HOMEWORK 1. NUMERICAL METHOD

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Problem 1

(1) Find intervals containing solutions to the following equations.

$$x - 2^{-x} = 0 \quad (1)$$

$$2xcos(2x) - (x+1)^2 = 0 \quad (2)$$

Solution:

In both problems the solution arise when is used the Bolzano's theorem, and observing that the equation is the intersecttion between

$$f(x)$$

and

$$y = 0$$

a) In the equation (1)

$$f(x) = x - 2^{-x}$$

We can to begin proving the number 0 being that this number will facility us the calculation so, when

$$x = 0$$

$$f(0) = -1$$

Now we will evaluate the function in 1, so,

$$f(1) = 0.5$$

therefore by the Bolzano's theorem the solution in (1) will be in the interval

$$\boxed{(0, 1)}$$

Now to ensure that is a unique solution, we can utilice the extreme value theorem, the reason for the use this theorem is because if extreme values exist the posibility of more roots in diferent intervals as well exist.

$$f^{'}(x) = 1 + ln(2)2^{-x} = 0$$

$$2^{-x} = -\frac{1}{ln(2)}$$

Here we can observe that the solution does not exist because a value negative evaluated in the function logaritmic is not defined in the real field. So the interval (0, 1) is the unique root.

b) In the equation (2)

$$f(x) = 2xcos(2x) - (x+1)^2$$

we will evaluate in several points.

$$f(0) = -1$$

$$f(\frac{\pi}{2}) = -\pi - (\frac{\pi}{2} + 1)^2 = -c$$

$$f(-\frac{\pi}{2}) = \pi - (-\frac{\pi}{2} + 1)^2$$

Founding the sign of the function we make:

$$\frac{\pi}{2} > 1$$

$$\frac{\pi}{2} - 1 > 0$$

$$-\frac{\pi}{2} + 1 < 0$$

$$(-\frac{\pi}{2} + 1)^2 < 0$$

$$\pi > (-\frac{\pi}{2} + 1)^2$$

so,

$$f(-\frac{\pi}{2}) = b$$

where b is a positive real number. Remembered that the principal thing in the Bolzano's theorem is the sign change, we inl have to observe if the result is negative or positive. Therefore c and b are real number and the exact value does not interest us.

Hence a interval with the solution is:

$$\boxed{(-\frac{\pi}{2}, 0)}$$

With this f(x), we could think that exist multiple roots because exist a term sinoidal and a term exponential.

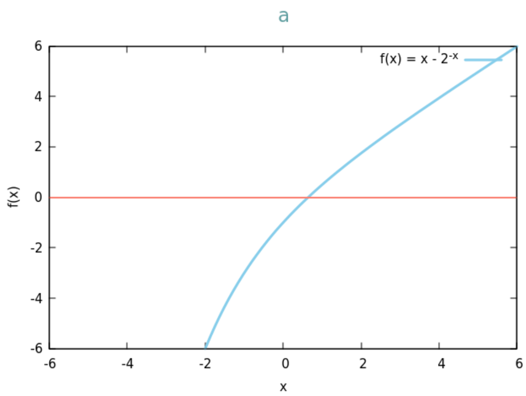
we will try to find and other interval that satisfies the Bolzano's theorem.

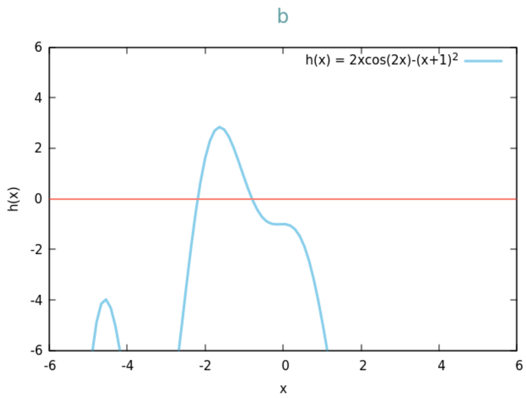
$$f(-\pi) = -2\pi - (-\pi + 1)^2$$
$$f(-\pi) = -d$$

the result is a negative value, so other intervale is

$$\boxed{(-\pi, -\frac{\pi}{2})}$$

Now we show chats with the functions





Now easily can see the intervals where the roots exist.

Problem 2

Find

$$\max_{a \leq x \leq b} f(x)$$

$$\min_{a \leq x \leq b} f(x)$$

$$\max_{a \leq x \leq b} |f(x)|$$

for the following functions and intervals

$$a : f(x) = \frac{2 - e^x + 2x}{3}, [0, 1]$$

$$b : f(x) = \frac{4x - 3}{x^2 - 2x}, [0.5, 1.5]$$

Solution:

In the function (a), we first will apply the extreme values theorem so

$$f^{'}(x) = \frac{-e^x + 2}{3} = 0$$

$$e^x = 2$$

$$x = ln(2)$$

so,

$$f(ln(2)) = \frac{2ln(2)}{3}$$

$$f(\frac{4ln(2)}{3}) = \frac{2 - e^{\frac{4ln(2)}{3}} + \frac{8ln(2)}{3}}{3} = \frac{ln(2)}{9} + \frac{2 - 16^{\frac{1}{3}}}{3}$$

we observe that the second term is negative, that the first term is one third of f(ln (2)), therefore.

$$f(ln(2)) > f(\frac{4ln(2)}{3})$$

$$\boxed{\max_{a \leq x \leq b} f(x) = \frac{2ln(2)}{3}, x = ln(2)}$$

Now we found that exist only one critical point and this point is a maximum, so the minimum should be in

$$x = 0$$

$$\boxed{\min_{a \leq x \leq b} f(x) = \frac{1}{3}, x = 0}$$

Finally the max value when aplies the absolute in the function should be the same that the max of the normal function, bacause all the values in the interval [0, 1] are positives. If in the interval would exist a critical point minimum, and the absoluute value would be grater than the absolute value of the maximum in f(x), in the

$$|f(x)|$$

, the maximum would be the minimum in f(x).

$$\boxed{\max_{a\leq x\leq b}|f(x)| = \frac{2ln(2)}{3}, x = ln(2)}$$

In the function (b) first we will utilice the first derivate criteria.

$$f^{'}(x) = \frac{4(x^2-2x)-(2x-2)(4x-3)}{(x^2-2x)^2} = 0$$

$$f^{'}(x) = \frac{-4x^2+6x-6}{(x^2-2x)^2} = 0$$

solving the equation

$$-4x^2+6x-6 = 0$$

$$x = -6 \pm \frac{\sqrt{36+16(-6)}}{-8} = \frac{-6 \pm 2\sqrt{-15}}{-8}$$

in the last termm we can observe that does not exist a solution in the Real field, so does not exist critical points in the function.

$$f(\frac{1}{2}) = \frac{\frac{4}{2}-3}{(\frac{1}{2})^2-1} = \frac{4}{3}$$

$$f(\frac{3}{2}) = \frac{\frac{4(3)}{2}-3}{(\frac{3}{2})^2-\frac{2(3)}{2}}) = -4$$

so

$$\boxed{\max_{a\leq x\leq b} f(x) = \frac{4}{3}, x = \frac{1}{2}}$$

$$\boxed{\min_{a\leq x\leq b} f(x) = -4, x = \frac{3}{2}}$$

For the

$$|f(x)|$$

first we will find the asintots of f(x)

$$x^2-2x = 0$$

asintots exist in

$$x_1 = 0, x_2 = 2$$

so in the interval [0.5, 1.5] the function is continous, then we calculate the root of the function

$$\frac{4x-3}{x^2-2x} = 0$$

$$x = \frac{3}{4}$$

This root will change the sign of the values when going to apply the absolute function, but also we have to considerate the denominator sign change, in the previously results we observe that the sign changes in the denominator when

$$x \geq 2$$

so, for all the values less than 2 the denominator is negative, considerating that we have

$$f(x) = \begin{cases} \frac{3-4x}{x^2-2x} & \text{if } x \geq \frac{3}{4} \\ \frac{4x-3}{x^2-2x} & \text{if } x < \frac{3}{4} \end{cases}$$

$$f(\frac{1}{2}) = \frac{2-3}{\frac{1}{4}-1} = \frac{4}{3}$$

$$f(\frac{3}{2}) = \frac{3-(\frac{4(3)}{2})}{\frac{9}{4}-3} = 4$$

$$\boxed{\max_{a\leq x\leq b}|f(x)| = 4, x = \frac{3}{2}}$$

Problem 3

Suppose $f \in C[a,b]$ and $x_1 and x_2$ are in $[a,b]$. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{f(x_1)+f(x_2)}{2}$$

Solution:

To verifyy the existence of ξ, we will use the intermediate value theorem. We know that the function is continuous in an interval $[a,b]$, now we comprobe the existence of $f(\xi)$

$$min(f(x_1), f(x_2)) < f(\xi) < max(f(x_1), f(x_2))$$

first we going to supose that $f(x_1) < f(x_2)$ so

$$f(x_1) < f(\xi) < f(x_2)$$

We can observe that the $f(\xi)$ is the intermediate value between $f(x_1)$ and $f(x_2)$

$$f(x_1) + \frac{f(x_2) - f(x_1)}{2} < f(x_2)$$

we can observe that the second term correspond to the medium distance between $f(x_1)$ and $f(x_2)$, so if we sum this to $f(x_1)$ the result is still less than $f(x_2)$.

For $f(x_1) < f(\xi)$ we can propose the same

$$f(x_1) < f(x_1) + \frac{f(x_2) - f(x_1)}{2}$$

This is correct because the new term added to $f(x_1)$ is positive so.

$$f(x_1) < f(x_1) + \frac{f(x_2) - f(x_1)}{2} < f(x_2)$$

$$f(x_1) < \frac{f(x_1) + f(x_2)}{2} < f(x_2)$$

A similar development ocurs if $f(x_2) < f(x_1)$, obtained the same result. so.

$$f(\xi) = \frac{f(x_1) + f(x_2)}{2}$$

$\therefore$

$$\boxed{\exists \xi \in (min(x_1, x_2), max(x_1, x_2))}$$

Problem 4

Assume you have n values x_i .

(a) Evaluate the sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

(b)Evaluate the sample variance.

(i) You can evaluate the sample variance using a two-pass algo- rithm, ie

$$Var(x) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

In this case, you need to evaluate the mean first, so you have to loop through the x i once to get the mean and a second time time to get the sample variance. (ii) You can also evaluate the sample variance using the one-pass algorithm, ie

$$Var(x) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

In this case, you can compute the sums of the x_i and the x_i^2 values at the same time and then perform only one subtraction at the end.

Write two functions, one for each algorithm, and test them on the two cases below:

$$x_i \in \{0, 0.01, 0.02, ..., 0.09\}$$

$$x_i \in \{123456789.0, 123456789.01, ..., 123456789.09\}$$

Solution:

En un inicio se habian utilizado tipos de dato float, pero para el segundo arreglo habia un error de representacion, asi que se opto por utilizar tipo de dato double (resuultado en figura 1), aunque para el segundo arreglo el one-pass algorithm no muestra un resultado del todo correcto, esto es porque al elevar al cuadrado cada uno de los elementos, genera un numero demasiado grande como para que pueda ser representado con un double.

Figure 1: Usando dato tipo double

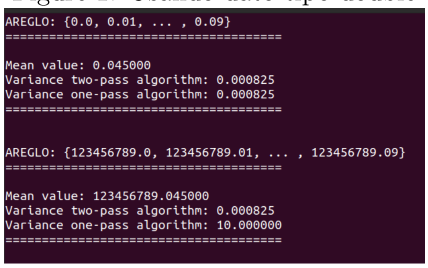
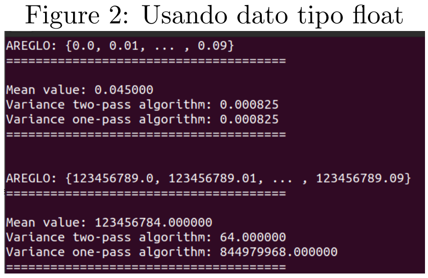


Figure 2: Usando dato tipo float



Code 1: Problem 4

```c
//Giovanny Encinia 15/08/2021
//Algoritmo para el problema 4, media, varianza

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#define SIZEOF(total) (sizeof(total)/sizeof(total[0]))

double mean(double vector[], int size);
double variance_alpha(double vector[], int size, double mean_v);
double variance_omega(double vector[], int size, double mean_v);
void print_result(double mean_, double var_a, double var_0, char *array_);

double mean(double vector[], int size)
{

    /*Calculate the mean of one array
    Parameters
    ==========
    float vector: the array
    int size: the size of the array

    Return
    ======
    double sum: the mean value of the array*/

    int i = 0;
    double sum = 0.0;

    while(i < size)
    {
        sum = sum + vector[i];
        i++;
    }

    sum = sum / size;// really is the mean value of the array
    return sum;
}

double variance_alpha(double vector[], int size, double mean_v)
{
    /*To evaluate the sample variance using a two-pass algorithm.
      In this case, you need to evaluate the mean first, so you have
      to loop through the x i once to get the mean and a second time
      time to get the sample variance.

      Parameters
      ==========
```

```
        float vector: the principal array
        int size: size of the array
        Return
        ======
        float sum: the variance of the array*/

    int i = 0;
    double sum = 0.0;

    while(i < size)
    {
        sum = sum + pow(vector[i] - mean_v, 2);
        i++;
    }

    sum = sum/size;

    return sum;
}

double variance_omega(double vector[], int size, double mean_v)
{
    /*To evaluate the sample variance using the one-pass
        algorithm
        Parameters
        ==========
        float vector: the principal array
        int size: size of the array
        Return
        ======
        float sum: the variance of the array**/

    int i = 0;
    double sum = 0.0;

    while(i < size)
    {
        sum += pow(vector[i], 2);
        i++;
    }

    sum = sum / size;
    sum = sum - pow(mean_v, 2);

    return sum;


}

void print_result(double mean_, double var_a, double var_o, char *array_)
```

```c
{

    printf("AREGLO: %s\n", array_);
    printf("========================================\n\n");
    printf("Mean value: %f\n", mean_);
    printf("Variance two-pass algorithm: %f\n", var_a);
    printf("Variance one-pass algorithm: %f\n", var_o);
    printf("========================================\n\n\n");
}

int main()
{
    double mean_value_1, mean_value_2, var_1_alpha, var_2_alpha;
    double var_1_omega, var_2_omega;
    double x_1[] = {
                    0.0, 0.01, 0.02,
                    0.03, 0.04, 0.05,
                    0.06, 0.07, 0.08, 0.09
                    };
    double x_2[] = {
                123456789.0, 123456789.01, 123456789.02,
                123456789.03, 123456789.04, 123456789.05,
                123456789.06, 123456789.07, 123456789.08,
                123456789.09
                };

    mean_value_1 = mean(x_1, SIZEOF(x_1));
    mean_value_2 = mean(x_2, SIZEOF(x_2));

    var_1_alpha = variance_alpha(x_1, SIZEOF(x_1), mean_value_1);
    var_1_omega = variance_omega(x_1, SIZEOF(x_1), mean_value_1);

    var_2_alpha = variance_alpha(x_2, SIZEOF(x_2), mean_value_2);
    var_2_omega = variance_omega(x_2, SIZEOF(x_2), mean_value_2);

    print_result(mean_value_1, var_1_alpha, var_1_omega, \
                 "{0.0, 0.01, ... , 0.09}");
    print_result(mean_value_2, var_2_alpha, var_2_omega, \
                 "{123456789.0, 123456789.01, ... , 123456789.09}");

    return 0;
}
```

Es de suma importancia saber o determinar con anticipacion que tipo de dato es el que se utilizara a la hora de implementar un algoritmo, ademas de esto tratar de hacer las operaciones aritmeticas de una manera tal que no genere errores al sumar o restar valores. en nuestro caso se pudo habeer solucionado el problema de los grandes numeros elevados al cuadrado usando un tipo de dato que los pudiese representar con mayor presicion.