

# Richard Bass: Real Analysis for Graduate Students

Notes for Chapters 1–4

Alex Skeldon

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# Chapter 1

## Preliminaries

### 1.1 Notation and terminology

We use  $A^c$  for the set of points not in  $A$ . Specifically, define

$$A^c = \{x \in X : x \notin A\}.$$

We write

$$A - B = A \cap B^c$$

(it is common to also see  $A \setminus B$ ) and

$$A \triangle B = (A - B) \cup (B - A)$$

The set  $A \triangle B$  is called the *symmetric difference* of  $A$  and  $B$  and is the set of points that are in one of the sets but not the other. If  $I$  is some non-empty index set, a collection of subsets  $\{A_\alpha\}_{\alpha \in I}$  is disjoint if  $A_\alpha \cap A_\beta = \emptyset$  whenever  $\alpha \neq \beta$ .

We write  $A_i \uparrow$  if  $A_1 \subset A_2 \subset \dots$  and write  $A_i \uparrow A$  if in addition  $A = \bigcup_{i=1}^{\infty} A_i$ . Similarly,  $A_i \downarrow$  means  $A_1 \supset A_2 \supset \dots$  and  $A_i \downarrow A$  means that in addition  $A = \bigcap_{i=1}^{\infty} A_i$ .

We use  $\log x$  to denote the natural logarithm of  $x$ , that is, the logarithm of  $x$  to the base  $e$ .

We use  $\mathbb{Q}$  to refer to the set of rational numbers,  $\mathbb{R}$  the set of real numbers, and  $\mathbb{C}$  the set of complex numbers. We use

$$x \vee y = \max(x, y) \quad \text{and} \quad x \wedge y = \min(x, y)$$

We can write a real number  $x$  in terms of its positive and negative parts:

$$x = x^+ \vee 0 \quad \text{and} \quad x^- = (-x) \vee 0.$$

If  $z$  is a complex number, then  $\bar{z}$  is the complex conjugate of  $z$ . The composition of two functions is defined by  $f \circ g(x) = f(g(x))$ .

If  $f$  is a function whose domain is the reals or a subset of the reals, then  $f(x+) = \lim_{y \rightarrow x+} f(y)$  and  $f(x-) = \lim_{y \rightarrow x-} f(y)$  are the right and left hand limits of  $f$  at  $x$ , resp.

We say a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *increasing* if  $x < y$  implies  $f(x) \leq f(y)$  and  $f$  is *strictly increasing* if  $x < y$  implies  $f(x) < f(y)$ . Decreasing and strictly decreasing are defined similarly. A function is monotone if  $f$  is either increasing or decreasing.

Given a sequence  $\{a_n\}$  of real numbers,

$$\limsup_{n \rightarrow \infty} a_n = \inf_n \sup_{m \geq n} a_m,$$

$$\liminf_{n \rightarrow \infty} a_n = \sup_n \inf_{m \geq n} a_m.$$

For example, if

$$a_n = \begin{cases} 1, & n \text{ even;} \\ -1/n, & n \text{ odd,} \end{cases} \quad (1.1)$$

then  $\limsup_{n \rightarrow \infty} a_n = 1$  and  $\liminf_{n \rightarrow \infty} a_n = 0$ . The sequence  $\{a_n\}$  has a limit if and only if  $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$  and both are finite. We use analogous definitions when we take a limit along the real numbers. For example,

$$\limsup_{y \rightarrow x} f(y) = \inf_{\delta > 0} \sup_{|y-x| < \delta} f(y).$$

## 1.2 Some undergraduate mathematics

We recall some definitions and facts from undergraduate topology, algebra, and analysis. Many proofs are omitted.

A set  $X$  is a *metric space* if there exists a function  $d : X \times X \rightarrow \mathbb{R}$ , called the metric, such that

- (1)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (2)  $d(x, y) \geq 0$  for all  $x, y \in X$  and  $d(x, y) = 0$  iff  $x = y$ ;
- (3)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

Condition (3) is called the triangle inequality.

Given a metric space  $X$ , let

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

be the *open ball* of radius  $r$  centered at  $x$ . If  $A \subset X$ , the *interior* of  $A$ , denoted  $A^\circ$ , is the set of  $x$  such that there exists  $r_x > 0$  with  $B(x, r_x) \subset A$ . The closure of  $A$ , denoted  $\overline{A}$ , is the set of  $x \in X$  such that every open ball centered at  $x$  contains at least one point of  $A$ . A set  $A$  is open if  $A = A^\circ$ , closed if  $A = \overline{A}$ . If  $f : X \rightarrow \mathbb{R}$ , the support of  $f$  is the closure of the set  $\{x : f(x) \neq 0\}$ .  $f$  is continuous at a point  $x$  if given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  whenever  $d(x, y) < \delta$ .  $f$  is continuous if it is continuous at every point of its domain.

Recall also the topological definition of continuity, convergence of sequences, Cauchy sequences, and completeness of a set  $X$ .

An open cover of a subset  $K$  of  $X$  is a non-empty collection  $\{G_\alpha\}_{\alpha \in I}$  of open sets such that  $K \subset \cup_{\alpha \in I} G_\alpha$ . The index set  $I$  can be finite or infinite. A set  $K$  is compact if every open cover contains a finite subcover, i.e. there exists  $G_1, \dots, G_n \in \{G_\alpha\}_{\alpha \in I}$  such that  $K \subset \cup_{i=1}^n G_i$ .

**Proposition 1.1** *If  $K$  is compact,  $F \subset K$ , and  $F$  is closed, then  $F$  is compact*

**Proposition 1.2** *If  $K$  is compact and  $f$  is continuous on  $K$ , then there exist  $x_1$  and  $x_2$  such that  $f(x_1) = \inf_{x \in K} f(x)$  and  $f(x_2) = \sup_{x \in K} f(x)$ . In other words,  $f$  takes on its maximum and minimum values.*

**Remark 1.3** If  $x \neq y$ , let  $r = d(x, y)$  and note that  $B(x, r/2)$  and  $B(y, r/2)$  are disjoint open sets containing  $x$  and  $y$ , resp. Therefore metric spaces are also what are called hausdorff spaces.

The eight vector space properties (AKA linear space), normed linear space when map  $x \rightarrow ||x||$  satisfying three props, metric induced by the norm, equivalence relationships.

Given an equivalence relationship,  $X$  can be written as the union of disjoint equivalence classes.  $x$  and  $y$  are in the same equivalence class if and only if  $x \sim y$ .

A set  $X$  has a partial order " $\leq$ " if

- (1)  $x \leq x$  for all  $x \in X$ ;
- (2) if  $x \leq y$  and  $y \leq x$ , then  $x = y$ ;
- (3) if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .

Note that given  $x, y \in X$ , it is not necessarily true that  $x \leq y$  or  $y \leq x$ . For an example, let  $Y$  be a set, let  $X$  be the collection of all subsets of  $Y$ , and say  $A \leq B$  if  $A, B \in X$  and  $A \subset B$ .

We need the following three facts about the real line.

**Proposition 1.4** *Suppose  $K \in \mathbb{R}$ ,  $K$  is closed, and  $K$  is contained in a finite interval. Then  $K$  is compact.*

**Proposition 1.5** *Suppose  $G \subset \mathbb{R}$  is open. Then  $G$  can be written as the countable union of disjoint open intervals.*

**Proposition 1.6** *Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing function. Then both  $\lim_{y \rightarrow x+} f(y)$  and  $\lim_{y \rightarrow x-} f(y)$  exist for every  $x$ . Moreover, the set of  $x$  where  $f$  is not continuous is countable.*

## 1.3 Proofs of Propositions

Skipped.

# Chapter 2

## Families of Sets

### 2.1 Algebras and $\sigma$ -algebras

When we turn to constructing measures in Chapter 4, we will see that we cannot in general define the measure of an arbitrary set. The class of sets that we will want to use are  $\sigma$ -algebras.

Let  $X$  be a set.

**Definition 2.1** An algebra is a collection  $\mathcal{A}$  of subsets of  $X$  such that

- (1)  $\emptyset \in \mathcal{A}$  and  $X \in \mathcal{A}$ ;
- (2) if  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$ ;
- (3) if  $A_1, \dots, A_n \in \mathcal{A}$ , then  $\cup_{i=1}^n A_i$  and  $\cap_{i=1}^n A_i$  are in  $\mathcal{A}$ .

$\mathcal{A}$  is a  $\sigma$ -algebra if in addition

- (4) whenever  $A_1, A_2, \dots$  are in  $\mathcal{A}$ , then  $\cup_{i=1}^{\infty} A_i$  and  $\cap_{i=1}^{\infty} A_i$  are in  $\mathcal{A}$ .

In (4) we allow countable unions and intersections only. The intersection requirement is redundant since it is the complement of an infinite union of complementary sets.

The pair  $(X, \mathcal{A})$  is called a measurable space. A set  $A$  is measurable or  $\mathcal{A}$  measurable if  $A \in \mathcal{A}$ .

**Example 2.2** Let  $X = \mathbb{R}$ , the set of real numbers, and let  $\mathcal{A}$  be the collection of all subsets of  $\mathbb{R}$ . Then  $\mathcal{A}$  is a  $\sigma$ -algebra.

**Lemma 2.7** If  $\mathcal{A}_\alpha$  is a  $\sigma$ -algebra for each  $\alpha$  in some non-empty index set  $I$ , then  $\cap_{\alpha \in I} \mathcal{A}_\alpha$  is a  $\sigma$ -algebra.

If we have a collection  $\mathcal{C}$  of subsets of  $X$ , define

$$\sigma(\mathcal{C}) = \cap \{ \mathcal{A}_\alpha : \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra, } \mathcal{C} \in \mathcal{A}_\alpha \},$$

We call  $\sigma(\mathcal{C})$  the  $\sigma$ -algebra generated by the collection  $\mathcal{C}$ . Since  $\sigma(\mathcal{C})$  is a sigma

algebra, then  $\sigma(\sigma(C)) = \sigma(C)$ .

If  $X$  has some additional structure (metric space), then we can talk about open sets. If  $\mathcal{G}$  is the collection of open subsets of  $X$ , then we call  $\sigma\mathcal{G}$  the Borel  $\sigma$ -algebra on  $X$ , and denote this  $\mathcal{B}$ . Elements of that are called Borel sets, said to be Borel measurable. We will see later that if  $X = \mathbb{R}$ , then  $\mathcal{B} \neq$  the collection of all subsets of  $X$ .

**Proposition 2.8** *If  $X = \mathbb{R}$ , then the Borel  $\sigma$ -algebra is generated by each of the following collections of sets:*

- (1)  $\mathcal{C}_1 = \{(a, b) : a, b \in \mathbb{R}\}$ ;
- (2)  $\mathcal{C}_2 = \{[a, b] : a, b \in \mathbb{R}\}$ ;
- (3)  $\mathcal{C}_3 = \{(a, b] : a, b \in \mathbb{R}\}$ ;
- (4)  $\mathcal{C}_4 = \{(a, \infty) : a \in \mathbb{R}\}$ ;

## 2.2 The monotone class theorem

**Definition 2.9** A monotone class is a collection of subsets  $\mathcal{M}$  of  $X$  such that

- (1) if  $A_i \uparrow A$  and each  $A_i \in \mathcal{M}$ , then  $A \in \mathcal{M}$
- (2) if  $A_i \downarrow A$  and each  $A_i \in \mathcal{M}$ , then  $A \in \mathcal{M}$ .

The intersection of monotone classes is a monotone class, and the intersection of all monotone classes containing a given collection of sets is the smallest monotone class containing that collection.

**Theorem 2.10 (The Monotone Class Theorem)** *Suppose  $\mathcal{A}_0$  is an algebra,  $\mathcal{A}$  is the smallest  $\sigma$ -algebra containing  $\mathcal{A}_0$ , and  $\mathcal{M}$  is the smallest monotone class containing  $\mathcal{A}_0$ . Then  $\mathcal{M} = \mathcal{A}$ .*

# Chapter 3

## Measures

In which we generalize length, Additivity property extended to countable unions (but not uncountable)

**Definition 3.1** Let  $X$  be a set and  $\mathcal{A}$  a sigma algebra consisting of subsets of  $X$ . A measure on  $(X, \mathcal{A})$  is a function  $\mu : \mathcal{A} \rightarrow [0, \infty]$  such that

- (1)  $\mu(\emptyset) = 0$ ;
- (2) if  $A_i \in \mathcal{A}, i = 1, 2, \dots$ , are pairwise disjoint, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 3.1(2) is known as countable additivity. We say a set function is finitely additive if the measure of the union commutes equals the sum of the measures for pairwise disjoint sets. The triple  $(X, \mathcal{A}, \mu)$  is called a measure space. Examples include the counting measure, assigning positive values to every member of the reals and then taking the sum of the elements in a sets' values under that assignment, and point mass.

**Proposition 3.5** The following hold:

- (1) If  $A, B \in \mathcal{A}$  with  $A \subset B$ , then  $\mu(A) \leq \mu(B)$
- (2) If  $A_i \in \mathcal{A}$  and  $A = \cup_{i=1}^{\infty} A_i$ , then  $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ .
- (3) Suppose  $A_i \in \mathcal{A}$  and  $A_i \uparrow A$ . Then  $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$ .
- (4) Suppose  $A_i \in \mathcal{A}$  and  $A_i \downarrow A$ . If  $\mu(A_1) < \infty$ , then we have  $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$ .

**Definition 3.7** A measure  $\mu$  is a finite measure if  $\mu(X) < \infty$ . A measure  $\mu$  is sigma finite if there exist sets  $E_i \in \mathcal{A}$  for  $i = 1, 2, \dots$  such that  $\mu(E_i) < \infty$  for each  $i$  and  $X = \cup_{i=1}^{\infty} E_i$ . If  $\mu$  is a finite measure, then  $(X, \mathcal{A}, \mu)$  is called a finite measure space, and similarly if  $\sigma$ -finite measure, then it's called a  $\sigma$ -finite measure space.

Let  $(X, \mathcal{A}, \mu)$  be a measure space. Call  $A \subset X$  a null set if it is a subset of some set  $B \in \mathcal{A}$  such that  $\mu(B) = 0$ .

A measure space is complete if all are null sets are in  $\mathcal{A}$ . The completion of  $\mathcal{A}$  is  $(X, \overline{\mathcal{A}}, \overline{\mu})$  such that  $\overline{\mathcal{A}}$  is the smallest  $\sigma$ -algebra such that there is an extension  $\overline{\mu}$  of  $\mu$  such that  $(X, \overline{\mathcal{A}}, \overline{\mu})$  is complete. That is,  $\overline{\mu}(B) = \mu(B)$  if  $B \in \mathcal{A}$ .

A probability measure is a measure  $\mu$  such that  $\mu(X) = 1$ .

# Chapter 4

## Construction of measures

Here we address a method for constructing measures, in the process introducing outer measures, the one-dimensional Lebesgue measure, results and examples related to that, and the inability to define Lebesgue measure for all subsets of the reals; finally the Carathéodory extension theorem.

### 4.1 Outer measures

**Definition 4.1** Let  $X$  be a set. An outer measure is a function  $\mu^*$  defined on the collection of all subsets of  $X$  satisfying

- (1)  $\mu^*(\emptyset) = 0$ ;
- (2) if  $A \subset B$ , then  $\mu^*(A) \leq \mu^*(B)$ ;
- (3)  $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$  whenever  $A_1, A_2, \dots$  are subsets of  $X$ .

**Definition 4.5** Let  $\mu^*$  be an outer measure. A set  $A \subset X$  is  $\mu^*$ -measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all  $E \subset X$

**Theorem 4.6 (Carathéodory criterion/theorem - NOT EXTENSION)**

*If  $\mu^*$  is an outer measure on  $X$ , then the collection  $\mathcal{A}$  of  $\mu^*$ -measurable sets is a  $\sigma$ -algebra. If  $\mu$  is the restriction of  $\mu^*$  to  $\mathcal{A}$ , then  $\mu$  is a measure. Moreover,  $\mathcal{A}$  contains all the null sets.*

### 4.2 Lebesgue-Stieltjes measures

Nothing yet.



### 4.3 Nonmeasurable sets

### 4.4 The Carathéodory extension theorem

**Theorem 4.17** Suppose  $\mathcal{A}_0$  is an algebra and  $l : \mathcal{A}_0 \rightarrow [0, \infty]$  is a measure on  $\mathcal{A}_0$ . Define

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} l(A_i) : \text{each } A_i \in \mathcal{A}_0, E \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

For  $E \subset X$ . Then

- (1)  $\mu^*$  is an outer measure;
- (2)  $\mu^*(A) = l(A)$  if  $A \in \mathcal{A}_0$ ;
- (3) Every set in  $\mathcal{A}_0$  and every  $\mu^*$ -null set is  $\mu^*$ -measurable;
- (4) if  $l$  is  $\sigma$ -finite, then there is a unique extension to  $\sigma(\mathcal{A}_0)$ .

# Chapter 8

## Properties of Lebesgue Integrals

Propositions about functions being zero a.e. and approximations.

### 4.5 criteria for a zero a.e. function

**Proposition 8.1** *Suppose  $f$  is measurable and non-negative and  $\int f d\mu = 0$ . Then  $f = 0$  almost everywhere.*

**Proposition 8.2** *Suppose  $f$  is real-valued and integrable and for every measurable set  $A$  we have  $\int_A f d\mu = 0$ . Then  $f = 0$  a.e.*