OPTIMAL CONTROL

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Preliminaries

- Our first goal is to handle the following optimization problem

$$\min_{x \in \mathcal{X}} J(x)$$

where:

  - $t = 0$ and $t = T$ are the initial and final times. $T > 0$ is given.
  - $x(t) : [0, T] \to \mathbb{R}^n$ is a trajectory defined $\forall t \in [0, T]$.
  - $\mathcal{X}$ is the set of all feasible trajectories. It may include constraints on $x(0)$ and $x(T)$.
  - $J(x) : \mathcal{X} \to \mathbb{R}$ is an objective function of the general type

$$J(x) = \int_0^T f(x(t), \dot{x}(t))dt$$

It is assumed that $f \in \mathbb{C}^1$. More sophisticated objective function will be addressed as well.

Preliminaries

- To show how to obtain the optimality conditions for the previous problem we consider, in addition, the initial and final time constraints

$$x(0) = x_0 \; . \; x(T) = x_T$$

- Assume a feasible trajectory $x^*(t)$ is optimal and is slightly perturbed

$$x(t) = x^*(t) + \epsilon\xi(t), \;\; \forall t \in [0, T]$$

where $\epsilon \in \mathbb{R}$ is an arbitrarily small parameter and $\xi(0) = \xi(T) = 0$. From Taylor's series development we obtain

$$J(x) = J(x^*) + \epsilon \int_0^T \left\{ \frac{\partial f'}{\partial x} \xi + \frac{\partial f'}{\partial \dot{x}} \dot{\xi} \right\} dt + \mathcal{O}(\epsilon^2)$$

Preliminaries

- The key observation is

$$\int_0^T \left\{ \frac{\partial f}{\partial \dot{x}}' \dot{\xi} \right\} dt = \frac{\partial f}{\partial \dot{x}}' \xi \Big|_0^T - \int_0^T \left\{ \frac{d}{dt} \frac{\partial f}{\partial \dot{x}}' \xi \right\} dt$$

which, imposing $\xi(0) = \xi(T) = 0$ yields

$$J(x) = J(x^*) + \epsilon \int_0^T \left\{ \frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \right\}' \xi dt + \mathcal{O}(\epsilon^2)$$

- Since $x(t)$ is optimal, we must have $J(x) \leq J(y)$ for all $\epsilon \in \mathbb{R}$ sufficiently small and all feasible trajectory $\xi(t)$. The consequence is that $x^*(t)$ must satisfy

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} = 0$$

Preliminaries

- The next theorem provides a necessary condition for optimality

Theorem (6)

Let $T > 0$ be given. If $x(t)$ satisfying $x(0) = x_0$ and $x(T) = x_T$ is a local minimum then it satisfies the so called Euler-Lagrange equation

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$$

- A trajectory candidate to be optimal can be determined by solving the Euler-Lagrange equation subject to the initial time $x(0) = x_0$ and final time $x(T) = x_T$ constraints.
- In general, the Euler-Lagrange equation requires a numerical procedure to be solved.

Preliminaries

- **Example 1 :** Consider $x(t) : [0,1] \to \mathbb{R}$ such that $x(0) = 0$ and $x(1) = 1$. We want to solve

$$\min_{x \in \mathcal{X}} \int_0^1 \sqrt{1 + \dot{x}(t)^2} dt$$

The Euler-Lagrange equation is written as

$$\frac{d}{dt} \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} = 0$$

which reduces to

$$\ddot{x} = 0$$

Solving it under the previous constraints, we get the unique solution $x^*(t) = t$ which is the line segment connecting the points $(0, x(0))$ and $(1, x(1))$, as expected.

Preliminaries

- **Example 2 :** In the plane (x, t) consider two points $(1, 0)$ and $(0, 1)$. Only under the gravity acceleration g, determine the curve $x(t)$, $0 \leq t \leq 1$ such that a mass starting from the rest in the first point will reach the second one in minimum time. We have to solve

$$\min_{x \in \mathcal{X}} \int_0^1 \sqrt{\frac{1 + \dot{x}(t)^2}{2gx(t)}} dt$$

Applying the Euler-Lagrange equation we obtain

$$2x\ddot{x} + (\dot{x})^2 + 1 = 0$$

subject to $x(0) = 1$ and $x(1) = 0$. This is a nonlinear second order differential equation. How to solve it, analytically or numerically?

A more general problem

- Our goal now is to handle the following optimization problem

$$\min_{x \in \mathcal{X}, T \geq 0} J(x, T)$$

where:

- $t = 0$ and $t = T$ are the initial and final times. $T > 0$ is free.
- $x(t) : [0, T] \to \mathbb{R}^n$ is a trajectory defined $\forall t \in [0, T]$.
- $\mathcal{X}$ is the set of all feasible trajectories. It includes the constraints $x(0) = x_0$ and $x(T) = x_T$.
- $J(x, T) : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ is an objective function of the general type

$$J(x, T) = \int_0^T f(x(t), \dot{x}(t))dt$$

It is assumed that $f \in \mathbb{C}^1$. The final time T is an optimization variable to be determined.

A more general problem

- Perturbing slightly the optimal trajectory as before and the optimal final time to $T = T^* + \epsilon\tau$ with some $\tau \in \mathbb{R}$, we have

$$J(x, T) = J(x, T^*) + \int_{T^*}^{T^*+\epsilon\tau} f(x(t), \dot{x}(t))dt$$

The first term has already been treated before, providing the first order approximation

$$\begin{aligned} J(x, T^*) \quad &= \quad J(x^*, T^*) + \epsilon \int_0^{T^*} \left\{ \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \right\}' \xi dt + \\ &+ \epsilon \left. \frac{\partial f}{\partial \dot{x}}' \xi \right|_0^{T^*} + \mathcal{O}(\epsilon^2) \end{aligned}$$

A more general problem

- Clearly $\xi(0) = 0$ since $x(0) = x^*(0) = 0$.
- However, the perturbed trajectory $x(t) = x^*(t) + \epsilon\xi(t)$, calculated at $t = T^*$ does not imply that $\xi(T^*) = 0$. Indeed, we can not impose $x(T^*) = x_T$ but we must impose that $x(T) = x(T^* + \epsilon\tau) = x_T$, that is

$$\begin{aligned} x(T^* + \epsilon\tau) &= x(T^*) + \epsilon\dot{x}(T^*)\tau + \mathcal{O}(\epsilon^2) \\ &= x^*(T^*) + \epsilon(\xi(T^*) + \dot{x}^*(T^*)\tau) + \mathcal{O}(\epsilon^2) \end{aligned}$$

which from the fact that $x^*(T^*) = x_T$ requires that

$$\xi(T^*) = -\dot{x}^*(T^*)\tau$$

A more general problem

- Putting these relations all together we have

$$\begin{aligned} J(x,T) &= J(x^*,T^*) + \epsilon \int_0^{T^*} \left\{ \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \right\}' \xi dt + \\ &+\epsilon \left\{ f(x^*(T^*), \dot{x}^*(T^*)) - \frac{\partial f}{\partial \dot{x}}' \dot{x}^*(T^*) \right\} \tau + \mathcal{O}(\epsilon^2) \end{aligned}$$

- Now, taking into account that the function $\xi(t)$, $\forall t \in [0, T^*]$ and the scalar $\tau \in \mathbb{R}$ are arbitrary, we obtain the next result by setting to zero the first order term of the development.

A more general problem

- The next theorem provides a necessary condition for optimality

Theorem (7)

If the pair $(x(t), T)$ satisfying $x(0) = x_0$, $x(T) = x_T$ and $T \geq 0$ is a local minimum then it satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$$

and the so called transversality condition

$$f(x(T), \dot{x}(T)) - \frac{\partial f}{\partial \dot{x}}' \dot{x}(T) = 0$$

- Compared to Theorem (6), the present one has one more constraint. It is necessary since now the final time $T > 0$ is an additional variable to be determined.

A more general problem

- **Example 3 :** We want to solve with $x(0) = 0$ and $x(T) = 1$

$$\min_{x \in \mathcal{X}, T \geq 0} \int_0^T (1 + x^2 + \dot{x}^2) dt$$

Applying the Euler-Lagrange equation we obtain $\ddot{x} - x = 0$ which yields

$$x(t) = \frac{\mathrm{senh}(t)}{\mathrm{senh}(T)}, \ \forall t \in [0, T]$$

while the transversality condition $\dot{x}(T) = \sqrt{2}$ allows us to determine the optimal final time as

$$\tanh(T) = \frac{1}{\sqrt{2}} \Longrightarrow T \approx 0.9$$

Discussion

- In the present case we have to handle trajectories $x(t) : [0, T] \to \mathbb{R}^n$. For $x(t)$ and $\xi(t)$ given we can evaluate the first order variation of $J(x + \epsilon\xi)$ in the general form

$$\delta J(x, \xi) = \lim_{\epsilon \to 0} \frac{J(x + \epsilon\xi) - J(x)}{\epsilon}$$

Hence, an extremal trajectory x^* satisfying the necessary condition for optimality is obtained from

$$\delta J(x^*, \xi) = 0, \ \forall \xi(t), \forall t \in [0, T]$$

This was the strategy used previously.

Discussion

- Once a candidate for minimum x^* has been obtained, we can go further to determine

$$\delta^2 J(x^*, \xi) = \lim_{\epsilon \to 0} \frac{J(x^* + \epsilon\xi) - J(x^*)}{\epsilon^2}$$

which characterizes a local minimum whenever

$$\delta^2 J(x^*, \xi) > \; 0, \; \forall \xi(t) \neq 0, \forall t \in [0, T]$$

This sufficient optimality test is accomplished by Taylor's series development. In general, the determination of $\delta^2 J(x^*, \xi)$ is not difficult but it is very time consuming.

Suggested problems

- **Problem 1:** Provide the necessary conditions for optimality of the problem

$$\min_{x\in\mathcal{X}} \int_0^T f(x(t),\dot{x}(t))dt + g(x(T))$$

where $x(0) = x_0$, $x(T)$ is free and $T > 0$ is given.

- **Problem 2:** Solve the problem

$$\min_{x\in\mathcal{X}} \int_0^1 \left(x(t)^2 + \dot{x}(t)^2\right) dt$$

subject to $x(0) = 1$ and $x(1) = 0$.

Suggested problems

- **Problem 3:** Solve the problem

$$\min_{x \in \mathcal{X}} \int_0^1 \left(x(t)^2 + \dot{x}(t)^2\right) dt + 2x(1)$$

subject to $x(0) = 1$.

- **Problem 4:** Solve the problem

$$\min_{x \in \mathcal{X}} \int_0^1 \left(x(t)^2 + \dot{x}(t)^2\right) dt + x(1)^2$$

subject to $x(0) = 1$.

Suggested problems

- **Problem 5:** Solve the problem

$$\min_{x \in \mathcal{X}} \int_0^T \left(x(t)^2 + \dot{x}(t)^2 \right) dt$$

subject to $x(0) = 1$, $x(T) = 1$ and $T > 0$ free.

- **Problem 6:** Solve the problem

$$\min_{x \in \mathcal{X}} \int_0^T \left(x(t)^2 - \dot{x}(t)^2 \right) dt$$

subject to $x(0) = 1$, $x(T) = 1$ and $T > 0$ free.