Risposta impulsiva

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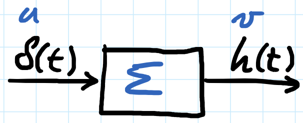
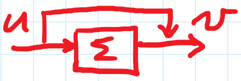
$$\Sigma: \quad \sum_{i=0}^{n} a_i \frac{d^i v(t)}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j u(t)}{dt^j}$$



$$h(t) = \underbrace{d_0\,\delta(t)}_{\text{se } m=n} + \left(\sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} d_{i,l}\, e^{\lambda_i t} \frac{t^l}{l!}\right)\delta_{-1}(t)$$



improprio $n < m$

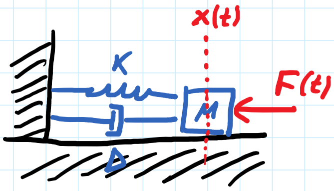
$v(t) = u(t) + \dot{u}(t)$

proprio $n = m$

$\dot{v}(t) + v(t) = u(t) + \dot{u}(t)$

Sistema MKD

$$\Sigma: \underbrace{M\ddot{x}(t) + D\dot{x}(t) + Kx(t)}_{n=2} = \underbrace{F(t)}_{m=0}$$



per semplicità, consideriamo $\Delta > 0$

1. Estraiamo le radici

$\Delta = D^2 - 4MK$

$$\lambda_{1,2} = \frac{-D \pm \sqrt{\Delta}}{2M} \rightsquigarrow \text{reali distinte}$$

e dalla definizione sopra

$$h(t) = (d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})\delta_{-1}(t)$$

2. Sostituiamo $x(t) \leftarrow h(t)$ e $F(t) \leftarrow \delta(t)$ in Σ

$$\Sigma': M\ddot{h}(t) + D\dot{h}(t) + Kh(t) = \delta(t)$$

e siccome abbiamo $\dot{h}(t)$ e $\ddot{h}(t)$, deriviamo $h(t)$ su t

per parti

$$\dot{h}(t) = \underbrace{(\lambda_1 d_1 e^{\lambda_1 t} + \lambda_2 d_2 e^{\lambda_2 t})}_{f}\underbrace{\delta_{-1}(t)}_{g} + \underbrace{(d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})}_{f}\underbrace{\delta(t)}_{g}$$

$$\ddot{h}(t) = \underbrace{(\lambda_1^2 d_1 e^{\lambda_1 t} + \lambda_2^2 d_2 e^{\lambda_2 t})}_{f'}\underbrace{\delta_{-1}(t)}_{g} + 2\underbrace{(\lambda_1 d_1 e^{\lambda_1 t} + \lambda_2 d_2 e^{\lambda_2 t})}_{f\ f'}\underbrace{\delta(t)}_{g'\ g} + \underbrace{(d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})}_{f}\underbrace{\dot{\delta}(t)}_{g'}$$

③ Sostituiamo $h(t)$, $\dot{h}(t)$ e $\ddot{h}(t)$ in Σ'

$$M\Big(\overbrace{(\lambda_1^2 d_1 e^{\lambda_1 t} + \lambda_2^2 d_2 e^{\lambda_2 t})\delta_{-1}(t) + 2(\lambda_1 d_1 e^{\lambda_1 t} + \lambda_2 d_2 e^{\lambda_2 t})\delta(t) + (d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})\dot{\delta}(t)}^{\ddot{h}}\Big) +$$

$$D\Big(\overbrace{(\lambda_1 d_1 e^{\lambda_1 t} + \lambda_2 d_2 e^{\lambda_2 t})\delta_{-1}(t) + (d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})\delta(t)}^{\dot{h}}\Big) +$$

$$K\Big(\overbrace{(d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})\delta_{-1}(t)}^{h}\Big) = \delta(t)$$

e raccogliamo le funzioni linearmente indipendenti $\delta_{-1}(t)$, $\delta(t)$ e $\dot{\delta}(t)$ a destra e a sinistra del $=$ e separiamo in un sistema

$$\begin{cases} \cancel{\delta_{-1}(t)(\cdots) = 0\,\delta_{-1}(t)} \quad \text{sarà sempre nulla} \\ \delta(t)\Big(2M(\lambda_1 d_1 e^{\lambda_1 t} + \lambda_2 d_2 e^{\lambda_2 t}) + D(d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})\Big) = 1\,\delta(t) \\ \dot{\delta}(t)\Big(M(d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t})\Big) = 0\,\dot{\delta}(t) \end{cases}$$

poi semplifichiamo i δ e impostiamo $t=0$

$$\begin{cases} 2M\lambda_1 d_1 + 2M\lambda_2 d_2 + Dd_1 + Dd_2 = 1 \\ Md_1 + Md_2 = 0 \end{cases}$$

④ Risolviamo per d_1 e d_2

$$\begin{cases} (2M\lambda_1 + D)d_1 + (2M\lambda_2 + D)d_2 = 1 \\ Md_1 + Md_2 = 0 \end{cases}$$

e sapendo che $\lambda_{1,2} = \dfrac{-D \pm \sqrt{\Delta}}{2M}$

$$\begin{cases} \sqrt{\Delta}\,d_1 - \sqrt{\Delta}\,d_2 = 1 \\ d_1 + d_2 = 0 \end{cases} \rightsquigarrow \begin{cases} d_1 = \dfrac{1}{2\sqrt{\Delta}} \\ d_2 = -\dfrac{1}{2\sqrt{\Delta}} \end{cases}$$

⑤ Scriviamo $h(t)$ al completo

$$\boxed{h(t) = \left(\frac{1}{2\sqrt{\Delta}} e^{\lambda_1 t} - \frac{1}{2\sqrt{\Delta}} e^{\lambda_2 t}\right)\delta_{-1}(t)}$$

$$\Delta = D^2 - 4MK > 0$$

Esercizi

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$n=2$ $m=1$

$$2\ddot{v}(t) + 5\dot{v}(t) - 3v(t) = \dot{u}(t)$$

1) Radici + $h(t)$

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{25+24}}{4} = \begin{cases} \frac{1}{2} \\ -3 \end{cases}$$

$$h(t) = \left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\delta_{-1}(t)$$

2) Σ_δ e $\dot{h}(t), \ddot{h}(t), \dots$

$$2\ddot{h} + 5\dot{h} - 3h = \dot{\delta}$$

$$\dot{h}(t) = \left(\tfrac{1}{2}d_1 e^{\frac{1}{2}t} - 3d_2 e^{-3t}\right)\delta_{-1}(t) + \left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\delta(t)$$

$$\ddot{h}(t) = \left(\tfrac{1}{4}d_1 e^{\frac{1}{2}t} + 9d_2 e^{-3t}\right)\delta_{-1}(t) + 2\left(\tfrac{1}{2}d_1 e^{\frac{1}{2}t} - 3d_2 e^{-3t}\right)\delta(t) + \left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\dot{\delta}(t)$$

3) Sostituzione

$$2\left(\left(\tfrac{1}{4}d_1 e^{\frac{1}{2}t} + 9d_2 e^{-3t}\right)\delta_{-1}(t) + 2\left(\tfrac{1}{2}d_1 e^{\frac{1}{2}t} - 3d_2 e^{-3t}\right)\delta(t) + \left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\dot{\delta}(t)\right) +$$
$$5\left(\left(\tfrac{1}{2}d_1 e^{\frac{1}{2}t} - 3d_2 e^{-3t}\right)\delta_{-1}(t) + \left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\delta(t)\right) +$$
$$-3\left(\left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\delta_{-1}(t)\right) = \dot{\delta}$$

$\downarrow$

$$\begin{cases} \delta_{-1}(t)(\cdots) = 0\,\delta_{-1}(t) \\ \delta(t)\left(4\left(\tfrac{1}{2}d_1 e^{\frac{1}{2}t} - 3d_2 e^{-3t}\right) + 5\left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\right) = 0\,\delta(t) \\ \dot{\delta}(t)\left(2\left(d_1 e^{\frac{1}{2}t} + d_2 e^{-3t}\right)\right) = 1\,\dot{\delta}(t) \end{cases}$$

$\downarrow$ $t=0$

4 $$\begin{cases} 2d_1 - 12d_2 + 5d_1 + 5d_2 = 0 \\ 2d_1 + 2d_2 = 1 \end{cases} \rightarrow \begin{cases} d_1 = \frac{1}{4} \\ d_2 = \frac{1}{4} \end{cases}$$

5 $$h(t) = \frac{1}{4}\left(e^{\frac{1}{2}t} + e^{-3t}\right)\delta_{-1}(t)$$

$$2\dot{v}(t) - \overset{n=1}{v(t)} = \overset{m=1}{\dot{u}(t)} + 3u(t)$$

1) $\lambda = \frac{1}{2}$

$$h(t) = \underbrace{d_0\,\delta(t)}_{se\ m=n} + \left(\sum_{i=1}^{r}\sum_{\ell=0}^{\mu_i-1} d_{i,\ell}\, e^{\lambda_i t}\frac{t^\ell}{\ell!}\right)\delta_{-1}(t)$$

$$= d_0\,\delta(t) + \left(\sum_{\ell=0}^{0} d_{1,\ell}\, e^{\lambda_1 t}\frac{t^\ell}{\ell!}\right)\delta_{-1}(t)$$

$$= d_0\,\delta(t) + \underset{d_1}{\cancel{d_{1,0}}}\, e^{\lambda_1 t}\,\delta_{-1}(t)$$

$$\downarrow$$

$$h(t) = \left(d_1 e^{\frac{1}{2}t}\right)\delta_{-1}(t) + d_0\,\delta(t)$$

2) $2\dot{h} - h = \dot{\delta} + 3\delta$

$$\dot{h}(t) = \tfrac{1}{2}d_1 e^{\frac{1}{2}t}\delta_{-1}(t) + d_1 e^{\frac{1}{2}t}\delta(t) + d_0\,\dot{\delta}(t)$$

3)

$$2\left(\tfrac{1}{2}d_1 e^{\frac{1}{2}t}\delta_{-1}(t) + d_1 e^{\frac{1}{2}t}\delta(t) + d_0\,\dot{\delta}(t)\right) +$$

$$-\left(\left(d_1 e^{\frac{1}{2}t}\right)\delta_{-1}(t) + d_0\,\delta(t)\right) = \dot{\delta}(t) + 3\,\delta(t)$$

$$\downarrow$$

$$\begin{cases} \delta_{-1}(t)(\dots) = 0\,\delta_{-1}(t) \\ \delta(t)\left(2d_1 e^{\frac{1}{2}t} - d_0\right) = 3\,\delta(t) \\ \dot{\delta}(t)(2d_0) = 1\,\dot{\delta}(t) \end{cases}$$

$$\downarrow t=0$$

4)

$$\begin{cases} 2d_1 - d_0 = 3 \\ 2d_0 = 1 \end{cases} \rightarrow \begin{cases} d_0 = \frac{1}{2} \\ d_1 = \frac{7}{4} \end{cases}$$

5) $h(t) = \frac{1}{2}\delta(t) + \frac{7}{4}e^{\frac{1}{2}t}\delta_{-1}(t)$

Risposta forzata e totale

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Risposta forzata

$$v_f(t) = (h * u)(t) \quad \left[= \int_{-\infty}^{+\infty} h(\tau)\, u(t-\tau)\, d\tau \right]$$

Risposta totale

$$v(t) = v_\ell(t) + v_f(t)$$

Esercizi

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$$\ddot{v}(t) - 4\dot{v}(t) + 4v(t) = \ddot{u}(t) + 3\dot{u}(t) + 2u(t) \qquad n=2 \quad m=2$$

$K=3$

$u(t) = 3\,\delta_{-1}(t)$

Risposta impulsiva

1) $$\lambda_1 = \frac{4 \pm \sqrt{16-16}}{2} = 2 \,, \quad \mu_1 = 2$$

$$h(t) = d_0\,\delta(t) + \cancel{d_{1,0}}^{\,d_1} e^{\lambda_1 t}\frac{t^0}{0!}\delta_{-1}(t) + \cancel{d_{1,1}}^{\,d_2} e^{\lambda_1 t}\frac{t^1}{1!}\delta_{-1}(t)$$

$$= \left(d_1 e^{2t} + d_2 t e^{2t}\right)\delta_{-1}(t) + d_0\,\delta(t)$$

2) Σ_δ: $\ddot{h} - 4\dot{h} + 4h = \ddot{\delta} + 3\dot{\delta} + 2\delta$

$$\dot{h}(t) = \left(2d_1 e^{2t} + d_2 e^{2t} + 2d_2 t e^{2t}\right)\delta_{-1}(t) + \left(d_1 e^{2t} + d_2 t e^{2t}\right)\delta(t) + d_0\,\dot{\delta}(t)$$

$$\ddot{h}(t) = \left(4d_1 e^{2t} + 2d_2 e^{2t} + 2d_2 e^{2t} + 4d_2 t e^{2t}\right)\delta_{-1}(t) +$$
$$2\left(2d_1 e^{2t} + d_2 e^{2t} + 2d_2 t e^{2t}\right)\delta(t) +$$
$$\left(d_1 e^{2t} + d_2 t e^{2t}\right)\dot{\delta}(t) +$$
$$d_0\,\ddot{\delta}(t)$$

3) $$\Big(\cancel{(4d_1 e^{2t} + 2d_2 e^{2t} + 2d_2 e^{2t} + 4d_2 t e^{2t})\delta_{-1}(t)} + 2(2d_1 e^{2t} + d_2 e^{2t} + 2d_2 t e^{2t})\delta(t) + (d_1 e^{2t} + d_2 t e^{2t})\dot{\delta}(t) + d_0\ddot{\delta}(t)\Big)$$
$$-4\Big(\cancel{(2d_1 e^{2t} + d_2 e^{2t} + 2d_2 t e^{2t})\delta_{-1}(t)} + (d_1 e^{2t} + d_2 t e^{2t})\delta(t) + d_0\dot{\delta}(t)\Big)$$
$$+4\Big(\cancel{(d_1 e^{2t} + d_2 t e^{2t})\delta_{-1}(t)} + d_0\delta(t)\Big) = \ddot{\delta}(t) + 3\dot{\delta}(t) + 2\delta(t)$$

$$\begin{cases} \delta(t)\Big(2(2d_1 e^{2t} + d_2 e^{2t} + 2d_2 t e^{2t}) - 4(d_1 e^{2t} + d_2 t e^{2t}) + 4d_0\Big) = 2\delta(t) \\ \dot{\delta}(t)\Big((d_1 e^{2t} + d_2 t e^{2t}) - 4d_0\Big) = 3\dot{\delta}(t) \\ \ddot{\delta}(t)\big(d_0\big) = 1\,\ddot{\delta}(t) \end{cases}$$

$t=0$

4) $$\begin{cases} 4d_1 + 2d_2 - 4d_1 + 4d_0 = 2 \\ d_1 - 4d_0 = 3 \\ d_0 = 1 \end{cases} \rightsquigarrow \begin{cases} d_0 = 1 \\ d_1 = 7 \\ d_2 = -1 \end{cases}$$

5) $$h(t) = \delta(t) + \left(7e^{2t} - te^{2t}\right)\delta_{-1}(t)$$

Risposta forzata

Ricordiamo che $u(t) = 3\delta_{-1}(t)$ $\left(u(t-\tau) = 3\delta_{-1}(t-\tau)\right)$

$$
\begin{aligned}
v_f(t) &= \int_{-\infty}^{+\infty} \left(\delta(\tau) + \left(7e^{2\tau} - \tau e^{2\tau}\right)\delta_{-1}(\tau)\right) 3\delta_{-1}(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\cancel{+\infty}\, t} \delta(\tau)\, 3\delta_{-1}(t-\tau)\, d\tau + \int_{\cancel{-\infty}\, 0}^{\cancel{+\infty}\, t} \left(7e^{2\tau} - \tau e^{2\tau}\right)\delta_{-1}(\tau)\, 3\delta_{-1}(t-\tau)\, d\tau \\
&= 3\int_{-\infty}^{t} \delta(t)\, d\tau + 21\int_0^t e^{2\tau} d\tau - 3\int_0^t \underbrace{\tau}_{f}\, \underbrace{e^{2\tau}}_{g'}\, d\tau \qquad \int f g'\, d\tau = fg - \int f' g\, d\tau \\
&= 3 + 21\left[\tfrac{1}{2} e^{2\tau}\right]_0^t - 3\left(\left[\tfrac{1}{2}\tau e^{2\tau}\right]_0^t - \int_0^t 1 \cdot e^{2\tau} d\tau\right) \\
&= 3 + 21\tfrac{1}{2}\left(e^{2t} - 1\right) - 3\left(\tfrac{1}{2} t e^{2t} - \cancel{0} - \left[\tfrac{1}{2} e^{2\tau}\right]_0^t\right) \\
&= 3 + \tfrac{21}{2}\left(e^{2t} - 1\right) - \tfrac{3}{2} t e^{2t} + \tfrac{3}{2}\left(e^{2t} - 1\right) \\
&= 3 + 12\left(e^{2t} - 1\right) - \tfrac{3}{2} t e^{2t} \\
&= -9 + 12 e^{2t} - \tfrac{3}{2} t e^{2t}
\end{aligned}
$$

$$
\boxed{v_f(t) = \left(-9 e^{0t} + 12 e^{2t} - \tfrac{3}{2} t e^{2t}\right)\delta_{-1}(t)}
$$

il -9 da solo non ha molto senso, ma in realtà è il coefficiente del nuovo modo elementare e^{0t}, che è stato introdotto dall'input $u(t)$