Week 12 Problems

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1. (Bass 2.3) We give a counterexample. Let $X = \mathbb{N}$. Let $\mathcal{A}_n$ be the σ-algebra consisting of all sets of natural numbers less than or equal to n, as well as the complements of each of those sets, and $\emptyset$ and $\mathbb{N}$. Then for every set A in the algebra, we can say that either A or A^c is finite. For $i \in \mathbb{N}$, every singleton set $\{2i\}$ lies in at least one of $\mathcal{A}_n$, but the infinite countable union of all such sets, which is all of the evens, does not lie in any of the $\mathcal{A}_n$ collections, so our nested union fails the property to be a σ-algebra.

2. (H1.4) Let the Lebesgue measure be defined as usual, with the set $X = \mathbb{R}$. By Theorem 4.16 in Bass there exists a subset $E \subset \mathbb{R}$ which is non-measurable. By Theorem 4.6, the collection of measurable sets under an outer measure is a σ-algebra, so the complement E^c must also be non-measurable. The union of these two sets is the entire set $\mathbb{R}$, which *is* measurable due to the aforementioned σ-algebra property.

3. (H3) Let $A \subset \mathbb{R}$. If A is open, then for any point $x \in A$, there is an open interval (an open ball in $\mathbb{R}$) around x of some size $\varepsilon > 0$ that is contained in A. This ball $(x - \varepsilon, x + \varepsilon)$ has lebesgue measure $2\varepsilon > 0$, and since it is a subset of A, then by property (2) of outer measures, $m(A) \geq 2\varepsilon > 0$ (this includes the case where $m(A) = \infty$, which we consider larger than 0 for inequality purposes).

   If A is compact, then by the Heine-Borel theorem, A is closed and bounded. Since A is bounded, there is some $n \in \mathbb{R}$ such that $A \subset [-n, +n]$. The measure of this bounding interval is $2n < \infty$, and by the same subset inequality property of outer measures, $m(A) \leq 2n < \infty$.