Chapter 11
Fiscal Policies in a Growth Model

11.1. Introduction

This chapter studies effects of technology and fiscal shocks on equilibrium outcomes in a nonstochastic growth model. We use the model to state some classic doctrines about the effects of various types of taxes and also as a laboratory to exhibit numerical techniques for approximating equilibria and to display the structure of dynamic models in which decision makers have perfect foresight about future government decisions. Foresight imparts effects on prices and allocations that precede government actions that cause them.

Following Hall (1971), we augment a nonstochastic version of the standard growth model with a government that purchases a stream of goods and that finances itself with an array of distorting flat-rate taxes. We take government behavior as exogenous,[1] which means that for us a *government* is simply a list of sequences for government purchases $\{g_t\}_{t=0}^{\infty}$ and taxes $\{\tau_{ct}, \tau_{kt}, \tau_{nt}, \tau_{ht}\}_{t=0}^{\infty}$. Here $\tau_{ct}, \tau_{kt}, \tau_{nt}$ are, respectively, time-varying flat-rate rates on consumption, earnings from capital, and labor earnings; and τ_{ht} is a lump-sum tax (a "head tax" or "poll tax").

Distorting taxes prevent a competitive equilibrium allocation from solving a planning problem. Therefore, to compute an equilibrium allocation and price system, we solve a system of nonlinear difference equations consisting of the first-order conditions for decision makers and the other equilibrium conditions. We first use a method called shooting. It produces an accurate approximation. Less accurate but in some ways more revealing approximations can be found by following Hall (1971), who solved a linear approximation to the equilibrium conditions. We apply the lag operators described in appendix A of chapter 2 to find and represent the solution in a way that is especially helpful in revealing the dynamic effects of perfectly foreseen alterations in taxes and expenditures and

[1] In chapter 16, we take up a version of the model in which the government chooses taxes to maximize the utility of a representative consumer.

how current values of endogenous variables respond to paths of future exogenous variables.[2]

11.2. Economy

11.2.1. Preferences, technology, information

There is no uncertainty, and decision makers have perfect foresight. A representative household has preferences over nonnegative streams of a single consumption good c_t and leisure $1-n_t$ that are ordered by

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t), \quad \beta \in (0,1) \tag{11.2.1}$$

where U is strictly increasing in c_t and $1-n_t$, twice continuously differentiable, and strictly concave. We require that $c_t \geq 0$ and $n_t \in [0,1]$. We'll typically assume that $U(c, 1-n) = u(c) + v(1-n)$. Common alternative specifications in the real business cycle literature are $U(c, 1-n) = \log c + \zeta \log(1-n)$ and $U(c, 1-n) = \log c + \zeta(1-n)$.[3] We shall also focus on another frequently studied special case that has $v = 0$ so that $U(c, 1-n) = u(c)$.

The technology is

$$g_t + c_t + x_t \leq F(k_t, n_t) \tag{11.2.2a}$$

$$k_{t+1} = (1-\delta)k_t + x_t \tag{11.2.2b}$$

where $\delta \in (0,1)$ is a depreciation rate, k_t is the stock of physical capital, x_t is gross investment, and $F(k, n)$ is a linearly homogeneous production function with positive and decreasing marginal products of capital and labor.[4] It is

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2 See Sargent (1987a) for a more comprehensive account of lag operators. By using lag operators, we extend Hall's results to allow arbitrary fiscal policy paths.

3 See Hansen (1985) for a comparison of these two specifications. Both of these specifications fulfill the necessary conditions for the existence of a balance growth path set forth by King, Plosser, and Rebelo (1988), which require that income and substitution effects cancel in an appropriate way.

4 In section 11.11, we modify the production function to admit labor augmenting technical change, a form that respects the King, Plosser, and Rebelo (1988) necessary conditions for the existence of a balance growth path.

sometimes convenient to eliminate x_t from (11.2.2) and express the technology as

$$g_t + c_t + k_{t+1} \leq F(k_t, n_t) + (1-\delta)k_t. \tag{11.2.3}$$

11.2.2. Components of a competitive equilibrium

There is a competitive equilibrium with all trades occurring at time 0. The household owns capital, makes investment decisions, and rents capital and labor to a representative production firm. The representative firm uses capital and labor to produce goods with the production function $F(k_t, n_t)$. A *price system* is a triple of sequences $\{q_t, \eta_t, w_t\}_{t=0}^{\infty}$, where q_t is the time 0 pretax price of one unit of investment or consumption at time t (x_t or c_t), η_t is the pretax price at time t that the household receives from the firm for renting capital at time t, and w_t is the pretax price at time t that the household receives for renting labor to the firm at time t. The prices w_t and η_t are expressed in terms of time t goods, while q_t is expressed in terms of the numeraire at time 0.

We extend the chapter 8 definition of a competitive equilibrium to include activities of a government. We say that a government expenditure and tax plan that satisfy a budget constraint is *budget feasible*. A set of competitive equilibria is indexed by alternative budget-feasible government policies.

The household faces the budget constraint:

$$\begin{aligned}&\sum_{t=0}^{\infty} q_t \left\{(1+\tau_{ct})c_t + [k_{t+1} - (1-\delta)k_t]\right\} \\ &\leq \sum_{t=0}^{\infty} q_t \left\{\eta_t k_t - \tau_{kt}(\eta_t - \delta_t)k_t + (1-\tau_{nt})w_t n_t - \tau_{ht}\right\}.\end{aligned} \tag{11.2.4}$$

Here we have assumed that the government gives a depreciation allowance δk_t from the gross rentals on capital $\eta_t k_t$ and so collects taxes $\tau_{kt}(\eta_t - \delta)k_t$ on rentals from capital. The government faces the budget constraint

$$\sum_{t=0}^{\infty} q_t g_t \leq \sum_{t=0}^{\infty} q_t \Big\{\tau_{ct} c_t + \tau_{kt}(\eta_t - \delta)k_t + \tau_{nt} w_t n_t + \tau_{ht}\Big\}. \tag{11.2.5}$$

There is a sense in which we have given the government access to too many kinds of taxes, because when lump-sum taxes are available, the government should not use any distorting taxes. We include all of these taxes because, like Hall (1971), we want a framework that is sufficiently general to allow us to analyze how the various taxes distort production and consumption decisions.

11.3. The term structure of interest rates

The price system $\{q_t\}_{t=0}^{\infty}$ evidently embeds within it a term structure of interest rates. It is convenient to represent q_t as

$$\begin{aligned} q_t &= q_0 \frac{q_1}{q_0} \frac{q_2}{q_1} \cdots \frac{q_t}{q_{t-1}} \\ &= q_0 m_{0,1} m_{1,2} \cdots m_{t-1,t} \end{aligned}$$

where $m_{t,t+1} = \frac{q_{t+1}}{q_t}$. We can represent the one-period *discount factor* $m_{t,t+1}$ as

$$m_{t,t+1} = R_{t,t+1}^{-1} = \frac{1}{1 + r_{t,t+1}} \approx \exp(-r_{t,t+1}). \tag{11.3.1}$$

Here $R_{t,t+1}$ is the gross one-period rate of interest between t and $t+1$ and $r_{t,t+1}$ is the net one-period rate of interest between t and $t+1$. Notice that q_t can also be expressed as

$$\begin{aligned} q_t &= q_0 \exp(-r_{0,1}) \exp(-r_{1,2}) \cdots \exp(-r_{t-1,t}) \\ &= q_0 \exp\big(-(r_{0,1} + r_{1,2} + \cdots + r_{t-1,t})\big) \\ &= q_0 \exp(-t r_{0,t}) \end{aligned}$$

where

$$r_{0,t} = t^{-1}(r_{0,1} + r_{1,2} + \cdots + r_{t-1,t}). \tag{11.3.2}$$

Here $r_{0,t}$ is the net t-period rate of interest between 0 and t. Since q_t is the time 0 price of one unit of time t consumption, $r_{0,t}$ is said to be the yield to maturity on a 'zero coupon bond' that matures at time t. A zero coupon bond promises no coupons before the date of maturity and pays only the principal due at the date of maturity. Equation (11.3.2) expresses the expectations theory of the term structure of interest rates, according to which interest rates on t-period (long) loans are averages of rates on one period (short) loans expected to prevail over the horizon of the long loan. More generally, the s-period long rate at time t is

$$r_{t,t+s} = \frac{1}{s}(r_{t,t+1} + r_{t+1,t+2} + \cdots + r_{t+s-1,t+s}). \tag{11.3.3}$$

A graph of $r_{t,t+s}$ against s for $s = 1, 2, \ldots, S$ is called the (real) yield curve at t.

An insight about the expectations theory of the term structure of interest rates can be gleaned from computing gross one-period holding period returns on zero coupon bonds of maturities $1, 2, \ldots$. Consider the gross return earned by someone who at time 0 purchases one unit of time t consumption for q_t units of the numeraire and then sells it at time 1. The person pays $\frac{q_t}{q_0}$ units of time 0 consumption goods to earn $\frac{q_t}{q_1}$ units of time 1 consumption goods. The gross rate of return from this trade measured in time 1 consumption goods per unit of time 0 consumption goods is $\frac{q_0}{q_1}$, which does not depend on the date t of the good bought at time 0 and then sold at time 1. Evidently, at time 0 the one-period return is *identical* for pure discount bonds of *all* maturities $t \geq 1$. More generally, at time t the one-period holding period gross return on zero coupon bonds of all maturities equals $\frac{q_t}{q_{t+1}}$.

A way to characterize the expectations theory of the term structure of interest rates is by the requirement that the price vector $\{q_t\}_{t=0}^{\infty}$ of zero coupon bonds must be such that one-period holding period yields are equated across zero coupon bonds of all maturities. Note also how the price system $\{q_t\}_{t=0}^{\infty}$ contains forecasts of one-period holding period yields on zero coupon bonds of all maturities at all dates $t \geq 0$.

In subsequent sections, we'll indicate how the growth model with taxes and government expenditures links the term structure of interest rates to aspects of government fiscal policy.

11.4. Digression: sequential version of government budget constraint

We have used the time 0 trading abstraction described in chapter 8. Sequential trading of one-period risk-free debt can also support the equilibrium allocations that we shall study in this chapter. It is especially useful explicitly to describe the sequence of one-period government debt that is implicit in the equilibrium tax policies here.

We presume that the government enters period 0 with no government debt.[5] Define total tax collections as $T_t = \tau_{ct} c_t + \tau_{kt}(\eta_t - \delta) k_t + w_t \tau_{nt} n_t + \tau_{ht}$ and express the government budget constraint as

$$\sum_{t=0}^{\infty} q_t (g_t - T_t) = 0. \tag{11.4.1}$$

This can be written as

$$g_0 - T_0 = \sum_{t=1}^{\infty} \frac{q_t}{q_0} (T_t - g_t),$$

which states that the government deficit $g_0 - T_0$ at time 0 equals the present value of future government surpluses. Here $B_0 \equiv \sum_{t=1}^{\infty} \frac{q_t}{q_0}(T_t - g_t)$ is the value of government debt issued at time 0, denominated in units of time 0 goods. We can use this definition of B_0 to deduce

$$B_0 \frac{q_0}{q_1} = T_1 - g_1 + \sum_{t=2}^{\infty} \frac{q_t}{q_1} (T_t - g_t)$$

or, by recalling from the previous subsection that $R_{0,1} \equiv \frac{q_0}{q_1}$ denotes the gross one-period real interest rate between time 0 and time 1,

$$B_0 R_{0,1} = T_1 - g_1 + B_1$$

where now

$$B_1 \equiv \sum_{t=2}^{\infty} \frac{q_t}{q_1} (T_t - g_t)$$

is the value of government debt issued in period 1 in units of time 1 consumption. Iterating this construction forward gives us a sequence of period-by-period government budget constraints

$$g_t + R_{t-1,t} B_{t-1} = T_t + B_t \tag{11.4.2}$$

for $t \geq 1$, where $R_{t-1,t} = \frac{q_{t-1}}{q_t}$ and

$$B_t \equiv \sum_{s=t+1}^{\infty} \frac{q_s}{q_t} (T_s - g_s). \tag{11.4.3}$$

[5] Letting $B_{-1} = 0$ be the government debt owed at time -1 allows us to apply equation (11.4.2) to date $t = 0$ too.

The left side of equation (11.4.2) is time t government expenditures including interest and principal payments on its debt, while the right side is total revenues including those raised by issuing new one-period debt in the amount B_t.

Thus, embedded in a government policy that satisfies (11.2.5) is a sequence of one-period government debts satisfying (11.4.3). The value of government debt at t is the present value of government surpluses from date $t+1$ onward. Equation (11.4.3) states that government *debts* at time t signal future *surpluses*.

Equation (11.4.2) can be represented as

$$B_t - B_{t-1} = g_t - T_t + r_{t-1,t} B_{t-1}. \tag{11.4.4}$$

Here $g_t - T_t$ is what is commonly called either the *net-of-interest* government deficit or the *operational* government deficit or the *primary* government deficit, while $r_{t-1,t}B_{t-1}$ are net interest payments on the government debt and $g_t - T_t + r_{t-1,t}B_{t-1}$ is the gross-of-interest government deficit. Equation (11.4.4) asserts that the change in government debt equals the gross-of-interest government deficit.

The Arrow-Debreu budget constraint (11.4.1) automatically enforces a 'no-Ponzi scheme' condition on the path of government debt $\{B_t\}$. To see this, first recall that $\frac{q_s}{q_t} = R_{t,t+1}^{-1} \cdots R_{s-1,s}^{-1}$ and write (11.4.3) as

$$B_t = \sum_{s=t+1}^{T} \frac{q_s}{q_t}(T_s - g_s) + \sum_{s=T+1}^{\infty} \frac{q_s}{q_t}(T_s - g_s)$$

or

$$B_t \equiv \sum_{s=t+1}^{T} \frac{q_s}{q_t}(T_s - g_s) + \frac{q_T}{q_t} B_T$$

or

$$B_t \equiv \sum_{s=t+1}^{T} \frac{q_s}{q_t}(T_s - g_s) + R_{t,t+1}^{-1} \cdots R_{T-1,T}^{-1} B_T.$$

An argument like that in subsection 11.5.1 can be applied to show that in an equilibrium $\lim_{T\to+\infty} q_T B_{T+1} = 0$.

11.4.1. Irrelevance of maturity structure of government debt

At time t, the government issues a list of *bonds* that in the aggregate promise to pay a stream $\{\xi_s^t\}_{s=1}^\infty$ of goods at time $s > t$ satisfying

$$B_t = \sum_{s=t+1}^{\infty} \frac{q_s}{q_t} \xi_s^t. \tag{11.4.5}$$

The only restriction that our model puts on the term structure of payments $\{\xi_s^t\}_{s=1}^\infty$ is that it must satisfy

$$\sum_{s=t+1}^{\infty} \frac{q_s}{q_t} \xi_s^t = \sum_{s=t+1}^{\infty} \frac{q_s}{q_t} (T_s - g_s) \equiv B_t \tag{11.4.6}$$

The model of this chapter asserts that one payment stream $\{\xi_s^t\}_{s=t+1}^\infty$ that satisfies (11.4.6) is as good as any other. The model pins down the total value of the continuation government IOU stream $\{\xi_s^t\}_{s=t+1}^\infty$ at each t, but it leaves the maturity structure of payments, whether early or late, for example, undetermined.[6] Two polar examples of maturity structures of the government debt are:

1. All debt consists of *one-period pure discount bonds* that are rolled over every period:

$$\xi_s^t = \begin{cases} \bar{\xi}^t & \text{if } s = t+1 \\ 0 & \text{if } s \geq t+2 \end{cases}$$

where $\bar{\xi}^t$ satisfies $\frac{q_{t+1}}{q_t} \bar{\xi}^t = B_t$.

2. All debt consists of *consols* that in the aggregate promise to pay a constant total coupon $\hat{\xi}^t$ for $s \geq t+1$, where $\hat{\xi}^t$ satisfies

$$\hat{\xi}^t \sum_{s=t+1}^{\infty} \frac{q_s}{q_t} = B_t.$$

[6] For models that restrict the maturity structure of government debt by imposing more imperfections than we analyze in this chapter, see Lucas and Stokey (1983), Angeletos (2002), Buera and Nicolini (2004), and Shin (2007). Lucas and Stokey show how to set the maturity structure of debt payments in a way that induces an authority responsible for sequentially choosing flat rate taxes on labor to implement a Ramsey plan. Angeletos (2002), Buera and Nicolini (2004), and Shin (2007) use variations over time in the maturity structure of risk-free government debt to complete markets.

The sequence of period-by-period net returns on the government debt $\{r_{t,t+1}B_t\}_{t=0}^{\infty}$ is independent of the government's choice of sequences $\{\{\xi_s^t\}_{s=t+1}^{\infty}\}_{t=0}^{\infty}$.

11.5. Competitive equilibria with distorting taxes

A representative household chooses a sequence $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize (11.2.1) subject to (11.2.4). A representative firm chooses $\{k_t, n_t\}_{t=0}^{\infty}$ to maximize $\sum_{t=0}^{\infty} q_t[F(k_t, n_t) - \eta_t k_t - w_t n_t]$.[7] A budget-feasible government policy is an expenditure plan $\{g_t\}_{t=0}^{\infty}$ and a tax plan that satisfy (11.2.5). A feasible allocation is a sequence $\{c_t, x_t, n_t, k_t\}_{t=0}^{\infty}$ that satisfies (11.2.3).

Definition: A *competitive equilibrium with distorting taxes* is a budget-feasible government policy, a feasible allocation, and a price system such that, given the price system and the government policy, the allocation solves the household's problem and the firm's problem.

11.5.1. The household: no-arbitrage and asset-pricing formulas

A no-arbitrage argument implies a restriction on prices and tax rates across time from which there emerges a formula for the "user cost of capital" (see Hall and Jorgenson, 1967). Collect terms in similarly dated capital stocks and thereby rewrite the household's budget constraint (11.2.4) as

$$\begin{aligned}\sum_{t=0}^{\infty} q_t\big[(1+\tau_{ct})c_t\big] &\le \sum_{t=0}^{\infty} q_t(1-\tau_{nt})w_t n_t - \sum_{t=0}^{\infty} q_t \tau_{ht} \\ &+ \sum_{t=1}^{\infty}\big[((1-\tau_{kt})(\eta_t-\delta)+1)q_t - q_{t-1}\big]k_t \\ &+ \big[(1-\tau_{k0})(\eta_0-\delta)+1\big]q_0 k_0 - \lim_{T\to\infty} q_T k_{T+1}\end{aligned} \tag{11.5.1}$$

The terms $\big[(1-\tau_{k0})(\eta_0-\delta)+1\big]q_0 k_0$ and $-\lim_{T\to\infty} q_T k_{T+1}$ remain after creating the weighted sum in k_t's for $t \ge 1$.

[7] Note the contrast with the setup in chapter 12, which has two types of firms. Here we assign to the household the physical investment decisions made by the type II firms of chapter 12.

The household inherits a given k_0 that it takes as an initial condition, and it is free to choose any sequence $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ that satisfies (11.5.1) where all prices and tax rates are taken as given. The objective of the household is to maximize lifetime utility (11.2.1), which is increasing in consumption $\{c_t\}_{t=0}^{\infty}$ and, for one of our preference specifications below, also increasing in leisure $\{1-n_t\}_{t=0}^{\infty}$.

All else equal, the household would be happier with larger values on the right side of (11.5.1), preferably plus infinity, which would enable it to purchase unlimited amounts of consumption goods. Because resources are finite, we know that the right side of the household's budget constraint must be bounded in an equilibrium. This fact leads to an important restriction on the price and tax sequences. If the right side of the household's budget constraint is to be bounded, then the terms multiplying k_t for $t \geq 1$ must all equal zero because if any of them were strictly positive (negative) for some date t, the household could make the right side of (11.5.1) an arbitrarily large positive number by choosing an arbitrarily large positive (negative) value of k_t. On the one hand, if one such term were strictly positive for some date t, the household could purchase an arbitrarily large capital stock k_t assembled at time $t-1$ with a present-value cost of $q_{t-1}k_t$ and then sell the rental services and the undepreciated part of that capital stock to be delivered at time t, with a present-value income of $[(1-\tau_{kt})(\eta_t - \delta) + 1]q_t k_t$. If such a transaction were to yield a strictly positive profit, it would offer the consumer a pure arbitrage opportunity and the right side of (11.5.1) would become unbounded. On the other hand, if there is one term multiplying k_t that is strictly negative for some date t, the household can make the right side of (11.5.1) arbitrarily large and positive by "short selling" capital by setting $k_t < 0$. The household could turn to purchasers of capital assembled at time $t-1$ and sell "synthetic" units of capital to them. Such a transaction need not involve any actual physical capital: the household could merely undertake trades that would give the other party to the transaction the same costs and incomes as those associated with purchasing capital assembled at time $t-1$. If such short sales of capital yield strictly positive profits, it would provide the consumer with a pure arbitrage opportunity and the right side of (11.5.1) would become unbounded. Therefore, the terms multiplying k_t must *equal* zero for all $t \geq 1$, so that

$$\frac{q_t}{q_{t+1}} = [(1-\tau_{kt+1})(\eta_{t+1} - \delta) + 1] \tag{11.5.2}$$

for all $t \geq 0$. These are zero-profit or no-arbitrage conditions. We have derived these conditions by using only the weak property that $U(c, 1-n)$ is increasing in consumption (i.e., that the household always prefers more to less).

It remains to be determined how the household sets the last term on the right side of (11.5.1), $-\lim_{T\to\infty} q_T k_{T+1}$. According to our preceding argument, the household would not purchase an amount of capital that would make this term strictly negative in the limit because that would reduce the right side of (11.5.1) and hence diminish the household's resources available for consumption. Instead, the household would like to make this term strictly positive and unbounded, so that the household could purchase unlimited amounts of consumption goods. But the market would stop the household from undertaking such a short sale in the limit, since no party would like to be on the other side of the transaction. This is obvious when considering a finite-horizon model where everyone would like to short sell capital in the very last period because there would then be no future period in which to fulfil the obligations of those short sales. Therefore, in our infinite-horizon model, as a condition of optimality, we impose the terminal condition that $-\lim_{T\to\infty} q_T k_{T+1} = 0$. Once we impose formula (11.5.5a) below linking q_t to U_{1t}, this terminal condition puts the following restriction on the equilibrium allocation:

$$-\lim_{T\to\infty} \beta^T \frac{U_{1T}}{(1+\tau_{cT})} k_{T+1} = 0. \tag{11.5.3}$$

The household's initial capital stock k_0 is given. According to (11.5.1), its value is $[(1-\tau_{k0})(\eta_0 - \delta) + 1] q_0 k_0$.

11.5.2. User cost of capital formula

The no-arbitrage conditions (11.5.2) can be rewritten as the following expression for the "user cost of capital" η_{t+1}:

$$\eta_{t+1} = \delta + \left(\frac{1}{1-\tau_{kt+1}}\right)\left(\frac{q_t}{q_{t+1}} - 1\right). \tag{11.5.4}$$

Recalling from (11.3.1) that $m_{t,t+1}^{-1} = R_{t,t+1} = (1 + r_{t,t+1}) = \frac{q_t}{q_{t+1}}$, equation (11.5.4) can be expressed as

$$\eta_{t+1} = \delta + \left(\frac{r_{t,t+1}}{1-\tau_{kt+1}}\right).$$

The user cost of capital takes into account the rate of taxation of capital earnings, the capital gain or loss from t to $t+1$, and a depreciation cost.[8]

11.5.3. Household first-order conditions

So long as the no-arbitrage conditions (11.5.2) prevail, households are indifferent about how much capital they hold. Recalling that the one-period utility function is $U(c, 1-n)$, let $U_1 = \frac{\partial U}{\partial c}$ and $U_2 = \frac{\partial U}{1-n}$ so that $\frac{\partial U}{\partial n} = -U_2$. Then we have that the household's first-order conditions with respect to c_t, n_t are:

$$\beta^t U_{1t} = \mu q_t (1 + \tau_{ct}) \tag{11.5.5a}$$

$$\beta^t U_{2t} \leq \mu q_t w_t (1 - \tau_{nt}), \quad = \text{if } n_t < 1, \tag{11.5.5b}$$

where μ is a nonnegative Lagrange multiplier on the household's budget constraint (11.2.4). Multiplying the price system by a positive scalar simply rescales the multiplier μ, so we are free to choose a numeraire by setting μ to an arbitrary positive number.

11.5.4. A theory of the term structure of interest rates

Equation (11.5.5a) allows us to solve for q_t as a function of consumption

$$\mu q_t = \beta^t U_{1t} / (1 + \tau_{ct}) \tag{11.5.6a}$$

or in the special case that $U(c_t, 1-n_t) = u(c_t)$

$$\mu q_t = \beta^t u'(c_t) / (1 + \tau_{ct}). \tag{11.5.6b}$$

In conjunction with the observations made in subsection 11.3, these formulas link the term structure of interest rates to the paths of c_t, τ_{ct}. The government policy $\{g_t, \tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$ affects the term structure of interest rates directly via τ_{ct} and indirectly via its impact on the path for $\{c_t\}_{t=0}^{\infty}$.

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8 This is a discrete-time version of a continuous-time formula derived by Hall and Jorgenson (1967).

11.5.5. Firm

Zero-profit conditions for the representative firm impose additional restrictions on equilibrium prices and quantities. The present value of the firm's profits is

$$\sum_{t=0}^{\infty} q_t \left[F(k_t, n_t) - w_t n_t - \eta_t k_t\right].$$

Applying Euler's theorem on linearly homogeneous functions to $F(k,n)$, the firm's present value is:

$$\sum_{t=0}^{\infty} q_t \left[(F_{kt} - \eta_t)k_t + (F_{nt} - w_t)n_t\right].$$

No-arbitrage (or zero-profit) conditions are:

$$\begin{aligned} \eta_t &= F_{kt} \\ w_t &= F_{nt}. \end{aligned} \tag{11.5.7}$$

11.6. Computing equilibria

The definition of a competitive equilibrium and the concavity conditions that we have imposed on preferences imply that an equilibrium is a price system $\{q_t, \eta_t, w_t\}$, a budget feasible government policy $\{g_t, \tau_t\} \equiv \{g_t, \tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{ht}\}$, and an allocation $\{c_t, n_t, k_{t+1}\}$ that solve the system of nonlinear difference equations consisting of (11.2.3), (11.5.2), (11.5.5), and (11.5.7) subject to the initial condition that k_0 is given and the terminal condition (11.5.3). In this chapter, we shall simplify things by treating $\{g_t, \tau_t\} \equiv \{g_t, \tau_{ct}, \tau_{nt}, \tau_{kt}\}$ as exogenous and then use $\sum_{t=0}^{\infty} q_t \tau_{ht}$ as a slack variable that we choose to balance the government's budget. We now attack this system of difference equations.

11.6.1. Inelastic labor supply

We'll start with the following special case. (The general case is just a little more complicated, and we'll describe it below.) Set $U(c, 1-n) = u(c)$, so that the household gets no utility from leisure, and set $n = 1$. We define $f(k) = F(k, 1)$ and express feasibility as

$$k_{t+1} = f(k_t) + (1-\delta)k_t - g_t - c_t. \tag{11.6.1}$$

Notice that $F_k(k,1) = f'(k)$ and $F_n(k,1) = f(k) - f'(k)k$. Substitute (11.5.5a), (11.5.7), and (11.6.1) into (11.5.2) to get

$$\begin{aligned} &\frac{u'\big(f(k_t) + (1-\delta)k_t - g_t - k_{t+1}\big)}{(1+\tau_{ct})} \\ &- \beta \frac{u'(f(k_{t+1}) + (1-\delta)k_{t+1} - g_{t+1} - k_{t+2})}{(1+\tau_{ct+1})} \times \\ &[(1-\tau_{kt+1})(f'(k_{t+1}) - \delta) + 1] = 0. \end{aligned} \tag{11.6.2}$$

Given the government policy sequences, (11.6.2) is a second-order difference equation in capital. We can also express (11.6.2) as

$$u'(c_t) = \beta u'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \left[(1-\tau_{kt+1})(f'(k_{t+1}) - \delta) + 1\right]. \tag{11.6.3}$$

To compute an equilibrium, we must find a solution of the difference equation (11.6.2) that satisfies two boundary conditions. As mentioned above, one boundary condition is supplied by the given level of k_0 and the other by (11.5.3). To determine a particular terminal value k_∞, we restrict the path of government policy so that it converges, a way to impose (11.5.3).

11.6.2. The equilibrium steady state

Tax rates and government expenditures serve as forcing functions for the difference equations (11.6.1) and (11.6.3). Let $z_t = [\,g_t \quad \tau_{kt} \quad \tau_{ct}\,]'$ and write (11.6.2) as

$$H(k_t, k_{t+1}, k_{t+2}; z_t, z_{t+1}) = 0. \tag{11.6.4}$$

To allow convergence to a steady state, we assume government policies that are eventually constant, i.e., that satisfy

$$\lim_{t\to\infty} z_t = \overline{z}. \tag{11.6.5}$$

When we actually solve our models, we'll set a date T after which all components of the forcing sequences that comprise z_t are constant. A terminal steady-state capital stock $\overline{k}$ evidently solves

$$H(\overline{k}, \overline{k}, \overline{k}, \overline{z}, \overline{z}) = 0. \tag{11.6.6}$$

For our model, we can solve (11.6.6) by hand. In a steady state, (11.6.3) becomes

$$1 = \beta[(1-\overline{\tau}_k)(f'(\overline{k})-\delta)+1].$$

Notice that an eventually constant consumption tax $\overline{\tau}_c$ does not distort $\overline{k}$ *vis-a-vis* its value in an economy without distorting taxes. Letting $\beta = \frac{1}{1+\rho}$, we can express the preceding equation as

$$\delta + \frac{\rho}{1-\tau_k} = f'(\overline{k}). \tag{11.6.7}$$

When $\tau_k = 0$, equation (11.6.7) becomes $(\rho+\delta) = f'(\overline{k})$, which is a celebrated formula for the so-called "augmented Golden Rule" capital-labor ratio.

When the exogenous sequence $\{g_t\}_{t=0}^\infty$ converges, the steady state capital-labor ratio that solves $(\rho+\delta) = f'(\overline{k})$ is the asymptotic value of the capital-labor ratio that would be approached by a benevolent planner who chooses $\{c_t, k_{t+1}\}_{t=0}^\infty$ to maximize $\sum_{t=0}^\infty \beta^t u(c_t)$ subject to k_0 given and the sequence of constraints $c_t + k_{t+1} + g_t \leq f(k_t) + (1-\delta)k_t$.

11.6.3. *Computing the equilibrium path with the shooting algorithm*

Having computed the terminal steady state, we are now in a position to apply the *shooting algorithm* to compute an equilibrium path that starts from an arbitrary initial condition k_0, assuming a possibly time-varying path of government policy.[9] The shooting algorithm solves the two-point boundary value problem by searching for an initial c_0 that makes the Euler equation (11.6.2) and the feasibility condition (11.2.3) imply that $k_S \approx \overline{k}$, where S is a finite but large time index meant to approximate infinity and $\overline{k}$ is the terminal steady value associated with the policy being analyzed. We let T be the value of t after which all components of z_t are constant. Here are the steps of the algorithm.[10]

1. Solve (11.6.4) for the terminal steady-state $\overline{k}$ that is associated with the permanent policy vector $\overline{z}$ (i.e., find the solution of (11.6.7)).

2. Select a large time index $S >> T$ and guess an initial consumption rate c_0. (A good guess comes from the linear approximation to be described in section 11.10.) Compute $u'(c_0)$ and solve (11.6.1) for k_1.

3. For $t = 0$, use (11.6.3) to solve for $u'(c_{t+1})$. Then invert u' and compute c_{t+1}. Use (11.6.1) to compute k_{t+2}.

4. Iterate on step 3 to compute candidate values $\hat{k}_t, t = 1, \ldots, S$.

5. Compute $\hat{k}_S - \overline{k}$.

6. If $\hat{k}_S > \overline{k}$, raise c_0 and compute a new $\hat{k}_t, t = 1, \ldots, S$.

7. If $\hat{k}_S < \overline{k}$, lower c_0.

8. In this way, search for a value of c_0 that makes $\hat{k}_S \approx \overline{k}$.

9. Compute $\sum_{t=0}^{\infty} q_t \tau_{ht}$ that satisfies the government budget constraint at equality.

---

[9] We recommend a suite of computer programs called `dynare`. We have used `dynare` to execute the numerical experiments described in this chapter. See Barillas, Bhandari, Bigio, Colacito, Juillard, Kitao, Matthes, Sargent, and Shin (2012) for dynare code that performs these and other calculations. See <http://www.dynare.org>.

[10] This algorithm proceeds in the spirit of the invariant-subspace method (implemented via a Schur decomposition) for solving the first-order conditions associated with the optimal linear regulator that we described in section 5.5 of chapter 5.

11.6.4. Other equilibrium quantities

After we solve (11.6.2) for an equilibrium $\{k_t\}$ sequence, we can recover other equilibrium quantities and prices from the following equations:

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1} - g_t \tag{11.6.8a}$$

$$q_t = \beta^t u'(c_t)/(1+\tau_{ct}) \tag{11.6.8b}$$

$$\eta_t = f'(k_t) \tag{11.6.8c}$$

$$w_t = f(k_t) - k_t f'(k_t) \tag{11.6.8d}$$

$$\begin{aligned}\bar{R}_{t+1} &= \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}\left[(1-\tau_{kt+1})(f'(k_{t+1})-\delta)+1\right] \\ &= \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}R_{t,t+1}\end{aligned} \tag{11.6.8e}$$

$$R_{t,t+1}^{-1} = m_{t,t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}\frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \tag{11.6.8f}$$

$$r_{t,t+1} \equiv R_{t,t+1} - 1 = (1-\tau_{k,t+1})(f'(k_{t+1})-\delta) \tag{11.6.8g}$$

It is convenient to express (11.6.3) as

$$u'(c_t) = \beta u'(c_{t+1})\bar{R}_{t+1} \tag{11.6.8h}$$

or

$$\bar{R}_{t+1}^{-1} = \beta u'(c_{t+1})/u'(c_t).$$

The left side of this equation is the rate which the market and the tax system allow the household to substitute consumption at t for consumption at $t+1$. The right side is the rate at which the household is willing to substitute consumption at t for consumption at $t+1$.

An equilibrium satisfies equations (11.6.8). In the case of constant relative risk aversion (CRRA) utility $u(c) = (1-\gamma)^{-1}c^{1-\gamma}, \gamma \geq 1$, (11.6.8$h$) implies

$$\log\left(\frac{c_{t+1}}{c_t}\right) = \gamma^{-1}\log\beta + \gamma^{-1}\log\bar{R}_{t+1}, \tag{11.6.9}$$

which shows that the log of consumption growth varies directly with $\bar{R}_{t+1}$. Variations in distorting taxes have effects on consumption and investment that are intermediated through this equation, as several experiments below highlight.

11.6.5. Steady-state $\bar{R}$

Using (11.6.7) and formula (11.6.8e), we can determine that the steady state value of $\bar{R}_{t+1}$ is[11]

$$\bar{R}_{t+1} = (1+\rho)\,. \tag{11.6.10}$$

11.6.6. Lump-sum taxes available

If the government can impose lump-sum taxes, we can implement the shooting algorithm for a specified g, τ_k, τ_c, solve for equilibrium prices and quantities, and then find an associated value for $q \cdot \tau_h = \sum_{t=0}^{\infty} q_t \tau_{ht}$ that balances the government budget. This calculation treats the present value of lump-sum taxes as a residual that balances the government budget. In calculations presented later in this chapter, we shall assume that lump-sum taxes are available and so shall use this procedure.

11.6.7. No lump-sum taxes available

If lump-sum taxes are not available, then an additional step is required to compute an equilibrium. In particular, we have to ensure that taxes and expenditures are such that the government budget constraint (11.2.5) is satisfied at an equilibrium price system with $\tau_{ht} = 0$ for all $t \geq 0$. Braun (1994) and McGrattan (1994b) accomplish this by employing an iterative algorithm that alters a particular distorting tax until (11.2.5) is satisfied. The idea is first to compute a candidate equilibrium for one arbitrary tax policy with possibly nonzero lump sum taxes, then to check whether the government budget constraint is satisfied. Usually we will find that lump sum taxes must be levied to balance the government budget in this candidate equilibrium. To find an equilibrium with zero lump sum taxes, we can proceed as follows. If the government budget would have have a deficit in present value without lump sum taxes (i.e., if the present value of lump sum taxes is positive in the candidate equilibrium), then either decrease some elements of the government expenditure sequence or increase some elements of the tax sequence and try again. Because there exist so

[11] To compute steady states, we assume that all tax rates and government expenditures are constant from some date T forward.

many equilibria, the class of tax and expenditure processes has to be restricted drastically to narrow the search for an equilibrium.[12]

11.7. A digression on back-solving

The shooting algorithm takes sequences for g_t and the various tax rates as given and finds paths of the allocation $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and the price system that solve the system of difference equations formed by (11.6.3) and (11.6.8). Thus, the shooting algorithm views government policy as exogenous and the price system and allocation as endogenous. Sims (1989) proposed another way to solve the growth model that exchanges the roles of some exogenous and endogenous variables. In particular, his *back-solving* approach takes a path $\{c_t\}_{t=0}^{\infty}$ as given, and then proceeds as follows.

Step 1: Given k_0 and sequences for the various tax rates, solve (11.6.3) for a sequence $\{k_{t+1}\}$.

Step 2: Given the sequences for $\{c_t, k_{t+1}\}$, solve the feasibility condition (11.6.8a) for a sequence of government expenditures $\{g_t\}_{t=0}^{\infty}$.

Step 3: Solve formulas (11.6.8b)–(11.6.8e) for an equilibrium price system.

The present model can be used to illustrate other applications of back-solving. For example, we could start with a given process for $\{q_t\}$, use (11.6.8b) to solve for $\{c_t\}$, and proceed as in steps 1 and 2 above to determine processes for $\{k_{t+1}\}$ and $\{g_t\}$, and then finally compute the remaining prices from the as yet unused equations in (11.6.8).

Sims recommended this method because it adopts a flexible or "symmetric" attitude toward exogenous and endogenous variables. Diaz-Giménez, Prescott, Fitzgerald, and Alvarez (1992), Sargent and Smith (1997), and Sargent and Velde (1999) have all used the method. We shall not use it in the remainder of this chapter, but it is a useful method to have in our toolkit.[13]

[12] See chapter 16 for theories about how to choose taxes in socially optimal ways.

[13] Constantinides and Duffie (1996) used back-solving to reverse engineer a cross-section of endowment processes that, with incomplete markets, would prompt households to consume their endowments at a given stochastic process of asset prices.

11.8. Effects of taxes on equilibrium allocations and prices

We use the model to analyze the effects of government expenditure and tax sequences. The household can alter his payments of a *distorting* by altering a decision. The household cannot alter his payments of a *nondistorting* tax. In the present model, τ_k, τ_c, τ_n are distorting taxes and the lump-sum tax τ_h is nondistorting. We can deduce the following outcomes from (11.6.8) and (11.6.7).

1. Lump-sum taxes and Ricardian equivalence. Suppose that the distorting taxes are all zero and that only lump-sum taxes are used to raise government revenues. Then the equilibrium allocation is identical with one that solves a version of a planning problem in which g_t is taken as an exogenous stream that is deducted from output. To verify this claim, notice that lump-sum taxes appear nowhere in formulas (11.6.8), and that these equations are identical with the first-order conditions and feasibility conditions for a planning problem. The timing of lump-sum taxes is irrelevant because only the present value of taxes $\sum_{t=0}^{\infty} q_t \tau_{ht}$ appears in the budget constraints of the government and the household.

2. When the labor supply is inelastic, constant τ_c and τ_n are not distorting. When the labor supply is inelastic, τ_n is not a distorting tax. A *constant* level of τ_c is not distorting.

3. Variations in τ_c over time are distorting. They affect the path of capital and consumption through equation (11.6.8g).

4. Capital taxation is distorting. Constant levels of the capital tax τ_k are distorting (see (11.6.8g) and (11.6.7)).

11.9. Transition experiments with inelastic labor supply

We continue to study the special case with $U(c, 1-n) = u(c)$. Figures 11.9.1 through 11.9.5 apply the shooting algorithm to an economy with $u(c) = (1-\gamma)^{-1}c^{1-\gamma}, f(k) = k^{\alpha}$ with parameter values $\alpha = .33, \delta = .2, \beta = .95$ and an initial constant level of g of $.2$. All of the experiments except one to be described in figure 11.9.2 set the critical utility curvature parameter $\gamma = 2$. We initially set all distorting taxes to zero and consider perturbations of them that we describe in the experiments below.

Figures 11.9.1 to 11.9.5 show responses to foreseen once-and-for-all increases in g, τ_c, and τ_k, that occur at time $T = 10$, where $t = 0$ is the initial time period. Prices induce effects that precede the policy changes that cause them. We start all of our experiments from an initial steady state that is appropriate for the pre-jump settings of all government policy variables. In each panel, a dashed line displays a value associated with the steady state at the initial constant values of the policy vector. A solid line depicts an equilibrium path under the new policy. It starts from the value that was associated with an initial steady state that prevailed before the policy change at $T = 10$ was announced. *Before* date $t = T = 10$, the response of each variable is entirely due to expectations about future policy changes. *After* date $t = 10$, the response of each variable represents a purely transient response to a new stationary level of the "forcing function" in the form of the exogenous policy variables. That is, before $t = T$, the forcing function is changing as date T approaches; after date T, the policy vector has attained its new permanent level, so that the only sources of dynamics are transient.

Discounted *future* values of fiscal variables impinge on current outcomes, where the discount rate in question is endogenous, while departures of the capital stock from its terminal steady-state value set in place a force for it to decay toward its steady state rate at a particular rate. These two forces, discounting of the future and transient decay back toward the terminal steady state, are evident in the experiments portrayed in Figures 11.9.1–11.9.5. In section 11.10.6, we express the decay rate as a function of the key curvature parameter γ in the one-period utility function $u(c) = (1-\gamma)^{-1}c^{1-\gamma}$, and we note that the endogenous rate at which future fiscal variables are discounted is tightly linked to that decay rate.

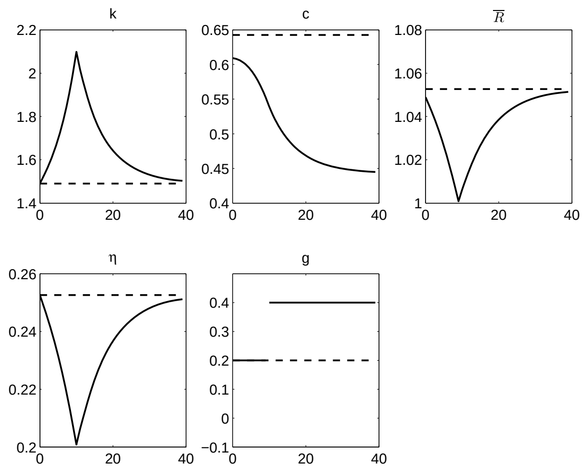

Figure 11.9.1: Response to foreseen once-and-for-all increase in g at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, g$. The dashed line is the original steady state.

Foreseen jump in g_t. Figure 11.9.1 shows the effects of a foreseen permanent increase in g at $t = T = 10$ that is financed by an increase in lump-sum taxes. Although the steady-state value of the capital stock is unaffected (this follows from the fact that g disappears from the steady state version of the Euler equation (11.6.2)), consumers make the capital stock vary over time. Consumers choose immediately to increase their saving in response to the adverse wealth effect that they suffer from the increase in lump-sum taxes that finances the permanently higher level of government expenditures. If the government consumes more, the household must consume less. The adverse wealth effect precedes the actual rise in government expenditures because consumers care about the present value of lump-sum taxes and are indifferent to their timing. Because the present value of lump-sum taxes jumps immediately, consumption also falls immediately in anticipation of the increase in government expenditures. This leads to a gradual build-up of capital in the dates between 0 and T, followed by a gradual fall after T. Variation over time in the capital stock helps smooth consumption over time, so that the main force at work is the consumption-smoothing motive featured in Milton Friedman's permanent income theory. The variation over time in $\bar{R}$ reconciles the consumer to a consumption path that is

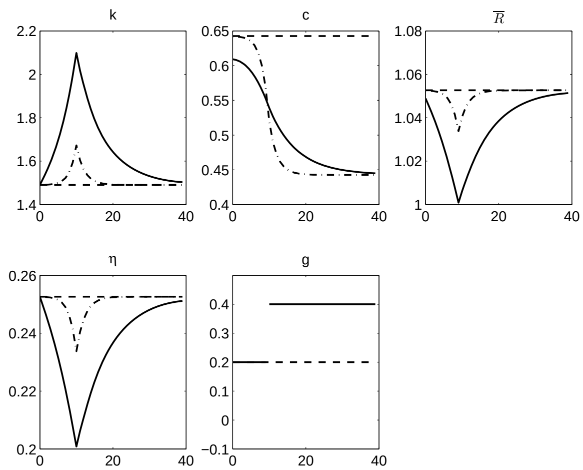

Figure 11.9.2: Response to foreseen once-and-for-all increase in g at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, g$. The dashed lines show the original steady state. The solid lines are for $\gamma = 2$, while the dashed-dotted lines are for $\gamma = .2$

not completely smooth. According to (11.6.9), the gradual increase and then the decrease in capital are inversely related to variations in the gross interest rate that reconcile the household to a consumption path that varies over time.

Figure 11.9.2 compares the responses to a foreseen increase in g at $t = 10$ for two economies, our original economy with $\gamma = 2$, shown in the solid line, and an otherwise identical economy with $\gamma = .2$, shown in the dashed-dotted line. The utility curvature parameter γ governs the household's willingness to substitute consumption across time. Lowering γ increases the household's willingness to substitute consumption across time. This shows up in the equilibrium outcomes in figure 11.9.2. For $\gamma = .2$, consumption is much less smooth than when $\gamma = 2$, and is closer to being a mirror image of the government expenditure path, staying high until government expenditures rise at $t = 10$. There are much smaller build ups and draw downs of capital, and this leads to smaller fluctuations in $\bar{R}$ and η. These two experiments reveal the dependence of the

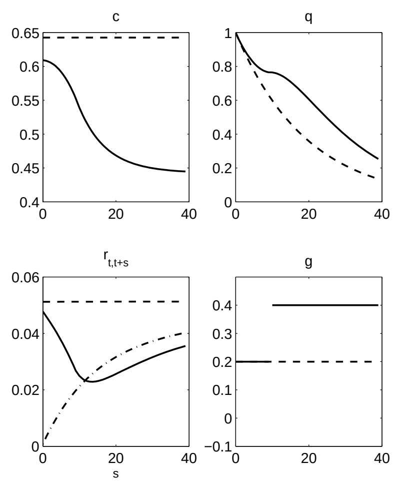

Figure 11.9.3: Response to foreseen once-and-for-all increase in g at $t = 10$. From left to right, top to bottom: $c, q, r_{t,t+1}$ and yield curves $r_{t,t+s}$ for $t = 0$ (solid line), $t = 10$ (dash-dotted line) and $t = 60$ (dashed line); term to maturity s is on the x axis for the yield curve, time t for the other panels.

strength of both the 'feedforward' anticipation effect and the 'feedback' transient effect that wears off initial conditions on the magnitude of γ. We discuss this more later in section 11.10.6 with the aid of equation (11.10.16).

For $\gamma = 2$ again, figure 11.9.3 describes the response of q_t and the term structure of interest rates to a foreseen increase in g_t at $t = 10$. The second panel on the top compares q_t for the initial steady state with q_t after the increase in g is foreseen at $t = 0$, while the third panel compares the implied short rate r_t computed via the section 11.3 formula $r_{t,t+1} = -\log(q_{t+1}/q_t) = -\log\bigl[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{(1+\tau_{ct})}{(1+\tau_{c,t+1})}\bigr]$ and the fourth panel reports the term structure of interest rates $r_{t,t+s}$ computed via formula (11.3.3) for $t = 0, 10$ and $t = 60$ in three separate yield curves for those three dates. In this panel, the term to maturity s is on the x axis, while in the other panels, calendar time t is on the x axis. In this model, $q_t = \beta^t c_t^{-\gamma}$ and $r_{t,t+1} = -\log \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$, so the term structure of interest rates reflects the equilibrium path for $\{c_t\}_{t=0}^{\infty}$.

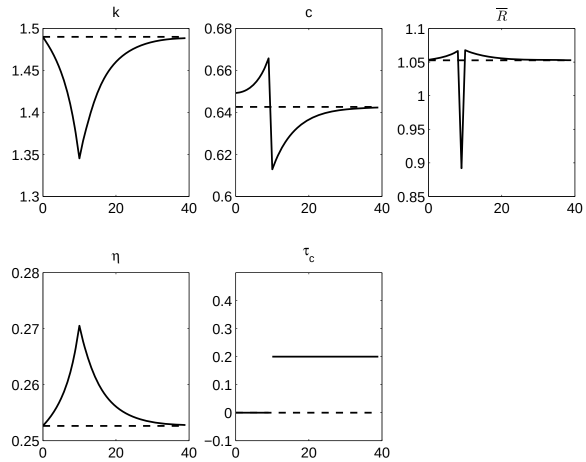

Figure 11.9.4: Response to foreseen once-and-for-all increase in τ_c at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \tau_c$.

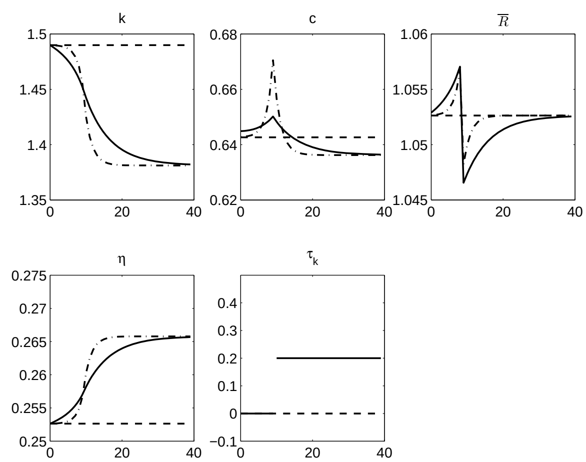

Figure 11.9.5: Response to foreseen increase in τ_k at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \tau_k$. The solid lines depict equilibrium outcomes when $\gamma = 2$, the dashed-dotted lines when $\gamma = .2$.

At $t = 60$, the system has converged to the new steady state and the term structure of interest rates is flat. At $t = 10$, the term structure of interest rates is upward sloping because, as the top left panel showing consumption reveals, the *rate of growth* of consumption is expected to increase over time. At $t = 0$, the term structure of interest rate is 'U-shaped', declining until maturity 10, then increasing for longer maturities. This pattern reflects the pattern for consumption growth, which declines at an increasing rate until $t = 10$, then at a decreasing rate after that.

Foreseen jump in τ_c. Figure 11.9.4 portrays the response to a foreseen increase in the consumption tax. As we have remarked, with an inelastic labor supply, the Euler equation (11.6.2) and the other equilibrium conditions show that *constant* consumption taxes do not distort decisions, but that anticipated *changes* in them do. Indeed, (11.6.2) or (11.6.3) indicates that a foreseen increase in τ_{ct} (i.e., a decrease in $\frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}$) operates like an *increase* in τ_{kt}. Notice that while all variables in Figure 11.9.4 eventually return to their initial steady-state values, the anticipated increase in τ_{ct} leads to an immediate jump in consumption at time 0, followed by a consumption binge that sends the capital stock downward until the date $t = T = 10$, at which τ_{ct} rises. The fall in capital causes $\bar{R}$ to rise over time, which via (11.6.9) requires the growth rate of consumption to rise until $t = T$. The jump in τ_c at $t = T = 10$ causes $\bar{R}$ to be depressed below 1, which via (11.6.9) accounts for the drastic fall in consumption at $t = 10$. From date $t = T$ onward, the effects of the *anticipated* distortion stemming from the fluctuation in τ_{ct} are over, and the economy is governed by the transient dynamic response associated with a capital stock that is now below the appropriate terminal steady-state capital stock. From date T onward, capital must rise. That requires austerity: consumption plummets at date $t = T = 10$. As the interest rate gradually falls, consumption grows at a diminishing rate along the path toward the terminal steady state.

Foreseen jump in τ_{kt}. For the two γ values 2 and .2, Figure 11.9.5 shows the response to a foreseen permanent jump in τ_{kt} at $t = T = 10$. Because the path of government expenditures is held fixed, the increase in τ_{kt} is accompanied by a reduction in the present value of lump-sum taxes that leaves the government budget balanced. The increase in τ_{kt} has effects that precede it. Capital starts declining immediately due to a rise in current consumption and a growing flow of consumption. The after-tax gross rate of return on capital starts rising at

$t = 0$ and increases until $t = 9$. It falls precipitously at $t = 10$ (see formula (11.6.8e) because of the foreseen jump in τ_k. Thereafter, $\bar{R}$ rises, as required by the transition dynamics that propel k_t toward its new lower steady state. Consumption is lower in the new steady state because the new lower steady-state capital stock produces less output. Consumption is smoother when $\gamma = 2$ than when $\gamma = .2$. Alterations in $\bar{R}$ accompany effects of the tax increase at $t = 10$ on consumption at earlier and later dates.

So far we have explored consequences of foreseen once-and-for-all changes in government policy. Next we describe some experiments in which there is a foreseen one-time change in a policy variable (a "pulse").

Foreseen one-time pulse in g_{10}. Figure 11.9.6 shows the effects of a foreseen one-time increase in g_t at date $t = 10$ that is financed entirely by alterations in lump sum taxes. Consumption drops immediately, then falls further over time in anticipation of the one-time surge in g. Capital is accumulated before $t = 10$. At $t = T = 10$, capital jumps downward because the government consumes it. The reduction in capital is accompanied by a jump in $\bar{R}$ above its steady-state value. The gross return $\bar{R}$ then falls toward its steady rate level and consumption rises at a diminishing rate toward its steady-state value. This experiment highlights what again looks like a version of a permanent income theory response to a foreseen decrease in the resources available for the public to spend (that is what the increase in g is about), with effects that are modified by the general equilibrium adjustments of the gross return $\bar{R}$.

11.10. Linear approximation

The present model is simple enough that it is very easy to apply the shooting algorithm. But for models with larger state spaces, it can be more difficult to apply the method. For those models, a frequently used procedure is to obtain a linear or log linear approximation around a steady state of the difference equation for capital, then to solve it to get an approximation of the dynamics in the vicinity of that steady state. The present model is a good laboratory for illustrating how to construct linear approximations. In addition to providing an easy way to approximate a solution, the method illuminates important features

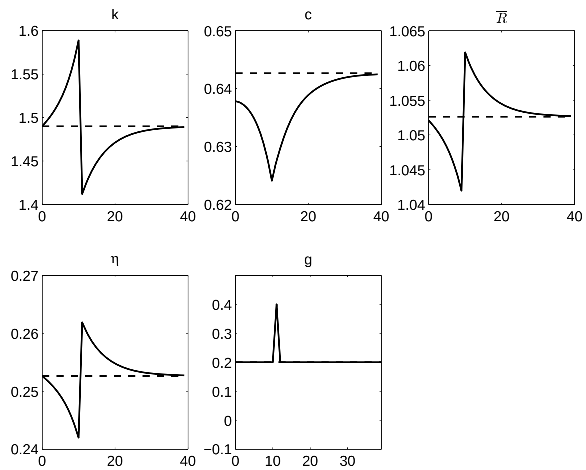

Figure 11.9.6: Response to foreseen one-time pulse increase in g at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, g$.

of the solution by partitioning it into two parts:[14] (1) a "feedback" part that portrays the transient response of the system to an initial condition k_0 that deviates from an asymptotic steady state, and (2) a "feedforward" part that shows the current effects of foreseen tax rates and expenditures.[15]

To obtain a linear approximation, perform the following steps:[16]

1. Set the government policy $z_t = \overline{z}$, a constant level. Solve $H(\overline{k}, \overline{k}, \overline{k}, \overline{z}, \overline{z}) = 0$ for a steady-state $\overline{k}$.

2. Obtain a first-order Taylor series approximation around $(\overline{k}, \overline{z})$:

$$\begin{aligned} &H_{k_t}\left(k_t - \overline{k}\right) + H_{k_{t+1}}\left(k_{t+1} - \overline{k}\right) + H_{k_{t+2}}\left(k_{t+2} - \overline{k}\right) \\ &+ H_{z_t}\left(z_t - \overline{z}\right) + H_{z_{t+1}}\left(z_{t+1} - \overline{z}\right) = 0 \end{aligned} \tag{11.10.1}$$

3. Write the resulting system as

$$\phi_0 k_{t+2} + \phi_1 k_{t+1} + \phi_2 k_t = A_0 + A_1 z_t + A_2 z_{t+1} \tag{11.10.2}$$

[14] Hall (1971) employed linear approximations to exhibit some of this structure.

[15] Vector autoregressions embed the consequences of both backward-looking (transient) and forward-looking (foresight) responses to government policies.

[16] For an extensive treatment of lag operators and their uses, see Sargent (1987a).

or

$$\phi(L)\, k_{t+2} = A_0 + A_1 z_t + A_2 z_{t+1} \tag{11.10.3}$$

where L is the lag operator (also called the backward shift operator) defined by $Lx_t = x_{t-1}$. Factor the characteristic polynomial on the left as

$$\phi(L) = \phi_0 + \phi_1 L + \phi_2 L^2 = \phi_0 (1 - \lambda_1 L)(1 - \lambda_2 L). \tag{11.10.4}$$

For most of our problems, it will turn out that one of the λ_i's exceeds unity and that the other is less than unity. We shall therefore adopt the convention that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. At this point, we ask the reader to accept that the values of λ_i split in this way. We discuss why they do so in section 11.10.2. Notice that equation (11.10.4) implies that $\phi_2 = \lambda_1 \lambda_2 \phi_0$. To obtain the factorization (11.10.4), we proceed as follows. Note that $(1 - \lambda_i L) = -\lambda_i \left(L - \frac{1}{\lambda_i}\right)$. Thus,

$$\phi(L) = \lambda_1 \lambda_2 \phi_0 \left(L - \frac{1}{\lambda_1}\right)\left(L - \frac{1}{\lambda_2}\right) = \phi_2 \left(L - \frac{1}{\lambda_1}\right)\left(L - \frac{1}{\lambda_2}\right) \tag{11.10.5}$$

because $\phi_2 = \lambda_1 \lambda_2 \phi_0$. Equation (11.10.5) identifies $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ as the zeros of the polynomial $\phi(\zeta)$, i.e., $\lambda_i = \zeta_0^{-1}$ where $\phi(\zeta_0) = 0$.[17] We want to operate on both sides of (11.10.3) with the inverse of $(1 - \lambda_1 L)$, but that inverse is unstable backward (i.e., the power series $\sum_{j=0}^{\infty} \lambda_1^j L^j$ has coefficients that diverge in higher powers of L). Fortunately $(1-\lambda_1 L)$ has a stable inverse in the *forward* direction, i.e., in terms of the forward shift operator L^{-1}.[18] In particular, notice that $(1 - \lambda_1 L) = -\lambda_1 L(1 - \lambda_1^{-1} L^{-1})$.[19] Using this result and $\phi_2 = \lambda_1 \lambda_2 \phi_0$, we can rewrite $\phi(L)$ as[20]

$$\phi(L) = -\frac{1}{\lambda_2} \phi_2 \left(1 - \lambda_1^{-1} L^{-1}\right)(1 - \lambda_2 L)\, L.$$

Represent equation (11.10.2) as

$$-\lambda_2^{-1} \phi_2 L \left(1 - \lambda_1^{-1} L^{-1}\right)(1 - \lambda_2 L)\, k_{t+2} = A_0 + A_1 z_t + A_2 z_{t+1}. \tag{11.10.6}$$

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17 The Matlab roots command `roots(phi)` finds zeros of polynomials, but you must arrange the polynomial as $\phi = [\,\phi_2 \quad \phi_1 \quad \phi_0\,]$.

18 See appendix A of chapter 2.

19 Notice that we can express $(1-\lambda_1 L)^{-1} = -\lambda_1^{-1} L^{-1}(1-\lambda^{-1} L^{-1})^{-1}$ as $-\lambda_1^{-1} L^{-1} \sum_{j=0}^{\infty} \lambda_1^{-j} L^{-j}$.

20 Justifications for these steps are described at length in Sargent (1987a) and with rigor in Gabel and Roberts (1973).

Operate on both sides of (11.10.6) by $-(\phi_2/\lambda_2)^{-1}(1-\lambda_1^{-1}L^{-1})^{-1}$ to get the following representation:[21]

$$(1-\lambda_2 L)\,k_{t+1} = \frac{-\lambda_2\phi_2^{-1}}{1-\lambda_1^{-1}L^{-1}}\left[A_0 + A_1 z_t + A_2 z_{t+1}\right]. \tag{11.10.7}$$

This concludes the procedure.

Equation (11.10.7) is our linear approximation to the equilibrium k_t sequence. It can be expressed as

$$k_{t+1} = \lambda_2 k_t - \lambda_2\phi_2^{-1}\sum_{j=0}^{\infty}(\lambda_1)^{-j}\left[A_0 + A_1 z_{t+j} + A_2 z_{t+j+1}\right]. \tag{11.10.8}$$

We can summarize the process of obtaining this approximation as solving stable roots backward and unstable roots forward. Solving the unstable root forward imposes the terminal condition (11.5.3). This step corresponds to the step in the shooting algorithm that adjusts the initial investment rate to ensure that the capital stock eventually approaches the terminal steady-state capital stock.[22]

The term $\lambda_2 k_t$ is sometimes called the "feedback" part. The coefficient λ_2 measures the transient response rate, in particular, the rate at which capital returns to a steady state when it starts away from it. The remaining terms on the right side of (11.10.8) are sometimes called the "feedforward" parts. They depend on the infinite *future* of the exogenous z_t (which for us contain the components of government policy) and measure the effect on the current capital stock k_t of perfectly foreseen paths of fiscal policy. The decay parameter λ_1^{-1} measures the rate at which expectations of future fiscal policies are discounted in terms of their effects on current investment decisions. To a linear approximation, every rational expectations model has embedded within it both feedforward and feedback parts. The decay parameters λ_2 and λ_1^{-1} of the feedback and feedforward parts are determined by the roots of the characteristic polynomial. Equation (11.10.8) thus neatly exhibits the mixture of the pure foresight and the pure transient responses that are reflected in our examples in Figures 11.9.1 through 11.9.5. The feedback part captures the purely transient response and the feedforward part captures the perfect foresight component.

[21] We have thus solved the stable root backward and the unstable root forward.

[22] The invariant subspace methods described in chapter 5 are also all about solving stable roots backward and unstable roots forward.

11.10.1. Relationship between the λ_i's

It is a remarkable fact that if an equilibrium solves a planning problem, then the roots are linked by $\lambda_1 = \frac{1}{\beta\lambda_2}$, where $\beta \in (0,1)$ is the planner's discount factor.[23] In this case, the feedforward decay rate $\lambda_1^{-1} = \beta\lambda_2$. Therefore, when the equilibrium allocation solves a planning problem, one of the λ_i's is less than $\frac{1}{\sqrt{\beta}}$ and the other exceeds $\frac{1}{\sqrt{\beta}}$ (this follows because $\lambda_1\lambda_2 = \frac{1}{\beta}$).[24] From this it follows that one of the λ_i's, say λ_1, satisfies $\lambda_1 > \frac{1}{\sqrt{\beta}} > 1$ and the other λ_i, say λ_2 satisfies $\lambda_2 < \frac{1}{\sqrt{\beta}}$. Thus, for β close to 1, the condition $\lambda_1\lambda_2 = \frac{1}{\beta}$ almost implies our earlier assumption that $\lambda_1\lambda_2 = 1$, but not quite. Having $\lambda_2 < \frac{1}{\sqrt{\beta}}$ is sufficient to allow our linear approximation for k_t to satisfy $\sum_{t=0}^{\infty} \beta^t k_t^2 < +\infty$ for all z_t sequences that satisfy $\sum_{t=0}^{\infty} \beta^t z_t \cdot z_t < +\infty$.

A relationship between the feedforward and feedback decay rates appears evident in the experiments depicted in Figure 11.9.2. In particular, when the utility curvature parameter $\gamma = 2$, the rates at which future events are discounted in influencing outcomes before $t = 10$ and the rates of convergence back to steady state after $t = 10$ are both lower than when $\gamma = .2$.

11.10.2. Conditions for existence and uniqueness

For equilibrium allocations that do not solve planning problems, it ceases to be true that $\lambda_1\lambda_2 = \frac{1}{\beta}$. In this case, the location of the zeros of the characteristic polynomial can be used to assess the existence and uniqueness of an equilibrium up to a linear approximation. If both λ_i's exceed $\frac{1}{\sqrt{\beta}}$, there exists no equilibrium allocation for which $\sum_{t=0}^{\infty} \beta^t k_t^2 < \infty$. If both λ_i's are less than $\frac{1}{\sqrt{\beta}}$, there exists a continuum of equilibria that satisfy that inequality. If the λ_i's split, with one exceeding and the other being less than $\frac{1}{\sqrt{\beta}}$, there exists a unique equilibrium.

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23 See Sargent (1987a, chap. XI) for a discussion.

24 Notice that this means that the solution (11.10.8) remains valid for those divergent z_t processes, provided that they satisfy $\sum_{t=0}^{\infty} \beta^t z_{jt}^2 < +\infty$.

11.10.3. Once-and-for-all jumps

Next we specialize (11.10.7) to capture some examples of foreseen policy changes that we have studied above. Consider the special case treated by Hall (1971) in which the jth component of z_t follows the path

$$z_{jt} = \begin{cases} 0 & \text{if } t \leq T-1 \\ \overline{z}_j & \text{if } t \geq T \end{cases} \tag{11.10.9}$$

We define

$$\begin{aligned} v_t &\equiv \sum_{i=0}^{\infty} \lambda_1^{-i} z_{t+i,j} \\ &= \begin{cases} \frac{\left(\frac{1}{\lambda_1}\right)^{T-t} \overline{z}_j}{1-\left(\frac{1}{\lambda_1}\right)} & \text{if } t \leq T \\ \frac{1}{1-\left(\frac{1}{\lambda_1}\right)} \overline{z}_j & \text{if } t \geq T \end{cases} \end{aligned} \tag{11.10.10}$$

$$\begin{aligned} h_t &\equiv \sum_{i=0}^{\infty} \left(\frac{1}{\lambda_1}\right)^i z_{t+i+1,j} \\ &= \begin{cases} \frac{\left(\frac{1}{\lambda_1}\right)^{T-(t+1)} \overline{z}_j}{1-\left(\frac{1}{\lambda_1}\right)} & \text{if } t \leq T-1 \\ \frac{1}{1-\left(\frac{1}{\lambda_1}\right)} \overline{z}_j & \text{if } t \geq T-1. \end{cases} \end{aligned} \tag{11.10.11}$$

Using these formulas, let the vector z_t follow the path

$$z_t = \begin{cases} 0 & \text{if } t \leq T-1 \\ \overline{z} & \text{if } t \geq T \end{cases}$$

where $\overline{z}$ is a vector of constants. Then applying (11.10.10) and (11.10.11) to (11.10.7) gives the formulas

$$k_{t+1} = \begin{cases} \lambda_2 k_t - \frac{(\phi_0 \lambda_1)^{-1} A_0}{1-\left(\frac{1}{\lambda_1}\right)} - \frac{(\phi_0 \lambda_1)^{-1} \left(\frac{1}{\lambda_1}\right)^{T-t}}{1-\left(\frac{1}{\lambda_1}\right)} (A_1 + A_2 \lambda_1) \overline{z} & \text{if } t \leq T-1 \\ \lambda_2 k_t - \frac{(\phi_0 \lambda_1)^{-1}}{1-\left(\frac{1}{\lambda_1}\right)} [A_0 + (A_1 + A_2) \overline{z}] & \text{if } t \geq T. \end{cases}$$

11.10.4. Simplification of formulas

These formulas can be simultaneously generalized and simplified by using the following trick. Let z_t be governed by the state-space system

$$\bar{x}_{t+1} = A_x \bar{x}_t \tag{11.10.12a}$$

$$z_t = G_z \bar{x}_t, \tag{11.10.12b}$$

with initial condition $\bar{x}_0$ given. In chapter 2, we saw that many finite-dimensional linear time series models could be represented in this form, so that we are accommodating a large class of tax and expenditure processes. Then notice that

$$\left(\frac{A_1}{1-\lambda_1^{-1}L^{-1}}\right) z_t = A_1 G_z \left(I - \lambda_1^{-1} A_x\right)^{-1} \bar{x}_t \tag{11.10.13a}$$

$$\left(\frac{A_2}{1-\lambda_1^{-1}L^{-1}}\right) z_{t+1} = A_2 G_z \left(I - \lambda_1^{-1} A_x\right)^{-1} A_x \bar{x}_t \tag{11.10.13b}$$

Substituting these expressions into (11.10.8) gives

$$\begin{aligned} k_{t+1} =& \lambda_2 k_t - \lambda_2 \phi_2^{-1} \big[(1-\lambda_1^{-1})^{-1} A_0 + A_1 G_z (I - \lambda_1^{-1} A_x)^{-1} \bar{x}_t \\ &+ A_2 G_z (I - \lambda_1^{-1} A_x)^{-1} A_x \bar{x}_t \big]. \end{aligned} \tag{11.10.13c}$$

Taken together, system (11.10.13) gives a complete description of the joint evolution of the exogenous state variables $\bar{x}_t$ driving z_t (our government policy variables) and the capital stock. System (11.10.13) concisely displays the cross-equation restrictions that are the hallmark of rational expectations models: nonlinear functions of the parameter occurring in G_z, A_x in the law of motion for the exogenous processes appear in the equilibrium representation (11.10.13c) for the endogenous state variables.

We can easily use the state space system (11.10.13) to capture the special case (11.10.9). In particular, to portray $\bar{x}_{j,t+1} = \bar{x}_{j+1,t}$, set the $T \times T$ matrix A to be

$$A = \begin{bmatrix} 0_{T-1\times 1} & I_{T-1\times T-1} \\ 0_{1\times T-1} & 1 \end{bmatrix} \tag{11.10.14}$$

and take the initial condition $\bar{x}_0 = [\,0 \quad 0 \quad \cdots \quad 0 \quad 1\,]'$. To represent an element of z_t that jumps once and for all from 0 to $\overline{z}_j$ at $T=0$, set the jth component of G_z equal to $G_{zj} = [\,\overline{z}_j \quad 0 \cdots \quad 0\,]$.

11.10.5. A one-time pulse

We can modify the transition matrix (11.10.14) to model a one-time pulse in a component of z_t that occurs at and only at $t = T$. To do this, we simply set

$$A = \begin{bmatrix} 0_{T-1\times 1} & I_{T-1\times T-1} \\ 0_{1\times T-1} & 0 \end{bmatrix}. \tag{11.10.15}$$

11.10.6. Convergence rates and anticipation rates

Equation (11.10.8) shows that up to a linear approximation, the feedback coefficient λ_2 equals the geometric rate at which the model returns to a steady state after a transient displacement away from a steady state. For our benchmark values of our other parameters $\delta = .2, \beta = .95, \alpha = .33$ and all distorting taxes set to zero, we can compute that λ_2 is the following function of the utility curvature parameter γ that appears in $u(c) = (1-\gamma)^{-1}c^{1-\gamma}$:[25]

$$\lambda_2 = \frac{\gamma}{a_1\gamma^{-1} + a_2 + a_3(\gamma^{-1} + a_4\gamma^{-2} + a_5)^{\frac{1}{2}}} \tag{11.10.16}$$

where $a_1 = .975, a_2 = .0329, a_3 = .0642, a_4 = .00063, a_5 = .0011$. Figure 11.10.1 plots this function. When $\gamma = 0$, the period utility function is linear and the household's willingness to substitute consumption over time is unlimited. In this case, $\lambda_2 = 0$, which means that in response to a perturbation of the capital stock away from a steady state, the return to a steady state is immediate. Furthermore, as mentioned above, because there are no distorting taxes in the initial steady state, we know that $\lambda_1 = \frac{1}{\beta\lambda_2}$, so that according to (11.10.8), the feedforward response to future z's is a discounted sum that decays at rate $\beta\lambda_2$. Thus, when $\gamma = 0$, anticipations of *future* z's have no effect on current k. This is the other side of the coin of the immediate adjustment associated with the feedback part.

As the curvature parameter γ increases, λ_2 increases, more rapidly at first, more slowly later. As γ increases, the household values a smooth consumption path more and more highly. Higher values of γ impart to the equilibrium capital sequence both a more sluggish feedback response and a feedforward response that puts relatively more weight on prospective values of the z's in the more distant future.

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[25] We used the Matlab symbolic toolkit to compute this expression.

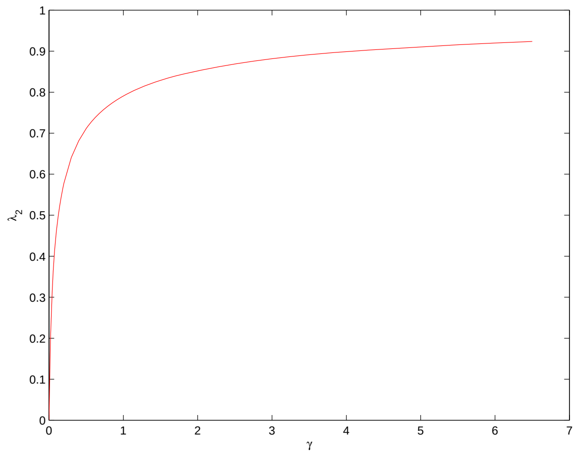

Figure 11.10.1: Feedback coefficient λ_2 as a function γ, evaluated at $\alpha = .33, \beta = .95, \delta = .2, g = .2$.

11.10.7. A remark about accuracy: Euler equation errors

It is important to estimate the accuracy of approximations. One simple diagnostic tool is to take a candidate solution for a sequence c_t, k_{t+1}, substitute them into the two Euler equations (11.12.1) and (11.12.2), and call the deviations between the left sides and the right sides the Euler equation errors.[26] An accurate method makes these errors small.[27]

[26] For more about this method, see Den Haan and Marcet (1994) and Judd (1998).

[27] Calculating Euler equation errors, but for a different purpose, goes back a long time. In chapter 2 of *The General Theory of Interest, Prices, and Money*, John Maynard Keynes noted that plugging in *data* (not a candidate *simulation*) into (11.12.2) gives big residuals. Keynes therefore assumed that (11.12.2) does not hold ("workers are off their labor supply curve").

11.11. Growth

It is straightforward to alter the model to allow for exogenous growth. We modify the production function to be

$$Y_t = F(K_t, A_t n_t) \tag{11.11.1}$$

where Y_t is aggregate output, N_t is total employment, A_t is labor-augmenting technical change, and $F(K, AN)$ is the same linearly homogeneous production function as before. We assume that A_t follows the process

$$A_{t+1} = \mu_{t+1} A_t \tag{11.11.2}$$

and will usually but not always assume that $\mu_{t+1} = \overline{\mu} > 1$. We exploit the linear homogeneity of (11.11.1) to express the production function as

$$y_t = f(k_t) \tag{11.11.3}$$

where $f(k) = F(k, 1)$ and now $k_t = \frac{K_t}{n_t A_t}, y_t = \frac{Y_t}{n_t A_t}$. We say that k_t and y_t are measured per unit of "effective labor" $A_t n_t$. We also let $c_t = \frac{C_t}{A_t n_t}$ and $g_t = \frac{G_t}{A_t n_t}$ where C_t and G_t are total consumption and total government expenditures, respectively. We consider the special case in which labor is inelastically supplied. Then feasibility can be summarized by the following modified version of (11.6.1):

$$k_{t+1} = \mu_{t+1}^{-1}[f(k_t) + (1-\delta)k_t - g_t - c_t]. \tag{11.11.4}$$

Noting that per capita consumption is $c_t A_t$, we obtain the following counterpart to equation (11.6.3):

$$\begin{aligned} u'(c_t A_t) =& \beta u'(c_{t+1} A_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \\ & [(1-\tau_{kt+1})(f'(k_{t+1}) - \delta) + 1]. \end{aligned} \tag{11.11.5}$$

We assume the power utility function $u'(c) = c^{-\gamma}$, which makes the Euler equation become

$$(c_t A_t)^{-\gamma} = \beta (c_{t+1} A_{t+1})^{-\gamma} \bar{R}_{t+1},$$

where $\bar{R}_{t+1}$ continues to be defined by (11.6.8e), except that now k_t is capital per effective unit of labor. The preceding equation can be represented as

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta \mu_{t+1}^{-\gamma} \bar{R}_{t+1}. \tag{11.11.6}$$

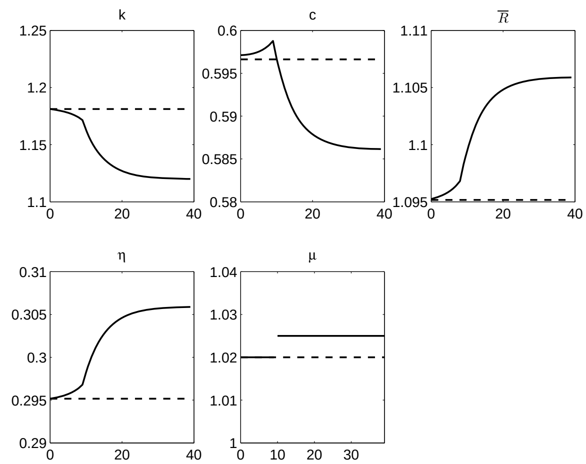

Figure 11.11.1: Response to foreseen once-and-for-all increase in rate of growth of productivity μ at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \mu$, where now k, c are measured in units of effective unit of labor.

In a steady state, $c_{t+1} = c_t$. Then the steady-state version of the Euler equation (11.11.5) is

$$1 = \mu^{-\gamma}\beta[(1-\tau_k)(f'(k)-\delta)+1], \tag{11.11.7}$$

which can be solved for the steady-state capital stock. It is easy to compute that the steady-state level of capital per unit of effective labor satisfies

$$f'(k) = \delta + \left(\frac{(1+\rho)\mu^{\gamma}-1}{1-\tau_k}\right) \tag{11.11.8}$$

and that

$$\bar{R} = (1+\rho)\mu^{\gamma}. \tag{11.11.9}$$

Equation (11.11.9) immediately shows that *ceteris paribus*, a jump in the rate of technical change raises $\bar{R}$.

Next we apply the shooting algorithm to compute equilibria. We augment the vector of forcing variables z_t by including μ_t, so that it becomes $z_t = [\, g_t \quad \tau_{kt} \quad \tau_{ct} \quad \mu_t \,]'$, where g_t is understood to be measured in effective units of labor, then proceed as before.

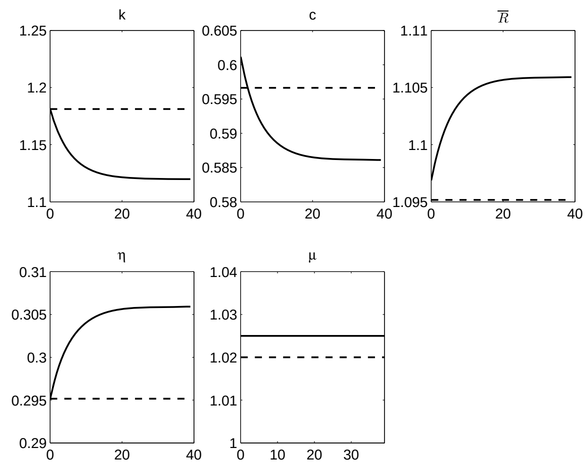

Figure 11.11.2: Response to increase in rate of growth of productivity μ at $t = 0$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \mu$, where now k, c are measured in units of effective unit of labor.

Foreseen jump in productivity growth at $t = 10$. Figure 11.11.1 shows effects of a permanent increase from 1.02 to 1.025 in the productivity gross growth rate μ_t at $t = 10$. This figure and also Figure 11.11.2 now measure c and k in effective units of labor. The steady-state Euler equation (11.11.7) guides main features of the outcomes, and implies that a permanent increase in μ will lead to a decrease in the steady-state value of capital per unit of effective labor. Because capital is more efficient, even with less of it, consumption per capita can be raised, and that is what individuals care about. Consumption jumps immediately because people are wealthier. The increased productivity of capital spurred by the increase in μ leads to an increase in the gross return $\bar{R}$. Perfect foresight makes the effects of the increase in the growth of capital precede it.

Immediate (unforeseen) jump in productivity growth at $t = 1$. Figure 11.11.2 shows effects of an immediate jump in μ at $t = 0$. It is instructive to compare these with the effects of the foreseen increase in Figure 11.11.1. In Figure 11.11.2, the paths of all variables are entirely dominated by the feedback part of the solution, while before $t=10$ those in Figure 11.11.1 have contributions from the feedforward part. The absence of feedforward effects makes the paths of all variables in Figure 11.11.2 smooth. Consumption per effective unit of labor jumps immediately then declines smoothly toward its steady state as the economy moves to a lower level of capital per unit of effective labor. The after-tax gross return $\bar{R}$ once again comoves with the consumption growth rate to verify the Euler equation (11.11.7).

11.12. Elastic labor supply

We now again shut down productivity growth by setting the gross productivity growth rate $\mu=1$, but we allow a possibly nonzero labor supply elasticity by specifying $U(c,1-n)$ to include a preference for leisure. Again, we let U_i be the partial derivative of U with respect to its ith argument. We assume an interior solution for $n \in (0,1)$. Now we have to carry along equilibrium conditions both for the intertemporal evolution of capital and for the labor-leisure choice. These are the two difference equations:

$$\begin{aligned}
&\frac{1}{(1+\tau_{ct})} U_1\big(F(k_t,n_t)+(1-\delta)k_t-g_t-k_{t+1},1-n_t\big)\\
&=\beta(1+\tau_{ct+1})^{-1} U_1\big(F(k_{t+1},n_{t+1})+(1-\delta)k_{t+1}-g_{t+1}-k_{t+2},1-n_{t+1}\big)\\
&\times [(1-\tau_{kt+1})(F_k(k_{t+1},n_{t+1})-\delta)+1]
\end{aligned} \tag{11.12.1}$$

$$\begin{aligned}
&\frac{U_2\big(F(k_t,n_t)+(1-\delta)k_t-g_t-k_{t+1},1-n_t\big)}{U_1\big(F(k_t,n_t)+(1-\delta)k_t-g_t-k_{t+1},1-n_t\big)}\\
&=\frac{(1-\tau_{nt})}{(1+\tau_{ct})} F_n(k_t,n_t).
\end{aligned} \tag{11.12.2}$$

The linear approximation method applies equally well to this more general setting with just one additional step. We obtain a linear approximation to this dynamical system by proceeding as follows. First, find steady-state values $(\overline{k},\overline{n})$ by solving the two steady-state versions of equations (11.12.1), (11.12.2). (Now $(\overline{k},\overline{n})$ are steady-state values of capital per person and labor supplied per person, respectively.) Then take the following linear approximations to

(11.12.1), (11.12.2), respectively, around the steady state:

$$\begin{aligned} &H_{k_t}(k_t-\overline{k})+H_{k_{t+1}}(k_{t+1}-\overline{k})+H_{n_{t+1}}(n_{t+1}-\overline{n})+H_{k_{t+2}}(k_{t+2}-\overline{k}) \\ &+H_{n_t}(n_t-\overline{n})+H_{z_t}(z_t-\overline{z})+H_{z_{t+1}}(z_{t+1}-\overline{z})=0 \end{aligned} \tag{11.12.3}$$

$$G_k(k_t-\overline{k})+G_{n_t}(n_t-\overline{n})+G_{k_{t+1}}(k_{t+1}-\overline{k})+G_z(z_t-\overline{z})=0 \tag{11.12.4}$$

Solve (11.12.4) for $(n_t-\overline{n})$ as functions of the remaining terms, substitute into (11.12.3) to get a version of equation (11.10.2), and proceed as before with a difference equation of the form (11.6.4).

11.12.1. Steady-state calculations

To compute a steady state for this version of the model, assume that government expenditures and all flat-rate taxes are constant over time. Steady-state versions of (11.12.1), (11.12.2) are

$$1=\beta[(1+(1-\tau_k)(F_k(\overline{k},\overline{n})-\delta)] \tag{11.12.5}$$

$$\frac{U_2(\overline{c},1-\overline{n})}{U_1(\overline{c},1-\overline{n})}=\frac{(1-\tau_n)}{(1+\tau_c)}F_n(\overline{k},\overline{n}) \tag{11.12.6}$$

and the steady state version of the feasibility condition (11.2.2) is

$$\overline{c}+\overline{g}+\delta\overline{k}=F(\overline{k},\overline{n}). \tag{11.12.7}$$

The linear homogeneity of $F(k,n)$ means that equation (11.12.5) by itself determines the steady-state capital-labor ratio $\frac{\overline{k}}{\overline{n}}$. In particular, where $\tilde{k}=\frac{k}{n}$, notice that $F(k,n)=nf(\tilde{k})$ and $F_k(k,n)=f'(\tilde{k})$. It is useful to use these facts to write (11.12.7) as

$$\frac{\overline{c}+\overline{g}}{\overline{n}}=f(\tilde{k})-\delta\tilde{k}. \tag{11.12.8}$$

Next, letting $\beta=\frac{1}{1+\rho}$, (11.12.5) can be expressed as

$$\delta+\frac{\rho}{(1-\tau_k)}=f'(\tilde{k}), \tag{11.12.9}$$

an equation that determines a steady-state capital-labor ratio $\tilde{k}$. An increase in $\frac{1}{(1-\tau_k)}$ decreases the capital-labor ratio, but the steady-state capital-labor ratio is independent of the steady state values of τ_c, τ_n. However, given the steady state value of the capital-labor ratio $\tilde{k}$, flat rate taxes on consumption and labor income influence the steady-state levels of consumption and labor via the steady state equations (11.12.6) and (11.12.7). Formula (11.12.6) reveals how both τ_c and τ_n distort the same labor-leisure margin.

If we define $\check{\tau}_c = \frac{\tau_n+\tau_c}{1+\tau_c}$ and $\check{\tau}_k = \frac{\tau_k}{1-\tau_k}$, then it follows that $\frac{(1-\tau_n)}{(1+\tau_c)} = 1 - \check{\tau}_c$ and $\frac{1}{(1-\tau_k)} = 1 + \check{\tau}_k$. The wedge $1 - \check{\tau}_c$ distorts the steady-state labor-leisure decision via (11.12.6) and the wedge $1 + \check{\tau}_k$ distorts the steady-state capital-labor ratio via (11.12.9).

11.12.2. Some experiments

To make things concrete, we use the following preference specification popularized by Hansen (1985) and Rogerson (1988):

$$U(c, 1-n) = \ln c + B(1-n) \tag{11.12.10}$$

where we set B substantially greater than 1 to assure an interior solution $n \in (0,1)$ for labor supply. In particular, we set $B = 3$ in the experiments below. In terms of steady states, equation (11.12.6) becomes

$$B\overline{c} = \frac{(1-\tau_n)}{(1+\tau_c)}\big[f(\tilde{k}) - \tilde{k}f'(\tilde{k})\big]. \tag{11.12.11}$$

It is useful to collect equations (11.12.9), (11.12.11), and (11.12.8), into the following system that recursively determines steady-state outcomes for $\tilde{k}, \overline{c}$, and $\overline{n}$ in the experiments to follow:

$$\delta + \frac{\rho}{(1-\tau_k)} = f'(\tilde{k}) \tag{11.12.12}$$

$$B\overline{c} = \frac{(1-\tau_n)}{(1+\tau_c)}\big[f(\tilde{k}) - \tilde{k}f'(\tilde{k})\big] \tag{11.12.13}$$

$$\overline{c} = \overline{n}\left(f(\tilde{k}) - \delta\tilde{k}\right) - \overline{g}. \tag{11.12.14}$$

Unforeseen jump in g. Figure 11.12.1 displays the consequences of an unforeseen and permanent jump in g at $t = 0$, financed entirely by adjustments in lump sum taxes. Equation (11.12.12) determines $\tilde{k}$, which is unaltered. Equation (11.12.13) then implies that $\overline{c}$ is unaltered. Equation (11.12.14) determines $\overline{k}$ and $\overline{n}$, their ratio having been determined by (11.12.12). The consequences of an unforeseen increase in g differ markedly from those analyzed above for the case in which the labor elasticity is zero. Then, the consequence was immediately and permanently to *lower* consumption per capita by the amount of the *increase* in government purchases per capita. Now the effect is to leave unaltered both steady state consumption per capita and the steady state capital/labor ratio. This is accomplished by raising the steady state levels of both capital and the labor supply. Thus, now the consequence of the increase in g is to 'grow the economy' enough eventually to leave consumption unaffected despite the increase in g.

These asymptotic outcomes immediately drop out of our steady state equations. The increase in g is accompanied by increases in k and n that leave the steady state capital/labor ratio unaltered, as required by equation (11.12.9). Equation (11.12.11) then dictates that steady-state consumption per capita also remain unaltered.

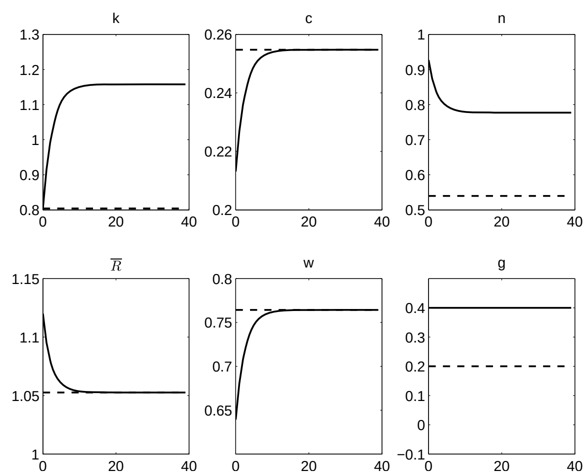

Figure 11.12.1: Elastic labor supply: response to unforeseen increase in g at $t = 0$. From left to right, top to bottom: $k, c, n, \bar{R}, w, g$. The dashed line is the original steady state.

Unforeseen jump in τ_n. Figure 11.12.2 shows outcomes from an unforeseen increase in the marginal tax rate on labor τ_n. Here the effect is to shrink the economy. As required by equation (11.12.9), the steady state capital labor ratio is unaltered. But equation (11.12.11) then requires that steady state consumption per capita must fall in response to the increase in τ_n. Both labor supplied n and capital fall in the new steady state.

Countervailing forces contributing to Prescott (2002) The preceding two experiments isolate forces that Prescott (2002) combines to reach his conclusion that Europe's economic activity has been depressed relative to the U.S. because its tax rates have been higher. Prescott's numerical calculations activate the forces that shrink the economy in our second experiment that increases τ_n while shutting down the force to grow the economy implied by a larger g. In particular, Prescott assumes that cross-country outcomes are generated by second experiment, with lump sum transfers being used to rebate the revenues raised from the larger labor tax rate τ_n that he estimates to prevail in Europe. If instead one assumes that higher taxes in Europe are used to pay for larger per capita government purchases, then forces to grow the economy identified in our first experiment are unleashed, making the adverse consequences for the level of economic activity of larger g, τ_n pairs in Europe become much smaller than Prescott calculated.

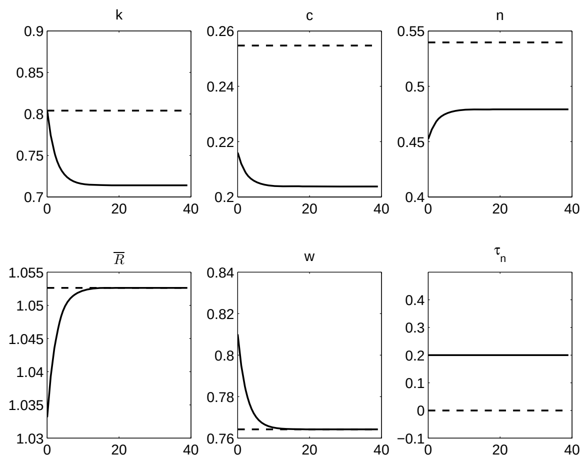

Figure 11.12.2: Elastic labor supply: response to unforeseen increase in τ_n at $t = 0$. From left to right, top to bottom: $k, c, n, \bar{R}, w, \tau_n$. The dashed line is the original steady state.

Foreseen jump in τ_n. Figure 11.12.3 describes consequences of a foreseen increase in τ_n that occurs at time $t = 10$. While the ultimate effects are identical with those described in the preceding experiment, transient outcomes differ. The immediate effect of the foreseen increase in τ_n is to spark a boom in employment and capital accumulation, while leaving consumption unaltered before time $t = 10$. People work more in response to the anticipation that rewards to working will decrease permanently at $t = 10$. Thus, the foreseen increase in τ_n sparks a temporary employment and investment boom.

To interpret what is going on here, we begin by noting that with preference specification (11.12.10), the following system of difference equations determines the dynamics of equilibrium allocations:

$$c_{t+1} = \beta \bar{R}_{t+1} c_t \tag{11.12.15a}$$

$$\bar{R}_{t+1} = \frac{1+\tau_{ct}}{1+\tau_{ct+1}} \left[1 + (1-\tau_{kt+1})(f'\big(k_{t+1}/n_{t+1}\big) - \delta)\right] \tag{11.12.15b}$$

$$Bc_t = \frac{(1-\tau_{nt})}{(1+\tau_{ct})} F_n(k_t, n_t) \tag{11.12.15c}$$

$$k_{t+1} = F(k_t, n_t) + (1-\delta)k_t - g_t - c_t \tag{11.12.15d}$$

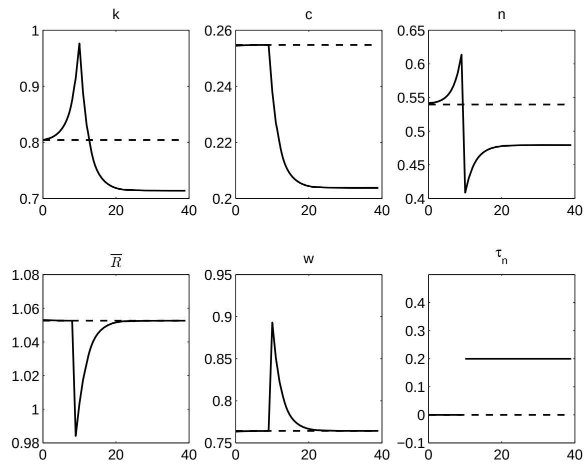

Figure 11.12.3: Elastic labor supply: response to foreseen increase in τ_n at $t = 10$. From left to right, top to bottom: $k, c, n, \bar{R}, w, \tau_n$. The dashed line is the original steady state.

These equations teach us that the foreseen increase in τ_n sparks a substantial rearrangement in how the household distributes its work over time. The effect of the permanent increase in τ_n at $t = 10$ is to reduce the after-tax wage from $t = 10$ onward, though initially the real wage falls by less than the decrease in $(1 - \tau_n)$ because of the increase in the capital labor ratio induced by the drastic fall in n at $t = 10$. Eventually, as the pre-tax real wage w returns to its initial value, the real wage falls by the entire amount of the decrease in $(1 - \tau_n)$. The decrease in the after-tax wage after $t = 10$ makes it relatively more attractive to work before $t = 10$. As a consequence, n_t rises above its initial steady state value before $t = 10$. The household uses the extra income to purchase enough capital to keep the capital-labor ratio and consumption equal to their respective initial steady state values for the first nine periods. This force increases n_t in the periods before $t = 10$. The effect of the build up of capital in the periods before $t = 0$ is to attenuate the decrease in the after tax wage that occurs at $t = 10$ because the equilibrium marginal product of labor has been raised higher than it would have been if capital had remained at its initial steady state value. From $t = 10$ onward, the capital stock is drawn down and the marginal product of labor falls, making the pre-tax real wage eventually return to its value in the initial steady state.

Mertens and Ravn (2011) use these to offer an interpretation of contractionary contributions that the Reagan tax cuts made to the U.S. recession of the early 1980s.

11.13. A two-country model

This section describes a two country version of the basic model of this chapter. The model has a structure similar to ones used in the international real business cycle literature (e.g., Backus, Kehoe, and Kydland (1992)) and is in the spirit of an analysis of distorting taxes by Mendoza and Tesar (1998), though our presentation differs from theirs. We paste two countries together and allow them freely to trade goods, claims on future goods, but not labor. We shall have to be careful in how we specify taxation of earnings by non residents.

There are now two countries like the one in previous sections. Objects for the first country are denoted without asterisks, while those for the second country bear asterisks. There is international trade in goods, capital, and debt, but not in labor. We assume that leisure generates utility in neither country. Preferences over consumption streams in the two countries are ordered by $\sum_{t=0}^{\infty} \beta^t u(c_t)$ and $\sum_{t=0}^{\infty} \beta^t u(c_t^*)$, respectively, where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$. Feasibility for the world economy is

$$(c_t+c_t^*)+(g_t+g_t^*)+(k_{t+1}-(1-\delta)k_t)+(k_{t+1}^*-(1-\delta)k_t^*) = f(k_t)+f(k_t^*) \tag{11.13.1}$$

where $f(k) = Ak^\alpha$ with $\alpha \in (0,1)$.

A consumer in country one can hold capital in either country, but pays taxes on rentals from foreign holdings of capital at the rate set by the foreign country. At time 0, residents in both countries can purchase consumption at date t at a common Arrow-Debreu price q_t. Let $\tilde{k}_t$ be capital in country 2 held by a representative consumer of country 1. Temporarily, in this paragraph only, let k_t denote the amount of domestic capital owned by the domestic consumer. (In all other paragraphs of our exposition of the two-country model, k_t denotes the amount of capital in country 1.) Let B_t^f be the amount of time t goods that the representative domestic consumer raises by issuing a one-period IOU to the representative foreign consumer; so $B_t^f > 0$ indicates that the domestic consumer is borrowing from abroad at t and $B_t^f < 0$ indicates that the domestic consumer is lending abroad at t. For $t \geq 1$ let $R_{t-1,t}$ be the gross return on a one-period loan from period $t-1$ to period t. Define $R_{-1,0} \equiv 1$ and let the domestic consumer's initial debt to the foreign consumer be $R_{-1,0} B_{-1}^f$. We assume that returns on loans are not taxed by either country. The budget constraint of a country 1 consumer is

$$\begin{aligned}
&\sum_{t=0}^{\infty} q_t \left(c_t + (k_{t+1} - (1-\delta)k_t) + (\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t) + R_{t-1,t} B_{t-1}^f \right) \\
&\leq \sum_{t=0}^{\infty} q_t \left((\eta_t - \tau_{kt}(\eta_t - \delta))k_t + (\eta_t^* - \tau_{kt}^*(\eta_t^* - \delta))\tilde{k}_t + (1-\tau_{nt})w_t n_t - \tau_{ht} + B_t^f \right).
\end{aligned} \tag{11.13.2}$$

A no-arbitrage condition for k_0 and $\tilde{k}_0$ is

$$(1-\tau_{k0})\eta_0 + \delta\tau_{k0} = (1-\tau_{k0}^*)\eta_0^* + \delta\tau_{k0}^*.$$

No-arbitrage conditions for k_t and $\tilde{k}_t$ for $t \geq 1$ imply

$$\begin{aligned}
q_{t-1} &= \left[(1-\tau_{kt})(\eta_t - \delta) + 1\right] q_t \\
q_{t-1} &= \left[(1-\tau_{kt}^*)(\eta_t^* - \delta) + 1\right] q_t,
\end{aligned} \tag{11.13.3}$$

which together imply that after-tax rental rates on capital are equalized across the two countries:

$$(1-\tau_{kt}^*)(\eta_t^* - \delta) = (1-\tau_{kt})(\eta_t - \delta). \tag{11.13.4}$$

No arbitrage conditions for B_t^f for $t \geq 0$ are $q_t = q_{t+1} R_{t,t+1}$, which implies that

$$q_{t-1} = q_t R_{t-1,t} \tag{11.13.5}$$

for $t \geq 1$.

Since domestic capital, foreign capital, and consumption loans bear the same rates of return by virtue of (11.13.4) and (11.13.5), portfolios are indeterminate. We are free to set holdings of foreign capital equal to zero in each country if we allow B_t^f to be nonzero. Adopting this way of resolving portfolio indeterminacy is convenient because it economizes on the number of initial conditions we have to specify. Therefore, we set holdings of foreign capital equal to zero in both countries but allow international lending. Then given an initial level B_{-1}^f of debt from the domestic country to the foreign country *, and where $R_{t-1,t} = \frac{q_{t-1}}{q_t}$, international debt dynamics satisfy

$$B_t^f = R_{t-1,t} B_{t-1}^f + c_t + (k_{t+1} - (1-\delta)k_t) + g_t - f(k_t) \tag{11.13.6}$$

and

$$c_t^* + (k_{t+1}^* - (1-\delta)k_t^*) + g_t^* - R_{t-1,t} B_{t-1}^f = f(k_t^*) - B_t^f. \tag{11.13.7}$$

Firms' first-order conditions in the two countries are:

$$\begin{aligned} \eta_t &= f'(k_t), \quad w_t = f(k_t) - k_t f'(k_t) \\ \eta_t^* &= f'(k_t^*), \quad w_t^* = f(k_t^*) - k_t^* f'(k_t^*). \end{aligned} \tag{11.13.8}$$

International trade in goods establishes

$$\frac{q_t}{\beta^t} = \frac{u'(c_t)}{1+\tau_{ct}} = \mu^* \frac{u'(c_t^*)}{1+\tau_{ct}^*}, \tag{11.13.9}$$

where μ^* is a nonnegative number that is a function of the Lagrange multiplier on the budget constraint for a consumer in country * and where we have normalized the Lagrange multiplier on the budget constraint of the domestic country to set the corresponding μ for the domestic country to unity. Equilibrium requires that the following two national Euler equations be satisfied for $t \geq 0$:

$$u'(c_t) = \beta u'(c_{t+1}) \left[(1-\tau_{kt+1})(f'(k_{t+1}) - \delta) + 1\right] \left[\frac{1+\tau_{ct+1}}{1+\tau_{ct}}\right] \tag{11.13.10}$$

$$u'(c_t^*) = \beta u'(c_{t+1}^*) \left[(1-\tau_{kt+1}^*)(f'(k_{t+1}^*) - \delta) + 1\right] \left[\frac{1+\tau_{ct+1}^*}{1+\tau_{ct}^*}\right] \tag{11.13.11}$$

Given that equation (11.13.9) holds for all $t \geq 0$, either equation (11.13.10) or equation (11.13.11) is redundant.

11.13.1. Initial conditions

As initial conditions, we take the pre-international-trade allocation of capital across countries $(\check{k}_0, \check{k}_0^*)$ and an initial level $B_{-1}^f = 0$ of international debt owed by the unstarred (domestic) country to the starred (foreign) country.

11.13.2. Equilibrium steady state values

The following two equations determine steady values for k and k^*.

$$f'(\overline{k}) = \delta + \frac{\rho}{1-\tau_k} \tag{11.13.12}$$

$$f'(\overline{k}^*) = \delta + \frac{\rho}{1-\tau_k^*} \tag{11.13.13}$$

Given the steady state capital-labor ratios $\overline{k}$ and $\overline{k}^*$, the following two equations determine steady state values of domestic and foreign consumption $\overline{c}$ and $\overline{c}^*$ as functions of a steady state value $\overline{B}^f$ of debt from the domestic country to country *:

$$(\overline{c} + \overline{c}^*) = f(\overline{k}) + f(\overline{k}^*) - \delta(\overline{k} + \overline{k}^*) - (\overline{g} + \overline{g}^*) \tag{11.13.14}$$

$$\overline{c} = f(\overline{k}) - \delta\overline{k} - \overline{g} - \rho\overline{B}^f \tag{11.13.15}$$

Equation (11.13.14) expresses feasibility at a steady state while (11.13.15) expresses trade balance, including interest payments, at a steady state.

11.13.3. Initial equilibrium values

Trade in physical capital and time 0 debt takes place before production and trade in other goods occurs at time 0. We shall always initialize international debt at zero: $B_{-1}^f = 0$, a condition that we use to express that international trade in capital begins at time 0. Given an initial total world-wide capital stock $\check{k}_0 + \check{k}_0^*$, initial values of k_0 and k_0^* satisfy

$$k_0 + k_0^* = \check{k}_0 + \check{k}_0^* \tag{11.13.16}$$

$$(1-\tau_{k0})f'(k_0) + \delta\tau_{k0} = (1-\tau_{k0}^*)f'(k_0^*) + \delta\tau_{k0}^*. \tag{11.13.17}$$

The price of a unit of capital in either country at time 0 is

$$p_{k0} = \left[(1-\tau_{k0})f'(k_0) + (1-\delta) + \delta\tau_{k0}\right]. \tag{11.13.18}$$

It follows that

$$B_{k0} = p_{k0}[k_0 - \check{k}_0], \tag{11.13.19}$$

which says that the domestic country finances imports of physical capital from abroad by borrowing from the foreign country *.

11.13.4. Shooting algorithm

To apply a shooting algorithm, we would search for pairs c_0, μ^* that yield a pair (k_0, k_0^*) and paths $\{c_t, c_t^*, k_t, k_t^*, B_t^f\}_{t=0}^T$ that solve equations (11.13.16), (11.13.17), (11.13.18), (11.13.19), (11.13.6), (11.13.9), and (11.13.18). The shooting algorithm 'aims' for $(\overline{k}, \overline{k}^*)$ that satisfy the steady-state equations (11.13.12), (11.13.13).

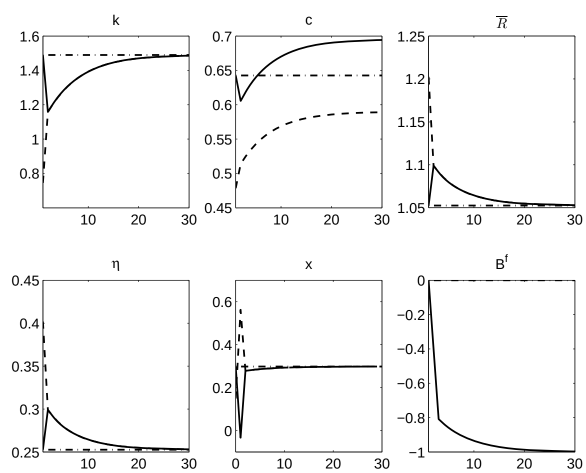

Figure 11.13.1: Response to unforeseen opening of trade at time 1. From left to right, top to bottom: $k, c, \bar{R}, \eta, x,$ and B^f. The solid line is the domestic country, the dashed line is the foreign country and the dashed dotted line is the original steady state.

11.13.5. Transition exercises

In the one-country exercises earlier in this chapter, announcements of new policies always occurred at time 0. In the two-country exercises to follow, we assume that announcements of new paths of tax rates and/or expenditures or trade regimes all occur at time 1. We do this to show some dramatic jumps in particular variables that occur at time 1 in response to announcements about changes that will occur at time 10 and later. Showing variables at times 0 and 1 helps display some of the outcomes on which we shall focus here. The production function is $f(k) = Ak^{\alpha}$. Parameter values are $\beta = .95, \gamma = 2, \delta = .2, \alpha = .33, A = 1$; g is initially $.2$ in both countries and all distorting taxes are initially 0 in both countries.

We describe outcomes from three exercises that illustrate two economic forces. The first force is consumers' desire to smooth consumption over time, expressed through households' consumption Euler equations. The second force is that equilibrium outcomes must offer no opportunities for arbitrage, expressed through equations that equate rates of returns on bonds and capital in both countries.

In the first two experiments, all taxes are lump sum in both countries. In the third experiment we activate a tax on capital in the domestic but not the foreign country. In all experiments, we allow lump sum taxes in both countries to adjust to satisfy government budget constraints in both countries.

11.13.5.1. Opening International Flows

In our first example, we study the transition dynamics for two countries when in period one newly produced output and stocks of capital, but not labor, suddenly become internationally mobile. The two economies are initially identical in all aspects except for one: we start the domestic economy at its autarkic steady state, while we start the foreign economy at an initial capital stock below its autarkic steady state. Because there are no distorting taxes on returns to physical capital, capital stocks in both economies converge to the same level.

In this experiment the domestic country is at its steady state capital stock while the poorer foreign country has a capital stock that is $.5$ less. This means that initially, before trade is opened at $t = 1$, the marginal product of capital in the foreign country exceeded the marginal product capital in the domestic country, that the foreign interest rate $R_{0,1}^*$ exceeded the domestic rate $R_{0,1}$, and that consequently the foreign consumption growth rate exceeded the domestic consumption growth rate. The disparity of interest rates before trade is opened is a force for physical capital to flow from the domestic country to the foreign country once when trade is opened at $t = 1$. Figure 11.13.2 presents the transitional dynamics. When countries become open to trade in goods and capital in period one, there occurs an immediate reallocation of capital from the capital-rich domestic country to the capital-poor foreign country. This transfer of capital has to take place because if it didn't, capital in different countries would yield different returns, providing consumers in both countries with arbitrage opportunities. Those cannot occur in equilibrium.

Before international trade had opened, rental rates on capital and interest rates differed across country because marginal products of capital differed and consumption growth rates differed. When trade opens at time 1 and capital is reallocated across countries to equalize returns, the interest rate in the domestic country jumps at time one. Because $\gamma = 2$, this means that consumption c in the domestic country must fall. The opposite is true for the foreign economy. Notice also that figure 11.13.2 shows an investment spike abroad while there is a large decline in investment in the domestic economy. This occurs because capital is reallocated from the domestic country to the foreign one. This transfer is feasible because investment in capital is reversible. The foreign country finances this import of physical capital by borrowing from the domestic country, so $-B^f$ increases. Foreign debt $-B^f$ continues to increase as both economies converge smoothly towards a steady-state with a positive level of $-\overline{B}^f$. Ultimately, these differences account for differences in steady-state consumption by $2\rho\overline{B^f}$.

Opening trade in goods and capital at time 1 benefits consumers in both economies. By opening up to capital flows, the foreign country achieves convergence to a steady-state consumption level at an accelerated rate. This steady-state consumption rate is lower than what it would be had the economy remained closed, but this reduction in long-run consumption is more than compensated by the rapid increase in consumption and output in the short-run. In contrast, domestic consumption falls in the short-run as trade allows domestic consumers to accumulate foreign assets that eventually support greater steady-state consumption.

This experiment shows the importance of studying transitional dynamics for welfare analysis. In this example, focusing only on steady-state consumption would lead to the false conclusion that opening markets are detrimental for poorer economies.

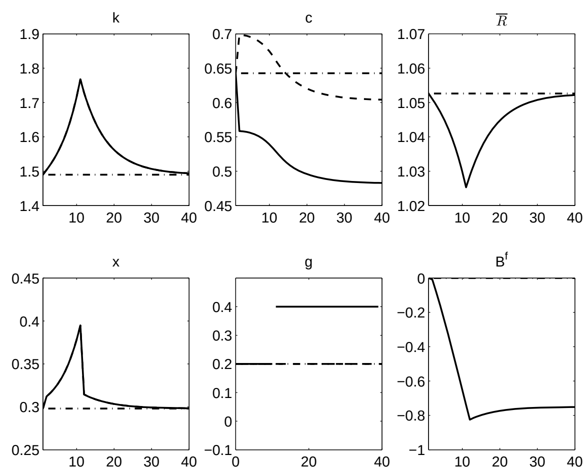

Figure 11.13.2: Response to increase in g at time 10 foreseen at time 1. From left to right, top to bottom: $k, c, \bar{R}, x, g, B^f$. The dashed-dotted line is the original steady state in the domestic country. The dashed line denotes the foreign country.

11.13.5.2. Foreseen Increase in g

Figure 11.13.2 presents transition dynamics after an increase in g in the domestic economy from .2 to .4 that is announced ten periods in advance. We start both economies from a steady-state with $B_0^f = 0$. When the new g path is announced at time 1, consumption smoothing motives induce domestic households to increase their savings in response to the adverse shock to domestic private wealth that is caused at time 1 by the foreseen increase in domestic government purchases g. Domestic households plan to use those savings to dampen the impact on consumption in periods after g will have increased ten periods ahead. Households save partly by accumulating more domestic capital in the short-run, their only source of assets in the closed economy version of this experiment. In an open economy, they have other ways to save, namely, by lending abroad.

The no-arbitrage conditions connect adjustments of both types of saving: the increase in savings by domestic households will reduce the equilibrium return on bonds and capital in the foreign economy to prevent arbitrage opportunities. Confronting the revised interest rate path that now begins with lower interest rates, foreign households increase their rates of consumption and investment in physical capital. These increases in foreign absorbtion are funded by increases in foreign consumers' external debt. After the announcement of the increase in g, the paths for consumption (and capital) in both countries follow the same patterns because no-arbitrage conditions equate the ratios of their marginal utilities of consumption. Both countries continue to accumulate capital until the increase in g occurs. After that, domestic households begin consuming some of their capital. Again by no-arbitrage conditions, when g actually increases both countries reduce their investment rates. The domestic economy, in turn, starts running current-account deficits partially to fund the increase in g. This means that foreign households begin repaying part of their external debt by reducing their capital stock. Although not plotted in figure 11.13.2, there is a sharp reduction in gross investment x in both countries when the increase in g occurs. After $t = 10$, all variable converge smoothly towards a new steady state where the domestic economy persists with positive asset holdings $-B^f$. Ultimately, this explains why the foreign country ends with lower steady state consumption than in the initial steady state. In the new steady state, minus the sum of the *decreases* of consumption rates across the two countries equals the *increase* in steady state government expenditures in the domestic country.[28]

The experiment teaches valuable economic lessons. First, it shows how the consequences of the foreseen increase in g will be distributed across time and households. Second, it tells how this distribution takes place: through time by accumulating or reducing the capital stock, and across households in different countries by running current-account deficits and surpluses.

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[28] Despite the decrease in its steady state consumption, we have calculated that $\sum_{t=0}^{\infty} \beta^t u(c_t^*)$ is higher in the new equilibrium than in the old.

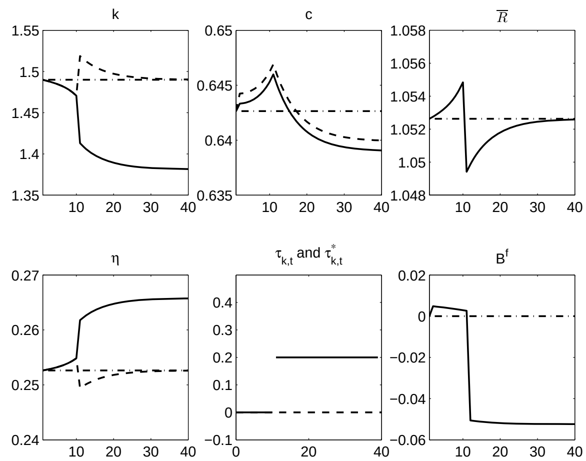

Figure 11.13.3: Response to once-and-for-all increase in τ_k at $t = 10$ foreseen at time $t = 1$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \tau_k$ and τ_k^*, and B^f. Domestic country (solid line), foreign country (dotted line) and steady-state values (dot-line).

11.13.5.3. Foreseen increase in τ_k

We now explore the impact of an increase in capital taxation in the domestic economy 10 periods after its announcement at $t = 1$. Figure 11.13.3 shows equilibrium outcomes. When the increase in τ_k is announced, domestic households become aware that the domestic capital stock will eventually decline to increase gross returns to equalize after-tax returns across countries despite a higher domestic tax rate on returns from capital. Domestic households will reduce their capital stock by increasing their rate of consumption. The consequent higher equilibrium world interest rates then also induces foreign households to increase consumption. Prior to the increase in τ_k, the domestic country runs a current account deficit. When τ_k is eventually increased, capital is rapidly reallocated across borders to preclude arbitrage opportunities, leading to a lower interest rate on bonds. The fall in the return on bonds occurs because the capital returns tax τ_k in the domestic country will reduce the after-tax return on capital, and because the foreign economy has a higher capital stock. Foreign households fund this large purchase of capital with a sharp increase in external debt, to be interpreted as a current account deficit. After τ_k has increased, the economies smoothly converge to a new steady state that features lower consumption rates in both countries and where the differences in the capital stock equate after-tax returns. It is useful to note that steady-state consumption in the foreign economy is higher than in the domestic country despite its perpetually having positive liabilities. This occurs because foreign output is larger because the capital stock held abroad is also larger.

This example shows how, via the no-arbitrage conditions, both countries share the impact of the shock and how fluctuations in capital stocks smooth over time the adjustments in consumption in both countries.

11.14. Concluding remarks

In chapter 12 we shall describe a stochastic version of the basic growth model and alternative ways of representing its competitive equilibrium.[29] Stochastic and nonstochastic versions of the growth model are widely used throughout aggregative economics to study a range of policy questions. Brock and Mirman (1972), Kydland and Prescott (1982), and many others have used a stochastic version of the model to approximate features of the business cycle. In much of the earlier literature on real business cycle models, the phrase "features of the business cycle" has meant "particular moments of some aggregate time series that have been filtered in a particular way to remove trends." Lucas (1990) uses a nonstochastic model like the one in this chapter to prepare rough quantitative estimates of the eventual consequences of lowering taxes on capital and raising those on consumption or labor. Prescott (2002) uses a version of the model in this chapter with leisure in the utility function together with some illustrative (high) labor supply elasticities to construct the argument that in the last two decades, Europe's economic activity has been depressed relative to that of the United States because Europe has taxed labor more highly that the United States. Ingram, Kocherlakota, and Savin (1994) and Hall (1997) use actual data to construct the errors in the Euler equations associated with stochastic versions

[29] It will be of particular interest to learn how to achieve a recursive representation of an equilibrium by finding an appropriate formulation of a state vector in terms of which to cast an equilibrium. Because there are endogenous state variables in the growth model, we shall have to extend the method used in chapter 8.

of the basic growth model and interpret them not as computational errors, as in the procedure recommended in section 11.10.7, but as measures of additional shocks that have to be added to the basic model to make it fit the data. In the basic stochastic growth model described in chapter 12, the technology shock is the only shock, but it cannot by itself account for the discrepancies that emerge in fitting all of the model's Euler equations to the data. A message of Ingram, Kocherlakota, and Savin (1994) and Hall (1997) is that more shocks are required to account for the data. Wen (1998) and Otrok (2001) build growth models with more shocks and additional sources of dynamics, fit them to U.S. time series using likelihood function-based methods, and discuss the additional shocks and sources of data that are required to match the data. See Christiano, Eichenbaum, and Evans (2003) and Christiano, Motto, and Rostagno (2003) for papers that add a number of additional shocks and measure their importance. Greenwood, Hercowitz, and Krusell (1997) introduced what seems to be an important additional shock in the form of a technology shock that impinges directly on the relative price of investment goods. Jonas Fisher (2006) develops econometric evidence attesting to the importance of this shock in accounting for aggregate fluctuations. Davig, Leeper, and Walker (2012) use stochastic versions of the types of models discussed in this chapter to study issues of intertemporal fiscal balance.

Schmitt-Grohe and Uribe (2004b) and Kim and Kim (2003) warn that the linear and log linear approximations described in this chapter can be treacherous when they are used to compare the welfare under alternative policies of economies, like the ones described in this chapter, in which distortions prevent equilibrium allocations from being optimal ones. They describe ways of attaining locally more accurate welfare comparisons by constructing higher order approximations to decision rules and welfare functions.

A. Log linear approximations

Following Christiano (1990), a widespread practice is to obtain log linear rather than linear approximations. Here is how this would be done for the model of this chapter.

Let $\log k_t = \tilde{k}_t$ so that $k_t = \exp \tilde{k}_t$; similarly, let $\log g_t = \tilde{g}_t$. Represent z_t as $z_t = [\exp(\tilde{g}_t) \quad \tau_{kt} \quad \tau_{ct}]'$ (note that only g_t has been replaced by it's log here). Then proceed as follows to get a log linear approximation.

1. Compute the steady state as before. Set the government policy $z_t = \overline{z}$, a constant level. Solve $H(\exp(\tilde{k}_\infty), \exp(\tilde{k}_\infty), \exp(\tilde{k}_\infty), \overline{z}, \overline{z}) = 0$ for a steady state $\tilde{k}_\infty$. (Of course, this will give the same steady state for the original unlogged variables as we got earlier.)

2. Take first-order Taylor series approximation around $(\tilde{k}_\infty, \overline{z})$:

$$\begin{aligned} &H_{\tilde{k}_t}(\tilde{k}_t - \tilde{k}_\infty) + H_{\tilde{k}_{t+1}}(\tilde{k}_{t+1} - \tilde{k}_\infty) + H_{\tilde{k}_{t+2}}(\tilde{k}_{t+2} - \tilde{k}_\infty) \\ &+ H_{z_t}(z_t - \overline{z}) + H_{z_{t+1}}(z_{t+1} - \overline{z}) = 0. \end{aligned} \tag{11.A.1}$$

(But please remember here that the first component of z_t is now $\tilde{g}_t$.)

3. Write the resulting system as

$$\phi_0 \tilde{k}_{t+2} + \phi_1 \tilde{k}_{t+1} + \phi_2 \tilde{k}_t = A_0 + A_1 z_t + A_2 z_{t+1} \tag{11.A.2}$$

or

$$\phi(L) \tilde{k}_{t+2} = A_0 + A_1 z_t + A_2 z_{t+1} \tag{11.A.3}$$

where L is the lag operator (also called the backward shift operator). Solve the linear difference equation $(11.A.3)$ exactly as before, but for the sequence $\{\tilde{k}_{t+1}\}$.

4. Compute $k_t = \exp(\tilde{k}_t)$, and also remember to exponentiate $\tilde{g}_t$, then use equations $(11.6.8)$ to compute the associated prices and quantities. Compute the Euler equation errors as before.

Exercises

Exercise 11.1 **Tax reform: I**

Consider the following economy populated by a government and a representative household. There is no uncertainty, and the economy and the representative household and government within it last forever. The government consumes a constant amount $g_t = g > 0, t \geq 0$. The government also sets sequences for two types of taxes, $\{\tau_{ct}, \tau_{ht}\}_{t=0}^{\infty}$. Here τ_{ct}, τ_{ht} are, respectively, a possibly time-varying flat-rate tax on consumption and a time-varying lump-sum or "head" tax. The preferences of the household are ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\beta \in (0,1)$ and $u(\cdot)$ is strictly concave, increasing, and twice continuously differentiable. The feasibility condition in the economy is

$$g_t + c_t + k_{t+1} \leq f(k_t) + (1-\delta) k_t$$

where k_t is the stock of capital owned by the household at the beginning of time t and $\delta \in (0,1)$ is a depreciation rate. At time 0, there are complete markets for dated commodities. The household faces the budget constraint:

$$\sum_{t=0}^{\infty} \left\{ q_t [(1+\tau_{ct}) c_t + k_{t+1} - (1-\delta) k_t] \right\} \leq \sum_{t=0}^{\infty} q_t \left\{ \eta_t k_t + w_t - \tau_{ht} \right\}$$

where we assume that the household inelastically supplies one unit of labor, and q_t is the price of date t consumption goods measured in the numeraire at time 0, η_t is the rental rate of date t capital measured in consumption goods at time t, and w_t is the wage rate of date t labor measured in consumption goods at time t. Capital is neither taxed nor subsidized.

A production firm rents labor and capital. The production function is $f(k)n$, where $f' > 0, f'' < 0$. The value of the firm is

$$\sum_{t=0}^{\infty} q_t \big[f(k_t) n_t - w_t n_t - \eta_t k_t \big],$$

where k_t is the firm's capital-labor ratio and n_t is the amount of labor it hires.

The government sets g_t exogenously and must set τ_{ct}, τ_{ht} to satisfy the budget constraint:

$$\sum_{t=0}^{\infty} q_t(\tau_{ct} c_t + \tau_{ht}) = \sum_{t=0}^{\infty} q_t g_t. \tag{1}$$

a. Define a competitive equilibrium.

b. Suppose that historically the government had unlimited access to lump-sum taxes and availed itself of them. Thus, for a long time the economy had $g_t = \overline{g} > 0, \tau_{ct} = 0$. Suppose that this situation had been expected to go on forever. Tell how to find the steady-state capital-labor ratio for this economy.

c. In the economy depicted in b, prove that the timing of lump-sum taxes is irrelevant.

d. Let $\bar{k}_0$ be the steady value of k_t that you found in part b. Let this be the initial value of capital at time $t = 0$ and consider the following experiment. Suddenly and unexpectedly, a court decision rules that lump-sum taxes are illegal and that starting at time $t = 0$, the government must finance expenditures using the consumption tax τ_{ct}. The value of g_t remains constant at $\overline{g}$. Policy advisor number 1 proposes the following tax policy: find a *constant* consumption tax that satisfies the budget constraint (1), and impose it from time 0 onward. Please compute the new steady-state value of k_t under this policy. Also, get as far as you can in analyzing the transition path from the old steady state to the new one.

e. Policy advisor number 2 proposes the following alternative policy. Instead of imposing the increase in τ_{ct} suddenly, he proposes to ease the pain by postponing the increase for 10 years. Thus, he/she proposes to set $\tau_{ct} = 0$ for $t = 0, \ldots, 9$, then to set $\tau_{ct} = \overline{\tau}_c$ for $t \geq 10$. Please compute the steady-state level of capital associated with this policy. Can you say anything about the transition path to the new steady-state k_t under this policy?

f. Which policy is better, the one recommended in d or the one in e?

Exercise 11.2 **Tax reform: II**

Consider the following economy populated by a government and a representative household. There is no uncertainty, and the economy and the representative household and government within it last forever. The government consumes a constant amount $g_t = g > 0, t \geq 0$. The government also sets sequences of two types of taxes, $\{\tau_{ct}, \tau_{kt}\}_{t=0}^{\infty}$. Here τ_{ct}, τ_{kt} are, respectively, a possibly time-varying flat-rate tax on consumption and a time-varying flat-rate tax on earnings from capital. The preferences of the household are ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\beta \in (0,1)$ and $u(\cdot)$ is strictly concave, increasing, and twice continuously differentiable. The feasibility condition in the economy is

$$g_t + c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

where k_t is the stock of capital owned by the household at the beginning of time t and $\delta \in (0,1)$ is a depreciation rate. At time 0, there are complete markets for commodities at all dates. The household faces the budget constraint:

$$\begin{aligned}
&\sum_{t=0}^{\infty} \{q_t[(1+\tau_{ct})c_t + k_{t+1} - (1-\delta)k_t]\} \\
&\leq \sum_{t=0}^{\infty} q_t \{\eta_t k_t - \tau_{kt}(\eta_t - \delta)k_t + w_t\}
\end{aligned}$$

where we assume that the household inelastically supplies one unit of labor, and q_t is the price of date t consumption goods in units of the numeraire at time 0, η_t is the rental rate of date t capital in units of time t goods, and w_t is the wage rate of date t labor in units of time t goods.

A production firm rents labor and capital. The value of the firm is

$$\sum_{t=0}^{\infty} q_t \big[f(k_t)n_t - w_t n_t - \eta_t k_t n_t\big],$$

where here k_t is the firm's capital-labor ratio and n_t is the amount of labor it hires.

The government sets $\{g_t\}$ exogenously and must set the sequences $\{\tau_{ct}, \tau_{kt}\}$ to satisfy the budget constraint:

$$\sum_{t=0}^{\infty} q_t\big(\tau_{ct}c_t + \tau_{kt}(\eta_t - \delta)k_t\big) = \sum_{t=0}^{\infty} q_t g_t. \tag{1}$$

a. Define a competitive equilibrium.

b. Assume an initial situation in which from time $t \geq 0$ onward, the government finances a constant stream of expenditures $g_t = \overline{g}$ entirely by levying a constant tax rate τ_k on capital and a zero consumption tax. Tell how to find steady-state levels of capital, consumption, and the rate of return on capital.

c. Let $\bar{k}_0$ be the steady value of k_t that you found in part b. Let this be the initial value of capital at time $t = 0$ and consider the following experiment. Suddenly and unexpectedly, a new party comes into power that repeals the tax on capital, sets $\tau_k = 0$ forever, and finances the same constant level of $\overline{g}$ with a flat-rate tax on consumption. Tell what happens to the new steady-state values of capital, consumption, and the return on capital.

d. Someone recommends comparing the two alternative policies of (1) relying completely on the taxation of capital as in the initial equilibrium and (2) relying completely on the consumption tax, as in our second equilibrium, by comparing the discounted utilities of consumption in steady state, i.e., by comparing $\frac{1}{1-\beta}u(\overline{c})$ in the two equilibria, where $\overline{c}$ is the steady-state value of consumption. Is this a good way to measure the costs or gains of one policy *vis-a-vis* the other?

Exercise 11.3 **Anticipated productivity shift**

An infinitely lived representative household has preferences over a stream of consumption of a single good that are ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad \beta \in (0,1)$$

where u is a strictly concave, twice continuously differentiable, one-period utility function, β is a discount factor, and c_t is time t consumption. The technology is:

$$c_t + x_t \leq f(k_t) n_t$$
$$k_{t+1} = (1-\delta) k_t + \psi_t x_t$$

where for $t \geq 1$

$$\psi_t = \begin{cases} 1 & \text{for } t < 4 \\ 2 & \text{for } t \geq 4. \end{cases}$$

Here $f(k_t)n_t$ is output, where $f > 0, f' > 0, f'' < 0$, k_t is capital per unit of labor input, and n_t is labor input. The household supplies one unit of labor inelastically. The initial capital stock k_0 is given and is owned by the representative household. In particular, assume that k_0 is at the optimal steady value for k presuming that ψ_t had been equal to 1 forever. There is no uncertainty. There is no government.

a. Formulate the planning problem for this economy in the space of sequences and form the pertinent Lagrangian. Find a formula for the optimal steady-state level of capital. How does a permanent increase in ψ affect the steady values of k, c, and x?

b. Formulate the planning problem for this economy recursively (i.e., compose a Bellman equation for the planner). Be careful to give a complete description of the state vector and its law of motion. ("Finding the state is an art.")

c. Formulate an (Arrow-Debreu) competitive equilibrium with time 0 trades, assuming the following decentralization. Let the household own the stocks of capital and labor and in each period let the household rent them to the firm. Let the household choose the investment rate each period. Define an appropriate price system and compute the first-order necessary conditions for the household and for the firm.

d. What is the connection between a solution of the planning problem and the competitive equilibrium in part c? Please link the prices in part c to corresponding objects in the planning problem.

e. Assume that k_0 is given by the steady-state value that corresponds to the assumption that ψ_t had been equal to 1 forever, and had been expected to remain equal to 1 forever. Qualitatively describe the evolution of the economy from time 0 on. Does the jump in ψ at $t = 4$ have any effects that precede it?

Exercise 11.4 **A capital levy**

A nonstochastic economy produces one good that can be allocated among consumption, c_t, government purchases, g_t, and gross investment, x_t. The economy-wide resource constraints are

$$c_t + g_t + x_t \leq f(k_t)$$
$$k_{t+1} = (1 - \delta)k_t + x_t,$$

where $\delta \in (0, 1)$ is a depreciation rate, k_t is the capital stock, and $f(k_t)$ gives production as a function of capital, where $f(k) = Ak^{\alpha}$ with $\alpha \in (0, 1)$. A single representative consumer owns the capital stock and one unit of labor. The consumer rents capital and labor to a competitive firm each period. The consumer ranks consumption plans according to

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, with $\gamma \geq 1$. The household supplies one unit of labor inelastically each period.

The government has only one tax at its disposal, a one-time capital levy in which it confiscates part of the capital stock from the private sector. When the government imposes a capital levy, we assume that it sends the consumer a tax bill for a fraction ϕ of the beginning of period capital stock. Below, you will be asked to compare consequences of levying this tax either at the beginning of time $T = 0$ or at the beginning of time $T = 10$. The fraction can exceed 1 if that is necessary to finance the government budget. The government is allowed to impose no other taxes. In particular, it cannot impose a direct lump sum or 'head' tax .

a. Define a competitive equilibrium with time 0 trading.

b. Suppose that before time 0 the economy had been in a steady state in which g had always been zero and had been expected always to equal zero. Find a formula for the initial steady state capital stock in a competitive equilibrium with time zero trading. Let this value be $\overline{k}_0$.

c. At time 0, everyone suddenly wakes up to discover that from time 0 on, government expenditures will be $\overline{g} > 0$, where $\overline{g} + \delta\overline{k}_0 < f(\overline{k}_0)$, which implies that the new level of government expenditures would be feasible in the old steady state. Suppose that the government finances the new path of expenditures by a capital levy at time $T = 0$. The government imposes a capital levy by sending the household a bill for a fraction of the value of its capital at the time indicated. Find the new steady state value of the capital stock in a competitive equilibrium. Describe an algorithm to compute the fraction of the capital stock that the government must tax away at time 0 to finance its budget. Find the new steady state value of the capital stock in a competitive equilibrium. Describe the time paths of capital, consumption, and the interest rate from $t = 0$ to $t = +\infty$ in the new equilibrium and compare them with their counterparts in the initial $g_t \equiv 0$ equilibrium.

d. Assume the same new path of government expenditures indicated in part **c**, but now assume that the government imposes the one-time capital levy at time $T = 10$, and that this is foreseen at time 0. Find the new steady state value of the capital stock in a competitive equilibrium that is associated with this tax policy. Describe an algorithm to compute the fraction of the capital stock that the government must tax away at time $T = 10$ to finance its budget. Describe the time paths of capital, consumption, and the interest rate in this new equilibrium and compare them with their counterparts in part **b** and in the initial $g_t \equiv 0$ equilibrium.

e. Define a competitive equilibrium with sequential trading of one-period Arrow securities. Describe how to compute such an equilibrium. Describe the time path of the consumer's holdings of one-period securities in a competitive equilibrium with one period Arrow securities under the government tax policy assumed in part **d**. Describe the time path of government debt.

Exercise 11.5

A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t \log(c_t) \quad , \quad 0 < \beta < 1$$

where $\beta = \frac{1}{1+\rho}, \rho > 0$, and c_t is the consumption per worker. The technology is

$$y_t = f(k_t) = z k_t^\alpha \quad , \quad 0 < \alpha < 1 \quad , \quad z > 0$$

where y_t is the output per unit labor and k_t is capital per unit labor

$$\begin{aligned} y_t &= c_t + x_t + g_t \\ k_{t+1} &= (1-\delta)k_t + x_t \quad , \quad 0 < \delta < 1 \end{aligned}$$

and x_t is gross investment per unit of labor and g_t is government expenditures per unit of labor. Assume a competitive equilibrium with a price system $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ and a government policy $\{g_t, \tau_{ht}\}_{t=0}^{\infty}$

Assume that the government finances its expenditures by levying lump sum taxes. There are no distorting taxes. Assume that at time 0, the economy starts out with a capital per unit of labor k_0 that equals the steady state value appropriate for an economy in which g_t had been zero forever.

a. Find a formula for the steady state capital stock when $g_t = 0\ \forall t$.

b. Compare the steady state capital labor ratio $\overline{k}$ in the competitive equilibrium with the capital labor ratio $\tilde{k}$ that maximizes steady state consumption per capita, i.e , $\tilde{k}$ solves

$$\tilde{c} = \max_k \bigl(f(k) - \delta k\bigr)$$

Is $\tilde{k}$ greater than or less that $\overline{k}$? If they differ, why ? Is $\tilde{c}$ greater or less than $\overline{c} = f(\overline{k}) - \delta\overline{k}$? Explain why.

c. Now assume that at time 0, g_t suddenly jumps to the value $g = \frac{1}{2}\overline{c}$ where $\overline{c}$ is the value of consumption per capita in the initial steady state in which g was zero forever. Starting from $k_0 = \overline{k}$ for the old $g = 0$ steady state, find the time paths of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ associated with the new path $g_t = g > 0\ \forall t$ for government expenditures per capita. Also show the time path for $\bar{R}_{t+1} \equiv (1-\delta) + f'(k_{t+1})$ Explain why the new paths are as they are.

Exercise 11.6 **Trade and growth**

Part I

Consider the problem of the planner in a small economy. When the economy is <u>closed to international trade</u>, the planner chooses $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1, \beta = \frac{1}{1+\rho}, \rho > 0$ subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \quad \delta \in (0,1)$$

where $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, $f(k_t) = z k_t^{\alpha}$, and $0 < \alpha < 1$.

Let $\overline{k}$ be the steady state value of k_t under the optimal plan.

a. Find a formula for $\overline{k}$.

b. Assume that $k_0 < \overline{k}$. Describe time paths for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $\bar{R}_{t+1} = (1-\delta) + f'(k_{t+1})$.

c. What is the steady state value of $\bar{R}_{t+1}$?

d. Is $\bar{R}_{t+1}$ less or greater than its steady state value when $k_{t+1} < \overline{k}$?

Part II

Now assume that the economy is <u>open to international trade</u> in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$ at which the planner can borrow or lend, what is often called a 'small open economy' assumption. The planner can use the proceeds of borrowing to purchase goods on the international market. These goods can be used to augment capital or to consume.

Let $\overline{k}_0$ be the level of initial capital ($\overline{k}_0 < \overline{k}$) just before the country opens up to trade just before time 0. Let $\overline{k}_0$ be the same initial capital per capita $\overline{k}_0 < \overline{k}$ studied in parts **a-d**.

At time $t = -1$, after $\overline{k}_0$ was set, trade opens up. At time $t = -1$, the planner can issue IOU's or bonds in amount B_{-1} and use the proceeds to purchase capital, thereby setting

$$k_0 = \overline{k}_0 + B_{-1}$$

where B_{-1} is denominated in time -1 consumption goods. The bonds are one period in duration and bear the constant world gross interest rate $R = \beta^{-1}$. For $t \geq 0$, the planner faces the constraints

$$c_t + k_{t+1} + RB_{t-1} = f(k_t) + (1-\delta)k_t + B_t$$

Here RB_{t-1} is the interest and the principal on the bonds issued at $t-1$ and B_t is the amount of one-period bonds issued at $t-1$.

The planning problem in the small open economy is now to choose $\{c_t, k_{t+1}, B_t\}_{t=0}^{\infty}$ and B_{-1}, subject to $\overline{k}_0$ given. (Notice that k_0 is a choice variable and that $\overline{k}_0$ is an initial condition.) Please solve the planning problem in the small open economy and compare the solution to the solution in the closed economy. Is welfare $\sum_{t=0}^{\infty} \beta^t u(c_t)$ higher in the "open" or "closed" economy.

The next several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1, \qquad \beta = \frac{1}{1+\rho}, \qquad \rho > 0$$

where c_t is the consumption per worker and where

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{for } \gamma > 0 \text{ and } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1. \end{cases}$$

The technology is $y_t = f(k_t) = zk_t^{\alpha}, 0 < \alpha < 1, z > 0$, where y_t is the output per unit labor and k_t is capital per unit labor and

$$\begin{aligned} y_t &= c_t + x_t + g_t \\ k_{t+1} &= (1-\delta)k_t + x_t \quad , \quad 0 < \delta < 1 \end{aligned}$$

where x_t is gross investment per unit of labor and g_t is government expenditures per unit of labor. The government finances its expenditures by levying some combination of a flat rate tax τ_{ct} on the value of consumption goods purchased at t , a flat rate tax τ_{nt} on the value of labor earnings at t, a flat rate tax τ_{kt} on earnings from capital at t and a lump sum tax of τ_{ht} in time t consumption goods per worker at time t.

Let $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ be a price system.

Exercise 11.7

Consider an economy in which $g_t = \bar{g} > 0 \ \ \forall \ t \geq 0$ and in which initially the government finances all expenditures by lump sum taxes.

a. Find a formula for the steady state capital labor ratio k_t for this economy. Find formulas for the steady state level of c_t and $\bar{R}_t = [(1-\delta) + (1-\tau_{kt+1})f'(k_{t+1})]$

b. Now suppose that starting from $k_0 = \bar{k}$, i.e., the steady state that you computed in part **a**, the government suddenly increases the tax on earnings from capital to a constant level $\tau_k > 0$. The government adjusts lump sum taxes to keep the government budget balances. Describe competitive equilibrium time paths for $c_t, k_{t+1}, \bar{R}_t$ and their relationship to corresponding values in the old steady state that you described in part **a**.

c. Describe how the <u>shapes</u> of the paths that you found in part **b** depend on the curvature parameter γ in the utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Higher values of γ imply higher curvature and more aversion to consumption path that fluctuate. Higher values of γ imply that the consumer values smooth consumption paths even more.

d. Starting from the steady state $\bar{k}$ that you computed in part a, now consider a situation in which the government announces at time 0 that starting in period 10 the tax on earnings from capital τ_k will rise permanently to $\overline{\tau_k} > 0$. The government adjusts its lump sum taxes to balance it's budget.

i) Find the new steady state values for $k_t, c_t, \bar{R}_t$.

ii) Describe the shapes of the transition paths from the initial steady states to the new one for $k_t, c_t, \bar{R}_t$.

iii) Describe how the shapes of the transition paths depend on the curvature parameter γ in the utility function u(c).

<u>Hint</u> : When γ is bigger, consumers more strongly prefer smoother consumption paths. Recall the forces behind formula (11.10.16) in section 11.10.6.

Exercise 11.8 **Trade and growth, version II**

Part I

Consider the problem of the planner in a small economy. When the economy is <u>closed to trade</u>, the planner chooses $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1, \quad \beta = \frac{1}{1+\rho}, \rho > 0$$

subject to

$$c_t + k_t + 1 = f(k_t) + (1-\delta)k_t, \delta \in (0,1)$$

where

Let $\overline{k}$ be the steady state value of k_t under the optimal plan.

a. Find a formula for $\overline{k}$.

b. Assume that $k_0 > \overline{k}$. Describe time paths for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $\bar{R}_{t+1} = (1-\delta) + f'(k_{t+1})$.

c. What is the steady state value of $\bar{R}_{t+1}$?

d. Is $\bar{R}_{t+1}$ less or greater than its steady state value when $k_{t+1} > \overline{k}$?

Part II

e. Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$

at which the planner can borrow or lend. In particular, the planner is free to use the following plan. The planner can sell all of its capital $\overline{k}_0$ and simply consume the interest payments. Let $\overline{k}_0$ be the level of initial capital ($\overline{k}_0 > \overline{k}$) just before the country opens up to trade just before time 0. Let $\overline{k}_0$ be the initial capital per capita $\overline{k}_0 > \overline{k}$ studied in parts **a- d**. At time $t = -1$, after $\overline{k}_0$ was set, trade opens up. At time $t = -1$, the planner sells $\overline{k}_0$ in exchange for IOU's or bonds from the rest of the world in the amount $A_{-1} = \overline{k}_0$.

The bonds A_{-1} are one-period in duration and bear the constant world gross interest rate $R = \beta^{-1}$. After the sale of $\overline{k}_0 = A_{-1}$, the planner has zero capital and so shuts down the technology. Instead, the planner uses the asset market to smooth consumption. The planner chooses $\{A_{t+1}, c_t\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1, \quad \beta = \frac{1}{1+\rho}, \quad \rho > 0$$

subject to $c_t + \beta A_{t+1} = A_t, A_0 = \overline{k}_0$. Please find the optimal path for consumption $\{c_t\}$.

f. Compare the path of $(c_t, k_{t+1}, \bar{R}_t)$ that you computed in parts **a-d** with "no trade" with the part **e** path "with trade" in which the government "shuts down the home technology" and lives entirely from the returns on foreign assets. Can you say which path the representative consumer will prefer?

g. Now return to the economy in part **e** with $\overline{k}_0 > \bar{k}$ from part **d**. Assume that the planner is free to borrow or lend capital at the fixed gross interest of $\beta^{-1} = 1+\rho$, as before. But now assume the planner chooses the optimal amount of $\overline{k}_0$ to sell off and so does not necessarily sell off the entire $\overline{k}_0$ and possibly continues to operate the technology.

i) Find the solution of the planning problem.

ii) Explain why it is optimal not to shut down the technology. **Hint**: Starting from having shut the technology down, think of putting a small amount ϵ of capital into the technology-this earns $z\epsilon^{\alpha}$ and costs $\rho\epsilon$ in terms of foregone interest. Because $0 < \alpha < 1$, $\rho\epsilon < z\epsilon^{\alpha}$ for small ϵ. Thus, the technology is very productive for small ϵ, so it is efficient to use it.

Exercise 11.9

Consider a consumer who wants to choose $\{c_t\}_{t=0}^T$ to maximize

$$\sum_{t=0}^{T} \beta^t c_t \quad , \quad 0 < \beta < 1$$

subject to the intertemporal budget constraint

$$\sum_{t=0}^{T} R^{-t}[c_t - y_t] = 0 \quad , \quad R > 1$$

where $c_t \geq 0$ and $y_t > 0$ for $t = 0, \cdots, T$. Here $R > 1$ is the gross interest rate $(1 + r)$, $r > 0$. Assume R is constant. Here $\{y_t\}_{t=0}^T$ is an exogenous sequence of "labor income".

a. Assume that $\beta R < 1$. Find the optimal path $\{c_t\}_{t=0}^T$.

b. Assume that $\beta R > 1$. Find the optimal path $\{c_t\}_{t=0}^T$.

c. Assume that $\beta R = 1$. Find the optimal path $\{c_t\}_{t=0}^T$.

Exercise 11.10 **Term Structure of Interest Rates**

This problem assumes the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 \quad , \quad \beta = \frac{1}{1+\rho} \quad , \quad \rho > 0$$

where c_t is the consumption per worker and where

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases}$$

The technology is

$$y_t = f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1 \quad , \quad z > 0$$

where y_t is the output per unit labor and k_t is capital per unit labor

$$\begin{aligned} y_t &= c_t + x_t + g_t \\ k_{t+1} &= (1-\delta)k_t + x_t \quad , \quad 0 < \delta < 1 \end{aligned}$$

where x_t is gross investment per unit of labor and g_t is government expenditures per unit of labor. The government finances its expenditures by levying some combination of a flat rate tax τ_{ct} on the value of consumption goods purchased at t, a flat rate tax τ_{nt} on the value of labor earnings at t, a flat rate tax τ_{kt} on earnings from capital at t and a lump sum tax of τ_{ht} in time t consumption goods per worker at time t. Define $\bar{R}_{t+1} = \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}\left[1 + (1-\tau_{kt+1})(f'(k_{t+1} - \delta)\right]$.

a. Recall that we can represent

$$q_t^0 = q_0^0 m_{0,1} m_{1,2} \cdots m_{t-1,t}$$

where $m_{t-1,t} = \frac{q_t^0}{q_{t-1}^0}$ and $m_{t-1,t} \equiv \exp(-r_{t-1,t}) \approx \frac{1}{1+r_{t-1,t}}$. Further, recall that the t period long yield satisfies $q_t^0 = \exp(-t r_{0,t})$ and $r_{0,t} = \frac{1}{t}[r_{0,1} + r_{1,2} + \cdots + r_{t-1,t}]$. Now suppose that at $t=0$, $k_0 = \bar{k}$, where $\bar{k}$ is the steady state appropriate for an economy with constant $g_t = \bar{g} > 0$ and all expenditures are financed by lump sum taxes. Find q_t^0 for this economy.

b. Plot $r_{t-1,t}$ for this economy for $t = 1, 2, \cdots, 10$.

c. Plot $r_{0,t}$ for this economy for $t = 1, 2, \cdots, 10$ (this is what Bloomberg plots).

d. Now assume that at time 0, starting from $k_0 = \bar{k}$ for the steady state you computed in part a, the government unexpectedly and permanently raises the tax rate on income from capital $\tau_{kt} = \tau_k > 0$ to a positive rate.

i) Plot r_{t-1t} for this economy for $t = 1, 2, \cdots, 10$. Explain how you got this outcome.

ii) Plot $r_{0,t}$ for this economy for $t = 1, 2, \cdots, 10$. Explain how you got this outcome.

Exercise 11.11

This problem assumes the same economic environment as the previous exercise i.e, the "growth model" with fiscal policy. Suppose that you observe the path for consumption per capita in figure 11.1. Say what you can about the likely behavior over time of k_t, $\bar{R}_t = [1 + (1-\tau_{kt})(f'(k_t) - \delta)]$, g_t and τ_{kt}.(You are free to make up any story that is consistent with the model.)

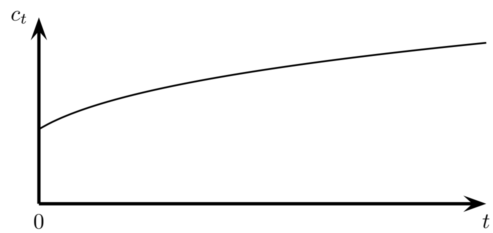

Figure 11.1: Consumption per capita.

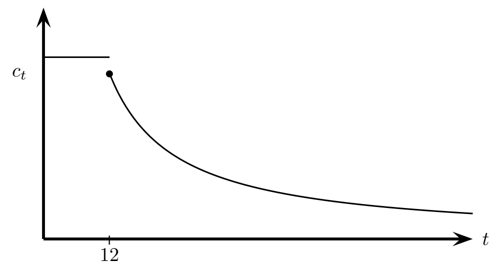

Figure 11.2: Consumption per capita.

Exercise 11.12

Assume the same economic environment as in the previous two problems. Assume that someone has observed the time path for c_t in figure 11.2:

a. Describe a consistent set of assumptions about the fiscal policy that explains this time path for c_t. In doing so, please distinguish carefully between changes in taxes and expenditures that are <u>foreseen</u> versus <u>unforeseen</u>.

b. Describe what is happening to $k_t, \bar{R}_t$ and g_t over time. Make whatever assumptions you must to get a complete but consistent story - here "consistent" means "consistent with the economic environment we have assumed".

Exercise 11.13

Consider the optimal growth model with a representative consumer with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t, \;\; 0 < \beta < 1$$

with technology

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \quad \delta \in (0,1)$$

$$c_t \geq 0, \quad k_0 > 0 \text{ given}$$

$$f' > 0, \quad f'' < 0, \quad \lim_{k\to 0} f'(k) = +\infty, \quad \lim_{k\to +\infty} f'(k) = 0$$

Let $\bar{k}$ be the steady state capital stock for the optimal planning problem.

a. For k_0 given, formulate and solve the optimal planning problem.

b. For $k_0 > \bar{k}$, describe the optimal time path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

c. For $k_0 < \bar{k}$, describe the optimal path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

d. Let the <u>saving rate</u> s_t be defined as the s_t that satisfies

$$k_{t+1} = s_t f(k_t) + (1-\delta)k_t.$$

Here s_t in general varies along a $\{c_t, k_{t+1}\}$ sequence. Say what you can about how s_t varies as a function of k_t.

Exercise 11.14

A representative consumer has preferences over consumption streams ordered by $\sum_{t=0}^{\infty} \beta^t u(c)$, $\quad 0 < \beta < 1$, where $\beta \equiv \frac{1}{1+\rho}$, c_t is consumption per worker, $\gamma > 0$, and

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1. \end{cases}$$

The consumer supplies one unit of labor inelastically. The technology is

$$y_t = f(k_t) = z k_t^{\alpha}, \quad 0 < \alpha < 1$$

where y_t is output per worker, k_t is capital per worker, x_t is gross investment per worker, g_t = government expenditures per worker, and

$$\begin{aligned} y_t &= c_t + x_t + g_t \\ k_{t+1} &= (1-\delta)k_t + x_t, \quad 0 < \delta < 1. \end{aligned}$$

The government finances its expenditure stream $\{g_t\}$ by levying a stream of flat rate taxes $\{\tau_{ct}\}$ on the value of the consumption good purchased at t, a stream of flat rate taxes $\{\tau_{kt}\}$ on earnings from capital at t, and a stream of lump sum taxes $\{\tau_{ht}\}$. Let $\{q_t, q_t\eta_t, q_t w_t\}_{t=0}^{\infty}$ be a *price system*, where q_t is the price of time t consumption and investment goods, $q_t\eta_t$ is the price of renting capital at time t, and $q_t w_t$ is the price of renting labor at time t. All trades occur at time 0 and all prices are measured in units of the time 0 consumption good. The initial capital stock k_0 is given.

a. Define a competitive equilibrium with taxes and government purchases.

b. Assume that $g_t = 0$ for all $t \geq 0$ and that all taxes are also zero. Find the value $\bar{k}$ of the steady state capital per worker. Find a formula for the saving rate $\frac{x_t}{f(k_t)}$ at the steady state value of the capital stock.

c. Suppose that the initial capital stock $k_0 = .5\bar{k}$, so that the economy starts below its steady state level. Describe (i.e., draw graphs showing) the time paths of $\{c_t, k_{t+1}, \bar{R}_{t+1}\}_{t=0}^{\infty}$ where $\bar{R}_{t+1} \equiv [(1-\delta) + f'(k_{t+1})]$.

d. Starting from the same initial k_0 as in part **c**, assume now that $g_t = \bar{g} = \phi f(k_0) > 0$ for all $t \geq 0$ where $\phi \in (0, 1-\delta)$. Assume that the government finances its purchases by imposing lump sum taxes. Describe (i.e., draw graphs) showing the time paths of $\{c_t, k_{t+1}, \bar{R}_{t+1}\}_{t=0}^{\infty}$ and compare them to the outcomes that you obtained in part **c**. What outcomes differ? What outcomes, if any, are identical across the two economies? Please explain.

e. Starting from the same initial k_0 assumed in part **c**, assume now that $g_t = \bar{g} = \phi f(k_0) > 0$ for all $t \geq 0$ where $\phi \in (0, 1-\delta)$. Assume that the government must now finance these purchases by imposing a time-invariant tax rate $\bar{\tau}_k$ on capital each period. The government *cannot* impose lump sum taxes or any other kind of taxes to balance its budget. Please describe how to find a competitive equilibrium.

Exercise 11.15

The structure of the economy is identical to that described in the previous exercise. Let $r_{0,t}$ be the yield to maturity on a t period bond at time 0, $t = 1, 2, \ldots,$. At time 0, Bloomberg reports the term structure of interest rates in figure 11.3. Please say what you can about the evolution of $\{c_t, k_{t+1}\}$ in this economy. Feel free to make any assumptions you need about fiscal policy $\{g_t, \tau_{kt}, \tau_{ct}, \tau_{ht}\}_{t=0}^{\infty}$ to make your answer coherent.

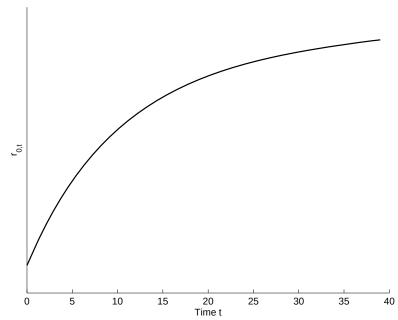

Figure 11.3: Yield to maturity $r_{0,t}$ at time 0 as a function of term to maturity t.

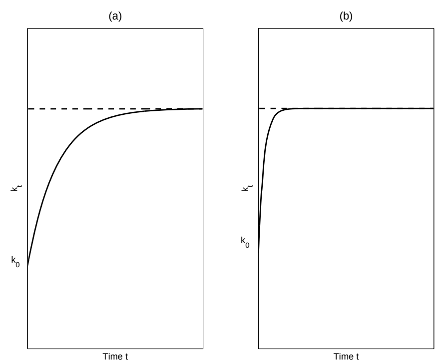

Figure 11.4: Capital stock as function of time in two economies with different values of γ.

Exercise 11.16

The structure of the economy is identical to that described in exercise 11.14. Assume that $\{g_t, \tau_{ct}, \tau_{kt}\}_{t=0}^{\infty}$ are all constant sequences (their values don't change over time). In this problem, we ask you to infer differences across two economies in which all aspects of the economy are identical *except* the parameter γ in the utility function.[30] In both economies, $\gamma > 0$. In one economy, $\gamma > 0$ is high

[30] It is possible that lump sum taxes differ across the two economies. Assume that lump sum taxes are adjusted to balance the government budget.

and in the other it is low. Among other identical features, the two economies have identical government policies and identical initial capital stocks.

a. Please look at figure 11.4. Please tell which outcome for $\{k_{t+1}\}_{t=0}^{\infty}$ describes the low γ economy, and which describes the high γ economy. Please explain your reasoning.

b. Please look at figure 11.5. Please tell which outcome for $\{f'(k_t)\}_{t=0}^{\infty}$ describes the low γ economy, and which describes the high γ economy. Please explain your reasoning.

c. Please plot time paths of consumption for the low γ and the high γ economies.

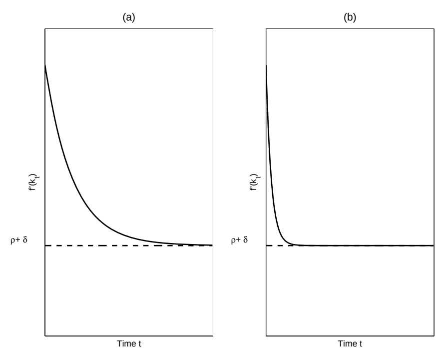

Figure 11.5: Marginal product of capital as function of time in two economies with different values of γ.

Exercise 11.17

The structure of the economy is identical to that described in exercise 11.16. Assume that $\{\tau_{ct}, \tau_{kt}\}_{t=0}^{\infty}$ are constant sequences (their values don't change over time) but that $\{g_t\}_{t=0}^{\infty}$ follows the path described in panel **c** of figure 11.6 – it takes a once and for all jump at time $t = 10$. Lump sum taxes adjust to balance the government budget. Panels **a** and **b** give consumption paths for two economies that are identical except in one respect. In one of the economies, the time 10 jump in g had been anticipated since time 0, while in the other, the jump in g that occurs at time 10 is completely unanticipated at time 10.

Please tell which panel corresponds to which view of the arrival of news about the path of g_t. Please say as much as you can about how $\{k_{t+1}\}_{t=0}^{\infty}$ and the interest rate behave in these two economies.

Exercise 11.18

A planner chooses sequences $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u\big(c_t - \alpha c_{t-1}\big), \quad \alpha \in (-1, 1), \quad \beta \in (0, 1)$$

subject to

$$k_{t+1} + c_t = f(k_t) + (1-\delta) k_t, \quad \delta \in (0,1), \quad c_t \geq 0$$

where

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1 \\ \log(x) & \text{if } \gamma = 1 \end{cases}$$

and (k_0, c_{-1}) are given initial conditions. Here $u' > 0, u'' < 0$, and $\lim_{c \downarrow 0} u'(c) = 0$. If $\alpha > 0$, it indicates that the consumer has a 'habit'; if $\alpha < 0$, it indicates that the consumption good is somewhat durable.

a. Find a complete set of first-order necessary conditions for the planner's problem.

b. Define an optimal *steady state.*

c. Find optimal steady state values for $(k, f'(k), c)$.

d. For given initial conditions (k_0, c_{-1}), describe as completely as you can an algorithm for computing a path $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ that solves the planning problem.

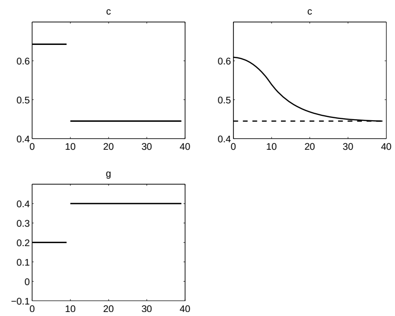

Figure 11.6: Panels **a** and **b**: consumption c_t as function of time in two economies. Panel **c**: government expenditures g_t as a function of time.