Lecturer: Dr. Amira Meddah
Email: amira.meddah@jku.at

Tutorial 4

Exercise 18

Homogeneous Poisson process I

a.) Implement an algorithm to generate a path of the homogeneous Poisson process $(N_t)_{t\in[0,T]}$ with rate λ.

b.) Plot a path for $\lambda = 3$ and $T = 10$.

c.) Write an algorithm to obtain $M = 10^4$ realizations of the random variable N_2. Compare the empirical pmf of the simulations with the true pmf of N_2.

d.) Write an algorithm to obtain $M = 10^4$ realizations of the jumping time T_4. Compare the empirical pdf (probability density function) of the simulations with the true pdf of T_4.

Exercise 19

Non-homogeneous Poisson process

a.) Implement an algorithm to generate a path of the non-homogeneous Poisson process $(N_t)_{t\in[0,T]}$ with rate function $\lambda(t) = 5\sin^2(t/2)$.

b.) Plot a path for $T = 30$.

c.) Produce $M = 10^4$ realizations of the random variable N_5. Compare the empirical pmf of the simulations with the true pmf of N_5.

Exercise 20

Wiener process with drift

Let $(W_t)_{t\geq 0}$ be a Wiener process. We consider the Wiener process with drift defined via $X_t = \mu t + \sigma W_t$, for $t \geq 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$.

a.) Write a function in R to simulate a path of the Wiener process with drift on a given interval $[0, T]$ and for a given time step $h > 0$.

b.) Generate 5 paths of the Wiener process with drift for $\mu = 5$ and $\sigma = 1$ on the interval $[0, 1]$ with $h = 10^{-3}$ and visualize them in one plot. Add the function $t \to \mathbb{E}[X_t]$ to the plot.

c.) Generate paths of the Wiener process with drift using $\mu = 0, -1, 1, -5, 5$ and $\sigma = 1$ on the interval $[0, 10]$ with $h = 10^{-3}$ and visualize them in one plot.

d.) Generate paths of the Wiener process with drift for $\mu = 0$ and $\sigma = 0, 1, 5, 10$ on the interval $[0, 10]$ with $h = 10^{-3}$ and visualize them in one plot.

Exercise 21

Geometric Brownian motion: Exact simulation

Consider the geometric Brownian motion which solves the SDE

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = X_0.$$

a.) Implement an exact simulation method to generate a path of the geometric Brownian motion.

b.) Consider $T = 1$, $h = 10^{-3}$, $X_0 = 5$, $\mu = 1$ and $\sigma = 0.4$. Use your algorithm to generate 10 paths and visualize them in one plot. Add the mapping $t \to \mathbb{E}[X(t)]$ to the plot.

Exercise 22

Geometric Brownian motion: Numerical methods

Consider the geometric Brownian motion which solves the SDE

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = X_0.$$

a.) Implement the Euler-Maruyama method to generate a path of the geometric Brownian motion. Consider $T = 1$, $h = 10^{-3}$, $X_0 = 5$, $\mu = 1$ and $\sigma = 0.4$. Use your algorithm to generate a path and plot it.

b.) Implement the Milstein method to generate a path of the geometric Brownian motion. Consider $T = 1$, $h = 10^{-3}$, $X_0 = 5$, $\mu = 1$ and $\sigma = 0.4$. Use your algorithm to generate a path and plot it.

Exercise 23

Geometric Brownian motion: mean-square convergence order

Consider again the setting of Exercise 24, 25 and 26. In order to illustrate the mean-square convergence order of the Euler-Maruyama and Milstein methods, we consider the *root mean-squared error* RMSE(h) (see the lecture slides of Week 10) as a function of the time step h. In particular, we use $t^* = 1$, $M = 10^3$ and $h = 2^{-l}$, for $l = 2, \dots, 10$. Moreover, we use now $X_0 = 5$, $\mu = -0.05$ and $\sigma = 0.2$.

Mimic the plot of slide 10 from Week 10, i.e., plot RMSE(h) of the Euler-Maruyama and Milstein methods in log-scale, and illustrate their mean-square convergence order.

Remark: Note that for every $k \in \{1, \dots, M\}$, the values $\widetilde{X}_k(t^*)$ (k-th path at time t^* obtained under the respective numerical method) and $X_k(t^*)$ (k-th path at time t^* obtained under the exact simulation method) have to be simulated with the same pseudo-random numbers.

Exercise 24

Monte Carlo integration

Consider the following integral

$$\theta = \int_0^2 (\cos(50x) + \sin(20x))^2 \, dx.$$

a.) Use standard numerical integration to approximate θ.

b.) Use Monte Carlo integration to obtain an estimate $\tilde{\theta}$ for θ. *Hint*: You can use that Y is uniformly distributed and consider $n = 10^4$ iid realizations of Y.

c.) Produce a plot to illustrate the dependence of your result on n. Add a horizontal line with the value for θ obtained via standard numerical integration to the plot.