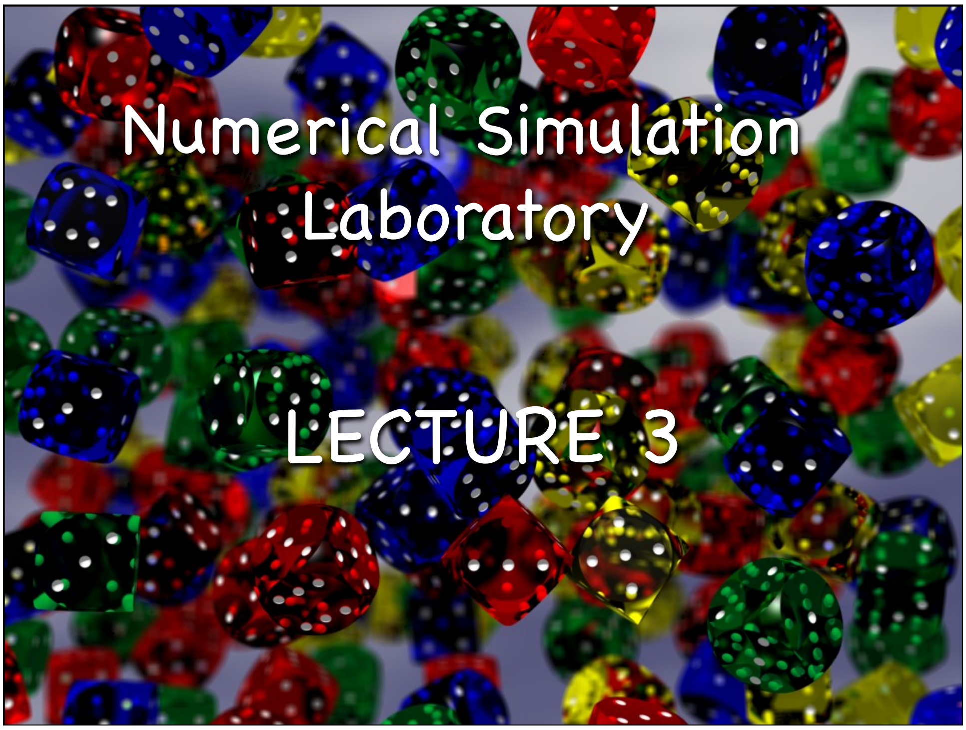

Outline

- Wiener process = Brownian motion, Brownian noise
- Stochastic integrals & Stochastic differential equations
- Itô formula
- Introduction to Financial markets
- Geometric Brownian Motion
- Derivatives
- Classical option pricing & Numerical pricing

2

Stochastic calculus

- When dealing with continuous stochastic processes, describing phenomena depending on time, one wishes to have tools to study functions of stochastic processes, typically performing "derivatives" or "integrals", and to characterize them via stochastic differential or integral equations. Stochastic calculus is the branch of mathematics dealing with this important topic.

- Traditional calculus is not suitable for stochastic processes, lies essentially in the fact that stochastic processes have non differentiable trajectories

- The stochastic calculus, moreover, allows to write down differential equations involving stochastic processes, providing thus a powerful generalization of ordinary differential equations to study phenomena evolving in time in a non deterministic way.

- We will introduce the formalism of stochastic differential equations, but before we do this let's introduce a concrete physical example: the Brownian motion

The most important idea in science?

- "If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis (or the atomic *fact*, or whatever you wish to call it) …

- …that *all things are made of atoms—little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another*." R.P. Feynman

- Not new! Lucippus/Democritos (around 440 B.C.)

Brownian motion

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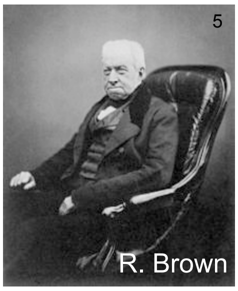

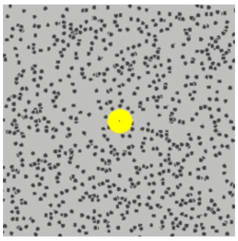

- Brownian motion provides some of the most spectacular evidence for the atomic nature of matter. This can be seen if one immerses a large particle (usually about one micron in diameter) in a fluid with the same density as the particle: When viewed under a microscope, the large particle (the Brownian particle) appears to be in a state of agitation, undergoing rapid and random movements. Early in the nineteenth century, the biologist R. Brown wrote a paper on this phenomenon (1827).
- Not new! Lucretius (De rerum natura) 99-55 B.C.
- The modern era in the theory of Brownian motion began with Einstein (1905), who, initially unaware of the widely observed phenomenon of Brownian motion, was looking for a way to confirm the atomic nature of matter.
- Not new! L. Bachelier (PhD thesis in Mathematics: "Théorie de la Spéculation", 1900): stochastic treatment of the price of financial assets.

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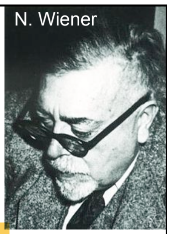
Wiener process



- A rigorous mathematical theory for the Brownian motion was constructed by N. Wiener in 1923, after which the Brownian motion became also known as the Wiener process
- Definition: The standard Brownian motion or Wiener process $\{W(t), t \geq 0\}$ is a stochastic process with the following properties:

1. $W(0) = 0$
2. The increments $W(t)-W(s)$ are stationary and independent
3. For $t>s$, $W(t)-W(s)$ has a normal dist.: $\mathcal{N}(0;t-s)$ [notation: $\mathcal{N}(\mu;\sigma^2)$]

Dati 2 intervalli (t1,s1) e (t2,s2) che non si intersecano le variabili casuali "incremento" W(t1)-W(s1) e W(t2)-W(s2) sono indipendenti.

La variabile casuale W(t)-W(s) è distribuita secondo una Gaussiana di varianza t-s

- The stationarity condition implies that the probability distribution for $W(t)-W(s)$, for $t > s$, depends only on the time difference $t-s$
- Conditions 2 and 3 imply that $W(t)$ is distributed according to $\mathcal{N}(0;t)$ for $t > 0$ (a diffusion process!). In particular, we have $\langle W(t)\rangle = 0$ for all $t>0$. Moreover, being $Z_i \sim \mathcal{N}(0;1)$ one can sample, without any discretization error, a Wiener process via:

$$W(t_{i+1}) = W(t_i) + Z_{i+1}\sqrt{t_{i+1} - t_i} \qquad i = 0,1,\cdots$$

X~N(mu,sigma), Z~N(0,1)
—> X=mu+sigma*Z

which is a (continuos in time!) random walk with i.i.d Normal increments.

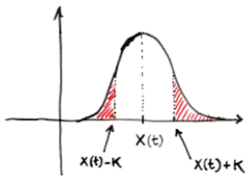
indipendent
identically distributed

Path properties

- The Brownian motion has continuous sample paths:

$$Prob\left[\left|W(t+\Delta t)-W(t)\right|>k\right]=2\int_k^\infty dx\,\frac{1}{\sqrt{2\pi\Delta t}}\exp\left[-\frac{x^2}{2\Delta t}\right]$$

$$\Rightarrow \lim_{\Delta t\to 0} Prob\left[\left|W(t+\Delta t)-W(t)\right|>k\right]=0 \qquad \forall k$$



- The sample paths are, with probability one, nowhere differentiable:

$$Prob\left[\left|\frac{W(t+\Delta t)-W(t)}{\Delta t}\right|>k\right]=2\int_{k\Delta t}^\infty dx\,\frac{1}{\sqrt{2\pi\Delta t}}\exp\left[-\frac{x^2}{2\Delta t}\right]$$

$$\lim_{\Delta t\to 0} Prob\left[\left|\frac{W(t+\Delta t)-W(t)}{\Delta t}\right|>k\right]=1 \qquad \forall k$$

- This opens a problem in the interpretation of quantities like:

$$\frac{dW(t)}{dt};\ dW(t)$$

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Fractal properties

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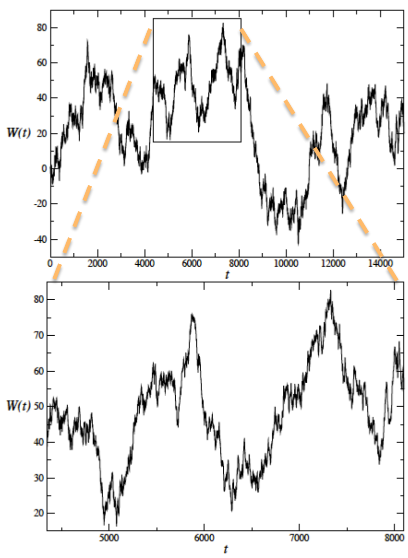
Moreover, the Wiener process W(t) is self-similar in the following sense:

$W(at) \underset{d}{=} a^{1/2}W(t)$ for all a > 0.

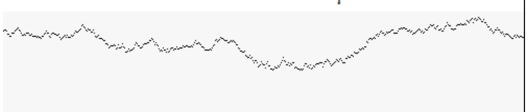
- This equality symbol is in the sense of probability distribution, that is, the two processes have the same joint distribution

⇒ any finite portion of W(t) when properly rescaled is (statistically) indistinguishable from the whole path

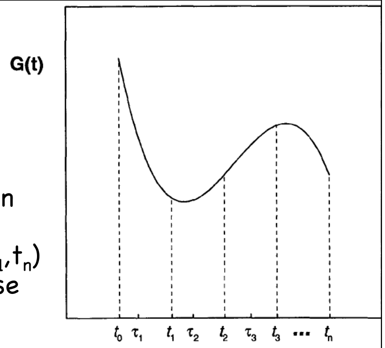
e.g. if we 'zoom in' in any given region of W(t), by rescaling the time axis by a factor of a and the vertical axis by a factor of √a, we obtain a curve similar (statistically speaking) to the original path.



The average features of W(t) do not change while zooming in, and it zooms in quadratically faster horizontally than vertically.



Mean-square limit



- Let us consider a function $G(t)$ over an interval (t_0,t_n)
- In the same way as is done for the Riemann integral, we consider a partitioning of this interval into subintervals $(t_0,t_1),(t_1,t_2),\ldots,(t_{n-1},t_n)$ and a set of points τ_i, $t_{i-1} < \tau_i < t_i$ inside these subintervals
- We define the mean-square limit in the following way:

$$\text{ms-}\lim_{n\to\infty} X_n = X \Leftrightarrow \lim_{n\to\infty} \left\langle (X_n - X)^2 \right\rangle = 0$$

- We then define: $\int_{t_0}^{t_n} G(t')\,dW(t') := \text{ms-}\lim_{n\to\infty} \sum_{i=1}^{n} G(\tau_i)\left[W(t_i) - W(t_{i-1})\right]$
- Convergence in the mean-square limit is a weaker requirement than the point-wise convergence used for the Riemann integral. This equation defines a stochastic integral, but not yet uniquely, because its value depends on the choice of points τ_i at which the integrand is evaluated

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Itô stochastic integral

- Setting $\tau_i = \alpha t_i + (1-\alpha)t_{i-1}$, with $0 \le \alpha \le 1$, the most used choice for α is: $\alpha = 0$, defining the Itô stochastic integral: $\tau_i = t_{i-1}$
- For example, with this choice one can show that (see supplementary material):

$$\int_{t_0}^{t_n} W(t')\,dW(t') = 0$$

- The Itô stochastic integral has the advantage that it is a martingale (see supplementary material); this fact leads to many helpful mathematical properties and makes usage of the Ito stochastic integral ubiquitous in the literature
- Definition: A sequence Y_n is a martingale iff $E\left[Y_{n+1} \mid Y_1,\ldots,Y_n\right] = Y_n$
- This means that for a martingale the best estimator for the next value, given all the information obtained through the past values, is the actual value.
- If the process is a martingale, its changes (increment process) are a "fair game", meaning that the expectation value of the increments is zero.

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Causality requirements

- A central concept for Itô stochastic integrals is that of non-anticipating functions
- Definition: a function G(t) is called a non-anticipating function (also: an "adapted process") of t when G(t) is statistically independent of W(s)-W(t) for all s,t with t < s
- This is a type of causality requirement, where one does not want G(t) to anticipate the future behaviour of the Wiener process W(t)
- Examples of non-anticipating functions are:
  - $W(t)$ $\quad \left(\left\langle W(t)\left[W(s)-W(t)\right]\right\rangle = \left\langle W(t)W(s)\right\rangle - \left\langle W(t)W(t)\right\rangle = t - t = 0\right)$
  - $\int_{t_0}^{t} dt'\, G(t')$
    If G itself is non-anticipating
  - $\int_{t_0}^{t} dW(t')\, G(t')$
  - $\int_{t_0}^{t} dt'\, F\left[W(t')\right]$ where F is a functional of the Wiener process.
  - $\int_{t_0}^{t} dW(t')\, F\left[W(t')\right]$

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Mnemonic equations

- For an arbitrary non-anticipating G(t') we have (see supplementary information):

$$\text{ms-}\lim_{n\to\infty}\left(\sum_{i=1}^{n}\Delta W_i^2 G(t_{i-1}) - \sum_{i=1}^{n}\Delta t_i G(t_{i-1})\right) = 0 \quad \Rightarrow \int_{t_0}^{t}\left[dW(t')\right]^2 G(t') = \int_{t_0}^{t} dt'\, G(t')$$

$$\Rightarrow \boxed{\left[dW(t)\right]^2 = dt}$$

- One can prove also that:

$$\text{ms-}\lim_{n\to\infty}\left(\sum_{i=1}^{n}\Delta W_i^m G(t_{i-1})\right)_{m\geq 3} = \int_{t_0}^{t}\left[dW(t')\right]^m G(t') = 0$$

$$\Rightarrow \boxed{\left[dW(t)\right]^m = 0\,,\ m \geq 3}$$

$$\text{ms-}\lim_{n\to\infty}\sum_{i=1}^{n}\Delta W_i \Delta t_i G(t_{i-1}) = \int_{t_0}^{t} dW(t')\, dt'\, G(t') = 0 \quad \Rightarrow \boxed{dW(t)dt = 0}$$

- One can interpret these equations to mean: $\boxed{dW(t) = O\left(dt^{1/2}\right)}$

Total differential

- As an important consequence, we are now able to derive the behaviour of the total differential of a functional of the Wiener process:

$$\mathrm{d}f\left[W(t),t\right] = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial W}\mathrm{d}W + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\mathrm{d}t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\mathrm{d}W^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t \partial W}\mathrm{d}t\,\mathrm{d}W + \cdots$$

- Making use of:

$$dt^m \to 0 \quad \forall m \geq 2 \qquad dW^2 = dt \qquad dtdW \to 0$$

$$dW^m \to 0 \quad \forall m \geq 3$$

- We get the result $\boxed{\mathrm{d}f\left[W(t),t\right] = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right]\mathrm{d}t + \frac{\partial f}{\partial W}\mathrm{d}W(t)}$
- This last equation already has the form of a stochastic differential equation, that we want to give a meaning.
- We now can proceed to define what is meant by a stochastic differential equation

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Itô stochastic differential equation

- Definition: A stochastic process x(t) obeys an Itô stochastic differential equation (SDE) written as

$$\mathrm{d}x(t) = a\left[x(t),t\right]dt + b\left[x(t),t\right]\mathrm{d}W(t)$$

when for all t_0 , t we have

$$x(t) = x(t_0) + \int_{t_0}^{t} dt'\, a\left[x(t'),t'\right] + \int_{t_0}^{t} dW(t')\, b\left[x(t'),t'\right]$$

i.e. we use the stochastic integral to define the stochastic differential equation

- Suppose now x(t) fulfils the previous SDE and that f is a functional of x and t. We want to compute the total differential df[x(t),t] and to obtain it we perform a Taylor expansion up to the second order:

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}x + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}\mathrm{d}t^2} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}x^2 + \frac{\partial^2 f}{\partial t \partial x}\mathrm{d}t\,\mathrm{d}x$$

- Making use of $dW^2 = dt$ and

$$dt^m \to 0\ \forall m \geq 2 \qquad dW^m \to 0\ \forall m \geq 3 \qquad dtdW \to 0$$

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Itô formula

- We obtain …

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x}\left[a\,dt + b\,dW\right] + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}\left[a\,dt + b\,dW\right]^2 + \frac{\partial^2 f}{\partial t \partial x} dt\left[a\,dt + b\,dW\right] =$$

$$= \frac{\partial f}{\partial t} dt + a\frac{\partial f}{\partial x} dt + b\frac{\partial f}{\partial x} dW + \tfrac{1}{2} b^2 \frac{\partial^2 f}{\partial x^2} dW^2 = \cdots$$

… the Itô formula: $df[x(t),t] =$

$$= \left\{\frac{\partial f[x(t),t]}{\partial t} + a[x(t),t]\frac{\partial f[x(t),t]}{\partial x} + \tfrac{1}{2} b^2[x(t),t]\frac{\partial^2 f[x(t),t]}{\partial x^2}\right\} dt +$$

$$+ b[x(t),t]\frac{\partial f[x(t),t]}{\partial x} dW(t)$$

- e.g. consider the SDE: $dx = a \cdot x\, dt + b \cdot x\, dW$ and $f(x) = \ln(x)$, it follows

$$d\ln[x(t)] = \left\{a \cdot x(t) \cdot \frac{1}{x(t)} + \tfrac{1}{2} b^2 x^2(t) \frac{-1}{x^2(t)}\right\} dt + b \cdot x(t) \frac{1}{x(t)} dW(t)$$

$$\Rightarrow \quad d\ln[x(t)] = \left(a - \tfrac{1}{2} b^2\right) dt + b\, dW(t)$$

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Generating sample paths

- Now we want to develops methods for simulating paths of a variety of stochastic processes. We first consider methods for exact simulation of continuous-time processes at a discrete set of dates
- The methods are exact in the sense that the joint distribution of the simulated values coincides with the joint distribution of the continuous-time process on the simulation time grid
- Exact methods rely on special features of a model and are generally available only for models that offer some tractability. More complex models must ordinarily be simulated through, e.g., discretization of stochastic differential equations. We begin with methods for simulating Brownian motion in one dimension
- Definition: for constants μ and $\sigma > 0$, we call a process X(t) a Brownian motion with drift μ and diffusion coefficient σ^2 (abbreviated X~BM(μ,σ^2)) if

$$\frac{X(t) - \mu t}{\sigma} \sim \mathrm{BM}(0,1)$$

i.e. it is a standard Brownian motion

- Thus, we may construct X from a standard Brownian motion W

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(non standard) Brownian motion

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- It follows: $X(t) = \mu t + \sigma W(t)$ X(t)=mu*t+sigma*rad(t)*Z con Z~N(0,1) —> X~N(mu*t,t*sigma^2)
- Moreover $X(t) \sim \mathcal{N}((\mu t, \sigma^2 t)$, and X(t) solves the SDE:

$$\mathrm{d}X(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$$

- For deterministic but time-varying $\mu(t)$ and $\sigma(t) > 0$, we may define a Brownian motion with drift $\mu(t)$ and diffusion coefficient $\sigma^2(t)$ through the SDE

$$\boxed{\begin{aligned}&\mathrm{d}X(t) = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t)\\ &X(t) = X_0 + \int_0^t \mu(t')\,\mathrm{d}t' + \int_0^t \sigma(t')\,\mathrm{d}W(t')\end{aligned}}$$

i.e. through

where X_0 is an arbitrary constant.
- The process X has continuous sample paths and independent increments X(t)-X(s) which are normally distributed with mean and variance:

$$\langle X(t) - X(s)\rangle = \int_s^t \mu(t')\,\mathrm{d}t' \qquad \mathrm{var}\left[X(t) - X(s)\right] = \int_s^t \sigma^2(t')\,\mathrm{d}t'$$

Sampling Brownian motion

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- Because Brownian motion has independent normally distributed increments, simulating the $W(t_i)$ or $X(t_i)$ from their increments is straightforward
- Let Z_1 , ..., Z_n be independent standard normal random variables, for a standard Brownian motion set $t_0 = 0$ and $W(0) = 0$. Subsequent values can be generated as follows:

$$W(t_{i+1}) = W(t_i) + Z_{i+1}\sqrt{t_{i+1} - t_i} \qquad i = 0,\cdots,n-1$$

- For $X(t)\sim BM(\mu,\sigma^2)$ with constant μ and $\sigma > 0$ and given $X(0)=X_0$ the path can be obtained as:

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma Z_{i+1}\sqrt{t_{i+1} - t_i} \qquad i = 0,\cdots,n-1$$

- Whereas with time-dependent coefficients, the recursion becomes

$$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(t')\,dt' + Z_{i+1}\sqrt{\int_{t_i}^{t_{i+1}} \sigma^2(t')dt'} \qquad i = 0,\cdots,n-1$$

- These methods are exact in the sense that the joint distribution of the simulated values coincides with the joint distribution of the corresponding Brownian motion

From Brownian motion to financial markets

- In 1900 L. Bachelier, essentially developed the mathematics of Brownian motion to model the time evolution of asset prices. Perhaps it was because of its application to the financial market that Bachelier's work was not recognized by the scientific community at that time. It seems as if it fell into complete oblivion until the early 1940s
- In 1944, Itô used it as a motivation to introduce his calculus and a variant of Brownian motion, geometric Brownian motion (GBM)
- GBM, in turn, became an important model for the financial market. Its economic significance was recognized in the work of P.A. Samuelson from 1965 (1970 Nobel Prize in Economics)

- In 1973, F. Black and M. Scholes and, independently, R. Merton used the geometric Brownian motion to construct a theory for determining the price of stock options, a milestone in the development of mathematical finance, (Nobel Prize in Economics to Scholes and Merton 1997)

Econophysics

- Though widely applied, the valuation formulas are not perfect and the discrepancy between theory and application is well documented in the financial literature.
- Recently, physicists have tried to understand these deviations with very diverse approaches, ranging from statistical analyses of the time evolution of asset prices to microscopic trader models. This young branch of physics constitutes an new independent field, which is called econophysics.
- The perception of the financial market as a complex many-body system offers, in fact, an interesting challenge for testing well-established or new physical concepts and methods in a new field.
- Moreover, Financial markets are continuously monitored - down to time scales of seconds. And further, virtually every economic transaction is recorded, and an increasing fraction of the total number of recorded economic data is becoming accessible to interested researchers …
- … a "big data" experiment where models/theories can be falsified a posteriori!

Financial market: basic notions

- Economists would define a market as a 'place', where buyers and sellers meet to exchange 'products'.
- At every time instant t, the products have a price, the spot price S(t).
- The spot price is determined by the interplay of supply and demand in a free-market economy. If demand in a given product increases and supply correspondingly decreases, S(t) increases, and vice versa
- A special kind of market is the financial market, where the product traded is, loosely speaking, money. On the financial market, large sums of money are lent, borrowed and invested in commodities (materie prime) and securities, such as bonds (obbligazioni) or stocks (titoli)
- To accomplish this aim many stringent rules are imposed in order to make trading more efficient and safer. But, these also limit the freedom.
- Market participants, the so called "agents", who don't want to renounce to freedom can resort to over-the-counter (OTC) trading. This implies a contract is established between two parties, the features of which are adapted to the individual needs of the partners

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Risk managment

- Such tailor-made deals can be very speculative and therefore much more risky than standardized contracts of a regulated exchange. But, traders are willing to accept (a certain amount of) risk
- If this were not the case, they could deposit all their money on a bank account and pocket the interest rate granted by the bank
- This return is certain, but low compared to the possible outcome of a highly profitable transaction. This expected gain is the main driving force for the market participants to trade and to deliberately tolerate a certain exposure to risk
- However, everybody obviously wants to make money at the end of the day; nobody wants to lose it
- The aim of modern risk management consists in finding the sources of potential losses and in devising strategies to limit the embedded risk as much as possible
- The risk management is the main driving force to involve Physicists and Mathematicians and their competences into finance

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Financial market, a complex system

- Financial time series look unpredictable, and their future values are essentially impossible to predict
- This property is not a manifestation of the fact that the time series does not reflect any valuable and important economic information
- Indeed, the opposite is true. The time series of the prices in a financial market carries a large amount of non redundant information
- But because the quantity of this information is so large, it is difficult to extract a subset of economic information associated with some specific aspect
- So, the difficulty in making predictions is thus related to an abundance of information in the financial data, not to a lack of it
- When a given piece of information affects the price in a market in a specific way, and this allows for the detection, from the time series of price, of the presence of this information then the market is said to be not completely efficient

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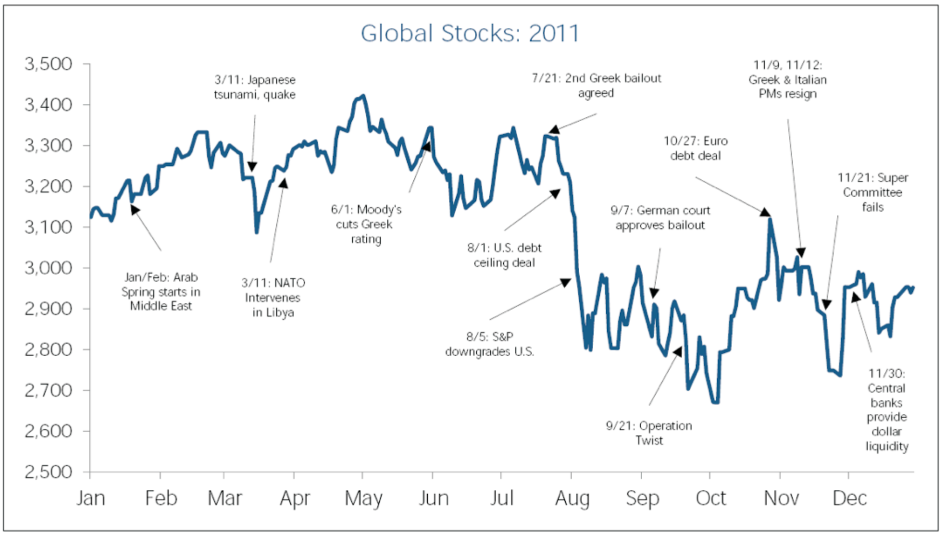
Information influences



Global stocks as represented by MSCI World Index

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Efficient Market Hypothesis

- Definition: a market is called efficient:
  - if the participants quickly and comprehensively obtain all information relevant to trading.
  - if it is liquid. i.e. any investor can easily buy or sell a financial product at any time. The more liquid a market is, the more secure it is to invest. The investor knows that he can always cash-in his assets. This easy exchange between money and financial products raises the attractivity of the market
  - if there is low market friction. Market friction is a collective expression for all kinds of trading costs. The sum of these costs is negligible compared with the transaction volume if the market friction is low.
- An efficient market 'digests' the new information so efficiently that all the current information about the market development is at all times completely contained in the present prices. The efficient market is an idealized system. Real markets are only approximately efficient
- No advantage is gained by taking into account all or part of the previous price evolution. This amounts to a Markov assumption.

Markov assumption

- Imagine that a time series exhibits a structure from which the rise of an asset price could be predicted in the near future. Certainly, investors would buy the asset now and sell it later in order to pocket the difference
- However, an efficient market immediately responds to the increased demand by increasing the price. The profitable opportunity vanishes due to competition between the many active traders
- This argument limits any correlations to a very short time range and advocates the random nature of the time series
- In fact, this Markov assumption seems very plausible and has a long history: after a detailed analysis of the French market, Bachelier drew the same conclusion (1900). He suggested a model for the time series $S(t)$ which we would today call a Wiener process with a drift term to account for the gradual increase of $S(t)$.
- His model suffers from the unrealistic property that it allows negative asset prices. Prices, however, are by definition always positive
- This fact led to a refined version of a random walk model – the geometric Brownian motion – which underlies many modern approaches to pricing derivative securities, such as options.

Modeling the financial market

- To get an idea of how to define a model for the evolution of S(t) let us first assume that at t = 0 we deposit a sum S(0) in a bank
- The bank grants a risk-free interest rate r for the deposit. If the interest is paid once at the end of time t, the initial sum has grown by

$$S(t) = S(0) + rtS(0) = S(0)\left(1 + rt\right)$$

- On the other hand, if it is paid twice, we have

$$S(t) = S(0) + \frac{rt}{2}S(0) + \frac{rt}{2}\left(S(0) + \frac{rt}{2}S(0)\right) = S(0)\left(1 + \frac{rt}{2}\right)^2$$

- Iterating this for n payments in time t, we obtain $S(t) = S(0)\left(1 + \frac{rt}{n}\right)^n$

which yields: $S(t) = S(0)\left(1 + \frac{rt}{n}\right)^n \xrightarrow[n\to\infty]{} \boxed{S(0)\,e^{rt}}$

in the limit of continuously compounded interest

- Now, we imagine the asset price to be similar to a bank deposit, but perturbed by stochastic fluctuations. Therefore, the price change dS in the small time interval dt should consist of two contributions: a deterministic and a random one ...

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Geometric Brownian motion

- A reasonable ansatz for the deterministic part is: $\boxed{d\,S(t) = \mu\,S(t)\,d\,t}$

  where μ is called drift and measures the average growth rate of the asset price. One can expect μ to be larger than r. No investor would assume the risk of losing money on the market if a bank account yielded the same or an even better profit.

- The second contribution models the stochastic nature of the price evolution. It should be Markovian according to the efficient market hypothesis. A possible choice is an ansatz symmetric to the previous one:

$$\boxed{d\,S(t) = \sigma\,S(t)\,d\,W(t)}$$

  this introduces a second phenomenological parameter, the volatility σ. The volatility measures the strength of the statistical price fluctuations.

- Combining both contributions, we obtain the following result:

$$\boxed{d\,S(t) = \mu\,S(t)\,d\,t + \sigma\,S(t)\,d\,W(t)}$$

This SDE defines a variant of Brownian motion, which is called geometric Brownian motion

Return relevance

- Geometric Brownian motion is a specific case of an Itô process
$$dS(t) = a\left[S(t),t\right]dt + b\left[S(t),t\right]dW(t)$$
where a[·] and b[·] are proportional to the random variable S(t)

- This suggests that not the absolute change dS = S(t+dt)–S(t), but the return dS/S(t) in the time interval dt is the relevant variable

- Financially, this is sensible. An absolute change of dS = \$10 in time dt is much more significant for a starting capital of S(t) = \$100 than for one of S(t) = \$10 000. The return dS/S clearly expresses this difference

- The interpretation of dS/S as the relevant variable suggests that should be rewritten in terms of ln S(t). Using Ito's formula we found:

$$\boxed{d\ln S(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW(t)}$$

- Therefore, it is the logarithm of the asset price, and not the price itself which performs a Wiener process with a (constant) drift.

Geometric Brownian motion properties

- Previous discussion implies that d ln S is normally distributed with a mean and variance:
$$\left\langle d\ln S(t)\right\rangle = \left(\mu - \frac{1}{2}\sigma^2\right)dt \qquad \mathrm{var}\left[d\ln S(t)\right] = \sigma^2\, dt$$

- and that the transition probability from (S,t) to (S′,t') is given by
$$p\left(S',t'\middle|S,t\right) = \frac{1}{\sqrt{2\pi\left(\sigma S'\right)^2(t'-t)}}\exp\left\{-\frac{\left[\ln\left(S'/S\right) - \left(\mu - \sigma^2/2\right)(t'-t)\right]^2}{2\pi\sigma^2(t'-t)}\right\}$$

…the price S(t) thus has a log-normal distribution.

- We will use the notation S~GBM(μ,σ^2) to indicate that S is geometric Brownian motion with drift μ and volatility σ

- If S~GBM(μ,σ^2) and if S has initial value S(0), then
$$S(t) = S(0)\exp\left[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W(t)\right]$$

or more generally, if t′<t
$$S(t) = S(t')\exp\left[(\mu - \tfrac{1}{2}\sigma^2)(t-t') + \sigma\left(W(t) - W(t')\right)\right]$$

Sampling a geometric Brownian motion

- Moreover, since the increments of W are independent and normally distributed, this provides a simple recursive procedure for sampling values of S~GBM(μ,σ^2) at $0 = t_0 < t_1 < \ldots < t_n$:

$$S(t_{i+1}) = S(t_i)\exp\left\{(\mu - \tfrac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma Z_{i+1}\sqrt{t_{i+1} - t_i}\right\} \quad i = 0,\cdots,n-1$$

with $Z_1, Z_2, \ldots, Z_n$ independent standard normals.

- In fact, this is equivalent to exponential version of the algorithmic sampling equation for X~BM(μ,σ^2) with μ replaced by $\mu-\sigma^2/2$
- This method is exact (no discretization error) in the sense that the sequence it produces has the joint distribution of the process S~GBM(μ,σ^2) at t_0 , t_1 , …, t_n
- Time-dependent parameters can be incorporated by taking the exponential of the relative algorithmic sampling equation of the Brownian motion with time dependent drift and volatility

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Real S(t): distribution of dS/S

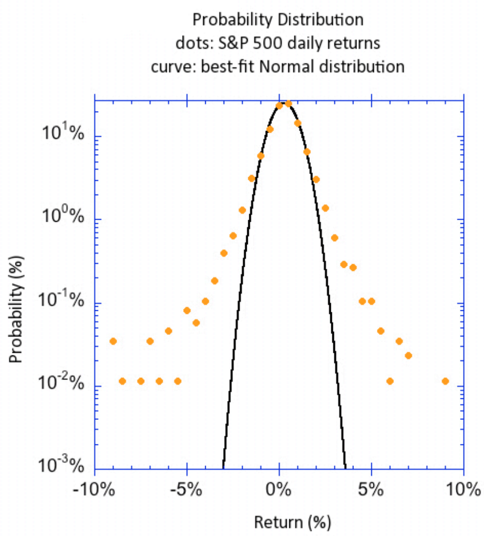
- There are many ways in which markets may deviate from GBM:

i) the returns may not be normally distributed

ii) non constant drift or volatility

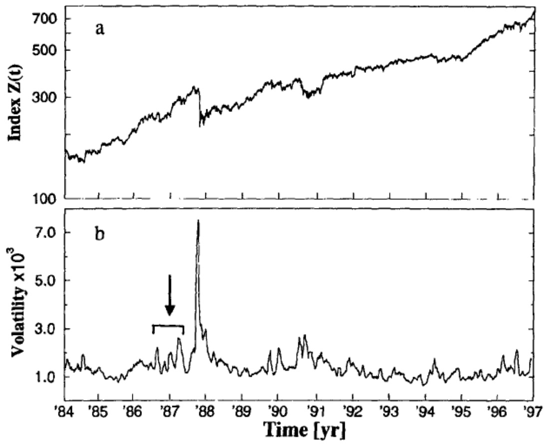
iii) long memory effects on the time series.

- There is strong evidences that the distribution of returns decays much slower than a Gaussian
- An example of this behaviour can be seen in the next figure, where the distribution for the daily returns of the S&P500 index is shown together with a Gaussian distribution with the variance of the data, which appears as a parabola in the linear-log scale of the graph



Real S(t): stochastic behaviour of σ

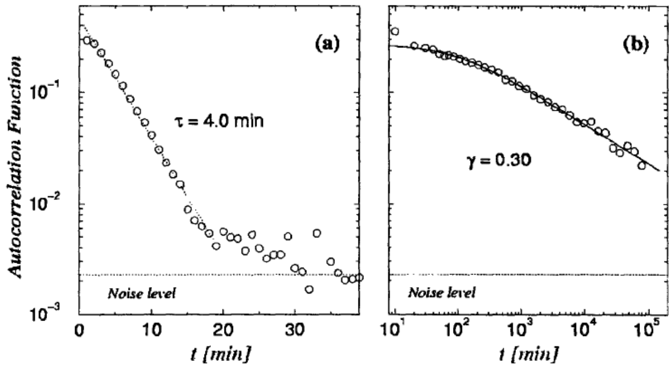
- Contrary to our assumption, the volatility is not constant in reality, it fluctuates with time
- This is shown in the next figure, which compares the time evolution of the S&P500 index to that of its volatility: the volatility depends on time and that periods of high volatility tend to be correlated.



- This is particularly pronounced for the stock market crash of October 19, 1987. The sharp drop of the S&P500 leads to a very high value of the volatility, which decreases to the normal level only after several months. This correlation is called volatility clustering: large changes of the asset price tend to follow one another, but they do not necessarily occur in the same direction

Real S(t): autocorrelations in the price

- This figure shows the disparity in the relaxation rate between (b) the autocorrelation of the modulus of the returns |dS/S| , which is represented well by a slowly decaying power law, contrary to the fast exponential decrease of (a) the returns dS/S autocorrelation function



- In (b) the autocorrelation extends over several days to months. A trading day corresponds to 390 min and there are about 21 trading days per month. All data refer to the S&P500 recorded between January 3, 1984, and December 31, 1996.

Contracts in idealized markets

- A prudent investor wants to avoid this risk and to protect himself against possible losses. How can he achieve that?
- A derivative is a financial product whose value 'derives' from the price of an underlying asset. Mathematically, if the underlying is characterized by its price then a derivative is just a function of $S(t)$
- First example: a forward is a contract between two parties in which:
  - one partner agrees to buy an underlying asset ('long position')
  - at a certain specified time T in the future, known as the delivery date

  - for a prescribed price K, known as the delivery price
  - and the other partner pledges to sell the underlying at time T for price K ('short position')
- The basic motivation to sign a contract, such as a forward, is to remove the market risk of the unpredictable price evolution

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Options

- Options are contracts in which only one partner assumes the obligation, whereas the other obtains a right. There are many different kinds of options. The <u>simplest</u> are European options
- A European option is a contract between two parties in which:
  - the seller of the option, known as the writer,
  - grants the buyer of the option, known as the holder,
  - the right to purchase (= call option) from the writer or to sell (= put option) to him an underlying (with a current spot price $S(t)$)
  - for a prescribed price K, called the exercise or strike price
  - at the expiry date T in the future.
- The key property of an option is that only the writer has an obligation. He must sell or buy the underlying asset for the strike price at time T.
- The holder will only exploit his right if he gains a profit, i.e., if $S(T) > K$ for a call option. Otherwise, he can buy the underlying for a cheaper price, $S(T) < K$, on the market:

  profit=$\max[0,S(T)-K]$ for a call; profit=$\max[0,K-S(T)]$ for a put

- Of course, the writer does not incur this financial obligation without requiring a compensation

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Exotic Options

- Colloquially, European option are called "plain vanilla" options to emphasize that, nowadays, they are ubiquitous and of simple structure
- Many further types of options have been invented:
  - An American option deviates from the European style only in that the holder may exercise his right at any time prior to expiry
  - A Bermudan option (like Bermuda islands) is in the middle: the privilege can be exercised at any given discrete time prior to expiry
  - An Asian option has a profit at expiry which depends on the whole time evolution of the price S(t) and not only by its value at expiry, more precisely on the average price ⟨S(t)⟩ measured between the contract sign date and the expiry:
    profit=max(0,⟨S⟩-K) for a call; profit=max(0,K-⟨S⟩) for a put
  - A Loopback option has a profit at expiry which depends on the maximum/minimum of the price measured between the contract sign date and the expiry
- Pricing these exotic options correctly is a great challenge and requires numerical (Monte Carlo) pricing techniques. We will only be concerned with European options in the remainder of this lecture

Classical Option Pricing: Black-Scholes Theory

- The central question, therefore, is: how much an option should cost?
- The Black-Scholes theory gives an answer to this question for European options. The Black-Scholes analysis is based on some assumptions:

1. The market is maximally efficient, i.e., it is infinitely liquid and does not exhibit any friction
2. The time evolution of the asset price is stochastic and exhibits geometric Brownian motion with constant risk-free interest rate r (which substitute the drift to avoid arbitrage) and volatility... etc. etc.
   ⇒ Black-Scholes analytic solution for the price at time t<T:

- CALL: $C\left[S(t),t\right] = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$
- PUT: $P\left[S(t),t\right] = S(t)\left[N(d_1) - 1\right] - Ke^{-r(T-t)}\left[N(d_2) - 1\right]$

where $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$ $\quad d_2 = d_1 - \sigma\sqrt{T-t}$

and $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} dz\, e^{-z^2/2}$ is the cumulative of a Gaussian

Numerical options pricing

- As soon as we abandon Black-Scholes assumptions, e.g. S(t) is not a geometric Brownian motion with constant drift and volatility, or if we want the fair price of an exotic option ⇒ no more analytic solution is available; as we shall see, we can use Monte Carlo methods

- Consider an asset with price S(t) and an European call option related to it; assume that the evolution of S(t) is given by $S \sim GBM(r,\sigma^2)$
- The profit at expiry (t=T) is given by: $\left(S(T)-K\right)^+ := \max\left[0, S(T)-K\right]$
- To obtain the present value of this profit, i.e. at time t, we have to discount it by a factor exp(-rT) due to the interest that a Bank would have guaranteed with a deposit at time $t_0=0$, in fact:

  $t=t_0$ ⇒ initial value: $\exp(-rT)\,(S(T)-K)^+$

  $t=T$ ⇒ final value: $\exp(rT)\exp(-rT)\,(S(T)-K)^+ = (S(T)-K)^+$

- Now, the calculation of the option price corresponds to the calculation of the expectation (average) value for the discounted profit on the distribution of the prices at expiry:

$$E\left[e^{-rT}\left(S(T)-K\right)^+\right] = \left\langle e^{-rT}\left(S(T)-K\right)^+\right\rangle$$

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- This quantity is, in fact, the estimate of what the holder of the option have to pay at t=0 on the basis of the expected profit at time t=T
- This average value can be easily obtained via a Monte Carlo simulation; algorithm:

```
do  i=1,N
   generate Z_i~N(0,1)
   S_i(T)=S(0)exp[(r-σ²/2)T+σZ_i√T]
   C_i=exp(-rT)max[0,S_i(T)-K]
enddo
C_N=(∑_{i=1,N}C_i)/N
end
```

```
do j=1,M
   S(t_{i+1})= …
enddo
S(T_i)=S(t_M)
```

- In this way we will have that: $C_N \underset{N\to\infty}{\to} E\left[e^{-rT}\left(S(T)-K\right)^+\right]$
- An estimate for the statistical uncertainty can be obtained via: $\Delta C_N = \left[\frac{1}{N(N-1)}\sum_{i=1}^N\left(C_i - C_N\right)^2\right]^{1/2}$
- By simulating S(t) when we do not know the solution of a SDE, Monte Carlo can be extended to compute any exotic option price

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Lecture 3: Suggested books

- R.N. Mantegna & H.E. Stanley, "Introduction to Econophysics" Cambridge
- P. Glasserman, "Monte Carlo Methods in Financial Engineering" Springer
- W. Paul & J. Baschnagel, "Stochastic Processes - From Physics to Finance" Springer
- C.W. Gardiner, *Handbook of Stochastic Methods* – Springer 2004
- E. Vitali, M. Motta, D.E. Galli, *Theory and Simulation of Random Phenomena* – Springer 2018

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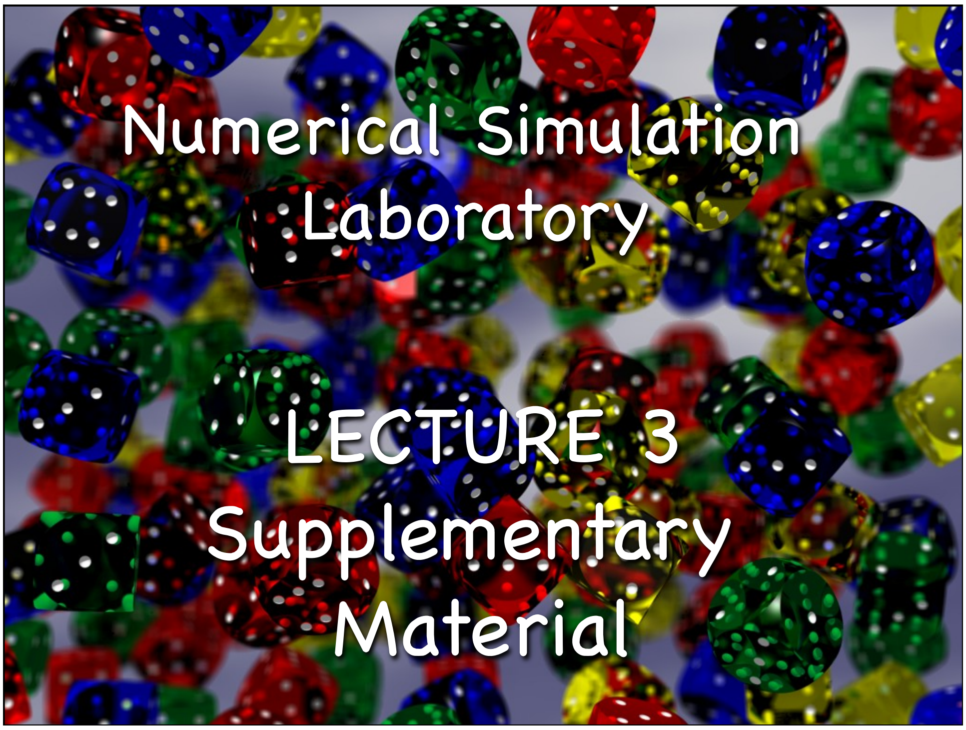

To see that $\int_{t_0}^{t_n} W(t')\,dW(t') \underset{\substack{It\hat{o}\\ \alpha=0}}{:=} \text{ms-}\lim_{n\to\infty} \sum_{i=1}^{n} W(t_{i-1})\left[W(t_i) - W(t_{i-1})\right] = 0$

We can compute: $\langle S_n \rangle = \sum_{i=1}^{n} \left\langle W(\tau_i)\left[W(t_i) - W(t_{i-1})\right]\right\rangle$

and set $\tau_i = \alpha t_i + (1-\alpha)t_{i-1}$, with $0 \le \alpha \le 1$. Then we have

$$\langle S_n \rangle = \sum_{i=1}^{n} \left\langle W(\alpha t_i + (1-\alpha)t_{i-1})\left[W(t_i) - W(t_{i-1})\right]\right\rangle = \cdots$$

we make use of the result for the covariance of W(t) (see the next slide): $\langle W(t)W(s) \rangle = \min(t,s)$

to obtain:

$$\cdots = \sum_{i=1}^{n} \left\langle \min\left[\alpha t_i + (1-\alpha)t_{i-1}; t_i\right] - \min\left[\alpha t_i + (1-\alpha)t_{i-1}; t_{i-1}\right]\right\rangle =$$

$$= \sum_{i=1}^{n} \left\langle \alpha t_i + (1-\alpha)t_{i-1} - t_{i-1}\right\rangle = \alpha \sum_{i=1}^{n} (t_i - t_{i-1}) = \alpha(t_n - t_0)$$

Thus $\langle S_n \rangle \underset{\alpha=0}{=} 0 \quad \Rightarrow \quad \int_{t_0}^{t_n} W(t')\,dW(t') \underset{\substack{It\hat{o}\\ \alpha=0}}{:=} ms - \lim_{n\to\infty}(0) = 0$

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Wiener process properties

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- From the probability distribution of a Wiener process one can prove that the covariance is $\boxed{\langle W(t)W(s) \rangle = \min(t,s)}$
- In fact, suppose t<s then:

$$\langle W(t)W(s) \rangle = \langle W(t)W(t) \rangle + \left\langle W(t)\left(W(s) - W(t)\right)\right\rangle = \cdots$$

$$\cdots = t + \langle W(t) \rangle \langle W(s) - W(t) \rangle = t + 0 = t$$

W(t) distributed as $\mathcal{N}(0;t)$

Stationary, independent & ~$\mathcal{N}(0;s-t)$

- Specularly, suppose t>s then:

$$\langle W(t)W(s) \rangle = \langle W(s)W(s) \rangle + \left\langle \left(W(t) - W(s)\right)W(s)\right\rangle = \cdots$$

$$\cdots = s + \left\langle W(s)\left(W(t) - W(s)\right)\right\rangle = s + \langle W(s) \rangle \langle W(t) - W(s) \rangle = s$$

Itô -> Martingale

- For a non-anticipating function G we can now show that

$$X(t) = \int_0^t dW(t')\,G(t')$$

(integrated following Itô rule) is a Martingale.

- In fact, for $t \geq s$ we have:

$$E\left[X(t)\middle|X(s)\right] = E\left[\int_0^t dW(t')\,G(t')\middle|X(s)\right] =$$

$$= E\left[\int_0^s dW(t')\,G(t')\middle|X(s)\right] + E\left[\int_s^t dW(t')\,G(t')\middle|X(s)\right] =$$

$$= E\left[X(s)\middle|X(s)\right] + E\left[\int_s^t dW(t')\,G(t')\right] =$$

since G(t') and dW(t') are independent of the past, in particular of X(s).

$$= X(s) + \int_s^t \underbrace{E\left[dW(t')\right]}_{=0} E\left[G(t')\right] = X(s) + 0 = X(s)$$

G: non-anticipating & Itô

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Other properties

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- To become more familiar with the properties of the Itô stochastic integral, we want to note the following result for an arbitrary bounded non-anticipating function G(t'):

$$\int_{t_0}^t \left[dW(t')\right]^2 G(t') = \int_{t_0}^t dt'\,G(t')$$

- In fact defining $\Delta W_i^2 := [W(t_i)-W(t_{i-1})]^2$ and $G_i := G(t_i)$ we have

$$I := \lim_{n\to\infty}\left\langle\left[\sum_{i=1}^n G_{i-1}\left(\Delta W_i^2 - \Delta t_i\right)\right]^2\right\rangle =$$

$$= \lim_{n\to\infty}\left[\left\langle\sum_{i=1}^n (G_{i-1})^2\left(\Delta W_i^2 - \Delta t_i\right)^2\right\rangle + \left\langle 2\sum_{i>j} G_{i-1}G_{j-1}\left(\Delta W_i^2 - \Delta t_i\right)\left(\Delta W_j^2 - \Delta t_j\right)\right\rangle\right] =$$

... but G and W are non-anticipating functions, thus G_{i-1} is independent from ΔW_i^2, moreover if i>j ($G_{i-1}G_{j-1}$) is independent from ΔW_i^2 but not from ΔW_j^2. We can write I as:

$$= \lim_{n\to\infty}\left[\sum_{i=1}^n \left\langle (G_{i-1})^2\right\rangle\left\langle\left(\Delta W_i^2 - \Delta t_i\right)^2\right\rangle + 2\sum_{i>j}\left\langle G_{i-1}G_{j-1}\left(\Delta W_j^2 - \Delta t_j\right)\right\rangle\left\langle\left(\Delta W_i^2 - \Delta t_i\right)\right\rangle\right] = \cdots$$

Moreover: $\left\langle \Delta W_i^2 \right\rangle = \left\langle W^2(t_i) \right\rangle + \left\langle W^2(t_{i-1}) \right\rangle - 2\left\langle W(t_i)W(t_{i-1}) \right\rangle = t_i + t_{i-1} - 2\min(t_i, t_{i-1}) = \Delta t_i$

and thus the second term of I vanishes. We have also that

$$\left\langle \left(\Delta W_i^2 - \Delta t_i \right)^2 \right\rangle = \left\langle \left(\Delta W_i^2 \right)^2 \right\rangle - 2\left\langle \Delta W_i^2 \right\rangle \Delta t_i + \Delta t_i^2 = \left\langle \left(\Delta W_i \right)^4 \right\rangle - \Delta t_i^2 = \cdots$$

- And using that for X normally distributed as $\mathcal{N}(0, \sigma^2)$ we have:

$$E\left[X^p\right] = \begin{cases} 0 & p \text{ odd} \\ \sigma^p (p-1)(p-3) \cdot \ldots \cdot 1 & p \text{ even} \end{cases}$$

And recalling that $\Delta W_i := [W(t_i) - W(t_{i-1})] \sim \mathcal{N}(0, \Delta t_i)$

we obtain: $\left\langle \left(\Delta W_i^2 - \Delta t_i \right)^2 \right\rangle = 3\Delta t_i^2 - \Delta t_i^2 = 2\Delta t_i^2$

- Using this result, we finally arrive at

$$I = 2 \cdot \lim_{n \to \infty} \sum_{i=1}^{n} \left\langle G_{i-1}^2 \right\rangle \Delta t_i^2 = 0$$

where the final equality is valid, for instance, when G is bounded,

because then $\lim_{n \to \infty} \sum_{i=1}^{n} \left\langle G_{i-1}^2 \right\rangle \Delta t_i^2 = \lim_{n \to \infty} \sum_{i=1}^{n} \left\langle G_{i-1}^2 \right\rangle \left(\frac{t - t_0}{n} \right)^2 \le (t - t_0)^2 \lim_{n \to \infty} \frac{G_{Max}^2}{n} = 0$