Lectures Notes

BV functions and sets of finite perimeter

Winter Term 2019/20

Lecture by Giovanni Comi

Notes taken by Dennis Schmeckpeper

Version: October 16, 2019

Contents

Chapter 1

Introduction?

1.0 Motivation

Let's consider

$$\inf\left\{\int_\Omega |\nabla u| dx \,:\, u\in W^{1,1}(\Omega), \|u\|_{L^1} = K > 0\right\} =: m_K,$$

where $W^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega,\mathbb{R}^n)\}$ is the Sobolev space. Then there exists a sequence $(u_j \in W^{1,1}(\Omega))_{j\in\mathbb{N}}$ such that $\|u_j\|_{L^1(\Omega)} = K$ and $\|\nabla u_j\|_{L^1(\Omega)} \to m_K$ for $j\to\infty$. Now, in general this does *not* imply that there is a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ which will convergence even only weakly to an $v\in W^{1,1}(\Omega)$ with $\|v\|_{L^1} = K$ and $\|\nabla v\|_{L^1} = m_k$.

The reason for this is essentially because L^1 is not a (topological) dual of any space, though it is contain in one. Lacks details

Another example is the ***Isoperimetric problem***:

$$\min\{\sigma_{n-1}(\partial F) \,:\, F \text{ with some regularity}, |F| = K > 0\} =: \gamma_k,$$

where $|F| =: \mathcal{L}^n(F)$ denotes the n-dimensional Lebesgue measure.

1.1 Measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*, that is, the collection of all subsets of X.

Definition ((outer) measure)**.** A mapping $\mu : \mathcal{P}(X) \to [0,+\infty]$ satisfying

(1) $\mu(\emptyset) = 0$

(2) $\mu(A) \le \sum_{k=1}^\infty \mu(A_k)$ if $A\subset \bigcup_{k=1}^\infty A_k$ (σ-subadditivity)

is called an (outer) measure.

Remark. The (outer) measure is not decreasing, that is, for $A\subset B$, where $A, B\in\mathcal{P}(X)$, we have $\mu(A)\le\mu(B)$.

Definition (Restriction of a measure)**.** If $Y\subset X$, the *restriction of* μ *to* Y, denoted by $\mu \llcorner Y$, is defined as $(\mu\llcorner Y)(A) := \mu(Y\cap A)$.

Definition (μ-measurable)**.** We call a subset $A\subset X$ μ-measurable if

$$\mu(B) = \mu(B\cap A) + \mu(B\setminus A) \qquad \text{for all} \quad B\subseteq X.$$

Remark. This definition is meaningful since *Vitali* found that there exists a set $E\subset\mathbb{R}$ which is *not* $\mathcal{L}^1$-measurable.

Definition (σ-algebra)**.** A subset $\mathfrak{F}\subset\mathcal{P}(X)$ is called a *σ-algebra of sets* if holds

(1) $\emptyset, X\in\mathfrak{F}$,

(2) for $A \in \mathfrak{F}$ also $X \setminus A \in \mathfrak{F}$,

(3) for a family $(A_i \in \mathfrak{F})_{i \in I}$ we have have $\bigcup_{i \in I} A_i \in \mathfrak{F}$.

Theorem. *Let μ be a (outer) measure on X, then the restriction to the σ-algebra of μ-measurable sets is σ-additive, that is, if $(A_j)_{j \in I}$ is a (at most) countable disjoint μ-measurable family of subsets of X, then*

$$\mu\left(\bigcup_{j \in I} A_j\right) = \sum_{j \in I} \mu(A_j).$$

Definition. Here we collect some important definitions

(1) Let $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ-algebra containing $\mathfrak{C}$, the *σ-algebra generated by* $\mathfrak{C}$.[♯]

(2) The *Borel-algebra* on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ-algebra generated by the family of open sets in $\mathbb{R}^n$ (in the standard topology). The elements of the Borel-algebra are called *Borel sets*.

(3) A (outer) measure μ in $\mathbb{R}^n$ is called a *Borel measure* if each Borel sets is μ-measurable.

(4) A (outer) measure μ in $\mathbb{R}^n$ is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

(5) A Borel regular measure μ which is locally finite (e.g. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure.*

Theorem. *Let μ be a Radon measure on $\mathbb{R}^n$. We have*

(1) for all $A \subseteq \mathbb{R}^n$ holds $\mu(A) = \inf\{\mu(U) : U \supset A,\ U \text{ open}\}$ (outer regularity),

(2) for all μ-measurable sets B holds $\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ compact}\}$ (inner regularity).

Theorem (Carathéodory's criteria)**.** *Let μ be a (outer) measure on $\mathbb{R}^n$. If for all $A, B \subset \mathbb{R}^n$ that satisfy* $\operatorname{dist}(A, B) > 0$ *we have $\mu(A \cup B) = \mu(A) + \mu(B)$, then μ is a Borel measure.*

Examples.

(1) For $x \in \mathbb{R}^n$ we can define that *dirac measure* by

$$\delta_x(A) := \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

This is in fact a Radon measure.

(2) We define the *counting measure* by

$$\#(E) = \begin{cases} \operatorname{card}(E) & \text{if } E \text{ is finite} \\ +\infty & \text{otherwise.} \end{cases}$$

This measure is Borel regular, but *not* a Radon measure (since it is clearly not locally finite).

(3) The also have the well-known *Lebesgue measure* defined by

$$\mathcal{L}^n(A) := \inf\left\{\sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i,\ Q_i \text{ cubes}\right\},$$

where $\mathcal{L}^n(Q_i)$ is equal to the side length of the cubes Q_i to the n-th power. In particular, we have

$$\mathcal{L}^1(A) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam} C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j,\ C_j \subset \mathbb{R}\right\}$$

and so we can characterize

$$\mathcal{L}^n = \underbrace{\mathcal{L}^1 \times \mathcal{L}^1 \times \cdots \times \mathcal{L}^1}_{n-\text{times}} = \mathcal{L}^{n-1} \times \mathcal{L}^1.$$

[♯] Here $\mathfrak{C}$ = \mathfrak{C}

(4) (**Hausdorff measure**) Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *Hausdorff α-dimensional content of A* as

$$\mathcal{H}^\alpha_\delta(A) := \inf \left\{ \sum_{j \in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2} \right)^\alpha \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j,\ \operatorname{diam} C_j \leq \delta, C_j \subseteq \mathbb{R}^n \right\},$$

where the infimum is taking over all the (at most countable) coverings $(C_j \subset \mathbb{R}^n)_{j \in I}$ of A, and set $w_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2}+1)}$. Since $\mathcal{H}^\alpha_\delta$ is a not-increasing function in δ the following limit

$$\mathcal{H}^\alpha(A) := \lim_{\delta \searrow 0} \mathcal{H}^\alpha_\delta(A) = \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(A)$$

always exists in the extended real numbers. This limit is defined to be the *Hausdorff measure.*

Theorem (Hausdorff measure is Borel regular)**.** *$\mathcal{H}^\alpha$ is a Borel regular measure on $\mathbb{R}^n$ for all $\alpha \geq 0$.*

Theorem (Basic properties of the Hausdorff measure)**.**

(1) $\mathcal{H}^0 = \#$

(2) $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$

(3) $\mathcal{H}^\alpha \equiv 0$ for all $\alpha > n$ in $\mathbb{R}^n$.

(4) $\mathcal{H}^\alpha(\lambda A) = \lambda^\alpha \mathcal{H}^\alpha(A)$ for all $A \subseteq \mathbb{R}^n$ and $\lambda > 0$

(5) $\mathcal{H}^\alpha(L(A)) = \mathcal{H}^\alpha(A)$ for all affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$.

Proof.

(1) Since $\omega_0 = 1$ we have

$$\begin{aligned}
\mathcal{H}^0(A) &= \lim_{\delta \searrow 0} \inf \left\{ \sum_{j \in I} \left(\frac{\operatorname{diam}(C_j)}{2} \right)^0 \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j,\ \operatorname{diam} C_j \leq \delta \right\} \\
&= \lim_{\delta \searrow 0} \inf \left\{ \sum_{j \in I} 1 \mid A \subset \bigcup_{j \in I} C_j,\ \operatorname{diam} C_j \leq \delta \right\} \\
&= \begin{cases} \operatorname{card}(A) & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}
\end{aligned}$$

(2) We estimate the Lebesgue measure $\mathcal{L}^1$ from both sides by the Hausdorff measure: Since $\omega_1 = 2 = |(-1,1)|$ we first get

$$\begin{aligned}
\mathcal{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\
&\leq \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j,\ \operatorname{diam} C_j \leq \delta \right\} = \mathcal{H}^1_\delta(A),
\end{aligned}$$

which is true for all $\delta > 0$ so we obtained $\mathcal{L}^1(A) \leq \mathcal{H}^1(A)$.
Now, we define a partition of $\mathbb{R}$ by $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$ and first fixed $\delta > 0$. These are intervals of diameter δ so for every $j \in I$ we get $\operatorname{diam}(C_j \cap I_{k,\delta}) \leq \delta$. Also we have $\sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap I_{k,\delta}) \leq \operatorname{diam} C_j$, since $I_{k,\delta}$ are a partition $\mathbb{R}$ of disjoint intervals in k. So we get

$$\mathcal{L}^1(A) = \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \geq \inf \left\{ \sum_{j \in I} \sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap I_{k,\delta}) \mid A \subset \bigcup_{j \in I} \bigcup_{k \in \mathbb{Z}} C_j \cap I_{k,\delta} \right\}$$

since $\operatorname{diam}(C_j \cap I_{k,\delta}) \leq \delta$ and after relabeling the index sets I and $\mathbb{Z}$ to an index set $I^{(k,\delta)}$ this last expressions reads

$$\cdots = \inf \left\{ \sum_{j \in I^{(k,\delta)}} C_j^{(k)} \mid A \subset \bigcup_{j \in I^{(k,\delta)}} C_j^{(k)},\, \operatorname{diam} C_j^{(k)} \leq \delta \right\} \geq \mathcal{H}^1_\delta.$$

And since this is true for every $\delta > 0$ we arrive at the claim.

(3) TODO

□