OPTIMAL CONTROL

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Contents

Problem formulation

- The Minimum Principle is a set of necessary conditions for optimality that can be applied to a wide class of optimal control problems formulated in $\mathbb{C}^1$. We first bring our attention to the following one:

$$\min_{x,u\in U}\int_0^T f(x(t),u(t))dt$$

subject to

$$\dot{x}(t)=F(x(t),u(t))$$

$$x(0)=x_0,\ \ x(T)=x_T$$

where the final time $T>0$ is given, $x(t)\in\mathcal{X}$ and $u(t)\in U$ for all $t\in[0,T]$.

Problem formulation

- Although more general problems can be considered, here we focus on those that satisfy the following assumptions:
  - For any $u(t) \in U$ the differential equation $\dot{x}(t) = F(x(t), u(t))$ admits one and only one solution $x(t) \in \mathcal{X}$ for all $t \in [0, T]$.
  - The set U is time-decoupled. This means that it may include a constraint of the type

    $$u(t)'u(t) \leq 1, \ \forall t \in [0, T]$$

    but it can not impose a constraint such as

    $$\int_0^T u(t)'u(t)dt \leq 1$$

    In other words, U must impose the same constraint for every time $t \in [0, T]$.

Problem formulation

- From the previous assumption the problem under consideration can be restated as

$$\min_{u(t)\in U} J(u)$$

where $J(\cdot) : \mathbb{R}^m \times [0, T] \to \mathbb{R}$. An important issue to face this optimal control problem is to interpret the differential equation as an equality constraint that must be satisfied for each $t \in [0, T]$ which allows us to associate to it a time-varying Lagrange multiplier $p(t) \in \mathbb{R}^n$. The Lagrangian becomes

$$L(x, u, p) = \int_0^T f(x, u)dt + \int_0^T p'(F(x, u) - \dot{x})dt$$

Problem formulation

- In addition, introducing the Hamiltonian

$$H(x,u,p) = f(x,u) + p'F(x,u)$$

we can write

$$L(x,u,p) = \int_0^T \{H(x,u,p) - p'\dot{x}\}dt$$

Hence, the necessary conditions for optimality of the original problem are obtained by simply applying to the Lagrangian the necessary conditions for optimality we have developed in the previous chapter. In a first step we suppose that $U \equiv \mathbb{R}^m$.

Minimum Principle

- To this end, we have to apply the conditions provided in the previous chapter to each independent variable.
  - With respect to the state variable $x(t) \in \mathbb{R}^n$:

$$\frac{\partial H}{\partial x}(x, u, p) + \dot{p} = 0$$

  - With respect to the multiplier $p(t) \in R^n$:

$$\frac{\partial H}{\partial p}(x, u, p) - \dot{x} = 0$$

  - With respect to the control variable $u(t) \in R^m$:

$$\frac{\partial H}{\partial u}(x, u, p) = 0$$

  - Initial and final time constraints $x(0) = x_0$ and $x(T) = x_T$.

Minimum Principle

- Remarks :
  - As it can be easily verified

$$\frac{\partial H}{\partial p}(x, u, p) = F(x, u)$$

    then the second condition is nothing else than the differential equation $\dot{x} = F(x, u)$.
  - If $U \subset \mathbb{R}^m$ then, as we have established before, the third condition must be replaced by

$$\min_{u \in U} \int_0^T H(x, u, p)dt \equiv \int_0^T \min_{u(t) \in U} H(x(t), u(t), p(t))dt$$

    which holds only if U is time-decoupled.

Minimum Principle

- We now provide a rigorous proof of Pontriagin's Minimum Principle. To this end, let us take two feasible control trajectories $u \in U$ and $v \in U$ such that

$$\|u - v\| < \epsilon$$

for some $\epsilon > 0$ arbitrarily small. Due to continuity, we may argue that the solution of $\dot{y} = F(y, v)$ with $y(0) = x_0$ and $y(T) = x_T$ is of the form

$$y(t) = x(t) + \epsilon\xi(t) + \mathcal{O}(\epsilon^2)$$

for some $\xi(t)$ satisfying $\xi(0) = \xi(T) = 0$.

Minimum Principle

- Consequently, we can write

$$\begin{aligned} J(v) - J(u) &= \int_0^T \{f(y,v) - f(x,u)\}dt \\ &= \int_0^T \{H(y,v,p) - H(x,u,p)\}dt - \epsilon \int_0^T p'\dot{\xi}dt \end{aligned}$$

and integration by parts together with the fact already indicated that $\xi(0) = \xi(T) = 0$, yields

$$\int_0^T p'\dot{\xi}dt = -\int_0^T \dot{p}'\xi dt$$

Minimum Principle

- On the other hand, we also have

$$\begin{aligned} H(y,v,p) &= H(x,v,p) + \epsilon \frac{\partial H}{\partial x}(x,v,p)'\xi + \mathcal{O}(\epsilon^2) \\ &= H(x,v,p) + \epsilon \frac{\partial H}{\partial x}(x,u,p)'\xi + \mathcal{O}(\epsilon^2) \end{aligned}$$

due to the assumption that $\|u - v\| < \epsilon$. Putting all together we finally obtain

$$\begin{aligned} J(v) - J(u) &= \int_0^T \{H(x,v,p) - H(x,u,p)\}dt + \\ &+\epsilon \int_0^T \left\{ \frac{\partial H}{\partial x}(x,u,p) + \dot{p} \right\}' \xi dt + \mathcal{O}(\epsilon^2) \end{aligned}$$

Minimum Principle

- Assuming that $u(t) \in U$ is a local minimum, in view of the previous formula we must have

$$\frac{\partial H}{\partial x}(x, u, p) + \dot{p} = 0$$

and, since x and p does not depend on v

$$H(x, u, p) \leq H(x, v, p), \ \forall v \in U, \ \forall t \in [0, T]$$

otherwise it is possible to determine a feasible trajectory $v \in U \cap \|u - v\| < \epsilon$ such that $J(v) < J(u)$ which is an impossibility.

Minimum Principle

- The next theorem formally states the celebrated Pontriagin's Minimum Principle.

Theorem (8)

Let $T > 0$ be given. If $u \in U$ is a local minimum of the functional $J(u)$ then (x, u, p) such that $x(0) = x_0$, $x(T) = x_T$ satisfy

$$\frac{\partial H}{\partial x}(x, u, p) + \dot{p} = 0$$

$$\frac{\partial H}{\partial p}(x, u, p) - \dot{x} = 0$$

$$\min_{u \in U} H(x, u, p)$$

- The Hamiltonian does not need to be differentiable with respect to u.

Minimum Principle

- It is clear that if $U \equiv \mathbb{R}^m$ and H is differentiable with respect to $u \in \mathbb{R}^m$, then the third condition can be replaced by

$$\frac{\partial H}{\partial u}(x, u, p) = 0$$

  and we obtain once again the optimality conditions provided before. This is done with no loss of generality.
- The necessity join sufficiency for convex problems, that is:
  - Linear differential equation

$$F(x, u) = Ax + Bu + d$$

  - Convex objective function $f(x, u)$ with respect to both variables.
  - Convex set $U \subset \mathbb{R}^m$.

A more general problem

- Consider now the same problem but with free final time, that is $T \geq 0$ is an additional variable to be determined.
- Following the same steps as before, we take two feasible control trajectories such that

$$\|u - v\| < \epsilon$$

and perturb the final time slightly to $T + \epsilon\tau$. Doing this, we can write the cost difference as

$$J(v) - J(u) = \int_0^T \{f(y, v) - f(x, u)\}dt + \int_T^{T+\epsilon\tau} f(y, v)dt$$

A more general problem

- Two points are important to proceed:
  - Due to the continuity of $f(x, u)$, we have

$$\begin{aligned}\int_T^{T+\epsilon\tau} f(y, v)dt &= \epsilon f(y(T), v(T))\tau + \mathcal{O}(\epsilon^2) \\ &= \epsilon f(x(T), u(T))\tau + \mathcal{O}(\epsilon^2)\end{aligned}$$

  - Although $\xi(0) = 0$, the same is not true for $\xi(T)$. To impose $y(T) = x_T$ we must have $\xi(T) = -\dot{x}(T)\tau$. Consequently, integration by parts yields

$$\begin{aligned}\int_0^T p'\dot{\xi}dt &= p(T)'\xi(T) - \int_0^T \dot{p}'\xi dt \\ &= -p(T)'\dot{x}(T)\tau - \int_0^T \dot{p}'\xi dt \\ &= -p(T)'F((x(T), u(T))\tau - \int_0^T \dot{p}'\xi dt\end{aligned}$$

A more general problem

- Hence, putting all these things together, we have

$$\begin{aligned} J(v) - J(u) &= \int_0^T \{H(x,v,p) - H(x,u,p)\}dt + \\ &+\epsilon \int_0^T \left\{ \frac{\partial H}{\partial x}(x,u,p) + \dot{p} \right\}' \xi dt + \\ &+\epsilon H(x(T),u(T),p(T))\tau + \mathcal{O}(\epsilon^2) \end{aligned}$$

where we notice that, compared to the previous case where T was fixed, the only difference in this development is the existence of an additional term due to the fact that now the final time is free.

A more general problem

- The next theorem formally states the necessary conditions of optimality for problems with free final time.

Theorem (9)

If the pair $(u \in U, T \geq 0)$ *is a local minimum of the functional* $J(u)$ *then* (x, u, p) *such that* $x(0) = x_0$, $x(T) = x_T$ *satisfy*

$$\frac{\partial H}{\partial x}(x, u, p) + \dot{p} = 0$$

$$\frac{\partial H}{\partial p}(x, u, p) - \dot{x} = 0$$

$$\min_{u \in U} H(x, u, p)$$

$$H(x(T), u(T), p(T)) = 0$$

A more general problem

- The previous calculations put in evidence that whenever $U \equiv \mathbb{R}^m$ if we determine $J_u(T) = \min_u J(u, T)$ with $T > 0$ fixed then

$$\frac{d}{dT} J_u(T) = H(x(T), u(T), p(T))$$

  This important property can be used to evaluate the cost variation with respect to the final time. For instance, the problem with free final time can be approached by the classical gradient method.

- This and other numerical issues will be treated in the sequel.

Discussion

- The theorems we have just presented enable us to put in evidence an important property of the Hamitonian in the particular case that $U \equiv \mathbb{R}^m$ and $H(\cdot)$ does not depend on $t \in [0, T]$ explicitly. Using the optimality conditions, we obtain

$$\begin{aligned} \frac{d}{dt}H(x(t), u(t), p(t)) &= \frac{\partial H'}{\partial x}\dot{x} + \frac{\partial H'}{\partial u}\dot{u} + \frac{\partial H'}{\partial p}\dot{p} \\ &= \frac{\partial H'}{\partial x}\frac{\partial H}{\partial p} + \frac{\partial H'}{\partial u}\dot{u} - \frac{\partial H'}{\partial p}\frac{\partial H}{\partial x} \\ &= 0 \end{aligned}$$

$$\Downarrow$$

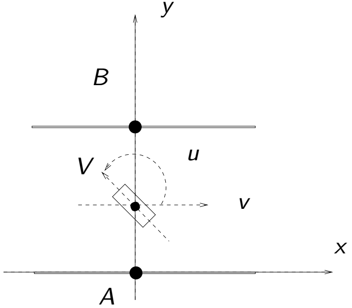
$$H(x(t), u(t), p(t)) = c = \mathrm{cte}, \ \forall t \in (0, T)$$

Discussion

- We want to stress that the assumption $U \in \mathbb{R}^m$ is extremely important to establish the previous result because we have used the fact that the optimality with respect to the control is characterized by
$$\frac{\partial H}{\partial u}(x, u, p) = 0$$
Clearly, if this assumption does not hold then the Hamiltonian may not be constant during all $t \in [0, T]$.
- For problems of this class with free final time, at the optimal solution, the Hamiltonian must satisfy
$$H(x(t), u(t), p(t)) = 0, \ \forall t \in (0, T)$$

Example

- A small boat with constant speed V with respect to the water must be controlled to cross a river of width ℓ from points A and B as indicated in the next figure. The rudder angle $u(t)$ is the control to be determined, v is the water speed with respect to the border and the goal is to minimize the total time needed to cross the river.



Example

- We need to solve the following free final time problem:

$$\min_{u(t)} \int_0^T dt$$

subject to

$$\begin{aligned} \dot{x} &= V\cos(u) + v, & x(0) = 0,\ x(T) = 0 \\ \dot{y} &= V\mathrm{sen}(u), & y(0) = 0,\ y(T) = \ell \end{aligned}$$

The Hamiltonian is written as

$$H = 1 + p_x(V\cos(u) + v) + p_y V\mathrm{sen}(u)$$

and the necessary conditions for optimality can be readily applied (see Theorem (9)) for details.

Example

- We immediately have

$$\frac{\partial H}{\partial x} + \dot{p}_x = 0, \;\rightarrow\; \dot{p}_x = 0$$

$$\frac{\partial H}{\partial y} + \dot{p}_y = 0, \;\rightarrow\; \dot{p}_y = 0$$

This means that p_x and p_y are constant for all $t \in [0, T]$.

$$\frac{\partial H}{\partial u} = 0, \;\rightarrow\; u(t) = \mathrm{tg}^{-1}(p_y/p_x) = u_0, \;\forall t \in [0, T]$$

Imposing the initial and final conditions, we obtain

$$x(t) = (V\cos(u_0) + v)t, \;\; y(t) = (V\mathrm{sen}(u_0))t, \;\; t \in [0, T]$$

Example

- The condition $x(T) = 0$ yields the optimal control

$$u(t) = \cos^{-1}(-v/V), \;\; t \in [0, T]$$

and $y(T) = \ell$ yields the minimum final time

$$T = \frac{\ell}{V \mathrm{sen}(u_0)} = \frac{\ell}{\sqrt{V^2 - v^2}}$$

As expected, this problem admits a solution only if

$$V > v$$

otherwise the final point B will never be attained.

Suggested problems

- **Problem 1:** Determine the necessary conditions for optimality of the problem

$$\min_{x,u\in U} \int_0^T f(x(t),u(t))dt + g(x(T))$$

subject to

$$\dot{x}(t) = F(x(t),u(t)), \; x(0) = x_0$$

where the final time $T > 0$ is given and the final state is free.

- **Problem 2:** Solve the above problem considering

$$F(x,u) = x + u, \; x_0 = 1, \; T = 2, \; g(x) = 0$$

$$f(x,u) = -3x + 3u + u^2, \; U = \{u \;:\; 0 \le u \le 1\}$$

Suggested problems

- **Problem** 3: Solve the problem

$$\max_{x,0\leq u\leq 5} \int_0^{12} (x(t) - u(t))dt + x(12)$$

subject to

$$\dot{x}(t) = -2x(t) + u(t), \;\; x(0) = 100$$

- **Problem** 4: Solve the problem

$$\min_{x,u} \int_0^{\infty} (x(t)^2 + u(t)^2)dt$$

subject to

$$\dot{x}(t) = x(t) + u(t), \;\; x(0) = 1$$