Homework 2

Giovanni Tortia, s328867

Problem 1

a) The random walk of the particle can be simulated by having a Poisson clock attached to each node in the graph, with rates given by the vector $\omega = \Lambda\mathbb{1}$. When the Poisson clock ticks, the particle jumps from its current node i to node j with probability P_{ij}, where $P = D^{-1}\Lambda$, once the particle comes back to node a the total time is measured and saved to be averaged. This results in an average return time of 6.056, computed over 10^6 simulations

b) The expected value of the return time can be computed as:

$$\mathbb{E}[T_a^+] = \frac{1}{\omega_a \overline{\pi}_a}$$

$\overline{\pi}$ is the invariant distribution of $\overline{P}$, equal to its eigenvector associated with the eigenvalue 1. Strong connectivity of the graph guarantees its uniqueness, while aperiodicity implies that, for each eigenvalue λ_i of $\overline{P}$, if $\lambda_i \neq 1$ then $\lambda_i < 1$.
Because of these two properties, the rows of $\overline{P}^t$, as t tends to infinity, converge to $\overline{\pi}$, which means that $\overline{\pi}$ can be approximated simply by raising $\overline{P}$ to a high enough power and taking any of its rows. With $t = 1500$:

$$\overline{\pi} \approx \begin{pmatrix} 0.231 & 0.165 & 0.277 & 0.182 & 0.146 \end{pmatrix}$$

which, when plugged in the expression before gives the value $\mathbb{E}[T_a^+] = 6.059$.
This is compatible with the result achieved earlier, and it can be seen that if the number of simulations is raised, the average converges to the theoretical result:

#Simulations	Average T_a^+
100	5.988
1000	6.004
10^4	6.012
10^5	6.046
10^6	6.056

c) According to the simulations, it takes on average 10.761 time units for the particle to travel from node o to node d. This number was obtained by averaging 10^6 simulations

d) From theory:

$$\begin{cases} \mathbb{E}_i[T_s] = \frac{1}{\omega_i} + \sum_j P_{ij}\mathbb{E}_j[T_s], i \notin s \\ \mathbb{E}_i[T_s] = 0, i \in s \end{cases}$$

The expected value of the hitting times on d starting from each node then are:

$$\begin{cases} \mathbb{E}_o[T_d] = \frac{1}{\omega_o} + \frac{2}{3}\mathbb{E}_a[T_d] + \frac{1}{3}\mathbb{E}_b[T_d] \\ \mathbb{E}_a[T_d] = \frac{1}{\omega_a} + \frac{3}{4}\mathbb{E}_b[T_d] + \frac{1}{4}\mathbb{E}_c[T_d] \\ \mathbb{E}_b[T_d] = \frac{1}{\omega_b} + \frac{3}{5}\mathbb{E}_o[T_d] + \frac{2}{5}\mathbb{E}_c[T_d] \\ \mathbb{E}_c[T_d] = \frac{1}{\omega_c} + \frac{1}{3}\mathbb{E}_b[T_d] + \frac{2}{3}\mathbb{E}_d[T_d] \\ \mathbb{E}_d[T_d] = 0 \end{cases}$$

Solving for $\mathbb{E}_o[T_d]$ yields a value of 10.6667 time units, which is compatible with the simulations. Once again, it can be observed that the result of the simulations converges to this number as the average is carried out on more experiments:

#Simulations	Average $\mathbb{E}_o[T_d]$
100	10.022
1000	10.825
10^4	10.629
10^5	10.747
10^6	10.761

e) Running a simple script shows that the graph is strongly connected and aperiodic, these conditions guarantee the convergence to a consensus for any initial condition $x(0)$. In particular:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} P^t x(0) = \alpha\mathbb{1}$$

$$\alpha = \pi x(0)$$

π is the invariant distribution of P, and can be computed in the same way as in point **a)**, since the same conditions are met. raising P to the power 1500:

$$\pi \approx \begin{pmatrix} 0.165 & 0.197 & 0.275 & 0.217 & 0.145 \end{pmatrix}$$

f) The case where each component of x is an independent variable with given variance, is akin to having a number of independent sensors measuring the same quantity, each with a given uncertainty. To simulate this I used five normally distributed random variables, with the same, arbitrary mean $\mu = 5$, and variance as specified; I then computed the consensus value as described in the previous point and computed the variance with respect to μ. Repeating this process 10^6 times results in an average variance of $\overline{\sigma}_\alpha = 0.3716$.
The theoretical value can be computed as:

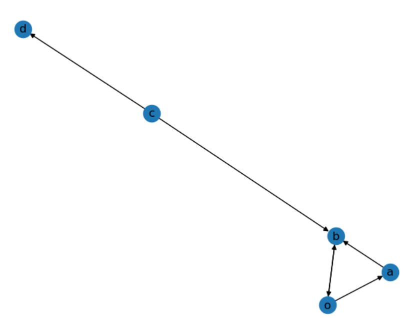
$$\sigma_\alpha = \sum_{i \in \mathcal{V}} \sigma_i^2 \pi_i^2 = 0.3722$$

As in the previous points, a higher number of experiments brings the average closer to the theoretical value:

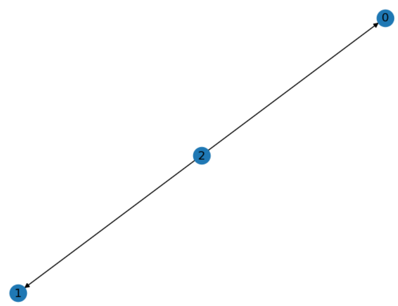
#Simulations	$\overline{\sigma}_\alpha$
100	0.3131
1000	0.4125
10^4	0.3697
10^5	0.3734
10^6	0.3716

Interestingly, the resulting variance is lower than the variance on any single node, due to the so-called "wisdom of the crowds".

g) Removing the specified arcs we obtain the following graph:



The resulting network is no longer strongly connected nor aperiodic, and by analyzing the condensation graph, we can see it has two sink components:



$$0 = \{o, a, b\}, 1 = \{c\}, 2 = \{d\}$$

Let's now analyze the dynamics of each separate component, starting from 0, which is aperiodic: the rows of P^t relative to the nodes in it, as t tends to infinity, converge to $\pi^{(0)} = \begin{pmatrix} 0.375 & 0.25 & 0.375 & 0 & 0 \end{pmatrix}$. As expected, since 0 is a sink, $\pi^{(0)}$ has support only on the nodes in 0, which implies that a local consensus within the component can be reached, with consensus value:

$$\alpha_0 = \pi^{(0)} x(0)$$

Component 2 instead is composed of a single, stubborn node, so its dynamics is simply:

$$x_d(t) = x_d(0)$$

Finally, the last component, which only comprises node c, does not influence the rest of the network, as it's the only source component in the condensation graph, and its dynamics are given by the weighted average of node c and node b according to matrix P:

$$x_c(t+1) = P_{cb} x_b(t) + P_{cd} x_d(t) = P_{cb} x_b(t) + P_{cd} x_d(0)$$

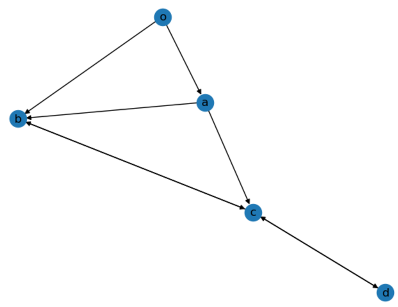
As t tends to infinity, since component 0 is guaranteed to reach a stable consensus, and component 1 has constant opinion $x_d(0)$, node c will also reach an equilibrium:

$$\overline{x}_c = \lim_{t \to \infty} x_c(t) = \lim_{t \to \infty} P_{cb} x_b(t) + P_{cd} x_d(0) = P_{cb} \alpha_0 + P_{cd} x_d(0)$$

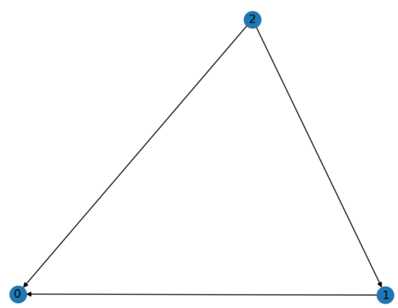
This means that consensus over the whole network is reached if and only if

$$x_d(0) = \alpha_0$$

h) Removing the specified arcs results in the following graph:



This is an aperiodic, non strongly connected network, with condensation graph shown in the next figure:



$$0 = \{b, c, d\}, 1 = \{a\}, 2 = \{o\}$$

It can be easily seen that the critical component to reach consensus is 0, as it's the only sink in the condensation graph, and analyzing it shows that it is periodic. This condition alone prevents a stable consensus to be reached: in facts, when computing the states over time we find the following dynamics:

$$x(t+1) = Px(t) = \begin{pmatrix} \frac{2}{3}x_a(t) + \frac{1}{3}x_b(t) \\ \\ \frac{3}{4}x_b(t) + \frac{1}{4}x_c(t) \\ \\ x_c(t) \\ \\ \frac{2}{3}x_b(t) + \frac{1}{3}x_d(t) \\ \\ x_c(t) \end{pmatrix}$$

Which, starting from

$$x(0) = \begin{pmatrix} x_{o,0} & x_{a,0} & x_{b,0} & x_{c,0} & x_{d,0} \end{pmatrix}$$

evolves as:

$$x(1) = \begin{pmatrix} \frac{2}{3}x_{o,0} + \frac{1}{3}x_{b,0} \\ \frac{3}{4}x_{b,0} + \frac{1}{4}x_{c,0} \\ x_{c,0} \\ \frac{2}{3}x_{b,0} + \frac{1}{3}x_{d,0} \\ x_{c,0} \end{pmatrix} \quad x(2) = \begin{pmatrix} \frac{1}{2}x_{b,0} + \frac{1}{2}x_{c,0} \\ \frac{1}{12}x_{b,0} + \frac{3}{4}x_{c,0} + \frac{1}{16}x_{d,0} \\ \frac{1}{3}x_{b,0} + \frac{2}{3}x_{d,0} \\ x_{c,0} \\ \frac{1}{3}x_{b,0} + \frac{2}{3}x_{d,0} \end{pmatrix}$$

$$x(3) = \begin{pmatrix} \frac{1}{6}x_{b,0} + \frac{1}{2}x_{c,0} + \frac{1}{3}x_{d,0} \\ \frac{1}{4}x_{b,0} + \frac{1}{4}x_{c,0} + \frac{1}{2}x_{d,0} \\ x_{c,0} \\ \frac{1}{3}x_{b,0} + \frac{2}{3}x_{d,0} \\ x_{c,0} \end{pmatrix} \quad x(4) = \begin{pmatrix} \frac{1}{6}x_{b,0} + \frac{1}{2}x_{c,0} + \frac{1}{3}x_{d,0} \\ \frac{1}{12}x_{b,0} + \frac{3}{4}x_{c,0} + \frac{1}{6}x_{d,0} \\ \frac{1}{3}x_{b,0} + \frac{2}{3}x_{d,0} \\ x_{c,0} \\ \frac{1}{3}x_{b,0} + \frac{2}{3}x_{d,0} \end{pmatrix}$$

$$x(5) = \begin{pmatrix} \frac{1}{6}x_{b,0} + \frac{1}{2}x_{c,0} + \frac{1}{3}x_{d,0} \\ \frac{1}{4}x_{b,0} + \frac{1}{4}x_{c,0} + \frac{1}{2}x_{d,0} \\ x_{c,0} \\ \frac{1}{3}x_{b,0} + \frac{2}{3}x_{d,0} \\ x_{c,0} \end{pmatrix} = x(3)$$

This is a cyclical behavior, where the state of the system alternates between $x(3)$ and $x(4)$, for $t \geq 3$, and the only nodes which contribute to the two oscillating states are those contained in component 0. From this, it follows that consensus in the whole network is reached if and only if component 0 starts from a local consensus, that is:

$$x_{b,0} = x_{c,0} = x_{d,0}$$

Problem 2

a) The proposed simulation is perfectly equivalent to repeating N times the simulation in point **a)** and **b)** of Problem 1: the random walk is a Poisson process, which implies that the single events (a node passing one particle to the next) are stochastically independent. This is confirmed by the simulations, which, with N = 100, yields an average return time $\overline{T_a^+} = 5.913$. If the number of particles traveling through the network is increased, the average converges to the expected value $\mathbb{E}_a[T_a^+]$:

N	Average T_a^+
100	5.913
1000	6.092
10^4	6.078
10^5	6.046

b) Simulating N particles starting from node a results in the following average number of particles, at each instant of time:

Node	Average #particles
o	23.07
a	17.28
b	26.72
c	18.73
d	14.20

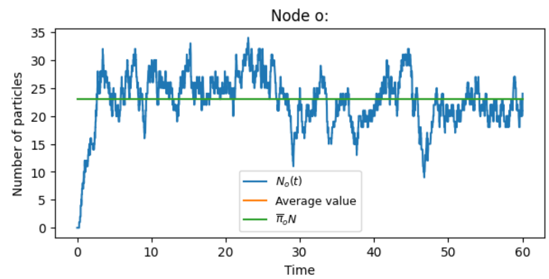
Now, since the graph is strongly connected and aperiodic, for the same reasons as in point **b)** of Problem 1, the stationary distribution of $\overline{P}$ can be approximated by taking any row of $\overline{P}^t$ for t sufficiently large. $t = 1500$ results in the vector:

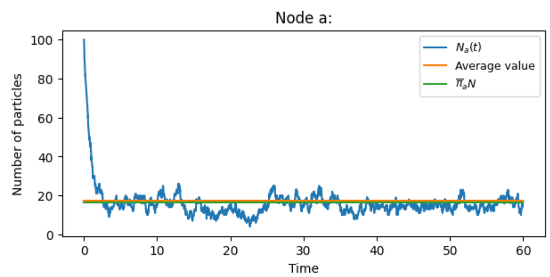
$$\overline{\pi} \approx \begin{pmatrix} 0.2306 & 0.1650 & 0.2767 & 0.1820 & 0.1456 \end{pmatrix}$$

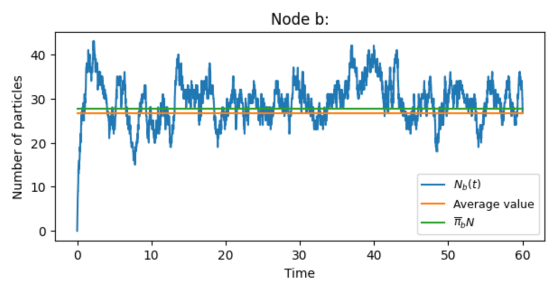
This is strikingly similar to the averages obtained through the simulations if multiplied by N, due to the fact that the graph of the walk is strongly connected and therefore, for every initial probability distribution $\overline{\pi}(0)$:

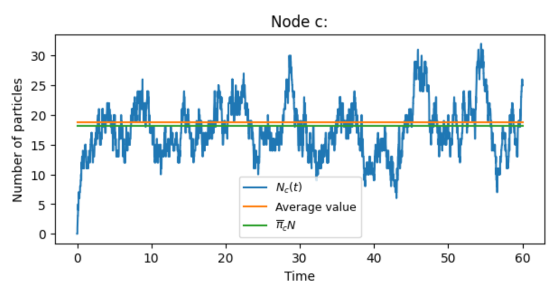
$$\lim_{t\to\infty} \overline{\pi}(t) = \overline{\pi}$$

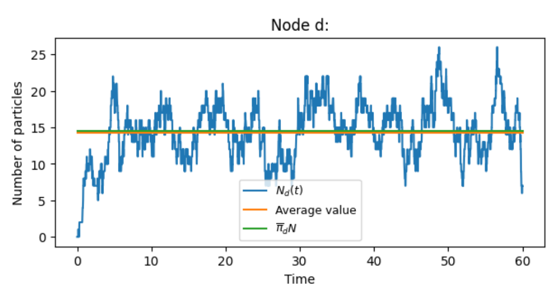
where $\overline{\pi}$ is the unique invariant distribution of $\overline{P}$. This property is particularly evident from the plots:


Node o:
Number of particles
Time
$N_o(t)$
Average value
$\overline{\pi}_o N$


Node a:
Number of particles
Time
$N_a(t)$
Average value
$\overline{\pi}_a N$


Node b:
Number of particles
Time
$N_b(t)$
Average value
$\overline{\pi}_b N$


Node c:
Number of particles
Time
$N_c(t)$
Average value
$\overline{\pi}_c N$


Node d:
Number of particles
Time
$N_d(t)$
Average value
$\overline{\pi}_d N$

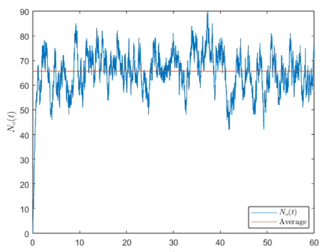
Problem 3

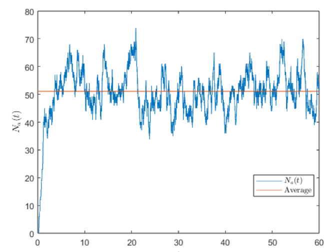
In this exercise, each node can be modeled as an M/M/1 queue, with arrival and departure rates which vary in the two parts

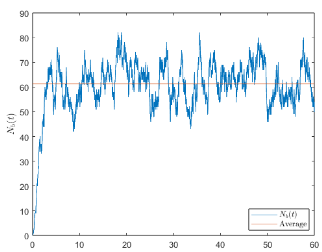
a) In the first case, each node has a departure rate μ_i proportional to the number of particles in it at any given time $N_i(t)$:

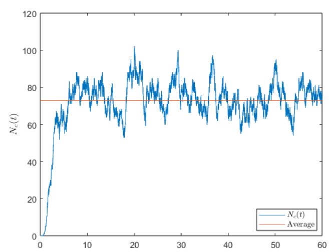
$$\mu_i(t) = \omega_i N_i(t)$$

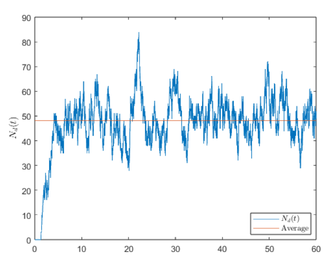
This ensures the stability of every node in the network regardless of the value of λ, as per Erlang's formula. If the evolution in time of N(t) is plotted for each node, we can indeed see that it remains bounded and stable around its average:



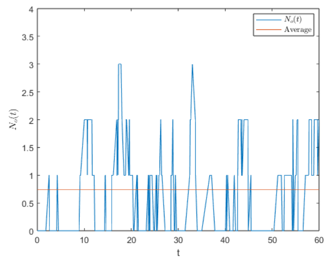


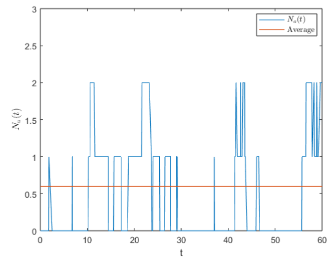


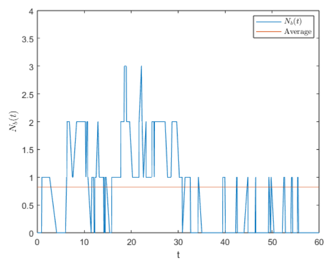


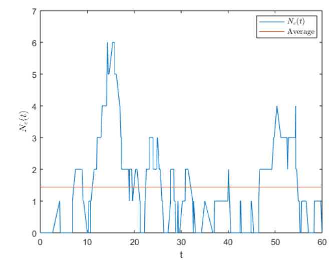


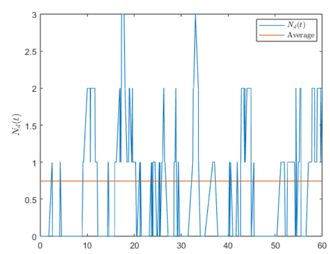
b) Running the simulation with a fixed rate of departure results in the following plots, which show that $\lambda = 1$ is a stable configuration:











If the nodes have a fixed departure rate $\mu_i = \omega_i$, stability is only guaranteed as long as $\lambda_i < \mu_i$. To find the highest admissible value of λ that keeps the network stable, we can compute the arrival rate of each node as a function of λ as if every node could pass particles to the next in no time, that is:

$$\lambda_i = \sum_{j \in \mathcal{V}} P_{ji} \lambda_j, i \in \mathcal{V} \setminus \{o, d\}$$

$$\lambda_o = \lambda_d = \lambda$$

By doing so we obtain the following result:

$$\lambda_o = \lambda \quad \lambda_a = \tfrac{1}{2}\lambda \quad \lambda_b = \tfrac{5}{8}\lambda \quad \lambda_c = \tfrac{3}{4}\lambda \quad \lambda_d = \lambda$$

The condition for stability of the whole network is $\lambda_i < \mu_i = \omega_i, \forall i \in \mathcal{V}$, where:

$$\omega = \begin{pmatrix} 1.5 & 1 & 1 & 1 & 2 \end{pmatrix}$$

The most stringent of these inequalities, which determine the maximum value of λ, is the one relative to c, which imposes $\lambda < \frac{4}{3}$