OPTIMAL CONTROL

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Linear-Quadratic Problem

- This is perhaps the most important problem in the framework of Optimal Control theory. It has consequences in many other areas as for instance, optimal filtering, optimal fitting and optimal estimation among others. It is stated as follows:

$$\min_{u(t)} \frac{1}{2} \int_0^T \{x(t)'Qx(t) + u(t)'Ru(t)\} \, dt$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \; x(0) = x_0$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $Q \geq 0$, $R > 0$ and $T > 0$ is fixed.

Linear-Quadratic Problem

- We first focus on the open-loop solution of this problem. The Hamiltonian is

$$H(x, u, p) = \frac{1}{2}\{x'Qx + u'Ru\} + p'(Ax + Bu)$$

which is convex with respect to the state and strictly convex with respect to the control. Applying the Minimum Principle we obtain

$$\begin{aligned} Qx + A'p + \dot{p} &= 0, \;\; p(T) = 0 \\ Ax + Bu - \dot{x} &= 0, \;\; x(0) = x_0 \\ Ru + B'p &= 0 \end{aligned}$$

Linear-Quadratic Problem

- The optimal control is $u(t) = -R^{-1}B'p(t),\ \forall t \in [0, T]$, and the pair (x, p) must satisfy the LTI differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ p \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ p(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

Notice the initial and final time conditions on $x(0)$ and $p(T)$. Partitioning $\Phi = e^{\mathcal{A}T} \in \mathbb{R}^{2n\times 2n}$ in four square blocks we verify that

$$p(0) = -\Phi_{22}^{-1}\Phi_{21}x_0 \Longrightarrow p(T) = 0$$

Linear-Quadratic Problem

- Consequently, the optimal trajectories are given by

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} I \\ -\Phi_{22}^{-1}\Phi_{21} \end{bmatrix} x_0, \quad t \in [0, T]$$

From this, the optimal control can be written as

$$u(t) = \begin{bmatrix} 0 & -R^{-1}B' \end{bmatrix} e^{\mathcal{A}t} \begin{bmatrix} I \\ -\Phi_{22}^{-1}\Phi_{21} \end{bmatrix} x_0, \quad t \in [0, T]$$

This is essentially an open-loop control. Notice that if the initial condition x_0 changes, the optimal control must be recalculated!

Linear-Quadratic Problem

- We now turn our attention to the solution of the HJBE. The main issue is to search for a solution of the form

$$\nu(x,t) = \frac{1}{2}x'P(t)x$$

where $P(t)$ is a symmetric and nonnegative definite matrix function to be determined. The HJBE becomes

$$x\dot{P}(t)x + \min_{u\in\mathbb{R}^m}\left\{\frac{1}{2}(x'Qx + u'Ru) + x'P(t)(Ax + Bu)\right\} = 0$$

where the inner minimization gives the optimal control

$$u(t) = \mu(x(t),t) = -R^{-1}B'P(t)x(t), \;\; t \in [0,T]$$

Linear-Quadratic Problem

- Plugging in the HJBE, it is seen that the symmetric solution $P(t) \geq 0$ must satisfy

$$-\dot{P}(t) = A'P(t) + P(t)A - P(t)BR^{-1}B'P(t) + Q$$

and $\nu(x, T) = (1/2)x'P(T)x = 0, \ \forall x \in \mathbb{R}^n$ imposes the final time condition

$$P(T) = 0$$

This is known as the Differential Riccati Equation. A solution $P(t) \geq 0, \ t \in [0, T]$ is determined by backwards integration starting from the final time condition $P(T) = 0$. Such a solution always exists and is unique!

Linear-Quadratic Problem

- The optimal solution is of the form

$$u(t) = -K(t)x(t), \;\; K(t) = R^{-1}B'P(t)$$

which is essentially a closed-loop control with a linear feedback gain $K(t)$. Under the optimal control, the closed-loop system is defined by the LTV differential equation

$$\dot{x} = (A - BK(t))x, \;\; x(0) = x_0$$

Finally, the minimum cost is given by

$$\nu(0,0) = \frac{1}{2}x_0' P(0) x_0$$

Linear-Quadratic Problem

- An important particular case is the Infinite Horizon LQP which corresponding to $T = +\infty$. The reasoning is that the imposition of $P(\infty) = 0$ and integration backwards will lead to (if any) an equilibrium (or stationary) solution characterized by $\dot{P}(t) = 0$. Hence $P \geq 0$ satisfies

$$A'P + PA - PBR^{-1}B'P + Q = 0$$

the Algebraic Riccati Equation. Under the assumption that the pair (A, B) is stabilizable, such a solution always exists and is unique. The optimal control is now of the form

$$u(t) = -Kx(t), \; K = R^{-1}B'P$$

of a constant feedback gain.

Linear-Quadratic Problem

- Assuming that $Q > 0$ and (A, B) stabilizable, the algebraic Riccati equation admits a positive definite solution $P > 0$. Hence, from the factorization

$$(A - BK)'P + P(A - BK) = -K'RK - Q$$

the Lyapunov function $v(x) = x'Px$ candidate is such that

$$\begin{aligned} \frac{dv}{dt}(x) &= \dot{x}'Px + x'P\dot{x} \\ &= -x'(Q + K'RK)x \\ &< 0 \ \forall x \neq 0 \in \mathbb{R}^n \end{aligned}$$

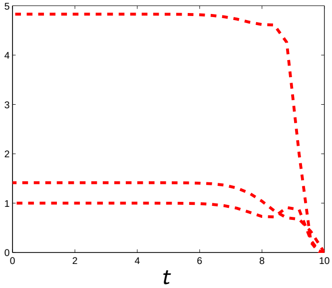
proving thus the optimal closed-loop system is asymptotically stable.

Example

- **Example 1 :** An LQP with $T = 10$ [s] and

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; R = 1$$

The next figure shows the elements of $P(t)$, $\forall t \in [0, T]$.



Example

- It is interesting to see that starting the integration backwards from $T = 10$ and $P(10) = 0$ at $t = 0$ we obtain

$$P(0) = \begin{bmatrix} 1.4142 & 1.0000 \\ 1.0000 & 4.8284 \end{bmatrix} \approx P = \lim_{t \to -\infty} P(t)$$

In addition, we determine the optimal state feedback gain

$$K = \begin{bmatrix} 1.0000 & 4.8284 \end{bmatrix}$$

and the eigenvalues of the closed-loop system $\dot{x} = (A - BK)x$

$$\{-1.4142, \ -1.4142\}$$

attesting, as expected, its asymptotic stability.

Linear-Linear Problem

- It is also called Linear-Dynamic Problem and has a particular importance in the context of positive linear systems. It is stated as follows:

$$\min_{u(t)\in U} \int_0^T \left\{c'x(t) + d'u(t)\right\} dt$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \ \ x(0) = x_0$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $T > 0$ is fixed and

$$U = \{u \in \mathbb{R}^m \ : \ \underline{u} \leq u \leq \overline{u}\}$$

Linear-Linear Problem

- We first discuss the use of Pontriagin's Minimum Principle. The Hamiltonian

$$H(x,u,p) = c'x + d'u + p'(Ax + Bu)$$

provides the optimality conditions

$$\begin{aligned} c + A'p + \dot{p} &= 0, \;\; p(T) = 0 \\ Ax + Bu - \dot{x} &= 0, x(0) = \;\; x_0 \\ u(t) &= \arg\min_{u \in U} H(x,u,p) \end{aligned}$$

where we notice that the set U is necessary to assure the existence of an optimal control since the Hamiltonian is a linear function.

Linear-Linear Problem

- Solving the first equation

$$p(t) = e^{-A't}p(0) - \int_0^T e^{-A'(t-\tau)}cd\tau, \;\; t \in [0, T]$$

imposing $p(T) = 0$, after some algebraic manipulations we have

$$p(t) = \left(\int_0^{T-t} e^{A'\xi}d\xi\right)c, \;\; t \in [0, T]$$

Defining $\lambda(t) = d + B'p(t), \forall t \in [0, T]$ we see that the third optimality condition provides the optimal control

$$u_i(t) = \begin{cases} \underline{u}_i & , \quad \lambda_i(t) > 0 \\ \overline{u}_i & , \quad \lambda_i(t) \leq 0 \end{cases}$$

for $i = 1, \cdots, m$.

Linear-Linear Problem

- Since the trajectory $\lambda(t), \ t \in [0, T]$ can be calculated a priori, there is no difficulty to determine the optimal control $u(t), t \in [0, T]$. This is essentially an open-loop solution.
- In continuous-time, the closed-loop solution provided by the HJBE é not simple to calculate.
- Adopting the procedure discussed before, the discrete-time version of the HJBE can be solved, but again, numerical difficulties due to large dimensions may appear.

Minimum Time Problem

- The Minimum Time Problem consists on the determination of a control law that transfers the initial state $x(0) = x_0$ to the origin $x(T) = 0$ in minimum time. A simplified version with a single control variable is presented.

$$\min_{u(t) \in U} \int_0^T dt$$

subject to

$$\dot{x}(t) = Ax(t) + bu(t), \; x(0) = x_0, \; x(T) = 0$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $T > 0$ is free and

$$U = \{u \in \mathbb{R} \; : \; |u| \leq 1\}$$

Minimum Time Problem

- The Hamiltonian is written in the form

$$H(x,u,p) = 1 + p'(Ax + bu)$$

and gives the optimality conditions

$$\begin{aligned} A'p + \dot{p} &= 0 \\ Ax + bu - \dot{x} &= 0, \;\; x(0) = x_0, \;\; x(T) = 0 \\ u &= \arg\min_{u \in U} H(x,u,p) \\ H(x(T), u(T), p(T)) &= 0 \end{aligned}$$

Minimum Time Problem

- We immediately have the optimal control

$$u(t) = \begin{cases} -1 & , \quad b'p(t) > 0 \\ 1 & , \quad b'p(t) \leq 0 \end{cases}$$

and consequently

$$\min_{u \in U} H(x, u, p) = 1 + p(t)'Ax - |b'p(t)|$$

which allows us to rewrite the last optimality condition as

$$|b'p(T)| = 1$$

Example

- In the general case, it is not simple to solve the previous optimality conditions. To put in evidence the difficulties we consider a well known example - the double integrator - defined as

$$\min_{|u(t)|\leq 1} \int_0^T dt$$

subject to

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u(t), \;\; x(T) = 0$$

where the initial condition $x(0) = x_0 \in \mathbb{R}^2$ is given but arbitrary.

Example

- Hence, from the optimality conditions with $p(t) \in \mathbb{R}^2$ we have

$$b'p(t) = p_2(t) = -p_{10}t + p_{20}, \;\; t \in [0, T]$$

where $p(0) = [p_{10} \;\; p_{20}]'$. Based on this, we have the following implications:

  - Optimality is characterized by presenting at most one control commutation in the time interval $[0, T]$.
  - The optimal control is readily determined from the commutation time t_c such that $b'p(t_c) = 0$, that is

$$t_c = p_{20}/p_{10}$$

The optimal control reduces to the determination of the initial condition $p(0) \in \mathbb{R}^2$. This is not a simple task.

Example

- Based on the fact that at most one commutation is allowed to the optimal control, we observe that, in the phase plane (x_1, x_2), the trajectories satisfy

$$u \int_{x_{10}}^{x_1} dx_1 - \int_{x_{20}}^{x_2} x_2 dx_2 = 0$$

whenever $u(t) = \pm 1$ is constant for all $t \in [0, T]$. The trajectories passing through the origin are such that

$$\begin{cases} x_1 + x_2^2/2 = 0 & , \quad u = -1 \\ x_1 - x_2^2/2 = 0 & , \quad u = 1 \end{cases}$$

Example

- Defining the state depending commutation function as being

$$S(x) = x_1 + \frac{1}{2}x_2|x_2|$$

we conclude that the state feedback control law given by

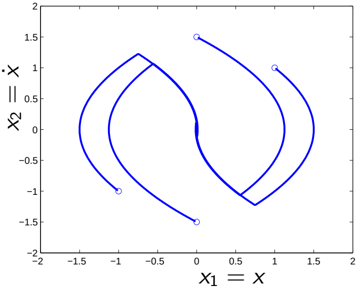
$$u(t) = \begin{cases} -1 & , \quad S(x(t)) > 0 \\ 1 & , \quad S(x(t)) \leq 0 \end{cases}$$

is optimal for the problem under consideration. The following conclusions are in order:

  - The optimal control is a kind of relay control. It is implemented from the output of an ideal relay having the signal $S(x(t))$ as input.
  - Clearly, the state of the system must be measured on-line.

Example

- The next figure shows the optimal trajectories for different initial conditions.



Notice that the origin is a globally asymptotically stable equilibrium point of the closed-loop system.

Suggested problems

- **Problem 1:** Solve the minimum time problem for the double integrator with arbitrary initial condition and final state given by

$$\begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- **Problem 2:** For the double integrator solve the Infinite Horizon LQP with cost

$$\int_0^\infty (x_1(t)^2 + \rho u(t)^2) dt$$

where $\rho \in [0.1, 10]$. Determine $P(\rho) > 0$, the gain $K(\rho)$ and the eigenvalues of $A - BK(\rho)$ for all ρ in the given interval.