Week 13 Problems

Alex Skeldon

1. **(Bass 5.7)** Because f is differentiable, f is continuous. Since the Borel σ-algebra contains all the open sets, then by Bass Proposition 5.6, f is measurable. Now, let

$$f_n = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}.$$

We can then express the derivative f' as the limit of f_n:

$$f' = \lim_{n\to\infty} f_n.$$

in which case the limits inferior and superior of f_n are equal to each other (and the limit itself). Then, by Bass Proposition 5.7, each f_n is measurable, and by Proposition 5.8, f' is measurable.

2. **(Bass 6.3)** We first take linearity of simple (non-negative) function's integrals as true from the first step of the proof of Theorem 7.4, and prove that simple functions obey the proposed equality. For any simple function s and disjoint measurable sets A_1, A_2, we have that

$$\int_{A_1} s + \int_{A_2} s = \sum_{n=1}^{m} a_n \mu(E_n \cap A_1) + \sum_{n=1}^{m} a_n \mu(E_n \cap A_2),$$

where the coefficients a_n and sets E_n are those defined in the construction of a simple function. We then have by basic set logic and additivity of measures that, since A_1 and A_2 are disjoint, that the latter expression is equal to

$$\sum_{n=1}^{m} a_n \mu(E_n \cap (A_1 \cup A_2)) = \int_{A_1 \cup A_2} s.$$

With this equality, we prove the first of the two equalities in question.
If s is a simple non-negative function and $s \leq f$, we have that

$$\int_{A_1 \cup A_2} s = \int_{A_1} s + \int_{A_2} s \leq \int_{A_1} f + \int_{A_2} f.$$

Since $\int_{A_1 \cup A_2} f = \sup \left\{ \int_{A_1 \cup A_2} s | s \leq f \right\}$, then

$$\int_{A_1 \cup A_2} f \leq \int_{A_1} f + \int_{A_2} f.$$

We make a similar argument to prove inequality in the other direction. Let $s_1 \leq f$ on A_1, and zero elsewhere. Define s_2 the same on A_2. Then we have

$$\int_{A_1 \cup A_2} f \geq \int_{A_1 \cup A_2} s_1 + s_2 = \int_{A_1} s_1 + \int_{A_2} s_2,$$

where we are invoking our established linearity. With the integrals of f defined over each of A_1 and A_2 each defined as the supremum of the integrals of the sets of functions s_1 and s_2, respectively, we get

$$\int_{A_1 \cup A_2} f \geq \int_{A_1} f + \int_{A_2} f,$$

satisfying equality. For the case where f is integrable, consider the component functions f^+ and f^-, and apply the non-negative case to each of those functions.

3. **(Additional Problem #1)** (We neglect the infinite-value cases in this proof) In the $\leq$ case, we first note that $f-g$ is measurable by Proposition 5.7 and consider the equivalent difference sets. Then, the set
$$\{x \in X | f(x) - g(x) \leq 0\}$$
is measurable by the equivalent definitions of a measurable function in Proposition 5.5. Since the collection of measurable sets is a σ-algebra, and the set
$$\{x \in X | f(x) - g(x) \geq 0\}$$
is also measurable, then the intersection of the $\leq$ and $\geq$ sets, which is equal to
$$\{x \in X | f(x) - g(X) = 0\},$$
is also measurable by the closure of σ-algebras under finite intersections.