Problem Set #5

MACS 40200, Dr. Evans
Due Thursday, Feb. 9 at 12:00pm

1. **Estimating the Brock and Mirman (1972) model by GMM (5 points).** You can observe time series data in an economy for the following variables: (c_t, k_t, w_t, r_t). Data on (c_t, k_t, w_t, r_t) can be loaded from the file `MacroSeries.txt`. This file is a comma separated text file with no labels. The variables are ordered as (c_t, k_t, w_t, r_t). These data have 100 periods, which are quarterly (25 years). Suppose you think that the data are generated by a process similar to the Brock and Mirman (1972). A simplified set of characterizing equations of the Brock and Mirman model are the following.

$$(c_t)^{-1} - \beta E\left[r_{t+1}(c_{t+1})^{-1}\right] = 0 \tag{1}$$

$$c_t + k_{t+1} - w_t - r_t k_t = 0 \tag{2}$$

$$w_t - (1-\alpha)e^{z_t}\left(k_t\right)^{\alpha} = 0 \tag{3}$$

$$r_t - \alpha e^{z_t}\left(k_t\right)^{\alpha-1} = 0 \tag{4}$$

$$z_t = \rho z_{t-1} + (1-\rho)\mu + \varepsilon_t \quad \text{where} \quad E\left[\varepsilon_t\right] = 0 \tag{5}$$

The variable c_t is aggregate consumption in period t, k_{t+1} is total household savings and investment in period t for which they receive a return in the next period (this model assumes full depreciation of capital). The wage per unit of labor in period t is w_t and the interest rate or rate of return on investment is r_t. Total factor productivity is z_t, which follows an AR(1) process given in (5). The rest of the symbols in the equations are parameters that must be estimated $(\alpha, \beta, \rho, \mu)$. The constraints on these parameters are the following.

$$\alpha, \beta \in (0,1), \quad \mu, \sigma > 0, \quad \rho \in (-1,1)$$

Assume that the first observation in the data file variables is $t = 1$. Let k_1 be the first observation in the data file for the variable k_t.

   (a) Estimate α, β, ρ, and μ by GMM using the unconditional moment conditions that $E[\varepsilon_t] = 0$ and $E[\beta r_{t+1} c_t / c_{t+1} - 1] = 0$. Use the identity matrix $I(4)$ as your estimator of the optimal weighting matrix. Use the following four moment conditions to estimate the four parameters.

$$E\Big[z_{t+1} - \rho z_t - (1-\rho)\mu\Big] = 0 \tag{6}$$

$$E\left[\Big(z_{t+1} - \rho z_t - (1-\rho)\mu\Big) z_t\right] = 0 \tag{7}$$

$$E\left[\beta\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1}\frac{c_t}{c_{t+1}} - 1\right] = 0 \tag{8}$$

$$E\left[\left(\beta\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1}\frac{c_t}{c_{t+1}} - 1\right) w_t\right] = 0 \tag{9}$$

The estimation inside each iteration of the minimizer of the GMM objective function is the following.

- Given a guess for $(\alpha, \beta, \rho, \mu)$ and data (c_t, k_t, w_t, r_t), use (4) to back out an implied series for z_t.
- Given z_t, parameters $(\alpha, \beta, \rho, \mu)$ and data (c_t, k_t, w_t, r_t), calculate four empirical analogues of the moment conditions (6), (7), (8), and (9).
- Update guesses for parameters $(\alpha, \beta, \rho, \mu)$ until minimum criterion value is found.

Report your estimated parameter values $(\hat{\alpha}, \hat{\beta}, \hat{\rho}, \hat{\mu})$ and the value of your minimized criterion function.

2. **Estimating the Brock and Mirman (1972) model by SMM (5 points).** One nice property of the Brock and Mirman (1972) model is that the household decision has a known analytical solution in which the optimal savings decision k_{t+1} is a function of the productivity shock today z_t and the amount of capital today k_t.

$$k_{t+1} = \alpha\beta e^{z_t} k_t^{\alpha} \tag{10}$$

With this solution, it is straightforward to simulate the data of the Brock and Mirman (1972) model given parameters $(\alpha, \beta, \rho, \mu, \sigma)$. First, assume that $z_1 = \mu$ and that $k_1 = \text{mean}(k_t)$ from the data. These are initial values that will not change across simulations. Next, draw $T = 100$ normally distributed values of $\varepsilon_t \sim N(0, \sigma)$. Note that for SMM we have to return to fully specifying the distributional assumptions. Then, you can use equation (5) to calculate the simulated series for z_t. Now, you can use the policy function for savings (10) recursively to solve for the entire k_t series. With the entire k_t and z_t simulated series, you can use (3) to solve for the w_t series and (4) to solve for the r_t series. Lastly, you use the budget constraint (2) to solve for the c_t series.

   (a) Estimate the five parameters of the Brock and Mirman (1972) model $(\alpha, \beta, \rho, \mu, \sigma)$ described by equations (1) through (5) by SMM. Choose the five parameters to match the following six moments from the 100 periods of empirical data $\{c_t, k_t, w_t, r_t\}_{t=1}^{100}$ in `MacroSeries.txt`: $\text{mean}(c_t)$, $\text{mean}(k_t)$, $\text{var}(c_t)$, $\text{var}(k_t)$, $\text{corr}(c_t, k_t)$, and $\text{corr}(k_t, k_{t+1})$. In your simulations of the model, set $T = 100$ and $S = 1,000$. Start each of your simulations from $k_1 = \text{mean}(k_t)$ from the `MacroSeries.txt` file and $z_1 = \mu$. Input the bounds to be $\alpha, \beta \in [0.01, 0.99]$, $\rho \in [-0.99, 0.99]$, $\mu \in [-0.5, 1]$, and $\sigma \in [0.001, 1]$. Also, use the identity matrix as your weighting matrix $\boldsymbol{W}$. Report your solution $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\rho}, \hat{\mu}, \hat{\sigma})$, the vector of moment differences at the optimum, and the criterion function value.

References

Brock, William A. and Leonard J. Mirman, "Optimal economic growth and uncertainty: The discounted case," *Journal of Economic Theory*, June 1972, *4* (3), 479–513.