# OPTIMAL CONTROL

**JOSÉ C. GEROMEL**

DSCE / School of Electrical and Computer Engineering
UNICAMP, 13083-852, Campinas, SP, Brazil,
geromel@dsce.fee.unicamp.br

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# Contents

# Introduction

- In this chapter, we provide some information about numerical methods that can be adopted to solve optimal control problems.
- Special attention will be given to the celebrated Hamilton-Jacob-Bellman Equation (HJBE). Whenever solved (in general,it is very hard to solve) it provides the global minimum of any optimal control problem. The control law is of the form

$$u(t) = \mu(x(t), t)$$

  called closed-loop structure. Remember that the local minimum obtained from the Pontriagin's Minimum Principle is of the form $u(t)$ called open-loop structure.

# Bellman's Principle of Optimality

- To introduce the celebrated Bellman's Principle of Optimality, we need to define the following cost-to-go function

$$\nu(\xi, \tau) \ : \ \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$$

which for the initial condition $x(\tau) = \xi$ equals the minimum value of the objective function calculated in the time interval $t \in [\tau, T]$.

$$\nu(\xi, \tau) = \min_{u \in U} \int_{\tau}^{T} f(x, u) dt + g(x(T))$$

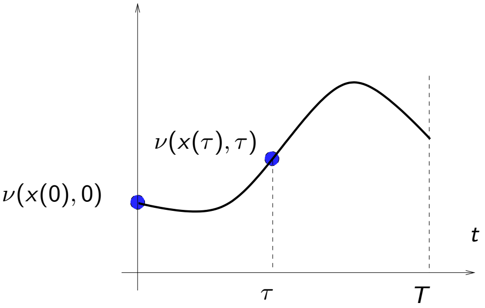
subject to

$$\dot{x} = F(x, u), \ \ x(\tau) = \xi$$

We are presenting a problem with terminal cost. Any other optimal control problem can be considered.

# Bellman's Principle of Optimality

- Bellman's Principle of Optimality states that any remaining part of an optimal trajectory is optimal. See the illustration in the next figure.




$$\nu(x(0),0) = \min_{u \in U}\left\{\int_0^{\tau} f(x,u)dt + \nu(x(\tau),\tau)\right\}$$

# Bellman's Principle of Optimality

- Or, equivalently for $t \in (0, T)$ and $\delta t \to 0^+$

$$\begin{aligned} \nu(x(t), t) &= \min_{u \in U} \left\{ \int_t^{t+\delta t} f(x, u) dt + \nu(x(t+\delta t), t+\delta t) \right\} \\ &= \min_{u \in U} \{ f(x, u)\delta t + \nu(x(t+\delta t), t+\delta t) \} \end{aligned}$$

Assuming that $\nu \in \mathbb{C}^1$ we may adopt the development

$$\begin{aligned} \nu(x(t+\delta t), t+\delta t) &= \nu(x + F(x, u)\delta t, t + \delta t) \\ &= \nu(x, t) + \frac{\partial \nu}{\partial t}\delta t + \frac{\partial \nu}{\partial x}' F(x, u)\delta t \end{aligned}$$

Notice that we have to impose $\delta t > 0$ as a consequence of Bellman's Principle of Optimality.

# Bellman's Principle of Optimality

- Putting all these things together we obtain

$$\frac{\partial \nu}{\partial t}(x,t) + \min_{u \in U}\left\{ H(x,u,\frac{\partial \nu}{\partial x}(x,t)) \right\} = 0$$

where we have to impose the value of the cost at the final time, that is

$$\nu(x,T) = g(x)$$

This is a partial differential equation denominated Hamilton-Jacobi-Bellman Equation (HJBE). In general, it is difficult to solve. A way to solve it numerically is by the adoption of discretization leading to the Dynamic Programming recursive equation.

# Bellman's Principle of Optimality

- It is important to keep in mind that once the HJBE is solved, it provides the minimum cost

$$\nu(x_0, 0)$$

and the associated optimal control law. Notice that it is given by

$$u(t) = \mu(x, t) = \arg\min_{u \in U} H(x, u, \frac{\partial \nu}{\partial x}(x, t))$$

This is a well known and important control structure called closed-loop. By construction, the solution of the HJBE always provides the optimal control with this structure.

# Lyapunov Theory

- An interesting particular case is characterized by infinity horizon $T = +\infty$. The HJBE starts with $\nu(x, +\infty)$ and proceeds backwards to calculate the optimal solution associated to $\nu(x_0, 0)$. We argue that this is equivalent to determine the stationary solution to the HJBE by imposing that the function $\nu(x)$ does not depend on time, that is

$$\min_{u \in U} \left\{ H\left( x, u, \frac{d\nu}{dx}(x) \right) \right\} = 0$$

As before, the optimal control law exhibits the closed-loop form $u(t) = \mu(x(t))$ for all $t \geq 0$.

# Lyapunov Theory

- Assuming that $f(x,u) > 0$ for all $(x,u) \neq (0,0)$ and $f(0,0) = 0$ it is simple to verify that

$$\nu(x) > 0 \ \forall x \neq 0$$

and $\nu(0) = 0$. In other words, $\nu(x)$ is a positive definite function over all $x \in \mathbb{R}^n$ and so it can be used as a Lyapunov function associated to the closed-loop system

$$\dot{x} = F(x, \mu(x))$$

Notice that we have to assume that the origin $x = 0$ is an equilibrium point.

# Lyapunov Theory

- The time derivative along an arbitrary trajectory is

$$\begin{aligned}
\dot{\nu}(x) &= \frac{d\nu}{dx}(x)'\dot{x} \\
&= \frac{d\nu}{dx}(x)'F(x,\mu(x)) \\
&= -f(x,\mu(x)) \\
&< 0, \ \forall x \neq 0
\end{aligned}$$

proving global asymptotic stability, that is $\lim_{t\to\infty} x(t) = 0$ for any initial condition $x(0) = x_0$. Moreover,

$$\int_0^\infty f(x(t),\mu(x(t)))dt = \nu(x_0)$$

evaluates the optimal performance.

# Numerical Solution

- The simplest way (but in general still hard) to solve the HJBE is by discretization. Adopting the notation $x(t_k) = x_k$ and $u(t_k) = u_k$, taking $\delta t > 0$ sufficiently small and $t_{k+1} - t_k = \delta t$ such that $t_0 = 0$ and $T = t_N = N\delta t$, we adopt the first order approximations
  - *Differential equation :* $x_{k+1} = \underbrace{x_k + F(x_k, u_k)\delta t}_{G(x_k, u_k)}, \ x_0 = x(0).$
  - *Objective function :* $J(u) = \sum_{k=0}^{N-1} \underbrace{f(x_k, u_k)\delta t}_{h(x_k, u_k)} + g(x_N).$

  The discrete-time version of the HJBE is

$$\nu(x_k, k) = \min_{u_k \in U} \{h(x_k, u_k) + \nu(x_{k+1}, k+1)\}$$

# Numerical Solution

- With the difference equation, it becomes

$$\begin{aligned} \nu(x,k) &= \min_{u\in U}\{h(x,u)+\nu(G(x,u),k+1)\}, \;\; k=N-1,\cdots,0 \\ \nu(x,N) &= g(x) \end{aligned}$$

In principle, this is a simple recursion that can be handled with no major difficulty. However, as the reader can verify, depending on the involved dimensions $n$, $m$ and $N$ the computation burden involved is prohibitive. As it will be clear in the sequel, the HJBE is relevant by its theoretical importance.

# Example

- The following problem is important in the framework of Networked Control Systems. For $T > 0$ and $a(t)$ given for all $t \in [0, T]$, our goal is to solve

$$\min_{u(t)\in\mathbb{R}} \int_0^T \frac{1}{u(t)-1} dt$$

subject to

$$\underbrace{\int_0^T u(t)dt = c}_{\dot{x}=u,\ x(0)=0,\ x(T)=c}$$

# Example

- As before, the discretization provides the problem to be solved of the form

$$\min_{v_k \in \mathbb{R}} \sum_{k=0}^{N-1} \frac{a^2}{v_k - a}$$

subject to

$$x_{k+1} = x_k + v_k, \;\; x(0) = 0, \;\; x(N) = c$$

where $a = \delta t > 0$ and $v_k = u(t_k)\delta t$. To solve the HJBE we move time backwards for $k = N-1, \cdots, 0$. Step by step we obtain the minimum cost and the control law for each time.

# Example

- For $k = N$ we have
$$\nu(x, N) = 0$$
- For $k = N - 1$ we have $\mu(x, N-1) = c - x$ and
$$\nu(x, N-1) = \min_{v \in \mathbb{R}} \left\{ \frac{a^2}{v-a}, \; c = x + v \right\} = \frac{a^2}{(c-x) - a}$$
- For $k = N - 2$ we have $\mu(x, N-2) = (c-x)/2$ and
$$\nu(x, N{-}2) = \min_{v \in \mathbb{R}} \left\{ \frac{a^2}{v-a} + \frac{a^2}{(c-(x+v)) - a} \right\} = \frac{2a^2}{(c-x)/2 - a}$$
- For $k = N - r$ we have $\mu(x, N-r) = (c-x)/r$
$$\nu(x, N-r) = \frac{ra^2}{(c-x)/r - a}$$

# Example

- The minimum cost is obtained by setting $k = 0$ and $x_0 = 0$

$$\nu(0,0) = \frac{Na^2}{c/N - a}$$

- The optimal solution is of the closed-loop form

$$v_k = \mu(x_k, k) = \frac{(c - x_k)}{(N - k)}, k = 0, \cdots, N - 1$$

which together with the difference equation allows us to determine

$$v_k = \frac{c}{N}, \; k = 0, \cdots, N - 1$$

$$x_k = k\frac{c}{N}, \; k = 0, \cdots, N$$

# Suggested problems

- **Problem** 1: Assume that $f(x,u) > 0, \ \forall (x,u) \neq (0,0)$. Prove that $u(t) = \mu(x(t))$ provided by

$$\min_{u \in U} \left\{ H\left( x, u, \frac{d\nu}{dx}(x) \right) \right\} \leq 0$$

is such that:

  - The origin $x = 0$ is a globally asymptotically stable equilibrium point of the closed-loop system $\dot{x} = F(x, \mu(x))$.
  - The cost function satisfies

$$\int_0^{\infty} f(x(t), \mu(x(t)))dt \leq \nu(x_0)$$

# Suggested problems

- **Problem** 2: Consider the scalar linear-quadratic problem

$$\min_u \int_0^\infty (x(t)^2 + u(t)^2)dt$$

subject to

$$\dot{x}(t) = x(t) + u(t),\ x(0) = 1$$

  - Solve the stationary HJB-equation by adopting a solution of the type $\nu(x) = px^2$.
  - Solve the HJB inequality

$$\min_u \left\{ H\left(x, u, \frac{d\nu}{dx}(x)\right) \right\} = q \leq 0$$

    by adopting a solution of the type $\nu(x) = px^2$. Plot the function $p(q)$ for $0 \leq q \leq 2$ and compare the solutions provided in both cases.