Homework 2. Numerical Methods

Carlos Giovanny Encinia González

August 2021

Problem 1

1) Use four-digit rounding arithmetic and the quadratic formulas to find the most accurate approximations to the roots of the following quadratic equations. Also use the form of the quadratic formula by rationalizing the numerator. Compute the absolute errors and relative errors.

$$\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0 \tag{1}$$

$$\frac{1}{3}x^2 + \frac{123}{4}x - \frac{1}{6} = 0 \tag{2}$$

Answer

First we will write the equations to find the root, so;

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{4}$$

And rationalizing the numerator we have:

$$x_1' = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\frac{-b + \sqrt{b^2 - 4ac}}{-b + \sqrt{b^2 - 4ac}}$$

$$x_1' = \frac{-2c}{b + \sqrt{b^2 - 4ac}} \tag{5}$$

$$x_2' = \frac{-b + \sqrt{b^2 - 4ac}}{2a}\frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

$$x_2' = \frac{2c}{-b + \sqrt{b^2 - 4ac}} \tag{6}$$

1) Now for the equation (1) we write a, b, c, using four-digit rounding arithmetic

$$a_1 = \frac{1}{3} = 0.3333$$

$$b_1 = \frac{-123}{4} = -30.75$$

$$c_1 = \frac{1}{6} = 0.1667$$

Now sustituing in the second term:

$$\sqrt{b_1^2 - 4a_1c_1} = \sqrt{(30.75)^2 - 4(0.3333)(0.1667)}$$

$$\sqrt{b_1^2 - 4a_1c_1} = \sqrt{945.6 - 4(0.0556)}$$

$$\sqrt{b_1^2 - 4a_1c_1} = \sqrt{945.6 - 0.2224}$$

$$\sqrt{b_1^2 - 4a_1c_1} = \sqrt{945.4}$$

$$\sqrt{b_1^2 - 4a_1c_1} = 30.75 \tag{7}$$

Now we can compute the $f_l(x_{1,1})$ and $f_l(x_{1,2})$

$$f_l(x_{1,1}) = \frac{30.75 + 30.75}{0.6666} = \frac{945.6}{0.6666}$$

$$f_l(x_{1,1}) = 92.26$$

$$f_l(x_{1,2}) = \frac{30.75 - 30.75}{0.6666} = \frac{0.0000}{0.6666}$$

$$f_l(x_{1,2}) = 0.0000$$

Now we compute using the quadratic formula by rationalizing the numerator equation (5) and (6):

$$f_l(x_{1,1}') = \frac{-2(0.1667)}{-30.75 + 30.75} = \frac{0.3334}{0.0000} = \infty$$

$$f_l(x_{1,2}') = \frac{2(0.1667)}{30.75 + 30.75} = \frac{0.3334}{61.50} = 0.0054$$

Now calculating the "real" value of $x_{1,1}$ and $x_{1,2}$

$$x_{1,1} = 92.2445796273123$$
$$x_{1,2} = 0.005420372687694908$$

Calculating the absolute error and relative error in the normal formula:

$$E_{a,x_{1,1}} = |x_{1,1} - f_l(x_{1,1})| = |92.24 - 92.26| = 0.0200$$

$$E_{r,x_{1,1}} = \frac{E_a}{|x_{1,1}|} = 0.0002$$

$$E_{a,x_{1,2}} = |x_{1,1} - f_l(x_{1,1})| = |0.0054 - 0.00| = 0.0054$$

$$E_{r,x_{1,2}} = \frac{E_a}{|x_{1,1}|} = 1.000$$

Calculating the absolute error and relative error in the rationalized formula:

$$E_{a,x'_{1,1}} = |x_{1,1} - f_l(x_{1,1})| = |92.24 - \infty| = \infty$$

$$E_{r,x'_{1,1}} = \frac{E_a}{|x_{1,1}|} = \infty$$

$$E_{a,x'_{1,2}} = |x_{1,1} - f_l(x_{1,1})| = |0.0054 - 0.0054| = 0.00$$

$$E_{r,x'_{1,2}} = \frac{E_a}{|x_{1,1}|} = 0.0000$$

2) Now for the equation (2) we write a, b, c, using four-digit rounding arithmetic

$$a_2 = \frac{1}{3} = 0.3333$$
$$b_2 = \frac{123}{4} = 30.75$$
$$c_2 = \frac{-1}{6} = -0.1667$$

Now sustituing in the second term:

$$\sqrt{b_2^2 - 4a_2c_2} = \sqrt{(30.75)^2 - 4(0.3333)(-0.1667)}$$
$$\sqrt{b_2^2 - 4a_2c_2} = \sqrt{945.6 - 4(-0.0556)}$$
$$\sqrt{b_2^2 - 4a_2c_2} = \sqrt{945.6 + 0.2224}$$
$$\sqrt{b_2^2 - 4a_2c_2} = \sqrt{945.8}$$

$$\sqrt{b_2^2 - 4a_2c_2} = 30.75 \tag{8}$$

Now we can compute the $f_l(x_{2,1})$ and $f_l(x_{2,2})$

$$f_l(x_{2,1}) = \frac{-30.75 + 30.75}{0.6666} = \frac{0.0000}{0.6666}$$
$$f_l(x_{2,1}) = 0.0000$$

$$f_l(x_{2,2}) = \frac{-30.75 - 30.75}{0.6666} = \frac{-945.6}{0.6666}$$
$$f_l(x_{2,2}) = -92.26$$

Now we compute using the quadratic formula by rationalizing the numerator equation (5) and (6):

$$f_l(x'_{2,1}) = \frac{-2(-0.1667)}{30.75 + 30.75} = \frac{0.3334}{61.50} = 0.0054$$

$$f_l(x'_{2,2}) = \frac{2(-0.1667)}{-30.75 + 30.75} = \frac{-0.3334}{0.0000} = -\infty$$

Now calculating the "real" value of $x_{2,1}$ and $x_{2,2}$

$$x_{2,1} = 0.005419735788228408$$
$$x_{2,2} = -92.25541973578822$$

Calculating the absolute error and relative error in the normal formula:

$$E_{a,x_{2,1}} = |x_{2,1} - f_l(x_{2,1})| = |0.0054 - -0.0| = 0.0054$$

$$E_{r,x_{2,1}} = \frac{E_{a,x_{2,1}}}{|x_{2,1}|} = 1.000$$

$$E_{a,x_{2,2}} = |x_{2,1} - f_l(x_{2,1})| = |-92.26 - -92.26| = 0.0$$

$$E_{r,x_{2,2}} = \frac{E_{a,x_{2,2}}}{|x_{2,1}|} = 0.000$$

Calculating the absolute error and relative error in the rationalized formula:

$$E_{a,x'_{2,1}} = |x_{2,1} - f_l(x'_{2,1})| = |0.0054 - 0.0054| = 0.0$$

$$E_{r,x'_{2,1}} = \frac{E_a}{|x_{2,1}|} = 0.0$$

$$E_{a,x'_{2,2}} = |x_{1,1} - f_l(x'_{2,1})| = | - 92.26 - -\infty| = \infty$$

$$E_{r,x'_{2,2}} = \frac{E_a}{|x'_{2,1}|} = \infty$$

Ahora resumiremos los resultados en la tabla (1) y en la tabla (2). Podemos observar como varían los resultados, para la ecuación (1) para encontrar x_1 la formula normal es buena, pero para x_2 lo formula racionalizada es mucho mejor que la normal, ya que esta ultima tiende a infinito. Por otro lado para la ecuación (2).

Table 1: $\frac{x^2}{3} - \frac{123x}{4} + \frac{1}{6}$

	x_1		x_2	
Type / Error	Normal	Rationalized	Normal	Rationalized
Absolute	0.0200	∞	0.0054	0.0000
Relative	0.0002	∞	1.0000	0.0000

Table 2: $\frac{x^2}{3} + \frac{123x}{4} - \frac{1}{6}$

	x_1		x_2	
Type / Error	Normal	Rationalized	Normal	Rationalized
Absolute	0.0054	0.0000	0.0000	∞
Relative	1.0000	0.0000	0.0000	∞

2

Let

$$\frac{xcos(x)sin(x)}{x - sin(x)} \tag{9}$$

(a) Find $\lim_{x\to 0} f(x)$.

(b) Use four-digit rounding arithmetic to evaluate $f(0.1)$.

(c) Replace each trigonometric function with its third Maclaurin polynomial and repeat part (b).

(d) The actual value is $f(0.1) = -1.99899998$. Find the relative error for the values obtained in parts (b) and (c).

Answer

a) We write some values of x around 0 in the table (3) and we can observe that the limit of the equation (9) when x to 0 does not exist, this is because a small value in the left is negative and a small value in the right is positive in other hand.

$$\lim_{x\to 0_-} = -\infty$$

and

$$\lim_{x\to 0_+} = \infty$$

Table 3: Diverge

x	f(x)
-1.000000	-2.867921
-0.100000	-59.630607
-0.050000	-119.815074
-0.030000	-199.889021
-0.020000	-299.926012
-0.001000	-5999.996019
-0.000500	-11999.997585
-0.000200	-29999.999837
-0.000010	-599997.744016
-0.000001	-6000465.640433
0.000001	6000465.640433
0.000010	599997.744016
0.000200	29999.999837
0.000500	11999.997585
0.001000	5999.996019
0.020000	299.926012
0.030000	199.889021
0.050000	119.815074
0.100000	59.630607
1.000000	2.867921

b) We have that

$$f(0.1000) \approx \frac{0.1000cos(0.1000)sin(0.1000)}{0.1000 - sin(0.1000)}$$

$$f(0.1000) \approx \frac{0.1000 * 0.9950 * 0.0998}{0.1000 - 0.0998}$$

$$f(0.1000) \approx \frac{0.1000 * 0.0993}{0.0002}$$

$$f(0.1000) \approx \frac{0.0099}{0.0002}$$

$f(0.1000) \approx$ [illegible]0

c) Firt we will find the Maclaurin series for sin(x) and cos(x) We know that the Maclaurin in third polynomial is;

$$G(x_0) = f(x_0) + f'(x_0)x + \frac{f''(x_0)x^2}{2} + \frac{f'''(x_0)x^3}{6}$$

Let

$$h(x) = sin(x)$$

$$h'(x) = cos(x)$$

$$h''(x) = -sin(x)$$

$$h'''(x) = -cos(x)$$

so

$$sin(x) \approx sin(0) + xcos(0) - \frac{x^2 sin(0)}{2} - \frac{x^3 cos(0)}{6}$$

$$sin(x) \approx x - \frac{x^3}{6} \tag{10}$$

And now for

$$g(x) = cos(x)$$
$$g'(x) = -sin(x)$$
$$g''(x) = -cos(x)$$
$$g'''(x) = sin(x)$$
$$cos(x) \approx cos(0) - xsin(0) - \frac{x^2 cos(0)}{2} + \frac{x^3 sin(0)}{6}$$

$$cos(x) \approx 1 - \frac{x^2}{2} \tag{11}$$

Now we going to replace equations (10) and (11) in equation (9), so.

$$f(x) \approx \frac{x(1-\frac{x^2}{2})(x-\frac{x^3}{6})}{\frac{x^3}{6}} \tag{12}$$

Now we evaluate in x=0.1 and repeat the part b)

$$f(0.1000) \approx \frac{0.1000(1.000 - \frac{0.10000^2}{2})(0.1000 - \frac{0.1000^3}{6})}{\frac{0.1000^3}{6}}$$

$$f(0.1000) \approx \frac{0.1000(1.000 - \frac{0.0100}{2})(0.1000 - \frac{0.0010}{6})}{\frac{0.0010}{6}}$$

$$f(0.1000) \approx \frac{0.1000(1.000 - .0050)(0.1000 - 0.0002)}{0.0002}$$

$$f(0.1000) \approx \frac{0.1000(0.9950)(0.0998)}{0.0002}$$

$$f(0.1000) \approx \frac{0.1000(0.0993)}{0.0002}$$

$$f(0.1000) \approx \frac{0.0099}{0.0002}$$

$$f(0.1000) \approx 49.50$$

Now compute the relative error, first we can observe that in b) and c) we have the exact value, so , we can compute only one relative error, also the "real" value is in the table (3):

$$E_r = \frac{|59.63 - 49.50|}{|59.63|}$$

$$E_r = \frac{|59.63 - 49.50|}{|59.63|}$$

$$E_r = \frac{|10.13|}{|59.63|}$$

$$E_r = 0.1699$$

3

Find the number of terms of the exponential series such that their sum gives the value of e^x correct to six decimal places at $x = 1$.

Answer

De tareas anteriores sabemos que la serie de Maclaurin de e^x es:

$$e^x \approx \sum_{n=0}^{m} \frac{x^n}{n!}$$

Y el error de truncamiento esta dado por

$$|R_n| \leq \frac{Mx^{n+1}}{(n+1)!}$$

$$M = |f^{n+1}(\xi(x))|$$

Primero podemos proponer $n = 3$ y sabiendo que este termino siempre es menor que e entonces tenemos que:

$$M = |f^4(\xi(x))| \leq e \leq 3$$

Entonces sustituyendo en la definición de error de truncamiento tenemos que:

$$|R_3| \leq \frac{3x^4}{(4)!}$$

pero $x = 1$ entonces

$$|R_3| \leq \frac{3}{24} = \frac{1}{8}$$

Podemos ver que el error a penas y llega a una cifra correcta, pero si observamos el error depende de que tanto crece el denominador, entonces en particular para esta función podemos hacer.

$$|R_n| \leq \frac{3}{(n+1)!}$$

Si queremos que la funcion tenga seis cifras significativas correctas entonces buscamos que el error se de la forma $\frac{1}{1000000}$

$$|R_n| \leq \frac{3}{(n+1)!} = \frac{1}{1000000}$$

tenemos que resolver

$$\frac{3}{(n+1)!} = \frac{1}{10000}$$

$$(n+1)! = 3000000$$

Ahora podemos proponer varios n y ver que resultado dan, por ejemplo empezamos con:

$$8! = 40320$$

$$9! = 362880$$

$$10! = 3628800$$

Observamos que 10! es mayor que 3000000, entonces podemos tomar $n + 1 = 10$, por lo que

$$n = 9$$

Entonces el numero de terminos que debemos usar para tener exactitud de 3 cifras es 9: Ahora calculando el resultado tenemos

$$e^1 \approx 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{9!}$$

$$e^1 \approx 2.71828152557$$

El valor "real" es:

$$e = 2.71828182846$$

El error absoluto es

$$E_a = |2.71828182846 - 2.71828152557|$$

$$E_a = 0.00000030$$