Richard Bass: Real Analysis for Graduate Students

Notes for Chapters 1-10

Alex Skeldon

November 21, 2024

Contents

Chapter 1

Preliminaries

1.1 Notation and terminology

We use A^c for the set of points not in A. Specifically, define

$$A^c = \{x \in X : x \notin A\}.$$

We write

$$A - B = A \cap B^c$$

(it is common to also see $A \backslash B$) and

$$A \triangle B = (A - B) \cup (B - A)$$

The set $A \triangle B$ is called the *symmetric difference* of A and B and is the set of points that are in one of the sets but not the other. If I is some non-empty index set, a collection of subsets $\{A_\alpha\}_{\alpha \in I}$ is disjoint if $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$.

We write $A_i \uparrow$ if $A_1 \subset A_2 \subset \cdots$ and write $A_i \uparrow A$ if in addition $A = \cup_{i=1}^\infty A_i$. Similarly, $A_i \downarrow$ means $A_1 \supset A_2 \supset \cdots$ and $A_i \downarrow A$ means that in addition $A = \cap_{i=1}^\infty A_i$.

We use $\log x$ to denote the natural logarithm of x, that is, the logarithm of x to the base e.

We use $\mathbb{Q}$ to refer to the set of rational nembers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. We use

$$x \vee y = \max(x, y) \qquad \text{and} x \wedge y = \min(x, y)$$

We can write a real number x in terms of its positive and negative parts:

$$x = x^+ \vee 0 \qquad \text{and} \qquad x^- = (-x) \vee 0.$$

If z is a complex number, then $\overline{z}$ is the complex conjugate of z. The composition of two functions is defined by $f \circ g(x) = f(g(x))$.

If f is a function whose domain is the reals or a subset of the reals, then $f(x+) = \lim_{y \to x+} f(y)$ and $f(x-) = \lim_{y \to x-} f(y)$ are the right and left hand limits of f at x, resp.

We say a function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* if $x < y$ implies $f(x) \leq f(y)$ and

f is *strictly increasing* if $x < y$ implies $f(x) < f(y)$. Decreasing and strictly decreasing are defined similarly. A function is monotone if f is either increasing or decreasing.
Given a sequence $\{a_n\}$ of real numbers,

$$\limsup_{x\to\infty} a_n = \inf_n \sup_{m\geq n} a_m,$$

$$\liminf_{x\to\infty} a_n = \sup_n \inf_{m\geq n} a_m.$$

For example, if

$$a_n = \begin{cases} 1, & \text{n even;} \\ -1/n, & \text{n odd,} \end{cases} \tag{1.1}$$

then $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} = 0$. The sequence $\{a_n\}$ has a limit if and only if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ and both are finite. We use analogous definitions when we take a limit along the real numbers. For example,

$$\limsup_{y\to x} f(y) = \inf_{\delta>0} \sup_{|y-x|<\delta} f(y).$$

1.2 Some undergraduate math

We recall some definitions and facts from undergraduate topology, algebra, and analysis. Many proofs are omitted.

A set X is a *metric space* if there exists a function $d : X \times X \to \mathbb{R}$, called the metric, such that
(1) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(2) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Condition (3) is called the triangle inequality.

Given a metric space X, let

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

be the *open ball* of radius r centered at x. If $A \subset X$, the *interior* of A, denoted A^o, is the set of x such that there exists $r_x > 0$ with $B(x, r_x) \subset A$. The closure of A, denoted $\overline{A}$, is the set of $x \in X$ such that every open ball centered at x contains at least one point of A. A set A is open if $A = A^o$, closed if $A = \overline{A}$. If $f : X \to \mathbb{R}$, the support of f is the closure of the set $\{x : f(x) \neq 0\}$. f is continuous at a point x if given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$. f is continuous if it is continuous at every point of its domain.

Recall also the topological definition of continuity, convergence of sequences, Cauchy sequences, and completeness of a set X.

An open cover of a subset K of X is a non-empty collection $\{G_\alpha\}_{\alpha\in I}$ of open sets such that $K \subset \cup_{\alpha\in I} G_\alpha$. The index set I can be finite or infinite. A set K is compact if every open cover contains a finite subcover, i.e. there exists $G_1, \cdots, G_n \in \{G_\alpha\}_{\alpha in I}$ such that $K \subset \cup_{i=1}^n G_i$.

Proposition 1.1 *If K is compact, $F \subset K$, and F is closed, then F is compact*

Proposition 1.2 *If K is compact and f is continuous on K, then there exist x_1 and x_2 such that $f(x_1) = \inf_{x\in K} f(x)$ and $f(x_2) = \sup_{x\in K} f(x)$. In other words, f takes on its maximum and minimum values.*

Remark 1.3 If $x \neq y$, let $r = d(x, y)$ and note that $B(x, r/2)$ and $B(y, r/2)$ are disjoint open sets containing x and y, resp. Therefore metric spaces are also what are called hausdorff spaces.

The eight vector space properties (AKA linear space), normed linear space when map $x \to ||x||$ satisfying three props, metric induced by the norm, equivalence relationships.

Given an equivalence relationship, X can be written as the union of disjoint equivalence classes. x and y are in the same equivalence class if and only if $x \sim y$.

A set X has a partial order "$\leq$" if
(1) $x \leq x$ for all $x \in X$;
(2) if $x \leq y$ and $y \leq x$, then $x = y$;
(3) if $x \leq y$ and $y \leq z$, then $x \leq z$.

Note that given $x, y \in X$, it is not necessarily true that $x \leq y$ or $y \leq x$. For an example, let Y be a set, let X be the collection of all subsets of Y, and say $A \leq B$ if $A, B \in X$ and $A \subset B$.

We need the following three facts about the real line.

Proposition 1.4 *Suppose $K \in \mathbb{R}$, K is closed, and K is contained in a finite interval. Then K is compact.*

Proposition 1.5 *Suppose $G \subset \mathbb{R}$ is open. Then G can be written as the countable union of disjoint open intervals.*

Proposition 1.6 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is an increasing function. Then both $\lim\limits_{y\to x+} f(y)$ and $\lim\limits_{y\to x-} f(y)$ exist for every x. Moreover, the set of x where f is not continuous is countable.*

1.3 Proofs of Propositions

Skipped.

Chapter 2

Families of Sets

2.1 Algebras and σ-algebras

When we turn to constructing measures in Chapter 4, we will see that we cannot in general define the measure of an arbitrary set. The class of sets that we will want to use are σ-algebras.

Let X be a set.

Definition 2.1 An algebra is a collection $\mathcal{A}$ of subsets of X such that
(1) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$;
(2) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
(3) if $A_1, \cdots, A_n \in \mathcal{A}$, then $\cup_{i=1}^n A_i = \cap_{i=1}^n A_i$ are in $\mathcal{A}$.

$\mathcal{A}$ is a *σ-algebra* if in addition
(4) whenever $A_1, A_2, \cdots$ are in $\mathcal{A}$, then $\cup_{i=1}^\infty A_i$ and $\cap_{i=1}^\infty A_i$ are in $\mathcal{A}$.

In (4) we allow countable unions and intersections only. The intersection requirement is redundant since it is the complement of an infinite union of complementary sets.

The pair $(X, \mathcal{A})$ is called a measurable space. A set A is measurable or $\mathcal{A}$ measurable if $A \in \mathcal{A}$.

Example 2.2 Let $X = \mathbb{R}$, the set of real numbers, and let $\mathcal{A}$ be the collection of all subsets of $\mathbb{R}$. Then $\mathcal{A}$ is a σ-algebra.

Lemma 2.7 *If $\mathcal{A}_\alpha$ is a σ-algebra for each α in some non-empty index set I, then $\cap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra.*

If we have a collection $\mathcal{C}$ of subsets of X, define

$$\sigma(\mathcal{C}) = \cap\{\mathcal{A}_\alpha : \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra}, \mathcal{C} \in \mathcal{A}_\alpha\},$$

We call $\sigma(\mathcal{C})$ the σ-algebra generated by the collection $\mathcal{C}$. Since $\sigma(C)$ is a sigma algebra, then $\sigma(\sigma(C)) = \sigma(C)$.

If X has some additional structure (metric space), then we can talk about open sets. If $\mathcal{G}$ is the collection of open subsets of X, then we call $\sigma\mathcal{G}$ the Borel σ-algebra on X, and denote this $\mathcal{B}$. Elements of that are called Borel sets, said

to be Borel measurable. We will see later that if $X = \mathbb{R}$, then $\mathcal{B} \neq$ the collection of all subsets of X.
Proposition 2.8 *If $X = \mathbb{R}$, then the Borel σ-algebra is generated by each of the following collections of sets:*
(1) $\mathcal{C}_1 = \{(a, b) : a, b \in \mathbb{R}\}$;
(2) $\mathcal{C}_2 = \{[a, b] : a, b \in \mathbb{R}\}$;
(3) $\mathcal{C}_3 = \{(a, b] : a, b \in \mathbb{R}\}$;
(4) $\mathcal{C}_4 = \{(a, \infty) : a \in \mathbb{R}\}$;

2.2 The monotone class theorem

Definition 2.9 A monotone class is a collection of subsets $\mathcal{M}$ of X such that
(1) if $A_i \uparrow A$ and each $A_i \in \mathcal{M}$, then $A \in \mathcal{M}$
(2) if $A_i \downarrow A$ and each $A_i \in \mathcal{M}$, then $A \in \mathcal{M}$.
The intersection of monotone classes is a monotone class, and the intersection of all monotone classes containing a given collection of sets is the smallest monotone class containg that collection.
Theorem 2.10 (The Monotone Class Theorem) *Suppose $\mathcal{A}_0$ is an algebra, $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{A}_0$, and $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}_0$. Then $\mathcal{M} = \mathcal{A}$.*

Chapter 3

Measures

In which we generalize length, Additivity property extended to countable unions (but not uncountable)
Definition 3.1 Let X be a set and $\mathcal{A}$ a sigma algebra consisting of subsets of X. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that
(1) $\mu(\emptyset) = 0$;
(2) if $A_i \in \mathcal{A}, i = 1, 2, \cdots$, are pairwise disjoint, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 3.1(2) is known as countable additivity. We say a set function is finitely additive if the measure of the union commutes equals the sum of the measures for pairwise disjoint sets. The triple $(X, \mathcal{A}, \mu)$ is called a measure space. Examples include the counting measure, assigning positive values to every member of the reals and then taking the sum of the elements in a sets' values under that assignment, and point mass.

Proposition 3.5 The following hold:
(1) If $A, B \in \mathcal{A}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$
(2) If $A_i \in \mathcal{A}$ and $A = \cup_{i=1}^{\infty} A_i$, then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
(3) Suppose $A_i \in \mathcal{A}$ and $A_i \uparrow A$. Then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.
(4) Suppose $A_i \in \mathcal{A}$ and $A_i \downarrow A$. If $\mu(A_1) < \infty$, then we have $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.
Definition 3.7 A measure μ is a finite measure if $\mu(X) < \infty$. A measure μ is sigma finite if there exist sets $E_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ such that $\mu(E_i) < \infty$ for each i and $X = \cup_{i=1}^{\infty} E_i$. If μ is a finite measure, then $X, \mathcal{A}, \mu)$ is called a finite measure space, and similarly if σ-finite measure, then it's called a σ-finite measure space.

Let $(X, \mathcal{A}, \mu)$ be a measure space. Call $A \subset X$ a null set if it is a subset of some set $B \in \mathcal{A}$ such that $\mu(B) = 0$.
A measure space is complete if all are null sets are in $\mathcal{A}$. The completion of $\mathcal{A}$ is $(X, \overline{\mathcal{A}}, \overline{\mu})$ such that $\overline{\mathcal{A}}$ is the smallest σ-algebra such that there is an extension $\overline{\mu}$ of μ such that $X, \overline{\mathcal{A}}, \overline{\mu}$ is complete. That is, $\overline{\mu}(B) = \mu(B)$ if $B \in \mathcal{A}$.

A probability measure is a measure μ such that $\mu(X) = 1$.

Chapter 4

Construction of measures

Here we address a method for constructing measures, in the process introducing outer measures, the one-dimensional Lebesgue measure, results and examples related to that, and the inability to define Lebesgue measure for all subsets of the reals; finally the Carathéodory extension theorem.

4.1 Outer measures

Definition 4.1 Let X be a set. An outer measure is a function μ^* defined on the collection of all subsets of X satisfying
(1) $\mu^*(\emptyset) = 0$;
(2) if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$;
(3) $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ whenever $A_1, A_2, \ldots$ are subsets of X.
Definition 4.5 Let μ^* be an outer measure. A set $A \subset X$ is $\mu^* - measurable$ if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $E \subset X$
Theorem 4.6 (Carathéodory criterion/theorem - NOT EXTENSION) *If μ^* is an outer measure on X, then the collection $\mathcal{A}$ of μ^*-measurable sets is a σ-algebra. If μ is the restriction of μ^* to $\mathcal{A}$, then μ is a measure. Morever, $\mathcal{A}$ contains all the null sets.*

4.4 The Carathéodory extension theorem

Theorem 4.17 Suppose $\mathcal{A}_0$ is an algebra and $l : \mathcal{A}_0 \to [0, \infty]$ is a measure on $\mathcal{A}_0$. Define

$$\mu^*(E) = \inf\{\sum_{i=1}^{\infty} l(A_i) : \textit{each } A_i \in \mathcal{A}_0, E \subset \cup_{i=1}^{\infty} A_i\}$$

For $E \subset X$. Then
(1) μ^* is an outer measure;
(2) $\mu^*(A) = l(A)$ if $A \in \mathcal{A}_0$;
(3) Every set in $\mathcal{A}_0$ and every μ^*-null set is μ^*-measurable;
(4) if l is σ-finite, then there is a unique extension to $\sigma(\mathcal{A}_0)$.

Chapter 8

Properties of Lebesgue Integrals

Propositions about functions being zero a.e. and approximations.

8.1 criteria for a zero a.e. function

Proposition 1. Suppose f is measurable and non-negative and $\int f \, d\mu = 0$. Then $f = 0$ almost everywhere.

Proof. If f is not equal to 0 almost everywhere, there exists an n such that $\mu(A_n) > 0$ where $A_n = \{x : f(x) > \frac{1}{n}\}$. But since f is non-negative,

$$0 = \int f \geq \int_{A_n} f \geq \frac{1}{n}\mu(A_n),$$

a contradiction. □

Proposition 2. Suppose f is real-valued and integrable and for every measurable set A we have $\int_A f \, d\mu = 0$. Then $f = 0$ a.e.

Corollary. Let m be Lebesgue measure and $a \in \mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $\int_a^x f(y) \, dy = 0$ for all x. Then $f = 0$ a.e.

8.2 An approximation result

Theorem 1. Suppose f is a Lebesgue measurable real-valued integrable function on $\mathbb{R}$. Let $\varepsilon > 0$. Then there exists a continuous function g with compact support such that

$$\int |f - g| < \varepsilon.$$

Chapter 9

Riemann integrals

In which we show that the Riemann integral of a function exists iff the set of discontinuities of the function of the function have Lebesgue measure zero; in this case, the two integrals agree.

9.1 Comparison with the Lebesgue integral

Recall the Riemann integral definition from Munkres. We only consider bounded functions from $[a, b]$ into $\mathbb{R}$. Here we denote Riemann integrals as $R(f)$.

$$\overline{R}(f) = \inf\{U(P, f) : P \text{ is a partition}\}.$$

and

$$\underline{R}(f) = \sup\{L(P, f) : P \text{ is a partition}\}.$$

The Riemann integral exists if the two are equal.

Theorem 2. A bounded real-valued function f on $[a, b]$ is Riemann integrable iff the set of points at which f is discontinuous has Lebesgue measure 0, and in that case, f is Lebesgue measurable and the Riemann integral of f is equal in value to the Lebesgue integral of f.

Example. Let $[a, b] = [0, 1]$ and $f = \chi_A$, where A is the set of irrational numbers in $[0, 1]$. If $x \in [0, 1]$, every neighborhood of x contains both rational and irrational points, so f is continuous at no point of $[0, 1]$. Therefore, f is not Riemann integrable.

Example. Define $f(x)$ on $[0, 1]$ to be 0 if x is irrational and to be $\frac{1}{q}$ if x is rational and equals $\frac{p}{q}$ when in reduced form. f is discontinuous at every rational. If x is irrational and $\varepsilon > 0$, there are only finitely many rationals r for which $f(r) \geq \varepsilon$,so taking δ less than the distance from x to any of this finite collection of rationals shows that $|f(y) - f(x)| < \varepsilon$ if $|y - x| < \delta$. Hence f is continuous at x. Therefore the set of discontinuities is a countable set, hence of measure 0, hence f is Riemann integrable.

Chapter 10

Types of convergence

There are various ways in which a sequence of functions f_n can converge, and we compare some of them. Assume all functions in this chapter to be measurable.

10.1 Definitions and examples

Definition 1. If μ is a measure, we say a sequence of measurable functions f_n *converges almost everywhere* to f and write $f_n \to f$ a.e. if there is a set of measure 0 such that for x not in this set we have $f_n(x) \to f(x)$.
We say f_n *converges in measure* to f if for each $\varepsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0.$$

as $n \to \infty$.
Let $1 \leq p < \infty$. We say f_n *converges in* L^p to f if

$$\int |f_n - f|^p \, d\mu \to 0.$$

as $n \to \infty$.

Definition 2. (L^p norms definition from chapter 15) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. For $1 \leq p < \infty$, define the L^p norm of f by

$$\|f\|_p = (\int |f(x)|^p \, d\mu)^{\frac{1}{p}}.$$

For $p = \infty$, define the L^∞ norm of f by

$$\|f\|_\infty = \inf\{M \geq 0 : \mu(\{x : |f(x)| \geq M\}) = 0\}.$$

If no such M exists, then $||f||_\infty = \infty$. Thus the L^∞ norm of a function f is the smallest number M such that $|f| \leq M$ a.e.

Proposition 3. (1) Suppose μ is a finite measure. If $f_n \to f$ a.e., then f_n converges to f in measure. (2) If μ is a measure, not necessarily, finite, and $f_n \to f$ in measure, there is a subsequence n_j such that $f_{n_j} \to f$ a.e.

Proof. (1) Let $\varepsilon > 0$ and suppose $f_n \to f$ a.e. If

$$A_n = \{x : |f_n(x) - f(x)| > \varepsilon\},$$

then $\chi_{A_n} \to 0$ a.e., and by the dominated convergence theorem,

$$\mu(A_n) = \int \chi_{A_n}(x)\mu(dx) \to 0.$$

This proves (1). □

Example. Part (1) of the above proposition is not true if $\mu(X) = \infty$. To see this, let $X = \mathbb{R}$ and let $f_n = \chi_{(n,n+1)}$. We have $f_n \to 0$ a.e., but f_n does not converge in measure.

The next proposition compares convergence in L^p to convergence in measure. Before we prove this, we prove an easy preliminary result know as Chebyshev's inequality.

Lemma 1. If $1 \le p < \infty$, then

$$\mu(\{x : |f(x)| \ge a\}) \le \frac{\int |f|^p d\mu}{a^p}.$$

Proof. Let $A = \{x : |f(x)| \ge a\}$. Since $\chi_A \le |f|^p \chi_A / a^p$, we have

$$\mu(A) \le \int_A \frac{|f|^p}{a^p}\, d\mu \le \frac{1}{a^p} \int |f|^p\, d\mu.$$

This is what we wanted. □

Proposition 4. If f_n converges to f in L^p, then it converges in measure.

Proof. If $\varepsilon > 0$, by Chebyshev's inequality

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \le \frac{\int |f_n - f|^p}{\varepsilon^p} \to 0.$$

as required. □

Example. Let $f_n = n^2 \chi_{(0,\frac{1}{n})}$ on $[0, 1]$ and let μ be Lebesgue measure. This gives an example where f_n converges to 0 a.e. and in measure, but does not converge in L^p for any $p \ge 1$.

Theorem 3. (Egorov's Theorem) Suppose μ is a finite measure, $\varepsilon > 0$, and $f_n \to f$ a.e. Then there exists a measurable set A such that $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly on A^c.

This type of convergence is sometimes known as *almost uniform convergence*. Egorov's theorem is not as useful for solving problems as one might expect, and students have a tendency to try to use it when other methods work much better.

Proof. Let

$$A_{nk} = \cup_{m=n}^{\infty}\{x : |f_m(x) - f(x)| > \frac{1}{k}\}.$$

For fixed k, A_{nk} decreases as n increases. The intersection $\cap_n A_{nk}$ has measure 0 because for almost every x, $|f_m(x) - f(x)| \le \frac{1}{k}$ if m is sufficiently large. Therefore $\mu(A_{nk}) \to 0$ as $n \to \infty$. We can thus find an integer n_k such that $\mu(A_{n_k k}) < \varepsilon_2^{-k}$. Let

$$A = \cup_{k=1}^{\infty} A_{n_k k}.$$

Hence $\mu(A) < \varepsilon$. If $x \notin A$, then $x \notin A_{n_k k}$, and so $|f_n(x) - f(x)| \le \frac{1}{k}$ if $n \ge n_k$. Thus $f_n \to f$ uniformly on A^c. $\square$