

Algebraic Topology

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Lecture 1: intro

22/4/23

- First part: fundamental groups. We follow Munkres:
 - Chap 9 ‘Fundamental group’
 - Chap 11 ‘The Seifert-Van Kampen thm’ and
 - Chap 12/13 ‘Classification covering spaces’
- Second part: Homology groups, via cudi, chapter of a book.
- 10 problems (solve them during the semester) No feedback during the semester, but asking questions is allowed. Working together is allowed.
- Exam (completely open book) First 4 questions: 1h30 prep. Last one 30 min, no prep
 1. Theoretical question (open book, explain the proof, ...) /4
 2. New problem (comparable to one of the 10 problems) /4
 3. Explain your solution n -th problem solved at home /4
 4. Explain your solution m -th problem solved at home /4
 5. 4 small questions /4

After the exam, hand in your solutions of the other problems. After a quick look, the points of 3. and 4. can be ± 1 (in extreme cases ± 2)

- No exercise classes for this course!

Chapter 9

Fundamental group

9.51 Homotopy of paths

Definition 1 (Homotopy). Let $f, g : X \rightarrow Y$ be maps (so continuous). Then a homotopy between f and g is a continuous map $H : X \times I \rightarrow Y$ such that

- $H(x, 0) = f(x)$, $H(x, 1) = g(x)$
- For all $t \in I$, define $f_t : X \rightarrow Y : x \mapsto H(x, t)$

We say that f is homotopic with g , we write $f \simeq g$. If g is a constant map, we say that f is null homotopic.

Definition 2 (Path homotopy). Let $f, g : I \rightarrow X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H : I \times I \rightarrow X$ is a path homotopy between f and g , if and only if

- $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ (homotopy between maps)
- $H(0, t) = x_0$ and $H(1, t) = x_1$ (start and end points fixed)

Notation: $f \simeq_p g$.

Lemma 1. \simeq and \simeq_p are equivalence relations.

Proof.

- Reflective: $F(x, t) = f(x)$
- Symmetric: $G(x, t) = H(x, 1 - t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x, t) = \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases} . \quad \square$$

Example (Trivial, but important). Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g : X \rightarrow C$ are homotopic.
- Any two paths $f, g : I \rightarrow C$ with $f(0) = g(0)$ and $g(1) = f(1)$ are path homotopic.

Choose $H : X \times I \rightarrow C : (x, t) \mapsto H(x, t) = (1 - t)f(x) + tg(x)$.

Product of paths

Let $f : I \rightarrow X, g : I \rightarrow X$ be paths, $f(1) = g(0)$. Define

$$f * g : I \rightarrow X : s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g : I \rightarrow X \mid g \simeq_p f\}$

Theorem 1.

1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
2. Let e_x denote the constant path $e_x : I \rightarrow X : s \mapsto x, x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Let $\bar{f} : I \rightarrow X : s \mapsto f(1 - s)$. Then $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$

Proof. First, two observations

- Suppose $f \simeq_p g$ via homotopy $H, f, g : I \rightarrow X$. Let $k : X \rightarrow Y$. Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.
- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

2. Take $e_0 : I \rightarrow I : s \mapsto 0$. Take $i : I \rightarrow I : s \mapsto s$. Then $e_0 * i$ is a path from 0 to 1 in I . The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 * i \simeq_p i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

3. Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \bar{f} &\simeq_p e_{x_0} \\ [f] * [\bar{f}] &= [e_{x_0}]. \end{aligned}$$

1. Remark: Only defined if $f(1) = g(0), g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

Assume the product is defined. Suppose $[a, b], [c, d]$ are intervals in \mathbb{R} . Then there is a unique positive (positive slope), linear map from $[a, b] \rightarrow [c, d]$. For any $a, b \in [0, 1]$ with $0 < a < b < 1$, we define a path

$$\begin{aligned} k_{a,b} : [0, 1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 1] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

Then $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$

Let γ be that path $\gamma : I \rightarrow I$ with the following graphs:

Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

which is what we wanted to show. □

9.52 Fundamental group

Definition 3. Let X be a space and $x_0 \in X$, then the fundamental group of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f : I \rightarrow X, f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\bar{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

Theorem 2 (52.1). Let X be a space, $x_0, x_1 \in X$ and $\alpha : I \rightarrow X$ a path from x_0 to x_1 . Then

$$\begin{aligned}\hat{\alpha} : \pi(X, x_0) &\longrightarrow \pi(X, x_1) \\ [f] &\longmapsto [\bar{\alpha}] * [f] * [\alpha].\end{aligned}$$

is an isomorphism of groups. Note however that this isomorphism depends on α .

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * [f] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}[f] * \hat{\alpha}[g].\end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic. \square

Remark. If $f : (x, x_0) \rightarrow (Y, y_0)$ is a map of pointed topology spaces ($f : X \rightarrow Y$ continuous and $f(x_0) = y_0$). Then

$$f_* : \pi(X, x_0) \rightarrow \pi(Y, y_0) : [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two ‘rules’ discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \quad (1_X)_* = 1_{\pi(X, x_0)}.$$