Problem Set #3

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Problem 1: A simplified set of characterizing equations of the Brock and Mirman model are the following:

$$(c_t)^{-1} - \beta E[r_{t+1}(c_{t+1}^{-1}] = 0 \tag{1}$$
$$c_t + k_{t+1} - w_t - r_t k_t = 0 \tag{2}$$
$$w_t - (1-\alpha)e^{z_t}(k_t)^\alpha = 0 \tag{3}$$
$$r_t - \alpha e^{z_t}(k_t)^{\alpha-1} = 0 \tag{4}$$
$$z_t = \rho z_{t-1} + (1-\rho)\mu + \epsilon_t \tag{5}$$
$$\epsilon_t \sim N(0, \sigma^2)$$

Part (a). Use the data (w_t, k_t) and equations (3) and (5) to estimate the four parameters $(\alpha, \rho, \mu, \sigma)$ by maximum likelihood.

According to equation (3):

$$\begin{aligned} w_t - (1-\alpha)e^{z_t}(k_t)^\alpha &= 0 \\ (1-\alpha)e^{z_t}(k_t)^\alpha &= w_t \\ e^{z_t} &= \frac{w_t}{(1-\alpha)(k_t)^\alpha} \\ z_t &= \ln\frac{w_t}{(1-\alpha)(k_t)^\alpha} \end{aligned}$$

Because we assume that $z_0 = \mu$, so that $z_1 = \mu$, and thus

$$z_t \sim N(\rho z_{t-1} + (1-\rho)\mu, \sigma^2)$$

We can claim that the likelihood function is:

$$\begin{aligned} \prod_{i=1}^{n} \Pr(z_t | w_t; \alpha, \rho, \mu, \sigma) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{[z_t - (\rho z_{t-1} + (1-\rho)\mu)]^2}{2\sigma^2}} \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{[\ln\frac{w_t}{(1-\alpha)(k_t)^\alpha} - (\rho z_{t-1} + (1-\rho)\mu)]^2}{2\sigma^2}} \end{aligned}$$

and then applying MLE, we are able to obtain estimated parameters:

- $\hat{\alpha}_{MLE} = 0.955450545204$
- $\hat{\rho}_{MLE} = 0.9999999999$
- $\hat{\mu}_{MLE} = 2.97221625164$
- $\hat{\sigma}_{MLE} = 0.325931136042$

and the inverse hessian variance-covariance matrix of this set of estimators is:

$$\begin{bmatrix} 3.77 \times 10^{-4} & 1.27 \times 10^{-3} & 9.51 \times 10^{-2} & -8.72 \times 10^{-3} \\ 1.27 \times 10^{-3} & 1.39 \times 10^{-2} & 8.24 \times 10^{-1} & -7.40 \times 10^{-2} \\ 9.51 \times 10^{-2} & 8.24 \times 10^{-1} & 5.48 \times 10^{1} & -4.94 \times 10^{0} \\ -8.72 \times 10^{-3} & -7.40 \times 10^{-2} & -4.94 \times 10^{0} & 4.47 \times 10^{-1} \end{bmatrix}$$

Part (b). Use the data (r_t, k_t) and equations (4) and (5) to estimate the four parameters $(\alpha, \rho, \mu, \sigma)$ by maximum likelihood.

According to equation (4):

$$\begin{aligned} r_t - \alpha e^{z_t}(k_t)^{\alpha-1} &= 0 \\ \alpha e^{z_t}(k_t)^{\alpha-1} &= r_t \\ e^{z_t} &= \frac{r_t}{\alpha(k_t)^{\alpha-1}} \\ z_t &= \ln \frac{r_t}{\alpha(k_t)^{\alpha-1}} \end{aligned}$$

and thus the new log-likelihood function becomes:

$$\begin{aligned} \prod_{i=1}^{n} \Pr(z_t|r_t;\alpha,\rho,\mu,\sigma) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{[z_t - (\rho z_{t-1} + (1-\rho)\mu)]^2}{2\sigma^2}} \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{[\ln \dfrac{r_t}{\alpha(k_t)^{\alpha-1}} - (\rho z_{t-1} + (1-\rho)\mu)]^2}{2\sigma^2}} \end{aligned}$$

and then applying MLE, we are able to obtain estimated parameters:

- $\hat{\alpha}_{MLE_2} = 0.669377643876$
- $\hat{\rho}_{MLE_2} = 0.751303162852$
- $\hat{\mu}_{MLE_2} = 6.08110700814$
- $\hat{\sigma}_{MLE_2} = 0.162155742949$

and the inverse hessian variance-covariance matrix of this set of estimators is:

$$\begin{bmatrix} 9.65 \times 10^{-1} & 9.36 \times 10^{-3} & -1.52 \times 10^{-1} & 3.92 \times 10^{0} \\ 9.36 \times 10^{-3} & 9.98 \times 10^{-1} & 2.64 \times 10^{-2} & -6.78 \times 10^{-1} \\ -1.52 \times 10^{-1} & 2.64 \times 10^{-2} & 1.01 \times 10^{0} & -4.02 \times 10^{-1} \\ 3.92 \times 10^{0} & -6.78 \times 10^{-1} & -4.02 \times 10^{-1} & 1.65 \times 10^{1} \end{bmatrix}$$

Part (c). According to your estimates from part (a), if investment/savings in the current period is $k_t = 7,500,000$ and the productivity shock in the previous period was $z_{t-1} = 10$, what is the probability that the interest rate this period will be greater than $r_t = 1$.

Use equation (4) to solve for the $z_t = z^*$, such that $r_t = 1$, and such

$$z^* = 0.750808568787$$

After obtaining

$$z^* = 0.75$$

having MLE estimators:

$$\hat{\alpha}_{MLE} = 0.96, \hat{\rho}_{MLE} = 0.99, \hat{\mu}_{MLE} = 2.97, and \hat{\sigma}_{MLE} = 0.33$$

and knowing

$$z_{t-1} = 10$$

$$z_t \sim N(\rho z_{t-1} + (1-\rho)\mu, \sigma^2)$$

we can derive that

$$z_t \sim N(0.99 \times 10 + (1-0.99) \times 2.97, 0.33^2)$$

$$z_t \sim N(\mu = 9.9297, \sigma^2 = 0.1089)$$

and thus

$$\begin{aligned} \Pr(z_t > z^* | \hat{\alpha}_{MLE}, \hat{\rho}_{MLE}, \hat{\mu}_{MLE}, \hat{\sigma}_{MLE}) \\ &= 1 - \Pr(z_t < z^*) \\ &= 1 - F_{z_t}(z^*) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$