+1} is often called the *costate* vector. Recall from the second equation of (5.4.1) that $\mu_{t+1} = P x_{t+1}$ where P is the matrix that solves the algebraic Riccati equation. Thus, μ_{t+1} is the gradient of the value function. Solve the first equation of (5.5.1) for u_t in terms of μ_{t+1}; substitute into the law of motion $x_{t+1} = Ax_t + Bu_t$; arrange the resulting equation and the second equation of (5.5.1) into the form

$$L \begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = N \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}\ ,\ t \geq 0,$$

where

$$L = \begin{pmatrix} I & BQ^{-1}B' \\ 0 & A' \end{pmatrix}, N = \begin{pmatrix} A & 0 \\ -R & I \end{pmatrix}.$$

When L is of full rank (i.e., when A is of full rank), we can write this system as

$$\begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = M \begin{pmatrix} x_t \\ \mu_t \end{pmatrix} \tag{5.5.2}$$

[10] Such formulations are recommended by Chow (1997) and Anderson, Hansen, McGrattan, and Sargent (1996).

[11] Blanchard and Kahn (1980); Whiteman (1983); Hansen, Epple, and Roberds (1985); and Anderson, Hansen, McGrattan and Sargent (1996) use and extend such methods.

where

$$M \equiv L^{-1}N = \begin{pmatrix} A + BQ^{-1}B'A'^{-1}R & -BQ^{-1}B'A'^{-1} \\ -A'^{-1}R & A'^{-1} \end{pmatrix}. \tag{5.5.3}$$

We seek to solve the difference equation system (5.5.2) for a sequence $\{x_t\}_{t=0}^{\infty}$ that satisfies the initial condition for x_0 and a terminal condition $\lim_{t\to+\infty} x_t = 0$ that expresses our wish for a *stable* solution. We inherit our wish for stability of the $\{x_t\}$ sequence in this sense for a desire to maximize $-\sum_{t=0}^{\infty}\left[x_t'Rx_t + u_t'Qu_t\right]$, which requires that $x_t'Rx_t$ converge to zero.

To proceed, we study properties of the $(2n \times 2n)$ matrix M. It is helpful to introduce a $(2n \times 2n)$ matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The rank of J is $2n$.

Definition: A matrix M is called *symplectic* if

$$MJM' = J. \tag{5.5.4}$$

It can be verified directly that M in equation (5.5.3) is symplectic.

It follows from equation (5.5.4) and from the fact $J^{-1} = J' = -J$ that for any symplectic matrix M,

$$M' = J^{-1}M^{-1}J. \tag{5.5.5}$$

Equation (5.5.5) states that M' is related to the inverse of M by a similarity transformation. For square matrices, recall that (a) similar matrices share eigenvalues; (b) the eigenvalues of the inverse of a matrix are the inverses of the eigenvalues of the matrix; and (c) a matrix and its transpose have the same eigenvalues. It then follows from equation (5.5.5) that the eigenvalues of M occur in reciprocal pairs: if λ is an eigenvalue of M, so is λ^{-1}.

Write equation (5.5.2) as

$$y_{t+1} = My_t \tag{5.5.6}$$

where $y_t = \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}$. Consider the following triangularization of M

$$V^{-1}MV = \begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{pmatrix}$$

where each block on the right side is $(n \times n)$, where V is nonsingular, and where W_{22} has all its eigenvalues exceeding 1 in modulus and W_{11} has all of its eigenvalues less than 1 in modulus. The *Schur decomposition* and the *eigenvalue decomposition* are two such decompositions.[12] Write equation (5.5.6) as

$$y_{t+1} = VWV^{-1}y_t. \tag{5.5.7}$$

The solution of equation (5.5.7) for arbitrary initial condition y_0 is evidently

$$y_t = V \begin{bmatrix} W_{11}^t & W_{12,t} \\ 0 & W_{22}^t \end{bmatrix} V^{-1}y_0 \tag{5.5.8}$$

where $W_{12,t} = W_{12}$ for $t = 1$ and for $t \geq 2$ obeys the recursion

$$W_{12,t} = W_{11}^{t-1}W_{12,t-1} + W_{12,t-1}W_{22}^{t-1}$$

and where W_{ii}^t is W_{ii} raised to the tth power.

Write equation (5.5.8) as

$$\begin{pmatrix} y_{1t}^* \\ y_{2t}^* \end{pmatrix} = \begin{bmatrix} W_{11}^t & W_{12,t} \\ 0 & W_{22}^t \end{bmatrix} \begin{pmatrix} y_{10}^* \\ y_{20}^* \end{pmatrix}$$

where $y_t^* = V^{-1}y_t$, and in particular where

$$y_{2t}^* = V^{21}x_t + V^{22}\mu_t, \tag{5.5.9}$$

and where V^{ij} denotes the (i,j) piece of the partitioned V^{-1} matrix.

Because W_{22} is an unstable matrix, unless $y_{20}^* = 0$, y_t^* will diverge. Let V^{ij} denote the (i,j) piece of the partitioned V^{-1} matrix. To attain stability, we must impose $y_{20}^* = 0$, which from equation (5.5.9) implies

$$V^{21}x_0 + V^{22}\mu_0 = 0$$

[12] Evan Anderson's Matlab program `schurg.m` attains a convenient Schur decomposition and is very useful for solving linear models with distortions. See McGrattan (1994) for examples of distorted economies whose equilibria can be computed using a Schur decomposition.

or

$$\mu_0 = -\left(V^{22}\right)^{-1} V^{21} x_0.$$

This equation replicates itself over time in the sense that it implies

$$\mu_t = -\left(V^{22}\right)^{-1} V^{21} x_t. \tag{5.5.10}$$

But notice that because $(V^{21}\ V^{22})$ is the second row block of the inverse of V,

$$\begin{pmatrix} V^{21} & V^{22} \end{pmatrix} \quad \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = 0$$

which implies

$$V^{21} V_{11} + V^{22} V_{21} = 0.$$

Therefore,

$$-\left(V^{22}\right)^{-1} V^{21} = V_{21} V_{11}^{-1}.$$

So we can write

$$\mu_0 = V_{21} V_{11}^{-1} x_0 \tag{5.5.11}$$

and

$$\mu_t = V_{21} V_{11}^{-1} x_t.$$

However, we know from equations (5.4.1) that $\mu_t = P x_t$, where P occurs in the matrix that solves the Riccati equation (5.2.6). Thus, the preceding argument establishes that

$$P = V_{21} V_{11}^{-1}. \tag{5.5.12}$$

This formula provides us with an alternative, and typically computationally very efficient, way of computing the matrix P.

This same method can be applied to compute the solution of any system of the form (5.5.2) if a solution exists, even if the eigenvalues of M fail to occur in reciprocal pairs. The method will typically work so long as the eigenvalues of M split half inside and half outside the unit circle.[13] Systems in which the eigenvalues (properly adjusted for discounting) fail to occur in reciprocal pairs arise when the system being solved is an equilibrium of a model in which there are distortions that prevent there being any optimum problem that the equilibrium solves. See Woodford (1999) for an application of such methods to solve

[13] See Whiteman (1983); Blanchard and Kahn (1980); and Anderson, Hansen, McGrattan, and Sargent (1996) for applications and developments of these methods.

for linear approximations of equilibria of a monetary model with distortions. See chapter 11 for some applications to an economy with distorting taxes.

5.6. The Kalman filter again

Suitably reinterpreted, the same recursion (5.2.7) that solves the optimal linear regulator also determines the celebrated *Kalman filter* that we derived in section 2.7 of chapter 2. Recall that the Kalman filter is a recursive algorithm for computing the mathematical expectation $E[x_t|y_{t-1},\ldots,y_0]$ of a hidden state vector x_t, conditional on observing a history $y_t,\ldots,y_0$ of a vector of noisy signals on the hidden state. The Kalman filter can be used to formulate or simplify a variety of signal-extraction and prediction problems in economics.

We briefly remind the reader that the setting for the Kalman filter is the following linear state-space system.[14] Given $x_0 \sim \mathcal{N}(\hat{x}_0, \Sigma_0)$, let

$$x_{t+1} = Ax_t + Cw_{t+1} \tag{5.6.1a}$$

$$y_t = Gx_t + v_t \tag{5.6.1b}$$

where x_t is an $(n \times 1)$ state vector, w_t is an i.i.d. sequence Gaussian vector with $Ew_tw_t' = I$, and v_t is an i.i.d. Gaussian vector orthogonal to w_s for all t,s with $Ev_tv_t' = R$; and A, C, and G are matrices conformable to the vectors they multiply. Assume that the initial condition x_0 is unobserved but is known to have a Gaussian distribution with mean $\hat{x}_0$ and covariance matrix Σ_0. At time t, the history of observations $y^t \equiv [y_t,\ldots,y_0]$ is available to estimate the location of x_t and the location of x_{t+1}. The Kalman filter is a recursive algorithm for computing $\hat{x}_{t+1} = E[x_{t+1}|y^t]$. The algorithm is

$$\hat{x}_{t+1} = (A - K_tG)\,\hat{x}_t + K_ty_t \tag{5.6.2}$$

where

$$K_t = A\Sigma_tG'\left(G\Sigma_tG' + R\right)^{-1} \tag{5.6.3a}$$

$$\Sigma_{t+1} = A\Sigma_tA' + CC' - A\Sigma_tG'\left(G\Sigma_tG' + R\right)^{-1}G\Sigma_tA. \tag{5.6.3b}$$

[14] We derived the Kalman filter as a recursive application of population regression in chapter 2, page 56.

Here $\Sigma_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$, and K_t is called the *Kalman gain*. Sometimes the Kalman filter is written in terms of the "innovation representation"

$$\hat{x}_{t+1} = A\hat{x}_t + K_t a_t \tag{5.6.4a}$$

$$y_t = G\hat{x}_t + a_t \tag{5.6.4b}$$

where $a_t \equiv y_t - G\hat{x}_t \equiv y_t - E[y_t|y^{t-1}]$. The random vector a_t is called the *innovation* in y_t, being the part of y_t that cannot be forecast linearly from its own past. Subtracting equation (5.6.4b) from (5.6.1b) gives $a_t = G(x_t - \hat{x}_t) + v_t$; multiplying each side by its own transpose and taking expectations gives the following formula for the innovation covariance matrix:

$$E a_t a_t' = G\Sigma_t G' + R. \tag{5.6.5}$$

Equations (5.6.3) display extensive similarities to equations (5.2.7), the recursions for the optimal linear regulator. Indeed, the mathematical structures are identical when viewed properly. Note that equation (5.6.3b) is a Riccati equation. With the judicious use of matrix transposition and reversal of time, the two systems of equations (5.6.3) and (5.2.7) can be made to match.[15] See chapter 2, especially section 2.8, for some applications of the Kalman filter.[16]

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[15] See Hansen and Sargent (ch. 4, 2008) for an account of how the LQ dynamic programming problem and the Kalman filter are connected through duality. That chapter formulates the Kalman filtering problem in terms of a Lagrangian, then judiciously transforms the first-order conditions into an associated optimal linear regulator.

[16] The Matlab program `kfilter.m` computes the Kalman filter. Matlab has several programs that compute the Kalman filter for discrete time and continuous time models.

5.7. Concluding remarks

In exchange for their restrictions, the linear quadratic dynamic optimization problems of this chapter acquire tractability. The Bellman equation leads to Riccati difference equations that are so easy to solve numerically that the curse of dimensionality loses most of its force. It is easy to solve linear quadratic control or filtering with many state variables. That it is difficult to solve those problems otherwise is why linear quadratic approximations are widely used. We describe those approximations in Appendix B to this chapter.

In chapter 7, we go beyond the single-agent optimization problems of this chapter to study systems with multiple agents who simultaneously solve linear quadratic dynamic programming problems, with the decision rules of some agents influencing transition laws of variables appearing in other agents' decision problems. We introduce two related equilibrium concepts to reconcile different agents' decisions.

A. Matrix formulas

Let (z, x, a) each be $n \times 1$ vectors, A, C, D, and V each be $(n \times n)$ matrices, B an $(m \times n)$ matrix, and y an $(m \times 1)$ vector. Then $\frac{\partial a'x}{\partial x} = a$, $\frac{\partial x'Ax}{\partial x} = (A + A')x$, $\frac{\partial^2 (x'Ax)}{\partial x \partial x'} = (A + A')$, $\frac{\partial x'Ax}{\partial A} = xx'$, $\frac{\partial y'Bz}{\partial y} = Bz$, $\frac{\partial y'Bz}{\partial z} = B'y$, $\frac{\partial y'Bz}{\partial B} = yz'$.

The equation

$$A'VA + C = V$$

to be solved for V is called a *discrete Lyapunov equation*, and its generalization

$$A'VD + C = V$$

is called the discrete *Sylvester equation*. The discrete Sylvester equation has a unique solution if and only if the eigenvalues $\{\lambda_i\}$ of A and $\{\delta_j\}$ of D satisfy the condition $\lambda_i \delta_j \neq 1 \ \forall \ i, \ j$.

B. Linear quadratic approximations

This appendix describes an important use of the optimal linear regulator: to approximate the solution of more complicated dynamic programs.[17] Optimal linear regulator problems are often used to approximate problems of the following form: maximize over $\{u_t\}_{t=0}^{\infty}$

$$E_0 \sum_{t=0}^{\infty} \beta^t r\left(z_t\right) \tag{5.B.1}$$

$$x_{t+1} = A x_t + B u_t + C w_{t+1} \tag{5.B.2}$$

where $\{w_{t+1}\}$ is a vector of i.i.d. random disturbances with mean zero and finite variance, and $r(z_t)$ is a concave and twice continuously differentiable function of $z_t \equiv \begin{pmatrix} x_t \\ u_t \end{pmatrix}$. All nonlinearities in the original problem are absorbed into the composite function $r(z_t)$.

5.B.1. An example: the stochastic growth model

Take a parametric version of Brock and Mirman's stochastic growth model, whose social planner chooses a policy for $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

where

$$\begin{aligned} c_t + i_t &= A a_t^{\alpha} \theta_t \\ a_{t+1} &= (1-\delta)\, a_t + i_t \\ \ln \theta_{t+1} &= \rho \ln \theta_t + w_{t+1} \end{aligned}$$

where $\{w_{t+1}\}$ is an i.i.d. stochastic process with mean zero and finite variance, θ_t is a technology shock, and $\tilde{\theta}_t \equiv \ln \theta_t$. To get this problem into the form

[17] Kydland and Prescott (1982) used such a method, and so do many of their followers in the real business cycle literature. See King, Plosser, and Rebelo (1988) for related methods of real business cycle models.

$(5.B.1)$–$(5.B.2)$, take $x_t = \begin{pmatrix} a_t \\ \tilde{\theta}_t \end{pmatrix}$, $u_t = i_t$, and $r(z_t) = \ln(A a_t^{\alpha} \exp \tilde{\theta}_t - i_t)$, and we write the laws of motion as

$$\begin{pmatrix} 1 \\ a_{t+1} \\ \tilde{\theta}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-\delta) & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} 1 \\ a_t \\ \tilde{\theta}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i_t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w_{t+1}$$

where it is convenient to add the constant 1 as the first component of the state vector.

5.B.2. Kydland and Prescott's method

We want to replace $r(z_t)$ by a quadratic $z_t' M z_t$. We choose a point $\bar{z}$ and approximate with the first two terms of a Taylor series:[18]

$$\begin{aligned} \hat{r}(z) = r(\bar{z}) + (z - \bar{z})' \frac{\partial r}{\partial z} \\ + \frac{1}{2} (z - \bar{z})' \frac{\partial^2 r}{\partial z \partial z'} (z - \bar{z}). \end{aligned} \tag{5.B.3}$$

If the state x_t is $n \times 1$ and the control u_t is $k \times 1$, then the vector z_t is $(n+k) \times 1$. Let e be the $(n+k) \times 1$ vector with 0's everywhere except for a 1 in the row corresponding to the location of the constant unity in the state vector, so that $1 \equiv e' z_t$ for all t.

Repeatedly using $z'e = e'z = 1$, we can express equation $(5.B.3)$ as

$$\hat{r}(z) = z' M z,$$

where

$$\begin{aligned} M = & e \left[r(\bar{z}) - \left(\frac{\partial r}{\partial z} \right)' \bar{z} + \frac{1}{2} \bar{z}' \frac{\partial^2 r}{\partial z \partial z'} \bar{z} \right] e' \\ & + \frac{1}{2} \left(\frac{\partial r}{\partial z} e' - e \bar{z}' \frac{\partial^2 r}{\partial z \partial z'} - \frac{\partial^2 r}{\partial z \partial z'} \bar{z} e' + e \frac{\partial r}{\partial z}' \right) \\ & + \frac{1}{2} \left(\frac{\partial^2 r}{\partial z \partial z'} \right) \end{aligned}$$

[18] This setup is taken from McGrattan (1994) and Anderson, Hansen, McGrattan, and Sargent (1996).

where the partial derivatives are evaluated at $\bar{z}$. Partition M, so that

$$\begin{aligned} z'Mz &\equiv \begin{pmatrix} x \\ u \end{pmatrix}' \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ &= \begin{pmatrix} x \\ u \end{pmatrix}' \begin{pmatrix} R & W \\ W' & Q \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}. \end{aligned}$$

5.B.3. Determination of $\bar{z}$

Usually, the point $\bar{z}$ is chosen as the (optimal) stationary state of the *non-stochastic* version of the original nonlinear model:

$$\begin{aligned} &\sum_{t=0}^{\infty} \beta^t r(z_t) \\ &x_{t+1} = Ax_t + Bu_t. \end{aligned}$$

This stationary point is obtained in these steps:

1. Find the Euler equations.
2. Substitute $z_{t+1} = z_t \equiv \bar{z}$ into the Euler equations and transition laws, and solve the resulting system of nonlinear equations for $\bar{z}$. This purpose can be accomplished, for example, by using the nonlinear equation solver `fsolve.m` in Matlab.

5.B.4. Log linear approximation

For some problems, Christiano (1990) has advocated a quadratic approximation in logarithms. We illustrate his idea with the stochastic growth example. Define

$$\tilde{a}_t = \log a_t \ , \ \tilde{\theta}_t = \log \theta_t .$$

Christiano's strategy is to take $\tilde{a}_t, \tilde{\theta}_t$ as the components of the state and write the law of motion as

$$\begin{pmatrix} 1 \\ \tilde{a}_{t+1} \\ \tilde{\theta}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{a}_t \\ \tilde{\theta}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w_{t+1}$$

where the control u_t is $\tilde{a}_{t+1}$.

Express consumption as

$$c_t = A \left(\exp \tilde{a}_t\right)^{\alpha} \left(\exp \tilde{\theta}_t\right) + (1-\delta) \exp \tilde{a}_t - \exp \tilde{a}_{t+1}.$$

Substitute this expression into $\ln c_t \equiv r(z_t)$, and proceed as before to obtain the second-order Taylor series approximation about $\bar{z}$.

5.B.5. Trend removal

It is conventional in the real business cycle literature to specify the law of motion for the technology shock θ_t by

$$\tilde{\theta}_t = \log \left(\frac{\theta_t}{\gamma^t}\right), \ \gamma > 1$$

$$\tilde{\theta}_{t+1} = \rho \tilde{\theta}_t + w_{t+1}, \qquad |\rho| < 1.$$

This inspires us to write the law of motion for capital as

$$\gamma \frac{a_{t+1}}{\gamma^{t+1}} = (1-\delta) \frac{a_t}{\gamma^t} + \frac{i_t}{\gamma^t}$$

or

$$\gamma \exp \tilde{a}_{t+1} = (1-\delta) \exp \tilde{a}_t + \exp \left(\tilde{i}_t\right)$$

where $\tilde{a}_t \equiv \log\left(\frac{a_t}{\gamma^t}\right), \tilde{i}_t = \log\left(\frac{i_t}{\gamma_t}\right)$. By studying the Euler equations for a model with a growing technology shock $(\gamma > 1)$, we can show that there exists a steady state for $\tilde{a}_t$, but not for a_t. Researchers often construct linear quadratic approximations around the nonstochastic steady state of $\tilde{a}$.

Exercises

Exercise 5.1 Consider the modified version of the optimal linear regulator problem where the objective is to maximize

$$-\sum_{t=0}^{\infty} \beta^t \left\{x_t' R x_t + u_t' Q u_t + 2 u_t' H x_t\right\}$$

subject to the law of motion:

$$x_{t+1} = A x_t + B u_t.$$

Here x_t is an $n \times 1$ state vector, u_t is a $k \times 1$ vector of controls, and x_0 is a given initial condition. The matrices R, Q are positive definite and symmetric. The maximization is with respect to sequences $\{u_t, x_t\}_{t=0}^{\infty}$.

a. Show that the optimal policy has the form

$$u_t = -\left(Q + \beta B' P B\right)^{-1} \left(\beta B' P A + H\right) x_t,$$

where P solves the algebraic matrix Riccati equation

$$P = R + \beta A' P A - \left(\beta A' P B + H'\right) \left(Q + \beta B' P B\right)^{-1} \left(\beta B' P A + H\right). \tag{1}$$

b. Write a Matlab program to solve equation (1) by iterating on P starting from P being a matrix of zeros.

Exercise 5.2 Verify that equations (5.2.10) and (5.2.11) implement the policy improvement algorithm for the discounted linear regulator problem.

Exercise 5.3 A household chooses $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize

$$-\sum_{t=0}^{\infty} \beta^t \left\{ (c_t - b)^2 + \gamma i_t^2 \right\}$$

subject to

$$\begin{aligned} c_t + i_t &= r a_t + y_t \\ a_{t+1} &= a_t + i_t \\ y_{t+1} &= \rho_1 y_t + \rho_2 y_{t-1}. \end{aligned}$$

Here c_t, i_t, a_t, y_t are the household's consumption, investment, asset holdings, and exogenous labor income at t; while $b > 0, \gamma > 0, r > 0, \beta \in (0,1)$, and ρ_1, ρ_2 are parameters, and a_0, y_0, y_{-1} are initial conditions. Assume that ρ_1, ρ_2 are such that $(1 - \rho_1 z - \rho_2 z^2) = 0$ implies $|z| > 1$.

a. Map this problem into an optimal linear regulator problem.

b. For parameter values $[\beta, (1+r), b, \gamma, \rho_1, \rho_2] = (.95, .95^{-1}, 30, 1, 1.2, -.3)$, compute the household's optimal policy function using your Matlab program from exercise *5.1*.

Exercise 5.4 Modify exercise *5.3* by assuming that the household seeks to maximize

$$-\sum_{t=0}^{\infty} \beta^t \left\{ (s_t - b)^2 + \gamma i_t^2 \right\}$$

Here s_t measures consumption services that are produced by durables or habits according to

$$\begin{aligned} s_t &= \lambda h_t + \pi c_t \\ h_{t+1} &= \delta h_t + \theta c_t \end{aligned}$$

where h_t is the stock of the durable good or habit, $(\lambda, \pi, \delta, \theta)$ are parameters, and h_0 is an initial condition.

a. Map this problem into a linear regulator problem.

b. For the same parameter values as in exercise *5.3* and $(\lambda, \pi, \delta, \theta) = (1, .05, .95, 1)$, compute the optimal policy for the household.

c. For the same parameter values as in exercise *5.3* and $(\lambda, \pi, \delta, \theta) = (-1, 1, .95, 1)$, compute the optimal policy.

d. Interpret the parameter settings in part b as capturing a model of durable consumption goods, and the settings in part c as giving a model of habit persistence.

Exercise 5.5 A household's labor income follows the stochastic process

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1} + w_{t+1} + \gamma w_t,$$

where w_{t+1} is a Gaussian martingale difference sequence with unit variance. Calculate

$$E \sum_{j=0}^{\infty} \beta^j \left[y_{t+j} | y^t, w^t \right], \tag{1}$$

where y^t, w^t denotes the history of y, w up to t.

a. Write a Matlab program to compute expression (1).

b. Use your program to evaluate expression (1) for the parameter values $(\beta, \rho_1, \rho_2, \gamma) = (.95, 1.2, -.4, .5)$.

Exercise 5.6 **Finding the state is an art**

For $t \geq 0$, the endowment for a one-good economy d_t is governed by the second order stochastic difference equation

$$d_{t+1} = \rho_0 + \rho_1 d_t + \rho_2 d_{t-1} + \sigma_d \epsilon_{t+1}$$

where ϵ_{t+1} is an i.i.d. process and $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$, ρ_0, ρ_1, and ρ_2 are scalars, and d_0, d_1 are given initial conditions. A *stochastic discount factor* is given by $s_t = \beta^t (b_0 - b_1 d_t)$, where b_0 is a positive scalar, $b_1 \geq 0$, and $\beta \in (0, 1)$. The value of the endowment at time 0 is defined to be

$$v_0 = E_0 \sum_{t=0}^{\infty} s_t d_t \tag*{(1)}$$

and E_0 is the mathematical expectation operator conditioned on d_0, d_{-1}.

a. Assume that v_0 in equation (1) is finite. Carefully describe a recursive algorithm for computing v_0.

b. Describe conditions on β, ρ_1, ρ_2 that are sufficient to make v_0 finite.

Exercise 5.7 **Dynamic Laffer curves**

The demand for currency in a small country is described by

$$M_t/p_t = \gamma_1 - \gamma_2 p_{t+1}/p_t, \tag{1}$$

where $\gamma_1 > \gamma_2 > 0$, M_t is the stock of currency held by the public at the end of period t, and p_t is the price level at time t. There is no randomness in the country, so that there is perfect foresight. Equation (1) is a Cagan-like demand function for currency, expressing real balances as an inverse function of the expected gross rate of inflation.

Speaking of Cagan, the government is running a permanent real deficit of g per period, measured in goods, all of which it finances by currency creation. The government's budget constraint at t is

$$\left(M_t - M_{t-1}\right)/p_t = g, \tag{2}$$

where the left side is the real value of the new currency printed at time t. The economy starts at time $t = 0$, with the initial level of nominal currency stock $M_{-1} = 100$ being given.

For this model, define an *equilibrium* as a pair of *positive* sequences $\{p_t > 0, M_t > 0\}_{t=0}^{\infty}$ that satisfy equations (1) and (2) (portfolio balance and the government budget constraint, respectively) for $t \geq 0$, and the initial condition assigned for M_{-1}.

a. Let $\gamma_1 = 100, \gamma_2 = 50, g = .05$. Write a computer program to compute equilibria for this economy. Describe your approach and display the program.

b. Argue that there exists a continuum of equilibria. Find the *lowest* value of the initial price level p_0 for which there exists an equilibrium. (*Hint 1:* Notice the positivity condition that is part of the definition of equilibrium. *Hint 2:* Try using the general approach to solving difference equations described in section 5.5.)

c. Show that for all of these equilibria except the one that is associated with the minimal p_0 that you calculated in part b, the gross inflation rate and the gross money creation rate both eventually converge to the *same* value. Compute this value.

d. Show that there is a unique equilibrium with a lower inflation rate than the one that you computed in part b. Compute this inflation rate.

e. Increase the level of g to $.075$. Compare the (eventual or asymptotic) inflation rate that you computed in part b and the inflation rate that you computed in part c. Are your results consistent with the view that "larger permanent deficits cause larger inflation rates"?

f. Discuss your results from the standpoint of the Laffer curve.

Hint: A Matlab program `dlqrmon.m` performs the calculations. It is available from the web site for the book.

Exercise 5.8 A government faces an exogenous stream of government expenditures $\{g_t\}$ that it must finance. Total government expenditures at t consist of two components:

$$g_t = g_{Tt} + g_{Pt} \tag{1}$$

where g_{Tt} is transitory expenditures and g_{Pt} is permanent expenditures. At the beginning of period t, the government observes the history up to t of both g_{Tt} and g_{Pt}. Further, it knows the stochastic laws of motion of both, namely,

$$\begin{aligned} g_{Pt+1} &= g_{Pt} + c_1 \epsilon_{1,t+1} \\ g_{Tt+1} &= (1-\rho)\,\mu_T + \rho g_{Tt} + c_2 \epsilon_{2t+1} \end{aligned} \tag{2}$$

where $\epsilon_{t+1} = \begin{bmatrix} \epsilon_{1t+1} \\ \epsilon_{2t+1} \end{bmatrix}$ is an i.i.d. Gaussian vector process with mean zero and identity covariance matrix. The government finances its budget with a distorting taxes. If it collects T_t total revenues at t, it bears a dead weight loss of $W(T_t)$ where $W(T) = w_1 T_t + .5 w_2 T_t^2$, where $w_1, w_2 > 0$. The government's loss functional is

$$E \sum_{t=0}^{\infty} \beta^t W(T_t), \quad \beta \in (0,1). \tag{3}$$

The government can purchase or issue one-period risk-free loans at a constant price q. Therefore, it faces a sequence of budget constraints

$$g_t + q b_{t+1} = T_t + b_t, \tag{4}$$

where q^{-1} is the gross rate of return on one-period risk-free government loans. Assume that $b_0 = 0$. The government also faces the terminal value condition

$$\lim_{t \to +\infty} \beta^t W'(T_t)\, b_{t+1} = 0,$$

which prevents it from running a Ponzi scheme. The government wants to design a tax collection strategy expressing T_t as a function of the history of g_{Tt}, g_{Pt}, b_t that *minimizes* (3) subject to (1), (2), and (4).

a. Formulate the government's problem as a dynamic programming problem. Please carefully define the state and control for this problem. Write the Bellman equation in as much detail as you can. Tell a computational strategy for solving the Bellman equation. Tell the forms of the optimal value function and the optimal decision rule.

b. Using objects that you computed in part **a**, please state the form of the law of motion for the joint process of $g_{Tt}, g_{Pt}, T_t, b_{t+1}$ under the optimal government policy.

Some background: Assume now that the optimal tax rule that you computed above has been in place for a very long time. A macroeconomist who is studying the economy observes time series on g_t, T_t, but *not* on b_t or the breakdown of g_t into its components g_{Tt}, g_{Pt}. The macroeconomist has a very long time series for $[g_t, T_t]$ and proceeds to compute a *vector autoregression* for this vector.

c. Define a population vector autoregression for the $[g_t, T_t]$ process. (Feel free to assume that lag lengths are infinite if this simplifies your answer.)

d. Please tell precisely how the vector autoregression for $[g_t, T_t]$ depends on the parameters $[\rho, \beta, \mu, q, w_1, w_2, c_1, c_2]$ that determine the joint $[g_t, T_t]$ process according to the economic theory you used in part a.

e. Now suppose that in addition to his observations on $[T_t, g_t]$, the economist gets an error-ridden time series on government debt b_t:

$$\tilde{b}_t = b_t + c_3 w_{3t+1}$$

where w_{3t+1} is an i.i.d. scalar Gaussian process with mean zero and unit variance that is orthogonal to w_{is+1} for $i = 1, 2$ for all s and t. Please tell how the vector autoregression for $[g_t, T_t, \tilde{b}_t]$ is related to the parameters $[\rho, \beta, \mu, q, w_1, w_2, c_1, c_2, c_3]$.

Is there any way to use the vector autoregression to make inferences about those parameters?

Exercise 5.9

A planner chooses a contingency plan for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$-.5E_0 \sum_{t=0}^{\infty} \beta^t \left[(c_t - b_t)^2 + e i_t^2 \right]$$

subject to the technology

$$\begin{aligned} c_t + i_t &= \gamma k_t + d_t \\ k_{t+1} &= (1-\delta) k_t + i_t, \end{aligned}$$

the laws of motion for the exogenous shock processes

$$\begin{aligned} b_{t+1} &= \mu_b (1-\rho_b) + \rho_b b_t + \sigma_b \epsilon_{b,t+1} \\ d_{t+1} &= \mu_d (1-\rho_d) + \rho_d d_t + \sigma_d \epsilon_{d,t+1}, \end{aligned}$$

and given initial conditions k_0, b_0, d_0. Here k_t is physical capital, c_t is consumption, b_t is a scalar stochastic process for bliss consumption, and d_t is an exogenous endowment process, $\beta \in (0,1)$, $e > 0$, $\delta \in (0,1)$, $\rho_b \in (0,1)$, $\rho_d \in (0,1)$, and the adjustment cost parameter $e > 0$. Also, $\begin{bmatrix} \epsilon_{b,t+1} \\ \epsilon_{d,t+1} \end{bmatrix}$ is an i.i.d. process that is distributed $\sim \mathcal{N}(0, I)$. We assume that $\beta\gamma(1-\delta) = 1$, a condition that Hall and Friedman imposed to form permanent income models of consumption. For convenience, group all parameters into the vector

$$\theta = [\, \beta \quad \delta \quad \gamma \quad e \quad \mu_b \quad \mu_d \quad \rho_b \quad \rho_d \quad \sigma_b \quad \sigma_d \,].$$

Part I. Assume that the planner knows all parameters of the model. At time t, the planner observes the history of d_s, b_s, k_s for $s \leq t$.

a. Formulate the planning problem as a discounted dynamic programming problem.

b. Use the Bellman equation for the planning problem to describe the effects on the decision rule for c_t and k_{t+1} of an increase in σ_b. Tell the effects of an increase in σ_d.

c. Describe an algorithm to solve the Bellman equation.

Part II. An econometrician observes a time series $\{c_t, i_t\}_{t=0}^T$ for the economy described in part I. (This economy is either a socialist economy with a benevolent planner or a competitive economy with complete markets.) The econometrician does not observe b_t, d_t, k_t for any t but believes that

$$\begin{bmatrix} k_0 \\ b_0 \\ d_0 \end{bmatrix} \sim \mathcal{N}\left(\mu_0, \Sigma_0\right).$$

The econometrician knows the value of β but not the remaining parameters in θ.

a. Describe as completely as you can how the econometrician can form maximum likelihood estimates of the remaining parameters in θ given his sample $\{c_t, i_t\}_{t=0}^T$. If possible, find a recursive representation of the likelihood function.

b. Suppose that the econometrician has a Bayesian prior distribution over the unknown parameters in θ. Please describe an algorithm for constructing the Bayesian posterior distribution for these parameters.

Exercise 5.10

A consumer values consumption, asset streams $\{c_t, k_{t+1}\}_{t=0}^\infty$ according to

$$-.5 E_0 \sum_{t=0}^\infty \beta^t \left(c_t - b\right)^2 \tag{1}$$

where $\beta \in (0,1)$ and

$$\begin{aligned} k_{t+1} &= R\left(k_t + y_t - c_t\right) \\ y_{t+1} &= \mu_y \left(1 - \rho_1 - \rho_2\right) + \rho_1 y_t + \rho_2 y_{t-1} + \sigma_y \epsilon_{t+1} \\ c_t &= \alpha y_t + \left(R - 1\right) k_t, \quad \alpha \in (0,1) \end{aligned}$$

and k_0, y_0, y_{-1} are given initial conditions, and ϵ_{t+1} is an i.i.d. shock with $\epsilon_{t+1} \sim \mathcal{N}(0,1)$.

a. Tell how to compute the value of the objective function (1) under the prescribed decision rule for c_t. In particular, write a Bellman equation and get as far as you can in solving it.

b. Tell how to use the Howard policy improvement algorithm to get a better decision rule.

Exercise 5.11 **Firm level adjustment costs**

A competitive firms sells output y_t at price p_t and chooses a production plan to maximize

$$\sum_{t=0}^{\infty} \beta^t R_t \tag{1}$$

where

$$R_t = p_t y_t - .5d\,(y_{t+1} - y_t)^2 \tag{2}$$

subject to y_0 being a given initial condition. Here $\beta \in (0,1)$ is a discount factor, and $d > 0$ measures a cost of adjusting the rate of output. The firm is a price taker. The price p_t lies on the demand curve

$$p_t = A_0 - A_1 Y_t \tag{3}$$

where $A_0 > 0, A_1 > 0$ and Y_t is the market-wide level of output, being the sum of output of n identical firms. The firm believes that market-wide output follows the law of motion

$$Y_{t+1} = H_0 + H_1 Y_t \tag{4}$$

where Y_0 is a known initial condition. The firm observes Y_t and y_t at time t when it chooses y_{t+1}.

a. Formulate a Bellman equation for the firm.

b. For parameter values $\beta = .95, d = 2, A_0 = 100, A_1 = 1, H_0 = 200, H_1 = .8$, compute the firm's optimal value function and optimal decision rule.

Exercise 5.12 **Firm level adjustment cost, II**

A competitive firms sells output y_t at price p_t and chooses a production plan to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t R_t \tag{1}$$

where E_0 denotes a mathematical expectation conditional on time 0 information,

$$R_t = p_t y_t - .5d\,(y_{t+1} - y_t)^2 \tag{2}$$

subject to y_0 being a given initial condition. Here $\beta \in (0,1)$ is a discount factor, and $d > 0$ measures a cost of adjusting the rate of output. The firm is a price taker. The price p_t lies on the demand curve

$$p_t = A_0 - A_1 Y_t + u_t \tag{3}$$

where $A_0 > 0, A_1 > 0$ and Y_t is the market-wide level of output, being the sum of output of n identical firms. In (3), u_t is a demand shock that follows the first-order autoregressive process

$$u_{t+1} = \rho u_t + \sigma_u \epsilon_{t+1} \tag{4}$$

where ϵ_{t+1} is an i.i.d. scalar process with $\epsilon_{t+1} \sim \mathcal{N}(0,1)$ and $|\rho| < 1$. The firm believes that market-wide output follows the law of motion

$$Y_{t+1} = H_0 + H_1 Y_t + H_2 u_t \tag{5}$$

where Y_0 is a known initial condition. The firm observes P_t, Y_t, and y_t at time t when it chooses y_{t+1}.

a. Formulate a Bellman equation for the firm.

b. For parameter values $\beta = .95, d = 2, A_0 = 100, A_1 = 1, H_0 = 200, H_1 = .8, H_2 = 2, \rho = .9, \sigma_u = .05$, compute the firm's optimal value function and optimal decision rule.

Exercise 5.13 **Permanent income model**

A household chooses a process $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \{-.5\,(c_t - b)^2 - .5\epsilon a_t^2\}, \quad \beta \in (0,1)$$

subject to

$$\begin{aligned} a_{t+1} + c_t &= Ra_t + y_t \\ y_{t+1} &= (1 - \rho_1 - \rho_2) + \rho_1 y_t + \rho_2 y_{t-1} + \sigma_y \epsilon_{t+1} \end{aligned}$$

where c_t is consumption, $b > 0$ is a bliss level of consumption, a_t is financial assets at the beginning of t, $R = \beta^{-1}$ is the gross rate of return on assets held from t to $t+1$, and ϵ_{t+1} is an i.i.d. scalar process with $\epsilon_{t+]} \sim \mathcal{N}(0,1)$. The household faces known initial conditions a_0, y_0, y_{-1}.

a. Write a Bellman equation for the household's problem.

b. Compute the household's value function and optimal decision rule for the following parameter values: $b = 1000, \beta = .95, R = \beta^{-1}, \rho_1 = 1.2, \rho_2 = -.4, \sigma_y = .05, \epsilon = .000001$.

c. Compute the eigenvalues of $A - BF$.

d. Compute the household's value function and optimal decision rule for the following parameter values: $b = 1000, \beta = .95, R = \beta^{-1}, \rho_1 = 1.2, \rho_2 = -.4, \sigma_y = .05, \epsilon = 0$. Compare what you obtain with your answers in part **b**.

Exercise 5.14 **Permanent income model again**

A household chooses a process $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \{-.5 \left(c_t - b\right)^2 - .5 \epsilon a_t^2\}, \quad \beta \in (0,1)$$

subject to

$$\begin{aligned} a_{t+1} + c_t &= R a_t + y_t \\ y_{t+1} &= (1 - \rho_1 - \rho_2) + \rho_1 y_t + \rho_2 y_{t-3} + \sigma_y \epsilon_{t+1} \end{aligned}$$

where c_t is consumption, $b > 0$ is a bliss level of consumption, a_t is financial assets at the beginning of t, $R = \beta^{-1}$ is the gross rate of return on assets held from t to $t+1$, and ϵ_{t+1} is an i.i.d. scalar process with $\epsilon_{t+1} \sim \mathcal{N}(0,1)$. The household faces known initial conditions $a_0, y_0, y_{-1}, y_{-2}, y_{-3}$.

a. Write a Bellman equation for the household's problem.

b. Compute the household's value function and optimal decision rule for the following parameter values: $b = 1000, \beta = .95, R = \beta^{-1}, \rho_1 = .55, \rho_2 = .3, \sigma_y = .05, \epsilon = .000001$.

c. Compute the eigenvalues of $A - BF$.

d. Compute the household's value function and optimal decision rule for the following parameter values: $b = 1000, \beta = .95, R = \beta^{-1}, \rho_1 = .55, \rho_2 = .3, \sigma_y = .05, \epsilon = 0$. Compare what you obtain with your answers in part **b**.

Chapter 6
Search, Matching, and Unemployment

6.1. Introduction

This chapter applies dynamic programming to a choice between only two actions, to accept or reject a take-it-or-leave-it job offer. An unemployed worker faces a probability distribution of wage offers or job characteristics, from which a limited number of offers are drawn each period. Given his perception of the probability distribution of offers, the worker must devise a strategy for deciding when to accept an offer.

The theory of search is a tool for studying unemployment. Search theory puts unemployed workers in a setting where sometimes they choose to reject available offers and to remain unemployed now because they prefer to wait for better offers later. We use the theory to study how workers respond to variations in the rate of unemployment compensation, the perceived riskiness of wage distributions, the probability of being fired, the quality of information about jobs, and the frequency with which the wage distribution can be sampled.

This chapter provides an introduction to the techniques used in the search literature and a sampling of search models. The chapter studies ideas introduced in two important papers by McCall (1970) and Jovanovic (1979a). These papers differ in the search technologies with which they confront an unemployed worker.[1] We also study a related model of occupational choice by Neal (1999).

We hope to convey some of the excitement that Robert E. Lucas, Jr. (1987, p.57) expressed when he wrote this about the McCall search model: "Questioning a McCall worker is like having a conversation with an out-of-work friend: 'Maybe you are setting your sights too high' or 'Why did you quit your old job before you had a new one lined up?' This is real social science: an attempt to

[1] Stigler's (1961) important early paper studied a search technology different from both McCall's and Jovanovic's. In Stigler's model, an unemployed worker has to choose in advance a number n of offers to draw, from which he takes the highest wage offer. Stigler's formulation of the search problem was not sequential.

model, to *understand*, human behavior by visualizing the situations people find themselves in, the options they face and the pros and cons as they themselves see them." The modifications of the basic McCall model by Jovanovic, Neal, and in the various sections and exercises of this chapter all come from visualizing aspects of the situations in which workers find themselves.

6.2. Preliminaries

This section describes elementary properties of probability distributions that are used extensively in search theory.

6.2.1. Nonnegative random variables

We begin with some properties of nonnegative random variables that possess first moments. Consider a random variable p with a cumulative probability distribution function $F(P)$ defined by $\text{Prob}\{p \leq P\} = F(P)$. We assume that $F(0) = 0$, that is, that p is nonnegative. We assume that F, a nondecreasing function, is continuous from the right. We also assume that there is an upper bound $B < \infty$ such that $F(B) = 1$, so that p is bounded with probability 1.

The mean of p, Ep, is defined by

$$Ep = \int_0^B p \; dF(p) . \tag{6.2.1}$$

Let $u = 1 - F(p)$ and $v = p$ and use the integration-by-parts formula $\int_a^b u \; dv = uv\Big|_a^b - \int_a^b v \; du$, to verify that

$$\int_0^B [1 - F(p)] \, dp = \int_0^B p \; dF(p) .$$

Thus, we have the following formula for the mean of a nonnegative random variable:

$$Ep = \int_0^B [1 - F(p)] \, dp = B - \int_0^B F(p) \, dp. \tag{6.2.2}$$

Now consider two independent random variables p_1 and p_2 drawn from the distribution F. Consider the event $\{(p_1 < p) \cap (p_2 < p)\}$, which by the independence assumption has probability $F(p)^2$. The event $\{(p_1 < p) \cap (p_2 < p)\}$ is equivalent to the event $\{\max(p_1, p_2) < p\}$, where "max" denotes the maximum. Therefore, if we use formula (6.2.2), the random variable $\max(p_1, p_2)$ has mean

$$E \max (p_1, p_2) = B - \int_0^B F(p)^2 \, dp. \tag{6.2.3}$$

Similarly, if $p_1, p_2, \ldots, p_n$ are n independent random variables drawn from F, we have $\mathrm{Prob}\{\max(p_1, p_2, \ldots, p_n) < p\} = F(p)^n$ and

$$M_n \equiv E \max (p_1, p_2, \ldots, p_n) = B - \int_0^B F(p)^n \, dp, \tag{6.2.4}$$

where M_n is defined as the expected value of the maximum of $p_1, \ldots, p_n$.

6.2.2. Mean-preserving spreads

Rothschild and Stiglitz introduced the idea of a mean-preserving spread as a convenient way to characterize the riskiness of two distributions with the same mean. Consider a class of distributions with the same mean. We index this class by a parameter r belonging to some set R. For the rth distribution we denote $\mathrm{Prob}\{p \leq P\} = F(P, r)$ and assume that $F(P, r)$ is differentiable with respect to r for all $P \in [0, B]$. We assume that there is a single finite B such that $F(B, r) = 1$ for all r in R and that $F(0, r) = 0$ for all r in R, so that we are considering a class of distributions R for nonnegative, bounded random variables.

From equation (6.2.2), we have

$$Ep = B - \int_0^B F(p, r) \, dp. \tag{6.2.5}$$

Therefore, two distributions with the same value of $\int_0^B F(\theta, r) d\theta$ have identical means. We write this as the identical means condition:

(i) $$\int_0^B [F(\theta, r_1) - F(\theta, r_2)] \, d\theta = 0.$$

Two distributions r_1, r_2 are said to satisfy the *single-crossing property* if there exists a $\hat{\theta}$ with $0 < \hat{\theta} < B$ such that

(ii) $$F(\theta, r_2) - F(\theta, r_1) \leq 0 \, (\geq 0) \qquad \text{when} \quad \theta \geq (\leq) \, \hat{\theta}.$$

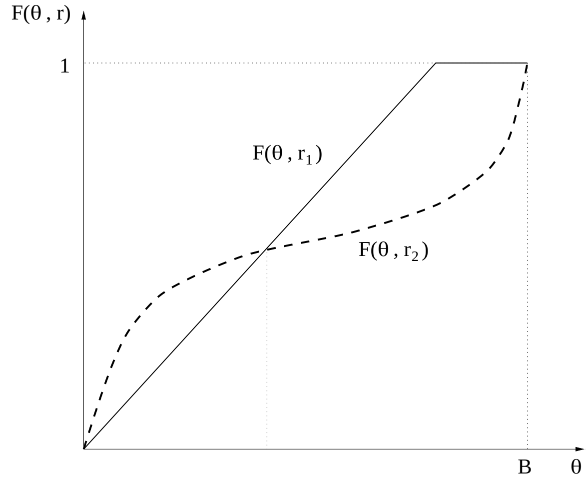

Figure 6.2.1: Two distributions, r_1 and r_2, that satisfy the single-crossing property.

Figure 6.2.1 illustrates the single-crossing property. If two distributions r_1 and r_2 satisfy properties (i) and (ii), we can regard distribution r_2 as having been obtained from r_1 by a process that shifts probability toward the tails of the distribution while keeping the mean constant.

Properties (i) and (ii) imply the following property:

$$\text{(iii)} \qquad \int_0^y \left[F(\theta, r_2) - F(\theta, r_1)\right] d\theta \geq 0, \qquad 0 \leq y \leq B\,.$$

Rothschild and Stiglitz regard properties (i) and (iii) as defining the concept of a "mean-preserving spread." In particular, a distribution indexed by r_2 is said to have been obtained from a distribution indexed by r_1 by a mean-preserving spread if the two distributions satisfy (i) and (iii).[2]

[2] Rothschild and Stiglitz (1970, 1971) use properties (i) and (iii) to characterize mean-preserving spreads rather than (i) and (ii) because (i) and (ii) fail to possess transitivity. That is, if $F(\theta, r_2)$ is obtained from $F(\theta, r_1)$ via a mean-preserving spread in the sense that the term has in (i) and (ii), and $F(\theta, r_3)$ is obtained from $F(\theta, r_2)$ via a mean-preserving spread in the sense of (i) and (ii), it does not follow that $F(\theta, r_3)$ satisfies the single-crossing property (ii) vis-à-vis distribution $F(\theta, r_1)$. A definition based on (i) and (iii), however, does provide a transitive ordering, which is a desirable feature for a definition designed to order distributions according to their riskiness.

For infinitesimal changes in r, Diamond and Stiglitz use the differential versions of properties (i) and (iii) to rank distributions with the same mean in order of riskiness. An increment in r is said to represent a mean-preserving increase in risk if

(iv) $$\int_0^B F_r(\theta, r)\, d\theta = 0$$

(v) $$\int_0^y F_r(\theta, r)\, d\theta \geq 0, \qquad 0 \leq y \leq B\ ,$$

where $F_r(\theta, r) = \partial F(\theta, r)/\partial r$.

6.3. McCall's model of intertemporal job search

We now consider an unemployed worker who is searching for a job under the following circumstances: Each period the worker draws one offer w from the same wage distribution $F(W) = \text{Prob}\{w \leq W\}$, with $F(0) = 0$, $F(B) = 1$ for $B < \infty$. The worker has the option of rejecting the offer, in which case he or she receives c this period in unemployment compensation and waits until next period to draw another offer from F; alternatively, the worker can accept the offer to work at w, in which case he or she receives a wage of w per period forever. Neither quitting nor firing is permitted.

Let y_t be the worker's income in period t. We have $y_t = c$ if the worker is unemployed and $y_t = w$ if the worker has accepted an offer to work at wage w. The unemployed worker devises a strategy to maximize the mathematical expectation of $\sum_{t=0}^{\infty} \beta^t y_t$ where $0 < \beta < 1$ is a discount factor.

Let $v(w)$ be the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ for a worker who has offer w in hand, who is deciding whether to accept or to reject it, and who behaves optimally. We assume no recall. The value function $v(w)$ satisfies the Bellman equation

$$v(w) = \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta}, c + \beta \int_0^B v(w')\, dF(w') \right\}, \tag{6.3.1}$$

where the maximization is over the two actions: (1) *accept* the wage offer w and work forever at wage w, or (2) *reject* the offer, receive c this period, and draw a new offer w' from distribution F next period. Figure 6.3.1 graphs the functional equation (6.3.1) and reveals that its solution is of the form

$$v(w) = \begin{cases} \dfrac{\overline{w}}{1-\beta} = c + \beta \displaystyle\int_0^B v(w')\,dF(w') & \text{if} \quad w \le \overline{w} \\ \dfrac{w}{1-\beta} & \text{if} \quad w \ge \overline{w}. \end{cases} \tag{6.3.2}$$

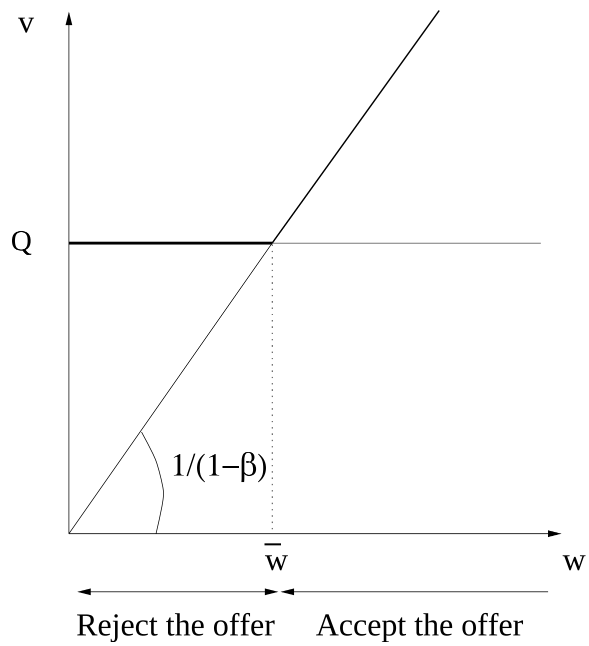

Figure 6.3.1: The function $v(w) = \max\{w/(1-\beta), c + \beta \int_0^B v(w')dF(w')\}$. The reservation wage $\overline{w} = (1-\beta)[c + \beta \int_0^B v(w')dF(w')]$.

Using equation (6.3.2), we can convert the functional equation (6.3.1) in the value function $v(w)$ into an ordinary equation in the reservation wage $\overline{w}$. Evaluating $v(\overline{w})$ and using equation (6.3.2), we have

$$\frac{\overline{w}}{1-\beta} = c + \beta \int_0^{\overline{w}} \frac{\overline{w}}{1-\beta} dF(w') + \beta \int_{\overline{w}}^B \frac{w'}{1-\beta} dF(w')$$

or

$$
\begin{aligned}
&\frac{\overline{w}}{1-\beta}\int_0^{\overline{w}} dF(w') + \frac{\overline{w}}{1-\beta}\int_{\overline{w}}^{B} dF(w') \\
&\qquad = c + \beta\int_0^{\overline{w}} \frac{\overline{w}}{1-\beta} dF(w') + \beta\int_{\overline{w}}^{B} \frac{w'}{1-\beta} dF(w')
\end{aligned}
$$

or

$$
\overline{w}\int_0^{\overline{w}} dF(w') - c = \frac{1}{1-\beta}\int_{\overline{w}}^{B} (\beta w' - \overline{w})\, dF(w').
$$

Adding $\overline{w}\int_{\overline{w}}^{B} dF(w')$ to both sides gives

$$
(\overline{w} - c) = \frac{\beta}{1-\beta}\int_{\overline{w}}^{B} (w' - \overline{w})\, dF(w'). \tag{6.3.3}
$$

Equation (6.3.3) is often used to characterize the reservation wage $\overline{w}$. The left side is the cost of searching one more time when an offer $\overline{w}$ is in hand. The right side is the expected benefit of searching one more time in terms of the expected present value associated with drawing $w' > \overline{w}$. Equation (6.3.3) instructs the agent to set $\overline{w}$ so that the cost of searching one more time equals the benefit.

6.3.1. *Characterizing reservation wage*

Let us define the function on the right side of equation (6.3.3) as

$$
h(w) = \frac{\beta}{1-\beta}\int_{w}^{B} (w' - w)\, dF(w'). \tag{6.3.4}
$$

Notice that $h(0) = Ew\beta/(1-\beta)$, that $h(B) = 0$, and that $h(w)$ is differentiable, with derivative given by[3]

$$
h'(w) = -\frac{\beta}{1-\beta}\left[1 - F(w)\right] < 0.
$$

---

[3] To compute $h'(w)$, we apply Leibniz's rule to equation (6.3.4). Let $\phi(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)dx$ for $t \in [c,d]$. Assume that f and f_t are continuous and that α, β are differentiable on $[c,d]$. Then Leibniz's rule asserts that $\phi(t)$ is differentiable on $[c,d]$ and

$$
\phi'(t) = f[\beta(t), t]\,\beta'(t) - f[\alpha(t), t]\,\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} f_t(x,t)\, dx.
$$

To apply this formula to the equation in the text, let w play the role of t.

We also have

$$h''(w) = \frac{\beta}{1-\beta} F'(w) > 0,$$

so that $h(w)$ is convex to the origin. Figure 6.3.2 graphs $h(w)$ against $(w-c)$ and indicates how $\overline{w}$ is determined. From Figure 6.3.2 it is apparent that an increase in unemployment compensation c leads to an increase in $\overline{w}$.

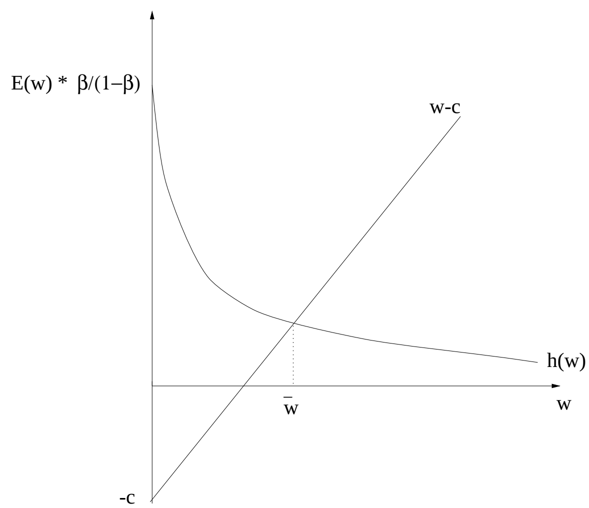

Figure 6.3.2: The reservation wage, $\overline{w}$, that satisfies $\overline{w} - c = [\beta/(1-\beta)] \int_{\overline{w}}^{B} (w' - \overline{w}) dF(w') \equiv h(\overline{w})$.

To get another useful characterization of $\overline{w}$, we express equation (6.3.3) as

$$\begin{aligned}
\overline{w} - c =& \frac{\beta}{1-\beta} \int_{\overline{w}}^{B} (w' - \overline{w})\, dF(w') + \frac{\beta}{1-\beta} \int_{0}^{\overline{w}} (w' - \overline{w})\, dF(w') \\
& - \frac{\beta}{1-\beta} \int_{0}^{\overline{w}} (w' - \overline{w})\, dF(w') \\
=& \frac{\beta}{1-\beta} Ew - \frac{\beta}{1-\beta} \overline{w} - \frac{\beta}{1-\beta} \int_{0}^{\overline{w}} (w' - \overline{w})\, dF(w')
\end{aligned}$$

or

$$\overline{w} - (1-\beta) c = \beta E w - \beta \int_{0}^{\overline{w}} (w' - \overline{w})\, dF(w').$$

Applying integration by parts to the last integral on the right side and rearranging, we have

$$\overline{w} - c = \beta \left(Ew - c \right) + \beta \int_0^{\overline{w}} F \left(w' \right) dw'. \tag{6.3.5}$$

At this point it is useful to define the function

$$g \left(s \right) = \int_0^s F \left(p \right) dp. \tag{6.3.6}$$

This function has the characteristics that $g(0) = 0$, $g(s) \geq 0$, $g'(s) = F(s) > 0$, and $g''(s) = F'(s) > 0$ for $s > 0$. Then equation (6.3.5) can be represented as $\overline{w} - c = \beta(Ew - c) + \beta g(\overline{w})$. Figure 6.3.3 uses equation (6.3.5) to determine $\overline{w}$.

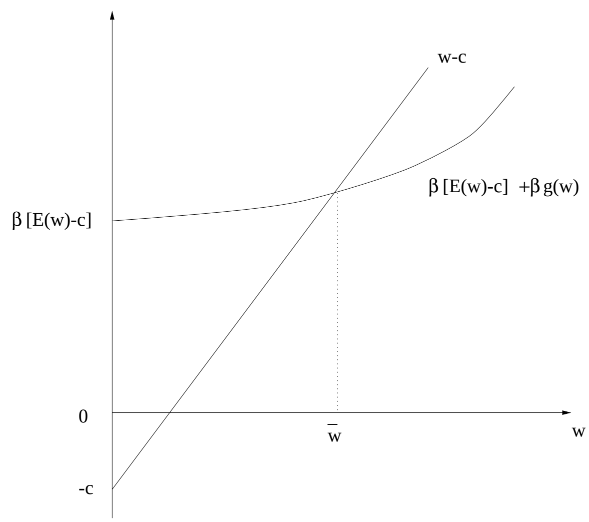

Figure 6.3.3: The reservation wage, $\overline{w}$, that satisfies $\overline{w} - c = \beta(Ew - c) + \beta \int_0^{\overline{w}} F(w')dw' \equiv \beta(Ew - c) + \beta g(\overline{w})$.

6.3.2. Effects of mean-preserving spreads

Figure 6.3.3 can be used to establish two propositions about $\overline{w}$. First, given F, $\overline{w}$ increases when the rate of unemployment compensation c increases. Second, given c, a mean-preserving increase in risk causes $\overline{w}$ to increase. This second proposition follows directly from Figure 6.3.3 and the characterization (iii) or (v) of a mean-preserving increase in risk. From the definition of g in equation (6.3.6) and the characterization (iii) or (v), a mean-preserving spread causes an upward shift in $\beta(Ew-c)+\beta g(w)$.

Since either an increase in unemployment compensation or a mean-preserving increase in risk raises the reservation wage, it follows from the expression for the value function in equation (6.3.2) that unemployed workers are also better off in those situations. It is obvious that an increase in unemployment compensation raises the welfare of unemployed workers but it might seem surprising that a mean-preserving increase in risk does too. Intuition for this latter finding can be gleaned from the result in option pricing theory that the value of an option is an increasing function of the variance in the price of the underlying asset. This is so because the option holder chooses to accept payoffs only from the right tail of the distribution. In our context, the unemployed worker has the option to accept a job and the asset value of a job offering wage rate w is equal to $w/(1-\beta)$. Under a mean-preserving increase in risk, the higher incidence of very good wage offers increases the value of searching for a job while the higher incidence of very bad wage offers is less detrimental because the option to work will not be exercised at such low wages.

6.3.3. Allowing quits

Thus far, we have supposed that the worker cannot quit. It happens that had we given the worker the option to quit and search again, after being unemployed one period, he would never exercise that option. To see this point, recall that the reservation wage $\overline{w}$ in (6.3.2) satisfies

$$v\left(\overline{w}\right)=\frac{\overline{w}}{1-\beta}=c+\beta\int v\left(w'\right)dF\left(w'\right). \tag{6.3.7}$$

Suppose the agent has in hand an offer to work at wage w. Assuming that the agent behaves optimally after any rejection of a wage w, we can compute the lifetime utility associated with three mutually exclusive alternative ways of responding to that offer:

A1. Accept the wage and keep the job forever:

$$\frac{w}{1-\beta}.$$

A2. Accept the wage but quit after t periods:

$$\frac{w-\beta^t w}{1-\beta}+\beta^t\left(c+\beta\int v\left(w'\right)dF\left(w'\right)\right)=\frac{w}{1-\beta}-\beta^t\frac{w-\overline{w}}{1-\beta}.$$

A3. Reject the wage:

$$c+\beta\int v\left(w'\right)dF\left(w'\right)=\frac{\overline{w}}{1-\beta}.$$

We conclude that if $w<\overline{w}$,

$$A1 \prec A2 \prec A3,$$

and if $w>\overline{w}$,

$$A1 \succ A2 \succ A3.$$

The three alternatives yield the same lifetime utility when $w=\overline{w}$.

6.3.4. Waiting times

It is straightforward to derive the probability distribution of the waiting time until a job offer is accepted. Let N be the random variable "length of time until a successful offer is encountered," with the understanding that $N=1$ if the first job offer is accepted. Let $\lambda=\int_0^{\overline{w}} dF(w')$ be the probability that a job offer is rejected. Then we have $\text{Prob}\{N=1\}=(1-\lambda)$. The event that $N=2$ is the event that the first draw is less than $\overline{w}$, which occurs with probability λ, and that the second draw is greater than $\overline{w}$, which occurs with probability $(1-\lambda)$. By virtue of the independence of successive draws, we have $\text{Prob}\{N=2\}=(1-\lambda)\lambda$. More generally, $\text{Prob}\{N=j\}=(1-\lambda)\lambda^{j-1}$, so the waiting time is geometrically distributed. The mean waiting time $\bar{N}$ is given by

$$\begin{aligned}\bar{N}&=\sum_{j=1}^{\infty} j\cdot\ \text{Prob}\{N=j\}=\sum_{j=1}^{\infty} j\left(1-\lambda\right)\lambda^{j-1}=\left(1-\lambda\right)\sum_{j=1}^{\infty}\sum_{k=1}^{j}\lambda^{j-1}\\&=\left(1-\lambda\right)\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\lambda^{j-1+k}=\left(1-\lambda\right)\sum_{k=0}^{\infty}\lambda^{k}\left(1-\lambda\right)^{-1}=\left(1-\lambda\right)^{-1}.\end{aligned}$$

That is, the mean waiting time to a successful job offer equals the reciprocal of the probability of an accepted offer on a single trial.[4]

As an illustration of the power of using a recursive approach, we can also compute the mean waiting time $\bar{N}$ as follows. First, given that our search environment is stationary and therefore is associated with a constant reservation wage and a constant probability of escaping unemployment, it follows that the "remaining" mean waiting time for all unemployed workers is equal to $\bar{N}$ in any given period. That is, all unemployed workers face a remaining mean waiting time of $\bar{N}$ regardless of how long of an unemployment spell they have suffered so far. Second, the mean waiting time $\bar{N}$ must then be equal to the weighted sum of two possible outcomes: either the worker accepts a job next period, with probability $(1-\lambda)$; or she remains unemployed in the next period, with probability λ. In the first case, the worker will have ended her unemployment after one last period of unemployment while in the second case, the worker will have suffered one period of unemployment *and* will face a remaining mean waiting time of $\bar{N}$ periods. Hence, the mean waiting time must satisfy the following recursive formula:

$$\bar{N} = (1-\lambda)\cdot 1 + \lambda \cdot \big(1+\bar{N}\big) \qquad \Longrightarrow \qquad \bar{N} = (1-\lambda)^{-1}.$$

We invite the reader to prove that, given F, the mean waiting time increases with increases in the rate of unemployment compensation, c.

[4] An alternative way of deriving the mean waiting time is to use the algebra of z transforms; we say that $h(z)=\sum_{j=0}^{\infty} h_j z^j$ and note that $h'(z)=\sum_{j=1}^{\infty} j h_j z^{j-1}$ and $h'(1)=\sum_{j=1}^{\infty} j h_j$. (For an introduction to z transforms, see Gabel and Roberts, 1973.) The z transform of the sequence $(1-\lambda)\lambda^{j-1}$ is given by $\sum_{j=1}^{\infty}(1-\lambda)\lambda^{j-1}z^j = (1-\lambda)z/(1-\lambda z)$. Evaluating $h'(z)$ at $z=1$ gives, after some simplification, $h'(1)=1/(1-\lambda)$. Therefore, we have that the mean waiting time is given by $(1-\lambda)\sum_{j=1}^{\infty} j\lambda^{j-1} = 1/(1-\lambda)$.

6.3.5. Firing

We now consider a modification of the job search model in which each period after the first period on the job the worker faces probability α of being fired, where $1 > \alpha > 0$. The probability α of being fired next period is assumed to be independent of tenure. A previously unemployed worker samples wage offers from a time-invariant and known probability distribution F. Unemployed workers receive unemployment compensation in the amount c. The worker receives a time-invariant wage w on a job until she is fired. A worker who is fired becomes unemployed for one period before drawing a new wage. Only previously employed workers are fired. A previously employed worker who is fired at the beginning of a period cannot draw a new wage offer that period but must be unemployed for one period.

We let $\hat{v}(w)$ be the expected present value of income of a previously unemployed worker who has offer w in hand and who behaves optimally. If she rejects the offer, she receives c in unemployment compensation this period and next period draws a new offer w', whose value to her now is $\beta \int \hat{v}(w')dF(w')$. If she rejects the offer, $\hat{v}(w) = c + \beta \int \hat{v}(w')dF(w')$. If she accepts the offer, she receives w this period; next period with probability $1-\alpha$, she is not fired and therefore what she receives is worth $\beta\hat{v}(w)$ today; with probability α, she is fired next period, which has the consequence that after one period of unemployment she draws a new wage, an outcome that today is worth $\beta[c + \beta \int \hat{v}(w')dF(w')]$. Therefore, if she accepts the offer, $\hat{v}(w) = w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha[c + \beta \int \hat{v}(w')dF(w')]$. Thus, the Bellman equation becomes[5]

$$\hat{v}(w) = \max_{\text{accept,reject}} \left\{ w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha[c + \beta E\hat{v}],\ c + \beta E\hat{v} \right\},$$

where $E\hat{v} = \int \hat{v}(w')dF(w')$. Here the appearance of $\hat{v}(w)$ on the right side recognizes that if the worker had accepted wage offer w last period with expected discounted present value $\hat{v}(w)$, the stationarity of the problem (i.e., the fact that

[5] If a worker who is fired at the beginning of a period were to have the opportunity to draw a new offer that same period, then the Bellman equation would instead be

$$\tilde{v}(w) = \max_{\text{accept,reject}} \left\{ w + \beta(1-\alpha)\tilde{v}(w) + \beta\alpha \int \tilde{v}(w')\,dF(w'),\, c + \beta \int \tilde{v}(w')\,dF(w') \right\}.$$

F, α, c are all fixed) makes $\hat{v}(w)$ also be the continuation value associated with retaining this job next period. This equation has a solution of the form[6]

$$\hat{v}(w) = \begin{cases} \frac{w + \beta\alpha [c + \beta E\hat{v}]}{1 - \beta(1-\alpha)}, & \text{if } w \geq \overline{w} \\ c + \beta E\hat{v}, & w \leq \overline{w} \end{cases}$$

where $\overline{w}$ solves

$$\frac{\overline{w} + \beta\alpha [c + \beta E\hat{v}]}{1 - \beta(1-\alpha)} = c + \beta E\hat{v},$$

which can be rearranged as

$$\frac{\overline{w}}{1-\beta} = c + \beta \int \hat{v}(w')\, dF(w'). \tag{6.3.8}$$

We can compare the reservation wage in (6.3.8) to the reservation wage in expression (6.3.7) when there was no risk of being fired. The two expressions look identical but the reservation wages differ because the value functions differ. In particular, $\hat{v}(w)$ is strictly less than $v(w)$. This is an immediate implication of our argument that it cannot be optimal to quit if you have accepted a wage strictly greater than the reservation wage in the situation without possible firings (see section 6.3.3). So even though workers who face no possible firings can mimic outcomes in situations where they would facing possible firings by occasionally "firing themselves" by quitting into unemployment, they choose not to do so because that would lower their expected present value of income. Since the employed workers in the situation where they face possible firings are worse off than employed workers in the situation without possible firings, it follows that $\hat{v}(w)$ lies strictly below $v(w)$ over the whole domain because, even at wages that are rejected, the value function partly reflects a stream of future outcomes whose expectation is less favorable in the situation in which workers face a chance of being fired.

Since the value function $\hat{v}(w)$ with firings lies strictly below the value function $v(w)$ without firings, it follows from (6.3.8) and (6.3.7) that the reservation wage $\overline{w}$ is strictly lower with firings. There is less of a reason to hold out for high-paying jobs when a job is expected to last for a shorter period of time.

[6] That it takes this form can be established by guessing that $\hat{v}(w)$ is nondecreasing in w. This guess implies the equation in the text for $\hat{v}(w)$, which is nondecreasing in w. This argument verifies that $\hat{v}(w)$ is nondecreasing, given the uniqueness of the solution of the Bellman equation.

That is, unemployed workers optimally invest less in search when the payoffs associated with wage offers have gone down because of the probability of being fired.

6.4. A lake model

Consider an economy consisting of a continuum of *ex ante* identical workers living in the environment described in the previous section. These workers move recurrently between unemployment and employment. The mean duration of each spell of employment is α^{-1} and the mean duration of unemployment is $[1 - F(\overline{w})]^{-1}$. The average unemployment rate U_t across the continuum of workers obeys the difference equation

$$U_{t+1} = \alpha\,(1 - U_t) + F\,(\overline{w})\,U_t,$$

where α is the hazard rate of escaping employment and $[1-F(\overline{w})]$ is the hazard rate of escaping unemployment. Solving this difference equation for a stationary solution, i.e., imposing $U_{t+1} = U_t = U$, gives

$$U = \frac{\alpha}{\alpha + 1 - F\,(\overline{w})} \qquad \Longrightarrow \qquad U = \frac{\dfrac{1}{1 - F\,(\overline{w})}}{\dfrac{1}{1 - F\,(\overline{w})} + \dfrac{1}{\alpha}}. \tag{6.4.1}$$

Equation (6.4.1) expresses the stationary unemployment rate in terms of the ratio of the average duration of unemployment to the sum of average durations of unemployment and employment. The unemployment rate, being an average across workers at each moment, thus reflects the average outcomes experienced by workers *across time*. This way of linking economy-wide averages at a point in time with the time-series average for a representative worker is our first encounter with a class of models sometimes referred to as Bewley models, which we shall study in depth in chapter 18.

This model of unemployment is sometimes called a lake model and can be depicted as in Figure 6.4.1, with two lakes denoted U and $1 - U$ representing volumes of unemployment and employment, and streams of rate α from the $1 - U$ lake to the U lake and of rate $1 - F(\overline{w})$ from the U lake to the $1 - U$ lake. Equation (6.4.1) allows us to study the determinants of the unemployment rate in terms of the hazard rate of becoming unemployed α and the hazard rate of escaping unemployment $1 - F(\overline{w})$.

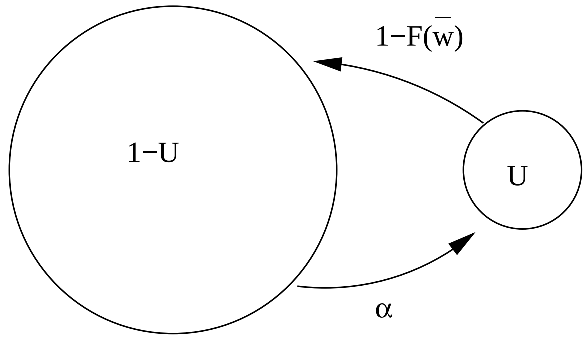

Figure 6.4.1: Lake model with flows of rate α from employment state $1-U$ to unemployment state U and of rate $[1-F(\overline{w})]$ from U to $1-U$.

6.5. A model of career choice

This section describes a model of occupational choice that Derek Neal (1999) used to study the employment histories of recent high school graduates. Neal wanted to explain why young men switch jobs *and* careers often early in their work histories, then later focus their searches for jobs within a single career, and finally settle down in a particular job. Neal's model can be regarded as a simplified version of Brian McCall's (1991) model.

A worker chooses career-job (θ, ϵ) pairs subject to the following conditions: There is no unemployment. The worker's earnings at time t equal $\theta_t + \epsilon_t$, where θ_t is a component specific to a *career* and ϵ_t is a component specific to a particular *job*. The worker maximizes $E\sum_{t=0}^{\infty}\beta^t(\theta_t+\epsilon_t)$. A *career* is a draw of θ from c.d.f. F; a *job* is a draw of ϵ from c.d.f. G. Successive draws are independent, and $G(0) = F(0) = 0$, $G(B_\epsilon) = F(B_\theta) = 1$. The worker can draw a new career only if he also draws a new job. However, the worker is free to retain his existing career (θ), and to draw a new job (ϵ'). The worker decides at the beginning of a period whether to stay in a career-job pair inherited from the past, stay in the inherited career but draw a new job, or draw a new career-job pair. There is no opportunity to recall past jobs or careers.

Let $v(\theta, \epsilon)$ be the optimal value of the problem at the beginning of a period for a worker currently having inherited career-job pair (θ, ϵ) and who is about to decide whether to draw a new career and or job. The Bellman equation is

$$v(\theta,\epsilon) = \max\left\{\theta+\epsilon+\beta v(\theta,\epsilon),\ \theta+\int\left[\epsilon'+\beta v(\theta,\epsilon')\right]dG(\epsilon'),\right.$$
$$\left.\int\int\left[\theta'+\epsilon'+\beta v(\theta',\epsilon')\right]dF(\theta')\,dG(\epsilon')\right\}. \tag{6.5.1}$$

The maximization is over the three possible actions: (1) retain the present job-career pair; (2) retain the present career but draw a new job; and (3) draw both a new job and a new career. We might nickname these three alternatives 'stay put', 'new job', 'new life'. The value function is increasing in both θ and ϵ.

Figures 6.5.1 and 6.5.2 display the optimal value function and the optimal decision rule for Neal's model where F and G are each distributed according to discrete uniform distributions on $[0,5]$ with 50 evenly distributed discrete values for each of θ and ϵ and $\beta = .95$. We computed the value function by iterating to convergence on the Bellman equation. The optimal policy is characterized by three regions in the (θ, ϵ) space. For high enough values of $\epsilon + \theta$, the worker stays put. For high θ but low ϵ, the worker retains his career but searches for a better job. For low values of $\theta + \epsilon$, the worker finds a new career and a new job. In figures 6.5.1 and 6.5.2, the decision to *retain* both job and career occurs in the high θ, high ϵ region of the state space; the decision to retain career θ but search for a new job ϵ occurs in the high θ and low ϵ region of the state space; and the decision to 'get a new life' by drawing both a new θ and a new ϵ occurs in the low θ, low ϵ region.[7]

When the career-job pair (θ, ϵ) is such that the worker chooses to stay put, the value function in (6.5.1) attains the value $(\theta+\epsilon)/(1-\beta)$. Of course, this happens when the decision to stay put weakly dominates the other two actions, which occurs when

$$\frac{\theta+\epsilon}{1-\beta} \geq \max\left\{C(\theta), Q\right\}, \tag{6.5.2}$$

where Q is the value of drawing both a new job and a new career,

$$Q \equiv \int\int\left[\theta'+\epsilon'+\beta v(\theta',\epsilon')\right]dF(\theta')\,dG(\epsilon'),$$

[7] The computations were performed by the Matlab program `neal2.m`.

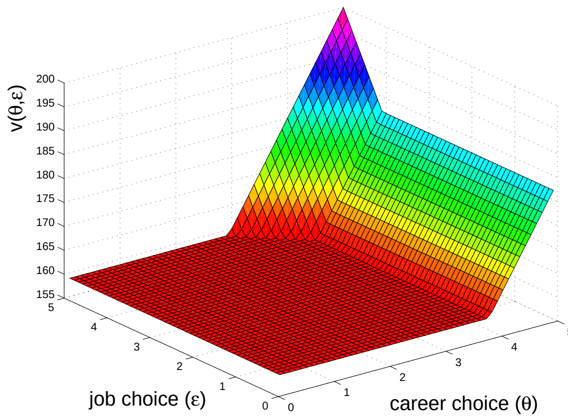

Figure 6.5.1: Optimal value function for Neal's model with $\beta = .95$. The value function is flat in the reject (θ, ϵ) region; increasing in θ only in the keep-career-but-draw-new-job region; and increasing in both θ and ϵ in the stay-put region.

and $C(\theta)$ is the value of drawing a new job but keeping θ:

$$C(\theta) = \theta + \int [\epsilon' + \beta v(\theta, \epsilon')] \, dG(\epsilon').$$

For a given career θ, a job $\overline{\epsilon}(\theta)$ makes equation (6.5.2) hold with equality. Evidently, $\overline{\epsilon}(\theta)$ solves

$$\overline{\epsilon}(\theta) = \max [(1-\beta) C(\theta) - \theta, (1-\beta) Q - \theta].$$

The decision to stay put is optimal for any career-job pair (θ, ϵ) that satisfies $\epsilon \geq \overline{\epsilon}(\theta)$. When this condition is not satisfied, the worker will draw either a new career-job pair (θ', ϵ') or only a new job ϵ'. Retaining the current career θ is optimal when

$$C(\theta) \geq Q. \tag{6.5.3}$$

We can solve (6.5.3) for the critical career value $\overline{\theta}$ satisfying

$$C(\overline{\theta}) = Q. \tag{6.5.4}$$

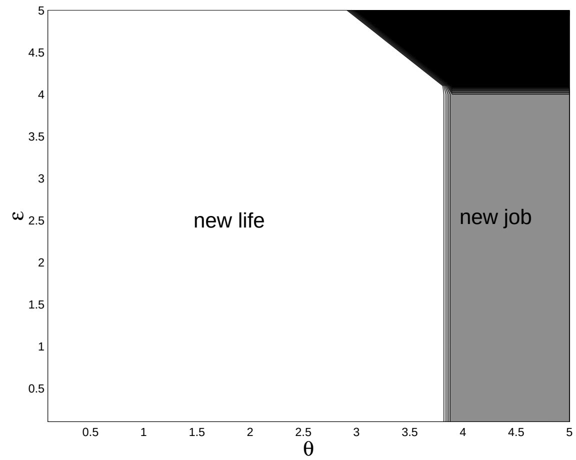

Figure 6.5.2: Optimal decision rule for Neal's model. For (θ,ϵ)'s within the white area, the worker changes both jobs and careers. In the grey area, the worker retains his career but draws a new job. The worker accepts (θ,ϵ) in the black area.

Thus, independently of ϵ, the worker will never abandon any career $\theta \geq \overline{\theta}$. The decision rule for accepting the current career can thus be expressed as follows: accept the current career θ if $\theta \geq \overline{\theta}$ or if the current career-job pair (θ,ϵ) satisfies $\epsilon \geq \overline{\epsilon}(\theta)$.

We can say more about the cutoff value $\overline{\epsilon}(\theta)$ in the retain-θ region $\theta \geq \overline{\theta}$. When $\theta \geq \overline{\theta}$, because we know that the worker will keep θ forever, it follows that

$$C\left(\theta\right) = \frac{\theta}{1-\beta} + \int J\left(\epsilon'\right) dG\left(\epsilon'\right),$$

where $J(\epsilon)$ is the optimal value of $\sum_{t=0}^{\infty} \beta^t \epsilon_t$ for a worker who has just drawn ϵ, who has already decided to keep his career θ, and who is deciding whether to try a new job next period. The Bellman equation for J is

$$J\left(\epsilon\right) = \max\left\{\frac{\epsilon}{1-\beta}, \epsilon + \beta \int J\left(\epsilon'\right) dG\left(\epsilon'\right)\right\}. \tag{6.5.5}$$

This resembles the Bellman equation for the optimal value function for the basic McCall model, with a slight modification. The optimal policy is of the reservation-job form: keep the job ϵ for $\epsilon \geq \bar{\epsilon}$, otherwise try a new job next period. The absence of θ from (6.5.5) implies that in the range $\theta \geq \overline{\theta}$, $\bar{\epsilon}$ is independent of θ.

These results explain some features of the value function plotted in Figure 6.5.1 At the boundary separating the "new life" and "new job" regions of the (θ, ϵ) plane, equation (6.5.4) is satisfied. At the boundary separating the "new job" and "stay put" regions, $\frac{\theta+\epsilon}{1-\beta} = C(\theta) = \frac{\theta}{1-\beta} + \int J(\epsilon')dG(\epsilon')$. Finally, between the "new life" and "stay put" regions, $\frac{\theta+\epsilon}{1-\beta} = Q$, which defines a diagonal line in the (θ, ϵ) plane (see Figure 6.5.2). The value function is the constant value Q in the "get a new life" region (i.e., the region in which the optimal decision is to draw a new (θ, ϵ) pair). Equation (6.5.3) helps us understand why there is a set of high θ's in Figure 6.5.2 for which $v(\theta, \epsilon)$ rises with θ but is flat with respect to ϵ.

Probably the most interesting feature of the model is that it is possible to draw a (θ, ϵ) pair such that the value of keeping the career (θ) and drawing a new job match (ϵ') exceeds both the value of stopping search, and the value of starting again to search from the beginning by drawing a new (θ', ϵ') pair. This outcome occurs when a large θ is drawn with a small ϵ. In this case, it can occur that $\theta \geq \overline{\theta}$ and $\epsilon < \bar{\epsilon}(\theta)$.

Viewed as a normative model for young workers, Neal's model tells them: don't shop for a firm until you have found a career you like. As a positive model, it predicts that workers will not switch careers after they have settled on one. Neal presents data indicating that while this stark prediction does not hold up perfectly, it is a good first approximation. He suggests that extending the model to include learning, along the lines of Jovanovic's model to be described in section 6.8, could help explain the later career switches that his model misses.[8]

[8] Neal's model can be used to deduce waiting times to the event $(\theta \geq \overline{\theta}) \cup (\epsilon \geq \bar{\epsilon}(\theta))$. The first event within the union is choosing a career that is never abandoned. The second event is choosing a permanent job. Neal used the model to approximate and interpret observed career and job switches of young workers.

6.6. Offer distribution unknown

Consider the following modification of the McCall search model. An unemployed worker wants to maximize the expected present value of $\sum_{t=0}^{\infty} \beta^t y_t$ where y_t equals wage w when employed and c when unemployed. Each period the worker receives one offer to work forever at a wage w drawn from one of two cumulative distribution functions F and G, where $F(0) = G(0) = 0$ and $F(B) = G(B) = 1$ for $B > 0$. Nature draws from the same distribution, either F or G, at all dates and the worker knows this, but he or she does not know whether it is F of G. At time 0 *before* drawing a wage offer, the worker attaches probability $\pi_{-1} \in (0,1)$ to the distribution being F. We assume that the distributions have densities f and g, respectively, and that they have common support. Before drawing a wage at time 0, the worker thus believes that the density of w_0 is $h(w_0;\pi_{-1}) = \pi_{-1} f(w_0) + (1-\pi_{-1}) g(w_0)$. After drawing w_0, the worker uses Bayes' law to deduce that the posterior probability that the density is $f(w)$ is[9]

$$\pi_0 = \frac{\pi_{-1} f(w_0)}{\pi_{-1} f(w_0) + (1-\pi_{-1})\, g(w_0)}.$$

More generally, after observing w_t for the tth draw, the worker believes that the probability that w_{t+1} is to be drawn from distribution F is

$$\pi_t = \frac{\pi_{t-1} f(w_t)/g(w_t)}{\pi_{t-1} f(w_t)/g(w_t) + (1-\pi_{t-1})} \tag{6.6.1}$$

and that the density of w_{t+1} is

$$h(w_{t+1};\pi_t) = \pi_t f(w_{t+1}) + (1-\pi_t)\, g(w_{t+1}). \tag{6.6.2}$$

Notice that

$$\begin{aligned} E(\pi_t|\pi_{t-1}) &= \int \Big[\frac{\pi_{t-1} f(w)}{\pi_{t-1} f(w) + (1-\pi_{t-1}) g(w)}\Big] \Big[\pi_{t-1} f(w) + (1-\pi_{t-1}) g(w)\Big] dw \\ &= \pi_{t-1} \int f(w) dw \\ &= \pi_{t-1}, \end{aligned}$$

[9] The worker's initial beliefs induce a joint probability distribution over a potentially infinite sequence of draws $w_0, w_1, \ldots$. Bayes' law is simply an application of the laws of probability to compute the conditional distribution of the tth draw w_t conditional on $[w_0, \ldots, w_{t-1}]$. Since we assume from the start that the decision maker *knows* the joint distribution and the laws of probability, one respectable view is that Bayes' law is less a 'theory of learning' than a statement about the consequences of information inflows for a decision maker who thinks he knows the truth (i.e., a joint probability distribution) from the beginning.

so that the process π_t is a *martingale* bounded by 0 and 1. (In the first line in the above string of equalities, the term in the first set of brackets is just π_t as a function of w_t, while the term in the second set of brackets is the density of w_t conditional on π_{t-1}.) Notice that here we are computing $E(\pi_t|\pi_{t-1})$ under the subjective density described in the second term in brackets. It follows from the martingale convergence theorem (see appendix A of chapter 17) that π_t converges almost surely to a random variable in $[0,1]$. Practically, this means if the probability attached to all sample paths $\{\pi_t\}_{t=0}^\infty$ that converge is unity. However, in general different sample paths converge to different limiting values. The limit points of $\{\pi_t\}_{t=0}^\infty$ as $t \to +\infty$ thus constitute a random variable with what is in general a non-trivial distribution.

Let $v(w_t, \pi_t)$ be the optimal value of the problem for a previously unemployed worker who has just drawn w and updated π according to (6.6.1). The Bellman equation is

$$v(w, \pi_t) = \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta}, c + \beta \int v(w', \pi_{t+1}(w'))\, h(w'; \pi_t)\, dw' \right\} \tag{6.6.3}$$

subject to (6.6.1) and (6.6.2). The state vector is the worker's current draw w and his post-draw estimate of the probability that the distribution is f. The second term on the right side of (6.6.3) integrates the value function evaluated at next period's state vector with respect to the worker's subjective distribution $h(w'; \pi_t)$ of next period's draw w'. The value function for next period recognizes that π_{t+1} will be updated in a way that depends on w' via Bayes' law as captured by equation (6.6.1). Evidently, the optimal policy is to set a reservation wage $\bar{w}(\pi_t)$ that depends on π_t.

As an example, we have computed the optimal policy by backward induction assuming that f is a uniform distribution on $[0,2]$ while g is a beta distribution with parameters (3,1.2).[10] We set unemployment compensation $c = .6$ and the discount factor $\beta = .95$.[11] The two densities are plotted in figure 6.6.1, which shows that the g density provides better prospects for the worker than does the uniform f density. It stands to reason that the worker's reservation wage falls as the posterior probability π that he places on density f rises, as figure 6.6.2 confirms.

[10] The beta distribution for w is characterized by a density $g(w; \alpha, \gamma) \propto w^{\alpha-1}(1-w)^{(\gamma-1)}$, where the factor of proportionality is chosen to make the density integrate to 1.

[11] The matlab programs `search_learn_francisco_3.m` and `search_learn_beta_2.m` perform these calculations.

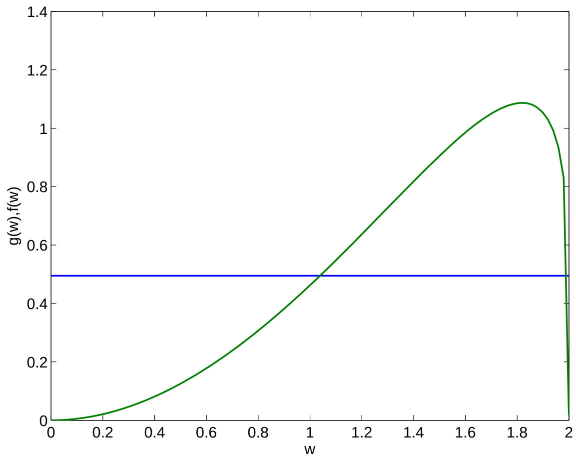

Figure 6.6.1: Two densities for wages, a uniform $f(w)$ and a $g(w)$ corresponding to a beta distribution with parameters 3, 1.2.

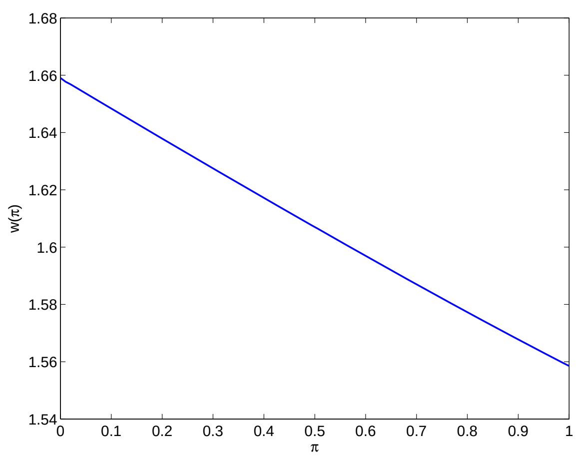

Figure 6.6.2: The reservation as a function of the posterior probability π that the worker thinks that the wage is drawn from the uniform density f.

Figure 6.6.3 shows empirical cumulative distribution functions for durations of unemployment and π at time of job acceptance under two alternative assumptions about whether the uniform distribution F or the beta distribution G permanently governs the wage. We constructed these by simulating the model 10,000 times at the parameter values just given, starting from a common initial condition for beliefs $\pi_{-1} = .5$ and assuming that, unbeknownst to the worker, either the uniform density $f(w)$ or the beta density $g(w)$ truly governs successive wage draws. Only when π_t approaches 1 will workers have learned that nature is drawing from f and not g. Evidently, most workers accept jobs long before a law of large numbers has enough time to teach them for sure which of the two densities from which nature draws wage offers. Thus, workers usually choose not to collect enough observations for them to learn for sure which distribution governs wage offers. In both panels, the lower line shows the cumulative distribution function when nature draws from F and the lower panel shows the c.d.f. when nature draws from G.[12]

A comparison of the CDF's when nature draws from F and G, respectively, is revealing. When G prevails, the cumulative distribution functions in the top panel reveal that workers typically accept jobs earlier than when F prevails. This captures what the interrogator of an unemployed McCall worker in the passage of Lucas cited in the introduction might have had in mind when he said 'Maybe you are setting your sights too high'. The bottom panel reveals that when nature permanently draws from G, employed workers put a higher probability on their having actually sampled from G than from F, while the reverse is true when nature draws permanently from F.

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12 It is a useful exercise to use recall formula (6.2.2) for the mean of a nonnegative random variable and then glance at the CDFs in the bottom panel to approximate the mean π_t at time of job acceptance.

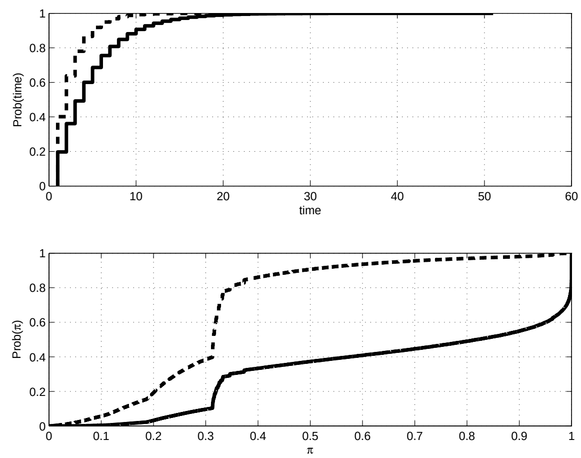

Figure 6.6.3: Top panel: CDF of duration of unemployment; bottom panel: CDF of π at time worker accepts wage and leaves unemployment. In each panel, the lower filled line is the CDF when nature permanently draws from the uniform density f while the dotted line is the CDF when nature permanently draws from the beta density g.

6.7. An equilibrium price distribution

The McCall search model confronts a worker with a given distribution of wages. In this section, we ask why firms might conceivably choose to confront an *ex ante* homogenous collection of workers with a nontrivial distribution of wages. Knowing that the workers have a reservation wage policy, why would a firm ever offer a worker *more* than the reservation wage? That question challenges us to think about whether it is possible to conceive of a coherent setting in which it would be optimal for a collection of profit maximizing firms somehow to make decisions that generate a distribution of wages.

In this section, we take up this question, but for historical reasons investigate it in the context of a sequential search model in which buyers seek the lowest price.[13] Buyers can draw additional offers from a known distribution at a fixed cost c for each additional *batch* of n independent draws from a known

[13] See Burdett and Mortensen (1998) for a parallel analysis of the analogous issues in a model of job search.

price distribution. Both within and across batches, successive draws are independent. The buyer's optimal strategy is to set a reservation price and to continue drawing until the first time a price less than the reservation price has been offered. Let $\tilde{p}$ be the reservation price.

Rothschild (1973) posed the following challenge for a model in which there is a large number of identical buyers each of whom has reservation price $\tilde{p}$. If all sellers know the reservation price $\tilde{p}$, why would any of them offer a price less than $\tilde{p}$? This cogent question points to a force for the price distribution to collapse, an outcome that would destroy the motive for search behavior on the part of buyers. Thus, the challenge is to construct an *equilibrium* version of a search model in which it is in firms' interest to generate the non-trivial price distribution that sustains buyers' search activities.

Burdett and Judd (1983) met this challenge by creating an environment in which *ex ante* identical buyers *ex post* receive differing numbers of price offers that are drawn from a common distribution set by firms. They construct an equilibrium in which a continuum of profit maximizing sellers are content to generate this distribution of prices. Sellers set their prices to maximize expected profit per customer. But sellers don't know the number of other offers that a prospective customer has received. Heterogeneity in the number of offers received by buyers together with seller's ignorance of the number and nature of *other* offers received by a *particular* customer creates a tradeoff between profit per customer and volume that makes possible a non-degenerate equilibrium price distribution. Firms that post higher prices are lower-volume sellers. Firms that post lower prices are higher-volume sellers. There exists an equilibrium distribution of prices in which all types of firms expect to earn the same profit per potential customer.

6.7.1. A Burdett-Judd setup

A continuum of buyers purchases a single good from one among a continuum of firms. Each firm contacts a fixed measure ν of potential buyers. The firms produce a homogeneous good at zero marginal cost. Each firm takes the c.d.f. of prices charged by other firms as given and chooses a price. The firm wants to maximize its expected profits per consumer. A firm's expected profit per consumer equals its price times the probability that its price is the minimum among the set of acceptable offers received by the buyer. The distribution of prices set by other firms impinges on a firm's expected profits because it affects the probability that its offer will be accepted by a buyer.

6.7.2. Consumer problem with noisy search

A consumer wants to purchase a good for a minimum price. Firms make offers that buyers can view as being drawn from a distribution of nonnegative prices with cumulative distribution function $G(P) = \text{Prob}(p \leq P)$ with $G(\underline{p}) = 0, G(B) = 1$. Assume that G is continuously differentiable and so has an associated probability density. A buyer's search activity is divided into batches. Within each batch the buyer receives a random number of offers drawn from the same distribution G. Burdett and Judd call this structure 'noisy search'. In particular, at a cost of $c > 0$ per search round, with probability $q \in (0,1)$ a buyer receives one offer drawn from G and with probability $1-q$ receives two offers. Thus, a 'round' consists of a 'compound lottery' first of a random number of draws, then that number of i.i.d. random draws price offers from the c.d.f. G. A buyer can recall offers within a round but not across rounds. Evidently, $\text{Prob}\{\min(p_1,p_2) \geq p\} = (1-G(p))^2$ and $\text{Prob}\{\min(p_1,p_2) \leq p\} = 1-(1-G(p))^2$. Then *ex ante* the c.d.f. of low prices drawn in a single round is

$$H(p) = qG(p) + (1-q)\left(1-(1-G(p))^2\right). \tag{6.7.1}$$

Let $v(p)$ be the expected price including future search costs of a consumer who has already paid c, has offer p in hand, and is about to decide whether to accept or reject the offer. The Bellman equation is

$$v(p) = \min_{\text{accept,reject}} \left\{p, c + \int_{\underline{p}}^{B} v(p')\, dH(p')\right\}. \tag{6.7.2}$$

The reservation price $\tilde{p}$ satisfies $v(\tilde{p}) = \tilde{p} = c + \int_{\underline{p}}^{\tilde{p}} p' dH(p')$, which implies[14]

$$c = \int_{\underline{p}}^{\tilde{p}} H(p)\, dp. \tag{6.7.3}$$

Combining equation (6.7.3) with the formula $Ep = \int_{\underline{p}}^{\tilde{p}} (1 - H(p)) dp$ for the mean of a nonnegative random variable implies that the reservation price $\tilde{p}$ satisfies

$$\tilde{p} = c + Ep,$$

which states that the reservation price equals the cost of one additional round of search plus the mean price drawn from one more round of noisy search. The challenge is to construct an equilibrium price distribution G, and thus an implied distribution H, in which most firms choose to post prices less than the buyer's reservation price $\tilde{p}$.

6.7.3. Firms

For simplicity and to focus our attention entirely on the search problem, we assume that the good costs firms nothing to produce. In setting its price, we assume that a firm seeks to maximize expected profit per customer. A firm makes an offer to a customer without knowing whether this is the only offer available to the customer or whether the customer, having drawn two offers, possibly has a lower offer in hand. The firm begins by computing the fraction of its customers who will have received one offer and the fraction of its customers who will have received only one offer. Let there be a large number ν of total potential buyers per batch, consisting of νq persons each of whom receives one offer and $\nu(1-q)$ people each of whom receives two offers. The total number of offers is evidently $\nu(1q + 2(1-q)) = \nu(2-q)$. Evidently, the fraction of all offers that is received by customers who have received one offer is $\frac{\nu q}{\nu(2-q)} = \frac{q}{2-q}$. This calculation induces a typical firm to believe that the fraction of its customers

[14] The Bellman equation implies $\tilde{p} = c + \int_{\tilde{p}}^{B} v(p') dH(p)$, which can be rearranged to become $\int_{\underline{p}}^{\tilde{p}} (\tilde{p} - p) dH(p) = c$. Let $u = \tilde{p} - p$ and $dv = dH(p)$ and apply the integration by parts formula $\int u dv = uv - \int v du$ to the previous equality to get $\int_{\underline{p}}^{\tilde{p}} H(p) dp = c$.

who receive one offer is

$$\hat{q} = \frac{q}{2-q} \tag{6.7.4}$$

and the fraction who receive two offers is $1 - \hat{q} = \frac{2(1-q)}{2-q}$. The firm regards $\hat{q}$ as its estimate of the probability that a given customer has received only its offer, while it thinks that a fraction $1 - \hat{q}$ of its customers has also received a competing offer from another firm.

There is a continuum of firms each of which takes as given a price offer distribution of other firms with c.d.f. $G(p)$, where $G(\underline{p}) = 0, G(\tilde{p}) = 1$. We have assume that G is differentiable.[15] This distribution satisfies the outcome that in equilibrium no firm makes an offer exceeding the buyer's reservation price $\tilde{p}$. Let $Q(p)$ be the probability that a consumer will accept an offer p, where $\underline{p} \leq p \leq \tilde{p}$. Evidently, a consumer who receives *one* offer $p < \tilde{p}$ will accept it with probability 1. But only a fraction $1 - G(p)$ of consumer who receive *two* offers will accept an offer $p < \tilde{p}$. Why? because $1 - G(p)$ is the fraction of consumers whose *other* offer exceeds p; so a fraction $G(p)$ of two-offer customers who receive offer p will reject it because they have received an offer lower than p. Therefore, the overall probability that a randomly encountered consumer will accept an offer $p \in [\underline{p}, \tilde{p}]$ is

$$Q(p) = \hat{q} + (1 - \hat{q})(1 - G(p)). \tag{6.7.5}$$

[15] Burdett and Judd (1983, p. 959, lemma 1) show that an equilibrium G is differentiable when $q \in (0,1)$ and $\tilde{p} > 0$. Their argument goes as follows. Suppose to the contrary that there is a positive probability attached to a single price $p' \in (0, \tilde{p})$. Consider a firm that contemplates charging p'. When $q < 1$, the firm knows that there is a positive probability that a prospective consumer has received another offer also of p'. If the firm lowers its offer infinitesimally, it can expect to steal that customer and thereby increase its expected profits. Therefore, a decision to charge p' can't maximize expected profits for a typical firm. We have been led to a contradiction by assuming that G has a discontinuity at p'.

6.7.4. Equilibrium

The objects in play are a reservation price $\tilde{p}$ and a value function $v(p)$ for a typical buyer; and a c.d.f. $G(p)$ of prices that is the outcome of the independent price-setting decisions of individual firms and that is taken as given by all buyers and sellers.

DEFINITION: An equilibrium is a c.d.f. of price offers $G(p)$ on domain $[\underline{p}, \tilde{p}]$, a c.d.f. of per-batch price offers to consumers $H(p)$, and a reservation price $\tilde{p}$ such that (i) the c.d.f. of offers to buyers $H(p)$ satisfies (6.7.1); (ii) $\tilde{p}$ is an optimal reservation price for buyers that satisfies $c = \int_{\underline{p}}^{\tilde{p}} dH(p)$; and (iii) firms are indifferent with respect to charging any $p \in [\underline{p}, \tilde{p}]$; therefore, firms choose p by randomizing using $G(p)$.

We confirm an equilibrium by using a guess-and-verify method. Make the following guess for an equilibrium c.d.f. $G(p)$.[16] First, set

$$\underline{p} = \hat{q}\tilde{p} \tag{6.7.6}$$

and then set

$$G(p) = \begin{cases} 0 & \text{if } p \le \underline{p} \\ 1 - \frac{\tilde{p}-p}{p}\frac{\hat{q}}{1-\hat{q}} & \text{if } p \in \left[\underline{p}, \tilde{p}\right] \\ 1 & \text{if } p > \tilde{p} \, . \end{cases} \tag{6.7.7}$$

Under this guess, $Q(p)$ becomes

$$Q(p) = \frac{\tilde{p}\hat{q}}{p} \quad \forall p \in \left[\underline{p}, \tilde{p}\right].$$

Therefore, the expected profit per customer for a firm that sets price $p \in [\underline{p}, \tilde{p}]$ is

$$pQ(p) = \tilde{p}\hat{q}, \tag{6.7.8}$$

which is evidently independent of the firm's choice of offer p in the interval $[\underline{p}, \tilde{p}]$. The firm is indifferent about the price it offers on this interval. In particular, notice that The right side of equality (6.7.8) is the product of the fraction of a firm's buyers receiving one offer, $\hat{q}$, times the reservation price $\tilde{p}$. This is the expected profit per customer of a firm that charges the reservation price. The

[16] We can make sure that the buyer's search problem is consistent with this guess by setting c to confirm (6.7.3).

left side of equality (6.7.8) is the product of the price p times probability $Q(p)$ that a buyer will accept price p, which as we have noted equals the expected profit per customer for a firm that sets price p.

We assume that firms randomize over choices of p in such a way that $G(p)$ given by (6.7.7) emerges as the c.d.f. for prices.

6.7.5. Special cases

The Burdett-Judd model isolates forces for the price distribution to collapse and countervailing forces that can sustain a nontrivial price distribution.

1. Consider the special case in which $q = 1$ (and therefore $\hat{q} = 1$). Here, $\underline{p} = \tilde{p}$. The formula (6.7.7) shows that the distribution of prices collapses. This case exhibits the Rothschild challenge with which we began.
2. Next, consider the opposite special case in which $q = 0$ (and therefore $\hat{q} = 0$). Here, $\underline{p} = 0$ and the c.d.f. $G(p) = 1 \forall p \in [\underline{p}, \tilde{p}]$. Bertrand competition drives all prices down to the marginal cost of production, which we have assumed to be zero. This case exhibits another force for the price distribution to collapse, again in the spirit of Rothschild's challenge.
3 Finally, consider the general case in which $q \in (0,1)$ and therefore $\hat{q} \in (0,1)$). When q is strictly in the interior of $[\underline{p}, \tilde{p}]$, we can sustain a nontrivial distribution of prices. Firms are indifferent between being high volume, low price sellers and high price, low volume sellers. The equilibrium price distribution $G(p)$ renders a firm's expected profits per prospective customer $pQ(p)$ independent of p.

6.8. Jovanovic's matching model

Another interesting effort to confront Rothschild's questions about the source of the equilibrium wage (or price) distribution comes from matching models, in which the main idea is to reinterpret w not as a wage but instead, more broadly, as a parameter characterizing the entire quality of a match occurring between a pair of agents. The variable w is regarded as a summary measure of the productivities or utilities jointly generated by the activities of the match. We can consider pairs consisting of a firm and a worker, a man and a woman, a house and an owner, or a person and a hobby. The idea is to analyze the way in which matches form and maybe also dissolve by viewing both parties to the match as being drawn from populations that are statistically homogeneous to an outside observer, even though the match is idiosyncratic from the perspective of the parties to the match.

Jovanovic (1979a) has used a model of this kind supplemented by a hypothesis that both sides of the match behave optimally but only gradually learn about the quality of the match. Jovanovic was motivated by a desire to explain three features of labor market data: (1) on average, wages rise with tenure on the job, (2) quits are negatively correlated with tenure (that is, a quit has a higher probability of occurring earlier in tenure than later), and (3) the probability of a subsequent quit is negatively correlated with the current wage rate. Jovanovic's insight was that each of these empirical regularities could be interpreted as reflecting the operation of a matching process with gradual learning about match quality. We consider a simplified version of Jovanovic's model of matching. (Prescott and Townsend, 1980, describe a discrete-time version of Jovanovic's model, which has been simplified here.) A market has two sides that could be variously interpreted as consisting of firms and workers, or men and women, or owners and renters, or lakes and fishermen. Following Jovanovic, we shall adopt the firm-worker interpretation here. An unmatched worker and a firm form a pair and jointly draw a random match parameter θ from a probability distribution with cumulative distribution function $\text{Prob}\{\theta \leq s\} = F(s)$. Here the match parameter reflects the marginal productivity of the worker in the match. In the first period, before the worker decides whether to work at this match or to wait and to draw a new match next period from the same distribution F, the worker and the firm both observe only $y = \theta + u$, where u is a random noise that is uncorrelated with θ. Thus, in the first period, the worker-firm pair receives only a noisy observation on θ. This situation corresponds to that when both sides of the market form only an error-ridden impression of the quality of the match at first. On the basis of this noisy observation, the firm, which is imagined to operate competitively under constant returns to scale, offers to pay the worker the conditional expectation of θ, given $(\theta + u)$, for the first period, with the understanding that in subsequent periods it will pay the worker the expected value of θ, depending on whatever additional information both sides of the match receive.[17] Given this policy of the firm, the worker decides whether to accept the match and to work this period for $E[\theta|(\theta+u)]$ or to refuse the offer and draw a new match parameter θ' and noisy observation on it, $(\theta' + u')$, next period. If the worker decides to accept the offer in the first period, then in the second period both the firm and the worker are assumed to observe the true value of θ. This situation corresponds to that in which both sides learn about each other and about the quality of the match. In the second period the firm offers to pay the worker θ then and forever more. The worker next decides whether to accept this offer or to quit, be unemployed this period, and draw a new match parameter and a noisy observation on it next period.

We can conveniently think of this process as having three stages. Stage 1 is the "predraw" stage, in which a previously unemployed worker has yet to draw the one match parameter and the noisy observation on it that he is entitled to draw after being unemployed the previous period. We let Q denote the expected present value of wages, before drawing, of a worker who was unemployed last period and who behaves optimally. The second stage of the process occurs after the worker has drawn a match parameter θ, has received the noisy observation of $(\theta + u)$ on it, and has received the firm's wage offer of $E[\theta|(\theta + u)]$ for this period. At this stage, the worker decides whether to accept this wage for this period and the prospect of receiving θ in all subsequent periods. The third stage occurs in the next period, when the worker and firm discover the true value of θ and the worker must decide whether to work at θ this period and in all subsequent periods that he remains at this job (match).

[17] Jovanovic assumed firms to be risk neutral and to maximize the expected present value of profits. They compete for workers by offering wage contracts. In a long-run equilibrium the payments practices of each firm would be well understood, and this fact would support the described implicit contract as a competitive equilibrium.

We now add some more specific assumptions about the probability distribution of θ and u. We assume that θ and u are independently distributed random variables. Both are normally distributed, θ being normal with mean μ and variance σ_0^2, and u being normal with mean 0 and variance σ_u^2. Thus, we write

$$\theta \sim N\left(\mu, \sigma_0^2\right), \qquad u \sim N\left(0, \sigma_u^2\right) . \tag{6.8.1}$$

In the first period, after drawing a θ, the worker and firm both observe the noise-ridden version of θ, $y = \theta + u$. Both worker and firm are interested in making inferences about θ, given the observation $(\theta + u)$. They are assumed to use Bayes' law and to calculate the posterior probability distribution of θ, that is, the probability distribution of θ conditional on $(\theta + u)$. The probability distribution of θ, given $\theta + u = y$, is known to be normal, with mean m_0 and variance σ_1^2. Using the Kalman filtering formula in chapter 2, we have[18]

$$\begin{aligned} m_0 = E(\theta|y) &= E(\theta) + \frac{\operatorname{cov}(\theta, y)}{\operatorname{var}(y)}\left[y - E(y)\right] \\ &= \mu + \frac{\sigma_0^2}{\sigma_0^2 + \sigma_u^2}(y - \mu) \equiv \mu + K_0(y - \mu) , \\ \sigma_1^2 = E\left[(\theta - m_0)^2 |y\right] &= \frac{\sigma_0^2}{\sigma_0^2 + \sigma_u^2}\,\sigma_u^2 = K_0 \sigma_u^2 . \end{aligned} \tag{6.8.2}$$

After drawing θ and observing $y = \theta + u$ the first period, the firm is assumed to offer the worker a wage of $m_0 = E[\theta|(\theta + u)]$ the first period and a promise to pay θ for the second period and thereafter. The worker has the choice of accepting or rejecting the offer.

From equation (6.8.2) and the property that the random variable $y - \mu = \theta + u - \mu$ is normal, with mean zero and variance $(\sigma_0^2 + \sigma_u^2)$, it follows that m_0 is itself normally distributed, with mean μ and variance $\sigma_0^4/(\sigma_0^2 + \sigma_u^2) = K_0\sigma_0^2$:

$$m_0 \sim N\left(\mu, K_0\sigma_0^2\right). \tag{6.8.3}$$

Note that $K_0\sigma_0^2 < \sigma_0^2$, so that m_0 has the same mean but a smaller variance than θ.

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18 In the special case in which random variables are jointly normally distributed, linear least-squares projections equal conditional expectations.

6.8.1. Recursive formulation and solution

The worker seeks to maximize the expected present value of wages. We now proceed to solve the worker's problem by working backward. At stage 3, the worker knows θ and is confronted by the firm with an offer to work this period and forever more at a wage of θ. We let $J(\theta)$ be the expected present value of wages of a worker at stage 3 who has a known match θ in hand and who behaves optimally. The worker who accepts the match this period receives θ this period and faces the same choice at the same θ next period. (The worker can quit next period, though it will turn out that the worker who does not quit this period never will.) Therefore, if the worker accepts the match, the value of match θ is given by $\theta + \beta J(\theta)$, where β is the discount factor. The worker who rejects the match must be unemployed this period and must draw a new match next period. The expected present value of wages of a worker who was unemployed last period and who behaves optimally is Q. Therefore, the Bellman equation is $J(\theta) = \max\{\theta + \beta J(\theta), \beta Q\}$. This equation is graphed in Figure 6.8.1 and evidently has the solution

$$J(\theta) = \begin{cases} \theta + \beta J(\theta) = \frac{\theta}{1-\beta} & \text{for } \theta \geq \overline{\theta} \\ \beta Q & \text{for } \theta \leq \overline{\theta}. \end{cases} \tag{6.8.4}$$

The optimal policy is a reservation wage policy: accept offers $\theta \geq \overline{\theta}$, and reject offers $\theta \leq \overline{\theta}$, where θ satisfies

$$\frac{\overline{\theta}}{1-\beta} = \beta Q. \tag{6.8.5}$$

We now turn to the worker's decision in stage 2, given the decision rule in stage 3. In stage 2, the worker is confronted with a current wage offer $m_0 = E[\theta|(\theta+u)]$ and a conditional probability distribution function that we write as $\text{Prob}\{\theta \leq s|\theta+u\} = F(s|m_0, \sigma_1^2)$. (Because the distribution is normal, it can be characterized by the two parameters m_0, σ_1^2.) We let $V(m_0)$ be the expected present value of wages of a worker at the second stage who has offer m_0 in hand and who behaves optimally. The worker who rejects the offer is unemployed this period and draws a new match parameter next period. The expected present value of this option is βQ. The worker who accepts the offer receives a wage of m_0 this period and a probability distribution of wages of $F(\theta'|m_0, \sigma_1^2)$ for next period. The expected present value of this option is $m_0 + \beta \int J(\theta') dF(\theta'|m_0, \sigma_1^2)$.

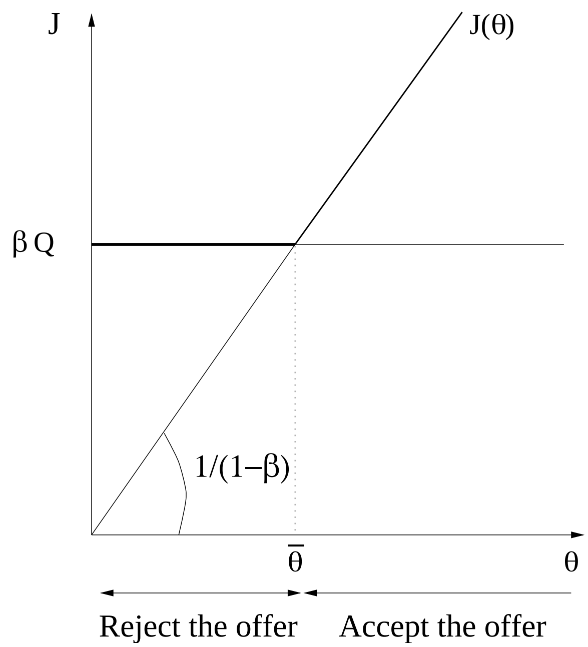

Figure 6.8.1: The function $J(\theta) = \max\{\theta + \beta J(\theta), \beta Q\}$. The reservation wage in stage 3, $\overline{\theta}$, satisfies $\overline{\theta}/(1-\beta) = \beta Q$.

The Bellman equation for the second stage therefore becomes

$$V(m_0) = \max\left\{m_0 + \beta \int J(\theta')\, dF\left(\theta' | m_0, \sigma_1^2\right), \beta Q\right\}. \tag{6.8.6}$$

Note that both m_0 and $\beta \int J(\theta') dF(\theta'|m_0, \sigma_1^2)$ are increasing in m_0, whereas βQ is a constant. For this reason a reservation wage policy will be an optimal one. The functional equation evidently has the solution

$$V(m_0) = \begin{cases} m_0 + \beta \int J(\theta')\, dF\left(\theta' | m_0, \sigma_1^2\right) & \text{for } m_0 \geq \overline{m}_0 \\ \beta Q & \text{for } m_0 \leq \overline{m}_0. \end{cases} \tag{6.8.7}$$

If we use equation (6.8.7), an implicit equation for the reservation wage $\overline{m}_0$ is then

$$V(\overline{m}_0) = \overline{m}_0 + \beta \int J(\theta')\, dF\left(\theta' | \overline{m}_0, \sigma_1^2\right) = \beta Q. \tag{6.8.8}$$

Using equations (6.8.8) and (6.8.4), we shall show that $\overline{m}_0 < \overline{\theta}$, so that the worker becomes choosier over time with the firm. This force makes wages rise with tenure.

Using equations (6.8.4) and (6.8.5) repeatedly in equation (6.8.8), we obtain

$$\begin{aligned}
&\overline{m}_0 + \beta \frac{\overline{\theta}}{1-\beta} \int_{-\infty}^{\overline{\theta}} dF\left(\theta'|\overline{m}_0, \sigma_1^2\right) + \frac{\beta}{1-\beta} \int_{\overline{\theta}}^{\infty} \theta' dF\left(\theta'|\overline{m}_0, \sigma_1^2\right) \\
&\quad = \frac{\overline{\theta}}{1-\beta} = \frac{\overline{\theta}}{1-\beta} \int_{-\infty}^{\overline{\theta}} dF\left(\theta'|\overline{m}_0, \sigma_1^2\right) \\
&\quad + \frac{\overline{\theta}}{1-\beta} \int_{\overline{\theta}}^{\infty} dF\left(\theta'|\overline{m}_0, \sigma_1^2\right).
\end{aligned}$$

Rearranging this equation, we get

$$\overline{\theta} \int_{-\infty}^{\overline{\theta}} dF\left(\theta'|\overline{m}_0, \sigma_1^2\right) - \overline{m}_0 = \frac{1}{1-\beta} \int_{\overline{\theta}}^{\infty} \left(\beta\theta' - \overline{\theta}\right) dF\left(\theta'|\overline{m}_0, \sigma_1^2\right). \tag{6.8.9}$$

Now note the identity

$$\overline{\theta} = \int_{-\infty}^{\overline{\theta}} \overline{\theta} dF\left(\theta'|\overline{m}_0, \sigma_1^2\right) + \left(\frac{1}{1-\beta} - \frac{\beta}{1-\beta}\right) \int_{\overline{\theta}}^{\infty} \overline{\theta} dF\left(\theta'|\overline{m}_0, \sigma_1^2\right). \tag{6.8.10}$$

Adding equation (6.8.10) to (6.8.9) gives

$$\overline{\theta} - \overline{m}_0 = \frac{\beta}{1-\beta} \int_{\overline{\theta}}^{\infty} \left(\theta' - \overline{\theta}\right) dF\left(\theta'|\overline{m}_0, \sigma_1^2\right). \tag{6.8.11}$$

The right side of equation (6.8.11) is positive. The left side is therefore also positive, so that we have established that

$$\overline{\theta} > \overline{m}_0. \tag{6.8.12}$$

Equation (6.8.11) resembles equation (6.3.3) and has a related interpretation. Given $\overline{\theta}$ and $\overline{m}_0$, the right side is the expected benefit of a match $\overline{m}_0$, namely, the expected present value of the match in the event that the match parameter eventually turns out to exceed the reservation match $\overline{\theta}$ so that the match endures. The left side is the one-period cost of temporarily staying in a match paying less than the eventual reservation match value $\overline{\theta}$: having remained unemployed for a period in order to have the privilege of drawing the match parameter θ, the worker has made an investment to acquire this opportunity and must make a similar investment to acquire a new one. Having only the noisy observation of $(\theta + u)$ on θ, the worker is willing to stay in matches m_0 with $\overline{m}_0 < m_0 < \overline{\theta}$ because it is worthwhile to speculate that the match is really better than it seems now and will seem next period.

Now turning briefly to stage 1, we have defined Q as the predraw expected present value of wages of a worker who was unemployed last period and who is about to draw a match parameter and a noisy observation on it. Evidently, Q is given by

$$Q = \int V(m_0)\, dG\left(m_0|\mu, K_0\sigma_0^2\right) \tag{6.8.13}$$

where $G(m_0|\mu, K_0\sigma_0^2)$ is the normal distribution with mean μ and variance $K_0\sigma_0^2$, which, as we saw before, is the distribution of m_0.

Collecting some of the equations, we see that the worker's optimal policy is determined by

$$J(\theta) = \begin{cases} \theta + \beta J(\theta) = \frac{\theta}{1-\beta} & \text{for } \theta \geq \overline{\theta} \\ \beta Q & \text{for } \theta \leq \overline{\theta} \end{cases} \tag{6.8.14}$$

$$V(m_0) = \begin{cases} m_0 + \beta \int J(\theta')\, dF\left(\theta'|m_0, \sigma_1^2\right) & \text{for } m_0 \geq \overline{m}_0 \\ \beta Q & \text{for } m_0 \leq \overline{m}_0 \end{cases} \tag{6.8.15}$$

$$\overline{\theta} - \overline{m}_0 = \frac{\beta}{1-\beta} \int_{\overline{\theta}}^{\infty} \left(\theta' - \overline{\theta}\right) dF\left(\theta'|\overline{m}_0, \sigma_1^2\right) \tag{6.8.16}$$

$$Q = \int V(m_0)\, dG\left(m_0|\mu, K_0\sigma_0^2\right). \tag{6.8.17}$$

To analyze formally the existence and uniqueness of a solution to these equations, one would proceed as follows. Use equations (6.8.14), (6.8.15), and (6.8.16) to write a single functional equation in V,

$$\begin{aligned} V(m_0) = \max\Bigg\{ & m_0 + \beta \int \max\Bigg[\frac{\theta}{1-\beta}, \\ & \beta \int V(m_1')dG\left(m_1'|\mu, K_0\sigma_0^2\right)\Bigg] dF(\theta|m_0, \sigma_1^2), \\ & \beta \int V(m_1')dG\left(m_1'|\mu, K_0\sigma_0^2\right)\Bigg\}. \end{aligned}$$

The expression on the right defines an operator, T, mapping continuous functions V into continuous functions TV. This functional equation can be expressed $V = TV$. The operator T can be directly verified to satisfy the following two properties: (1) it is monotone, that is, $v(m) \geq z(m)$ for all m implies $(Tv)(m) \geq (Tz)(m)$ for all m; (2) for all positive constants c, $T(v+c) \leq Tv + \beta c$. These are Blackwell's sufficient conditions for the functional equation $Tv = v$ to have a unique continuous solution. See Appendix A on functional analysis (see Technical Appendixes).

6.8.2. Endogenous statistics

We now proceed to calculate probabilities and expectations of some interesting events and variables. The probability that a previously unemployed worker accepts an offer is given by

$$\text{Prob}\{m_0 \geq \overline{m}_0\} = \int_{\overline{m}_0}^{\infty} dG\left(m_0|\mu, K_0\sigma_0^2\right).$$

The probability that a previously unemployed worker accepts an offer and then quits the second period is given by

$$\text{Prob}\{\left(\theta \leq \overline{\theta}\right) \cap (m_0 \geq \overline{m}_0)\} = \int_{\overline{m}_0}^{\infty}\int_{-\infty}^{\overline{\theta}} dF\left(\theta|m_0, \sigma_1^2\right) dG\left(m_0|\mu, K_0\sigma_0^2\right).$$

The probability that a previously unemployed worker accepts an offer the first period and also elects not to quit the second period is given by

$$\text{Prob}\{\left(\theta \geq \overline{\theta}\right) \cap (m_0 \geq \overline{m})\} = \int_{\overline{m}_0}^{\infty}\int_{\overline{\theta}}^{\infty} dF\left(\theta|m_0, \sigma_1^2\right) dG\left(m_0|\mu, K_0\sigma_0^2\right).$$

The mean wage of those employed the first period is given by

$$\overline{w}_1 = \frac{\displaystyle\int_{\overline{m}_0}^{\infty} m_0\, dG\left(m_0|\mu, K_0\sigma_0^2\right)}{\displaystyle\int_{\overline{m}_0}^{\infty} dG\left(m_0|\mu, K_0\sigma_0^2\right)}, \tag{6.8.18}$$

whereas the mean wage of those workers who are in the second period of tenure is given by

$$\overline{w}_2 = \frac{\displaystyle\int_{\overline{m}_0}^{\infty}\int_{\overline{\theta}}^{\infty} \theta\, dF\left(\theta|m_0, \sigma_1^2\right)\, dG\left(m_0|\mu, K_0\sigma_0^2\right)}{\displaystyle\int_{\overline{m}_0}^{\infty}\int_{\overline{\theta}}^{\infty} dF\left(\theta|m_0, \sigma_1^2\right)\, dG\left(m_0|\mu, K_0\sigma_0^2\right)}. \tag{6.8.19}$$

We shall now prove that $\overline{w}_2 > \overline{w}_1$, so that wages rise with tenure. After substituting $m_0 \equiv \int \theta dF(\theta|m_0, \sigma_1^2)$ into equation (6.8.18),

$$\begin{aligned}
\overline{w}_1 &= \frac{\int_{\overline{m}_0}^{\infty} \int_{-\infty}^{\infty} \theta \, dF\left(\theta|m_0, \sigma_1^2\right) \, dG\left(m_0|\mu, K_0\sigma_0^2\right)}{\int_{\overline{m}_0}^{\infty} dG\left(m_0|\mu, K_0\sigma_0^2\right)} \\
&= \frac{1}{\int_{\overline{m}_0}^{\infty} dG\left(m_0|\mu, K_0\sigma_0^2\right)} \left\{ \int_{\overline{m}_0}^{\infty} \int_{-\infty}^{\overline{\theta}} \theta \, dF\left(\theta|m_0, \sigma_1^2\right) \, dG\left(m_0|\mu, K_0\sigma_0^2\right) \right. \\
&\qquad \left. + \; \overline{w}_2 \int_{\overline{m}_0}^{\infty} \int_{\overline{\theta}}^{\infty} dF\left(\theta|m_0, \sigma_1^2\right) \, dG\left(m_0|\mu, K_0\sigma_0^2\right) \right\} \\
&< \frac{\int_{\overline{m}_0}^{\infty} \left\{ \overline{\theta} \, F\left(\overline{\theta}|m_0, \sigma_1^2\right) \, + \, \overline{w}_2 \left[1 - F\left(\overline{\theta}|m_0, \sigma_1^2\right)\right] \right\} dG\left(m_0|\mu, K_0\sigma_0^2\right)}{\int_{\overline{m}_0}^{\infty} dG\left(m_0|\mu, K_0\sigma_0^2\right)} \\
&< \overline{w}_2 .
\end{aligned}$$

It is quite intuitive that the mean wage of those workers who are in the second period of tenure must exceed the mean wage of all employed in the first period. The former group is a subset of the latter group where workers with low productivities, $\theta < \overline{\theta}$, have left. Since the mean wages are equal to the true average productivity in each group, it follows that $\overline{w}_2 > \overline{w}_1$.

The model thus implies that "wages rise with tenure," both in the sense that mean wages rise with tenure and in the sense that $\overline{\theta} > \overline{m}_0$, which asserts that the lower bound on second-period wages exceeds the lower bound on first-period wages. That wages rise with tenure was observation 1 that Jovanovic sought to explain.

Jovanovic's model also explains observation 2, that quits are negatively correlated with tenure. The model implies that quits occur between the first and second periods of tenure. Having decided to stay for two periods, the worker never quits.

The model also accounts for observation 3, namely, that the probability of a subsequent quit is negatively correlated with the current wage rate. The probability of a subsequent quit is given by

$$\text{Prob}\{\theta' < \overline{\theta}|m_0\} = F\left(\overline{\theta}|m_0, \sigma_1^2\right),$$

which is evidently negatively correlated with m_0, the first-period wage. Thus, the model explains each observation that Jovanovic sought to interpret. In the version of the model that we have studied, a worker eventually becomes permanently matched with probability 1. If we were studying a population of such workers of fixed size, all workers would eventually be absorbed into the state of being permanently matched. To provide a mechanism for replenishing the stock of unmatched workers, one could combine Jovanovic's model with the "firing" model in section 6.3.5. By letting matches θ "go bad" with probability λ each period, one could presumably modify Jovanovic's model to get the implication that, with a fixed population of workers, a fraction would remain unmatched each period because of the dissolution of previously acceptable matches.

6.9. A longer horizon version of Jovanovic's model

Here we consider a $T+1$ period version of Jovanovic's model, in which learning about the quality of the match continues for T periods before the quality of the match is revealed by "nature." (Jovanovic assumed that $T=\infty$.) We use the recursive projection technique (the Kalman filter) of chapter 2 to handle the firm's and worker's sequential learning. The prediction of the true match quality can then easily be updated with each additional noisy observation.

A firm-worker pair jointly draws a match parameter θ at the start of the match, which we call the beginning of period 0. The value θ is revealed to the pair only at the beginning of the $(T+1)$th period of the match. After θ is drawn but before the match is consummated, the firm-worker pair observes $y_0 = \theta + u_0$, where u_0 is random noise. At the beginning of each period of the match, the worker-firm pair draws another noisy observation $y_t = \theta + u_t$ on the match parameter θ. The worker then decides whether or not to continue the match for the additional period. Let $y^t = \{y_0, \dots, y_t\}$ be the firm's and worker's information set at time t. We assume that θ and u_t are independently distributed random variables with $\theta \sim \mathcal{N}(\mu, \Sigma_0)$ and $u_t \sim \mathcal{N}(0, \sigma_u^2)$. For $t \geq 0$ define $m_t = E[\theta | y^t]$ and $m_{-1} = \mu$. The conditional means m_t and variances $E(\theta - m_t)^2 = \Sigma_{t+1}$ can be computed with the Kalman filter via the formulas from chapter 2:

$$m_t = (1 - K_t)\, m_{t-1} + K_t y_t \tag{6.9.1a}$$

$$K_t = \frac{\Sigma_t}{\Sigma_t + R} \tag{6.9.1b}$$

$$\Sigma_{t+1} = \frac{\Sigma_t R}{\Sigma_t + R}, \tag{6.9.1c}$$

where $R = \sigma_u^2$ and Σ_0 is the unconditional variance of θ. The recursions are to be initiated from $m_{-1} = \mu$, and given Σ_0.

Using the formulas from chapter 2, we have that conditional on y^t, $m_{t+1} \sim \mathcal{N}(m_t, K_{t+1}\Sigma_{t+1})$ and $\theta \sim \mathcal{N}(m_t, \Sigma_{t+1})$ where Σ_0 is the unconditional variance of θ.

6.9.1. The Bellman equations

For $t \geq 0$, let $v_t(m_t)$ be the value of the worker's problem at the beginning of period t for a worker who optimally estimates that the match value is m_t after having observed y^t. At the start of period $T+1$, we suppose that the value of the match is revealed without error. Thus, at time T, $\theta \sim \mathcal{N}(m_T, \Sigma_{T+1})$. The firm-worker pair estimates θ by m_t for $t = 0, \ldots, T$, and by θ for $t \geq T+1$. Then the following functional equations characterize the solution of the problem:

$$v_{T+1}(\theta) = \max\left\{\frac{\theta}{1-\beta}, \beta Q\right\}, \tag{6.9.2}$$

$$v_T(m) = \max\left\{m + \beta \int v_{T+1}(\theta)\, dF(\theta \mid m, \Sigma_{T+1}),\, \beta Q\right\}, \tag{6.9.3}$$

$$v_t(m) = \max\left\{m + \beta \int v_{t+1}(m')\, dF(m' | m, K_{t+1}\Sigma_{t+1}),\, \beta Q\right\}, \quad t = 0, \ldots, T-1, \tag{6.9.4}$$

$$Q = \int v_0(m)\, dF(m | \mu, K_0 \Sigma_0), \tag{6.9.5}$$

with K_t and Σ_t from the Kalman filter. Starting from v_{T+1} and reasoning backward, it is evident that the worker's optimal policy is to set reservation wages $\overline{m}_t, t = 0, \ldots, T$ that satisfy

$$\begin{aligned}
&\overline{m}_{T+1} = \overline{\theta} = \beta(1-\beta) Q, \\
&\overline{m}_T + \beta \int v_{T+1}(\theta)\, dF(\theta | \overline{m}_T, \Sigma_{T+1}) = \beta Q, \\
&\overline{m}_t + \beta \int v_{t+1}(m')\, dF(m' \mid \overline{m}_t, K_{t+1}\Sigma_{t+1}) = \beta Q, \qquad t = 1, \ldots, T-1.
\end{aligned} \tag{6.9.6}$$

To compute a solution to the worker's problem, we can define a mapping from Q into itself, with the property that a fixed point of the mapping is the optimal value of Q. Here is an algorithm:

a. Guess a value of Q, say Q^i with $i = 1$.
b. Given Q^i, compute sequentially the value functions in equations (6.9.2) through (6.9.4). Let the solutions be denoted $v^i_{T+1}(\theta)$ and $v^i_t(m)$ for $t = 0, \ldots, T$.
c. Given $v^i_1(m)$, evaluate equation (6.9.5) and call the solution $\tilde{Q}^i$.
d. For a fixed "relaxation parameter" $g \in (0,1)$, compute a new guess of Q from

$$Q^{i+1} = gQ^i + (1-g)\,\tilde{Q}^i\,.$$

e. Iterate on this scheme to convergence.

We now turn to the case where the true θ is never revealed by nature, that is, $T = \infty$. Note that $(\Sigma_{t+1})^{-1} = (\sigma_u^2)^{-1} + (\Sigma_t)^{-1}$, so $\Sigma_{t+1} < \Sigma_t$ and $\Sigma_{t+1} \to 0$ as $t \to \infty$. In other words, the accuracy of the prediction of θ becomes arbitrarily good as the information set y^t becomes large. Consequently, the firm and worker eventually learn the true θ, and the value function "at infinity" becomes

$$v_\infty(\theta) = \max\left\{\frac{\theta}{1-\beta}, \beta Q\right\},$$

and the Bellman equation for any finite tenure t is given by equation (6.9.4), and Q in equation (6.9.5) is the value of an unemployed worker. The optimal policy is a reservation wage $\overline{m}_t$, one for each tenure t. In fact, in the absence of a final date $T+1$ when θ is revealed by nature, the solution is actually a time-invariant policy function $\overline{m}(\sigma_t^2)$ with an acceptance and a rejection region in the space of (m, σ^2).

To compute a numerical solution when $T = \infty$, we would still have to rely on the procedure that we have outlined based on the assumption of some finite date when the true θ is revealed, say in period $\hat{T}+1$. The idea is to choose a sufficiently large $\hat{T}$ so that the conditional variance of θ at time $\hat{T}$, $\sigma^2_{\hat{T}}$, is close to zero. We then examine the approximation that $\sigma^2_{\hat{T}+1}$ is equal to zero. That is, equations (6.9.2) and (6.9.3) are used to truncate an otherwise infinite series of value functions.

6.10. Concluding remarks

The situations analyzed in this chapter are ones in which a currently unemployed worker rationally chooses to refuse an offer to work, preferring to remain unemployed today in exchange for better prospects tomorrow. The worker is voluntarily unemployed in one sense, having chosen to reject the current draw from the distribution of offers. In this model, the activity of unemployment is an investment incurred to improve the situation faced in the future. A theory in which unemployment is voluntary permits an analysis of the forces impinging on the choice to remain unemployed. Thus we can study the response of the worker's decision rule to changes in the distribution of offers, the rate of unemployment compensation, the number of offers per period, and so on.

Chapter 22 studies the optimal design of unemployment compensation. That issue is a trivial one in the present chapter with risk-neutral agents and no externalities. Here the government should avoid any policy that affects the workers' decision rules since it would harm efficiency, and the first-best way of pursuing distributional goals is through lump-sum transfers. In contrast, chapter 22 assumes risk-averse agents and incomplete insurance markets, which together with information asymmetries, make for an intricate contract design problem in the provision of unemployment insurance.

Chapter 28 presents various equilibrium models of search and matching. We study workers searching for jobs in an island model, workers and firms forming matches in a model with a "matching function," and how a medium of exchange can overcome the problem of "double coincidence of wants" in a search model of money.

A. More numerical dynamic programming

This appendix describes two more examples using the numerical methods of chapter 4.

6.A.1. Example 4: search

An unemployed worker wants to maximize $E_0 \sum_{t=0}^{\infty} \beta^t y_t$ where $y_t = w$ if the worker is employed at wage w, $y_t = 0$ if the worker is unemployed, and $\beta \in (0,1)$. Each period an unemployed worker draws a positive wage from a discrete-state Markov chain with transition matrix P. Thus, wage offers evolve according to a Markov process with transition probabilities given by

$$P(i,j) = \text{Prob}\left(w_{t+1} = \tilde{w}_j | w_t = \tilde{w}_i\right).$$

Once he accepts an offer, the worker works forever at the accepted wage. There is no firing or quitting. Let v be an $(n \times 1)$ vector of values v_i representing the optimal value of the problem for a worker who has offer $w_i, i = 1, \ldots, n$ in hand and who behaves optimally. The Bellman equation is

$$v_i = \max_{\text{accept,reject}} \left\{ \frac{w_i}{1-\beta}, \beta \sum_{j=1}^{n} P_{ij} v_j \right\}$$

or

$$v = \max\{\tilde{w}/\left(1-\beta\right), \beta P v\}.$$

Here $\tilde{w}$ is an $(n \times 1)$ vector of possible wage values. This matrix equation can be solved using the numerical procedures described earlier. The optimal policy depends on the structure of the Markov chain P. Under restrictions on P making w positively serially correlated, the optimal policy has the following reservation wage form: there is a $\overline{w}$ such that the worker should accept an offer w if $w \geq \overline{w}$.

6.A.2. Example 5: a Jovanovic model

Here is a simplified version of the search model of Jovanovic (1979a). A newly unemployed worker draws a job offer from a distribution given by $\mu_i = \text{Prob}(w_1 = \tilde{w}_i)$, where w_1 is the first-period wage. Let μ be the $(n \times 1)$ vector with ith component μ_i. After an offer is drawn, subsequent wages associated with the job evolve according to a Markov chain with time-varying transition matrices

$$P_t\left(i, j\right) = \text{Prob}\left(w_{t+1} = \tilde{w}_j | w_t = \tilde{w}_i\right),$$

for $t = 1, \ldots, T$. We assume that for times $t > T$, the transition matrices $P_t = I$, so that after T a job's wage does not change anymore with the passage of time. We specify the P_t matrices to capture the idea that the worker-firm pair is learning more about the quality of the match with the passage of time. For example, we might set

$$P_t = \begin{bmatrix} 1-q^t & q^t & 0 & 0 & \ldots & 0 & 0 \\ q^t & 1-2q^t & q^t & 0 & \ldots & 0 & 0 \\ 0 & q^t & 1-2q^t & q^t & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1-2q^t & q^t \\ 0 & 0 & 0 & 0 & \ldots & q^t & 1-q^t \end{bmatrix},$$

where $q \in (0, 1)$. In the following numerical examples, we use a slightly more general form of transition matrix in which (except at endpoints of the distribution),

$$\begin{aligned} \text{Prob}\left(w_{t+1} = \tilde{w}_{k \pm m} | w_t = \tilde{w}_k\right) = P_t\left(k, k \pm m\right) = q^t \\ P_t\left(k, k\right) = 1 - 2q^t. \end{aligned} \tag{6.A.1}$$

Here $m \geq 1$ is a parameter that indexes the spread of the distribution.

At the beginning of each period, a previously matched worker is exposed with probability $\lambda \in (0, 1)$ to the event that the match dissolves. We then have a set of Bellman equations

$$v_t = \max\{\tilde{w} + \beta\left(1-\lambda\right) P_t v_{t+1} + \beta\lambda Q, \beta Q + \overline{c}\}, \tag{6.A.2a}$$

for $t = 1, \ldots, T$, and

$$v_{T+1} = \max\{\tilde{w} + \beta\left(1-\lambda\right) v_{T+1} + \beta\lambda Q, \beta Q + \overline{c}\}, \tag{6.A.2b}$$

$$Q = \mu' v_1 \otimes \mathbf{1}$$
$$\overline{c} = c \otimes \mathbf{1}$$

where $\otimes$ is the Kronecker product, and $\mathbf{1}$ is an $(n \times 1)$ vector of ones. These equations can be solved by using calculations of the kind described previously. The optimal policy is to set a sequence of reservation wages $\{\overline{w}_j\}_{j=1}^T$.

Wage distributions

We can use recursions to compute probability distributions of wages at tenures $1, 2, \ldots, n$. Let the reservation wage for tenure j be $\overline{w}_j \equiv \tilde{w}_{\rho(j)}$, where $\rho(j)$ is the index associated with the cutoff wage. For $i \geq \rho(1)$, define

$$\delta_1(i) = \text{Prob}\{w_1 = \tilde{w}_i \mid w_1 \geq \overline{w}_1\} = \frac{\mu_i}{\sum_{h=\rho(1)}^n \mu_h}.$$

Then

$$\gamma_2(j) = \text{Prob}\{w_2 = \tilde{w}_j \mid w_1 \geq \overline{w}_1\} = \sum_{i=\rho(1)}^n P_1(i,j)\,\delta_1(i).$$

For $i \geq \rho(2)$, define

$$\delta_2(i) = \text{Prob}\{w_2 = \tilde{w}_i \mid w_2 \geq \overline{w}_2 \cap w_1 \geq \overline{w}_1\}$$

or

$$\delta_2(i) = \frac{\gamma_2(i)}{\sum_{h=\rho(2)}^n \gamma_2(h)}.$$

Then

$$\gamma_3(j) = \text{Prob}\{w_3 = \tilde{w}_j \mid w_2 \geq \overline{w}_2 \cap w_1 \geq \overline{w}_1\} = \sum_{i=\rho(2)}^n P_2(i,j)\,\delta_2(i).$$

Next, for $i \geq \rho(3)$, define $\delta_3(i) = \text{Prob}\{w_3 = \tilde{w}_i \mid (w_3 \geq \overline{w}_3) \cap (w_2 \geq \overline{w}_2) \cap (w_1 \geq \overline{w}_1)\}$. Then

$$\delta_3(i) = \frac{\gamma_3(i)}{\sum_{h=\rho(3)}^n \gamma_3(h)}.$$

Continuing in this way, we can define the wage distributions $\delta_1(i), \delta_2(i), \delta_3(i), \ldots$. The mean wage at tenure k is given by

$$\sum_{i \geq \rho(k)} \tilde{w}_i \delta_k(i).$$

Separation probabilities

The probability of rejecting a first period offer is $Q(1) = \sum_{h<\rho(1)} \mu_h$. The probability of separating at the beginning of period $j \geq 2$ is $Q(j) = \sum_{h<\rho(j)} \gamma_j(h)$.

Numerical examples

Figures 6.A.1, 6.A.2, and 6.A.3 report some numerical results for three versions of this model. For all versions, we set $\beta = .95, c = 0, q = .5$, and $T+1 = 21$. For all three examples, we used a wage grid with 60 equispaced points on the interval $[0, 10]$.

For the initial distribution μ we used the uniform distribution. We used a sequence of transition matrices of the form $(6.A.1)$, with a "gap" parameter of m. For the first example, we set $m = 6$ and $\lambda = 0$, while the second sets $m = 10$ and $\lambda = 0$ and third sets $m = 10$ and $\lambda = .1$.

Figure 6.A.1 shows the reservation wage falls as m increases from 6 to 10, and that it falls further when the probability of being fired λ rises from zero to .1. Figure 6.A.2 shows the same pattern for average wages. Figure 6.A.3 displays quit probabilities for the first two models. They fall with tenure, with shapes and heights that depend to some degree on m, λ.

Exercises

Exercise 6.1 **Being unemployed with a chance of an offer**

An unemployed worker samples wage offers on the following terms: each period, with probability ϕ, $1 > \phi > 0$, she receives no offer (we may regard this as a wage offer of zero forever). With probability $(1-\phi)$ she receives an offer to work for w forever, where w is drawn from a cumulative distribution function $F(w)$. Assume that $F(0) = 0, F(B) = 1$ for some $B > 0$. Successive draws across periods are independently and identically distributed. The worker chooses a strategy to maximize

$$E \sum_{t=0}^{\infty} \beta^t y_t, \qquad \text{where} \quad 0 < \beta < 1,$$

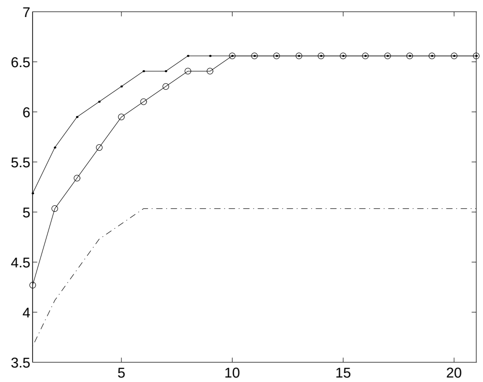

Figure 6.A.1: Reservation wages as a function of tenure for model with three different parameter settings $[m = 6, \lambda = 0]$ (the dots), $[m = 10, \lambda = 0]$ (the line with circles), and $[m = 10, \lambda = .1]$ (the dashed line).

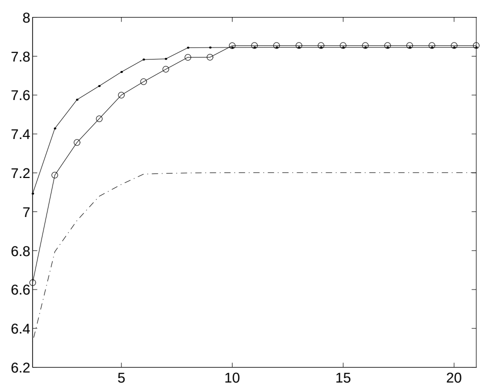

Figure 6.A.2: Mean wages as a function of tenure for model with three different parameter settings $[m = 6, \lambda = 0]$ (the dots), $[m = 10, \lambda = 0]$ (the line with circles), and $[m = 10, \lambda = .1]$ (the dashed line).

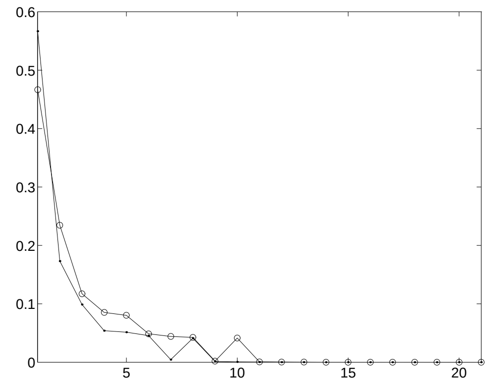

Figure 6.A.3: Quit probabilities as a function of tenure for Jovanovic model with $[m = 6, \lambda = 0]$ (line with dots) and $[m = 10, \lambda = .1]$ (the line with circles).

$y_t = w$ if the worker is employed, and $y_t = c$ if the worker is unemployed. Here c is unemployment compensation, and w is the wage at which the worker is employed. Assume that, having once accepted a job offer at wage w, the worker stays in the job forever.

Let $v(w)$ be the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ for an unemployed worker who has offer w in hand and who behaves optimally. Write the Bellman equation for the worker's problem.

Exercise 6.2 **Two offers per period**

Consider an unemployed worker who each period can draw *two* independently and identically distributed wage offers from the cumulative probability distribution function $F(w)$. The worker will work forever at the same wage after having once accepted an offer. In the event of unemployment during a period, the worker receives unemployment compensation c. The worker derives a decision rule to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$, where $y_t = w$ or $y_t = c$, depending on whether she is employed or unemployed. Let $v(w)$ be the value of $E \sum_{t=0}^{\infty} \beta^t y_t$ for a currently unemployed worker who has best offer w in hand.

a. Formulate the Bellman equation for the worker's problem.

b. Prove that the worker's reservation wage is *higher* than it would be had the worker faced the same c and been drawing only *one* offer from the same distribution $F(w)$ each period.

Exercise 6.3 **A random number of offers per period**

An unemployed worker is confronted with a random number, n, of job offers each period. With probability π_n, the worker receives n offers in a given period, where $\pi_n \geq 0$ for $n \geq 1$, and $\sum_{n=1}^{N} \pi_n = 1$ for $N < +\infty$. Each offer is drawn independently from the same distribution $F(w)$. Assume that the number of offers n is independently distributed across time. The worker works forever at wage w after having accepted a job and receives unemployment compensation of c during each period of unemployment. He chooses a strategy to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$ where $y_t = c$ if he is unemployed, $y_t = w$ if he is employed.

Let $v(w)$ be the value of the objective function of an unemployed worker who has best offer w in hand and who proceeds optimally. Formulate the Bellman equation for this worker.

Exercise 6.4 **Cyclical fluctuations in number of job offers**

Modify exercise *6.3* as follows: Let the number of job offers n follow a Markov process, with

$$\begin{aligned}
&\text{Prob}\{\text{Number of offers next period} = m | \text{Number of offers this period} = n\}\\
&\qquad = \pi_{mn}, \qquad m = 1, \ldots, N, \quad n = 1, \ldots, N\\
&\sum_{m=1}^{N} \pi_{mn} = 1 \qquad \text{for} \quad n = 1, \ldots, N.
\end{aligned}$$

Here $[\pi_{mn}]$ is a "stochastic matrix" generating a Markov chain. Keep all other features of the problem as in exercise *6.3*. The worker gets n offers per period, where n is now generated by a Markov chain so that the number of offers is possibly correlated over time.

a. Let $v(w,n)$ be the value of $E \sum_{t=0}^{\infty} \beta^t y_t$ for an unemployed worker who has received n offers this period, the best of which is w. Formulate the Bellman equation for the worker's problem.

b. Show that the optimal policy is to set a reservation wage $\overline{w}(n)$ that depends on the number of offers received this period.

Exercise 6.5 **Choosing the number of offers**

An unemployed worker must choose the number of offers n to solicit. At a cost of $k(n)$ the worker receives n offers this period. Here $k(n+1) > k(n)$ for $n \geq 1$. The number of offers n must be chosen in advance at the beginning of the period and cannot be revised during the period. The worker wants to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$. Here y_t consists of w each period she is employed but not searching, $[w - k(n)]$ the first period she is employed but searches for n offers, and $[c - k(n)]$ each period she is unemployed but solicits and rejects n offers. The offers are each independently drawn from $F(w)$. The worker who accepts an offer works forever at wage w.

Let Q be the value of the problem for an unemployed worker who has not yet chosen the number of offers to solicit. Formulate the Bellman equation for this worker.

Exercise 6.6 **Mortensen externality**

Two parties to a match (say, worker and firm) jointly draw a match parameter θ from a c.d.f. $F(\theta)$. Once matched, they stay matched forever, each one deriving a benefit of θ per period from the match. Each unmatched pair of agents can influence the number of offers received in a period in the following way. The worker receives n offers per period, where $n = f(c_1 + c_2)$ and c_1 represents the resources the worker devotes to searching and c_2 represents the resources the typical firm devotes to searching. Symmetrically, the representative firm receives n offers per period where $n = f(c_1 + c_2)$. (We shall define the situation so that firms and workers have the same reservation θ so that there is never unrequited love.) Both c_1 and c_2 must be chosen at the beginning of the period, prior to searching during the period. Firms and workers have the same preferences, given by the expected present value of the match parameter θ, net of search costs. The discount factor β is the same for worker and firm.

a. Consider a Nash equilibrium in which party i chooses c_i, taking c_j, $j \neq i$, as given. Let Q_i be the value for an unmatched agent of type i before the level of c_i has been chosen. Formulate the Bellman equation for agents of types 1 and 2.

b. Consider the social planning problem of choosing c_1 and c_2 sequentially so as to maximize the criterion of λ times the utility of agent 1 plus $(1-\lambda)$ times the utility of agent 2, $0 < \lambda < 1$. Let $Q(\lambda)$ be the value for this problem for two unmatched agents before c_1 and c_2 have been chosen. Formulate the Bellman equation for this problem.

c. Comparing the results in a and b, argue that, in the Nash equilibrium, the optimal amount of resources has not been devoted to search.

Exercise 6.7 **Variable labor supply**

An unemployed worker receives each period a wage offer w drawn from the distribution $F(w)$. The worker has to choose whether to accept the job—and therefore to work forever—or to search for another offer and collect c in unemployment compensation. The worker who decides to accept the job must choose the number of hours to work in each period. The worker chooses a strategy to maximize

$$E\sum_{t=0}^{\infty}\beta^t u\left(y_t, l_t\right), \qquad \text{where} \quad 0<\beta<1,$$

and $y_t = c$ if the worker is unemployed, and $y_t = w(1-l_t)$ if the worker is employed and works $(1-l_t)$ hours; l_t is leisure with $0 \leq l_t \leq 1$.

Analyze the worker's problem. Argue that the optimal strategy has the reservation wage property. Show that the number of hours worked is the same in every period.

Exercise 6.8 **Wage growth rate and the reservation wage**

An unemployed worker receives each period an offer to work for wage w_t forever, where $w_t = w$ in the first period and $w_t = \phi^t w$ after t periods on the job. Assume $\phi > 1$, that is, wages increase with tenure. The initial wage offer is drawn from a distribution $F(w)$ that is constant over time (entry-level wages are stationary); successive drawings across periods are independently and identically distributed.

The worker's objective function is to maximize

$$E\sum_{t=0}^{\infty}\beta^t y_t, \qquad \text{where} \quad 0<\beta<1,$$

and $y_t = w_t$ if the worker is employed and $y_t = c$ if the worker is unemployed, where c is unemployment compensation. Let $v(w)$ be the optimal value of the objective function for an unemployed worker who has offer w in hand. Write the Bellman equation for this problem. Argue that, if two economies differ only in the growth rate of wages of employed workers, say $\phi_1 > \phi_2$, the economy with the higher growth rate has the smaller reservation wage. *Note:* Assume that $\phi_i \beta < 1$, $i = 1, 2$.

Exercise 6.9 **Search with a finite horizon**

Consider a worker who lives two periods. In each period the worker, if unemployed, receives an offer of lifetime work at wage w, where w is drawn from a distribution F. Wage offers are identically and independently distributed over time. The worker's objective is to maximize $E\{y_1 + \beta y_2\}$, where $y_t = w$ if the worker is employed and is equal to c—unemployment compensation—if the worker is not employed.

Analyze the worker's optimal decision rule. In particular, establish that the optimal strategy is to choose a reservation wage in each period and to accept any offer with a wage at least as high as the reservation wage and to reject offers below that level. Show that the reservation wage decreases over time.

Exercise 6.10 **Finite horizon and mean-preserving spread**

Consider a worker who draws every period a job offer to work forever at wage w. Successive offers are independently and identically distributed drawings from a distribution $F_i(w)$, $i = 1, 2$. Assume that F_1 has been obtained from F_2 by a mean-preserving spread. The worker's objective is to maximize

$$E \sum_{t=0}^{T} \beta^t y_t, \qquad 0 < \beta < 1,$$

where $y_t = w$ if the worker has accepted employment at wage w and is zero otherwise. Assume that both distributions, F_1 and F_2, share a common upper bound, B.

a. Show that the reservation wages of workers drawing from F_1 and F_2 coincide at $t = T$ and $t = T - 1$.

b. Argue that for $t \leq T - 2$ the reservation wage of the workers that sample wage offers from the distribution F_1 is higher than the reservation wage of the workers that sample from F_2.

c. Now introduce unemployment compensation: the worker who is unemployed collects c dollars. Prove that the result in part a no longer holds; that is, the reservation wage of the workers that sample from F_1 is higher than the one corresponding to workers that sample from F_2 for $t = T - 1$.

Exercise 6.11 **Pissarides' analysis of taxation and variable search intensity**

An unemployed worker receives each period a zero offer (or no offer) with probability $[1 - \pi(e)]$. With probability $\pi(e)$ the worker draws an offer w from the distribution F. Here e stands for effort—a measure of search intensity—and $\pi(e)$ is increasing in e. A worker who accepts a job offer can be fired with probability α, $0 < \alpha < 1$. The worker chooses a strategy, that is, whether to accept an offer or not and how much effort to put into search when unemployed, to maximize

$$E \sum_{t=0}^{\infty} \beta^t y_t, \qquad 0 < \beta < 1,$$

where $y_t = w$ if the worker is employed with wage w and $y_t = 1 - e + z$ if the worker spends e units of leisure searching and does not accept a job. Here z is unemployment compensation. For the worker who searched and accepted a job, $y_t = w - e - T(w)$; that is, in the first period the wage is net of search costs. Throughout, $T(w)$ is the amount paid in taxes when the worker is employed. We assume that $w - T(w)$ is increasing in w. Assume that $w - T(w) = 0$ for $w = 0$, that if $e = 0$, then $\pi(e) = 0$—that is, the worker gets no offers—and that $\pi'(e) > 0$, $\pi''(e) < 0$.

a. Analyze the worker's problem. Establish that the optimal strategy is to choose a reservation wage. Display the condition that describes the optimal choice of e, and show that the reservation wage is independent of e.

b. Assume that $T(w) = t(w - a)$ where $0 < t < 1$ and $a > 0$. Show that an increase in a decreases the reservation wage and increases the level of effort, increasing the probability of accepting employment.

c. Show under what conditions a change in t has the opposite effect.

Exercise 6.12 **Search and financial income**

An unemployed worker receives every period an offer to work forever at wage w, where w is drawn from the distribution $F(w)$. Offers are independently and identically distributed. Every agent has another source of income, which we denote ϵ_t, that may be regarded as financial income. In every period all agents get a realization of ϵ_t, which is independently and identically distributed over time, with distribution function $G(\epsilon)$. We also assume that w_t and ϵ_t are independent. The objective of a worker is to maximize

$$E\sum_{t=0}^{\infty}\beta^t y_t, \qquad 0<\beta<1,$$

where $y_t = w+\phi\epsilon_t$ if the worker has accepted a job that pays w, and $y_t = c+\epsilon_t$ if the worker remains unemployed. We assume that $0<\phi<1$ to reflect the fact that an employed worker has less time to collect financial income. Assume $1 > \text{Prob}\{w \geq c+(1-\phi)\epsilon\} > 0$.

Analyze the worker's problem. Write down the Bellman equation, and show that the reservation wage increases with the level of financial income.

Exercise 6.13 **Search and asset accumulation**

A previously unemployed worker receives an offer to work forever at wage w, but only if he chooses to do so, where w is drawn from the distribution $F(w)$. Previously employed workers receive no offers to work. But a previously employed worker is free to quit in any period, receive unemployment compensation that period, and so become a previously unemployed worker in the following period. Wage offers are identically and independently distributed over time. The worker maximizes

$$E\sum_{t=0}^{\infty}\beta^t\left(u\left(c_t\right)+v\left(l_t\right)\right), \qquad 0<\beta<1,$$

where c_t is consumption and l_t is leisure. Assume that $u(c)$ is strictly increasing, twice continuously differentiable, bounded, and strictly concave, while $v(l)$ is strictly increasing, twice continuously differentiable, and strictly concave; that $c_t \geq 0$; and that $l_t \in \{0,1\}$, so that the person can either work full time (here $l_t = 0$) or not at all (here $l_t = 1$). A gross return on assets a_t held between t and $t+1$ is R_{t+1} and is i.i.d. with c.d.f. $H(R)$. The budget constraint is given by

$$a_{t+1} \leq R_{t+1}\left(a_t + w_t - c_t\right)$$

if the worker has a job that pays w_t. The random gross return R_{t+1} is observed at the beginning of period $t+1$ before the worker chooses n_{t+1}, c_{t+1}. If the worker is unemployed, the budget constraint is $a_{t+1} \leq R_{t+1}(a_t + z - c_t)$ and

$l_t = 1$. Here z is unemployment compensation. It is assumed that a_t, the worker's asset position, cannot be negative. This is a no-borrowing assumption. Write a Bellman equation for this problem.

Exercise 6.14 **Temporary unemployment compensation**

Each period an unemployed worker draws one, and only one, offer to work forever at wage w. Wages are i.i.d. draws from the c.d.f. F, where $F(0) = 0$ and $F(B) = 1$. The worker seeks to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$, where y_t is the sum of the worker's wage and unemployment compensation, if any. The worker is entitled to unemployment compensation in the amount $\gamma > 0$ only during the *first* period that she is unemployed. After one period on unemployment compensation, the worker receives none.

a. Write the Bellman equations for this problem. Prove that the worker's optimal policy is a time-varying reservation wage strategy.

b. Show how the worker's reservation wage varies with the duration of unemployment.

c. Show how the worker's "hazard of leaving unemployment" (i.e., the probability of accepting a job offer) varies with the duration of unemployment.

Now assume that the worker is also entitled to unemployment compensation if she quits a job. As before, the worker receives unemployment compensation in the amount of γ during the first period of an unemployment spell, and zero during the remaining part of an unemployment spell. (To qualify again for unemployment compensation, the worker must find a job and work for at least one period.)

The timing of events is as follows. At the very beginning of a period, a worker who was employed in the previous period must decide whether or not to quit. The decision is irreversible; that is, a quitter cannot return to an old job. If the worker quits, she draws a new wage offer as described previously, and if she accepts the offer she immediately starts earning that wage without suffering any period of unemployment.

d. Write the Bellman equations for this problem. *Hint*: At the very beginning of a period, let $v^e(w)$ denote the value of a worker who was employed in the previous period with wage w (before any wage draw in the current period). Let $v_1^u(w')$ be the value of an unemployed worker who has drawn wage offer

w' and who is entitled to unemployment compensation, if she rejects the offer. Similarly, let $v_+^u(w')$ be the value of an unemployed worker who has drawn wage offer w' but who is not eligible for unemployment compensation.

e. Characterize the three reservation wages, $\overline{w}^e$, $\overline{w}_1^u$, and $\overline{w}_+^u$, associated with the value functions in part d. How are they related to γ? (*Hint*: Two of the reservation wages are straightforward to characterize, while the remaining one depends on the actual parameterization of the model.)

Exercise 6.15 **Seasons, I**

An unemployed worker seeks to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$, where $\beta \in (0,1)$, y_t is her income at time t, and E is the mathematical expectation operator. The person's income consists of one of two parts: unemployment compensation of c that she receives each period she remains unemployed, or a fixed wage w that the worker receives if employed. Once employed, the worker is employed forever with no chance of being fired. Every odd period (i.e., $t = 1, 3, 5, \ldots$) the worker receives one offer to work forever at a wage drawn from the c.d.f. $F(W) = \text{Prob}(w \leq W)$. Assume that $F(0) = 0$ and $F(B) = 1$ for some $B > 0$. Successive draws from F are independent. Every even period (i.e., $t = 0, 2, 4, \ldots$), the unemployed worker receives two offers to work forever at a wage drawn from F. Each of the two offers is drawn independently from F.

a. Formulate the Bellman equations for the unemployed person's problem.

b. Describe the form of the worker's optimal policy.

Exercise 6.16 **Seasons, II**

Consider the following problem confronting an unemployed worker. The worker wants to maximize

$$E_0 \sum_0^{\infty} \beta^t y_t, \quad \beta \in (0,1),$$

where $y_t = w_t$ in periods in which the worker is employed and $y_t = c$ in periods in which the worker is unemployed, where w_t is a wage rate and c is a constant level of unemployment compensation. At the start of each period, an unemployed worker receives one and only one offer to work at a wage w drawn from a c.d.f. $F(W)$, where $F(0) = 0, F(B) = 1$ for some $B > 0$. Successive draws from F are identically and independently distributed. There is no recall of past offers. Only unemployed workers receive wage offers. The wage is fixed as long as the worker remains in the job. The only way a worker can leave a job is if she is fired. At the *beginning* of each odd period ($t = 1, 3, \ldots$), a previously employed worker faces the probability of $\pi \in (0,1)$ of being fired. If a worker is fired, she immediately receives a new draw of an offer to work at wage w. At each even period ($t = 0, 2, \ldots$), there is no chance of being fired.

a. Formulate a Bellman equation for the worker's problem.

b. Describe the form of the worker's optimal policy.

Exercise 6.17 **Gittins indexes for beginners**

At the end of each period,[19] a worker can switch between two jobs, A and B, to begin the following period at a wage that will be drawn at the beginning of next period from a wage distribution specific to job A or B, and to the worker's history of past wage draws from jobs of either type A or type B. The worker must decide to stay or leave a job at the end of a period after his wage for this period on his current job has been received, but before knowing what his wage would be next period in either job. The wage at either job is described by a job-specific n-state Markov chain. Each period the worker works at either job A or job B. At the end of the period, before observing next period's wage on either job, he chooses which job to go to next period. We use lowercase letters ($i, j = 1, \ldots, n$) to denote states for job A, and uppercase letters ($I, J = 1, \ldots n$) for job B. There is no option of being unemployed.

Let $w_a(i)$ be the wage on job A when state i occurs and $w_b(I)$ be the wage on job B when state I occurs. Let $A = [A_{ij}]$ be the matrix of one-step transition probabilities between the states on job A, and let $B = [B_{ij}]$ be the matrix for job B. If the worker leaves a job and later decides to return to it, he draws the wage for his first new period on the job from the conditional distribution determined by his last wage working at that job.

The worker's objective is to maximize the expected discounted value of his lifetime earnings, $E_0 \sum_{t=0}^{\infty} \beta^t y_t$, where $\beta \in (0,1)$ is the discount factor, and where y_t is his wage from whichever job he is working at in period t.

a. Consider a worker who has worked at both jobs before. Suppose that $w_a(i)$ was the last wage the worker receives on job A and $w_b(I)$ the last wage on job B. Write the Bellman equation for the worker.

[19] See Gittins (1989) for more general versions of this problem.

b. Suppose that the worker is just entering the labor force. The first time he works at job A, the probability distribution for his initial wage is $\pi_a = (\pi_{a1}, \ldots, \pi_{an})$. Similarly, the probability distribution for his initial wage on job B is $\pi_b = (\pi_{b1}, \ldots, \pi_{bn})$ Formulate the decision problem for a new worker, who must decide which job to take initially. *Hint*: Let $v_a(i)$ be the expected discounted present value of lifetime earnings for a worker who was last in state i on job A and has never worked on job B; define $v_b(I)$ symmetrically.

Exercise 6.18 **Jovanovic (1979b)**

An employed worker in the tth period of tenure on the current job receives a wage $w_t = x_t(1 - \phi_t - s_t)$ where x_t is job-specific human capital, $\phi_t \in (0,1)$ is the fraction of time that the worker spends investing in job-specific human capital, and $s_t \in (0,1)$ is the fraction of time that the worker spends searching for a new job offer. If the worker devotes s_t to searching at t, then with probability $\pi(s_t) \in (0,1)$ at the beginning of $t+1$ the worker receives a new job offer to begin working at new job-specific capital level μ' drawn from the c.d.f. $F(\cdot)$. That is, searching for a new job offer promises the prospect of instantaneously reinitializing job-specific human capital at μ'. Assume that $\pi'(s) > 0, \pi''(s) < 0$. While on a given job, job-specific human capital evolves according to

$$x_{t+1} = G(x_t, \phi_t) = g(x_t \phi_t) - \delta x_t,$$

where $g'(\cdot) > 0, g''(\cdot) < 0$, $\delta \in (0,1)$ is a depreciation rate, and $x_0 = \mu$ where t is tenure on the job, and μ is the value of the "match" parameter drawn at the start of the current job. The worker is risk neutral and seeks to maximize $E_0 \sum_{\tau=0}^{\infty} \beta^\tau y_\tau$, where y_τ is his wage in period τ.

a. Formulate the worker's Bellman equation.

b. Describe the worker's decision rule for deciding whether to accept an offer μ' at the beginning of next period.

c. Assume that $g(x\phi) = A(x\phi)^\alpha$ for $A > 0, \alpha \in (0,1)$. Assume that $\pi(s) = s^{.5}$. Assume that F is a discrete n-valued distribution with probabilities f_i; for example, let $f_i = n^{-1}$. Write a Matlab program to solve the Bellman equation. Compute the optimal policies for ϕ, s and display them.

Exercise 6.19 **Value function iteration and policy improvement algorithm**, donated by Pierre-Olivier Weill

The goal of this exercise is to study, in the context of a specific problem, two methods for solving dynamic programs: value function iteration and Howard's policy improvement. Consider McCall's model of intertemporal job search. An unemployed worker draws one offer from a c.d.f. F, with $F(0) = 0$ and $F(B) = 1$, $B < \infty$. If the worker rejects the offer, she receives unemployment compensation c and can draw a new wage offer next period. If she accepts the offer, she works forever at wage w. The objective of the worker is to maximize the expected discounted value of her earnings. Her discount factor is $0 < \beta < 1$.

a. Write the Bellman equation. Show that the optimal policy is of the reservation wage form. Write an equation for the reservation wage w^*.

b. Consider the value function iteration method. Show that at each iteration, the optimal policy is of the reservation wage form. Let w_n be the reservation wage at iteration n. Derive a recursion for w_n. Show that w_n converges to w^* at rate β.

c. Consider Howard's policy improvement algorithm. Show that at each iteration, the optimal policy is of the reservation wage form. Let w_n be the reservation wage at iteration n. Derive a recursion for w_n. Show that the rate of convergence of w_n towards w^* is locally quadratic. Specifically use a Taylor expansion to show that, for w_n close enough to w^*, there is a constant K such that $w_{n+1} - w^* \cong K(w_n - w^*)^2$.

Exercise 6.20

Different types of unemployed workers are identical, except that they sample from different wage distributions. Each period an unemployed worker of type α draws a single new offer to work forever at a wage w from a cumulative distribution function F_α that satisfies $F_\alpha(w) = 0$ for $w < 0$, $F_\alpha(0) = \alpha, F_\alpha(B) = 1$, where $B > 0$ and F_α is a right continuous function mapping $[0, B]$ into $[0, 1]$. The c.d.f. of a type α worker is given by

$$F_\alpha(w) = \begin{cases} \alpha & \text{for } 0 \le w \le \alpha B \ ; \\ w/B & \text{for } \alpha B < w < B - \alpha B \ ; \\ 1-\alpha & \text{for } B - \alpha B \le w < B; \\ 1 & \text{for } w = B \end{cases}$$

where $\alpha \in [0, .5)$. An unemployed α worker seeks to maximize the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$, where $\beta \in (0,1)$ and $y_t = w$ if the worker is employed and $y_t = c$ if he or she is unemployed, where $0 < c < B$ is a constant level of unemployment compensation. By choosing a strategy for accepting or rejecting job offers, the worker affects the distribution with respect to which the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ is calculated. The worker cannot recall past offers. If a previously unemployed worker accepts an offer to work at wage w this period, he must work forever at that wage (there is neither quitting nor firing nor searching for a new job while employed).

a. Formulate a Bellman equation for a type α worker. Prove that the worker's optimal strategy is to set a time-invariant reservation wage.

b. Consider two types of workers, $\alpha = 0$ and $\alpha = .3$. Can you tell which type of worker has a higher reservation wage?

c. Which type of worker would you expect to find a job more quickly?

Exercise 6.21 **Searching for the lowest price**

A buyer wants to purchase an item at the lowest price, net of total search costs. At a cost of $c > 0$ per draw, the buyer can draw an offer to buy the item at a price p that is drawn from the c.d.f. $F(P) = \text{Prob}(p \leq P)$ where P is a non-decreasing, right-continuous function with $F(\underline{B}) = 0, F(\overline{B}) = 1$, where $0 < \underline{B} < \overline{B} < +\infty$. All search occurs within one period.

a. Find the buyer's optimal strategy.

b. Find an expression for the expected value of the purchase price net of all search costs.

Exercise 6.22 **Quits**

Each period an unemployed worker draws one offer to work at a nonnegative wage w, where w is governed by a c.d.f F that satisfies $F(0) = 0$ and $F(B) = 1$ for some $B > 0$. The worker seeks to maximize the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ where $y_t = w$ if the worker is employed and c if the worker is unemployed. At the beginning of each period a worker employed at wage w the previous period is exposed to a probability of $\alpha \in (0,1)$ of having his job reclassified, which means that he will be given a new wage w' drawn from F. A reclassified worker has the option of working at wage w' until reclassified again, or quitting, receiving unemployment compensation of c this period, and drawing a new wage offer the next period.

a. Formulate the Bellman equation for an unemployed worker.

b. Describe the decision rule for a previously *unemployed* worker.

c. Describe the decision rule for quitting or staying for a previously *employed* worker.

d. Describe how to compute the probability that a previously employed worker will quit.

Exercise 6.23 **A career ladder**

Each period a previously unemployed worker draws one offer to work at a non-negative wage w, where w is governed by a c.d.f F that satisfies $F(0) = 0$ and $F(B) = 1$ for some $B > 0$. The worker seeks to maximize the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ where $\beta \in (0,1)$ and $y_t = w$ if the worker is employed and c if the worker is unemployed. At the beginning of each period a worker employed at wage w the previous period is exposed to a probability of $\alpha \in (0,1)$ of getting a promotion, which means that he will be given a new wage γw where $\gamma > 1$. This new wage will prevail until a next promotion.

a. Formulate a Bellman equation for a previously employed worker.

b. Formulate a Bellman equation for a previously unemployed worker.

c. Describe the decision rule for an unemployed worker.

d. Describe the decision rule for a previously employed worker.

Exercise 6.24 **Human capital**

A previously unemployed worker draws one offer to work at a wage wh, where h is his level of human capital and w is drawn from a c.d.f. F where $F(0) = 0, F(B) = 1$ for $B > 0$. The worker retains w, but not h, so long as he remains in his current job (or employment spell). The worker knows his current level of h before he draws w. Wage draws are independent over time. When employed, the worker's human capital h evolves according to a discrete state Markov chain on the space $[\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n]$ with transition density H^e where $H^e(i,j) = \text{Prob}[h_{t+1} = \bar{h}_j | h_t = \bar{h}_i]$. When unemployed, the worker's human capital h evolves according to a discrete state Markov chain on the same space $[\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n]$ with transition density H^u where $H^u(i,j) = \text{Prob}[h_{t+1} = \bar{h}_j | h_t = \bar{h}_i]$. The two transition matrices H^e and H^u are such that an employed worker's human capital grows probabilistically (meaning that it is more likely that next period's human capital will be higher than this period's) and an unemployed worker's human capital decays probabilistically (meaning that it is more likely that next period's human capital will be lower than this period's). An unemployed worker receives unemployment compensation of c per period. The worker wants to maximize the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ where $y_t = w h_t$ when employed, and c when unemployed. At the beginning of each period, employed workers receive their new human capital realization from the Markov chain H^e. Then they are free to quit, meaning that they surrender their previous w, retain their newly realized level of human capital but immediately become unemployed, and can immediately draw a new w from F. They can accept that new draw immediately or else choose to be unemployed for at least one period while waiting for new opportunities to draw one w offer per period from F.

a. Obtain a Bellman equation or Bellman equations for the worker's problem.

b. Describe qualitatively the worker's optimal decision rule. Do you think employed workers might ever decide to quit?

c. Describe an algorithm to solve the Bellman equation or equations.

Exercise 6.25 **Markov wages**

Each period, a previously unemployed worker draws one offer to work forever at wage w. The worker wants to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$, where $\beta \in (0,1)$ and $y_t = c > 0$ if the worker is unemployed, and $y_t = w$ if the worker is employed. Quitting is not allowed and once hired the worker cannot be fired. Successive draws of the wage are from a Markov chain with transition probabilities arranged in the $n \times n$ transition matrix P with (i,j) element $P_{ij} = \text{Prob}(w_{t+1} = \overline{w}_j | w_t = \overline{w}_i)$ where $\overline{w}_1 < \overline{w}_2 < \cdots < \overline{w}_n$.

a. Construct a Bellman equation for the worker.

b. Can you prove that the worker's optimal strategy is to set a reservation wage?

c. Assume that $\beta = .95$, $c = 1$, $[\,\overline{w}_1 \quad \overline{w}_2 \quad \cdots \quad \overline{w}_n\,] = [\,1 \quad 2 \quad 3 \quad 4 \quad 5\,]$ and

$$P = \begin{bmatrix} .8 & .2 & 0 & 0 & 0 \\ .18 & .8 & .02 & 0 & 0 \\ .25 & .25 & 0 & .25 & .25 \\ 0 & 0 & .02 & .8 & .18 \\ 0 & 0 & 0 & .2 & .8 \end{bmatrix}.$$

Please write a Matlab or R or C++ program to solve the Bellman equation. Show the optimal policy function and the value function.

d. Assume that all parameters are the same as in part **c** except for β, which now equals .99. Please find the optimal policy function and the optimal value function.

e. Please discuss whether, why, and how your answers to parts **c** and **d** differ.

Exercise 6.26 **Neal model with unemployment**

Consider the following modification of the Neal (1999) model. A worker chooses career-job (θ, ϵ) pairs subject to the following conditions. If employed, the worker's earnings at time t equal $\theta_t + \epsilon_t$, where θ_t is a component specific to a *career* and ϵ_t is a component specific to a particular *job*. If unemployed, the worker receives unemployment compensation equal to c. The worker maximizes $E \sum_{t=0}^{\infty} \beta^t y_t$ where $y_t = (\theta_t + \epsilon_t)$ if the worker is employed and $y_t = c$ if the worker is unemployed. A *career* is a draw of θ from c.d.f. F; a *job* is a draw of ϵ from c.d.f. G. Successive draws are independent, and $G(0) = F(0) = 0$, $G(B_\epsilon) = F(B_\theta) = 1$. The worker can draw a new career only if he also draws a new job. However, the worker is free to retain his existing career (θ), and to draw a new job (ϵ') next period. The worker decides at the beginning of a period whether to stay in a career-job pair inherited from the past, stay in the inherited career but draw a new job *for next period*, or draw a new career-job pair (θ', ϵ') *for next period*. If the worker decides to draw either a new θ' or a new ϵ' for next period, he or she must become unemployed this period.

a. Let $v(\theta, \epsilon)$ be the optimal value of the problem at the beginning of a period for a worker currently having inherited career-job pair (θ, ϵ) and who is about to decide whether to decide whether to become unemployed in order to draw a new career and or job next period. Formulate a Bellman equation.

b. Characterize the worker's optimal policy.

Part III

Competitive equilibria and applications

Chapter 7
Recursive (Partial) Equilibrium

7.1. An equilibrium concept

This chapter formulates competitive and oligopolistic equilibria in some dynamic settings. Up to now, we have studied single-agent problems where components of the state vector not under the control of the agent were taken as given. In this chapter, we describe multiple-agent settings in which components of the state vector that one agent takes as exogenous are determined by the decisions of other agents. We study partial equilibrium models of a kind applied in microeconomics.[1] We describe two closely related equilibrium concepts for such models: a rational expectations or recursive competitive equilibrium, and a Markov perfect equilibrium. The first equilibrium concept jointly restricts a Bellman equation and a transition law that is taken as given in that Bellman equation. The second equilibrium concept leads to pairs (in the duopoly case) or sets (in the oligopoly case) of Bellman equations and transition equations that are to be solved by simultaneous backward induction.

Though the equilibrium concepts introduced in this chapter transcend linear quadratic setups, we choose to present them in the context of linear quadratic examples because this renders the Bellman equations tractable.

[1] For example, see Rosen and Topel (1988) and Rosen, Murphy, and Scheinkman (1994)

7.2. Example: adjustment costs

This section describes a model of a competitive market with producers who face adjustment costs.[2] In the course of the exposition, we introduce and exploit a version of the 'big K, little k' trick that is widely used in macroeconomics and applied economic dynamics.[3] The model consists of n identical firms whose profit function makes them want to forecast the aggregate output decisions of other firms just like them in order to choose their own output. We assume that n is a large number so that the output of any single firm has a negligible effect on aggregate output and, hence, firms are justified in treating their forecast of aggregate output as unaffected by their own output decisions. Thus, one of n competitive firms sells output y_t and chooses a production plan to maximize

$$\sum_{t=0}^{\infty} \beta^t R_t \tag{7.2.1}$$

where

$$R_t = p_t y_t - .5d\left(y_{t+1} - y_t\right)^2 \tag{7.2.2}$$

subject to y_0 being a given initial condition. Here $\beta \in (0,1)$ is a discount factor, and $d > 0$ measures a cost of adjusting the rate of output. The firm is a price taker. The price p_t lies on the inverse demand curve

$$p_t = A_0 - A_1 Y_t \tag{7.2.3}$$

where $A_0 > 0, A_1 > 0$ and Y_t is the market-wide level of output, being the sum of output of n identical firms. The firm believes that market-wide output follows the law of motion

$$Y_{t+1} = H_0 + H_1 Y_t \equiv H\left(Y_t\right), \tag{7.2.4}$$

where Y_0 is a known initial condition. The belief parameters H_0, H_1 are equilibrium objects, but for now we proceed on faith and take them as given. The firm observes Y_t and y_t at time t when it chooses y_{t+1}. The adjustment cost $d(y_{t+1} - y_t)^2$ gives the firm the incentive to forecast the market price, but since

[2] The model is a version of one analyzed by Lucas and Prescott (1971) and Sargent (1987a). The recursive competitive equilibrium concept was used by Lucas and Prescott (1971) and described further by Prescott and Mehra (1980).

[3] Also see section 12.8 of chapter 12.

the market price is a function of market output Y_t via the demand equation (7.2.3), this in turn motivates the firm to forecast future values of Y. To state the firm's optimization problem completely requires that we specify laws of motion for all state variables, including ones like Y that it cares about but does not control. For this reason, the perceived law of motion (7.2.4) for Y is among the constraints that the firm faces.

Substituting equation (7.2.3) into equation (7.2.2) gives

$$R_t = (A_0 - A_1 Y_t)\, y_t - .5d\, (y_{t+1} - y_t)^2 .$$

The firm's incentive to forecast the market price translates into an incentive to forecast the level of market output Y. We can write the Bellman equation for the firm as

$$v\,(y, Y) = \max_{y'} \left\{ A_0 y - A_1 y Y - .5d\, (y' - y)^2 + \beta v\, (y', Y') \right\} \tag{7.2.5}$$

where the maximization is subject to the perceived law of motion $Y' = H(Y)$. Here $'$ denotes next period's value of a variable. The Euler equation for the firm's problem is

$$-d\,(y' - y) + \beta v_y\, (y', Y') = 0. \tag{7.2.6}$$

Noting that for this problem the control is y' and applying the Benveniste-Scheinkman formula from chapter 3 gives

$$v_y\,(y, Y) = A_0 - A_1 Y + d\,(y' - y)\,.$$

Substituting this equation into equation (7.2.6) gives

$$-d\,(y_{t+1} - y_t) + \beta\,[A_0 - A_1 Y_{t+1} + d\,(y_{t+2} - y_{t+1})] = 0. \tag{7.2.7}$$

In the process of solving its Bellman equation, the firm sets an output path that satisfies equation (7.2.7), taking equation (7.2.4) as given, subject to the initial conditions (y_0, Y_0) as well as an extra terminal condition. The terminal condition is

$$\lim_{t \to \infty} \beta^t y_t v_y\,(y_t, Y_t) = 0. \tag{7.2.8}$$

This is called the transversality condition and acts as a first-order necessary condition "at infinity." The firm's decision rule solves the difference equation (7.2.7) subject to the given initial condition y_0 and the terminal condition

(7.2.8). Solving Bellman equation (7.2.5) by backward induction automatically incorporates both equations (7.2.7) and (7.2.8).

The firm's optimal policy function is

$$y_{t+1} = h\left(y_t, Y_t\right). \tag{7.2.9}$$

Then with n identical firms, setting $Y_t = n y_t$ makes the actual law of motion for output for the market

$$Y_{t+1} = nh\left(Y_t/n, Y_t\right). \tag{7.2.10}$$

Thus, when firms believe that the law of motion for market-wide output is equation (7.2.4), their optimizing behavior makes the actual law of motion equation (7.2.10).

For this model, we adopt the following definition:

DEFINITION: A recursive competitive equilibrium[4] of the model with adjustment costs is a value function $v(y, Y)$, an optimal policy function $h(y, Y)$, and a law of motion $H(Y)$ such that

a. Given H, $v(y, Y)$ satisfies the firm's Bellman equation and $h(y, Y)$ is the optimal policy function.

b. The law of motion H satisfies $H(Y) = nh(Y/n, Y)$.

A recursive competitive equilibrium equates the actual and perceived laws of motion (7.2.4) and (7.2.10). The firm's optimum problem induces a mapping $\mathcal{M}$ from a perceived law of motion for output H to an actual law of motion $\mathcal{M}(H)$. The mapping is summarized in equation (7.2.10). The H component of a rational expectations equilibrium is a fixed point of the operator $\mathcal{M}$.

This is a special case of a recursive competitive equilibrium, to be defined more generally in section 7.3. How might we find an equilibrium? The mapping $\mathcal{M}$ is not a contraction and there is no guarantee that direct iterations on $\mathcal{M}$ will converge.[5] In fact, in many contexts, including the present one, there exist admissible parameter values for which divergence of iterations on $\mathcal{M}$ prevails.

[4] This is also often called a rational expectations equilibrium.

[5] A literature that studies whether models populated with agents who learn can converge to rational expectations equilibria features iterations on a modification of the mapping $\mathcal{M}$ that can be approximated as $\gamma\mathcal{M} + (1-\gamma)I$ where I is the identity operator and $\gamma \in (0,1)$ is a relaxation parameter. See Marcet and Sargent (1989) and Evans and Honkapohja (2001) for

The next subsection shows another method that works when the equilibrium solves an associated planning problem. For convenience, we'll assume from now on that the number of firms n is one, while retaining the assumption of price-taking behavior.

7.2.1. A planning problem

Our approach to computing an equilibrium is to seek to match the Euler equations of the market problem with those for a planning problem that can be posed as a single-agent dynamic programming problem. The optimal quantities from the planning problem are then the recursive competitive equilibrium quantities, and the equilibrium price is a shadow price in the planning problem.

For convenience we set $n = 1$. To construct a planning problem, we first compute the sum S_t of consumer and producer surplus at time t, defined as

$$S_t = S(Y_t, Y_{t+1}) = \int_0^{Y_t} (A_0 - A_1 x)\, dx - .5d (Y_{t+1} - Y_t)^2 . \tag{7.2.11}$$

The first term is the area under the demand curve. The planning problem is to choose a production plan to maximize

$$\sum_{t=0}^{\infty} \beta^t S(Y_t, Y_{t+1}) \tag{7.2.12}$$

subject to an initial condition Y_0. The Bellman equation for the planning problem is

$$V(Y) = \max_{Y'} \left\{ A_0 Y - \frac{A_1}{2} Y^2 - .5d (Y' - Y)^2 + \beta V(Y') \right\} . \tag{7.2.13}$$

The Euler equation is

$$-d(Y' - Y) + \beta V'(Y') = 0. \tag{7.2.14}$$

statements and applications of this approach to establish conditions under which collections of adaptive agents who use least squares learning converge to a rational expectations equilibrium. The Marcet-Sargent-Evans-Honkapohja approach provides foundations for a method that Krusell and Smith (1998) use to approximation a rational expectations equilibrium of an incomplete-markets economy. See chapter 18.

Applying the Benveniste-Scheinkman formula gives

$$V'(Y) = A_0 - A_1 Y + d(Y' - Y). \tag{7.2.15}$$

Substituting this into equation (7.2.14) and rearranging gives

$$\beta A_0 + dY_t - [\beta A_1 + d(1+\beta)] Y_{t+1} + d\beta Y_{t+2} = 0. \tag{7.2.16}$$

Return to equation (7.2.7) and set $y_t = Y_t$ for all t. (Remember that we have set $n = 1$. When $n \neq 1$ we have to adjust pieces of the argument for n.) Notice that with $y_t = Y_t$, equations (7.2.16) and (7.2.7) are identical. The Euler equation for the planning problem matches the second-order difference equation that we derived by first finding the Euler equation of the representative firm and substituting into it the expression $Y_t = ny_t$ that "makes the representative firm representative". Thus, if it is appropriate to apply the same terminal conditions for these two difference equations, which it is, then we have verified that a solution of the planning problem also is an equilibrium. Setting $y_t = Y_t$ in equation (7.2.7) amounts to dropping equation (7.2.4) and instead solving for the coefficients H_0, H_1 that make $y_t = Y_t$ true and that jointly solve equations (7.2.4) and (7.2.7).

It follows that for this example we can compute an equilibrium by forming the optimal linear regulator problem corresponding to the Bellman equation (7.2.13). The optimal policy function for this problem is the law of motion $Y' = H(Y)$ that a firm faces within a rational expectations equilibrium.[6]

[6] Lucas and Prescott (1971) used the method of this section. The method exploits the connection between equilibrium and Pareto optimality expressed in the fundamental theorems of welfare economics. See Mas-Colell, Whinston, and Green (1995).

7.3. Recursive competitive equilibrium

The equilibrium concept of the previous section is widely used. Following Prescott and Mehra (1980), it is useful to define the equilibrium concept more generally as a *recursive competitive equilibrium.* Let x be a vector of state variables under the control of a representative agent and let X be the vector of those same variables chosen by "the market." Let Z be a vector of other state variables chosen by "nature," that is, determined outside the model. The representative agent's problem is characterized by the Bellman equation

$$v(x, X, Z) = \max_u \{R(x, X, Z, u) + \beta v(x', X', Z')\} \tag{7.3.1}$$

where $'$ denotes next period's value, and where the maximization is subject to the restrictions:

$$x' = g(x, X, Z, u) \tag{7.3.2}$$

$$X' = G(X, Z) \tag{7.3.3}$$

$$Z' = \zeta(Z). \tag{7.3.4}$$

Here g describes the impact of the representative agent's controls u on his state x'; G and ζ describe his beliefs about the evolution of the aggregate state. The solution of the representative agent's problem is a decision rule

$$u = h(x, X, Z). \tag{7.3.5}$$

To make the representative agent representative, we impose $X = x$, but only "after" we have solved the agent's decision problem. Substituting equation (7.3.5) and $X = x_t$ into equation (7.3.2) gives the *actual* law of motion

$$X' = G_A(X, Z), \tag{7.3.6}$$

where $G_A(X, Z) \equiv g[X, X, Z, h(X, X, Z)]$. We are now ready to propose a definition:

DEFINITION: A *recursive competitive equilibrium* is a policy function h, an actual aggregate law of motion G_A, and a perceived aggregate law G such that (a) given G, h solves the representative agent's optimization problem; and (b) h implies that $G_A = G$.

This equilibrium concept is also sometimes called a *rational expectations equilibrium*. The equilibrium concept makes G an outcome. The functions giving the representative agent's expectations about the aggregate state variables contribute no free parameters and are *outcomes* of the analysis. There are no free parameters that characterize expectations.[7]

7.4. Equilibrium human capital accumulation

As an example of a recursive competitive equilibrium, we formulate what we regard as a schooling model of the type used by Sherwin Rosen. A household chooses an amount of labor to send to a school that takes four periods to produce an educated worker. Time is a principal input into the schooling technology.

7.4.1. Planning problem

A planner chooses a contingency plan for new entrants n_t to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ f_0 + (f_1 + \theta_t) N_t - \frac{f_2}{2} N_t^2 - \frac{d}{2} n_t^2 \right\}$$

subject to the laws of motion

$$\begin{aligned} \theta_{t+1} &= \rho \theta_t + \sigma_\theta \epsilon_{t+1} \\ N_{t+1} &= \delta N_t + n_{t-3}, \end{aligned} \tag{7.4.1}$$

where N_t is the stock of educated labor at time t, n_t is the number of new entrants into school at time t, $\delta \in (0,1)$ is one minus a depreciation rate, θ_t is a technology shock, and ϵ_{t+1} is an i.i.d. random process distributed as $\mathcal{N}(0,1)$. The planner confronts initial conditions $\theta_0, N_0, n_{-1}, n_{-2}, n_{-3}$. Notice how (7.4.1) incorporates a four period time to build stocks of labor. The planner's problem can be formulated as a stochastic discounted optimal linear regulator problem, i.e., a linear-quadratic dynamic programming problem of the type studied in chapter 5. We ask the reader to verify that it suffices to take $X_t = \begin{bmatrix} \theta_t \\ N_{t+3} \end{bmatrix}$ as the state for the planner's problem. A solution of the planner's

[7] This is the sense in which rational expectations models make expectations disappear.

problem is then a law of motion $X_{t+1} = (A - BF)X_t + C\epsilon_{t+1}$ and a decision rule $n_t = -FX_t$.

For the purpose of defining a recursive competitive equilibrium, it is useful to note that it is also possible to define the state for the planner's problem more profligately as $\tilde{X}_t = [\theta \quad N_t \quad n_{t-1} \quad n_{t-2} \quad n_{t-3}]'$ with associated decision rule $n_t = -\tilde{F}\tilde{X}_t$ and law of motion

$$\tilde{X}_{t+1} = \left(\tilde{A} - \tilde{B}\tilde{F}\right)\tilde{X}_t + \tilde{C}\epsilon_{t+1}. \tag{7.4.2}$$

We can use this representation to express a shadow wage $\tilde{w}_t = f_1 - f_2 N_t + \theta_t$ as $\tilde{w}_t = S_w \tilde{X}_t$.

7.4.2. Decentralization

A firm and a representative household are price takers in a recursive competitive equilibrium. The firm faces a competitive wage process $\{w_t\}_{t=0}^{\infty}$ as a price taker and chooses a contingency plan for $\{N_t\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ f_0 + (f_1 + \theta_t) N_t - \frac{f_2}{2} N_t^2 - w_t N_t \right\}.$$

The first-order condition for the firm's problem is

$$w_t = f_1 - f_2 N_t + \theta_t, \tag{7.4.3}$$

which we can regard as an inverse demand function for the stock of labor.

A representative household chooses a contingency plan for $\{n_t, N_{t+4}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ w_t N_t - \frac{d}{2} n_t^2 \right\} \tag{7.4.4}$$

subject to (7.4.1) and initial conditions in the form of given values for N_t for $t = 0, 1, 2, 3$. To deduce first order conditions for this problem, it is helpful first to notice that (7.4.1) implies that for $j \geq 4$,

$$N_{t+j} = \delta^{j-3} N_{t+1} + \delta^{j-4} n_t + \delta^{j-3} n_{t+1} + \ldots \delta n_{t+j-5} + n_{t+j-4}, \tag{7.4.5}$$

so that

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j w_{t+j} N_{t+j}}{\partial n_t} = \beta^4 \sum_{j=0}^{\infty} (\beta\delta)^j \, w_{t+j+4}.$$

It follows that the first-order conditions for maximizing (7.4.4) subject to (7.4.1) are

$$n_t = d^{-1} E_t \beta^4 \sum_{j=0}^{\infty} (\beta\delta)^j \, w_{t+j+4}, \quad t \geq 0 \tag{7.4.6}$$

We can regard (7.4.6) as a supply curve for a flow of new entrants into the schooling technology. It expresses the supply of new entrants into school n_t as a linear function of the expected present value of wages.

A rational expectations equilibrium is a stochastic process $\{w_t, N_t, n_t\}$ such that (a) given the w_t process, N_t, n_t solves the household's problem, and (b) given the w_t process, N_t solves the firms' problem. Evidently, a rational expectations equilibrium can also be characterized as a $\{w_t, N_t, n_t\}$ process that equates demand for labor (equation (7.4.3)) to supply of labor (equations (7.4.5) and (7.4.6)).

To formulate the firm's and household's problems within a recursive competitive equilibrium, we can guess that the shadow wage $\tilde{w}_t$ mentioned above equals the competitive equilibrium wage. We can then confront the household with an exogenous wage governed by the stochastic process for w_t governed by the state space representation

$$\begin{aligned} \tilde{X}_{t+1} &= \left(\tilde{A} - \tilde{B}\tilde{F}\right) \tilde{X}_t + \tilde{C}\epsilon_{t+1} \\ w_t &= S_w \tilde{X}_t. \end{aligned}$$

7.5. Equilibrium occupational choice

As another example of a recursive competitive equilibrium, we formulate a modification of a Rosen schooling model designed to focus on occupational choice.[8] Like the model in the previous section, this one focuses on the cost of acquiring human capital via a time-to-build technology. Investment times now differ across occupations.

Output of a single good is produced via the following production function:

$$Y_t = f_0 + f_1 \begin{bmatrix} U_t \\ S_t \end{bmatrix} - \begin{bmatrix} U_t \\ S_t \end{bmatrix}' f_2 \begin{bmatrix} U_t \\ S_t \end{bmatrix} \tag{7.5.1}$$

[8] For applications see Siow (1984) and Ryoo and Rosen (2004).

where U_t is a stock of skilled labor and S_t is a stock of unskilled labor, and f_2 is a positive semi-definite matrix parameterizing whether skilled and unskilled labor are complements or substitutes in production. Stocks of the two types of labor evolve according to the laws of motion

$$\begin{aligned} U_{t+1} &= \delta_U U_t + n_{Ut} \\ S_{t+1} &= \delta_S S_t + n_{St-2} \end{aligned} \tag{7.5.2}$$

where flows into the two types of skills are restricted by

$$n_{Ut} + n_{St} = n_t, \tag{7.5.3}$$

where n_t is an exogenous flow of new entrants into the labor market governed by the stochastic process

$$n_{t+1} = \mu_n (1-\rho) + \rho n_t + \sigma_n \epsilon_{t+1} \tag{7.5.4}$$

where ϵ_{t+1} is an i.i.d. scalar stochastic process with time $t+1$ component distributed as $\mathcal{N}(0,1)$. Equations (7.5.2), (7.5.3), (7.5.4) express a time-to-build or schooling technology for converting new entrants n_t into increments in stocks of unskilled labor (this takes one period of waiting) and of skilled labor (this takes three periods of waiting). Stocks of skilled and unskilled labors depreciate, say through death or retirement, at the rates $(1-\delta_S), (1-\delta_U)$, respectively, where $\delta_S \in (0,1)$ and $\delta_U \in (0,1)$. In addition, we assume that there is an output cost of $\frac{e}{2} n_{st}^2$ associated with allocating new workers (or 'students') to the skilled worker pool.

7.5.1. A planning problem

Let's start with a planning problem, then construct a competitive equilibrium. Given initial conditions $(U_0, S_0, n_{S,-1}, n_{S,-2}, n_0)$, a planner chooses n_{St}, n_{Ut} to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ f_0 + f_1 \begin{bmatrix} U_t \\ S_t \end{bmatrix} - .5 \begin{bmatrix} U_t \\ S_t \end{bmatrix}' f_2 \begin{bmatrix} U_t \\ S_t \end{bmatrix} - \frac{e}{2} n_{St}^2 \right\} \tag{7.5.5}$$

subject to (7.5.2), (7.5.3), (7.5.4). This is a stochastic discounted optimal linear regulator problem. Define the state as $X_t = [\,U_t \quad S_t \quad 1 \quad n_{S,t-1} \quad n_{S,t-2} \quad n_t\,]$ and the control as n_{St}. An optimal decision rule has the form $n_{St} = -F X_t$ and the law of motion of the state under the optimal decision is

$$X_{t+1} = (A - BF)\, X_t + C\epsilon_{t+1}. \tag{7.5.6}$$

Define shadow wages

$$\begin{bmatrix} \tilde{w}_{Ut} \\ \tilde{w}_{St} \end{bmatrix} = f_1 - f_2 \begin{bmatrix} U_t \\ S_t \end{bmatrix} \equiv \begin{bmatrix} S_U \\ S_S \end{bmatrix} X_t, \tag{7.5.7}$$

where S_U and S_S are the appropriate selector vectors. The expected present value of entering school to become an unskilled worker is evidently

$$E_t \beta \sum_{j=1}^{\infty} (\beta\delta_U)^{j-1}\, \tilde{w}_{U,t+j} = \beta S_U \left(I - (A - BF)\, \beta\delta_U\right)^{-1} (A - BF)\, X_t$$

and the expected present value of entering school at t to become a skilled worker is

$$E_t \beta^3 \sum_{j=3}^{\infty} (\beta\delta_S)^{j-3}\, \tilde{w}_{S,t+j} = \beta^3 S_S \left(I - (A - BF)\, \beta\delta_S\right)^{-1} (A - BF)^3\, X_t.$$

7.5.2. Decentralization

We can decentralize the planning problem by finding a recursive competitive equilibrium whose allocation matches that associated with the planning problem. A competitive firm hires stocks of skilled and unskilled workers at competitive wages w_{St}, w_{Ut} each period. Taking those wages as given, it chooses S_t, U_t to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ f_0 + f_1 \begin{bmatrix} U_t \\ S_t \end{bmatrix} - .5 \begin{bmatrix} U_t \\ S_t \end{bmatrix}' f_2 \begin{bmatrix} U_t \\ S_t \end{bmatrix} - w_{Ut} U_t - w_{St} S_t \right\}. \tag{7.5.8}$$

Notice that the absence of intertemporal linkages in this problem makes it break into a sequence of static problems. The firm doesn't have to know the law of motion for wages. The firm equates the marginal products of each type of labor to that type's wage.

A representative family faces wages $\{w_{St}, w_{Ut}\}$ as a price taker and chooses contingency plans for $\{n_{St}, U_{t+1}, S_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ w_{Ut} U_t + w_{st} S_t - \frac{e}{2} n_{St}^2 \right\} \tag{7.5.9}$$

subject to the perceived law of motion for w_{Ut}, w_{St}

$$\begin{gathered} \begin{bmatrix} w_{Ut} \\ w_{St} \end{bmatrix} = U_w \tilde{X}_t \\ \tilde{X}_{t+1} = \tilde{A} \tilde{X}_t + \tilde{C} \epsilon_{t+1} \end{gathered} \tag{7.5.10}$$

and (7.5.2) and (7.5.3). According to (7.5.9), the family allocates n_t between n_{Ut} and n_{St} to maximize the expected present value of earnings from both types of labor, minus the present value of 'adjustment costs' $\frac{e}{2} n_{St}^2$. The state vector confronting the representative family is $[U_t \quad S_t \quad \tilde{X}_t]$ where $\tilde{X}_t$ has dimension comparable to X_t; $\tilde{A}X + C\varepsilon$ is a perceived law of motion for $\tilde{X}$. In a recursive competitive equilibrium, it will turn out that $\tilde{A} = A - BF$, where $A - BF$ is the optimal law of motion obtained from the planning problem.

In the spirit of Siow (1984) and Sherwin Rosen, it is interesting to focus on the special case in which $e = 0$. Here the competitive equilibrium features the outcome that

$$\beta E_t \sum_{j=1}^{\infty} (\beta \delta_U)^j \, w_{U,t+j} = E_t \beta^3 \sum_{j=3}^{\infty} (\beta \delta_S)^{j-3} \, w_{S,t+j}. \tag{7.5.11}$$

This condition says that the family allocates new entrants to equate the present values of earnings across occupations, a calculation that takes into account that it takes longer to train for some occupations than for others. The laws of motion of competitive equilibrium quantities adjust to equalize the present values of wages in the two occupations.

7.6. Markov perfect equilibrium

It is instructive to consider a dynamic model of duopoly. A market has two firms. Each firm recognizes that its output decision will affect the aggregate output and therefore influence the market price. Thus, we drop the assumption of price-taking behavior.[9] The one-period return function of firm i is

$$R_{it} = p_t y_{it} - .5d\left(y_{it+1} - y_{it}\right)^2 . \tag{7.6.1}$$

There is a demand curve

$$p_t = A_0 - A_1\left(y_{1t} + y_{2t}\right). \tag{7.6.2}$$

Substituting the demand curve into equation (7.6.1) lets us express the return as

$$R_{it} = A_0 y_{it} - A_1 y_{it}^2 - A_1 y_{it} y_{-i,t} - .5d\left(y_{it+1} - y_{it}\right)^2 , \tag{7.6.3}$$

where $y_{-i,t}$ denotes the output of the firm other than i. Firm i chooses a decision rule that sets y_{it+1} as a function of $(y_{it}, y_{-i,t})$ and that maximizes

$$\sum_{t=0}^{\infty} \beta^t R_{it}.$$

Temporarily assume that the maximizing decision rule is $y_{it+1} = f_i(y_{it}, y_{-i,t})$.

Given the function f_{-i}, the Bellman equation of firm i is

$$v_i\left(y_{it}, y_{-i,t}\right) = \max_{y_{it+1}} \left\{R_{it} + \beta v_i\left(y_{it+1}, y_{-i,t+1}\right)\right\}, \tag{7.6.4}$$

[9] One consequence of departing from the price-taking framework is that the market outcome will no longer maximize welfare, measured as the sum of consumer and producer surplus. See exercise *7.4* for the case of a monopoly.

where the maximization is subject to the perceived decision rule of the other firm

$$y_{-i,t+1} = f_{-i}\left(y_{-i,t}, y_{it}\right). \tag{7.6.5}$$

Note the cross-reference between the two problems for $i = 1, 2$.

We now advance the following definition:

DEFINITION: A Markov perfect equilibrium is a pair of value functions v_i and a pair of policy functions f_i for $i = 1, 2$ such that

a. Given f_{-i}, v_i satisfies the Bellman equation (7.6.4).

b. The policy function f_i attains the right side of the Bellman equation (7.6.4).

The adjective Markov denotes that the equilibrium decision rules depend on the current values of the state variables y_{it} only, not other parts of their histories. Perfect means 'complete', i.e., that the equilibrium is constructed by backward induction and therefore builds in optimizing behavior for each firm for all possible future states, including many that will not be realized when we iterate forward on the pair of equilibrium strategies f_i.

7.6.1. Computation

If it exists, a Markov perfect equilibrium can be computed by iterating to convergence on the pair of Bellman equations (7.6.4). In particular, let v_i^j, f_i^j be the value function and policy function for firm i at the jth iteration. Then imagine constructing the iterates

$$v_i^{j+1}\left(y_{it}, y_{-i,t}\right) = \max_{y_{i,t+1}} \left\{ R_{it} + \beta v_i^j\left(y_{it+1}, y_{-i,t+1}\right) \right\}, \tag{7.6.6}$$

where the maximization is subject to

$$y_{-i,t+1} = f_{-i}^j\left(y_{-i,t}, y_{it}\right). \tag{7.6.7}$$

In general, these iterations are difficult.[10] In the next section, we describe how the calculations simplify for the case in which the return function is quadratic and the transition laws are linear.

[10] See Levhari and Mirman (1980) for how a Markov perfect equilibrium can be computed conveniently with logarithmic returns and Cobb-Douglas transition laws. Levhari and Mirman construct a model of fish and fishers.

7.7. Linear Markov perfect equilibria

In this section, we show how the optimal linear regulator can be used to solve a model like that in the previous section. That model should be considered to be an example of a dynamic game. A dynamic game consists of these objects: (a) a list of players; (b) a list of dates and actions available to each player at each date; and (c) payoffs for each player expressed as functions of the actions taken by all players.

The optimal linear regulator is a good tool for formulating and solving dynamic games. The standard equilibrium concept—subgame perfection—in these games requires that each player's strategy be computed by backward induction. This leads to an interrelated pair of Bellman equations. In linear quadratic dynamic games, these "stacked Bellman equations" become "stacked Riccati equations" with a tractable mathematical structure.

We now consider the following two-player, linear quadratic *dynamic game*. An $(n \times 1)$ state vector x_t evolves according to a transition equation

$$x_{t+1} = A_t x_t + B_{1t} u_{1t} + B_{2t} u_{2t} \tag{7.7.1}$$

where u_{jt} is a $(k_j \times 1)$ vector of controls of player j. We start with a finite horizon formulation, where t_0 is the initial date and t_1 is the terminal date for the common horizon of the two players. Player 1 maximizes

$$-\sum_{t=t_0}^{t_1-1} \left(x_t^T R_1 x_t + u_{1t}^T Q_1 u_{1t} + u_{2t}^T S_1 u_{2t} \right) \tag{7.7.2}$$

where R_1 and S_1 are positive semidefinite and Q_1 is positive definite. Player 2 maximizes

$$-\sum_{t=t_0}^{t_1-1} \left(x_t^T R_2 x_t + u_{2t}^T Q_2 u_{2t} + u_{1t}^T S_2 u_{1t} \right) \tag{7.7.3}$$

where R_2 and S_2 are positive semidefinite and Q_2 is positive definite.

We formulate a Markov perfect equilibrium as follows. Player j employs linear decision rules

$$u_{jt} = -F_{jt} x_t, \quad t = t_0, \ldots, t_1 - 1$$

where F_{jt} is a $(k_j \times n)$ matrix. Assume that player i knows $\{F_{-i,t}; t = t_0, \ldots, t_1 - 1\}$. Then player 1's problem is to maximize expression (7.7.2) subject to the known law of motion (7.7.1) *and* the known control law $u_{2t} = -F_{2t} x_t$

of player 2. Symmetrically, player 2's problem is to maximize expression (7.7.3) subject to equation (7.7.1) and $u_{1t} = -F_{1t} x_t$. A Markov perfect equilibrium is a pair of sequences $\{F_{1t}, F_{2t};\ t = t_0, t_0+1, \ldots, t_1-1\}$ such that $\{F_{1t}\}$ solves player 1's problem, given $\{F_{2t}\}$, and $\{F_{2t}\}$ solves player 2's problem, given $\{F_{1t}\}$. We have restricted each player's strategy to depend only on x_t, and not on the *history* $h_t = \{(x_s, u_{1s}, u_{2s}), s = t_0, \ldots, t\}$. This restriction on strategy spaces accounts for the adjective "Markov" in the phrase "Markov perfect equilibrium."

Player 1's problem is to maximize

$$-\sum_{t=t_0}^{t_1-1} \left\{ x_t^T \left(R_1 + F_{2t}^T S_1 F_{2t}\right) x_t + u_{1t}^T Q_1 u_{1t} \right\}$$

subject to

$$x_{t+1} = \left(A_t - B_{2t} F_{2t}\right) x_t + B_{1t} u_{1t}.$$

This is an optimal linear regulator problem, and it can be solved by working backward. Evidently, player 2's problem is also an optimal linear regulator problem.

The solution of player 1's problem is given by

$$F_{1t} = \left(B_{1t}^T P_{1t+1} B_{1t} + Q_1\right)^{-1} B_{1t}^T P_{1t+1} \left(A_t - B_{2t} F_{2t}\right) \tag{7.7.4}$$

$$t = t_0, t_0+1, \ldots, t_1-1$$

where P_{1t} is the solution of the following matrix Riccati difference equation with terminal condition $P_{1t_1} = 0$:

$$\begin{aligned} P_{1t} &= \left(A_t - B_{2t} F_{2t}\right)^T P_{1t+1} \left(A_t - B_{2t} F_{2t}\right) + \left(R_1 + F_{2t}^T S_1 F_{2t}\right) \\ &- \left(A_t - B_{2t} F_{2t}\right)^T P_{1t+1} B_{1t} \left(B_{1t}^T P_{1t+1} B_{1t} + Q_1\right)^{-1} B_{1t}^T P_{1t+1} \left(A_t - B_{2t} F_{2t}\right). \end{aligned} \tag{7.7.5}$$

The solution of player 2's problem is

$$F_{2t} = \left(B_{2t}^T P_{2t+1} B_{2t} + Q_2\right)^{-1} B_{2t}^T P_{2t+1} \left(A_t - B_{1t} F_{1t}\right) \tag{7.7.6}$$

where P_{2t} solves the following matrix Riccati difference equation, with terminal condition $P_{2t_1} = 0$:

$$\begin{aligned} P_{2t} &= \left(A_t - B_{1t} F_{1t}\right)^T P_{2t+1} \left(A_t - B_{1t} F_{1t}\right) + \left(R_2 + F_{1t}^T S_2 F_{1t}\right) \\ &\quad - \left(A_t - B_{1t} F_{1t}\right)^T P_{2t+1} B_{2t} \\ &\quad \left(B_{2t}^T P_{2t+1} B_{2t} + Q_2\right)^{-1} B_{2t}^T P_{2t+1} \left(A_t - B_{1t} F_{1t}\right). \end{aligned} \tag{7.7.7}$$

The equilibrium sequences $\{F_{1t}, F_{2t}; t = t_0, t_0+1, \ldots, t_1-1\}$ can be calculated from the pair of coupled Riccati difference equations (7.7.5) and (7.7.7). In particular, we use equations (7.7.4), (7.7.5), (7.7.6), and (7.7.7) to "work backward" from time $t_1 - 1$. Notice that given P_{1t+1} and P_{2t+1}, equations (7.7.4) and (7.7.6) are a system of $(k_2 \times n) + (k_1 \times n)$ *linear* equations in the $(k_2 \times n) + (k_1 \times n)$ unknowns in the matrices F_{1t} and F_{2t}.

Notice how j's control law F_{jt} is a function of $\{F_{is}, s \geq t, i \neq j\}$. Thus, agent i's choice of $\{F_{it}; t = t_0, \ldots, t_1-1\}$ influences agent j's choice of control laws. However, in the Markov perfect equilibrium of this game, each agent is assumed to ignore the influence that his choice exerts on the other agent's choice.[11]

We often want to compute the solutions of such games for infinite horizons, in the hope that the decision rules F_{it} settle down to be time invariant as $t_1 \to +\infty$. In practice, we usually fix t_1 and compute the equilibrium of an infinite horizon game by driving $t_0 \to -\infty$. Judd followed that procedure in the following example.

7.7.1. An example

This section describes the Markov perfect equilibrium of an infinite horizon linear quadratic game proposed by Kenneth Judd (1990). The equilibrium is computed by iterating to convergence on the pair of Riccati equations defined by the choice problems of two firms. Each firm solves a linear quadratic optimization problem, taking as given and known the sequence of linear decision rules used by the other player. The firms set prices and quantities of two goods interrelated through their demand curves. There is no uncertainty. Relevant variables are defined as follows:

I_{it} = inventories of firm i at beginning of t.
q_{it} = production of firm i during period t.
p_{it} = price charged by firm i during period t.
S_{it} = sales made by firm i during period t.
E_{it} = costs of production of firm i during period t.

[11] In an equilibrium of a *Stackelberg* or *dominant player* game, the timing of moves is so altered relative to the present game that one of the agents called the *leader* takes into account the influence that his choices exert on the other agent's choices. See chapter 19.

C_{it} = costs of carrying inventories for firm i during t.

The firms' cost functions are

$$C_{it} = c_{i1} + c_{i2} I_{it} + .5 c_{i3} I_{it}^2$$
$$E_{it} = e_{i1} + e_{i2} q_{it} + .5 e_{i3} q_{it}^2$$

where e_{ij}, c_{ij} are positive scalars.
Inventories obey the laws of motion

$$I_{i,t+1} = (1 - \delta) I_{it} + q_{it} - S_{it}$$

Demand is governed by the linear schedule

$$S_t = d p_{it} + B$$

where $S_t = [\, S_{1t} \quad S_{2t} \,]'$, d is a (2×2) negative definite matrix, and B is a vector of constants. Firm i maximizes the undiscounted sum

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \left(p_{it} S_{it} - E_{it} - C_{it} \right)$$

by choosing a decision rule for price and quantity of the form

$$u_{it} = -F_i x_t$$

where $u_{it} = [\, p_{it} \quad q_{it} \,]'$, and the state is $x_t = [\, I_{1t} \quad I_{2t} \,]$.

In the web site for the book, we supply a Matlab program `nnash.m` that computes a Markov perfect equilibrium of the linear quadratic dynamic game in which player i maximizes

$$-\sum_{t=0}^{\infty} \{ x_t' r_i x_t + 2 x_t' w_i u_{it} + u_{it}' q_i u_{it} + u_{jt}' s_i u_{jt} + 2 u_{jt}' m_i u_{it} \}$$

subject to the law of motion

$$x_{t+1} = a x_t + b_1 u_{1t} + b_2 u_{2t}$$

and a control law $u_{jt} = -f_j x_t$ for the other player; here a is $n \times n$; b_1 is $n \times k_1$; b_2 is $n \times k_2$; r_1 is $n \times n$; r_2 is $n \times n$; q_1 is $k_1 \times k_1$; q_2 is $k_2 \times k_2$; s_1 is $k_2 \times k_2$; s_2 is $k_1 \times k_1$; w_1 is $n \times k_1$; w_2 is $n \times k_2$; m_1 is $k_2 \times k_1$; and m_2 is $k_1 \times k_2$. The equilibrium of Judd's model can be computed by filling in the matrices appropriately. A Matlab tutorial `judd.m` uses `nnash.m` to compute the equilibrium.

7.8. Concluding remarks

This chapter has introduced two equilibrium concepts and illustrated how dynamic programming algorithms are embedded in each. For the linear models we have used as illustrations, the dynamic programs become optimal linear regulators, making it tractable to compute equilibria even for large state spaces. We chose to define these equilibria concepts in partial equilibrium settings that are more natural for microeconomic applications than for macroeconomic ones. In the next chapter, we use the recursive equilibrium concept to analyze a general equilibrium in an endowment economy. That setting serves as a natural starting point for addressing various macroeconomic issues.

Exercises

These problems aim to teach about (1) mapping problems into recursive forms, (2) different equilibrium concepts, and (3) using Matlab. Computer programs are available from the web site for the book.[12]

Exercise 7.1 **A competitive firm**

A competitive firm seeks to maximize

$$\sum_{t=0}^{\infty} \beta^t R_t \tag{1}$$

where $\beta \in (0,1)$, and time t revenue R_t is

$$R_t = p_t y_t - .5d\left(y_{t+1} - y_t\right)^2, \quad d > 0, \tag{2}$$

where p_t is the price of output, and y_t is the time t output of the firm. Here $.5d(y_{t+1} - y_t)^2$ measures the firm's cost of adjusting its rate of output. The firm starts with a given initial level of output y_0. The price lies on the market demand curve

$$p_t = A_0 - A_1 Y_t, A_0, A_1 > 0 \tag{3}$$

[12] The web site is <https://files.nyu.edu/ts43/public/books.html>.

where Y_t is the market level of output, which the firm takes as exogenous, and which the firm believes follows the law of motion

$$Y_{t+1} = H_0 + H_1 Y_t, \tag{4}$$

with Y_0 as a fixed initial condition.

a. Formulate the Bellman equation for the firm's problem.

b. Formulate the firm's problem as a discounted optimal linear regulator problem, being careful to describe all of the objects needed. What is the *state* for the firm's problem?

c. Use the Matlab program `olrp.m` to solve the firm's problem for the following parameter values: $A_0 = 100, A_1 = .05, \beta = .95, d = 10, H_0 = 95.5$, and $H_1 = .95$. Express the solution of the firm's problem in the form

$$y_{t+1} = h_0 + h_1 y_t + h_2 Y_t, \tag{5}$$

giving values for the h_j's.

d. If there were n identical competitive firms all behaving according to equation (5), what would equation (5) imply for the *actual* law of motion (4) for the market supply Y?

e. Formulate the Euler equation for the firm's problem.

Exercise 7.2 **Rational expectations**

Now assume that the firm in problem 1 is "representative." We implement this idea by setting $n = 1$. In equilibrium, we will require that $y_t = Y_t$, but we don't want to impose this condition at the stage that the firm is optimizing (because we want to retain competitive behavior). Define a rational expectations equilibrium to be a pair of numbers H_0, H_1 such that if the representative firm solves the problem ascribed to it in problem 1, then the firm's optimal behavior given by equation (5) implies that $y_t = Y_t \ \forall\, t \geq 0$.

a. Use the program that you wrote for exercise *7.1* to determine which if any of the following pairs (H_0, H_1) is a rational expectations equilibrium: (i) (94.0888, .9211); (ii) (93.22, .9433), and (iii) (95.08187459215024, .95245906270392)?

b. Describe an iterative algorithm that uses the program that you wrote for exercise *7.1* to compute a rational expectations equilibrium. (You are not being asked actually to use the algorithm you are suggesting.)

Exercise 7.3 **Maximizing welfare**

A planner seeks to maximize the welfare criterion

$$\sum_{t=0}^{\infty} \beta^t S_t, \tag{1}$$

where S_t is "consumer surplus plus producer surplus" defined to be

$$S_t = S(Y_t, Y_{t+1}) = \int_0^{Y_t} (A_0 - A_1 x)\, d\, x - .5d\,(Y_{t+1} - Y_t)^2 .$$

a. Formulate the planner's Bellman equation.

b. Formulate the planner's problem as an optimal linear regulator, and, for the same parameter values in exercise 7.1, solve it using the Matlab program `olrp.m`. Represent the solution in the form $Y_{t+1} = s_0 + s_1 Y_t$.

c. Compare your answer in part b with your answer to part a of exercise *7.2*.

Exercise 7.4 **Monopoly**

A monopolist faces the industry demand curve (3) and chooses Y_t to maximize $\sum_{t=0}^{\infty} \beta^t R_t$ where $R_t = p_t Y_t - .5d(Y_{t+1} - Y_t)^2$ and where Y_0 is given.

a. Formulate the firm's Bellman equation.

b. For the parameter values listed in exercise *7.1*, formulate and solve the firm's problem using `olrp.m`.

c. Compare your answer in part b with the answer you obtained to part b of exercise *7.3*.

Exercise 7.5 **Duopoly**

An industry consists of two firms that jointly face the industry-wide inverse demand curve $p_t = A_0 - A_1 Y_t$, where now $Y_t = y_{1t} + y_{2t}$. Firm $i = 1, 2$ maximizes

$$\sum_{t=0}^{\infty} \beta^t R_{it} \tag{1}$$

where $R_{it} = p_t y_{it} - .5d(y_{i,t+1} - y_{it})^2$.

a. Define a Markov perfect equilibrium for this industry.

b. Formulate the Bellman equation for each firm.

c. Use the Matlab program `nash.m` to compute an equilibrium, assuming the parameter values listed in exercise *7.1*.

Exercise 7.6 **Self-control**

This is a model of a human who has time inconsistent preferences, of a type proposed by Phelps and Pollak (1968) and used by Laibson (1994).[13] The human lives from $t = 0, \ldots, T$. Think of the human as actually consisting of $T+1$ personalities, one for each period. Each personality is a distinct agent (i.e., a distinct utility function and constraint set). Personality T has preferences ordered by $u(c_T)$ and personality $t < T$ has preferences that are ordered by

$$u\left(c_t\right) + \delta \sum_{j=1}^{T-t} \beta^j u\left(c_{t+j}\right),$$

where $u(\cdot)$ is a twice continuously differentiable, increasing, and strictly concave function of consumption of a single good; $\beta \in (0,1)$, and $\delta \in (0,1]$. When $\delta < 1$, preferences of the sequence of personalities are time inconsistent (that is, not recursive). At each t, let there be a savings technology described by

$$k_{t+1} + c_t \leq f\left(k_t\right),$$

where f is a production function with $f' > 0, f'' \leq 0$.

a. Define a Markov perfect equilibrium for the $T+1$ personalities.

b. Argue that the Markov perfect equilibrium can be computed by iterating on the following functional equations:

$$\begin{aligned} V_{j+1}\left(k\right) &= \max_c \left\{u\left(c\right) + \beta\delta W_j\left(k'\right)\right\} \\ W_{j+1}\left(k\right) &= u\left[c_{j+1}\left(k\right)\right] + \beta W_j\left[f\left(k\right) - c_{j+1}\left(k\right)\right] \end{aligned}$$

where $c_{j+1}(k)$ is the maximizer of the right side of the first equation above for $j+1$, starting from $W_0(k) = u[f(k)]$. Here $W_j(k)$ is the value of $u(c_{T-j}) + \beta u(c_{T-j+1}) + \ldots + \beta^{T-j} u(c_T)$, taking the decision rules $c_h(k)$ as given for $h = 0, 1, \ldots, j$.

[13] See Gul and Pesendorfer (2000) for a single-agent recursive representation of preferences exhibiting temptation and self-control.

c. State the optimization problem of the time 0 person who is given the power to dictate the choices of all subsequent persons. Write the Bellman equations for this problem. The time 0 person is said to have a commitment technology for "self-control" in this problem.

Exercise 7.7 **Equilibrium search**

An economy consists of a continuum of *ex ante* identical workers each of whom is either employed or unemployed. A worker wants to maximize the expected value of $\sum_{t=0}^{\infty}\beta^t y_t$ where $\beta \in (0,1)$ and

$$y_t = \begin{cases} w & \text{if employed} \\ c\,(U) & \text{if unemployed}\ . \end{cases}$$

Each period, an unemployed worker draws one and only one offer to work (until fired) at a wage w drawn from a c.d.f. $F(W) = \text{Prob}(w \leq W)$ where $F(0) = 0, F(B) = 1$ for $B > 0$. Successive draws from F are i.i.d. If a worker accepts a job, he receives w this period and enters the beginning of next period as 'employed'. At the beginning of each period, each such previously employed worker is exposed to a probability of $\lambda \in (0,1)$ of being fired; with probability $1-\lambda$ he is not fired and again receives the previously drawn w as a wage. If fired, the worker becomes newly unemployed and has the same opportunity as all other unemployed workers, i.e., he draws an offer w from c.d.f. F. If an unemployed worker rejects that offer, he receives unemployment compensation $c(U) = c\big[\frac{1}{1+\exp(-6U)} - .5\big]$ and enters next period unemployed. Here U is the aggregate unemployment rate at the beginning of the period. The unemployment rate tomorrow U^* is related to the unemployment rate U today by the law of motion

$$U^* = \lambda\,(1-U) + (1-\phi\,(U))\,U,$$

where $\phi(U)$ is the fraction of unemployed workers who accept a wage offer this period.

a. Write a Bellman equation for an unemployed worker.

b. Describe the form of an unemployed worker's optimal decision rule.

c. Describe how $\phi(U)$ is implied by a typical worker's optimal decision rule.

d. Define a recursive competitive equilibrium for this environment.

Chapter 8
Equilibrium with Complete Markets

8.1. Time 0 versus sequential trading

This chapter describes competitive equilibria of a pure exchange infinite horizon economy with stochastic endowments. These are useful for studying risk sharing, asset pricing, and consumption. We describe two systems of markets: an *Arrow-Debreu* structure with complete markets in dated contingent claims all traded at time 0, and a sequential-trading structure with complete one-period *Arrow securities*. These two entail different assets and timings of trades, but have identical consumption allocations. Both are referred to as complete markets economies. They allow more comprehensive sharing of risks than do the incomplete markets economies to be studied in chapters 17 and 18, or the economies with imperfect enforcement or imperfect information, studied in chapters 20 and 21.

8.2. The physical setting: preferences and endowments

In each period $t \geq 0$, there is a realization of a stochastic event $s_t \in S$. Let the history of events up and until time t be denoted $s^t = [s_0, s_1, \ldots, s_t]$. The unconditional probability of observing a particular sequence of events s^t is given by a probability measure $\pi_t(s^t)$. For $t > \tau$, we write the probability of observing s^t conditional on the realization of s^τ as $\pi_t(s^t|s^\tau)$. In this chapter, we shall assume that trading occurs after observing s_0, which we capture by setting $\pi_0(s_0) = 1$ for the initially given value of s_0.[1]

In section 8.10 we shall follow much of the literatures in macroeconomics and econometrics and assume that $\pi_t(s^t)$ is induced by a Markov process. We wait to impose that special assumption until section 8.10 because some important findings do not require making that assumption.

[1] Most of our formulas carry over to the case where trading occurs before s_0 has been realized; just postulate a nondegenerate probability distribution $\pi_0(s_0)$ over the initial state.

There are I agents named $i = 1, \ldots, I$. Agent i owns a stochastic endowment of one good $y_t^i(s^t)$ that depends on the history s^t. The history s^t is publicly observable. Household i purchases a history-dependent consumption plan $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ and orders these consumption streams by[2]

$$U\left(c^i\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t^i\left(s^t\right)\right] \pi_t\left(s^t\right), \tag{8.2.1}$$

where $0 < \beta < 1$. The right side is equal to $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$, where E_0 is the mathematical expectation operator, conditioned on s_0. Here $u(c)$ is an increasing, twice continuously differentiable, strictly concave function of consumption $c \geq 0$ of one good. The utility function satisfies the Inada condition[3]

$$\lim_{c \downarrow 0} u'(c) = +\infty.$$

Notice that in assuming (8.2.1), we are imposing identical preference orderings across all individuals i that can be represented in terms of discounted expected utility with common β, common utility function $u(\cdot)$, and common probability distributions $\pi_t(s^t)$. As we proceed through this chapter, watch for results that would evaporate if we were instead to allow $\beta, u(\cdot)$, or $\pi_t(s^t)$ to depend on i.

A *feasible allocation* satisfies

$$\sum_i c_t^i\left(s^t\right) \leq \sum_i y_t^i\left(s^t\right) \tag{8.2.2}$$

for all t and for all s^t.

---

[2] Exercises 8.13 - 8.17 consider examples in which we replace (8.2.1) with

$$U^i\left(c^i\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t^i\left(s^t\right)\right] \pi_t^i\left(s^t\right),$$

where $\pi^i(s^t)$ is a personal probability distribution specific to agent i. Blume and Easley (2006) study such settings, focusing particularly on which agents' beliefs ultimately influence the tails of allocations and prices. Throughout most of this chapter, we adopt the assumption, routinely employed in much of macroeconomics, that all agents share probabilities.

[3] One role of this Inada condition is to make the consumption of each agent strictly positive in every date-history pair. A related role is to deliver a state-by-state borrowing limit to impose in economies with sequential trading of Arrow securities.

8.3. Alternative trading arrangements

For a two-event stochastic process $s_t \in S = \{0,1\}$, the trees in Figures 8.3.1 and 8.3.2 give two portraits of how histories s^t unfold. From the perspective of time 0 given $s_0 = 0$, Figure 8.3.1 portrays all prospective histories possible up to time 3. Figure 8.3.2 portrays a *particular* history that it is known the economy has indeed followed up to time 2, together with the two possible one-period continuations into period 3 that can occur after that history.

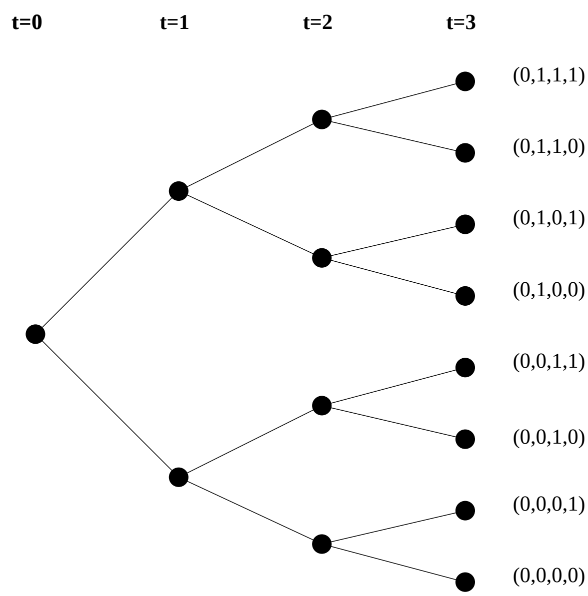

Figure 8.3.1: The Arrow-Debreu commodity space for a two-state Markov chain. At time 0, there are trades in time $t = 3$ goods for each of the eight nodes that signify histories that can possibly be reached starting from the node at time 0.

In this chapter we shall study two distinct trading arrangements that correspond, respectively, to the two views of the economy in Figures 8.3.1 and 8.3.2. One is what we shall call the Arrow-Debreu structure. Here markets meet at time 0 to trade claims to consumption at all times $t > 0$ and that are contingent on all possible histories up to t, s^t. In that economy, at time 0 and

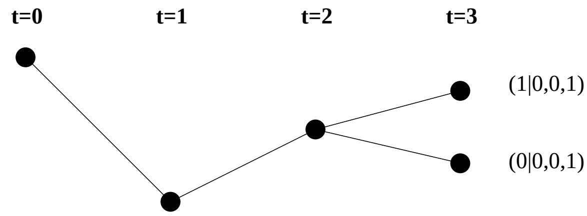

Figure 8.3.2: The commodity space with Arrow securities. At date $t = 2$, there are trades in time 3 goods for only those time $t = 3$ nodes that can be reached from the realized time $t = 2$ history $(0, 0, 1)$.

for all $t \geq 1$, households trade claims on the time t consumption good *at all nodes* s^t. After time 0, no further trades occur. The other economy has *sequential* trading of only one-period-ahead state-contingent claims. Here trades of one-period ahead state-contingent claims occur at each date $t \geq 0$. Trades for history s^{t+1}–contingent date $t+1$ goods occur only at the *particular* date t history s^t that has been reached at t, as in Figure 8.3.2. It turns out that these two trading arrangements support identical equilibrium allocations. Those allocations share the notable property of being functions only of the *aggregate* endowment realization $\sum_{i=1}^{I} y_t^i(s^t)$ and time-invariant parameters describing the initial distribution of wealth.

8.3.1. History dependence

Before trading, the situation of household i at time t depends on the history s^t. A natural measure of household i's luck in life is $\{y_0^i(s_0), y_1^i(s^1), \ldots, y_t^i(s^t)\}$, which evidently in general depends on the history s^t. A question that will occupy us in this chapter and in chapters 18 and 20 is whether, after trading, the household's consumption allocation at time t is also history dependent. Remarkably, in the complete markets models of this chapter, the consumption allocation at time t depends only on the aggregate endowment realization at time t and some time-invariant parameters that describe the time 0 *initial* distribution of wealth. The market incompleteness of chapter 18 and the information and enforcement frictions of chapter 20 will break that result and put history dependence into equilibrium allocations.

8.4. Pareto problem

As a benchmark against which to measure allocations attained by a market economy, we seek efficient allocations. An allocation is said to be efficient if it is Pareto optimal: it has the property that any reallocation that makes one household strictly better off also makes one or more other households worse off. We can find efficient allocations by posing a Pareto problem for a fictitious social planner. The planner attaches nonnegative *Pareto weights* $\lambda_i, i = 1, \ldots, I$ to the consumers' utilities and chooses allocations $c^i, i = 1, \ldots, I$ to maximize

$$W = \sum_{i=1}^{I} \lambda_i U\left(c^i\right) \tag{8.4.1}$$

subject to (8.2.2). We call an allocation *efficient* if it solves this problem for some set of nonnegative λ_i's. Let $\theta_t(s^t)$ be a nonnegative Lagrange multiplier on the feasibility constraint (8.2.2) for time t and history s^t, and form the Lagrangian

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta^t u\left(c_t^i\left(s^t\right)\right) \pi_t\left(s^t\right) + \theta_t\left(s^t\right) \sum_{i=1}^{I} \left[y_t^i\left(s^t\right) - c_t^i\left(s^t\right)\right] \right\}.$$

The first-order condition for maximizing L with respect to $c_t^i(s^t)$ is

$$\beta^t u'\left(c_t^i\left(s^t\right)\right) \pi_t\left(s^t\right) = \lambda_i^{-1} \theta_t\left(s^t\right) \tag{8.4.2}$$

for each i, t, s^t. Taking the ratio of (8.4.2) for consumers i and 1, respectively, gives

$$\frac{u'\left(c_t^i\left(s^t\right)\right)}{u'\left(c_t^1\left(s^t\right)\right)} = \frac{\lambda_1}{\lambda_i} \tag{8.4.3}$$

which implies

$$c_t^i\left(s^t\right) = u'^{-1}\left(\lambda_i^{-1} \lambda_1 u'\left(c_t^1\left(s^t\right)\right)\right). \tag{8.4.4}$$

Substituting (8.4.4) into feasibility condition (8.2.2) at equality gives

$$\sum_i u'^{-1}\left(\lambda_i^{-1} \lambda_1 u'\left(c_t^1\left(s^t\right)\right)\right) = \sum_i y_t^i\left(s^t\right). \tag{8.4.5}$$

Equation (8.4.5) is one equation in the one unknown $c_t^1(s^t)$. The right side of (8.4.5) is the realized aggregate endowment, so the left side is a function only of the aggregate endowment. Thus, given $\{\lambda_i\}_{i=1}^I$, $c_t^1(s^t)$ depends only on the current realization of the aggregate endowment and not separately either on the date t or on the specific history s^t leading up to that aggregate endowment or the cross-section distribution of individual endowments realized at t. Equation (8.4.4) then implies that for all i, $c_t^i(s^t)$ depends only on the aggregate endowment realization. We thus have:

PROPOSITION 1: An efficient allocation is a function of the realized aggregate endowment and does not depend separately on either the specific history s^t leading up to that aggregate endowment or on the cross-section distribution of individual endowments realized at t: $c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau)$ for s^t and $\tilde{s}^\tau$ such that $\sum_j y_t^j(s^t) = \sum_j y_\tau^j(\tilde{s}^\tau)$.

To compute the optimal allocation, first solve (8.4.5) for $c_t^1(s^t)$, then solve (8.4.4) for $c_t^i(s^t)$. Note from (8.4.4) that only the ratios of the Pareto weights matter, so that we are free to normalize the weights, e.g., to impose $\sum_i \lambda_i = 1$.

8.4.1. Time invariance of Pareto weights

Through equations (8.4.4) and (8.4.5), the allocation $c_t^i(s^t)$ assigned to consumer i depends in a time-invariant way on the aggregate endowment $\sum_j y_t^j(s^t)$. Consumer i's share of the aggregate endowment varies directly with his Pareto weight λ_i. In chapter 20, we shall see that the constancy through time of the Pareto weights $\{\lambda_j\}_{j=1}^I$ is a telltale sign that there are no enforcement- or information-related incentive problems in this economy. In chapter 20, when we inject those imperfections into the environment, the time invariance of the Pareto weights evaporates.

8.5. Digression on heterogenous beliefs

Debreu (1954, 1959) used a much more general representation of the preferences of consumer i. Instead of our highly restrictive specification (8.2.1), namely, $U^i(c^i) = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u(c^i(s^t))\pi_t(s^t)$, Debreu would just posit I distinct preference orderings $U^i(c^i)$ over consumption plans $\{c^i_t(s^t)\}_{t\geq 0}$. Debreu would not use the time-separable, discounted expected utility form of (8.2.1) and, in particular, would not mention probabilities $\pi_t(s^t)$ over histories s^t. Instead, his $U^i(c^i)$ would just directly specify tradeoffs among consumption goods indexed by different times and histories. For Debreu, there is no need to say anything directly about how likely the s^∞ paths are from the planner's point of view, or from anyone else's point of view either. To the Pareto planner, all that matters are aggregate resources available at different dates and histories and the preferences $U^i(c^i)$ of the I consumers. The outcome of a Pareto problem is an allocation of available consumption across people for all time and all histories. In the broader construction of Debreu, the theory is silent about which history $\{s^\infty\}$ is likely actually to occur.

We break that silence about which history $\{s^\infty\}$ is likely actually to occur when we adopt our restrictive expected utility specification (8.2.1) and proceed to impute a common probability specification $\{\pi_t(s^t)\}$ not only to all agents $i = 1,\ldots,I$ but also to 'nature', also known as the actual history-generating mechanism. Imputing common and, in the end, accurate beliefs to everyone inside the model, to nature, and to the author of the model too, imposes 'rational expectations'. This 'communism of probability models' assumption, which is widely used in macroeconomics, finance, and public finance, imposes substantially more structure than did Debreu. The rational expectations assumption acquires empirical power – meaning that it eliminates all of the free parameters that would be required to describe heterogeneous beliefs – by equating probability distributions in this way. It does so by confounding the distinct roles that the perceived probabilities $\pi_t(s^t)$ have in influencing individuals' preferences, and through them, Pareto optimal allocations (and as we shall see soon, competitive equilibrium prices). To indicate some of these distinct roles, it is useful temporarily to abandon the assumption of common beliefs and to follow Blume and Easley (2006) in considering a setting with disparate beliefs. This will also allow us to make contact briefly with the conjecture of Milton Friedman (1953, p. 22) that, by redistributing wealth towards people with more accurate beliefs, market forces would drive outcomes to look as if they are generated by a rational expectations equilibrium.

Thus, suppose that we modify the utility function for person i to allow for different probabilities

$$U^i\left(c^i\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c^i\left(s^t\right)\right) \pi_t^i\left(s^t\right). \tag{8.5.1}$$

Assume the common utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for all individuals i. With preferences altered in this way, assume that the planner continues to maximize the same welfare function W defined in (8.4.1). Then the counterparts to first-order conditions (8.4.3) become

$$c_t^i\left(s^t\right)^{\gamma} = \left(\frac{\lambda_i}{\lambda_1}\right)\left(\frac{\pi_t^i\left(s^t\right)}{\pi_t^1\left(s^t\right)}\right) c_t^1\left(s^t\right)^{\gamma}. \tag{8.5.2}$$

The likelihood ratio $\left(\frac{\pi_t^i(s^t)}{\pi_t^1(s^t)}\right)$, which is absent when there are common beliefs, now influences optimal allocations. For a given vector or Pareto weights $\{\lambda_i\}_{i=1}^I$, the Pareto planner assigns relatively more consumption at a given date and history to people who place higher probabilities on that date-history pair.

8.5.1. Example 1: one person knows the truth

This example illustrates that if one person's probability specification matches nature's, while everyone else's doesn't, then asymptotically the ratio of the knowledgeable person's consumption to that of a less knowledgeable person will diverge to $+\infty$.

Suppose that $S = \{0,1\}, I = 2$ and that agent i believes that s_t is independently and identically distributed with $\text{Prob}(s_t = 0) = \alpha_i$. Then for a history after $t = 0$ of $s_t, s_{t-1}, \ldots, s_1$ with $s_t = 1$ $t-n$ times and $s_t = 0$ n times, we have that

$$\frac{\pi_t^i\left(s^t\right)}{\pi_t^1\left(s^t\right)} = \left(\frac{1-\alpha_i}{1-\alpha_1}\right)^{t-n}\left(\frac{\alpha_i}{\alpha_1}\right)^{n}. \tag{8.5.3}$$

Suppose that $\{s_t\}_{t=1}^{\infty}$ is actually independently and identically distributed with $\text{Prob}(s_t = 0) = \tilde{\alpha}$. To compute the *actual* distribution of consumption governed by (8.5.2), we then want to compute the likelihood ratio (8.5.3) under the actual distribution. Under the actual distribution, $\frac{n}{t}$ is the fraction of 0's in a sample of length t. A Law of Large Numbers assures us that

$$\lim_{t\to\infty} \frac{n}{t} = \tilde{\pi}.$$

So we want to evaluate

$$\lim_{t\to\infty} L_t^i = \lim_{t\to\infty} \left(\frac{1-\alpha_i}{1-\alpha_1}\right)^{(1-\tilde{\alpha})t} \left(\frac{\alpha_i}{\alpha_1}\right)^{\tilde{\alpha}t}.$$

For example, assume that $I = 2, \alpha_2 = \tilde{\alpha}, \alpha_1 \neq \tilde{\alpha}$. Also, set $\gamma = 1$, so that $u(c) = \log(c)$. Then

$$L_t^2 = \left(\frac{1-\tilde{\alpha}}{1-\alpha_1}\right)^{(1-\tilde{\alpha})t} \left(\frac{\tilde{\alpha}}{\alpha_1}\right)^{\tilde{\alpha}t},$$

which implies that

$$\frac{1}{t}\log L_t = (1-\tilde{\alpha})\log\left(\frac{1-\tilde{\alpha}}{1-\alpha_1}\right) + \tilde{\alpha}\log\left(\frac{\tilde{\alpha}}{\alpha_1}\right). \tag{8.5.4}$$

The right side of (8.5.4) is the expected log likelihood ratio evaluated under the true distribution. This object is called the entropy of the actual distribution parameterized by $\tilde{\alpha}$ relative to consumer 1's distribution, call it $\operatorname{ent}(\tilde{\alpha}, \alpha_1)$. Given $\tilde{\alpha}$, relative entropy is a nonnegative function of α_1 on the interval $\alpha_1 \in [0,1]$; it attains its minimum value of 0 at the value $\alpha_1 = \tilde{\alpha}$. Relative entropy indicates how far one probability distribution is from another.[4]

From equation (8.5.4), we conclude that

$$L_t^2 = e^{\operatorname{ent}(\tilde{\alpha},\alpha_1)t} \tag{8.5.5}$$

which evidently diverges to $+\infty$ if $\operatorname{ent}(\tilde{\alpha}, \alpha_1) > 0$, i.e., whenever $\alpha_1 \neq \tilde{\alpha}$. For our example,

$$c_t^2\left(s^t\right) = \left(\frac{\lambda_2}{\lambda_1}\right) L_t^2 c_t^1\left(s^t\right), \tag{8.5.6}$$

so that when entropy is positive, the ratio of person 2's consumption to person 1's consumption diverges to $+\infty$ so long as $\left(\frac{\lambda_2}{\lambda_1}\right) > 0$.

[4] It governs statistical properties of likelihood ratio tests designed to discriminate one statistical model from another.

8.5.2. Example 2: one person is closer to the truth

This example illustrates that if one person's probability specification is wrong but 'better' than that of other consumers, then what can be expected actually to occur is that asymptotically the ratio of the consumption of the person with the better approximation to that of the person with the worse approximation will diverge to $+\infty$.

Modify the preceding example now to assume that $\alpha_2 \neq \tilde{\alpha}$ and $\alpha_1 \neq \tilde{\alpha}$, so that both consumers have beliefs that differ from the true data generating process. Proceeding as before, we deduce that under the true process

$$\frac{1}{t}\log L_t^2 = (1-\tilde{\alpha})\log\left(\frac{1-\alpha_2}{1-\alpha_1}\right) + \tilde{\alpha}\log\left(\frac{\alpha_2}{\alpha_1}\right)$$

which can be rewritten as

$$\begin{aligned}\frac{1}{t}\log L_t^2 &= (1-\tilde{\alpha})\log\left(\frac{1-\tilde{\alpha}}{1-\alpha_1}\right) + \tilde{\alpha}\log\left(\frac{\tilde{\alpha}}{\alpha_1}\right)\\ &\quad - (1-\tilde{\alpha})\log\left(\frac{1-\tilde{\alpha}}{1-\alpha_2}\right) + \tilde{\alpha}\log\left(\frac{\tilde{\alpha}}{\alpha_2}\right)\end{aligned}$$

or

$$\frac{1}{t}L_t^2 = \operatorname{ent}(\tilde{\alpha},\alpha_1) - \operatorname{ent}(\tilde{\alpha},\alpha_2)\,.$$

It follows that

$$L_t^2 = e^{(\operatorname{ent}(\tilde{\alpha},\alpha_1)-\operatorname{ent}(\tilde{\alpha},\alpha_2))t}$$

and so that (8.5.6) now implies that the ratio of person 2's consumption to person 1's diverges to $+\infty$ if $\operatorname{ent}(\tilde{\alpha},\alpha_1) - \operatorname{ent}(\tilde{\alpha},\alpha_2) > 0$.

This example is a special case of Blume and Easley's (2006) more general proposition that in the limit those consumers with beliefs closest to the truth, as measured by relative entropy, acquire the entire allocation.

8.5.3. Dependence on complete markets

In succeeding sections of this chapter, we show Pareto optimal consumption allocations are also competitive equilibrium allocations for two famous types of trading structures, one with only time 0 trading of many securities, and another with trading each period $t \geq 0$ of far fewer one-period securities. In studying these structures, we shall return to our earlier homogeneous beliefs specification of preferences (8.2.1). But it will be possible reader for the reader to extend the assertions about equivalences of Pareto optimal allocations to competitive equilibrium allocations to the heterogenous beliefs setting of this section.

The assertions about limiting allocations that we have made in this section all come from manipulating the first order condition (8.5.2) for our Pareto problem. While these assertions characterize outcomes for complete markets economies, they won't carry over to incomplete market economies, for example of the type to be analyzed in chapter 18. Indeed, examples of incomplete markets economies exist in which the consumption of the consumer with *less* accurate beliefs grows over time.[5]

8.6. Time 0 trading: Arrow-Debreu securities

We now describe how an optimal allocation can be attained by a competitive equilibrium with the Arrow-Debreu timing. Households trade dated history-contingent claims to consumption. There is a complete set of securities. Trades occur at time 0, after s_0 has been realized. At $t = 0$, households can exchange claims on time t consumption, contingent on history s^t at price $q_t^0(s^t)$, measured in some unit of account. The superscript 0 refers to the date at which trades occur, while the subscript t refers to the date that deliveries are to be made. The household's budget constraint is

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0 \left(s^t\right) c_t^i \left(s^t\right) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0 \left(s^t\right) y_t^i \left(s^t\right). \tag{8.6.1}$$

The household's problem is to choose c^i to maximize expression (8.2.1) subject to inequality (8.6.1).

[5] See Blume and Easley (2005) and Cogley, Sargent, and Tsyrennikov (2013).

Underlying the *single* budget constraint (8.6.1) is the fact that multilateral trades are possible through a clearing operation that keeps track of net claims.[6] All trades occur at time 0. After time 0, trades that were agreed to at time 0 are executed, but no more trades occur.

Attach a Lagrange multiplier μ_i to each household's budget constraint (8.6.1). We obtain the first-order conditions for the household's problem:

$$\frac{\partial U\left(c^i\right)}{\partial c_t^i\left(s^t\right)} = \mu_i q_t^0\left(s^t\right), \tag{8.6.2}$$

for all i, t, s^t. The left side is the derivative of total utility with respect to the time t, history s^t component of consumption. Each household has its own Lagrange multiplier μ_i that is independent of time. With specification (8.2.1) of the utility functional, we have

$$\frac{\partial U\left(c^i\right)}{\partial c_t^i\left(s^t\right)} = \beta^t u'\left[c_t^i\left(s^t\right)\right]\pi_t\left(s^t\right). \tag{8.6.3}$$

This expression implies that equation (8.6.2) can be written

$$\beta^t u'\left[c_t^i\left(s^t\right)\right]\pi_t\left(s^t\right) = \mu_i q_t^0\left(s^t\right). \tag{8.6.4}$$

We use the following definitions:

Definitions: A *price system* is a sequence of functions $\{q_t^0(s^t)\}_{t=0}^\infty$. An *allocation* is a list of sequences of functions $c^i = \{c_t^i(s^t)\}_{t=0}^\infty$, one for each i.

Definition: A *competitive equilibrium* is a feasible allocation and a price system such that, given the price system, the allocation solves each household's problem.

Notice that equation (8.6.4) implies

$$\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_t^j\left(s^t\right)\right]} = \frac{\mu_i}{\mu_j} \tag{8.6.5}$$

for all pairs (i, j). Thus, ratios of marginal utilities between pairs of agents are constant across all histories and dates.

[6] In the language of modern payments systems, this is a system with net settlements, not gross settlements, of trades.

An equilibrium allocation solves equations (8.2.2), (8.6.1), and (8.6.5). Note that equation (8.6.5) implies that

$$c_t^i\left(s^t\right) = u'^{-1}\left\{u'\left[c_t^1\left(s^t\right)\right]\frac{\mu_i}{\mu_1}\right\}. \tag{8.6.6}$$

Substituting this into equation (8.2.2) at equality gives

$$\sum_i u'^{-1}\left\{u'\left[c_t^1\left(s^t\right)\right]\frac{\mu_i}{\mu_1}\right\} = \sum_i y_t^i\left(s^t\right). \tag{8.6.7}$$

The right side of equation (8.6.7) is the current realization of the aggregate endowment. Therefore, the left side, and so $c_t^1(s^t)$, must also depend only on the current aggregate endowment, as well as on the ratio $\frac{\mu_i}{\mu_1}$. It follows from equation (8.6.6) that the equilibrium allocation $c_t^i(s^t)$ for each i depends only on the economy's aggregate endowment as well as on $\frac{\mu_i}{\mu_1}$. We summarize this analysis in the following proposition:

PROPOSITION 2: The competitive equilibrium allocation is a function of the realized aggregate endowment and does not depend on time t or the specific history or on the cross section distribution of endowments: $c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau)$ for all histories s^t and $\tilde{s}^\tau$ such that $\sum_j y_t^j(s^t) = \sum_j y_\tau^j(\tilde{s}^\tau)$.

8.6.1. Equilibrium pricing function

Suppose that c^i, $i = 1, \ldots, I$ is an equilibrium allocation. Then the marginal condition (8.6.2) or (8.6.4) can be regarded as determining the price system $q_t^0(s^t)$ as a function of the equilibrium allocation assigned to household i, for any i. But to exploit this fact in computation, we need a way first to compute an equilibrium allocation without simultaneously computing prices. As we shall see soon, solving the planning problem provides a convenient way to do that.

Because the units of the price system are arbitrary, one of the prices can be normalized at any positive value. We shall set $q_0^0(s_0) = 1$, putting the price system in units of time 0 goods. This choice implies that $\mu_i = u'[c_0^i(s_0)]$ for all i.

8.6.2. Optimality of equilibrium allocation

A competitive equilibrium allocation is a particular Pareto optimal allocation, one that sets the Pareto weights $\lambda_i = \mu_i^{-1}$. These weights are unique up to multiplication by a positive scalar. Furthermore, at a competitive equilibrium allocation, the *shadow prices* $\theta_t(s^t)$ for the associated planning problem equal the prices $q_t^0(s^t)$ for goods to be delivered at date t contingent on history s^t associated with the Arrow-Debreu competitive equilibrium. That allocations for the planning problem and the competitive equilibrium are identical reflects the two fundamental theorems of welfare economics (see Mas-Colell, Whinston, and Green (1995)). The first welfare theorem states that a competitive equilibrium allocation is efficient. The second welfare theorem states that any efficient allocation can be supported by a competitive equilibrium with an appropriate initial distribution of wealth.

8.6.3. Interpretation of trading arrangement

In the competitive equilibrium, all trades occur at $t = 0$ in one market. Deliveries occur after $t = 0$, but no more trades. A vast clearing or credit system operates at $t = 0$. It ensures that condition (8.6.1) holds for each household i. A symptom of the once-and-for-all and net-clearing trading arrangement is that each household faces one budget constraint that accounts for trades across all dates and histories.

In section 8.9, we describe another trading arrangement with more trading dates but fewer securities at each date.

8.6.4. Equilibrium computation

To compute an equilibrium, we have somehow to determine ratios of the Lagrange multipliers, μ_i/μ_1, $i = 1, \ldots, I$, that appear in equations (8.6.6) and (8.6.7). The following *Negishi algorithm* accomplishes this.[7]

1. Fix a positive value for one μ_i, say μ_1, throughout the algorithm. Guess some positive values for the remaining μ_i's. Then solve equations (8.6.6) and (8.6.7) for a candidate consumption allocation $c^i, i = 1, \ldots, I$.

2. Use (8.6.4) for any household i to solve for the price system $q_t^0(s^t)$.

3. For $i = 1, \ldots, I$, check the budget constraint (8.6.1). For those i's for which the cost of consumption exceeds the value of their endowment, raise μ_i, while for those i's for which the reverse inequality holds, lower μ_i.

4. Iterate to convergence on steps 1-3.

Multiplying all of the μ_i's by a positive scalar simply changes the units of the price system. That is why we are free to normalize as we have in step 1.

In general, the equilibrium price system and distribution of wealth are mutually determined. Along with the equilibrium allocation, they solve a vast system of simultaneous equations. The Negishi algorithm provides one way to solve those equations. In applications, it can be complicated to implement. Therefore, in order to simplify things, most of the examples and exercises in this chapter specialize preferences in a way that eliminates the dependence of equilibrium prices on the distribution of wealth.

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[7] See Negishi (1960).

8.7. Simpler computational algorithm

The preference specification in the following example enables us to avoid iterating on Pareto weights as in the Negishi algorithm.

8.7.1. Example 1: risk sharing

Suppose that the one-period utility function is of the constant relative risk-aversion (CRRA) form

$$u(c) = (1-\gamma)^{-1} c^{1-\gamma}, \ \gamma > 0.$$

Then equation (8.6.5) implies

$$\left[c_t^i\left(s^t\right)\right]^{-\gamma} = \left[c_t^j\left(s^t\right)\right]^{-\gamma} \frac{\mu_i}{\mu_j}$$

or

$$c_t^i\left(s^t\right) = c_t^j\left(s^t\right)\left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}}. \tag{8.7.1}$$

Equation (8.7.1) states that time t elements of consumption allocations to distinct agents are constant fractions of one another. With a power utility function, it says that individual consumption is perfectly correlated with the aggregate endowment or aggregate consumption.[8]

The fractions of the aggregate endowment assigned to each individual are independent of the realization of s^t. Thus, there is extensive cross-history and cross-time consumption sharing. The constant-fractions-of-consumption characterization comes from two aspects of the theory: (1) complete markets and (2) a homothetic one-period utility function.

[8] Equation (8.7.1) implies that conditional on the history s^t, time t consumption $c_t^i(s^t)$ is independent of the household's individual endowment at t, s^t, $y_t^i(s^t)$. Mace (1991), Cochrane (1991), and Townsend (1994) have tested and rejected versions of this conditional independence hypothesis. In chapter 20, we study how particular impediments to trade explain these rejections.

8.7.2. Implications for equilibrium computation

Equation (8.7.1) and the pricing formula (8.6.4) imply that an equilibrium price vector satisfies

$$q_t^0\left(s^t\right)=\mu_i^{-1}\alpha_i^{-\gamma}\beta^t\left(\overline{y}_t\left(s^t\right)\right)^{-\gamma}\pi_t\left(s^t\right), \tag{8.7.2}$$

where $c_t^i(s_t)=\alpha_i\overline{y}_t(s^t)$, $\overline{y}_t(s^t)=\sum_i y_t^i(s^t)$, and α_i is consumer i's fixed consumption share of the aggregate endowment. We are free to normalize the price system by setting $\mu_i\alpha_i^{-\gamma}$ for one consumer to an arbitrary positive number.

The homothetic CRRA preference specification that leads to equation (8.7.2) allows us to compute an equilibrium using the following steps:

1. Use (8.7.2) to compute an equilibrium price system.
2. Use this price system and consumer i's budget constraint to compute

$$\alpha_i=\frac{\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)y_t^i\left(s^t\right)}{\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)\overline{y}_t\left(s^t\right)}.$$

   Thus, consumer i's fixed consumption share α_i equals its share of aggregate wealth evaluated at the competitive equilibrium price vector.

8.7.3. Example 2: no aggregate uncertainty

In this example, the endowment structure is sufficiently simple that we can compute an equilibrium without assuming a homothetic one-period utility function. Let the stochastic event s_t take values on the unit interval $[0,1]$. There are two households, with $y_t^1(s^t)=s_t$ and $y_t^2(s^t)=1-s_t$. Note that the aggregate endowment is constant, $\sum_i y_t^i(s^t)=1$. Then equation (8.6.7) implies that $c_t^1(s^t)$ is constant over time and across histories, and equation (8.6.6) implies that $c_t^2(s^t)$ is also constant. Thus, the equilibrium allocation satisfies $c_t^i(s^t)=\bar{c}^i$ for all t and s^t, for $i=1,2$. Then from equation (8.6.4),

$$q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u'\left(\bar{c}^i\right)}{\mu_i}, \tag{8.7.3}$$

for all t and s^t, for $i=1,2$. Household i's budget constraint implies

$$\frac{u'\left(\bar{c}^i\right)}{\mu_i}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\left[\bar{c}^i-y_t^i\left(s^t\right)\right]=0.$$

Solving this equation for $\bar{c}^i$ gives

$$\bar{c}^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t \left(s^t\right) y_t^i \left(s^t\right). \tag{8.7.4}$$

Summing equation (8.7.4) verifies that $\bar{c}^1 + \bar{c}^2 = 1$.[9]

8.7.4. Example 3: periodic endowment processes

Consider the special case of the previous example in which s_t is deterministic and alternates between the values 1 and 0; $s_0 = 1$, $s_t = 0$ for t odd, and $s_t = 1$ for t even. Thus, the endowment processes are perfectly predictable sequences $(1, 0, 1, \ldots)$ for the first agent and $(0, 1, 0, \ldots)$ for the second agent. Let $\tilde{s}^t$ be the history of $(1, 0, 1, \ldots)$ up to t. Evidently, $\pi_t(\tilde{s}^t) = 1$, and the probability assigned to all other histories up to t is zero. The equilibrium price system is then

$$q_t^0 \left(s^t\right) = \begin{cases} \beta^t, & \text{if } s^t = \tilde{s}^t; \\ 0, & \text{otherwise;} \end{cases}$$

when using the time 0 good as numeraire, $q_0^0(\tilde{s}_0) = 1$. From equation (8.7.4), we have

$$\bar{c}^1 = (1-\beta) \sum_{j=0}^{\infty} \beta^{2j} = \frac{1}{1+\beta}, \tag{8.7.5a}$$

$$\bar{c}^2 = (1-\beta) \beta \sum_{j=0}^{\infty} \beta^{2j} = \frac{\beta}{1+\beta}. \tag{8.7.5b}$$

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9 If we let $\beta^{-1} = 1 + r$, where r is interpreted as the risk-free rate of interest, then note that (8.7.4) can be expressed as

$$\bar{c}^i = \left(\frac{r}{1+r}\right) E_0 \sum_{t=0}^{\infty} (1+r)^{-t} y_t^i \left(s^t\right).$$

Hence, equation (8.7.4) is a version of Friedman's permanent income model, which asserts that a household with zero financial assets consumes the annuity value of its human wealth defined as the expected discounted value of its labor income (which for present purposes we take to be $y_t^i(s^t)$). In the present example, the household completely smooths its consumption across time and histories, something that the household in Friedman's model typically cannot do. See chapter 17.

Consumer 1 consumes more every period because he is richer by virtue of receiving his endowment earlier.

8.7.5. Example 4

In this example, we assume that the one-period utility function is $\frac{c^{1-\gamma}}{1-\gamma}$. There are two consumers named $i = 1, 2$. Their endowments are $y_t^1 = y_t^2 = .5$ for $t = 0, 1$ and $y_t^1 = s_t$ and $y_t^2 = 1 - s_t$ for $t \geq 2$. The state space $s_t = \{0, 1\}$ and s_t is governed by a Markov chain with probability $\pi(s_0 = 1) = 1$ for the initial state and time-varying transition probabilities $\pi_1(s_1 = 1|s_0 = 1) = 1, \pi_2(s_2 = 1|s_1 = 1) = \pi_2(s_2 = 0|s_1 = 1) = .5, \pi_t(s_t = 1|s_{t-1} = 1) = 1, \pi_t(s_t = 0|s_{t-1} = 0) = 1$ for $t > 2$. This specification implies that $\pi_t(1, 1, \ldots, 1, 1, 1) = .5$ and $\pi_t(0, 0, \ldots, 0, 1, 1) = .5$ for all $t > 2$.

We can apply the method of subsection 8.7.2 to compute an equilibrium. The aggregate endowment is $\overline{y}_t(s^t) = 1$ for all t and all s^t. Therefore, an equilibrium price vector is $q_1^0(1, 1) = \beta, q_2^0(0, 1, 1) = q_2^0(1, 1, 1) = .5\beta^2$ and $q_t^0(1, 1, \ldots, 1, 1) = q_t^0(0, 0, \ldots, 1, 1) = .5\beta^t$ for $t > 2$. Use these prices to compute the value of agent i's endowment: $\sum_t \sum_{s^t} q_t^0(s^t) y_t^i(s^t) = \sum_t \beta^t .5[.5 + .5 + 0 + \ldots + 0] + \sum_t \beta^t .5[.5 + .5 + 1 + \ldots + 1] = 2 \sum_t \beta^t .5[.5 + .5 + \ldots + .5] = .5 \sum_t \beta^t = \frac{.5}{1-\beta}$. Consumer i's budget constraint is satisfied when he consumes a constant consumption of .5 each period in each state: $c_t^i(s^t) = .5$ for all t for all s^t.

In subsection 8.10.4, we shall use the equilibrium allocation from the Arrow-Debreu economy in this example to synthesize an equilibrium in an economy with sequential trading.

8.8. Primer on asset pricing

Many asset-pricing models assume complete markets and price an asset by breaking it into a sequence of history-contingent claims, evaluating each component of that sequence with the relevant "state price deflator" $q_t^0(s^t)$, then adding up those values. The asset is *redundant*, in the sense that it offers a bundle of history-contingent dated claims, each component of which has already been priced by the market. While we shall devote chapters 13 and 14 entirely to asset-pricing theories, it is useful to give some pricing formulas at this point because they help illustrate the complete market competitive structure.

8.8.1. Pricing redundant assets

Let $\{d_t(s^t)\}_{t=0}^{\infty}$ be a stream of claims on time t, history s^t consumption, where $d_t(s^t)$ is a measurable function of s^t. The price of an asset entitling the owner to this stream must be

$$p_0^0\left(s_0\right) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right) d_t\left(s^t\right). \tag{8.8.1}$$

If this equation did not hold, someone could make unbounded profits by synthesizing this asset through purchases or sales of history-contingent dated commodities and then either buying or selling the asset. We shall elaborate this arbitrage argument below and later in chapter 13 on asset pricing.

8.8.2. Riskless consol

As an example, consider the price of a *riskless consol*, that is, an asset offering to pay one unit of consumption for sure each period. Then $d_t(s^t) = 1$ for all t and s^t, and the price of this asset is

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right). \tag{8.8.2}$$

8.8.3. Riskless strips

As another example, consider a sequence of *strips* of payoffs on the riskless consol. The time t strip is just the payoff process $d_\tau = 1$ if $\tau = t \geq 0$, and 0 otherwise. Thus, the owner of the strip is entitled to the time t coupon only. The value of the time t strip at time 0 is evidently

$$\sum_{s^t} q_t^0 \left(s^t\right).$$

Compare this to the price of the consol (8.8.2). We can think of the t-period riskless strip as a t-period zero-coupon bond. See appendix B of chapter 14 for an account of a closely related model of yields on such bonds.

8.8.4. Tail assets

Return to the stream of dividends $\{d_t(s^t)\}_{t\geq 0}$ generated by the asset priced in equation (8.8.1). For $\tau \geq 1$, suppose that we strip off the first $\tau - 1$ periods of the dividend and want the time 0 value of the remaining dividend stream $\{d_t(s^t)\}_{t\geq\tau}$. Specifically, we seek the value of this asset for a particular possible realization of s^τ. Let $p_\tau^0(s^\tau)$ be the time 0 price of an asset that entitles the owner to dividend stream $\{d_t(s^t)\}_{t\geq\tau}$ if history s^τ is realized,

$$p_\tau^0\left(s^\tau\right) = \sum_{t\geq\tau} \sum_{s^t|s^\tau} q_t^0\left(s^t\right) d_t\left(s^t\right), \tag{8.8.3}$$

where the summation over $s^t|s^\tau$ means that we sum over all possible subsequent histories $\tilde{s}^t$ such that $\tilde{s}^\tau = s^\tau$. When the units of the price are time 0, state s_0 goods, the normalization is $q_0^0(s_0) = 1$. To convert the price into units of time τ, history s^τ consumption goods, divide by $q_\tau^0(s^\tau)$ to get

$$p_\tau^\tau\left(s^\tau\right) \equiv \frac{p_\tau^0\left(s^\tau\right)}{q_\tau^0\left(s^\tau\right)} = \sum_{t\geq\tau} \sum_{s^t|s^\tau} \frac{q_t^0\left(s^t\right)}{q_\tau^0\left(s^\tau\right)} d_t\left(s^t\right). \tag{8.8.4}$$

Notice that[10]

$$\begin{aligned} q_t^\tau\left(s^t\right) \equiv \frac{q_t^0\left(s^t\right)}{q_\tau^0\left(s^\tau\right)} &= \frac{\beta^t u'\left[c_t^i\left(s^t\right)\right]\pi_t\left(s^t\right)}{\beta^\tau u'\left[c_\tau^i\left(s^\tau\right)\right]\pi_\tau\left(s^\tau\right)} \\ &= \beta^{t-\tau}\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_\tau^i\left(s^\tau\right)\right]}\pi_t\left(s^t|s^\tau\right). \end{aligned} \tag{8.8.5}$$

10 Because the marginal conditions hold for all consumers, this condition holds for all i.

Here $q_t^\tau(s^t)$ is the price of one unit of consumption delivered at time t, history s^t in terms of the date τ, history s^τ consumption good; $\pi_t(s^t|s^\tau)$ is the probability of history s^t conditional on history s^τ at date τ. Thus, the price at time τ, history s^τ for the "tail asset" is

$$p_\tau^\tau\left(s^\tau\right) = \sum_{t\geq\tau}\sum_{s^t|s^\tau} q_t^\tau\left(s^t\right) d_t\left(s^t\right). \tag{8.8.6}$$

When we want to create a time series of, say, equity prices, we use the "tail asset" pricing formula (8.8.6). An equity purchased at time τ entitles the owner to the dividends from time τ forward. Our formula (8.8.6) expresses the asset price in terms of prices with time τ, history s^τ good as numeraire.

8.8.5. One-period returns

The one-period version of equation (8.8.5) is

$$q_{\tau+1}^\tau\left(s^{\tau+1}\right) = \beta\frac{u'\left[c_{\tau+1}^i\left(s^{\tau+1}\right)\right]}{u'\left[c_\tau^i\left(s^\tau\right)\right]}\pi_{\tau+1}\left(s^{\tau+1}|s^\tau\right).$$

The right side is the one-period *pricing kernel* at time τ. If we want to find the price at time τ at history s^τ of a claim to a random payoff $\omega(s_{\tau+1})$, we use

$$p_\tau^\tau\left(s^\tau\right) = \sum_{s_{\tau+1}} q_{\tau+1}^\tau\left(s^{\tau+1}\right)\omega\left(s_{\tau+1}\right)$$

or

$$p_\tau^\tau\left(s^\tau\right) = E_\tau\left[\beta\frac{u'\left(c_{\tau+1}\right)}{u'\left(c_\tau\right)}\omega\left(s_{\tau+1}\right)\right], \tag{8.8.7}$$

where E_τ is the conditional expectation operator. We have deleted the i superscripts on consumption, with the understanding that equation (8.8.7) is true for any consumer i; we have also suppressed the dependence of c_τ on s^τ, which is implicit.

Let $R_{\tau+1} \equiv \omega(s_{\tau+1})/p_\tau^\tau(s^\tau)$ be the one-period gross *return* on the asset. Then for any asset, equation (8.8.7) implies

$$1 = E_\tau\left[\beta\frac{u'\left(c_{\tau+1}\right)}{u'\left(c_\tau\right)}R_{\tau+1}\right] \equiv E_\tau\left[m_{\tau+1}R_{\tau+1}\right]. \tag{8.8.8}$$

The term $m_{\tau+1} \equiv \beta u'(c_{\tau+1})/u'(c_\tau)$ functions as a *stochastic discount factor*. Like $R_{\tau+1}$, it is a random variable measurable with respect to $s_{\tau+1}$, given s^τ. Equation (8.8.8) is a restriction on the conditional moments of returns and m_{t+1}. Applying the law of iterated expectations to equation (8.8.8) gives the unconditional moments restriction

$$1 = E\left[\beta \frac{u'(c_{\tau+1})}{u'(c_\tau)} R_{\tau+1}\right] \equiv E\left[m_{\tau+1} R_{\tau+1}\right]. \tag{8.8.9}$$

In chapters 13 and 14 we shall many more instances of this equation.

In the next section, we display another market structure in which the one-period pricing kernel $q_{t+1}^t(s^{t+1})$ also plays a decisive role. This structure uses the celebrated one-period "Arrow securities," the sequential trading of which substitutes perfectly for the comprehensive trading of long horizon claims at time 0.

8.9. Sequential trading: Arrow securities

This section describes an alternative market structure that preserves both the equilibrium allocation and the key one-period asset-pricing formula (8.8.7).

8.9.1. Arrow securities

We build on an insight of Arrow (1964) that one-period securities are enough to implement complete markets, provided that new one-period markets are re-opened for trading each period and provided that time t, history s^t wealth is properly assigned to each agent. Thus, at each date $t \geq 0$, but only at the history s^t actually realized, trades occur in a set of claims to one-period-ahead state-contingent consumption. We describe a competitive equilibrium of this sequential-trading economy. With a full array of these one-period-ahead claims, the sequential-trading arrangement attains the same allocation as the competitive equilibrium that we described earlier.

8.9.2. Financial wealth as an endogenous state variable

A key step in constructing a sequential-trading arrangement is to identify a variable to serve as the state in a value function for the household at date t. We find this state by taking an equilibrium allocation and price system for the (Arrow-Debreu) time 0 trading structure and applying a guess-and-verify method. We begin by asking the following question. In the competitive equilibrium where all trading takes place at time 0, what is the implied continuation wealth of household i at time t after history s^t, but before adding in its time t, history s^t endowment $y_t^i(s^t)$? To answer this question, in period t, conditional on history s^t, we sum up the value of the household's purchased claims to current and future goods net of its outstanding liabilities. Since history s^t has been realized, we discard all claims and liabilities contingent on time t histories $\tilde{s}^t \neq s^t$ that were not realized. Household i's net claim to delivery of goods in a future period $\tau \geq t$ contingent on history $\tilde{s}^\tau$ whose time t partial history $\tilde{s}^t = s^t$ is $[c_\tau^i(\tilde{s}^\tau) - y_\tau^i(\tilde{s}^\tau)]$. Thus, the household's financial wealth, or the value of all its current and future net claims, expressed in terms of the date t, history s^t consumption good is

$$\Upsilon_t^i\left(s^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t\left(s^\tau\right)\left[c_\tau^i\left(s^\tau\right) - y_\tau^i\left(s^\tau\right)\right]. \tag{8.9.1}$$

Notice that feasibility constraint $(8.2.2)$ at equality implies that

$$\sum_{i=1}^{I} \Upsilon_t^i\left(s^t\right) = 0, \qquad \forall t, s^t.$$

8.9.3. Financial and non-financial wealth

Define $\Upsilon_t^i(s^t)$ as *financial wealth* and $\sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$ as *non-financial wealth.*[11] In terms of these concepts, (8.9.1) implies

$$\Upsilon_t^i\left(s^t\right) + \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^t\left(s^\tau\right) y_\tau^i\left(s^\tau\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^t\left(s^\tau\right) c_\tau^i\left(s^\tau\right), \tag{8.9.2}$$

which states that at each time and each history, the sum of financial and non-financial wealth equals the present value of current and future consumption claims. At time 0, we have set $\Upsilon_t^i(s^0) = 0$ for all i. At $t > 0$, financial wealth $\Upsilon_t^i(s^t)$ typically differs from zero for individual i, but it sums to zero across i.

8.9.4. Reopening markets

Formula (8.8.5) takes the form of a pricing function for a complete markets economy with date- and history-contingent commodities whose markets can be regarded as having been reopened at date τ, history s^τ, starting from wealth levels implied by the tails of each household's endowment and consumption streams for a complete markets economy that originally convened at $t = 0$. We leave it as an exercise to the reader to prove the following proposition.

Proposition 3: Start from the distribution of time t, history s^t wealth that is implicit in a time 0 Arrow-Debreu equilibrium. If markets are reopened at date t after history s^t, no trades occur. That is, given the price system (8.8.5), all households choose to continue the tails of their original consumption plans.

[11] In some applications, financial wealth is also called 'non-human wealth' and non-financial wealth is called 'human wealth'.

8.9.5. Debt limits

In moving from the Arrow-Debreu economy to one with sequential trading, we propose to match the time t, history s^t wealth of the household in the sequential economy with the equilibrium tail wealth $\Upsilon_t^i(s^t)$ from the Arrow-Debreu economy computed in equation (8.9.2). But first we have to say something about debt limits, a feature that was only implicit in the time 0 budget constraint (8.6.1) in the Arrow-Debreu economy. In moving to the sequential formulation, we restrict asset trades to prevent Ponzi schemes. We want the weakest possible restrictions. We synthesize restrictions that work by starting from the equilibrium allocation of the Arrow-Debreu economy (with time 0 markets), and find some state-by-state debt limits that support the equilibrium allocation that emerged from the Arrow-Debreu economy under a sequential trading arrangement. Often we'll refer to these weakest possible debt limits as the "natural debt limits." These limits come from the common sense requirement that it has to be *feasible* for the consumer to repay his state contingent debt in every possible state. Together with our assumption that $c_t^i(s^t)$ must be nonnegative, that feasibility requirement leads to the natural debt limits.

Let $q_\tau^t(s^\tau)$ be the Arrow-Debreu price, denominated in units of the date t, history s^t consumption good. Consider the value of the tail of agent i's endowment sequence at time t in history s^t:

$$A_t^i\left(s^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t\left(s^\tau\right) y_\tau^i\left(s^\tau\right). \tag{8.9.3}$$

We call $A_t^i(s^t)$ the *natural debt limit* at time t and history s^t. It is the maximal value that agent i can repay starting from that period, assuming that his consumption is zero always. With sequential trading, we shall require that household i at time $t-1$ and history s^{t-1} cannot promise to pay more than $A_t^i(s^t)$ conditional on the realization of s_t tomorrow, because it will not be feasible to repay more. Household i at time $t-1$ faces one such borrowing constraint for each possible realization of s_t tomorrow.

8.9.6. Sequential trading

There is a sequence of markets in one-period-ahead state-contingent claims. At each date $t \geq 0$, households trade claims to date $t+1$ consumption, whose payment is contingent on the realization of s_{t+1}. Let $\tilde{a}_t^i(s^t)$ denote the claims to time t consumption, other than its time t endowment $y_t^i(s^t)$, that household i brings into time t in history s^t. Suppose that $\tilde{Q}_t(s_{t+1}|s^t)$ is a *pricing kernel* to be interpreted as follows: $\tilde{Q}_t(s_{t+1}|s^t)$ is the price of one unit of time $t+1$ consumption, contingent on the realization s_{t+1} at $t+1$, when the history at t is s^t. The household faces a sequence of budget constraints for $t \geq 0$, where the time t, history s^t budget constraint is

$$\tilde{c}_t^i\left(s^t\right) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i\left(s_{t+1}, s^t\right) \tilde{Q}_t\left(s_{t+1}|s^t\right) \leq y_t^i\left(s^t\right) + \tilde{a}_t^i\left(s^t\right). \tag{8.9.4}$$

At time t, a household chooses $\tilde{c}_t^i(s^t)$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$, where $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$ is a vector of claims on time $t+1$ consumption, there being one element of the vector for each value of the time $t+1$ realization of s_{t+1}. To rule out Ponzi schemes, we impose the state-by-state borrowing constraints

$$-\tilde{a}_{t+1}^i\left(s^{t+1}\right) \leq A_{t+1}^i\left(s^{t+1}\right), \tag{8.9.5}$$

where $A_{t+1}^i(s^{t+1})$ is computed in equation (8.9.3).

Let $\eta_t^i(s^t)$ and $\nu_t^i(s^t; s_{t+1})$ be nonnegative Lagrange multipliers on the budget constraint (8.9.4) and the borrowing constraint (8.9.5), respectively, for time t and history s^t. Form the Lagrangian

$$\begin{aligned} L^i = \sum_{t=0}^{\infty} \sum_{s^t} \Big\{ & \beta^t u(\tilde{c}_t^i(s^t))\pi_t(s^t) \\ & + \eta_t^i(s^t)\Big[y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t)\Big] \\ & + \sum_{s_{t+1}} \nu_t^i(s^t; s_{t+1})\left[A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1})\right] \Big\}, \end{aligned}$$

for a given initial wealth $\tilde{a}_0^i(s_0)$. The first-order conditions for maximizing L^i with respect to $\tilde{c}_t^i(s^t)$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}$ are

$$\beta^t u'(\tilde{c}_t^i(s^t))\pi_t(s^t) - \eta_t^i(s^t) = 0, \tag{8.9.6a}$$

$$-\eta_t^i(s^t)\tilde{Q}_t(s_{t+1}|s^t) + \nu_t^i(s^t; s_{t+1}) + \eta_{t+1}^i(s_{t+1}, s^t) = 0, \tag{8.9.6b}$$

for all s_{t+1}, t, s^t. In the optimal solution to this problem, the natural debt limit (8.9.5) will not be binding, and hence the Lagrange multipliers $\nu_t^i(s^t; s_{t+1})$ all equal zero for the following reason: if there were any history s^{t+1} leading to a binding natural debt limit, the household would from then on have to set consumption equal to zero in order to honor its debt. Because the household's utility function satisfies the Inada condition $\lim_{c \downarrow 0} u'(c) = +\infty$, that would mean that all future marginal utilities would be infinite. Thus, it would be easy to find alternative affordable allocations that yield higher expected utility by postponing earlier consumption to periods after such a binding constraint.

After setting $\nu_t^i(s^t; s_{t+1}) = 0$ in equation (8.9.6*b*), the first-order conditions imply the following restrictions on the optimally chosen consumption allocation,

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'(\tilde{c}_{t+1}^i(s^{t+1}))}{u'(\tilde{c}_t^i(s^t))} \pi_t(s^{t+1}|s^t), \tag{8.9.7}$$

for all s_{t+1}, t, s^t.

Definition: A *distribution of wealth* is a vector $\vec{\tilde{a}}_t(s^t) = \{\tilde{a}_t^i(s^t)\}_{i=1}^I$ satisfying $\sum_i \tilde{a}_t^i(s^t) = 0$.

Definition: A *competitive equilibrium with sequential trading of one-period Arrow securities* is an initial distribution of wealth $\vec{\tilde{a}}_0(s_0)$, a collection of borrowing limits $\{A_t^i(s^t)\}$ satisfying (8.9.3) for all i, for all t, and for all s^t, a feasible allocation $\{\tilde{c}^i\}_{i=1}^I$, and pricing kernels $\tilde{Q}_t(s_{t+1}|s^t)$ such that
(a) for all i, given $\tilde{a}_0^i(s_0)$, the borrowing limits $\{A_t^i(s^t)$, and the pricing kernels, the consumption allocation $\tilde{c}^i$ solves the household's problem for all i;
(b) for all realizations of $\{s^t\}_{t=0}^\infty$, the households' consumption allocations and implied portfolios $\{\tilde{c}_t^i(s^t), \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}\}_i$ satisfy $\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t)$ and $\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$ for all s_{t+1}.

This definition leaves open the initial distribution of wealth. We'll say more about the initial distribution of wealth soon.

8.9.7. Equivalence of allocations

By making an appropriate guess about the form of the pricing kernels, it is easy to show that a competitive equilibrium allocation of the complete markets model with time 0 trading is also an allocation for a competitive equilibrium with sequential trading of one-period Arrow securities, one with a particular initial distribution of wealth. Thus, take $q_t^0(s^t)$ as given from the Arrow-Debreu equilibrium and suppose that the pricing kernel $\tilde{Q}_t(s_{t+1}|s^t)$ makes the following recursion true:

$$q_{t+1}^0(s^{t+1}) = \tilde{Q}_t(s_{t+1}|s^t) q_t^0(s^t),$$

or

$$\tilde{Q}_t(s_{t+1}|s^t) = q_{t+1}^t(s^{t+1}), \tag{8.9.8}$$

where recall that $q_{t+1}^t(s^{t+1}) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$.

Let $\{c_t^i(s^t)\}$ be a competitive equilibrium allocation in the Arrow-Debreu economy. If equation (8.9.8) is satisfied, that allocation is also a sequential-trading competitive equilibrium allocation. To show this fact, take the household's first-order conditions (8.6.4) for the Arrow-Debreu economy from two successive periods and divide one by the other to get

$$\frac{\beta u'[c_{t+1}^i(s^{t+1})]\pi(s^{t+1}|s^t)}{u'[c_t^i(s^t)]} = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \tilde{Q}_t(s_{t+1}|s^t). \tag{8.9.9}$$

If the pricing kernel satisfies equation (8.9.8), this equation is equivalent with the first-order condition (8.9.7) for the sequential-trading competitive equilibrium economy. It remains for us to choose the initial wealth of the sequential-trading equilibrium so that the sequential-trading competitive equilibrium duplicates the Arrow-Debreu competitive equilibrium allocation.

We conjecture that the initial wealth vector $\vec{\tilde{a}}_0(s_0)$ of the sequential-trading economy should be chosen to be the zero vector. This is a natural conjecture, because it means that each household must rely on its own endowment stream to finance consumption, in the same way that households are constrained to finance their history-contingent purchases for the infinite future at time 0 in the Arrow-Debreu economy. To prove that the conjecture is correct, we must show that the zero initial wealth vector enables household i to finance $\{c_t^i(s^t)\}$ and leaves no room to increase consumption in any period after any history.

The proof proceeds by guessing that, at time $t \geq 0$ and history s^t, household i chooses a portfolio given by $\tilde{a}^i_{t+1}(s_{t+1}, s^t) = \Upsilon^i_{t+1}(s^{t+1})$ for all s_{t+1}. The value of this portfolio expressed in terms of the date t, history s^t consumption good is

$$\begin{aligned}\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) &= \sum_{s^{t+1}|s^t} \Upsilon^i_{t+1}(s^{t+1}) q^t_{t+1}(s^{t+1}) \\ &= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q^t_\tau(s^\tau)\left[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)\right], \quad (8.9.10)\end{aligned}$$

where we have invoked expressions (8.9.2) and (8.9.8).[12] To demonstrate that household i can afford this portfolio strategy, we now use budget constraint (8.9.4) to compute the implied consumption plan $\{\tilde{c}^i_\tau(s^\tau)\}$. First, in the initial period $t = 0$ with $\tilde{a}^i_0(s_0) = 0$, the substitution of equation (8.9.10) into budget constraint (8.9.4) at equality yields

$$\tilde{c}^i_0(s_0) + \sum_{t=1}^{\infty} \sum_{s^t} q^0_t(s^t)\left[c^i_t(s^t) - y^i_t(s^t)\right] = y^i_t(s_0) + 0\,.$$

This expression together with budget constraint (8.6.1) at equality imply $\tilde{c}^i_0(s_0) = c^i_0(s_0)$. In other words, the proposed portfolio is affordable in period 0 and the associated consumption plan is the same as in the competitive equilibrium of the Arrow-Debreu economy. In all consecutive future periods $t > 0$ and histories s^t, we replace $\tilde{a}^i_t(s^t)$ in constraint (8.9.4) by $\Upsilon^i_t(s^t)$, and after noticing that the value of the asset portfolio in (8.9.10) can be written as

$$\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \Upsilon^i_t(s^t) - \left[c^i_t(s^t) - y^i_t(s^t)\right], \qquad (8.9.11)$$

it follows immediately from (8.9.4) that $\tilde{c}^i_t(s^t) = c^i_t(s^t)$ for all periods and histories.

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12 We have also used the following identities,

$$q^{t+1}_\tau(s^\tau) q^t_{t+1}(s^{t+1}) = \frac{q^0_\tau(s^\tau)}{q^0_{t+1}(s^{t+1})} \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = q^t_\tau(s^\tau) \text{ for } \tau > t.$$

We have shown that the proposed portfolio strategy attains the same consumption plan as in the competitive equilibrium of the Arrow-Debreu economy, but what precludes household i from further increasing current consumption by reducing some component of the asset portfolio? The answer lies in the debt limit restrictions to which the household must adhere. In particular, if the household wants to ensure that consumption plan $\{c^i_\tau(s^\tau)\}$ can be attained starting next period in all possible future states, the household should subtract the value of this commitment to future consumption from the natural debt limit in (8.9.3). Thus, the household is facing a state-by-state borrowing constraint that is more restrictive than restriction (8.9.5): for any s^{t+1},

$$-\tilde{a}^i_{t+1}(s^{t+1}) \leq A^i_{t+1}(s^{t+1}) - \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}_\tau(s^\tau) c^i_\tau(s^\tau) = -\Upsilon^i_{t+1}(s^{t+1}),$$

or

$$\tilde{a}^i_{t+1}(s^{t+1}) \geq \Upsilon^i_{t+1}(s^{t+1}).$$

Hence, household i does not want to increase consumption at time t by reducing next period's wealth below $\Upsilon^i_{t+1}(s^{t+1})$ because that would jeopardize attaining the preferred consumption plan that satisfies first-order conditions (8.9.7) for all future periods and histories.

8.10. Recursive competitive equilibrium

We have established that equilibrium allocations are the same in the Arrow-Debreu economy with complete markets in dated contingent claims all traded at time 0 and in a sequential-trading economy with a complete set of one-period Arrow securities. This finding holds for arbitrary individual endowment processes $\{y^i_t(s^t)\}_i$ that are measurable functions of the history of events s^t, which in turn are governed by some arbitrary probability measure $\pi_t(s^t)$. At this level of generality, the pricing kernels $\tilde{Q}_t(s_{t+1}|s^t)$ and the wealth distributions $\vec{\tilde{a}}_t(s^t)$ in the sequential-trading economy both depend on the history s^t, so both are time-varying functions of all past events $\{s_\tau\}_{\tau=0}^t$. This can make it difficult to formulate an economic model that can be used to confront empirical observations. We want a framework in which economic outcomes are functions of a limited number of "state variables" that summarize the effects of past events and current information. This leads us to make the following specialization of the exogenous forcing processes that facilitates a recursive formulation of the sequential-trading equilibrium.

8.10.1. Endowments governed by a Markov process

Let $\pi(s'|s)$ be a Markov chain with given initial distribution $\pi_0(s)$ and state space $s \in S$. That is, $\text{Prob}(s_{t+1} = s'|s_t = s) = \pi(s'|s)$ and $\text{Prob}(s_0 = s) = \pi_0(s)$. As we saw in chapter 2, the chain induces a sequence of probability measures $\pi_t(s^t)$ on histories s^t via the recursions

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_1|s_0)\pi_0(s_0). \tag{8.10.1}$$

In this chapter, we have assumed that trading occurs after s_0 has been observed, which we capture by setting $\pi_0(s_0) = 1$ for the initially given value of s_0.

Because of the Markov property, the conditional probability $\pi_t(s^t|s^\tau)$ for $t > \tau$ depends only on the state s_τ at time τ and does not depend on the history before τ,

$$\pi_t(s^t|s^\tau) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_{\tau+1}|s_\tau). \tag{8.10.2}$$

Next, we assume that households' endowments in period t are time invariant measurable functions of s_t, $y_t^i(s^t) = y^i(s_t)$ for each i. Of course, all of our previous results continue to hold, but the Markov assumption for s_t imparts further structure to the equilibrium.

8.10.2. Equilibrium outcomes inherit the Markov property

Proposition 2 asserted a particular kind of history independence of the equilibrium allocation that prevails under any stochastic process for the endowments. In particular, each individual's consumption is a function only of the current realization of the aggregate endowment and does not depend on the specific history leading to that outcome.[13] Now, under our present assumption that $y_t^i(s^t) = y^i(s_t)$ for each i, it follows immediately that

$$c_t^i(s^t) = \bar{c}^i(s_t). \tag{8.10.3}$$

Substituting (8.10.2) and (8.10.3) into (8.9.7) shows that the pricing kernel in the sequential-trading equilibrium is a function only of the current state,

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'(\bar{c}^i(s_{t+1}))}{u'(\bar{c}^i(s_t))} \pi(s_{t+1}|s_t) \equiv Q(s_{t+1}|s_t). \tag{8.10.4}$$

After similar substitutions with respect to equation (8.8.5), we can also establish history independence of the relative prices in the Arrow-Debreu economy:

Proposition 4: If time t endowments are a function of a Markov state s_t, the Arrow-Debreu equilibrium price of date-$t \geq 0$, history s^t consumption goods expressed in terms of date τ $(0 \leq \tau \leq t)$, history s^τ consumption goods is not history dependent: $q_t^\tau(s^t) = q_k^j(\tilde{s}^k)$ for $j,k \geq 0$ such that $t - \tau = k - j$ and $[s_\tau, s_{\tau+1}, \ldots, s_t] = [\tilde{s}_j, \tilde{s}_{j+1}, \ldots, \tilde{s}_k]$.

Using this proposition, we can verify that both the natural debt limits (8.9.3) and households' wealth levels (8.9.2) exhibit history independence,

$$A_t^i(s^t) = \bar{A}^i(s_t)\,, \tag{8.10.5}$$

$$\Upsilon_t^i(s^t) = \bar{\Upsilon}^i(s_t)\,. \tag{8.10.6}$$

The finding concerning wealth levels (8.10.6) conveys a useful insight into how the sequential-trading competitive equilibrium attains the first-best outcome in which no idiosyncratic risk is borne by individual households. In particular, each household enters every period with a wealth level that is independent of past realizations of his endowment. That is, his past trades have fully insured

[13] Of course, the equilibrium allocation also depends on the distribution of $\{y_t^i(s^t)\}$ *processes* across agents i, as reflected in the relative values of the Lagrange multipliers μ_i.

him against the idiosyncratic outcomes of his endowment. And from that very same insurance motive, the household now enters the present period with a wealth level that is a function of the current state s_t. It is a state-contingent wealth level that was chosen by the household in the previous period $t-1$, and this wealth will be just sufficient to continue a trading strategy previously designed to insure against future idiosyncratic risks. The optimal holding of wealth is a function of s_t alone because the current state s_t determines the current endowment and the current pricing kernel and contains all information relevant for predicting future realizations of the household's endowment process as well as future prices. It can be shown that a household especially wants higher wealth levels for those states next period that either make his next period endowment low or more generally signal poor future prospects for its endowment into the more distant future. Of course, individuals' desires are tempered by differences in the economy's aggregate endowment across states (as reflected in equilibrium asset prices). Aggregate shocks cannot be diversified away but must be borne somehow by all of the households. The pricing kernel $Q(s_t|s_{t-1})$ and the assumed clearing of all markets set into action an "invisible hand" that coordinates households' transactions at time $t-1$ in such a way that only aggregate risk and no idiosyncratic risk is borne by the households.

8.10.3. Recursive formulation of optimization and equilibrium

The fact that the pricing kernel $Q(s'|s)$ and the endowment $y^i(s)$ are functions of a Markov process s motivates us to seek a recursive formulation of the household's optimization problem. Household i's state at time t is its wealth a_t^i and the current realization s_t. We seek a pair of optimal policy functions $h^i(a,s)$, $g^i(a,s,s')$ such that the household's optimal decisions are

$$c_t^i = h^i(a_t^i, s_t), \tag{8.10.7a}$$

$$a_{t+1}^i(s_{t+1}) = g^i(a_t^i, s_t, s_{t+1}). \tag{8.10.7b}$$

Let $v^i(a,s)$ be the optimal value of household i's problem starting from state (a,s); $v^i(a,s)$ is the maximum expected discounted utility household that household i with current wealth a can attain in state s. The Bellman equation for the household's problem is

$$v^i(a,s) = \max_{c,\hat{a}(s')} \left\{ u(c) + \beta \sum_{s'} v^i[\hat{a}(s'), s']\pi(s'|s) \right\} \tag{8.10.8}$$

where the maximization is subject to the following version of constraint (8.9.4):

$$c + \sum_{s'} \hat{a}(s')Q(s'|s) \leq y^i(s) + a \tag{8.10.9}$$

and also

$$c \geq 0, \tag{8.10.10a}$$

$$-\hat{a}(s') \leq \bar{A}^i(s'), \qquad \forall s'. \tag{8.10.10b}$$

Let the optimum decision rules be

$$c = h^i(a,s), \tag{8.10.11a}$$

$$\hat{a}(s') = g^i(a,s,s'). \tag{8.10.11b}$$

Note that the solution of the Bellman equation implicitly depends on $Q(\cdot|\cdot)$ because it appears in the constraint (8.10.9). In particular, use the first-order conditions for the problem on the right of equation (8.10.8) and the Benveniste-Scheinkman formula and rearrange to get

$$Q(s_{t+1}|s_t) = \frac{\beta u'(c^i_{t+1})\pi(s_{t+1}|s_t)}{u'(c^i_t)}, \tag{8.10.12}$$

where it is understood that $c^i_t = h^i(a^i_t, s_t)$ and $c^i_{t+1} = h^i(a^i_{t+1}(s_{t+1}), s_{t+1}) = h^i(g^i(a^i_t, s_t, s_{t+1}), s_{t+1})$.

DEFINITION: A *recursive competitive equilibrium* is an initial distribution of wealth $\vec{a}_0$, a set of borrowing limits $\{\bar{A}^i(s)\}_{i=1}^I$, a pricing kernel $Q(s'|s)$, sets of value functions $\{v^i(a,s)\}_{i=1}^I$, and decision rules $\{h^i(a,s), g^i(a,s,s')\}_{i=1}^I$ such that

(a) The state-by-state borrowing constraints satisfy the recursion

$$\bar{A}^i(s) = y^i(s) + \sum_{s'} Q(s'|s)\bar{A}^i(s'|s). \tag{8.10.13}$$

(b) For all i, given a_0^i, $\bar{A}^i(s)$, and the pricing kernel, the value functions and decision rules solve the household's problem;

(c) For all realizations of $\{s_t\}_{t=0}^\infty$, the consumption and asset portfolios $\{\{c_t^i, \{\hat{a}_{t+1}^i(s')\}_{s'}\}_i\}_t$ implied by the decision rules satisfy $\sum_i c_t^i = \sum_i y^i(s_t)$ and $\sum_i \hat{a}_{t+1}^i(s') = 0$ for all t and s'.

We shall use the recursive competitive equilibrium concept extensively in our discussion of asset pricing in chapter 13.

8.10.4. Computing an equilibrium with sequential trading of Arrow-securities

We use example 4 from subsection 8.7.5 to illustrate the following algorithm for computing an equilibrium in an economy with sequential trading of a complete set of Arrow securities:

1. Compute an equilibrium of the Arrow-Debreu economy with time 0 trading.
2. Set the equilibrium allocation for the sequential trading economy to the equilibrium allocation from Arrow-Debreu time 0 trading economy.
3. Compute equilibrium prices from formula (8.10.12) for a Markov economy or the corresponding formula (8.9.9) for a non-Markov economy.
4. Compute debt limits from (8.10.13).
5. Compute portfolios of one-period Arrow securities by first computing implied time t, history s^t wealth $\Upsilon_t^i(s^t)$ from (8.9.2) evaluated at the Arrow-Debreu equilibrium prices, then set $a_t^i(s_t) = \Upsilon_t^i(s^t)$.

Applying this procedure to example 4 from section 8.7.5 gives us the price system $Q_0(s_1 = 1|s_0 = 1) = \beta, Q_0(s_1 = 0|s_0 = 1) = 0, Q_1(s_2 = 1|s_1 = 1) = .5\beta, Q_1(s_2 = 0|s_1 = 0) = .5\beta$ and $Q_t(s_{t+1} = 1|s_t = 1) = Q_t(s_{t+1} = 0|s_t = 0) = \beta$ for $t \geq 2$. Also, $\Upsilon_t^i(s^t) = 0$ for $i = 1, 2$ and $t = 0, 1$. For $t \geq 2$, $\Upsilon_t^1(s_t = 1) = \sum_{\tau \geq t} \beta^{\tau - t}[.5 - 1] = \frac{-.5}{1-\beta}$ and $\Upsilon_t^2(s_t = 1) = \sum_{\tau \geq t} \beta^{\tau - t}[.5 - 0] = \frac{.5}{1-\beta}$. Therefore, in period 1, the first consumer trades Arrow securities in amounts $a_2^1(s_2 = 1) = \frac{-.5}{1-\beta}, a_2^1(s_2 = 0) = \frac{.5}{1-\beta}$, while the second consumer trades Arrow securities in amounts $a_2^1(s_2 = 1) = \frac{.5}{1-\beta}, a_2^1(s_2 = 0) = \frac{-.5}{1-\beta}$ After period 2, the consumers perpetually roll over their debts or assets of either $\frac{.5}{1-\beta}$ or $\frac{-.5}{1-\beta}$.

8.11. j-step pricing kernel

We are sometimes interested in the price at time t of a claim to one unit of consumption at date $\tau > t$ contingent on the time τ state being s_τ, *regardless* of the particular history by which s_τ is reached at τ. We let $Q_j(s'|s)$ denote the j-step pricing kernel to be interpreted as follows: $Q_j(s'|s)$ gives the price of one unit of consumption j periods ahead, contingent on the state in that future period being s', given that the current state is s. For example, $j=1$ corresponds to the one-step pricing kernel $Q(s'|s)$.

With markets in all possible j-step-ahead contingent claims, the counterpart to constraint (8.9.4), the household's budget constraint at time t, is

$$c_t^i + \sum_{j=1}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j}|s_t) z_{t,j}^i(s_{t+j}) \leq y^i(s_t) + a_t^i. \tag{8.11.1}$$

Here $z_{t,j}^i(s_{t+j})$ is household i's holdings at the end of period t of contingent claims that pay one unit of the consumption good j periods ahead at date $t+j$, contingent on the state at date $t+j$ being s_{t+j}. The household's wealth in the next period depends on the chosen asset portfolio and the realization of s_{t+1},

$$a_{t+1}^i(s_{t+1}) = z_{t,1}^i(s_{t+1}) + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j}|s_{t+1}) z_{t,j}^i(s_{t+j}).$$

The realization of s_{t+1} determines which element of the vector of one-period-ahead claims $\{z_{t,1}^i(s_{t+1})\}$ pays off at time $t+1$, and also the capital gains and losses inflicted on the holdings of longer horizon claims implied by equilibrium prices $Q_j(s_{t+j+1}|s_{t+1})$.

With respect to $z_{t,j}^i(s_{t+j})$ for $j > 1$, use the first-order condition for the problem on the right of (8.10.8) and the Benveniste-Scheinkman formula and rearrange to get

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} \frac{\beta u'[c_{t+1}^i(s_{t+1})] \pi(s_{t+1}|s_t)}{u'(c_t^i)} Q_{j-1}(s_{t+j}|s_{t+1}). \tag{8.11.2}$$

This expression, evaluated at the competitive equilibrium consumption allocation, characterizes two adjacent pricing kernels.[14] Together with first-order

[14] According to expression (8.10.3), the equilibrium consumption allocation is not history dependent, so that $(c_t^i, \{c_{t+1}^i(s_{t+1})\}_{s_{t+1}}) = (\bar{c}^i(s_t), \{\bar{c}^i(s_{t+1})\}_{s_{t+1}})$. Because marginal conditions hold for all households, the characterization of pricing kernels in (8.11.2) holds for any i.

condition (8.10.12), formula (8.11.2) implies that the kernels $Q_j, j = 2, 3, \ldots,$ can be computed recursively:

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t) Q_{j-1}(s_{t+j}|s_{t+1}). \tag{8.11.3}$$

8.11.1. Arbitrage-free pricing

It is useful briefly to describe how arbitrage free pricing theory deduces restrictions on asset prices by manipulating budget sets with redundant assets. We now present an arbitrage argument as an alternative way of deriving restriction (8.11.3) that was established above by using households' first-order conditions evaluated at the equilibrium consumption allocation. In addition to j-step-ahead contingent claims, we illustrate the arbitrage pricing theory by augmenting the trading opportunities in our Arrow securities economy by letting the consumer also trade an ex-dividend Lucas tree. Because markets are already complete, these additional assets are redundant. They have to be priced in a way that leaves the budget set unaltered.[15]

Assume that at time t, in addition to purchasing a quantity $z_{t,j}(s_{t+j})$ of j-step-ahead claims paying one unit of consumption at time $t+j$ if the state takes value s_{t+j} at time $t+j$, the consumer also purchases N_t units of a stock or Lucas tree. Let the ex-dividend price of the tree at time t be $p(s_t)$. Next period, the tree pays a dividend $d(s_{t+1})$ depending on the state s_{t+1}. Ownership of the N_t units of the tree at the beginning of $t+1$ entitles the consumer to a claim on $N_t[p(s_{t+1}) + d(s_{t+1})]$ units of time $t+1$ consumption.[16] As before, let a_t be the wealth of the consumer, apart from his endowment, $y(s_t)$. In this setting, the augmented version of constraint (8.11.1), the consumer's budget constraint, is

$$c_t + \sum_{j=1}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j}|s_t) z_{t,j}(s_{t+j}) + p(s_t) N_t \leq a_t + y(s_t) \tag{8.11.4a}$$

15 That the additional assets are redundant follows from the fact that trading Arrow securities is sufficient to complete markets.

16 We calculate the price of this asset using a different method in chapter 13.

and

$$a_{t+1}(s_{t+1}) = z_{t,1}(s_{t+1}) + [p(s_{t+1}) + d(s_{t+1})]\, N_t + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j}|s_{t+1}) z_{t,j}(s_{t+j}). \tag{8.11.4b}$$

Multiply equation $(8.11.4b)$ by $Q_1(s_{t+1}|s_t)$, sum over s_{t+1}, solve for $\sum_{s_{t+1}} Q_1(s_{t+1}|s_t) z_{t,1}(s_t)$, and substitute this expression in $(8.11.4a)$ to get

$$\begin{aligned} c_t + & \left\{ p(s_t) - \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)[p(s_{t+1}) + d(s_{t+1})] \right\} N_t \\ & + \sum_{j=2}^{\infty} \sum_{s_{t+j}} \left\{ Q_j(s_{t+j}|s_t) - \sum_{s_{t+1}} Q_{j-1}(s_{t+j}|s_{t+1}) Q_1(s_{t+1}|s_t) \right\} z_{t,j}(s_{t+j}) \\ & + \sum_{s_{t+1}} Q_1(s_{t+1}|s_t) a_{t+1}(s_{t+1}) \leq a_t + y(s_t). \end{aligned} \tag{8.11.5}$$

If the two terms in braces are not zero, the consumer can attain unbounded consumption and future wealth by purchasing or selling either the stock (if the first term in braces is not zero) or a state-contingent claim (if any of the terms in the second set of braces is not zero). Therefore, so long as the utility function has no satiation point, in any equilibrium, the terms in the braces must be zero. Thus, we have the arbitrage pricing formulas

$$p(s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)[p(s_{t+1}) + d(s_{t+1})], \tag{8.11.6a}$$

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} Q_{j-1}(s_{t+j}|s_{t+1}) Q_1(s_{t+1}|s_t). \tag{8.11.6b}$$

These are called *arbitrage pricing formulas* because if they were violated, there would exist an *arbitrage*. An arbitrage is defined as a risk-free transaction that earns positive profits.

8.12. Recursive version of Pareto problem

At the very outset of this chapter, we characterized Pareto optimal allocations. This section considers how to formulate a Pareto problem recursively, which will give a preview of things to come in chapters 20 and 23. For this purpose, we consider a special case of the earlier section 8.7.3 example 2 of an economy with a constant aggregate endowment and two types of household with $y_t^1 = s_t, y_t^2 = 1 - s_t$. We now assume that the s_t process is i.i.d., so that $\pi_t(s^t) = \pi(s_t)\pi(s_{t-1})\cdots\pi(s_0)$. Also, let's assume that s_t has a discrete distribution so that $s_t \in [\overline{s}_1, \ldots, \overline{s}_S]$ with probabilities $\Pi_i = \text{Prob}(s_t = \overline{s}_i)$ where $\overline{s}_{i+1} > \overline{s}_i$ and $\overline{s}_1 \geq 0$ and $\overline{s}_S \leq 1$.

In our recursive formulation, each period a planner delivers a pair of previously promised discounted utility streams by assigning a state-contingent consumption allocation today and a pair of state-contingent promised discounted utility streams starting tomorrow. Both the state-contingent consumption today and the promised discounted utility tomorrow are functions of the initial promised discounted utility levels.

Define v as the expected discounted utility of a type 1 person and $P(v)$ as the maximal expected discounted utility that can be offered to a type 2 person, given that a type 1 person is offered at least v. Each of these expected values is to be evaluated before the realization of the state at the initial date.

The Pareto problem is to choose stochastic processes $\{c_t^1(s^t), c_t^2(s^t)\}_{t=0}^\infty$ to maximize $P(v)$ subject to the utility constraint $\sum_{t=0}^\infty \sum_{s^t} \beta^t u(c_t^1(s^t))\pi_t(s^t) \geq v$ and $c_t^1(s^t) + c_t^2(s^t) = 1$. In terms of the competitive equilibrium allocation calculated for the section 8.7.3 example 2 economy above, let $\overline{c} = \overline{c}^1$ be the constant consumption allocated to a type 1 person and $1 - \overline{c} = \overline{c}^2$ be the constant consumption allocated to a type 2 person. Since we have shown that the competitive equilibrium allocation is a Pareto optimal allocation, we already know one point on the Pareto frontier $P(v)$. In particular, when a type 1 person is promised $v = u(\overline{c})/(1-\beta)$, a type 2 person attains life-time utility $P(v) = u(1-\overline{c})/(1-\beta)$.

We can express the discounted values v and $P(v)$ recursively[17] as

$$v = \sum_{i=1}^{S} [u(c_i) + \beta w_i] \, \Pi_i$$

and

$$P(v) = \sum_{i=1}^{S} [u(1-c_i) + \beta P(w_i)] \, \Pi_i,$$

where c_i is consumption of the type 1 person in state i, w_i is the continuation value assigned to the type 1 person in state i; and $1-c_i$ and $P(w_i)$ are the consumption and the continuation value, respectively, assigned to a type 2 person in state i. Assume that the continuation values $w_i \in V$, where V is a set of admissible discounted values of utility. In this section, we assume that $V = [u(\epsilon)/(1-\beta), \, u(1)/(1-\beta)]$ where $\epsilon \in (0,1)$ is an arbitrarily small number.

In effect, before the realization of the current state, a Pareto optimal allocation offers the type 1 person a state-contingent vector of consumption c_i in state i and a state-contingent vector of continuation values w_i in state i, with each w_i itself being a present value of one-period future utilities. In terms of the pair of values $(v, P(v))$, we can express the Pareto problem recursively as

$$P(v) = \max_{\{c_i, w_i\}_{i=1}^{S}} \sum_{i=1}^{S} [u(1-c_i) + \beta P(w_i)] \Pi_i \tag{8.12.1}$$

where the maximization is subject to

$$\sum_{i=1}^{S} [u(c_i) + \beta w_i] \Pi_i \geq v \tag{8.12.2}$$

where $c_i \in [0,1]$ and $w_i \in V$.

To solve the Pareto problem, form the Lagrangian

$$L = \sum_{i=1}^{S} \Pi_i [u(1-c_i) + \beta P(w_i) + \theta(u(c_i) + \beta w_i)] - \theta v$$

[17] This is our first example of a 'dynamic program squared'. We call it that because the state variable v that appears in the Bellman equation for $P(v)$ itself satisfies another Bellman equation.

where θ is a Lagrange multiplier on constraint (8.12.2). First-order conditions with respect to c_i and w_i, respectively, are

$$-u'(1-c_i) + \theta u'(c_i) = 0, \tag{8.12.3a}$$

$$P'(w_i) + \theta = 0. \tag{8.12.3b}$$

The envelope condition is $P'(v) = -\theta$. Thus, (8.12.3b) becomes $P'(w_i) = P'(v)$. But $P(v)$ happens to be strictly concave, so this equality implies $w_i = v$. Therefore, any solution of the Pareto problem leaves the continuation value w_i independent of the state i. Equation (8.12.3a) implies that

$$\frac{u'(1-c_i)}{u'(c_i)} = -P'(v). \tag{8.12.4}$$

Since the right side of (8.12.4) is independent of i, so is the left side, and therefore c is independent of i. And since v is constant over time (because $w_i = v$ for all i), it follows that c is constant over time.

Notice from (8.12.4) that $P'(v)$ serves as a relative Pareto weight on the type 1 person. The recursive formulation brings out that, because $P'(w_i) = P'(v)$, the relative Pareto weight remains constant over time and is independent of the realization of s_t. The planner imposes complete risk sharing.

In chapter 20, we shall encounter recursive formulations again. Impediments to risk sharing that occur in the form either of enforcement or of information constraints will impel the planner sometimes to make continuation values respond to the current realization of shocks to endowments or preferences.

8.13. Concluding remarks

The framework in this chapter serves much of macroeconomics either as foundation or straw man ("benchmark model" is a kinder phrase than "straw man"). It is the foundation of extensive literatures on asset pricing and risk sharing. We describe the literature on asset pricing in more detail in chapters 13 and 14. The model also serves as benchmark, or point of departure, for a variety of models designed to confront observations that seem inconsistent with complete markets. In particular, for models with exogenously imposed incomplete markets, see chapters 17 on precautionary saving and 18 on incomplete markets. For models with endogenous incomplete markets, see chapters 20 and 21 on enforcement and information problems. For models of money, see chapters 26 and 27. To take monetary theory as an example, complete markets models dispose of any need for money because they contain an efficient multilateral trading mechanism, with such extensive netting of claims that no medium of exchange is required to facilitate bilateral exchanges. Any modern model of money introduces frictions that impede complete markets. Some monetary models (e.g., the cash-in-advance model of Lucas, 1981) impose minimal impediments to complete markets, to preserve many of the asset-pricing implications of complete markets models while also allowing classical monetary doctrines like the quantity theory of money. The shopping time model of chapter 26 is constructed in a similar spirit. Other monetary models, such as the Townsend turnpike model of chapter 27 or the Kiyotaki-Wright search model of chapter 28, impose more extensive frictions on multilateral exchanges and leave the complete markets model farther behind. Before leaving the complete markets model, we'll put it to work in several of the following chapters.

A. Gaussian asset-pricing model

The theory of this chapter is readily adapted to a setting in which the state of the economy evolves according to a continuous-state Markov process. We use such a version in chapter 14. Here we give a taste of how such an adaptation can be made by describing an economy in which the state follows a linear stochastic difference equation driven by a Gaussian disturbance. If we supplement this with the specification that preferences are quadratic, we get a setting in which asset prices can be calculated swiftly.

Suppose that the state evolves according to the stochastic difference equation

$$s_{t+1} = As_t + Cw_{t+1} \tag{8.A.1}$$

where A is a matrix whose eigenvalues are bounded from above in modulus by $1/\sqrt{\beta}$ and w_{t+1} is a Gaussian martingale difference sequence adapted to the history of s_t. Assume that $Ew_{t+1}w_{t+1} = I$. The conditional density of s_{t+1} is Gaussian:

$$\pi(s_t|s_{t-1}) \sim \mathcal{N}(As_{t-1}, CC'). \tag{8.A.2}$$

More precisely,

$$\pi(s_t|s_{t-1}) = K \exp\left\{-.5(s_t - As_{t-1})(CC')^{-1}(s_t - As_{t-1})\right\}, \tag{8.A.3}$$

where $K = (2\pi)^{-\frac{k}{2}} \det(CC')^{-\frac{1}{2}}$ and s_t is $k \times 1$. We also assume that $\pi_0(s_0)$ is Gaussian.[18]

If $\{c^i_t(s_t)\}_{t=0}^{\infty}$ is the equilibrium allocation to agent i, and the agent has preferences represented by (8.2.1), the equilibrium pricing function satisfies

$$q^0_t(s^t) = \frac{\beta^t u'[c^i_t(s_t)]\pi(s^t)}{u'[c^i_0(s_0)]}. \tag{8.A.4}$$

Once again, let $\{d_t(s_t)\}_{t=0}^{\infty}$ be a stream of claims to consumption. The time 0 price of the asset with this dividend stream is

$$p_0 = \sum_{t=0}^{\infty} \int_{s^t} q^0_t(s^t) d_t(s_t) d\, s^t.$$

[18] If s_t is stationary, $\pi_0(s_0)$ can be specified to be the stationary distribution of the process.

Substituting equation $(8.A.4)$ into the preceding equation gives

$$p_0 = \sum_t \int_{s^t} \beta^t \frac{u'[c_t^i(s_t)]}{u'[c_0^i(s_0)]} d_t(s_t)\pi(s^t)ds^t$$

or

$$p_0 = E\sum_{t=0}^{\infty} \beta^t \frac{u'[c_t(s_t)]}{u'[c_0(s_0)]} d_t(s_t). \tag{8.A.5}$$

This formula expresses the time 0 asset price as an inner product of a discounted marginal utility process and a dividend process.[19]

This formula becomes especially useful in the case that the one-period utility function $u(c)$ is quadratic, so that marginal utilities become linear, and the dividend process d_t is linear in s_t. In particular, assume that

$$u(c_t) = -.5(c_t - b)^2 \tag{8.A.6}$$

$$d_t = S_d s_t, \tag{8.A.7}$$

where $b > 0$ is a bliss level of consumption. Furthermore, assume that the equilibrium allocation to agent i is

$$c_t^i = S_{ci} s_t, \tag{8.A.8}$$

where S_{ci} is a vector conformable to s_t.

The utility function $(8.A.6)$ implies that $u'(c_t^i) = b - c_t^i = b - S_{ci}s_t$. Suppose that unity is one element of the state space for s_t, so that we can express $b = S_b s_t$. Then $b - c_t = S_f s_t$, where $S_f = S_b - S_{ci}$, and the asset-pricing formula becomes

$$p_0 = \frac{E_0 \sum_{t=0}^{\infty} \beta^t s_t' S_f' S_d s_t}{S_f s_0}. \tag{8.A.9}$$

Thus, to price the asset, we have to evaluate the expectation of the sum of a discounted quadratic form in the state variable. This is easy to do by using results from chapter 2.

In chapter 2, we evaluated the conditional expectation of the geometric sum of the quadratic form

$$\alpha_0 = E_0 \sum_{t=0}^{\infty} \beta^t s_t' S_f' S_d s_t.$$

[19] For two scalar stochastic processes x, y, the inner product is defined as $< x, y > = E\sum_{t=0}^{\infty} \beta^t x_t y_t$.

We found that it could be written in the form

$$\alpha_0 = s_0'\mu s_0 + \sigma, \tag{8.A.10}$$

where μ is an $(n \times n)$ matrix and σ is a scalar that satisfy

$$\begin{aligned} \mu &= S_f' S_d + \beta A' \mu A \\ \sigma &= \beta\sigma + \beta \text{ trace } (\mu C C') \end{aligned} \tag{8.A.11}$$

The first equation of $(8.A.11)$ is a *discrete Lyapunov equation* in the square matrix μ, and can be solved by using one of several algorithms.[20] After μ has been computed, the second equation can be solved for the scalar σ.

B. The permanent income model revisited

This appendix is a variation on the theme that 'many single agent models can be reinterpreted as general equilibrium models'.

8.B.1. Reinterpreting the single-agent model

In this appendix, we cast the single-agent linear quadratic permanent income model of section 2.12 of chapter 2 as a competitive equilibrium with time 0 trading of a complete set of history-contingent securities. We begin by reformulating the model in that chapter as a planning problem. The planner has utility functional

$$E_0 \sum_{t=0}^{\infty} \beta^t u(\bar c_t) \tag{8.B.1}$$

where E_t is the mathematical expectation conditioned on the consumer's time t information, $\bar c_t$ is time t consumption, $u(c) = -.5(\gamma - \bar c_t)^2$, and $\beta \in (0,1)$ is a discount factor. The planner maximizes $(8.B.1)$ by choosing a consumption, borrowing plan $\{\bar c_t, b_{t+1}\}_{t=0}^{\infty}$ subject to the sequence of budget constraints

$$\bar c_t + b_t = R^{-1} b_{t+1} + y_t \tag{8.B.2}$$

[20] The Matlab control toolkit has a program called `dlyap.m`; also see a program called `doublej.m`.

where y_t is an exogenous stationary endowment process, R is a constant gross risk-free interest rate, $-R^{-1}b_t \equiv \bar{k}_t$ is the stock of an asset that bears a risk free one-period gross return of R, and b_0 is a given initial condition. We assume that $R^{-1} = \beta$ and that the endowment process has the state-space representation

$$z_{t+1} = A_{22}z_t + C_2 w_{t+1} \tag{8.B.3a}$$

$$y_t = U_y z_t \tag{8.B.3b}$$

where w_{t+1} is an i.i.d. process with mean zero and identity contemporaneous covariance matrix, A_{22} is a stable matrix, its eigenvalues being strictly below unity in modulus, and U_y is a selection vector that identifies y with a particular linear combination of z_t. As shown in chapter 2, the solution of what we now interpret as a planning problem can be represented as the following versions of equations (2.12.9) and (2.12.20), respectively:

$$\bar{c}_t = (1-\beta)\left[U_y(I-\beta A_{22})^{-1}z_t - R\bar{k}_t\right] \tag{8.B.4}$$

$$\bar{k}_{t+1} = \bar{k}_t + RU_y(I-\beta A_{22})^{-1}(A_{22}-I)z_t. \tag{8.B.5}$$

We can represent the optimal consumption, capital accumulation path compactly as

$$\begin{bmatrix} \bar{k}_{t+1} \\ z_{t+1} \end{bmatrix} = A \begin{bmatrix} \bar{k}_t \\ z_t \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \end{bmatrix} w_{t+1} \tag{8.B.6}$$

$$\bar{c}_t = S_c \begin{bmatrix} \bar{k}_t \\ z_t \end{bmatrix} \tag{8.B.7}$$

where the matrices A, S_c can readily be constructed from the solutions and specifications just mentioned. In addition, it is useful to have at our disposal the marginal utility of consumption process $p_t^0 \equiv (\gamma - \bar{c}_t)$, which can be represented as

$$p_t^0(z^t) = S_p \begin{bmatrix} \bar{k}_t \\ z_t \end{bmatrix} \tag{8.B.8}$$

and where S_p can be constructed easily from S_c. Solving equation (8.B.5) recursively shows that k_{t+1} is a function $k_{t+1}(z^t; k_0)$ of history z^t. In equation (8.B.8), $\bar{k}_t$ encodes the history dependence of $p_t^0(z^t)$.

Equations (8.B.6), (8.B.7), (8.B.8) together with the equation $r_t^0 = \alpha$ to be explained below turn out to be representations of the equilibrium price system in the competitive equilibrium to which we turn next.

8.B.2. Decentralization and scaled prices

Let $q_t^0(z^t)$ the time 0 price of a unit of time t consumption at history z^t. Let $\pi_t(z^t)$ the probability density of the history z^t induced by the state-space representation $(8.B.3)$. Define the adjusted Arrow-Debreu price scaled by discounting and probabilities as

$$p_t^0(z^t) = \frac{q_t^0(z^t)}{\beta^t \pi_t(z^t)}. \tag{8.B.9}$$

We find it convenient to express a representative consumer's problem and a representative firm's problem in terms of these scaled Arrow-Debreu prices.

Evidently, the present value of consumption, for example, can be represented as

$$\begin{aligned}\sum_{t=0}^{\infty}\sum_{z^t} q_t^0(z^t)c_t(z^t) &= \sum_{t=0}^{\infty}\sum_{z^t} \beta^t p_t^0(z^t)c_t(z^t)\pi_t(z^t)\\ &= E_0 \sum_{t=0}^{\infty} \beta^t p_t(z^t)c_t(z^t).\end{aligned}$$

Below, it will be convenient for us to represent present values as conditional expectations of discounted sums as is done in the second line.

We let $r_t^0(z^t)$ be the rental rate on capital, again scaled analogously to $(8.B.9)$. Both the consumer and the firm take these processes as given.

The consumer owns and operates the technology for accumulating capital. The consumer owns the endowment process $\{y_t\}_{t=0}^{\infty}$, which it sells to a firm that operates a production technology. The consumer rents capital to the firm. The firm uses the endowment and capital to produce output that it sells to the consumer at a competitive price. The consumer divides his time t purchases between consumption c_t and gross investment x_t.

8.B.2.1. The consumer

Let $\{p_t^0(z^t), r_t^0(z^t)\}_{t=0}^{\infty}$ be a price system, each component of which takes the form of a 'scaled Arrow-Debreu price' (attained by dividing a time-0 Arrow-Debreu price by a discount factor times a probability, as in the previous subsection). The representative consumer's problem is to choose processes $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$-.5E_0 \sum_{t=0}^{\infty} \beta^t (\gamma - c_t)^2 \tag{8.B.10}$$

subject to

$$E_0 \sum_{t=0}^{\infty} \beta^t p_t^0(z^t) o_t(z^t) = E_0 \sum_{t=0}^{\infty} \beta^t \big(p_t^0(z^t) y_t + r_t^0(z^t) k_t(z^t)\big) \tag{8.B.11}$$

$$k_{t+1} = (1-\delta) k_t + x_t \tag{8.B.12}$$

$$o_t(z^t) = c_t(z^t) + x_t(z^t) \tag{8.B.13}$$

where k_0 is a given initial condition. Here x_t is gross investment and k_t is physical capital owned by the household and rented to firms. The consumer purchases output $o_t = c_t + x_t$ from competitive firms. The consumer sells its endowment y_t and rents its capital k_t to firms at prices $p_t^0(z^t)$ and $r_t^0(z^t)$. Equation $(8.B.12)$ is the law of motion for physical capital, where $\delta \in (0,1)$ is a depreciation rate.

8.B.2.2. The firm

A competitive representative firm chooses processes $\{k_t, c_t, x_t\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \big\{p_t^0(z^t) o_t(z^t) - p_t^0(z^t) y_t - r_t^0(z^t) k_t\big\} \tag{8.B.14}$$

subject to the physical technology

$$o_t(z^t) = \alpha k_t + y_t(z_t), \tag{8.B.15}$$

where $\alpha > 0$. Since the marginal product of capital is α, a good guess is that

$$r_t^0(z^t) = \alpha. \tag{8.B.16}$$

8.B.3. Matching equilibrium and planning allocations

We impose the condition

$$\alpha + (1 - \delta) = R. \tag{8.B.17}$$

This makes the gross rates of return in investment identical in the planning and decentralized economies. In particular, if we substitute equation $(8.B.12)$ into equation $(8.B.15)$ and remember that $b_t \equiv Rk_t$, we obtain $(8.B.2)$.

It is straightforward to verify that the allocation $\{\bar{k}_{t+1}, \bar{c}_t\}_{t=0}^{\infty}$ that solves the planning problem is a competitive equilibrium allocation.

As in chapter 7, we have distinguished between the planning allocation $\{\bar{k}_{t+1}, \bar{c}_t\}_{t=0}^{\infty}$ that determines the equilibrium price functions defined in subsection 8.B.1 and the allocation chosen by the representative firm and the representative consumer who face those prices as price takers. This is yet another example of the 'big K, little k' device from chapter 7.

8.B.4. Interpretation

As we saw in section 2.12 of chapter 2 and also in representation $(8.B.4)$ $(8.B.5)$ here, what is now *equilibrium* consumption is a random walk. Why, despite his preference for a *smooth* consumption path, does the representative consumer accept fluctuations in his consumption? In the complete markets economy of this appendix, the consumer believes that it is possible for him completely to smooth consumption over time and across histories by purchasing and selling history contingent claims. But at the equilibrium prices facing him, the consumer prefers to tolerate fluctuations in consumption over time and across histories.

Exercises

Exercise 8.1 **Existence of representative consumer**

Suppose households 1 and 2 have one-period utility functions $u(c^1)$ and $w(c^2)$, respectively, where u and w are both increasing, strictly concave, twice differentiable functions of a scalar consumption rate. Consider the Pareto problem:

$$v_\theta(c) = \max_{\{c^1,c^2\}} \left[\theta u(c^1) + (1-\theta)w(c^2)\right]$$

subject to the constraint $c^1 + c^2 = c$. Show that the solution of this problem has the form of a concave utility function $v_\theta(c)$, which depends on the Pareto weight θ. Show that $v'_\theta(c) = \theta u'(c^1) = (1-\theta)w'(c^2)$.

The function $v_\theta(c)$ is the utility function of the *representative consumer*. Such a representative consumer always lurks within a complete markets competitive equilibrium even with heterogeneous preferences. At a competitive equilibrium, the marginal utilities of the representative agent and each and every agent are proportional.

Exercise 8.2 **Term structure of interest rates**

Consider an economy with a single consumer. There is one good in the economy, which arrives in the form of an exogenous endowment obeying[21]

$$y_{t+1} = \lambda_{t+1} y_t,$$

where y_t is the endowment at time t and $\{\lambda_{t+1}\}$ is governed by a two-state Markov chain with transition matrix

$$P = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix},$$

and initial distribution $\pi_\lambda = \begin{bmatrix} \pi_0 & 1-\pi_0 \end{bmatrix}$. The value of λ_t is given by $\bar\lambda_1 = .98$ in state 1 and $\bar\lambda_2 = 1.03$ in state 2. Assume that the history of y_s, λ_s up to t is observed at time t. The consumer has endowment process $\{y_t\}$ and has preferences over consumption streams that are ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

[21] Such a specification was made by Mehra and Prescott (1985).

where $\beta \in (0,1)$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma \geq 1$.

a. Define a competitive equilibrium, being careful to name all of the objects of which it consists.

b. Tell how to compute a competitive equilibrium.

For the remainder of this problem, suppose that $p_{11} = .8$, $p_{22} = .85$, $\pi_0 = .5$, $\beta = .96$, and $\gamma = 2$. Suppose that the economy begins with $\lambda_0 = .98$ and $y_0 = 1$.

c. Compute the (unconditional) average growth rate of consumption, computed before having observed λ_0.

d. Compute the time 0 prices of three risk-free discount bonds, in particular, those promising to pay one unit of time j consumption for $j = 0,1,2$, respectively.

e. Compute the time 0 prices of three bonds, in particular, ones promising to pay one unit of time j consumption contingent on $\lambda_j = \bar{\lambda}_1$ for $j = 0,1,2$, respectively.

f. Compute the time 0 prices of three bonds, in particular, ones promising to pay one unit of time j consumption contingent on $\lambda_j = \bar{\lambda}_2$ for $j = 0,1,2$, respectively.

g. Compare the prices that you computed in parts d, e, and f.

Exercise 8.3 An economy consists of two infinitely lived consumers named $i = 1,2$. There is one nonstorable consumption good. Consumer i consumes c_t^i at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i),$$

where $\beta \in (0,1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_t^i = 1,0,0,1,0,0,1,\ldots$. Consumer 2 is endowed with a stream of the consumption good $0,1,1,0,1,1,0,\ldots$. Assume that there are complete markets with time 0 trading.

a. Define a competitive equilibrium.

b. Compute a competitive equilibrium.

c. Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of that asset be?

Exercise 8.4 Consider a pure endowment economy with a single representative consumer; $\{c_t, d_t\}_{t=0}^{\infty}$ are the consumption and endowment processes, respectively. Feasible allocations satisfy

$$c_t \le d_t.$$

The endowment process is described by[22]

$$d_{t+1} = \lambda_{t+1} d_t.$$

The growth rate λ_{t+1} is described by a two-state Markov process with transition probabilities

$$P_{ij} = \mathrm{Prob}(\lambda_{t+1} = \bar\lambda_j | \lambda_t = \bar\lambda_i).$$

Assume that

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix},$$

and that

$$\bar\lambda = \begin{bmatrix} .97 \\ 1.03 \end{bmatrix}.$$

In addition, $\lambda_0 = .97$ and $d_0 = 1$ are both known at date 0. The consumer has preferences over consumption ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

where E_0 is the mathematical expectation operator, conditioned on information known at time 0, $\gamma = 2, \beta = .95$.

Part I

At time 0, after d_0 and λ_0 are known, there are complete markets in date- and history-contingent claims. The market prices are denominated in units of time 0 consumption goods.

[22] See Mehra and Prescott (1985).

a. Define a competitive equilibrium, being careful to specify all the objects composing an equilibrium.

b. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (.97, .97, 1.03, .97, 1.03)$. Please give a numerical answer.

c. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (1.03, 1.03, 1.03, 1.03, .97)$.

d. Give a formula for the price at time 0 of a claim on the entire endowment sequence.

e. Give a formula for the price at time 0 of a claim on consumption in period 5, contingent on the growth rate λ_5 being .97 (regardless of the intervening growth rates).

Part II

Now assume a different market structure. Assume that at each date $t \geq 0$ there is a complete set of one-period forward Arrow securities.

f. Define a (recursive) competitive equilibrium with Arrow securities, being careful to define all of the objects that compose such an equilibrium.

g. For the representative consumer in this economy, for each state compute the "natural debt limits" that constrain state-contingent borrowing.

h. Compute a competitive equilibrium with Arrow securities. In particular, compute both the pricing kernel and the allocation.

i. An entrepreneur enters this economy and proposes to issue a new security each period, namely, a risk-free two-period bond. Such a bond issued in period t promises to pay one unit of consumption at time $t+1$ for sure. Find the price of this new security in period t, contingent on λ_t.

Exercise 8.5

An economy consists of two consumers, named $i = 1, 2$. The economy exists in discrete time for periods $t \geq 0$. There is one good in the economy, which is not storable and arrives in the form of an endowment stream owned by each consumer. The endowments to consumers $i = 1, 2$ are

$$\begin{aligned} y_t^1 &= s_t \\ y_t^2 &= 1 \end{aligned}$$

where s_t is a random variable governed by a two-state Markov chain with values $s_t = \bar{s}_1 = 0$ or $s_t = \bar{s}_2 = 1$. The Markov chain has time invariant transition probabilities denoted by $\pi(s_{t+1} = s'|s_t = s) = \pi(s'|s)$, and the probability distribution over the initial state is $\pi_0(s)$. The *aggregate endowment* at t is $Y(s_t) = y_t^1 + y_t^2$.

Let c^i denote the stochastic process of consumption for agent i. Household i orders consumption streams according to

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)] \pi_t(s^t),$$

where $\pi_t(s^t)$ is the probability of the history $s^t = (s_0, s_1, \ldots, s_t)$.

a. Give a formula for $\pi_t(s^t)$.

b. Let $\theta \in (0, 1)$ be a Pareto weight on household 1. Consider the planning problem

$$\max_{c^1, c^2} \left\{ \theta \ln(c^1) + (1 - \theta) \ln(c^2) \right\}$$

where the maximization is subject to

$$c_t^1(s^t) + c_t^2(s^t) \leq Y(s_t).$$

Solve the Pareto problem, taking θ as a parameter.

c. Define a *competitive equilibrium* with history-dependent Arrow-Debreu securities traded once and for all at time 0. Be careful to define all of the objects that compose a competitive equilibrium.

d. Compute the competitive equilibrium price system (i.e., find the prices of all of the Arrow-Debreu securities).

e. Tell the relationship between the solutions (indexed by θ) of the Pareto problem and the competitive equilibrium allocation. If you wish, refer to the two welfare theorems.

f. Briefly tell how you can compute the competitive equilibrium price system *before* you have figured out the competitive equilibrium allocation.

g. Now define a recursive competitive equilibrium with trading every period in one-period Arrow securities only. Describe all of the objects of which such an equilibrium is composed. (Please denominate the prices of one-period time $t+1$ state-contingent Arrow securities in units of time t consumption.) Define the "natural borrowing limits" for each consumer in each state. Tell how to compute these natural borrowing limits.

h. Tell how to compute the prices of one-period Arrow securities. How many prices are there (i.e., how many numbers do you have to compute)? Compute all of these prices in the special case that $\beta = .95$ and $\pi(s_j|s_i) = P_{ij}$ where $P = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$.

i. Within the one-period Arrow securities economy, a new asset is introduced. One of the households decides to market a one-period-ahead riskless claim to one unit of consumption (a one-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$. Justify your formula for these prices in terms of first principles.

j. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides to market a two-period-ahead riskless claim to one unit of consumption (a two-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$.

k. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides at time t to market five-period-ahead claims to consumption at $t+5$ contingent on the value of s_{t+5}. Compute the equilibrium prices of these securities when $s_t = 0$ and $s_t = 1$ and $s_{t+5} = 0$ and $s_{t+5} = 1$.

Exercise 8.6 **Optimal taxation**

The government of a small country must finance an exogenous stream of government purchases $\{g_t\}_{t=0}^{\infty}$. Assume that g_t is described by a discrete-state Markov chain with transition matrix P and initial distribution π_0. Let $\pi_t(g^t)$ denote the probability of the history $g^t = g_t, g_{t-1}, \ldots, g_0$, conditioned on g_0. The state of the economy is completely described by the history g^t. There are complete markets in date-history claims to goods. At time 0, after g_0 has been realized, the government can purchase or sell claims to time t goods contingent on the history g^t at a price $p_t^0(g^t) = \beta^t \pi_t(g^t)$, where $\beta \in (0,1)$. The date-state prices are exogenous to the small country. The government finances its expenditures by raising history-contingent tax revenues of $R_t = R_t(g^t)$ at time t. The present value of its expenditures must not exceed the present value of its revenues.

Raising revenues by taxation is distorting. The government confronts a dead weight loss function $W(R_t)$ that measures the distortion at time t. Assume that W is an increasing, twice differentiable, strictly convex function that satisfies $W(0) = 0, W'(0) = 0, W'(R) > 0$ for $R > 0$ and $W''(R) > 0$ for $R \geq 0$. The government devises a state-contingent taxation and borrowing plan to minimize

$$E_0 \sum_{t=0}^{\infty} \beta^t W(R_t), \tag{1}$$

where E_0 is the mathematical expectation conditioned on g_0.

Suppose that g_t takes two possible values, $\bar{g}_1 = .2$ (peace) and $\bar{g}_2 = 1$ (war) and that $P = \begin{bmatrix} .8 & .2 \\ .5 & .5 \end{bmatrix}$. Suppose that $g_0 = .2$. Finally, suppose that $W(R) = .5R^2$.

a. Please write out (1) long hand, i.e., write out an explicit expression for the mathematical expectation E_0 in terms of a summation over the appropriate probability distribution.

b. Compute the optimal tax and borrowing plan. In particular, give analytic expressions for $R_t = R_t(g^t)$ for all t and all g^t.

c. There is an equivalent market setting in which the government can buy and sell one-period Arrow securities each period. Find the price of one-period Arrow securities at time t, denominated in units of the time t good.

d. Let $B_t(g_t)$ be the one-period Arrow securities at t that the government issued for state g_t at time $t-1$. For $t > 0$, compute $B_t(g_t)$ for $g_t = \bar{g}_1$ and $g_t = \bar{g}_2$.

e. Use your answers to parts b and d to describe the government's optimal policy for taxing and borrowing.

Exercise 8.7 **A competitive equilibrium**

An endowment economy consists of two type of consumers. Consumers of type 1 order consumption streams of the one good according to

$$\sum_{t=0}^{\infty} \beta^t c_t^1$$

and consumers of type 2 order consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t^2)$$

where $c_t^i \geq 0$ is the consumption of a type i consumer and $\beta \in (0,1)$ is a common discount factor. The consumption good is tradable but nonstorable. There are equal numbers of the two types of consumer. The consumer of type 1 is endowed with the consumption sequence

$$y_t^1 = \mu > 0 \quad \forall t \geq 0$$

where $\mu > 0$. The consumer of type 2 is endowed with the consumption sequence

$$y_t^2 = \begin{cases} 0 & \text{if } t \geq 0 \text{ is even} \\ \alpha & \text{if } t \geq 0 \text{ is odd} \end{cases}$$

where $\alpha = \mu(1 + \beta^{-1})$.

a. Define a competitive equilibrium with time 0 trading. Be careful to include definitions of all of the objects of which a competitive equilibrium is composed.

b. Compute a competitive equilibrium allocation with time 0 trading.

c. Compute the time 0 wealths of the two types of consumers using the competitive equilibrium prices.

d. Define a competitive equilibrium with sequential trading of Arrow securities.

e. Compute a competitive equilibrium with sequential trading of Arrow securities.

Exercise 8.8 **Corners**

A pure endowment economy consists of two type of consumers. Consumers of type 1 order consumption streams of the one good according to

$$\sum_{t=0}^{\infty} \beta^t c_t^1$$

and consumers of type 2 order consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t^2)$$

where $c_t^i \geq 0$ is the consumption of a type i consumer and $\beta \in (0,1)$ is a common discount factor. Please note the nonnegativity constraint on consumption of each person (the force of this is that c_t^i is *consumption*, not *production*). The consumption good is tradable but nonstorable. There are equal numbers of the two types of consumer. The consumer of type 1 is endowed with the consumption sequence

$$y_t^1 = \mu > 0 \quad \forall t \geq 0$$

where $\mu > 0$. The consumer of type 2 is endowed with the consumption sequence

$$y_t^2 = \begin{cases} 0 & \text{if } t \geq 0 \text{ is even} \\ \alpha & \text{if } t \geq 0 \text{ is odd} \end{cases}$$

where

$$\alpha = \mu(1 + \beta^{-1}). \tag{1}$$

a. Define a competitive equilibrium with time 0 trading. Be careful to include definitions of all of the objects of which a competitive equilibrium is composed.

b. Compute a competitive equilibrium allocation with time 0 trading. Compute the equilibrium price system. Please also compute the sequence of one-period gross interest rates. Do they differ between odd and even periods?

c. Compute the time 0 wealths of the two types of consumers using the competitive equilibrium prices.

d. Now consider an economy identical to the preceding one except in one respect. The endowment of consumer 1 continues to be 1 each period, but we assume that the endowment of consumer 2 is larger (though it continues to be zero in every even period). In particular, we alter the assumption about endowments in condition (1) to the new condition

$$\alpha > \mu(1 + \beta^{-1}).$$

Compute the competitive equilibrium allocation and price system for this economy.

e. Compute the sequence of one-period interest rates implicit in the equilibrium price system that you computed in part d. Are interest rates higher or lower than those you computed in part b?

Exercise 8.9 **Equivalent martingale measure**

Let $\{d_t(s_t)\}_{t=0}^{\infty}$ be a stream of payouts. Suppose that there are complete markets. From (8.6.4) and (8.8.1), the price at time 0 of a claim on this stream of dividends is

$$a_0 = \sum_{t=0} \sum_{s^t} \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t) d_t(s_t).$$

Show that this a_0 can also be represented as

$$\begin{aligned} a_0 &= \sum_t b_t \sum_{s^t} d_t(s_t) \tilde{\pi}_t(s^t) \\ &= \tilde{E}_0 \sum_{t=0}^{\infty} b_t d_t(s_t) \end{aligned} \tag{1}$$

where $\tilde{E}$ is the mathematical expectation with respect to the twisted measure $\tilde{\pi}_t(s^t)$ defined by

$$\begin{aligned} \tilde{\pi}_t(s^t) &= b_t^{-1} \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t) \\ b_t &= \sum_{s^t} \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t). \end{aligned}$$

Prove that $\tilde{\pi}_t(s^t)$ is a probability measure. Interpret b_t itself as a price of particular asset. Note: $\tilde{\pi}_t(s^t)$ is called an *equivalent martingale measure*. See chapters 13 and 14.

Exercise 8.10 **Harrison-Kreps prices**

Show that the asset price in (1) of the previous exercise can also be represented as

$$\begin{aligned} a_0 &= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t p_t^0(s^t) d_t(s^t) \pi_t(s^t) \\ &= E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 d_t \end{aligned}$$

where $p_t^0(s^t) = q_t^0(s^t)/[\beta^t \pi_t(s^t)]$.

Exercise 8.11 **Early resolution of uncertainty**

An economy consists of two households named $i = 1, 2$. Each household evaluates streams of a single consumption good according to $\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t)$. Here $u(c)$ is an increasing, twice continuously differentiable, strictly concave function of consumption c of one good. The utility function satisfies the Inada condition $\lim_{c \downarrow 0} u'(c) = +\infty$. A feasible allocation satisfies $\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)$. The households' endowments of the one nonstorable good are both functions of a state variable $s_t \in \mathbf{S} = \{0, 1, 2\}$; s_t is described by a time invariant Markov chain with initial distribution $\pi_0 = [\,0 \quad 1 \quad 0\,]'$ and transition density defined by the stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .5 & 0 & .5 \\ 0 & 0 & 1 \end{bmatrix}.$$

The endowments of the two households are

$$\begin{aligned} y_t^1 &= s_t/2 \\ y_t^2 &= 1 - s_t/2. \end{aligned}$$

a. Define a competitive equilibrium with Arrow securities.

b. Compute a competitive equilibrium with Arrow securities.

c. By hand, simulate the economy. In particular, for every possible realization of the histories s^t, describe time series of c_t^1, c_t^2 and the wealth levels a_t^i of the households. (Note: Usually this would be an impossible task by hand, but this problem has been set up to make the task manageable.)

Exercise 8.12 donated by Pierre-Olivier Weill

An economy is populated by a continuum of infinitely lived consumers of types $j \in \{0,1\}$, with a measure one of each. There is one nonstorable consumption good arriving in the form of an endowment stream owned by each consumer. Specifically, the endowments are

$$y_t^0(s_t) = (1-s_t)\bar{y}^0$$
$$y_t^1(s_t) = s_t\bar{y}^1,$$

where s_t is a two-state time-invariant Markov chain valued in $\{0,1\}$ and $\bar{y}^0 < \bar{y}^1$. The initial state is $s_0 = 1$. Transition probabilities are denoted $\pi(s'|s)$ for $(s,s') \in \{0,1\}^2$, where $'$ denotes a next period value. The aggregate endowment is $y_t(s_t) \equiv (1-s_t)\bar{y}^0 + s_t\bar{y}^1$. Thus, this economy fluctuates stochastically between recessions $y_t(0) = \bar{y}^0$ and booms $y_t(1) = \bar{y}^0$. In a recession, the aggregate endowment is owned by type 0 consumers, while in a boom it is owned by a type 1 consumers. A consumer orders consumption streams according to:

$$U(c^j) = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t|s_0)\frac{c_t^j(s^t)^{1-\gamma}}{1-\gamma},$$

where $s^t = (s_t, s_{t-1}, \ldots, s_0)$ is the history of the state up to time t, $\beta \in (0,1)$ is the discount factor, and $\gamma > 0$ is the coefficient of relative risk aversion.

a. Define a competitive equilibrium with time 0 trading. Compute the price system $\{q_t^0(s^t)\}_{t=0}^{\infty}$ and the equilibrium allocation $\{c^j(s^t)\}_{t=0}^{\infty}$, for $j \in \{0,1\}$.

b. Find a utility function $\bar{U}(c) = E_0\left(\sum_{t=0}^{\infty}\beta^t u(c_t)\right)$ such that the price system $q_t^0(s^t)$ and the aggregate endowment $y_t(s_t)$ is an equilibrium allocation of the single-agent economy $\left(\bar{U}, \{y_t(s_t)\}_{t=0}^{\infty}\right)$. How does your answer depend on the initial distribution of endowments $y_t^j(s_t)$ among the two types $j \in \{0,1\}$? How would you defend the representative agent assumption in this economy?

c. Describe the equilibrium allocation under the following three market structures: (i) at each node s^t, agents can trade only claims on their entire endowment streams; (ii) at each node s^t, there is a complete set of one-period ahead Arrow securities; and (iii) at each node s^t, agents can only trade two risk-free assets, namely, a one-period zero-coupon bond that pays one unit of consumption for sure at $t+1$ and a two-period zero-coupon bond that pays one unit of the consumption good for sure at $t+2$. How would you modify your answer in the absence of aggregate uncertainty?

d. Assume that $\pi(1|0) = 1$, $\pi(0|1) = 1$, and as before $s_0 = 1$. Compute the allocation in an equilibrium with time 0 trading. Does the type $j = 1$ agent always consume the largest share of the aggregate endowment? How does it depend on parameter values? Provide economic intuition for your results.

e. Assume that $\pi(1|0) = 1$ and $\pi(0|1) = 1$. Remember that $s_0 = 1$. Assume that at $t = 1$ agent $j = 0$ is given the option to default on her financial obligation. For example, in the time 0 trading economy, these obligations are deliveries of goods. Upon default, it is assumed that the agent is excluded from the market and has to consume her endowment forever. Will the agent ever exercise her option to default?

Exercise 8.13 **Diverse beliefs, I**

A pure endowment economy is populated by two consumers. Consumer i has preferences over history-contingent consumption sequences $\{c_t^i(s^t)\}$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t^i(s^t),$$

where $u(c) = \ln(c)$ and where $\pi_t^i(s^t)$ is a density that consumer i assigns to history s^t. The state space is time invariant. In particular, $s_t \in S = \{0, .5, 1\}$ for all $t \geq 0$. Only two histories are possible for $t = 0, 1, 2, \ldots$:

$$\text{history } 1 : .5, 1, 1, 1, 1, \ldots$$
$$\text{history } 2 : .5, 0, 0, 0, 0, \ldots$$

Consumer 1 assigns probability 1/3 to history 1 and probability 2/3 to history 2, while consumer 2 assigns probability 2/3 to history 1 and probability 1/3 to history 2. Nature assigns equal probabilities to the two histories. The endowments of the two consumers are:

$$y_t^1 = s_t$$
$$y_t^2 = 1 - s_t.$$

a. Define a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

b. Compute a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

c. Is the equilibrium allocation Pareto optimal?

Exercise 8.14 **Diverse beliefs, II**

Consider the following I person pure endowment economy. There is a state variable $s_t \in S$ for all $t \geq 0$. Let s^t denote a history of s from 0 to t. The time t aggregate endowment is a function of the history, so $Y_t = Y_t(s^t)$. Agent i attaches a personal probability of $\pi_t^i(s^t)$ to history s^t. The history s^t is observed by all I people at time t. Assume that for all i, $\pi_t^i(s_t) > 0$ if and only if $\pi_t^1(s_t) > 0$ (so the consumers agree about which histories have positive probability). Consumer i ranks consumption plans $c_t^i(s^t)$ that are measurable functions of histories via the expected utility functional

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln(c_t^i(s^t)) \pi_t^i(s^t) \tag{1}$$

The ownership structure of the economy is not yet determined.

A planner puts positive Pareto weights $\lambda_i > 0$ on consumers $i = 1, \ldots, I$ and solves a time 0 Pareto problem that respects each consumer's preferences as represented by (1).

a. Show how to solve for a Pareto optimal allocation. Display an expression for $c_t^i(s^t)$ as a function of $Y_t(s^t)$ and other pertinent variables.

b. Under what circumstances does the Pareto plan imply complete risk-sharing among the I consumers?

c. Under what circumstances does the Pareto plan imply an allocation that is not history dependent? By 'not history dependent', we mean that $Y_t(s^t) = Y_t(\tilde{s}^t)$ would imply the same allocation at time t?

d. For a given set of Pareto weights, find an associated equilibrium price vector and an initial distribution of wealth among the I consumers that makes the Pareto allocation be the allocation associated with a competitive equilibrium with time 0 trading of history-contingent claims on consumption.

e. Find a formula for the equilibrium price vector in terms of equilibrium quantities and the beliefs of consumers.

f. Suppose that $I = 2$. Show that as $\lambda_2/\lambda_1 \to +\infty$, the planner would distribute initial wealth in a way that makes consumer 2's beliefs more and more influential in determining equilibrium prices.

Exercise 8.15 **Diverse beliefs, III**

An economy consists of two consumers named $i = 1, 2$. Each consumer evaluates streams of a single nonstorable consumption good according to

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)] \pi_t^i(s^t).$$

Here $\pi_t^i(s^t)$ is consumer i's subjective probability over history s^t. A feasible allocation satisfies $\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)$ for all $t \geq 0$ and for all s^t. The consumers' endowments of the one good are functions of a state variable $s_t \in \mathbf{S} = \{0, 1, 2\}$. In truth, s_t is described by a time invariant Markov chain with initial distribution $\pi_0 = [\,0 \quad 1 \quad 0\,]'$ and transition density defined by the stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .5 & 0 & .5 \\ 0 & 0 & 1 \end{bmatrix}$$

where $P_{ij} = \text{Prob}[s_{t+1} = j - 1 | s_t = i - 1]$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. The endowments of the two consumers are

$$\begin{aligned} y_t^1 &= s_t/2 \\ y_t^2 &= 1 - s_t/2. \end{aligned}$$

In part I, both consumers know the true probabilities over histories s^t (i.e., they know both π_0 and P). In part II, the two consumers have different subjective probabilities.

Part I:

Assume that both consumers know (π_0, P), so that $\pi_t^1(s^t) = \pi_t^2(s^t)$ for all $t \geq 0$ for all s^t.

a. Show how to deduce $\pi_t^i(s^t)$ from (π_0, P).

b. Define a competitive equilibrium with sequential trading of Arrow securities.

c. Compute a competitive equilibrium with sequential trading of Arrow securities.

d. By hand, simulate the economy. In particular, for every possible realization of the histories s^t, describe time series of c_t^1, c_t^2 and the wealth levels for the two consumers.

Part II:

Now assume that while consumer 1 knows (π_0, P), consumer 2 knows π_0 but thinks that P is

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 \\ .4 & 0 & .6 \\ 0 & 0 & 1 \end{bmatrix}.$$

e. Deduce $\pi_t^2(s^t)$ from $(\pi_0, \hat{P})$ for all $t \geq 0$ for all s^t.

f. Formulate and solve a Pareto problem for this economy.

g. Define an equilibrium with time 0 trading of a complete set of Arrow-Debreu history-contingent securities.

h. Compute an equilibrium with time 0 trading of a complete set of Arrow-Debreu history-contingent securities.

i. Compute an equilibrium with sequential trading of Arrow securities. For every possible realization of s^t for all $t \geq 0$, please describe time series of c_t^1, c_t^2 and the wealth levels for the two consumers.

Exercise 8.16 **Diverse beliefs, IV**

A pure exchange economy is populated by two consumers. Consumer i has preferences over history-contingent consumption sequences $\{c_t^i(s^t)\}$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t^i(s^t),$$

where $u(c) = \ln(c)$, $\beta \in (0,1)$, and $\pi_t^i(s^t)$ is a density that consumer i assigns to history s^t. The state space is time invariant. In particular, $s_t \in S = \{0, .5, 1\}$ for all $t \geq 0$. Only two histories are possible for $t = 0, 1, 2, \ldots$:

$$\text{history } 1 : .5, 1, 1, 1, 1, \ldots$$
$$\text{history } 2 : .5, 0, 0, 0, 0, \ldots$$

Consumer 1 assigns probability 1 to history 1 and probability 0 to history 2, while consumer 2 assigns probability 0 to history 1 and probability 1 to history

2. Nature assigns equal probabilities to the two histories. The endowments of the two consumers are:

$$\begin{aligned} y_t^1 &= s_t \\ y_t^2 &= 1 - s_t. \end{aligned}$$

a. Formulate and solve a Pareto problem for this economy.

b. Define a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

c. Does a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities exist for this economy? If it does, compute it. If it does not, explain why.

Exercise 8.17 **Diverse beliefs, V**

A pure exchange economy is populated by two consumers. Consumer i has preferences over history-contingent consumption sequences $\{c_t^i(s^t)\}$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t^i(s^t),$$

where $u(c) = \ln(c)$, $\beta \in (0,1)$, and $\pi_t^i(s^t)$ is a density that consumer i assigns to history s^t. The state space is time invariant. In particular, $s_t \in S = \{0, .5, 1\}$ for all $t \geq 0$. Only two histories are possible for $t = 0, 1, 2, \ldots$:

$$\begin{aligned} &\text{history } 1 : .5, 1, 1, 1, 1, \ldots \\ &\text{history } 2 : .5, 0, 0, 0, 0, \ldots \end{aligned}$$

Consumer 1 assigns probability 1 to history 1 and probability 0 to history 2, while consumer 2 assigns probability 0 to history 1 and probability 1 to history 2. Nature assigns equal probabilities to the two histories. The endowments of the two consumers are:

$$\begin{aligned} y_t^1 &= 1 - s_t \\ y_t^2 &= s_t. \end{aligned}$$

a. Formulate and solve a Pareto problem for this economy.

b. Define a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

c. Does a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities exist for this economy? If it does, compute it. If it does not, explain why.

Exercise 8.18 **Risk-free bonds**

An economy consists of a single representative consumer who ranks streams of a single nonstorable consumption good according to $\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t(s^t)]\pi_t(s^t)$. Here $\pi_t(s^t)$ is the subjective probability that the consumer attaches to a history s^t of a Markov state s_t, where $s_t \in \{1, 2\}$. Assume that the subjective probability $\pi_t(s^t)$ equals the objective probability. Feasibility for this pure endowment economy is expressed by the condition $c_t \leq y_t$, where y_t is the endowment at time t. The endowment is exogenous and governed by

$$y_{t+1} = \lambda_{t+1}\lambda_t \cdots \lambda_1 y_0$$

for $t \geq 0$ where $y_0 > 0$. Here λ_t is a function of the Markov state s_t. Assume that $\lambda_t = 1$ when $s_t = 1$ and $\lambda_t = 1 + \zeta$ when $s_t = 2$, where $\zeta > 0$. States $s^t = [s_t, \ldots, s_0]$ are known at time t, but future states are not. The state s_t is described by a time invariant Markov chain with initial probability distribution $\pi_0 = [\,1 \quad 0\,]'$ and transition density defined by the stochastic matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{ij} = \text{Prob}[s_{t+1} = j | s_t = i]$ for $i = 1, 2$ and $j = 1, 2$. Assume that $P_{ij} \geq 0$ for all pairs (i, j).

a. Show how to deduce $\pi_t(s^t)$ from (π_0, P).

b. Define a competitive equilibrium with sequential trading of Arrow securities.

c. Compute a competitive equilibrium with sequential trading of Arrow securities.

d. Let p_t^b be the time t price of a risk-free claim to one unit of consumption at time $t+1$. Define a competitive equilibrium with sequential trading of risk-free claims to consumption one period ahead.

e. Let $R_t = (p_t^b)^{-1}$ be the one-period risk-free gross interest rate. Give a formula for R_t and tell how it depends on the history s^t.