Politecnico
di Torino

Department of Control and Computer Engineering
Master degree in Data Science and Engineering
Computational Linear Algebra for Large Scale Problems

THE \$25,000,000,000 EIGENVECTOR
The Linear Algebra Behind Google

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Contents

1 Boosting Page Importance by Adding Links

"Suppose the people who own page 3 in the web of Figure 1 are infuriated by the fact that its importance score, computed using formula (2.1), is lower than the score of page 1. In an attempt to boost page 3's score, they create a page 5 that links to page 3; page 3 also links to page 5. Does this boost page 3's score above that of page 1?"

In Figure 1 is shown the graph after the addition of the page 5. The link matrix **A1** obtained is

$$\mathbf{A1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \tag{0.1}$$

Since the score of page j is the sum of the scores of all pages linking to page j, adding a page that links only to page 3 will boost page 3's importance. It is also possible to verify the property by computing the (normalized) eigenvector x[1], which gives

$$x = \begin{bmatrix} 0.2449 \\ 0.0816 \\ \mathbf{0.3673} \\ 0.1224 \\ 0.1837 \end{bmatrix} \tag{0.2}$$

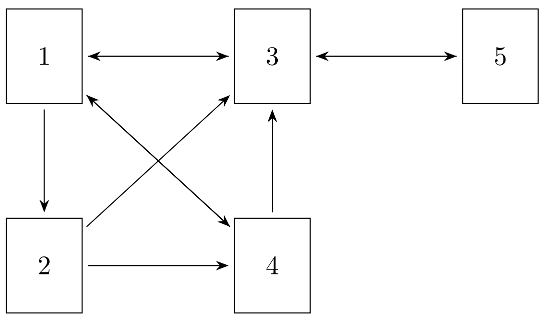

Figure 1: Graph of Exercise 0 after the addition of the page 5

As a result, the 3rd page has a *higher rank* than before, resulting a "higher score" page than page 1.

[1]It is always referring to the linear equation $Ax = x$

2 Analyzing the Eigenspace Dimension in a Disconnected Web

Construct a web consisting of three or more subwebs and verify that dim($V_1(\boldsymbol{A})$) equals (or exceeds) the number of the components in the web.

The link matrix **A2** was constructed as shown below:

$$\mathbf{A2} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{0.3}$$

The eigenvalues of **A2** are:

$$\lambda = \{1, 1, 1, -0.3606 \pm 0.4110i, -0.2788, -0.5 \pm 0.8660i, -1\} \tag{0.4}$$

It is worth noting that the presence of *complex eigenvalues* is expected in this context due to the non-symmetric nature of the link matrix **A2**. However, the key result is that there are three real eigenvalues equal to 1, corresponding to the three disconnected subwebs in the web. This confirms that the dimension of $V_1(\mathbf{A})$ equals **3**, as anticipated.

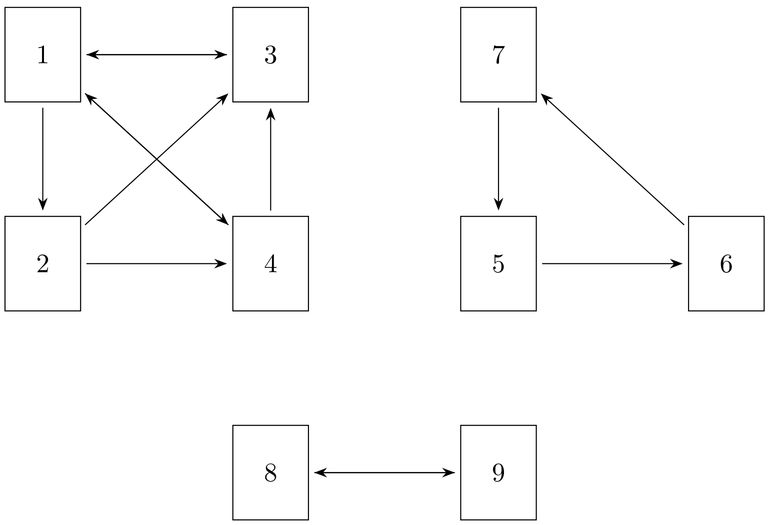

Figure 2: Graph of Exercise 0. In this graph, there is a web of 9 pages consisting of three disconnected "subwebs" W1 (pages 1, 2, 3, 4), W2 (pages 5, 6, 7) and W3 (pages 8 and 9)

3 Connecting Subwebs and Its Impact on Eigenspace

"Add a link from page 5 to page 1 in the web of Figure 2. The resulting web, considered as an undirected graph, is connected. What is the dimension of $V_1(\boldsymbol{A})$?"

The link matrix **A3** was constructed by adding a link from page 5 to page 1:

$$\mathbf{A3} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} & 1 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{0.5}$$

The following eigenvalues were calculated:

$$\lambda = \{1, -0.3606 \pm 0.4110i, -0.2788, 0\} \tag{0.6}$$

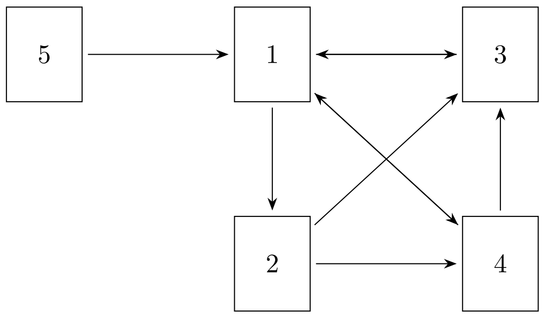

Figure 3: Graph of the link matrix **A3** after adding a link from page 5 to page 1

As expected, exactly one eigenvalue equal to 1 was found, indicating that the dimension of $V_1(\mathbf{A})$ is **1**. It's important to note that the eigenvalue corresponding to page 5 is equal to 0, which occurs because no other page links to page 5. This result reflects the fact that page 5 does not contribute to the ranking of other pages, confirming its isolation in terms of incoming links.

4 Handling Dangling Nodes and the Perron Eigenvalue

"In the web of Figure 2.1, remove the link from page 3 to page 1. In the resulting web page 3 is now a dangling node. Set up the corresponding substochastic matrix and find its largest positive (Perron) eigenvalue. Find a non-negative Perron eigenvector for this eigenvalue, and scale the vector so that components sum to one. Does the resulting ranking seem reasonable?"

The link matrix **A4** was constructed by removing the link from page 3 to page 1, making page 3 a dangling node.

$$\mathbf{A4} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \tag{0.7}$$

The resulting matrix is *substochastic*, with the column corresponding to page 3 summing to less than 1.

The Perron-Frobenius theorem guarantees that for a non-negative matrix, there exists a largest real positive eigenvalue, known as the Perron eigenvalue. Corresponding to this eigenvalue is the Perron eigenvector, which can be scaled to have non-negative components summing to 1 and can be used to rank the web pages.

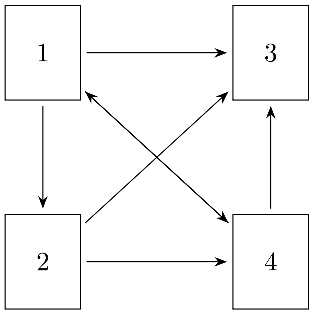

Figure 4: Graph corresponding to the link matrix **A4**. Because page 3 has no outgoing links, it is considered a dangling node.

In this case, the largest positive eigenvalue (Perron eigenvalue) was found to be $\lambda_{\text{Perron}} = 0.5614$, and the corresponding Perron eigenvector, scaled to sum to 1, is:

$$\mathbf{v}_{\text{Perron}} = \begin{bmatrix} 0.2066 \\ 0.1227 \\ 0.4386 \\ 0.2320 \end{bmatrix} \tag{0.8}$$

The resulting ranking effectively captures the influence of the dangling node (page 3), which, having no outgoing links, does not contribute to the ranking of other pages.

5 Proving Zero Importance for Pages with No Backlinks - A Matrix

"Prove that in any web the importance score of a page with no backlinks is zero."

The importance score of a page is computed using the formula:

$$x_k = \sum_{j \in L_k} \frac{x_j}{n_j} \tag{0.9}$$

where:

- L_k is the set of pages that link to page k (the backlinks).
- x_j is the importance score of page j.
- n_j is the number of outbound links from page j.

Now, consider a page k with no backlinks, i.e., L_k is an empty set. For this page, the formula for its importance score becomes:

$$x_k = \sum_{j \in L_k} \frac{x_j}{n_j} = \sum_{j \in \emptyset} \frac{x_j}{n_j} = 0 \tag{0.10}$$

Since the sum is over an empty set (there are no pages linking to page k), the sum itself is zero. Therefore, the importance score x_k of a page with no backlinks is **zero**.

6 Proving the Invariance of Page Importance Under Index Transposition

"Implicit in our analysis up to this point is the assertion that the manner in which the pages of a web W are indexed has no effect on the importance score assigned to any given page. Prove this, as follows: Let W contain n pages, each page assigned an index 1 through n, and let A be the resulting link matrix. Suppose we then transpose the indices of pages i and j (so page i is now page j and vice-versa). Let $\tilde{\boldsymbol{A}}$ be the link matrix for the relabelled web."

6.1 Matrix Representation after Index Transposition

- *Argue that $\tilde{\boldsymbol{A}} = \boldsymbol{PAP}$, where $\boldsymbol{P}$ is the elementary matrix obtained by transposing rows i and j of the $n \times n$ identity matrix. Note that the operation $\boldsymbol{A} \to \boldsymbol{PA}$ has the effect of swapping rows i and j of $\boldsymbol{A}$, while $\boldsymbol{A} \to \boldsymbol{AP}$ swaps columns i and j. Also, $\boldsymbol{P}^2 = \boldsymbol{I}$, the identity matrix.*

Let $\mathbf{P}$ be the elementary permutation matrix that results from swapping rows i and j of the identity matrix $\mathbf{I}$.

When the indices of pages i and j are swapped, the new link matrix $\tilde{\mathbf{A}}$ can be represented as:

$$\tilde{\mathbf{A}} = \mathbf{PAP} \tag{0.11}$$

- Multiplying $\mathbf{A}$ by $\mathbf{P}$ on the right ($\mathbf{AP}$) swaps the columns i and j of $\mathbf{A}$, corresponding to reassigning the page indices.
- Multiplying $\mathbf{A}$ by $\mathbf{P}$ on the left ($\mathbf{PA}$) swaps the rows i and j of $\mathbf{A}$, corresponding to adjusting the links between the pages after the indices have been swapped.

Since $\mathbf{P}$ is a permutation matrix, it has the property that $\mathbf{P}^2 = \mathbf{I}$, meaning applying the permutation twice returns the matrix to its original form.

6.2 Invariance of Eigenvalues under Permutation

- *"Suppose that x is an eigenvector for $\boldsymbol{A}$, so $\boldsymbol{A}x = \lambda x$ for some λ. Show that $y = \boldsymbol{P}x$ is an eigenvector for $\tilde{\boldsymbol{A}}$ with eigenvalue λ."*

Starting with the eigenvalue equation for $\mathbf{A}$:

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{0.12}$$

Now, multiply both sides by $\mathbf{P}$ (left-side multiplication):

$$\mathbf{PAx} = \lambda\mathbf{Px} \tag{0.13}$$

Since $\tilde{\mathbf{A}} = \mathbf{PAP}$, means that $\mathbf{P}\tilde{\mathbf{A}} = \mathbf{AP}$ (remember the property where $\mathbf{P^2} = \mathbf{I}$). Substitute this into equation 0.13:

$$\tilde{\mathbf{A}}\mathbf{Px} = \lambda\mathbf{Px} \tag{0.14}$$

Finally, since $\mathbf{y} = \mathbf{Px}$, it's possible to write:

$$\tilde{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y} \tag{0.15}$$

Therefore, $\mathbf{y} = \mathbf{Px}$ is indeed an eigenvector of $\tilde{\mathbf{A}}$ with eigenvalue λ.

6.3 Invariance of Importance Scores under Index Transposition

- *Explain why this shows that transposing the indices of any two pages leaves the importance scores unchanged, and use this result to argue that any permutation of the page indices leaves the importance scores unchanged.*

$\tilde{\mathbf{A}} = \mathbf{PAP}$ represents the link matrix after swapping the indices of pages i and j. It was shown that this transformation preserves the eigenvalues and eigenvectors of the original matrix $\mathbf{A}$ (Exercise 6.2), as $\mathbf{y} = \mathbf{Px}$ remains an eigenvector of $\tilde{\mathbf{A}}$ with the same eigenvalue λ.

This preservation implies that the importance scores are **unaffected** by the relabeling of the pages. Therefore, the importance scores of the pages in a web are invariant under any permutation of their indices.

7 Non-Negative Linear Combination of Column-Stochastic Matrices

*Prove that if **A** is an $n \times n$ column-stochastic matrix and $0 \leq m \leq 1$, then **M**=(1-m)**A** +m **S** is also a column-stochastic matrix.*

A matrix $\mathbf{A}$ is column-stochastic if all entries are non-negative and the sum of the entries in each column is 1.

Given:

- $\mathbf{A}$ is column-stochastic, meaning $\sum_{i=1}^{n} \mathbf{A}_{ij} = 1$ for each column j.
- $\mathbf{S}$ is also a column-stochastic matrix, so $\sum_{i=1}^{n} \mathbf{S}_{ij} = 1$ for each column j.
- $0 \leq m \leq 1$.

Since both $\mathbf{A}$ and $\mathbf{S}$ are column-stochastic, all entries in $\mathbf{A}$ and $\mathbf{S}$ are non-negative.

All entries of $\mathbf{M}$ are non-negative because:

$$\mathbf{M}_{ij} = (1-m)\mathbf{A}_{ij} + m\mathbf{S}_{ij}$$

which is a non-negative linear combination of non-negative numbers.

For any column j, the sum of the entries of column j in $\mathbf{M}$ is:

$$\sum_{i=1}^{n} \mathbf{M}_{ij} = \sum_{i=1}^{n} [(1-m)\mathbf{A}_{ij} + m\mathbf{S}_{ij}] \tag{0.16}$$

This can be expanded as:

$$\sum_{i=1}^{n} \mathbf{M}_{ij} = (1-m)\sum_{i=1}^{n} \mathbf{A}_{ij} + m\sum_{i=1}^{n} \mathbf{S}_{ij} \tag{0.17}$$

Since $\mathbf{A}$ and $\mathbf{S}$ are column-stochastic:

$$\sum_{i=1}^{n} \mathbf{A}_{ij} = 1 \quad \text{and} \quad \sum_{i=1}^{n} \mathbf{S}_{ij} = 1$$

Thus:

$$\sum_{i=1}^{n} \mathbf{M}_{ij} = (1-m) \times 1 + m \times 1 = 1$$

Therefore, $\mathbf{M}$ is a column-stochastic matrix since each column of it sums to 1.

8 Product of Two Column-Stochastic Matrices

"Show that the product of two column-stochastic matrices is also column-stochastic."

A matrix $\mathbf{A}$ is column-stochastic if all its entries are non-negative and the sum of the entries in each column is 1. Similarly, another matrix $\mathbf{B}$ is also column-stochastic if all its entries are non-negative and the sum of the entries in each column is 1.

In this demonstration it is considered the matrix

$$\mathbf{M} = \mathbf{AB}$$

Given:

- $\mathbf{A}$ is an $n \times n$ column-stochastic matrix, meaning $\sum_{i=1}^{n} \mathbf{A}_{ij} = 1$ for each column j.
- $\mathbf{B}$ is an $n \times n$ column-stochastic matrix, meaning $\sum_{i=1}^{n} \mathbf{B}_{ij} = 1$ for each column j.

First, consider the ij-th entry of the matrix $\mathbf{M}$, which is given by:

$$\mathbf{M}_{ij} = \sum_{k=1}^{n} \mathbf{A}_{ik}\mathbf{B}_{kj}$$

Since all entries $\mathbf{A}_{ik}$ and $\mathbf{B}_{kj}$ are non-negative (because $\mathbf{A}$ and $\mathbf{B}$ are column-stochastic), it follows that $\mathbf{M}_{ij}$ is also non-negative.

Next, for any fixed column j of $\mathbf{M}$:

$$\sum_{i=1}^{n} \mathbf{M}_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{n} \mathbf{A}_{ik}\mathbf{B}_{kj}$$

Summation is commutative, so:

$$\sum_{i=1}^{n} \mathbf{M}_{ij} = \sum_{k=1}^{n} \mathbf{B}_{kj} \left(\sum_{i=1}^{n} \mathbf{A}_{ik} \right)$$

Since $\mathbf{A}$ is column-stochastic, $\sum_{i=1}^{n} \mathbf{A}_{ik} = 1$ for each k. Therefore:

$$\sum_{i=1}^{n} \mathbf{M}_{ij} = \sum_{k=1}^{n} \mathbf{B}_{kj} \times 1 = \sum_{k=1}^{n} \mathbf{B}_{kj}$$

Because $\mathbf{B}$ is also column-stochastic, $\sum_{k=1}^{n} \mathbf{B}_{kj} = 1$ for each j. Hence:

$$\sum_{i=1}^{n} \mathbf{M}_{ij} = 1$$

Therefore, $\mathbf{M}$ is **column-stochastic**, because the sum of the entries in each column is 1.

9 Importance Score of a Page with No Backlinks - M Matrix

"Show that a page with no backlinks is given importance score $\frac{m}{n}$ by formula (3.2)."

Given:

- $\mathbf{A}$ the $n \times n$ link matrix
- $\mathbf{s}$ the $n \times 1$ vector with each entry equal to $\frac{1}{n}$.
- m a scalar such that $0 \leq m \leq 1$.
- $\mathbf{x}$ the importance score vector.

The formula number (3.2) in the pdf file ("*The Linear Algebra Behind Google*") is given by:

$$\mathbf{x} = (1 - m)\mathbf{A}\mathbf{x} + m\mathbf{s}$$

For a page j with no backlinks, the j-th column of $\mathbf{A}$ is all zeros, so the contribution from $\mathbf{A}\mathbf{x}$ for page j is zero (showed in Exercise 3 and 5)

Thus, the importance score x_j for page j is:

$$x_j = (1 - m) \times 0 + m \times \frac{1}{n} = \frac{m}{n}$$

Finally, it is possible to state that the importance score for a page with no backlinks is $\frac{m}{n}$.

10 Reachability, Positivity, and Eigenstructure in a Strongly Connected Web

"Suppose that $\mathbf{A}$ is the link matrix for a strongly connected web of n pages (any page can be reached from any other page by following a finite number of links). Show that dim($V_1(\mathbf{A})$) as follows. Let $(\mathbf{A}^k)_{ij}$ denote the (i,j)-entry of $\mathbf{A}^k$"

10.1 Reachability in Two Steps and the Power of the Link Matrix

- *"Note that page i can be reached from page j in one step if and only if $\mathbf{A}_{ij} > 0$ (since $\mathbf{A}_{ij} > 0$ means there's a link from j to i!). Show that $(\mathbf{A}^2)_{ij} > 0$ if and only if page i can be reached from page j in exactly two steps. Hint: $(\mathbf{A}^2)_{ij} = \sum_k \mathbf{A}_{ik}\mathbf{A}_{kj}$; all $\mathbf{A}_{ij}$ are non-negative, so $(\mathbf{A}^2)_{ij} > 0$ implies that for some k both $\mathbf{A}_{ik}$ and $\mathbf{A}_{kj}$ are positive."*

Given the link matrix $\mathbf{A}$, the entry $(\mathbf{A}^2)_{ij}$ represents the number of paths of length two from page j to page i. Specifically:

$$(\mathbf{A}^2)_{ij} = \sum_k \mathbf{A}_{ik}\mathbf{A}_{kj}$$

This sum implies that for $(\mathbf{A}^2)_{ij} > 0$, there must exist at least one intermediate page k such that both $\mathbf{A}_{ik} > 0$ and $\mathbf{A}_{kj} > 0$.

Sufficiency: Suppose $(\mathbf{A}^2)_{ij} > 0$. Then, there exists at least one page k such that both $\mathbf{A}_{ik} > 0$ and $\mathbf{A}_{kj} > 0$. This implies a path from j to i via k in exactly two steps.

Necessity: If page i can be reached from page j in exactly two steps, there must exist some intermediate page r such that there is a link from j to r ($\mathbf{A}_{rj} > 0$) and a link from r to i ($\mathbf{A}_{ir} > 0$). Thus, $(\mathbf{A}^2)_{ij}$ will include the positive term $\mathbf{A}_{ir}\mathbf{A}_{rj} > 0$, meaning $(\mathbf{A}^2)_{ij} > 0$.

Therefore, it is possible to conclude that $(\mathbf{A}^2)_{ij} > 0$ **if and only if** page i can be reached from page j in exactly two steps.

10.2 Generalization to p-Step Reachability

- *"Show more generally that $(\mathbf{A}^p)_{ij} > 0$ if and only if page i can be reached from page j in EXACTLY p steps."*

Given the link matrix $\mathbf{A}$, the entry $(\mathbf{A}^p)_{ij}$ represents the number of paths of length p from page j to page i:

$$(\mathbf{A}^p)_{ij} = \sum_{k_1, k_2, \ldots, k_{p-1}} \mathbf{A}_{ik_1} \mathbf{A}_{k_1 k_2} \cdots \mathbf{A}_{k_{p-1} j}$$

This sum implies that for $(\mathbf{A}^p)_{ij} > 0$, there must exist at least one sequence of pages $k_1, k_2, \ldots, k_{p-1}$ such that the product $\mathbf{A}_{ik_1} \mathbf{A}_{k_1 k_2} \cdots \mathbf{A}_{k_{p-1} j}$ is positive.

Sufficiency: Suppose $(\mathbf{A}^p)_{ij} > 0$. Then, there exists at least one sequence of pages $k_1, k_2, \ldots, k_{p-1}$ such that the product $\mathbf{A}_{ik_1} \mathbf{A}_{k_1 k_2} \cdots \mathbf{A}_{k_{p-1} j} > 0$. This implies a path from j to i through exactly p steps.

Necessity: If page i can be reached from page j in exactly p steps, there must exist a sequence of pages $k_1, k_2, \ldots, k_{p-1}$ such that each step from j to k_{p-1}, from k_{p-1} to k_{p-2}, and so on, finally from k_1 to i has a corresponding positive entry in matrix $\mathbf{A}$. This sequence contributes positively to $(\mathbf{A}^p)_{ij}$, making $(\mathbf{A}^p)_{ij} > 0$.

Therefore, $(\mathbf{A}^p)_{ij} > 0$ **if and only if** page i can be reached from page j in exactly p steps.

10.3 Cumulative Reachability within p-Steps

- *"Argue that $\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^p\right)_{ij} > 0$ if and only if page i can be reached from page j in p or fewer steps (note $p = 0$ is a legitimate choice—any page can be reached from itself in zero steps!)."*

The matrix $\mathbf{I}$ is the identity matrix, where $\mathbf{I}_{ij} > 0$ only when $i = j$. This indicates that any page can reach itself in zero steps. Each matrix $\mathbf{A}^k$ (for $k = 1, 2, \ldots, p$) represents the number of ways to reach page i from page j in exactly k steps. Therefore, the sum $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^p$ represents the total number of ways to reach page i from page j in p or fewer steps.

Sufficiency: Suppose $\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^p\right)_{ij} > 0$. This implies that there exists at least one k such that $(\mathbf{A}^k)_{ij} > 0$ for $0 \le k \le p$. Thus, page i can be reached from page j in k steps, where k is at most p.

Necessity: If page i can be reached from page j in p or fewer steps, then there exists a k such that $0 \le k \le p$ and $(\mathbf{A}^k)_{ij} > 0$. Hence, the corresponding term in the sum $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^p$ will be positive, making the entire sum $\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^p\right)_{ij} > 0$.

Special Case for $p = 0$: When $p = 0$, the sum reduces to just the identity matrix $\mathbf{I}$, where $\mathbf{I}_{ij} > 0$ only if $i = j$, meaning each page can reach itself in zero steps, confirming the base case.

Thus, $\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^p\right)_{ij} > 0$ if and only if page i can be reached from page j in p or fewer steps.

10.4 Positivity of the Summed Power Matrix in a Strongly Connected Web

- *"Explain why $\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}\right)$ is a positive matrix if the web is strongly connected."*

The matrix $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$ represents the cumulative effect of all paths from page j to page i that take 0 to $n-1$ steps. Since the web is strongly connected, for every pair (i, j), there exists some $p \le n-1$ such that $(\mathbf{A}^p)_{ij} > 0$. Therefore, the sum $\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}\right)_{ij}$ will be positive for all i and j.

Thus, $\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}\right)$ is a positive matrix when the web is strongly connected.

10.5 Column-Stochasticity and Positivity of Matrix B

- *"Use the last part (and Exercise 8) so show that* $\mathbf{B} = \frac{1}{n}\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}\right)$ *is positive and column-stochastic (and hence by Lemma 3.2, dim(*$V_1(\mathbf{B})$*) = 1"*

In the previous part, it was demonstrated that the matrix $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$ is positive if the web is strongly connected. This means that each entry of this matrix is greater than zero. Since $\mathbf{B}$ is defined as $\mathbf{B} = \frac{1}{n}\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}\right)$, and multiplying a positive matrix by a positive scalar $\frac{1}{n}$ results in another positive matrix, $\mathbf{B}$ is also positive.

Now, consider the sum of the elements in any column of $\mathbf{B}$:

$$\mathbf{B}_{ij} = \frac{1}{n}\left(\mathbf{I}_{ij} + \mathbf{A}_{ij} + (\mathbf{A}^2)_{ij} + \cdots + (\mathbf{A}^{n-1})_{ij}\right)$$

Since each of the matrices $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^{n-1}$ is column-stochastic (as proven in Exercise 8), the sum of each column of the matrix $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$ is n. Dividing by n, each column of $\mathbf{B}$ sums to 1. Therefore, $\mathbf{B}$ is column-stochastic.

Finally, according to Lemma 3.2, if a matrix is both positive (as shown in the previous part) and column-stochastic, the eigenspace corresponding to the eigenvalue 1 has dimension 1. Since $\mathbf{B}$ is both positive and column-stochastic, it follows that $\dim(V_1(\mathbf{B})) = 1$.

10.6 Inclusion of $\dim(V_1(\mathbf{A}))$ in $\dim(V_1(\mathbf{B}))$ and Its Dimensionality Implication

- *"Show that if* $\mathbf{x} \in V_1(\mathbf{A})$ *then* $\mathbf{x} \in V_1(\mathbf{B})$*. Why does this imply that dim(*$V_1(\mathbf{A})$*) = 1?"*

If $\mathbf{x} \in V_1(\mathbf{A})$, then $\mathbf{A}\mathbf{x} = \mathbf{x}$. Consider now:

$$\mathbf{B}\mathbf{x} = \frac{1}{n}\left(\mathbf{I}\mathbf{x} + \mathbf{A}\mathbf{x} + \mathbf{A}^2\mathbf{x} + \cdots + \mathbf{A}^{n-1}\mathbf{x}\right)$$

Since $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 1:

$$\mathbf{A}^k\mathbf{x} = \mathbf{x} \text{ for all } k \geq 1$$

Therefore:

$$\mathbf{B}\mathbf{x} = \frac{1}{n}(n\mathbf{x}) = \mathbf{x}$$

Hence, $\mathbf{x}$ is also an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue 1, implying $\mathbf{x} \in V_1(\mathbf{B})$.

From the previous part (Part 5), it was showed that $\mathbf{B}$ is a positive, column-stochastic matrix and by Lemma 3.2, $\dim(V_1(\mathbf{B})) = 1$. Since $\mathbf{x} \in V_1(\mathbf{A})$ implies $\mathbf{x} \in V_1(\mathbf{B})$, and given that $V_1(\mathbf{B})$ is one-dimensional, it follows that $V_1(\mathbf{A})$ cannot have dimension greater than 1. Therefore, $\dim(V_1(\mathbf{A})) = 1$.

11 PageRank with M Matrix

"Consider again the web in Figure 2.1, with the addition of a page 5 that links to page 3, where page 3 also links to page 5. Calculate the new ranking by finding the eigenvector of $\mathbf{M}$ *(corresponding to* λ*=1) that has positive components summing to one. Use* m *= 0.15."*

The link matrix $\mathbf{A11}$ is the same as (0.1) and the graph was already shown in Figure 1

To calculate the new PageRank values, the matrix $\mathbf{M}$ is constructed as

$$\mathbf{M11} = (1 - m)\mathbf{A11} + m\mathbf{S11} \tag{2.1}$$

where $m = 0.15$, and **S11** is a matrix with each entry equal to $\frac{1}{n}$, with n being the number of pages. The resulting matrix is:

$$\mathbf{M11} = \begin{bmatrix} 0.0300 & 0.0300 & 0.4550 & 0.4550 & 0.0300 \\ 0.3133 & 0.0300 & 0.0300 & 0.0300 & 0.0300 \\ 0.3133 & 0.4550 & 0.0300 & 0.4550 & 0.8800 \\ 0.3133 & 0.4550 & 0.0300 & 0.0300 & 0.0300 \\ 0.0300 & 0.0300 & 0.4550 & 0.0300 & 0.0300 \end{bmatrix} \tag{2.2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$, normalized so that the components sum to one, is found to be:

$$\mathbf{x} = \begin{bmatrix} 0.2371 \\ 0.0972 \\ \mathbf{0.3489} \\ 0.1385 \\ 0.1783 \end{bmatrix} \tag{2.3}$$

As expected, the PageRank of page 3 has increased, reflecting the additional link from page 5.

However, when compared to (0.2) it is possible to notice that the rank of page 3 is slightly lower when calculated using **M11**. A further analysis will be performed in Exercise 12.

12 PageRank Stability with the Addition of a Dangling Node

"Add a sixth page that links to every page of the web in the previous exercise, but to which no other page links. Rank the pages using **A***, then using* **M** *with m = 0.15, and compare the results."*

To construct the new link matrix **A12**, it is added a sixth page that links to every other page in the web, but no page links back to it (Figure 5). The new link matrix **A12** is:

$$\mathbf{A12} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.1}$$

The matrix **M12** is constructed using the formula (2.1) with the damping factor $m = 0.15$:

$$\mathbf{M12} = \begin{bmatrix} 0.0250 & 0.0250 & 0.4500 & 0.4500 & 0.0250 & 0.1950 \\ 0.3083 & 0.0250 & 0.0250 & 0.0250 & 0.0250 & 0.1950 \\ 0.3083 & 0.4500 & 0.0250 & 0.4500 & 0.8750 & 0.1950 \\ 0.3083 & 0.4500 & 0.0250 & 0.0250 & 0.0250 & 0.1950 \\ 0.0250 & 0.0250 & 0.4500 & 0.0250 & 0.0250 & 0.1950 \\ 0.0250 & 0.0250 & 0.0250 & 0.0250 & 0.0250 & 0.0250 \end{bmatrix} \tag{3.2}$$

The eigenvector of **A12** corresponding to the eigenvalue $\lambda = 1$ is:

$$\mathbf{x_A} = \begin{bmatrix} 0.2449 \\ 0.0816 \\ 0.3673 \\ 0.1224 \\ 0.1837 \\ 0.0000 \end{bmatrix} \tag{3.3}$$

The eigenvector of **M12** corresponding to the eigenvalue $\lambda = 1$ is:

$$\mathbf{x_M} = \begin{bmatrix} 0.2312 \\ 0.0948 \\ \mathbf{0.3402} \\ 0.1350 \\ 0.1738 \\ 0.0250 \end{bmatrix} \tag{3.4}$$

As in Exercise 11, the PageRank for page 3 remains the highest in both matrices, although slightly lower in **M12** compared to **A12**. Page 6, with no backlinks, has a score of zero in **A12** but a small positive score in **M12**, showing its minimal non-zero importance due to the damping factor. Moreover, the ranking using **M12** is more stable, as it incorporates the probability of random jumps, ensuring that even pages with few or no backlinks contribute to the overall ranking.

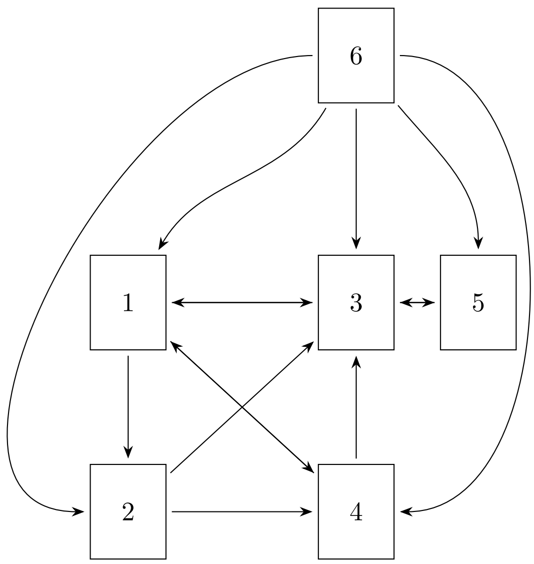

Figure 5: Graph of Exercise 3 after the addition of the page 6 linking to all other pages of the graph of Figure 1

13 PageRank for Disconnected Web with M Matrix

"Construct a web consisting of two or more subwebs and determine the ranking given by formula (3.1)."

It is possible to use the web constructed in Exercise 2 (refer to **A2** and Figure 2). The matrix **M13** is constructed using formula (3.1) with $m = 0.15$, applying the damping factor to the link matrix **A2**:

$$
\mathbf{M13} = \begin{bmatrix}
0.0167 & 0.0167 & 0.8667 & 0.4417 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 \\
0.3000 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 \\
0.3000 & 0.4417 & 0.0167 & 0.4417 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 \\
0.3000 & 0.4417 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 \\
0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.8667 & 0.0167 & 0.0167 & 0.0167 & 0.0167 \\
0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.8667 & 0.0167 & 0.0167 & 0.0167 \\
0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.8667 & 0.0167 & 0.0167 \\
0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.8667 & 0.0167 \\
0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.0167 & 0.8667
\end{bmatrix} \tag{4.1}
$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is:

$$
\mathbf{x} = \begin{bmatrix}
0.1636 \\ 0.0630 \\ 0.1280 \\ 0.0898 \\ 0.1111 \\ 0.1111 \\ 0.1111 \\ 0.1111 \\ 0.1111
\end{bmatrix} \tag{4.2}
$$

This eigenvector represents the ranking of the pages. As expected, the pages in the first subweb (1, 2, 3, 4) receive the highest ranks, with page 1 having the highest importance score. Pages 5 through 9, forming the other subwebs, have equal ranks within their subwebs, due to their isolated nature.

14 Convergence and Eigenvalue Analysis for PageRank Iterations

"For the web in Exercise 11, compute the values of $\|\mathbf{M}^k\mathbf{x}_0 - \mathbf{q}\|_1$ and $\frac{\|\mathbf{M}^k\mathbf{x}_0-\mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0-\mathbf{q}\|_1}$ for $k = 1, 5, 10, 50$, using an initial guess $\mathbf{x}_0$ not too close to the actual eigenvector $\mathbf{q}$ (so that you can watch the convergence). Determine $c = \max_{1\leq j\leq n} |1 - 2\min_{1\leq i\leq n} M_{ij}|$ and the absolute value of the second largest eigenvalue of $\mathbf{M}$."

Using the link matrix **M11** from Exercise 11, the values of $\|\mathbf{M}^k\mathbf{x}_0 - \mathbf{q}\|_1$ and $\frac{\|\mathbf{M}^k\mathbf{x}_0-\mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0-\mathbf{q}\|_1}$ are calculated for various iterations k. The initial guess $\mathbf{x}_0$ was chosen not too close to the actual eigenvector $\mathbf{q}$ to observe the convergence behavior.

The matrix **M14** is identical to **M11** (equation 2.2), and the initial vector $\mathbf{q}$ is:

$$
\mathbf{q} = \begin{bmatrix}
0.2371 \\ 0.0972 \\ 0.3489 \\ 0.1385 \\ 0.1783
\end{bmatrix} \tag{5.1}
$$

The initialized random vector[2] $\mathbf{x}_0$ was:

$$\mathbf{x}_0 = \begin{bmatrix} 0.1333 \\ 0.2667 \\ 0.1333 \\ 0.3000 \\ 0.1667 \end{bmatrix} \tag{5.2}$$

In Table 1 is shown the convergence behavior as expected.
The value of $c = \max_{1\leq j\leq n} |1 - 2\min_{1\leq i\leq n} M_{ij}|$ was calculated as:

$$c = 0.940 \tag{5.3}$$

Finally, the second largest eigenvalue λ_2 of **M14** was found to be:

$$\lambda_2 = 0.6113 \tag{5.4}$$

k	$\|\mathbf{M}^k\mathbf{x}_0 - \mathbf{q}\|_1$	Eigenvalue
0	0.66	-
1	2.879986×10^{-1}	0.4351
5	4.907671×10^{-2}	0.5727
10	3.938801×10^{-3}	0.6150
50	1.109848×10^{-11}	0.6113

Table 1: Convergence of $\|\mathbf{M}^k\mathbf{x}_0 - \mathbf{q}\|_1$ and corresponding eigenvalues for selected values of k.

15 The Second Largest Eigenvalue

"To see why the second largest eigenvalue plays a role in bounding $\frac{\|\mathbf{M}^k\mathbf{x}_0-\mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0-\mathbf{q}\|_1}$, consider an $n\times n$ positive column-stochastic matrix $\mathbf{M}$ that is diagonalizable. Let $\mathbf{x}_0$ be any vector with non-negative components that sum to one. Since $\mathbf{M}$ is diagonalizable, we can create a basis of eigenvectors $\{\mathbf{q}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$, where $\mathbf{q}$ is the steady state vector, and then write $\mathbf{x}_0 = a\mathbf{q} + \sum_{k=1}^{n-1} b_k\mathbf{v}_k$. Determine $\mathbf{M}^k\mathbf{x}_0$, and then show that $a = 1$ and the sum of the components of each $\mathbf{v}_k$ must equal 0. Next apply Proposition 4 to prove that, except for the non-repeated eigenvalue $\lambda = 1$, the other eigenvalues are all strictly less than one in absolute value. Use this to evaluate $\lim_{k\to\infty} \frac{\|\mathbf{M}^k\mathbf{x}_0-\mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0-\mathbf{q}\|_1}$."

Given an $n \times n$ positive column-stochastic matrix $\mathbf{M}$, and assuming $\mathbf{M}$ is diagonalizable, we can express the matrix as $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{P}$ is the matrix of eigenvectors, and $\mathbf{D}$ is the diagonal matrix of eigenvalues.

Let $\mathbf{x}_0$ be any vector with non-negative components that sum to one, such that:

$$\mathbf{x}_0 = a\mathbf{q} + \sum_{k=1}^{n-1} b_k\mathbf{v}_k \tag{6.1}$$

where $\mathbf{q}$ is the steady-state vector corresponding to the eigenvalue $\lambda = 1$, and $\mathbf{v}_k$ are the eigenvectors corresponding to the other eigenvalues.

[2] Notice that each time the Matlab code is run, a different random vector is initialized

Applying $\mathbf{M}^k$ to $\mathbf{x}_0$:

$$\mathbf{M}^k\mathbf{x}_0 = a\mathbf{q} + \sum_{j=1}^{n-1} b_j \lambda_j^k \mathbf{v}_j \tag{6.2}$$

Let's prove that $\sum_{k=1}^{n} \mathbf{v}_k = 0$.
Since M is diagonalizable, it is possible to start with the linear equation:

$$\mathbf{M}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

Taking the sum of all components of the vector $\mathbf{M}\mathbf{v}_k$, we obtain:

$$\sum_{i=1}^{n} (\mathbf{M}\mathbf{v}_k)_i = \lambda_k \sum_{i=1}^{n} (\mathbf{v}_k)_i$$

Expanding $\mathbf{M}\mathbf{v}_k$, we have:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} (\mathbf{v}_k)_j = \lambda_k \sum_{i=1}^{n} (\mathbf{v}_k)_i$$

where M_{ij} are the elements of the matrix $\mathbf{M}$.

Since $\mathbf{M}$ is column-stochastic, the sum of each column is 1. Therefore, the sum of $\mathbf{M}\mathbf{v}_k$ simplifies to:

$$\sum_{i=1}^{n} (\mathbf{v}_k)_i = \lambda_k \sum_{i=1}^{n} (\mathbf{v}_k)_i$$

The only way this equation can hold is if the eigenvalue λ_k that is equal to 1 (we will reject this assumption) or the sum $\sum_{i=1}^{n} (\mathbf{v}_k)_i$ equals 0. Thus:

$$\sum_{i=1}^{n} (\mathbf{v}_k)_i = 0 \tag{6.3}$$

Now, let's check $a = 1$.
Since $\mathbf{x}_0$ is a probability vector, its components are non-negative and sum to 1:

$$\sum_{i=1}^{n} (\mathbf{x}_0)_i = 1 \tag{6.4}$$

The steady-state vector $\mathbf{q}$ also satisfies:

$$\sum_{i=1}^{n} \mathbf{q}_i = 1 \tag{6.5}$$

Taking the sum of the components on both sides of the equation for $\mathbf{x}_0$:

$$\sum_{i=1}^{n} (\mathbf{x}_0)_i = a \sum_{i=1}^{n} \mathbf{q}_i + \sum_{k=1}^{n-1} b_k \sum_{i=1}^{n} (\mathbf{v}_k)_i \tag{6.6}$$

Substituting in equation (15.6) the sums (15.4) and (15.5):

$$1 = a \times 1 + \sum_{k=1}^{n-1} b_k \sum_{i=1}^{n} (\mathbf{v}_k)_i \tag{6.7}$$

Substituting what we get from equation (6.3), it is possible to write:

$$a = 1 \tag{6.8}$$

To prove that, except for the non-repeated eigenvalue $\lambda = 1$, the other eigenvalues λ_k are strictly less than 1 in absolute value (*Proposition* 4), we proceed starting from:

$$\|\mathbf{M}\mathbf{v}_k\|_1 \leq c \cdot \|\mathbf{v}_k\|_1$$

where $c = \max_{1\leq j\leq n} |1 - 2\min_{1\leq i\leq n} M_{ij}|$ and $c < 1$.

Thus, we have:

$$|\lambda_k| \cdot \|\mathbf{v}_k\|_1 \leq c \cdot \|\mathbf{v}_k\|_1$$

Since $\|\mathbf{v}_k\|_1$ is non-zero, it follows that:

$$|\lambda_k| \leq c < 1$$

Therefore, $|\lambda_k|$ is strictly less than 1.

Now, it's possible to evaluate the limit:

$$\begin{aligned}
\lim_{k\to\infty} \frac{\|\mathbf{M}^k\mathbf{x}_0 - \mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0 - \mathbf{q}\|_1} &= \lim_{k\to\infty} \frac{\|\sum_{i=2}^{n-1} \lambda_i^k \mathbf{b}_i\|_1}{\|\sum_{i=2}^{n-1} \lambda_i^{k-1} \mathbf{b}_i\|_1} \\
&= \lim_{k\to\infty} \frac{|\lambda_2|^k \|\sum_{i=2}^{n-1} \left(\frac{\lambda_i}{\lambda_2}\right)^k \mathbf{b}_i\|_1}{|\lambda_2|^{k-1} \|\sum_{i=2}^{n-1} \left(\frac{\lambda_i}{\lambda_2}\right)^{k-1} \mathbf{b}_i\|_1} \\
&= |\lambda_2| \lim_{k\to\infty} \frac{\|\sum_{i=2}^{n-1} \left(\frac{\lambda_i}{\lambda_2}\right)^k \mathbf{b}_i\|_1}{\|\sum_{i=2}^{n-1} \left(\frac{\lambda_i}{\lambda_2}\right)^{k-1} \mathbf{b}_i\|_1} \\
&= |\lambda_2| \\
&\Rightarrow \text{Second Largest Eigenvector}
\end{aligned}$$

16 Case of Non-diagonalizability of M

"Consider the link matrix

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix}.$$

Show that $\mathbf{M} = (1-m)\mathbf{A} + m\mathbf{S}$ *(where all* $S_{ij} = \frac{1}{3}$*) is not diagonalizable for* $0 < m < 1$.*"*

The matrix **M16** is constructed as:

$$\mathbf{M16} = (1-m)\mathbf{A16} + m\mathbf{S16}$$

where

$$\mathbf{A16} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix}, \quad \mathbf{S16} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Expanding the formula, **M16** can be written explicitly as:

$$\mathbf{M16} = \begin{bmatrix} \frac{m}{3} & \frac{1}{2} - \frac{m}{6} & \frac{1}{2} - \frac{m}{6} \\ \frac{m}{3} & \frac{m}{3} & \frac{1}{2} - \frac{m}{6} \\ 1 - \frac{2m}{3} & \frac{1}{2} - \frac{m}{6} & \frac{m}{3} \end{bmatrix}.$$

The eigenvalues of **M16** were computed to be 1 with algebraic multiplicity 1, and $m/2 - 1/2$ with algebraic multiplicity 2.

For the eigenvalue $\lambda_1 = 1$, the output of the Matlab code provides a geometric multiplicity equal to 1 and the corresponding eigenvector $\mathbf{v}_1$:

$$\mathbf{v}_1 = \begin{bmatrix} \frac{m-3}{2(m-2)} \\ -\frac{1}{m-2} \\ 1 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = m/2 - 1/2$ instead, it is found an algebraic multiplicity of 2 and a geometric multiplicity of 1:

$$\mathbf{v}_{m/2-1/2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

It is evident from the results that the matrix **M16** is not diagonalizable for $0 < m < 1$. This is indicated by the fact that the geometric multiplicity of the eigenvalue λ_2 is 1, which is less than its algebraic multiplicity of 2. Therefore, **M16** does not have a complete set of linearly independent eigenvectors, confirming the non-diagonalizability of the matrix.

17 The Damping Factor m and Its Impact on PageRank

"How should the value of m be chosen? How does this choice affect the rankings and the computation time?"

A lower value of m emphasizes the web's actual link structure, which can lead to rank instability in disconnected web (as it was shown in Exercise 2), while a higher m (closer to 1) promotes more equal treatment of all pages, leading to more stable and unique rankings, but ignoring the real behaviour of a user. Additionally, a higher m typically accelerates convergence in iterative methods, reducing computation time.