③ Using the definition of inner product

$$\langle a | b \rangle = \sum_{j=0}^{N-1} a_j^* b_j$$

$$\langle z^k | z^{k'} \rangle = \sum_{j=0}^{N-1} (z_j^k)^* (z_j^{k'}) = \sum_{j=0}^{N-1} e^{-i2\pi k j/N} * e^{i2\pi k' j/N}$$

$$= \sum_{j=0}^{N-1} e^{i2\pi (k-k') j/N} \quad (A)$$

if $k' = k \Rightarrow$ the exponent become 1, then the sum is N

if $k' \neq k$, and ~~r~~ $r = e^{i2\pi (k'-k)/N}$ and S is the sum, then, equation (A) become

$$S = 1 + r + r^2 + r^3 + \ldots + r^{N-1} \quad (B)$$

multiplying by r on both sides

$$Sr = r + r^2 + r^3 + \ldots + r^N \quad (C)$$

Substracting (B) - (C) I obtain:

$$S - Sr = (1 + r + r^2 + \ldots r^{N-1}) - (r + r^2 + \ldots + r^N)$$

$$S = \frac{1 - r^N}{1 - r} \longrightarrow r^N = 1 \text{ because}$$

$$r^N = \left(e^{i2\pi(k'-k)}\right)^N = e^{i2\pi(k'-k)}$$

$$e^{i2\pi m} = \cos(2\pi m) + i\sin(2\pi m)$$

$$= 1 + i \cdot 0 = 1$$

$$r^N = 1$$

then ~~then~~ I can say that $S = 0$, then

$$\frac{1}{N} \langle z^k | z^{k'} \rangle = \delta_{kk'}$$

$\Rightarrow$ orthogonal

④ $f(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$ $\quad F(q) = \int_{-\infty}^{\infty} f(x)\, e^{-iqx}\, dx$

exponential:

$$-\frac{x^2}{2\sigma^2} - iqx = -\frac{1}{2\sigma^2}\left(\underbrace{x^2 + 2i\sigma^2 q x}\right)$$

$$\to x^2 + 2i\sigma^2 q x = (x + i\sigma^2 q)^2 - (i\sigma^2 q)^2 = (x + i\sigma^2 q)^2 + \sigma^4 q^2$$

therefore

$$-\frac{x^2}{2\sigma^2} - iqx = -\frac{(x + i\sigma^2 q)^2}{2\sigma^2} - \frac{\sigma^2 q^2}{2}$$

I can do the transform

$$F(q) = \int_{-\infty}^{\infty} \exp\left(-\frac{(x + i\sigma^2 q)^2}{2\sigma^2}\right) \exp\left(-\frac{\sigma^2 q^2}{2}\right) dx = e^{-\frac{\sigma^2 q^2}{2}} \underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{(x + i\sigma^2 q)^2}{2\sigma^2}\right) dx}$$

$\to$ the integral is a Gaussian integral, translated by $a = i\sigma^2 q$ and scaled by σ

$$\int_{-\infty}^{\infty} e^{-\left(\frac{x+a}{2\sigma^2}\right)^2} dx = \sigma\sqrt{2} \int_{-\infty}^{+\infty} e^{-y^2} = \sigma\sqrt{2} \cdot \sqrt{\pi}$$

$$F(q) = e^{-\sigma^2 q^2/2} \cdot \sqrt{2\pi} \cdot \sigma$$