Homework 1

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Exercise 1

a) There are 8 possible o-d cuts:

$\mathcal{U}$	$\mathcal{V} \backslash \mathcal{U}$	boundary	cut-capacity
$\{d\}$	$\{o,\ a,\ b,\ c\}$	{e2, e4, e6}	5
$\{a,\ d\}$	$\{o,\ b,\ c\}$	{e1, e4, e6}	6
$\{b,\ d\}$	$\{o,\ a,\ c\}$	{e2, e3, e7, e6}	7
$\{c,\ d\}$	$\{o,\ a,\ b\}$	{e2, e4, e5}	7
$\{a,\ b,\ d\}$	$\{o,\ c\}$	{e1, e3, e6}	7
$\{a,\ c,\ d\}$	$\{o,\ b\}$	{e1, e4, e5}	8
$\{b,\ c,\ d\}$	$\{o,\ a\}$	{e2, e3, e7}	6
$\{a,\ b,\ c,\ d\}$	$\{o\}$	{e1, e3}	6

The smallest capacity which must be removed to have no feasible flow from o to d corresponds to the min-cut capacity, which in this case is equal to 5

b) Given $x > 0$ units of additional capacity, there are several ways to distribute it to obtain the maximum possible flow. This is an optimization problem: since the min-cut capacity is a linear combination of the capacities of each edge, it holds:

$$\tau_{od}(x) = \min\ \mathbf{K}(c + a(x))$$

where c is the capacity column vector, a(x) is the vector which represents how the additional capacity is distributed, and is such that: $||a||_1 = x, a > 0$, and

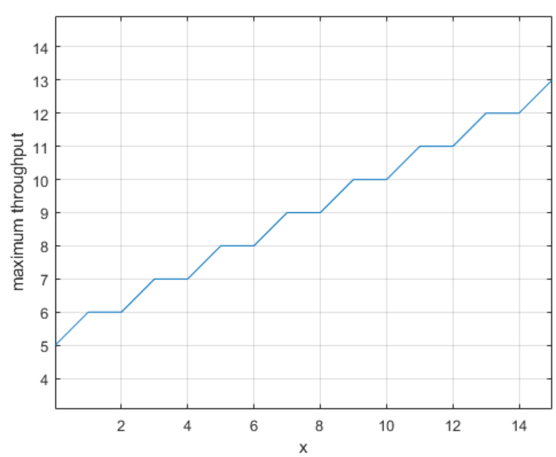
$$\mathbf{K} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Finding the best distribution can now be rewritten as a maximization problem:

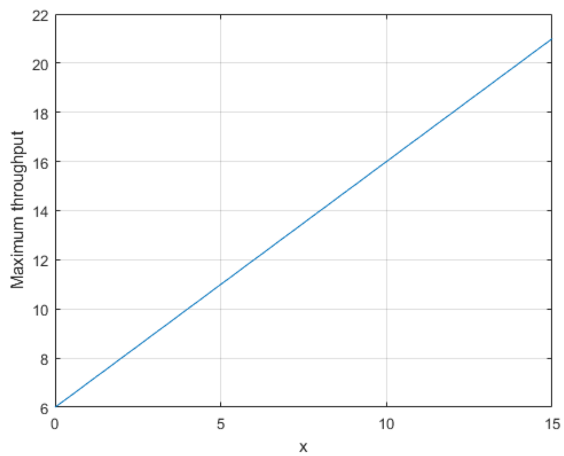
$$\tau^*_{od}(x) = \max_{a \in \mathbb{Z}^{|\mathcal{E}|}} \min \mathbf{K}(c + a(x)) \text{ s.t.: } ||a||_1 = x, a > 0$$

This is a non-convex, non-linear problem, however as long as x remains small, since the feasible set has finite cardinality equal to $|\mathcal{E}|^x$, brute force is a feasible solution.

Using brute force, the optimum is reached when the additional capacity is put in equal amounts on edges e1 and e2: this is due to the fact that e1 or e2 appear in the boundary of all the possible o-d cuts, therefore raising their capacity will raise the capacity of the min cut, whichever it may be. My solution, therefore, was alternating between increasing e1's capacity by 1 and then e2's capacity by 1. This method produces the following plot:

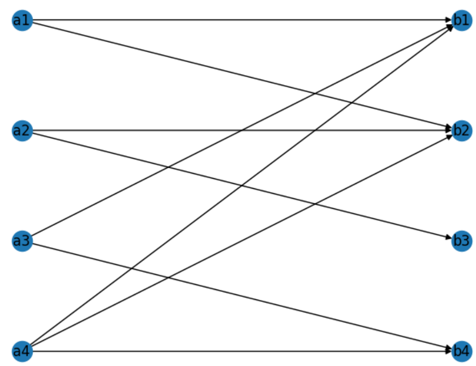


c) Given also the possibility of adding an edge between any two nodes, the best possible choice would be an edge that connects o to d directly: for obvious reasons, one such edge would be in the boundary of any o-d cut, therefore distributing all the additional capacity available on it would increase by the same amount the throughput of the whole network. In this case, the plot of the maximum throughput as a function of x is:

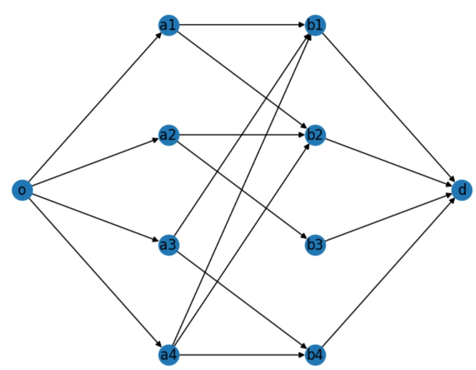


Exercise 2

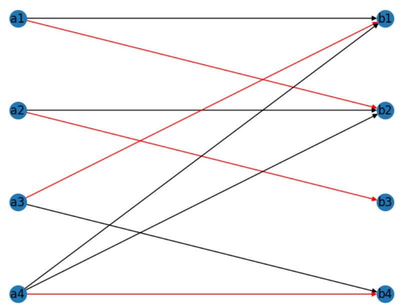
a) The situation can be modeled using a bipartite graph, where one subset of the nodes, $\mathcal{A} = \{a1, a2, a3, a4\}$, represents the people, and one subset of the nodes, $\mathcal{B} = \{b1, b2, b3, b4\}$, represents the food. Each edge represents the interest of a person towards a certain food, and any amount of flow through it represents that person taking the same amount of that food. The graph used to model this problem follows:



A perfect match can be found by adding two nodes (o and d), one connected to each node in subset $\mathcal{A}$ and the other connected to each node of subset $\mathcal{B}$ in the following manner:

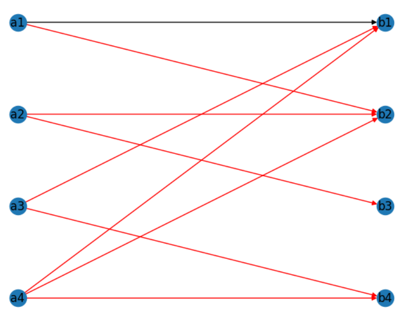


Now the problem of finding a perfect match can be solved by setting the capacity of each edge to 1 and by finding the max-flow. The perfect match is:



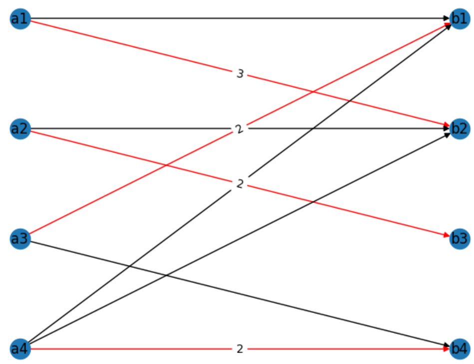
$$a1 \to b2,\ a2 \to b3,\ a3 \to b1,\ a4 \to b4$$

b) The case where more than one portion of each food is present can be represented by changing the capacity of the edges going from subset $\mathcal{B}$ to the additional node d to the amount of portions available, the capacity of each edge going from node o to any node of subset $\mathcal{A}$ to 4, and by keeping the capacity of each edge between the two subsets equal to 1. By doing so, the flow coming in the nodes in subset $\mathcal{B}$ will be at most equal to the number of portions available, due to the conservation of mass. Under the conditions described, a maximum of 8 portions of food can be served:



$$a1 \to \{b2\},\ a2 \to \{b2, b3\},\ a3 \to \{b1, b4\},\ a4 \to \{b1, b2, b4\}$$

c) By following the same principle as before, to impose taking at most a certain number of portions to each person, the capacity of each edge from o to $\mathcal{A}$ must be changed to the desired amount, the capacity of edges from $\mathcal{A}$ to $\mathcal{B}$ can then have infinite capacity, as the conservation of mass will still guarantee a finite flow following the requirements. It can then be seen that in the proposed scenario, 9 units of food can be served, as shown in the following figure:

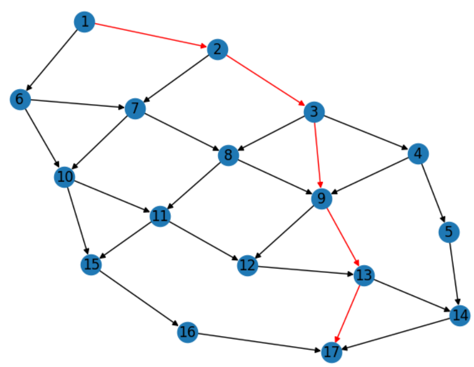


Exercise 3

a) Finding the shortest path corresponds to finding the path which minimizes the cost function. In this specific case, the cost of each edge is constant, as it is assumed that the delay is equal to the minimum travel time. The optimization problem is:

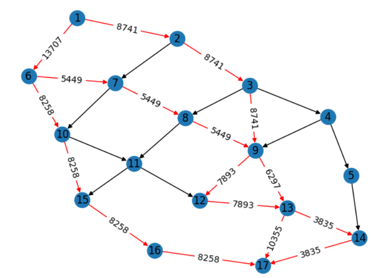
$$\min_{f \in \{0,1\}^{\mathcal{E}}} l^T f \text{ s.t.: } \mathbf{B} f = \nu = \delta^{(1)} - \delta^{(17)}$$

where $\mathbf{B}$ is the node-edge adjacency matrix, l is the vector containing the minimum travel times, and ν is the exogenous flow vector corresponding to traveling from node o to node d. The shortest path, resulting from this problem, is:



$$\gamma_{1-17} = (e_1, e_2, e_{12}, e_9, e_{25})$$

b) Running a simple script to find the min-cut capacity, it can be seen that it is equal to $c_{\mathcal{U}} = 22448$, therefore any arbitrary flow vector achieving this throughput is a max-flow. The one achieving the shortest travel time, which can be obtained by minimizing the sum of the cost function $\psi_e(f_e)$ on each edge, is the following:



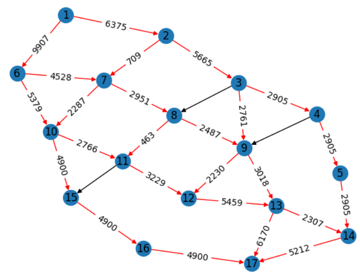
$$f = (8741,8741,0,0,13707,5449,5449,5449,6297,0,0,8741,0,0,8258,0,8258,0,0,7893,0,7893,3835,0,10355,3835,8258,8258)$$

c) $\nu = \mathbf{B}f = (16282,\ 9094,\ 19448,\ 4957,\ -746,\ 4768,\ 413,\ -2,\ -5671,\ 1169,\ -5,\ -7131,\ -380,\ -7412,\ -7810,\ -3430,\ -23544)$

d) The social optimum f^* can be computed by solving the optimization problem:

$$f^* = \arg \min_{f \in \mathbb{R}_+^{\mathcal{E}}} \sum_{e \in \mathcal{E}} f_e \tau_e(f_e) \text{ s.t.: } \mathbf{B}f = \nu, f \leq c$$

where c is the capacity vector, ν is the exogenous flow vector, as specified by the problem, and: $\tau_e(f_e) = \frac{l_e}{1 - f_e/c_e}$ is the delay function. Solving the problem yields the following result:



f= (6375,5665,2905,2905,9907,4528,2951,2487,3018,709,0,2761,0,
2905,5379,2766,4900,2287,463,2230,3229,5459,2307,0,6170,5212,4900,4900)

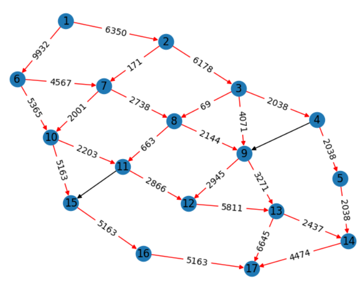
e) Finding the Wardrop equilibrium corresponds to solving the minimization problem:

$$f^{(0)} = \arg\min_{f \in \mathbb{R}_+^{\mathcal{E}}} \sum_{e \in \mathcal{E}} \psi_e(f_e) \text{ s.t.: } \mathbf{B}f = \nu, f \leq c$$

where the cost function $\psi_e(f_e)$ is chosen as:

$$\psi_e(f_e) = \int_0^{f_e} \tau_e(s)\, ds = \int_0^{f_e} \frac{l_e}{1 - s/c_e}\, ds = -c_e l_e \log(1 - f_e/c_e)$$

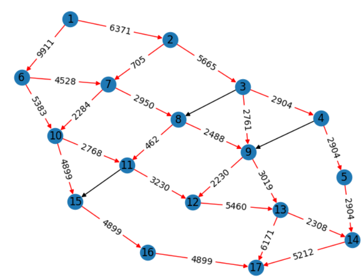
Resulting in the flow:



f= (6350,6178,2038,2038,9932,4567,2738,2144,3271,171,69,4071,0,
2038,5365,2203,5163,2001,663,2945,2866,5811,2437,0,6645,4474,5163,5163)

The total travel time ($\sum_{e \in \mathcal{E}} f_e \tau_e(f_e)$) of the Wardrop equilibrium is 24342, which is greater than the cost which achieved at the social optimum (by definition of social optimum), which instead is 23997, for a reasonably low price of anarchy of 1.014

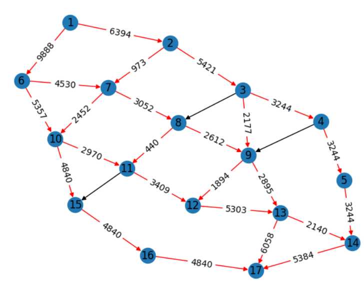
f) Computing the tolls as specified in the problem, results in a new Wardrop equilibrium f^ω, which has a travel time equal to the social optimum, but a slightly different flow:



$$f = (6371,5665,2904,2904,9911,4528,2950,2488,3019,705,0,2761,0,2904,5383,2768,4899,2284,462,2230,3230,5460,2308,0,6171,5212,4899,4899)$$

This is to be expected, as marginal tolls were introduced.

g) The system optimum can be computed in the same manner as in the previous exercises, and results in the following flow:

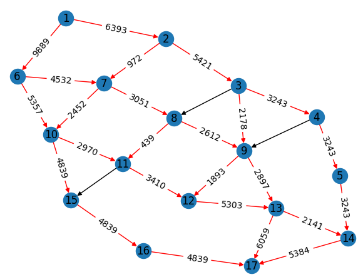


$$f = (6394,5421,3244,3244,9888,4530,3052,2612,2895,973,0,2177,0,3244,5357,2970,4840,2452,440,1894,3409,5303,2140,0,6058,5384,4840,4840)$$

The tolls which result in the Wardrop equilibrium aligning to the system optimum are the marginal tolls, computed as in point **f**):

$$\omega_e^* = \psi_e'(f_e^*) - \tau_e(f_e^*) = \frac{\partial}{\partial f_e}(f_e(\frac{l_e}{1 - f_e/c_e} - l_e)|_{f_e = f_e^*} - \tau_e(f_e^*) = \frac{f_e^* l_e(2c_e - f_e)}{(c_e - f_e^*)^2} - \frac{l_e}{1 - f_e^*/c_e}$$

After introducing the tolls, the Wardrop equilibrium becomes:



f= (6393,5421,3243,3243,9889,4532,3051,2612,2897,972,0,
2178,0,3243,5357,2970,4839,2452,439,1893,3410,5303,2141,0,6059,5384,4839,4839)