Validated Solution of Initial Value Problem for Ordinary Differential Equations based on Explicit and Implicit Runge-Kutta Schemes

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Abstract

We present in this report our tool based on Ibex library which provides an innovative and generic procedure to simulate an ordinary differential equation with any Runge-Kutta scheme (explicit or implicit). Our validated approach is based on the classical two steps integration: the Picard-Lindelöf operator to enclose all the solutions on a one step, and the computation of the approximated solution and its Local Truncation Error. This latter is computed with a generic and elegant approach using interval arithmetic and Fréchêt derivatives. We perform a strong experimentation through many numerical experiments coming from three different benchmarks and the results are shown and compared with competition.

Chapter 1

Introduction

Many scientific applications in physical fields such as mechanics, robotics, chemistry or electronics require differential equations. This kind of equations appears when only the velocity and/or the acceleration are available in the modeling of a system. In the general case, these differential equations cannot be formally integrated, i.e., closed form solution are not available, and a numerical integration scheme is used to approximate the state of the system. In this report, we focus on ordinary differential equations for which we develop a new method to solve them and validate the solution.

Notations $\dot{y}$ denotes the time derivative of the function y, i.e., $\frac{dy}{dt}$. x denotes a real values while $\mathbf{x}$ represents a vector of real values. $[x]$ represents an interval values and $[\mathbf{x}]$ represents a vector of interval values.

1.1 Solving ODE with Numerical Methods

An *ordinary differential equation* (ODE for short) is a relation between a function $y : \mathbb{R} \to \mathbb{R}^n$ and its derivative $\dot{y} = \frac{dy}{dt}$, written as $\dot{y} = f(t, y)$. An *initial value problem* (IVP for short) is an ODE together with an initial condition and a final time

$$\dot{y} = f(t, y) \quad \text{with} \quad y(0) = y_0,\ y_0 \in \mathbb{R}^n \quad \text{and} \quad t \in [0, t_{\text{end}}] \ . \tag{1.1}$$

We do not address here the problem of existence of the solution and we shall always assume that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in t and globally Lipschitz in y, so Equation (1.1) admits a unique solution on $\mathbb{R}$, see [11] for more details. As the exact solution $y(t)$ of Equation (1.1) is usually unknown, numerical methods are used to approximate $y(t)$ on a time grid.

1.2 Classical Runge-Kutta methods

We now recall the principles of numerical integration of ordinary differential equations. Solving the IVP means finding a continuous and differentiable function y_∞ such that $y_\infty(0) = y_0$ and

$$\forall t \in [0, t_{\text{end}}], \quad \dot{y}_\infty(t) = f\big(t, y_\infty(t)\big) \ .$$

Note that, higher order differential equations can be translated into first-order ODEs by introducing additional variables for the derivatives of y. We denote the solution at time t of Equation (1.1) with initial condition y_0 at $t = 0$ by $y(t; y_0)$.

An exact solution of Equation (1.1) is rarely computable so that in practice, approximation algorithms are used. The goal of an approximation algorithm is to compute a sequence of $n + 1$ time instants

$$0 = t_0 < t_1 < \cdots < t_n = t_{\text{end}},$$

and a sequence of $n + 1$ values $y_0, \ \ldots, \ y_n$ such that

$$\forall i \in [0, n], \quad y_i \approx y_\infty(t_i; y_0) \ .$$

There is a huge set of numerical methods to solve Equation (1.1). In this report, we focus on single-step methods member of the Runge-Kutta family, that is these methods only use y_i and approximations of $\dot{y}(t)$ to compute y_{i+1}.

A Runge-Kutta method, starting from an initial value y_n at time t_n and a finite time horizon h, the *step-size*, produces an approximation y_{n+1} at time t_{n+1}, with $t_{n+1} - t_n = h$, of the solution $y(t_{n+1}; y_n)$. Furthermore, to compute y_{n+1}, a Runge-Kutta method computes s evaluations of f at predetermined time instants. The number s is known as the number of *stages* of a Runge-Kutta method. More precisely, a Runge-Kutta method is defined by

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i \ , \tag{1.2}$$

with k_i defined by

$$k_i = f\left(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j\right) \ . \tag{1.3}$$

The coefficient c_i, a_{ij} and b_i, for $i, j = 1, 2, \cdots, s$, fully characterize the Runge-Kutta methods and their are usually synthesized in a *Butcher tableau* of the form

$$\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\
c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\
\hline
 & b_1 & b_2 & \ldots & b_s
\end{array}$$

In function of the form of the matrix A, made of the coefficients a_{ij}, a Runge-Kutta method can be

- *explicit*, e.g., the classical Runge-Kutta method of order 4 given in Figure 1.1(a). In other words, the computation of an intermediate k_i only depends on the previous steps k_j for $j < i$;

- *diagonally implicit*, e.g., a diagonally implicit method of order 4 given in Figure 1.1(b). In this case, the computation of an intermediate step k_i involves the value k_i and so non-linear systems in k_i must be solved;

- *fully implicit*, e.g., the Runge-Kutta method with a Lobatto quadrature formula of order 4 given in Figure 1.1(c). In this last case, the computation of intermediate steps involves the solution of a non-linear system of equations in all the values k_i for $i = 1, 2, \cdots, s$.

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}$$

(a) RK4

$$\begin{array}{c|ccccc}
\frac{1}{4} & \frac{1}{4} & & & & \\
\frac{3}{4} & \frac{1}{2} & \frac{1}{4} & & & \\
\frac{11}{20} & \frac{17}{50} & \frac{-1}{25} & \frac{1}{4} & & \\
\frac{1}{2} & \frac{371}{1360} & \frac{-137}{2720} & \frac{15}{544} & \frac{1}{4} & \\
1 & \frac{25}{24} & \frac{-49}{48} & \frac{125}{16} & \frac{-85}{12} & \frac{1}{4} \\
\hline
 & \frac{25}{24} & \frac{-49}{48} & \frac{125}{16} & \frac{-85}{12} & \frac{1}{4}
\end{array}$$

(b) SDIRK4

$$\begin{array}{c|ccc}
0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\
\frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\
1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array}$$

(c) Lobatto3c

Figure 1.1: Different kinds of Runge-Kutta methods

Note that in case of implicit Runge-Kutta methods the non-linear systems of n equations must be solved at each integration step. Usually, a Newton-like method is used for this purpose. Nevertheless, such implicit methods have very good stability properties, see [11, Chap. II] for more details, which make them very useful in case of *stiff* ODE.

1.3 Computing with Sets

To take into account numerical approximation coming from floating-point arithmetic and approximation due to numerical integration scheme, set-based computation is required. In this case, we transform an IVP into an interval initial value problem (IIVP for short) that is

$$\dot{y} = f(t, y) \quad \text{with} \quad y(0) = \mathcal{Y}_0,\ \mathcal{Y}_0 \subseteq \mathbb{R}^n \quad \text{and} \quad t \in [0, t_{\text{end}}] \ . \tag{1.4}$$

In Equation (1.4), the initial value is given by a set $\mathcal{Y}_0$ of values, i.e., we do not know exactly the initial value. In other terms, we want to compute the set of solutions $\mathcal{Y}_\infty(t; \mathcal{Y}_0)$ of IIVP such that

$$\mathcal{Y}_\infty(t; \mathcal{Y}_0) = \{y_\infty(t; y_0) : \forall y_0 \in \mathcal{Y}_0\} \ .$$

Note that the set $\mathcal{Y}_\infty$ should guarantee to contain the true solution y_∞. For the past decades IIVP have been solved using tools coming from interval analysis. The guaranteed solution of IIVP using interval arithmetic is mainly based on two kinds of methods:

i) Interval Taylor series methods [16, 15, 1, 17, 12, 20, 7, 14],

ii) Interval Runge-Kutta methods [9, 3, 2].

The former is the oldest method used in this context. Indeed, R. Moore [16] already applied this method in the sixties and until now it is the most used method to solve Equation (1.4). The latter is more recent, see in particular [3, 2], but Runge-Kutta methods have many interesting properties as strong stability that we would like to exploit in the context of validated solution of ODEs.

We present new guaranteed numerical integration schemes based on implicit Runge-Kutta methods. This work is an extension of [3, 2] which only considered explicit Runge-Kutta methods.

1.3.1 Interval arithmetic

The simplest and most common way to represent and manipulate sets of values is *interval arithmetic* [16]. An interval $[x_i] = [\underline{x_i}, \overline{x_i}]$ defines the set of reels x_i such that $\underline{x_i} \leq x_i \leq \overline{x_i}$. $\mathbb{IR}$ denotes the set of all intervals. The size or the width of $[x_i]$ is denoted by $w([x_i]) = \overline{x_i} - \underline{x_i}$. The center of an interval is denoted by $\text{Mid}([x])$ denotes the middle of $[x]$. A vector of intervals, or a *box*, $[\mathbf{x}]$ is the Cartesian product of intervals $[x_1] \times ... \times [x_i] \times ... \times [x_n]$. The width of a box is defined by $w([\mathbf{x}]) = \max_i w([x_i])$.

Interval arithmetic [16] extends to $\mathbb{IR}$ elementary functions over $\mathbb{R}$. For instance, the interval sum (i.e., $[x_1]+[x_2] = [\underline{x_1}+\underline{x_2}, \overline{x_1}+\overline{x_2}]$) encloses the image of the sum function over its arguments, and this enclosing property basically defines what is called an *interval extension* or an *inclusion function.*

Definition 1 (Extension of a function to $\mathbb{IR}$)**.** *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, then $[f]:\mathbb{IR}^n \to \mathbb{IR}$ is said to be an* **extension** *of f to intervals if*

$$\begin{aligned} \forall [x] \in \mathbb{IR}^n, \quad & [f]([x]) \supseteq \{f(x),\ x \in [x]\}, \\ \forall x \in \mathbb{R}^n, \quad & f(x) = [f](x) \ . \end{aligned}$$

In our context, the expression of a function f is always a composition of elementary functions. The **natural extension** $[f]_N$ is then simply a composition of the corresponding interval operators.

Definition 2 (Overestimation of a set)**.** *Consider the set $\mathcal{F} = \{f(x),\ x \in [x]\}$, the interval extension $[f]([x])$ is an overestimation of $\mathcal{F}$ and we note*

$$[f]([x]) = \Box\mathcal{F} \ .$$

Definition 3 (Integration)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and $[a] \subset \mathbb{IR}^n$, then the components of $\int_{\underline{a}}^{\overline{a}} f(s)ds$ are*

$$\left\{ \int_{\underline{a}}^{\overline{a}} f(s)ds \right\}_i = \int_{\underline{a}}^{\overline{a}} \{f(s)\}_i \, ds \ .$$

where $\{\}_i$ denotes the i-th component of a vector. Obviously, see [16],

$$\int_{\underline{a}}^{\overline{a}} f(s)ds \in (\underline{a} - \overline{a}) f([a]) = w([a])[f]([a]) \ .$$

The *interval arithmetic* is a powerful tool to deal with sets. Nevertheless, this representation usually produces too much over-approximated results, because it cannot take dependencies between variables in account: for instance, if $x = [0, 1]$, then $x - x = [-1, 1] \neq 0$. More generally, it can be shown for most integration schemes that the width of the result can only grow if we interpret sets of values as intervals.

Example 1.3.1. *Consider the ordinary differential equation $\dot{x}(t) = -x$ solved with the Euler's method with an initial value ranging in the interval $[0, 1]$ and with a step-size of $h = 0.5$. For one step of integration, we have to compute with interval arithmetic the expression $e = x + h \times (-x)$ which produces as a result the interval $[-0.5, 1]$. Rewriting the expression e such that $e' = x(1 - h)$, we obtain the interval $[0, 0.5]$ which is the exact result. Unfortunately, we cannot in general rewrite expressions with only one occurrence of each variable. More generally, it can be shown that for most integration schemes the width of the result can only grow if we interpret sets of values as intervals [18].* ■

1.3.2 Affine arithmetic

To avoid or limit the problem of dependency, we use an improvement over interval arithmetic named *affine arithmetic* [8] which can track linear correlations between variables.

A set of values in this domain is represented by an *affine form* $\hat{x}$, which is a formal expression of the form

$$\hat{x} = \alpha_0 + \sum_{i=1}^{n} \alpha_i \varepsilon_i,$$

where the coefficients α_i are real numbers, α_0 being called the *center* of the affine form, and the ε_i are formal variables ranging over the interval $[-1, 1]$ called *noise symbols*.

Obviously, an interval $a = [a_1, a_2]$ can be seen as the affine form $\hat{x} = \alpha_0 + \alpha_1 \varepsilon$ with $\alpha_0 = (a_1 + a_2)/2$ and $\alpha_1 = (a_2 - a_1)/2$. Moreover, affine forms encode linear dependencies between variables: if $x \in [a_1, a_2]$ and y is such that $y = 2x$, then x will be represented by the affine form $\hat{x}$ above and y will be represented as $\hat{y} = 2\alpha_0 + 2\alpha_1 \varepsilon$.

Usual operations on real numbers extend to affine arithmetic in the expected way. For instance, if we have two affine forms $\hat{x} = \alpha_0 + \sum_{i=1}^{n} \alpha_i \varepsilon_i$ and $\hat{y} = \beta_0 + \sum_{i=1}^{n} \beta_i \varepsilon_i$, then with $a, b, c \in \mathbb{R}$, we have

$$a\hat{x} \pm b\hat{y} \pm c = (a\alpha_0 \pm b\beta_0 \pm c) + \sum_{i=1}^{n} (a\alpha_i \pm b\beta_i)\varepsilon_i \ .$$

However, unlike the affine operations, most operations create new noise symbols. Multiplication for example is defined by

$$\hat{x} \times \hat{y} = \alpha_0 \alpha_1 + \sum_{i=1}^{n} (\alpha_i \beta_0 + \alpha_0 \beta_i)\varepsilon_i + \nu \varepsilon_{n+1},$$

where

$$\nu = \left(\sum_{i=1}^{n} |\alpha_i| \right) \times \left(\sum_{i=1}^{n} |\beta_i| \right),$$

over-approximates the error between the linear approximation of multiplication and multiplication itself.

Other operations, as sin or exp, are evaluated using two kinds of algorithm: *min range* method and *Tchebychev* method, see [8] for more details. Note that more recent work exists on increasing the accuracy of affine arithmetic [10, 19] but it is not mandatory to consider them in this work.

Example 1.3.2. *Consider again $e = x + h \times (-x)$ with $h = 0.5$ and $x = [0, 1]$ which is associated to the affine form $\hat{x} = 0.5 + 0.5\varepsilon_1$. Evaluating e with affine arithmetic without rewriting the expression, we obtain $[0, 0.5]$ as a result.* ■

The set-based evaluation of an expression only consists in interpreting all the mathematical operators (such as + or sin) by their counterpart in affine arithmetic. We will denote by $\text{Aff}(e)$ the evaluation of the expression e using affine arithmetic, see [4] for practical implementation details.

1.4 Scope of the report

In next chapter, we will describe the tool. After a short overview on the verified simulation process (Section 2.1), we will explain our new way to compute the truncation error in Section 2.2. Then, the algorithm used to compute the implicit Runge-Kutta schemes is described (Section 2.3). The chapter 3 gathers a large experimentation in order to compare us to the competition and validated our approach.

Chapter 2

Description of the tool

We describe in this chapter the main contribution of this article that is a new validated method to compute solution of Equation (1.1). Before presenting this new result we recall some results of the validated numerical integration based on Taylor series.

2.1 Overview on validated numerical integration

In the classical approach [15, 17] to define validated method for IVP, each step of an integration scheme consists in two steps: *a priori enclosure* and *solution tightening*. Starting from a valid enclosure $[y]_j$ at time t_j, the two following steps are applied

Step 1. Compute an *a priori* enclosure $[\tilde{y}]_j$ of the solution using Banach's theorem and the Picard-Lindelöf operator. This enclosure has the three major properties:

- $y(t, [y]_j)$ is guaranteed to exist for all $t \in [t_j, t_{j+1}]$, i.e., along the current step, and for all $y_j \in [y]_j$.
- $y(t, [y_j]) \subseteq [\tilde{y}]_j$ for all $t \in [t_j, t_{j+1}]$.
- the step-size $h_j = t_{j+1} - t_j$ is as larger as possible in terms of accuracy and existence proof for the IVP solution.

Step 2. Compute a tighter enclosure of $[y]_{j+1}$ such that $y(t_{j+1}, [y]_j) \subseteq [y]_{j+1}$. The main issue in this phase is how to counteract the well known wrapping effect [16, 15, 17]. This phenomenon appears when we try to enclose a set with an interval vector (geometrically a box). The arising overestimation creates a false dynamic for the next step, and, with accumulation, can lead to intervals with an unacceptably large width.

The different enclosures computed during each step are shown on Figure 2.1.

Some algorithms useful to perform these two steps are described in the following.

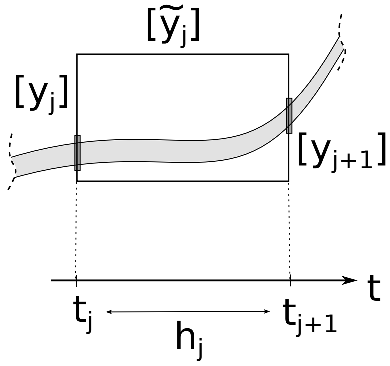


Figure 2.1: Enclosures appeared during one step

2.1.1 A priori solution enclosure

In order to compute the *a priori* enclosure, we use the Picard-Lindelöf operator. This operator is based on the following theorem.

Theorem 2.1.1 (Banach fixed-point theorem)**.** *Let* (K, d) *a complete metric space and let* $g : K \to K$ *a contraction that is for all* x*,* y *in* K *there exists* $c \in]0, 1[$ *such that*

$$d\left(g(x), g(y)\right) \leq c \cdot d(x, y) \ ,$$

then g *has a unique fixed-point in* K*.*

In context of IVP, we consider the space of continuously differentiable functions $C^0([t_j, t_{j+1}], \mathbb{R}^n)$ and the Picard-Lindelöf operator

$$P_f(y) = t \mapsto y_j + \int_{t_n}^{t} f(s, y(s))ds \ . \tag{2.1}$$

Note that this operator is associated to the integral form of Equation (1.1). So the solution of this operator is also the solution of Equation (1.1).

The Picard-Lindelöf operator is used to check the contraction of the solution on a integration step in order to prove the existence and the uniqueness of the solution of Equation (1.1) as stated by the Banach's fixed-point theorem. Furthermore, this operator is used to compute an enclosure of the solution of IVP over a time interval $[t_j, t_{j+1}]$.

Rectangular method for a priori enclosure

Using interval analysis and with a first order integration scheme we can define a simple interval Picard-Lindelöf operator such that

$$P_f([R]) = [y]_j + [0, h] \cdot f([R]), \tag{2.2}$$

with $h = t_{j+1} - t_j$ the step-size. Theorem 2.1.1 says that if we can find $[R]$ such that $P_f([R]) \subseteq [R]$ then the operator is contracting and Equation (1.1) has a unique solution. Furthermore,

$$\forall t \in [t_j, t_{j+1}], \quad \{y(t; y_j) : \forall y_j \in [y]_j\} \subseteq [R],$$

then $[R]$ is the *a priori* enclosure of the solution of Equation (1.1).

Remark that the operator defined in Equation (2.2) can also define a contractor (in a sens of interval analysis [6]) on $[R]$ after the fixed-point reached such that

$$[R] \leftarrow [R] \cap [y]_j + [0,h].f([R]) \ . \tag{2.3}$$

Hence, we can reduce the width of the *a priori* enclosure in order to increase the accuracy of the integration.

The operator defined in Equation (2.2) and its associated contractor defined in Equation (2.3) can be defined over a more accurate integration scheme (at the condition that it is a guaranteed scheme like the interval rectangle rule). For example, the evaluation of $\int_{t_j}^{t} f(s)ds$ can be easily improved with a Taylor or a Runge-Kutta scheme.

A priori enclosure with Taylor series

Interval version of Taylor series for ODE integration gives

$$[y]_{j+1} \subset \sum_{k=0}^{N} f^{[k]}([y]_j)h^k + f^{[N+1]}([\tilde{y}]_j)h^{N+1}, \tag{2.4}$$

with $f^{[0]} = [y]_j$, $f^{[1]} = f([y]_j)$,..., $f^{[k]} = \frac{1}{k}(\frac{\partial f^{[k-1]}}{\partial y} f)([y]_j)$.

By replacing h with interval $[0,h]$, this scheme becomes an efficient Taylor Picard-Lindelöf operator, with a parametric order N such that

$$y_{j+1}([t_j,t_{j+1}];[R]) = y_j + \sum_{k=0}^{N} f^{[k]}([y]_j)[0,h^k] + f^{[N+1]}([R])[0,h^{N+1}] \ . \tag{2.5}$$

In consequence, if $[R] \supseteq y_{j+1}([t_j,t_{j+1}],[R])$, $[R]$ then Equation (2.5) defined a contraction map and Theorem 2.1.1 can be applied.

In our tool, we use it at order 3 by default, it seems to be a good compromise between efficiency and computation quickness.

Note that the scheme defined in Equation (2.4) is usually evaluated in a centered form for a more accurate result

$$[y]_{j+1} \subset \sum_{k=0}^{N} f^{[k]}(\hat{y}_j)h^k + f^{[N+1]}([\tilde{y}]_j)h^{N+1} + \left(\sum_{k=0}^{N} J(f^{[k]},[y]_j)h^i)([y]_j - \hat{y}_j\right), \tag{2.6}$$

with $\hat{y}_j \in [y]_j$ $J(f^{[k]},[y]_j)$ is the Jacobian of $f^{[k]}$ evaluated at $[y]_j$. This scheme can also be combined with a QR-factorization to increase stability and counteract wrapping [17]. These two “tricks”, with a strong computational cost, can be avoided by using the affine arithmetic.

Picard-Lindelöf operator, as defined in Equation (2.5), gives an *a priori* enclosure $[R]$, using Theorem 2.1.1. Picard-Lindelöf operator is proven to be contracting on $[R]$, we can then use this operator to contract the box $[R]$ till a fixpoint is reached

In our tool, the default contractor uses a Taylor expansion as follow

$$[R] \cap x_j + \sum_{k=0}^{N} f^{[k]}([x]_j)[0,h^k] + f^{[N+1]}([R])[0,h^{N+1}]$$

It is very important to contract as much as possible this box $[R]$ because the Taylor remainder is function of $[R]$ and the step-size is function of the Taylor remainder.

2.1.2 Tighter enclosure and truncation error

Suppose that Step 1 has been done for the current step and that we dispose of the enclosure $[\tilde{y}]_j$ such that

$$y(t, t_j, [y]_j) \subseteq [\tilde{y}]_j \quad \forall t \in [t_j, t_{j+1}] \ .$$

In particular, we have $y(t_{j+1}, t_j, [y]_j) \subseteq [\tilde{y}]_j$. The goal of Step 2 is thus to compute the tighter enclosure $[y]_{j+1}$ such that

$$y(t_{j+1}, t_j, [y]_j) \subseteq [y]_{j+1} \subseteq [\tilde{y}]_j \ .$$

One way to do that consists in computing an approximate solution $y_{j+1} \approx y(t_{j+1}, t_j, [y]_j)$ with an integration scheme $\Phi(t_{j+1}, t_j, [y]_j)$, and then the associated local truncation error $LTE_\Phi(t, t_j, [y]_j)$. Indeed, a guaranteed integration scheme has the property that there exists a time $\xi \in [t_j, t_{j+1}]$ such that

$$y(t_{j+1}, t_j, [y]_j) \subseteq \Phi(t_{j+1}, t_j, [y]_j) + LTE_\Phi(\xi, t_j, [y]_j) \subseteq [\tilde{y}]_j \ .$$

So $[y]_{j+1} = \Phi(t_{j+1}, t_j, [y]_j) + LTE_\Phi(\xi, t_j, [y]_j)$ is an acceptable tight enclosure.

2.1.3 Wrapping effect

The problem of reducing the wrapping effect has been studied in many different ways. One of the most known and effective is the QR-factorization [15]. This method improves the stability of the Taylor series in the Vnode-LP tool [17]. An other way is to modify the geometry of the enclosing set (parallelepipeds with Eijgenram and moore, ellipsoids with Neumaier, convex polygons with Rihm and zonotopes with Stewart and chapoutot).

An efficient affine arithmetic allows us to counteract the wrapping effect as shown in Figure 2.1.3 while keeping a fast computation.

Example 2.1.1. *Consider the following IVP*

$$\dot{y} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} \tag{2.7}$$

with initial values: $[y_0] = ([-1, 1], [10, 11])$. *The exact solution of Equation* (2.7) *is*

$$y(t) = A(t)y_0 \ \textit{with} \ A(t) = \begin{pmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{pmatrix}$$

We compute periodically at $t = \frac{\pi}{4}n$ *with* $n = 1, \dots, 4$ *the solution of Equation* (2.7). ■

2.2 Validated Runge-Kutta Methods

We present in this section our main conctribution that is the way we validate *all* kinds of Runge-Kutta methods. The main challenge is to compute the local truncation error of each Runge-Kutta method. Moreover, based on Runge-Kutta methods we can also define a new way to compute *a priori* enclosure.

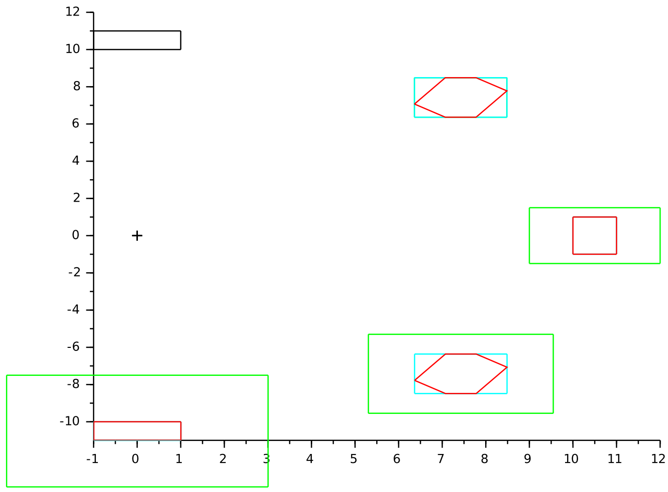


Figure 2.2: Wrapping effect comparison (black: initial, green: interval, blue: interval from QR, red: zonotope from affine)

2.2.1 The Local Truncation Error for Explicit Runge-Kutta Methods

The local truncation error, or LTE, is the error due to the integration scheme on one step j, i.e.,

$$y(t_j; y_{j-1}) - y_j \quad .$$

This error can be bound on each step of integration [11]. The truncation error of a Runge-Kutta scheme $\phi(t) = x_n + (t - t_n) \sum_{i=1}^{s} b_i k_i(t)$ is obtained by the *order condition* respected by each Runge-Kutta method, and it can be defined by

$$y(t_n; y_{j-1}) - y_j = \frac{h_n^{p+1}}{(p+1)!} \left(f^{(p)}\left(\xi, y(\xi)\right) - \frac{d^{p+1}\phi}{dt^{p+1}}(\eta) \right) \quad .$$

This error is exact for one $\xi \in]t_k, t_{k+1}[$ and one $\eta \in]t_n, t_{n+1}[$. In other terms, the LTE of Runge-Kutta methods can be expressed as the difference between the remainders of the Taylor expansion of the exact solution of Equation (1.1) and of the Taylor expansion of the numerical solution given by equations (1.2) and (1.3).

The main issues are then to bound the terms $\frac{d^{p+1}\phi}{dt^{p+1}}(\eta)$ and $f^{(p)}\left(\xi, x(\xi)\right)$, without knowing ξ and η. Nevertheless, the Picard-Lindelöf operator provides to us the box $y(t, t_j, [y_j]) \subseteq [\tilde{y}_j]$ for all $t \in [t_j, t_{j+1}]$, and so $x(\xi) \in [\tilde{y}_j]$. Obviously, $\eta \in]t_n, t_{n+1}[$, which is well-known.

This approach has given good results, see [2], with $\frac{d^{p+1}\phi}{dt^{p+1}}(\eta)$ computed symbolically. Unfortunately, this computation may take a long time. Moreover, in case of implicit Runge-Kutta method, it is not easy to express ϕ so this approach cannot be applied in that case. We propose an other approach for the computation of the derivatives, based on rooted trees to solve these problems.

2.2.2 Elementary Differentials

To build new Runge-Kutta methods, John Butcher in [5] expressed the Taylor expansions of the exact solution and the numerical solution from *elementary differentials*. These differentials are in fact the Fréchet derivatives of f and a combination of them composed a particular element of the Taylor expansion.

Let $z, f(z) \in \mathbb{R}^m$, the M-th Fréchet derivative of f, see [13] for more details, is defined by

$$f^{(M)}(z)(K_1, K_2, \ldots, K_M) = \sum_{i=1}^{m} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_M=1}^{m} {}^{i}f_{j_1 j_2 \ldots j_M} \, {}^{j_1}K_1 \, {}^{j_2}K_2 \, \ldots \, {}^{j_M}K_M e_i$$

where

$$\begin{aligned} {}^{i}f_{j_1 j_2 \ldots j_M} &= \frac{\partial^M}{\partial^{j_1} z \partial^{j_2} z \, \ldots \, \partial^{j_M} z} \\ K_k &= [{}^{1}K_1, {}^{2}K_2, \ldots, {}^{M}K_M] \in \mathbb{R}^m, \quad \text{for} \quad k = 1, \ldots, M \ . \end{aligned}$$

The notation ${}^{\ell}x$ stands for the ℓ-th component of x.

Example 2.2.1. *Let $m = 2$ with $\dot{y} = y^{(1)} = f(y)$ and $M = 1$ then*

$$\begin{aligned} f^{(1)}(z)(K_1) &= \sum_{i=1}^{2} \sum_{j_1=1}^{2} {}^{i}f_{j_1}({}^{j_1}K_1) e_i \\ &= \begin{bmatrix} {}^{1}f_1({}^{1}K_1) + {}^{1}f_2({}^{2}K_2) \\ {}^{2}f_1({}^{1}K_1) + {}^{2}f_2({}^{2}K_2) \end{bmatrix} \end{aligned}$$

with ${}^{i}f_1 = \frac{\partial^i f}{\partial^1 z}$ *and* ${}^{i}f_2 = \frac{\partial^i f}{\partial^2 z}$ *with* $i = 1, 2$
Replacing z by y and K_1 by $f(y)$ we get

$$f^{(1)}(y)(f(y)) = \begin{bmatrix} {}^{1}f_1({}^{1}f) + {}^{1}f_2({}^{2}f) \\ {}^{2}f_1({}^{1}f) + {}^{2}f_2({}^{2}f) \end{bmatrix} = y^{(2)}$$

Hence the second derivative of y is the first Fréchet derivative of f operating on f. ■

The elementary differentials $F_s : \mathbb{R}^m \to \mathbb{R}^m$ of f and their order are defined recursively by

1. f is the only elementary differential of order 1
2. if F_s, $s = 1, 2, \ldots, M$ are elementary differentials of order r_s then the Fréchet derivative $f^{(M)}(F_1, F_2, \ldots, F_m)$ is an elementary differential of order $1 + \sum_{s=1}^{M} r_s$

Example 2.2.2. *Let see different Fréchet derivatives:*

- *Order 1:* f
- *Order 2:* $f^{(1)}(f)$
- *Order 3:* $f^{(2)}(f, f) \quad f^{(1)}(f^{(1)}(f))$

- *Order 4:* $f^{(3)}(f, f, f) \quad f^{(2)}(f, f^{(1)}(f)) \quad f^{(1)}(f^{(2)}(f, f)) \quad f^{(1)}(f^{(1)}(f^{(1)}(f)))$

In consequence, the second and third time derivative of y associated to Equation (1.1) *are*

$$
\begin{aligned}
y^{(2)} &= f^{(1)}(f), \\
y^{(3)} &= f^{(2)}(f, f) + f^{(1)}(f^{(1)}(f)) \ .
\end{aligned}
$$

■

The great idea of John Butcher in [5] is to connect elementary derivatives to rooted trees. Indeed, an imporant question to answer is to know to a given order n of derivatives, how many elementary differentials do we have to consider. The answer is the same that counting the number of rooted tree with a given number of nodes. Furthermore, for each tree we can associate an elementary differential that is enumerating rooted trees of given order we have formula to express associated elementary derivatives. In Table 2.1 we gives to the fourth first time derivatives of y the number and the form of rooted trees. As in high order, the number of trees of the same form can be more than one due to symmetry, it is important to characterize rooted trees, it is the purpose of Table 2.2. Note that the number of trees increases very quickly, see Example 2.2.3.

Example 2.2.3. *The number of rooted trees up to order* 11*, from left* 11 *to right* 0 *is*

```
1842 719 286 115 48 20 9 4 2 1 1  (total 3047)
```

■

The link between rooted trees and elementary differentials is given in Table 2.3.

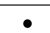
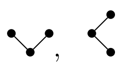
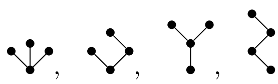
Order	Trees	Number of trees
1		1
2		1
3		2
4		4

Table 2.1: Rooted trees

One of the main results in [5] is let $\dot{y} = f(y)$, $f : \mathbb{R}^m \to \mathbb{R}^m$, then

$$
y^{(q)} = \sum_{r(\tau)=q} \alpha(\tau) F(\tau) \ .
$$

The second main results in [5] is let the a Runge-Kutta defined by a Butcher table then

$$
\frac{d^q}{dh^q} x_n|_{h=0} = \sum_{r(\tau)=q} \alpha(\tau)\gamma(\tau)\psi(\tau)F(\tau)
$$

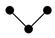
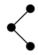
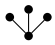
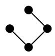
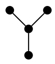
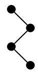
Tree	Name	$r(t)$	$\sigma(t)$	$\gamma(t)$	$\alpha(t)$
•	τ	1	1	1	1
	$[\tau]$	2	1	2	1
	$[\tau^2]$	3	2	3	1
	$[[\tau]]$	3	1	6	1
	$[\tau^3]$	4	6	4	1
	$[\tau[\tau]]$	4	1	8	3
	$[[\tau^2]]$	4	2	12	1
	$[[[\tau]]]$	4	1	24	1

Table 2.2: Rooted trees characteristics

The link between trees and coefficients of Bucther table is given in Table 2.4. Basically, a Runge-Kutta method has order p if $\psi(\tau) = \frac{1}{\gamma(\tau)}$ holds for all trees of order $r(\tau) \leq p$ and does not hold for some tree of order $p+1$.

2.2.3 Local truncation error

From the results presented in Section 2.2.2, we can use an unified approach to express LTE for explicit and implicit Runge-Kutta methods. More precisely, for a Runge-Kutta of order p we have

$$\begin{aligned}&\mathrm{LTE}(t, y(\xi)) := y(t_n; y_{n-1}) - y_n = \\ &\quad \frac{h^{p+1}}{(p+1)!} \sum_{r(\tau)=p+1} \alpha(\tau)\left[1 - \gamma(\tau)\psi(\tau)\right] F(\tau)(y(\xi)) \quad \xi \in [t_n, t_{n+1}]\end{aligned} \tag{2.8}$$

with

- τ is a **rooted tree**
- $F(\tau)$ is the **elementary differential** associated to τ
- $r(\tau)$ is the order of τ (number of nodes)
- $\gamma(\tau)$ is the density
- $\alpha(\tau)$ is the number of equivalent trees
- $\psi(\tau)$

Note that $y(\xi)$ is a particular solution of Equation (1.1) at a time instant ξ. This solution can be over-approximated using Picard-Lindelöf operator as for Taylor series approach.

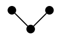
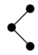
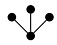
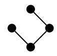
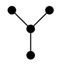
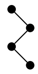
Order	Tree	t	$F(t)$
1	•	τ	f
2		$[\tau]$	$\{f\}$
3		$[\tau^2]$	$\{f^2\}$
		$[_2\tau]_2$	$\{_2f\}_2$
4		$[\tau^3]$	$\{f^3\}$
		$[\tau[\tau]_2$	$\{f\{f\}_2$
		$[[_2\tau^2]_2$	$\{_2f^2\}_2$
		$[_3\tau]_3$	$\{_3f\}_3$

Table 2.3: Rooted trees versus elementary differentials

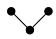
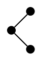
Tree	t	$\psi(\tau)$
•	τ	$\sum_i b_i$
	$[\tau]$	$\sum_i b_i c_i \quad \text{with} \quad c_i = \sum_j a_{ij}$
	$[\tau^2]$	$\sum_i b_i c_i^2$
	$[_2\tau]_2$	$\sum_{ij} b_i a_{ij} c_j$

Table 2.4: Rooted trees versus coefficients of Runge-Kutta methods

2.2.4 A priori enclosure with Runge-Kutta

A novelty of our approach is that we can define a new *a priori* enclosure based on Runge-Kutta methods. We can define a new enclosure such that scheme such that

$$k_i(t, y_j) = f\left(t_j + c_i(t - t_j), y_j + (t - t_j)\sum_{n=1}^{s} a_{in}k_n\right),$$

$$y_{j+1}(t, \xi) = y_j + (t - t_j)\sum_{i=1}^{s} b_i k_i(t, y_j) + \text{LTE(t,y}(\xi)) \ .$$

An inclusion function with $h = t_{j+1} - t_j$ is then defined with

$$y_{j+1}([t_j, t_{j+1}], [R]) = x_j + [0, h]\sum_{i=1}^{s} b_i k_i\left([t_j, t_{j+1}], y_j\right) + \text{LTE}([t_j, t_{j+1}], [R]) \ .$$

Proving the contraction of such scheme, that is

$$[R] \supseteq x_{j+1}\left([t_j, t_{j+1}], [R]\right)$$

can prove the existence and the uniqueness of the solution of Equation (1.1) using Theorem 2.1.1. In the sequel of this chapter we present a computable formula of the LTE for any explicit or implicit Runge-Kutta formula.

Remark. A the time of writing this report, we face a complexity issue in the computation of the local truncation error of Runge-Kutta methods. Until now, this new computation of a priori enclosure is not yet used in our tool.

2.3 Validated Implicit Runge-Kutta Methods

2.3.1 Implicit Runge-Kutta methods

In our tool we implemented the following implicit Runge-Kutta methods.

Implicit Euler The backward Euler method is first order. Unconditionally stable and non-oscillatory for linear diffusion problems.

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

Implicit midpoint The implicit midpoint method is of second order. It is the simplest method in the class of collocation methods known as the Gauss methods. It is a symplectic integrator.

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$$

Radau IIA Radau methods are fully implicit methods (matrix A of such methods can have any structure). Radau methods attain order $2s-1$ for s stages. Radau methods are A-stable, but expensive to implement. Also they can suffer from order reduction. The first order Radau method is similar to backward Euler method.

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

Lobatto IIIC There are three families of Lobatto methods, called IIIA, IIIB and IIIC. These are named after Rehuel Lobatto. All are implicit methods, have order $2s-2$ and they all have $c1=0$ and $cs=1$. Unlike any explicit method, it's possible for these methods to have the order greater than the number of stages. Lobatto lived before the classic fourth-order method was popularized by Runge and Kutta.

$$\begin{array}{c|ccc} 0 & 1/6 & -1/3 & 1/6 \\ 1/2 & 1/6 & 5/12 & -1/12 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

SDIRK4 For the so-called DIRK methods, also known as SDIRK or semi-explicit or semi-implicit methods, A has a lower triangular structure where the constant in diagonal is chosen for stability reasons. In cases in which the solution of integration in the current step is identical with the final stage, it is possible that $a11$ is equal to 0 rather than to the diagonal value, without taking away from the essential nature of a DIRK method.

$$
\begin{array}{c|ccccc}
1/4 & 1/4 & 0 & 0 & 0 & 0 \\
3/4 & 1/2 & 1/4 & 0 & 0 & 0 \\
11/20 & 17/50 & -1/25 & 1/4 & 0 & 0 \\
1/2 & 371/1360 & -137/2720 & 15/544 & 1/4 & 0 \\
1 & 25/24 & -49/48 & 125/16 & -85/12 & 1/4 \\
\hline
 & 25/24 & -49/48 & 125/16 & -85/12 & 1/4
\end{array}
$$

2.3.2 Solving an implicit Runge-Kutta scheme

Using an implicit Runge-Kutta in an integration scheme needs to solve a system of non-linear equations (Section 1.2). In classical numerical methods, it is done with a Newton-like solving procedure which provides generally a good approximation of the k_i. While some interval Newton-like procedure exists for solving systems of non-linear interval equations [16], we propose a lighter appraoch described in the following.

Naturally Contracting Form

First of all, it is interesting to note that each stages of an implicit Runge-Kutta allowing us to compute the intermediate k_i can be used as a contractor [6].

Proposition 2.3.1. *Each stage of an implicit Runge-Kutta is a natural contractor for k_i, $i = 1, \dots, s$.*

Proof. We recall the form of an intermediate stage:

$$k_i = f(y_n + h\sum_{j=1}^{s} a_{i,j}k_j, t_n + c_i h) \ . \tag{2.9}$$

We also know that for all the Runge-Kutta methods

$$c_i = \sum_{j=1}^{s} a_{i,j} \leq 1, \quad \forall i = 1, \dots, s \ .$$

Moreover, by the Picard-Lindelöf operator, we have $k_i \in [\tilde{y}_n]$, $i = 1, \dots, s$, because $t_n + c_i h \leq t_n + h$. Inserting this inside Equation (2.9) leads to

$$\sum_{j=1}^{s} a_{i,j}k_j \in \sum_{j=1}^{s} a_{i,j}[\tilde{y}_n] = c_i[\tilde{y}_n] \ .$$

Then, we can write

$$y_n + h\sum_{j=1}^{s} a_{i,j}k_j \in y_n + h[\tilde{y}_n] \ .$$

By Theorem 2.1.1 and propertie of $[\tilde{y}_n]$ obtained by Picard-Lindelöf operator, f is contracting on $y_n + h[\tilde{y}_n]$, and also on $y_n + h\sum_{j=1}^{s} a_{i,j}k_j$. □

Algorithm

By using the previous proposition, we write the contractor scheme

$$k_i = k_i \cap f\left(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{i,j}k_j\right) \quad .$$

This contractor is used inside a fixpoint to form the following solver for the implicit Runge-Kutta:

Algorithm 1 Solving an implicit RK

Require: $[\tilde{y}_n]$, $a_{i,j}$ of an implicit RK
  $k_i = [\tilde{y}_n]$, $\forall i = 1, \ldots, s$
  while at least one k_i is contracted **do**
    $k_1 = k_1 \cap f(y_n + h\sum_{j=1}^{s} a_{1,j}k_j)$
    $\vdots$
    $k_s = k_s \cap f(y_n + h\sum_{j=1}^{s} a_{s,j}k_j)$
  end while

This algorithm is light and, according to our tests, as efficient than a Newton-like method.

2.4 Complete algorithm

Now, we gather all the previous parts in Algorithm 2 for the simulation of an ODE with Runge-Kutta schemes, explicit or implicit. In this algorithm we have:

- RKe: a non guaranteed explicit Runge-Kutta method (RK4 for example)
- RKx: a guaranteed explicit, by an affine evaluation, or implicit, with Algorithm 1, Runge-Kutta method (RK4 or LC3 for examples)
- LTE: the local truncature error associated to RKx (see Section 2.2.3)
- PL: the Picard-Lindelöf operator based on an integration scheme (rectangular, Taylor or Runge-Kutta, see Section 2.1.1)

Algorithm 2 Simulation algorithm

Require: $f, y_0, t_{end}, h, atol, rtol$

$t_n = t_0$, $y_n = y_0$, $factor = 1$
while $(t_n < t_{end})$ **do**
  $h = h * factor$
  $h = min(h, t_{end} - t_n)$
  Loop:
  Initialize $\tilde{y}_0 = y_n \cup RKe(y_n, h)$
  Inflate $\tilde{y}_0$ by 10%
  Compute $\tilde{y}_1 = PL(\tilde{y}_0)$
  while $(\tilde{y}_1 \not\subset \tilde{y}_0)$ and $(iter < size(f) + 1)$ **do**
    $\tilde{y}_0 = \tilde{y}_1$
    Compute $\tilde{y}_1$ with $PL(\tilde{y}_0)$
  end while
  if $(\tilde{y}_1 \subset \tilde{y}_0)$ **then**
    while $(||\tilde{y}_1 - \tilde{y}_0|| < 1e - 18)$ **do**
      $\tilde{y}_0 = \tilde{y}_1$
      $\tilde{y}_1 = \tilde{y}_1 \cap PL(\tilde{y}_0)$
    end while
    Compute $lte = LTE(\tilde{y}_1)$
    $test = ||lte||/(atol + ||\tilde{y}_1|| * rtol)$
    if $(test \leq 1)$ or $(h < h_{min})$ **then**
      $factor = min(1.8, max(0.4, 0.9 * (1/test)^{1/p}))$
    else
      $h = max(h_{min}, h/2)$
      Goto Loop
    end if
  else
    $h = max(h_{min}, h/2)$
    Goto Loop
  end if
  Compute $y_{n+1} = RKx(y_n, h) + lte$
  $t_n = t_n + h$
end while

Chapter 3

Experimentation

3.1 Vericomp benchmark

3.1.1 Disclaimer

This section reports the results of the solution of various problems coming from the VERICOMP benchmark[1]. For each problem, different validated methods of Runge-Kutta of order 4 are applied among: the classical formula of Runge-Kutta (explicit), the Lobatto-3a formula (implicit) and the Lobatto-3c formula (implicit). Moreover, an homemade version of Taylor series, limited to order 5 and using affine arithmetic, is also applied on each problem.

For each problem, we report the following metrics:

- c5t: user time taken to simulate the problem for 1 second.
- c5w: the final diameter of the solution (infinity norm is used).
- c6t: the time to breakdown the method with a maximal limit of 10 seconds.
- c6w: the diameter of the solution a the breakdown time.

After the results listing, a discussion is done.

3.1.2 Results

[1] http://vericomp.inf.uni-due.de

Table 3.1: Simulation results of Problem 1

Problems	Methods	c5t	c5w	c6t	c6w
system_1	TAYLOR4 (TP8)	0.040	5.8147	10.000	9.6379e+08
system_1	TAYLOR4 (TP9)	0.050	5.8147	10.000	9.6379e+08
system_1	TAYLOR4 (TP10)	0.060	5.8147	10.000	9.6379e+08
system_1	TAYLOR4 (TP11)	0.110	5.8147	10.000	9.6379e+08
system_1	TAYLOR4 (TP12)	0.160	5.8147	10.000	9.6379e+08
system_1	TAYLOR4 (TP13)	0.220	5.8147	10.000	9.6379e+08
system_1	TAYLOR4 (TP14)	0.270	5.8147	10.000	9.6379e+08
system_1	RK4 (TP8)	0.030	5.8147	10.000	9.6379e+08
system_1	RK4 (TP9)	0.020	5.8147	10.000	9.6379e+08
system_1	RK4 (TP10)	0.040	5.8147	10.000	9.6379e+08
system_1	RK4 (TP11)	0.080	5.8147	10.000	9.6379e+08
system_1	RK4 (TP12)	0.100	5.8147	10.000	9.6379e+08
system_1	RK4 (TP13)	0.170	5.8147	10.000	9.6379e+08
system_1	RK4 (TP14)	0.230	5.8147	10.000	9.6379e+08
system_1	LA3 (TP8)	0.020	5.8323	10.000	9.8667e+08
system_1	LA3 (TP9)	0.040	5.8253	10.000	9.774e+08
system_1	LA3 (TP10)	0.050	5.8212	10.000	9.7205e+08
system_1	LA3 (TP11)	0.070	5.8187	10.000	9.6888e+08
system_1	LA3 (TP12)	0.100	5.8172	10.000	9.6695e+08
system_1	LA3 (TP13)	0.150	5.8163	10.000	9.6577e+08
system_1	LA3 (TP14)	0.200	5.8157	10.000	9.6503e+08
system_1	LC3 (TP8)	0.020	5.8753	10.000	1.046e+09
system_1	LC3 (TP9)	0.040	5.8521	10.000	1.013e+09
system_1	LC3 (TP10)	0.050	5.8378	10.000	9.9387e+08
system_1	LC3 (TP11)	0.080	5.8291	10.000	9.8239e+08
system_1	LC3 (TP12)	0.120	5.8237	10.000	9.7538e+08
system_1	LC3 (TP13)	0.160	5.8204	10.000	9.7105e+08
system_1	LC3 (TP14)	0.220	5.8183	10.000	9.6835e+08
system_1	Riot (02, 1e-11)	0m1.973s	10.059	10.000	1.2112e+10
system_1	Riot (03, 1e-11)	0m2.043s	10.059	10.000	1.2111e+10
system_1	Riot (04, 1e-11)	0m2.102s	10.059	10.000	1.2111e+10
system_1	Riot (05, 1e-11)	0m2.120s	10.059	10.000	1.2111e+10
system_1	Riot (06, 1e-11)	0m2.186s	10.059	10.000	1.2111e+10
system_1	Riot (07, 1e-11)	0m2.270s	10.059	10.000	1.2111e+10
system_1	Riot (09, 1e-11)	0m23.421s	10.059	-0.000	1.2111e+10
system_1	Riot (10, 1e-11)	0m2.524s	10.059	10.000	1.2111e+10
system_1	Riot (11, 1e-11)	0m24.797s	10.059	-0.000	1.2111e+10
system_1	Riot (15, 1e-11)	0m2.874s	10.059	10.000	1.2111e+10
system_1	Riot (18, 1e-11)	0m30.750s	10.059	-0.000	1.2111e+10
system_1	Valencia-IVP (0.00025)	0m1.690s	4.6755	3.469	999.98
system_1	Valencia-IVP (0.0025)	0m0.157s	4.7177	3.460	999.19
system_1	Valencia-IVP (0.025)	0m0.022s	5.1586	3.375	995.68
system_1	Valencia-IVP (0.25)	0m0.010s	14.082	2.250	516.32
system_1	VNODE-LP (12, 1e-1)	0m0.005s	6.2022	10.000	1.6902e+09
system_1	VNODE-LP (13, 1e-1)	0m0.008s	6.9272	10.000	1.7303e+09
system_1	VNODE-LP (14, 1e-1)	0m0.005s	5.4997	10.000	1.0761e+09
system_1	VNODE-LP (15, 1e-14,1e-14)	0m0.006s	6.6718	10.000	1.2705e+09
system_1	VNODE-LP (20, 1e-14,1e-14)	0m0.002s	6.8406	10.000	1.9442e+09
system_1	VNODE-LP (25, 1e-14,1e-14)	0m0.006s	4.6708	10.000	4.8518e+08

Table 3.2: Simulation results of Problem 2

Problems	Methods	c5t	c5w	c6t	c6w
system_2	TAYLOR4 (TP8)	0.840	0.23254	10.000	0.00040944
system_2	TAYLOR4 (TP9)	1.160	0.23254	10.000	0.00040873
system_2	TAYLOR4 (TP10)	1.660	0.23254	10.000	0.00040865
system_2	TAYLOR4 (TP11)	2.530	0.23254	10.000	0.00040861
system_2	TAYLOR4 (TP12)	3.930	0.23254	10.000	0.0004086
system_2	TAYLOR4 (TP13)	6.170	0.23254	10.000	0.0004086
system_2	TAYLOR4 (TP14)	9.770	0.23254	10.000	0.0004086
system_2	RK4 (TP8)	0.640	0.23255	10.000	0.00040939
system_2	RK4 (TP9)	0.890	0.23254	10.000	0.00040875
system_2	RK4 (TP10)	1.360	0.23254	10.000	0.00040866
system_2	RK4 (TP11)	2.100	0.23254	10.000	0.00040861
system_2	RK4 (TP12)	3.240	0.23254	10.000	0.0004086
system_2	RK4 (TP13)	5.060	0.23254	10.000	0.0004086
system_2	RK4 (TP14)	8.020	0.23254	10.000	0.0004086
system_2	LA3 (TP8)	0.500	0.26111	10.000	0.12375
system_2	LA3 (TP9)	0.730	0.25154	10.000	0.02491
system_2	LA3 (TP10)	1.040	0.24447	10.000	0.010686
system_2	LA3 (TP11)	1.600	0.24009	10.000	0.0074653
system_2	LA3 (TP12)	2.440	0.23734	10.000	0.0039061
system_2	LA3 (TP13)	3.850	0.23554	10.000	0.0074742
system_2	LA3 (TP14)	6.100	0.23442	10.000	0.002063
system_2	LC3 (TP8)	0.480	0.2641	10.000	0.14326
system_2	LC3 (TP9)	0.790	0.25281	10.000	0.014229
system_2	LC3 (TP10)	1.130	0.24513	10.000	0.0094465
system_2	LC3 (TP11)	1.730	0.24048	10.000	0.011631
system_2	LC3 (TP12)	2.700	0.23746	10.000	0.0080097
system_2	LC3 (TP13)	4.370	0.23561	10.000	0.0078812
system_2	LC3 (TP14)	6.700	0.2345	10.000	0.0017907
system_2	Riot (03, 1e-11)	35m43.710s	0.24697	0.000	0
system_2	Riot (05, 1e-11)	0m0.734s	0.23588	10.000	3.4736e+08
system_2	Riot (06, 1e-11)	0m0.342s	0.2417	-0.000	0.2417
system_2	Riot (07, 1e-11)	0m9.268s	0.2417	-0.000	0.42672
system_2	Riot (10, 1e-11)	0m0.297s	0.2417	10.000	0.43053
system_2	Riot (15, 1e-11)	0m0.438s	0.2417	10.000	0.69667
system_2	Valencia-IVP (0.00025)	0m3.878s	6.372	2.668	999.81
system_2	Valencia-IVP (0.0025)	0m0.382s	6.4647	2.655	992.41
system_2	Valencia-IVP (0.025)	0m0.046s	7.5087	2.550	986.22
system_2	VNODE-LP (13, 1e-1)	0m0.009s	0.23255	10.000	0.013215
system_2	VNODE-LP (15, 1e-14,1e-14)	0m0.004s	0.23254	10.000	0.013205
system_2	VNODE-LP (20, 1e-14,1e-14)	0m0.003s	0.23254	10.000	0.013205
system_2	VNODE-LP (25, 1e-14,1e-14)	0m0.004s	0.23254	10.000	0.013205

Table 3.3: Simulation results of Problem 3

Problems	Methods	c5t	c5w	c6t	c6w
system_3	TAYLOR4 (TP8)	0.060	0.48874	10.000	0.068846
system_3	TAYLOR4 (TP9)	0.100	0.48163	10.000	0.065318
system_3	TAYLOR4 (TP10)	0.150	0.47729	10.000	0.063275
system_3	TAYLOR4 (TP11)	0.200	0.47456	10.000	0.062043
system_3	TAYLOR4 (TP12)	0.280	0.47286	10.000	0.06129
system_3	TAYLOR4 (TP13)	0.400	0.47179	10.000	0.060825
system_3	TAYLOR4 (TP14)	0.000	1	0.000	1
system_3	RK4 (TP8)	0.020	0.47001	10.000	0.060058
system_3	RK4 (TP9)	0.050	0.46999	10.000	0.060051
system_3	RK4 (TP10)	0.090	0.46998	10.000	0.060047
system_3	RK4 (TP11)	0.070	0.46998	10.000	0.060046
system_3	RK4 (TP12)	0.160	0.46998	10.000	0.060046
system_3	RK4 (TP13)	0.220	0.46998	10.000	0.060046
system_3	RK4 (TP14)	0.310	0.46998	10.000	0.060045
system_3	LA3 (TP8)	0.040	0.4851	10.000	0.068211
system_3	LA3 (TP9)	0.050	0.47954	10.000	0.064964
system_3	LA3 (TP10)	0.070	0.476	10.000	0.063061
system_3	LA3 (TP11)	0.110	0.47374	10.000	0.061905
system_3	LA3 (TP12)	0.150	0.47235	10.000	0.061203
system_3	LA3 (TP13)	0.200	0.47147	10.000	0.060771
system_3	LA3 (TP14)	0.280	0.47092	10.000	0.0605
system_3	LC3 (TP8)	0.040	0.49094	10.000	0.071732
system_3	LC3 (TP9)	0.060	0.4831	10.000	0.066956
system_3	LC3 (TP10)	0.080	0.47815	10.000	0.064212
system_3	LC3 (TP11)	0.100	0.4751	10.000	0.062606
system_3	LC3 (TP12)	0.150	0.47319	10.000	0.061632
system_3	LC3 (TP13)	0.210	0.472	10.000	0.061037
system_3	LC3 (TP14)	0.300	0.47125	10.000	0.060666
system_3	Riot (05, 1e-11)	0m3.197s	0.44827	10.000	0.13094
system_3	Riot (10, 1e-11)	0m12.763s	0.44389	10.000	0.057421
system_3	Riot (15, 1e-11)	0m40.607s	0.44387	10.000	0.055362
system_3	Valencia-IVP (0.00025)	0m2.780s	2.8979	1.191	3.7768
system_3	Valencia-IVP (0.0025)	0m0.282s	2.9052	1.175	3.694
system_3	Valencia-IVP (0.025)	0m0.042s	2.9872	1.300	5.8585
system_3	VNODE-LP (15, 1e-14,1e-14)	0m0.009s	0.88761	6.361	151.77
system_3	VNODE-LP (20, 1e-14,1e-14)	0m0.007s	0.98714	3.815	218.19
system_3	VNODE-LP (25, 1e-14,1e-14)	0m0.009s	1.1388	2.597	270.43

Table 3.4: Simulation results of Problem 4

Problems	Methods	c5t	c5w	c6t	c6w
system_4	TAYLOR4 (TP8)	0.390	0.070037	9.074	85948
system_4	TAYLOR4 (TP9)	0.580	0.070009	9.320	62850
system_4	TAYLOR4 (TP10)	0.830	0.06993	8.853	85022
system_4	TAYLOR4 (TP11)	1.310	0.069876	7.474	67079
system_4	TAYLOR4 (TP12)	2.050	0.069864	8.570	70345
system_4	TAYLOR4 (TP13)	3.190	0.069834	8.542	64978
system_4	TAYLOR4 (TP14)	4.950	0.069829	7.852	73737
system_4	RK4 (TP8)	0.240	0.069785	9.617	78366
system_4	RK4 (TP9)	0.320	0.069787	9.191	62143
system_4	RK4 (TP10)	0.460	0.069801	8.962	77711
system_4	RK4 (TP11)	0.670	0.069802	9.178	81171
system_4	RK4 (TP12)	1.020	0.069819	8.626	64394
system_4	RK4 (TP13)	1.560	0.069798	8.298	82798
system_4	RK4 (TP14)	2.370	0.06983	8.973	65817
system_4	LA3 (TP8)	0.230	0.07624	5.512	83953
system_4	LA3 (TP9)	0.300	0.073963	5.626	82664
system_4	LA3 (TP10)	0.390	0.072495	5.722	86373
system_4	LA3 (TP11)	0.600	0.071545	5.928	60730
system_4	LA3 (TP12)	0.900	0.070933	5.969	81847
system_4	LA3 (TP13)	1.360	0.07052	6.916	79535
system_4	LA3 (TP14)	2.130	0.070275	5.983	63808
system_4	LC3 (TP8)	0.200	0.077751	5.516	97508
system_4	LC3 (TP9)	0.280	0.074792	5.726	88836
system_4	LC3 (TP10)	0.380	0.073062	5.658	74922
system_4	LC3 (TP11)	0.570	0.071849	5.816	95737
system_4	LC3 (TP12)	0.790	0.071113	6.249	82501
system_4	LC3 (TP13)	1.290	0.070648	6.607	67028
system_4	LC3 (TP14)	1.980	0.070313	7.398	68298
system_4	Riot (05, 1e-11)	0m37.601s	0.06757	0.000	0
system_4	Riot (10, 1e-11)	0m3.171s	0.06757	10.000	0.18331
system_4	Riot (15, 1e-11)	0m9.102s	0.06757	10.000	0.30021
system_4	Valencia-IVP (0.00025)	0m5.231s	10.971	1.140	910.02
system_4	Valencia-IVP (0.0025)	0m0.679s	13.023	1.105	154.09
system_4	Valencia-IVP (0.025)	0m0.063s	3.2425	0.600	3.2425
system_4	VNODE-LP (15, 1e-14,1e-14)	0m0.012s	0.073974	5.055	10185
system_4	VNODE-LP (20, 1e-14,1e-14)	0m0.014s	0.075043	4.977	21260
system_4	VNODE-LP (25, 1e-14,1e-14)	0m0.012s	0.076265	4.913	30511

Table 3.5: Simulation results of Problem 7

Problems	Methods	c5t	c5w	c6t	c6w
system_7	TAYLOR4 (TP8)	0.000	5.4885e-09	10.000	5.2398e-09
system_7	TAYLOR4 (TP9)	0.000	5.6577e-10	10.000	5.4977e-10
system_7	TAYLOR4 (TP10)	0.010	5.8386e-11	10.000	5.3574e-11
system_7	TAYLOR4 (TP11)	0.010	5.9324e-12	10.000	5.5432e-12
system_7	TAYLOR4 (TP12)	0.020	6.4071e-13	10.000	5.8407e-13
system_7	TAYLOR4 (TP13)	0.030	1.3856e-13	10.000	5.8756e-14
system_7	TAYLOR4 (TP14)	0.050	1.2923e-13	10.000	5.9005e-15
system_7	RK4 (TP8)	0.000	6.9766e-09	10.000	6.05e-09
system_7	RK4 (TP9)	0.000	7.3286e-10	10.000	6.93e-10
system_7	RK4 (TP10)	0.000	7.5791e-11	10.000	7.3548e-11
system_7	RK4 (TP11)	0.010	7.7225e-12	10.000	7.2765e-12
system_7	RK4 (TP12)	0.010	7.8859e-13	10.000	7.4488e-13
system_7	RK4 (TP13)	0.020	1.0791e-13	10.000	7.5389e-14
system_7	RK4 (TP14)	0.030	5.6066e-14	10.000	7.6827e-15
system_7	LA3 (TP8)	0.000	5.199e-09	10.000	5.0889e-09
system_7	LA3 (TP9)	0.000	5.4665e-10	10.000	4.8474e-10
system_7	LA3 (TP10)	0.000	5.792e-11	10.000	5.61e-11
system_7	LA3 (TP11)	0.000	5.7909e-12	10.000	5.4252e-12
system_7	LA3 (TP12)	0.010	6.0674e-13	10.000	5.8379e-13
system_7	LA3 (TP13)	0.020	8.2267e-14	10.000	5.7728e-14
system_7	LA3 (TP14)	0.030	4.13e-14	10.000	5.8007e-15
system_7	LC3 (TP8)	0.000	5.362e-09	10.000	5.0148e-09
system_7	LC3 (TP9)	0.000	5.611e-10	10.000	5.5022e-10
system_7	LC3 (TP10)	0.000	5.8373e-11	10.000	5.2443e-11
system_7	LC3 (TP11)	0.010	5.8898e-12	10.000	5.6076e-12
system_7	LC3 (TP12)	0.010	6.0607e-13	10.000	5.6303e-13
system_7	LC3 (TP13)	0.020	8.4266e-14	10.000	5.7818e-14
system_7	LC3 (TP14)	0.040	4.4076e-14	10.000	5.8898e-15
system_7	Riot (05, 1e-11)	0m0.073s	1.8582e-11	1.000	1.8582e-11
system_7	Riot (10, 1e-11)	0m0.106s	1.199e-14	10.000	1.061e-12
system_7	Riot (15, 1e-11)	0m0.189s	1.7097e-14	0.000	0
system_7	Valencia-IVP (0.00025)	0m1.491s	0.00029389	10.000	2.7571
system_7	Valencia-IVP (0.0025)	0m0.132s	0.0029465	10.000	27.915
system_7	Valencia-IVP (0.025)	0m0.016s	0.030251	10.000	316.61
system_7	VNODE-LP (15, 1e-14,1e-14)	0m0.005s	1.6653e-16	10.000	4.6756e-19
system_7	VNODE-LP (20, 1e-14,1e-14)	0m0.003s	2.7756e-16	10.000	4.0658e-19
system_7	VNODE-LP (25, 1e-14,1e-14)	0m0.007s	1.6653e-16	10.000	2.9138e-19

Table 3.6: Simulation results of Problem 8

Problems	Methods	c5t	c5w	c6t	c6w
system_8	TAYLOR4 (TP8)	0.630	6.2392e-08	10.000	2.6753e-07
system_8	TAYLOR4 (TP9)	0.900	6.8627e-09	10.000	7.328e-08
system_8	TAYLOR4 (TP10)	1.340	7.1243e-10	10.000	1.0083e-08
system_8	TAYLOR4 (TP11)	2.100	7.4399e-11	10.000	1.343e-09
system_8	TAYLOR4 (TP12)	3.380	7.6358e-12	10.000	1.7369e-10
system_8	TAYLOR4 (TP13)	5.260	1.0223e-12	10.000	2.2065e-11
system_8	TAYLOR4 (TP14)	8.140	5.7332e-13	10.000	3.1279e-12
system_8	RK4 (TP8)	0.510	8.0492e-08	10.000	4.8703e-07
system_8	RK4 (TP9)	0.760	8.8927e-09	10.000	9.2522e-08
system_8	RK4 (TP10)	1.140	9.2505e-10	10.000	1.1545e-08
system_8	RK4 (TP11)	1.810	9.6979e-11	10.000	1.3574e-09
system_8	RK4 (TP12)	2.810	9.8163e-12	10.000	1.8886e-10
system_8	RK4 (TP13)	4.420	1.0665e-12	10.000	2.5177e-11
system_8	RK4 (TP14)	6.910	2.8466e-13	10.000	3.3497e-12
system_8	LA3 (TP8)	0.410	6.3861e-08	10.000	1.9173e-06
system_8	LA3 (TP9)	0.590	6.8303e-09	10.000	2.1645e-07
system_8	LA3 (TP10)	0.870	7.1757e-10	10.000	2.0083e-08
system_8	LA3 (TP11)	1.320	7.3416e-11	10.000	1.9068e-09
system_8	LA3 (TP12)	2.100	7.5049e-12	10.000	2.0342e-10
system_8	LA3 (TP13)	3.280	8.1635e-13	10.000	2.2924e-11
system_8	LA3 (TP14)	5.150	2.1383e-13	10.000	2.7943e-12
system_8	LC3 (TP8)	0.430	6.3703e-08	10.000	3.2935e-06
system_8	LC3 (TP9)	0.630	6.9067e-09	10.000	2.6899e-07
system_8	LC3 (TP10)	0.950	7.17e-10	10.000	2.3447e-08
system_8	LC3 (TP11)	1.460	7.3931e-11	10.000	2.107e-09
system_8	LC3 (TP12)	2.300	7.5591e-12	10.000	2.1838e-10
system_8	LC3 (TP13)	3.630	8.2462e-13	10.000	2.4242e-11
system_8	LC3 (TP14)	5.610	2.2604e-13	10.000	2.9331e-12
system_8	Riot (05, 1e-11)	0m0.296s	9.0226e-11	10.000	8.8003e-05
system_8	Riot (10, 1e-11)	0m0.207s	1.299e-14	10.000	1.3371e-10
system_8	Riot (15, 1e-11)	0m0.253s	1.8319e-14	10.000	8.3085e-15
system_8	Valencia-IVP (0.00025)	0m4.114s	0.0026387	5.269	999.48
system_8	Valencia-IVP (0.0025)	0m0.402s	0.026723	4.485	996.18
system_8	Valencia-IVP (0.025)	0m0.048s	0.30489	3.575	963.25
system_8	VNODE-LP (15, 1e-14,1e-14)	0m0.006s	2.1094e-15	10.000	2.3327e-16
system_8	VNODE-LP (20, 1e-14,1e-14)	0m0.005s	1.1102e-15	10.000	1.0988e-16
system_8	VNODE-LP (25, 1e-14,1e-14)	0m0.003s	8.8818e-16	10.000	8.5489e-17

Table 3.7: Simulation results of Problem 10

Problems	Methods	c5t	c5w	c6t	c6w
system_10	TAYLOR4 (TP8)	0.010	2.1154e-08	10.000	2.5347e-08
system_10	TAYLOR4 (TP9)	0.020	2.2594e-09	10.000	2.6471e-09
system_10	TAYLOR4 (TP10)	0.030	2.3767e-10	10.000	2.7776e-10
system_10	TAYLOR4 (TP11)	0.050	2.4321e-11	10.000	2.8726e-11
system_10	TAYLOR4 (TP12)	0.090	2.5682e-12	10.000	2.9864e-12
system_10	TAYLOR4 (TP13)	0.140	3.8791e-13	10.000	4.0901e-13
system_10	TAYLOR4 (TP14)	0.230	2.4336e-13	10.000	2.105e-13
system_10	RK4 (TP8)	0.000	4.9113e-08	10.000	6.3159e-08
system_10	RK4 (TP9)	0.010	5.258e-09	10.000	6.5608e-09
system_10	RK4 (TP10)	0.010	5.1864e-10	10.000	6.569e-10
system_10	RK4 (TP11)	0.010	4.895e-11	10.000	6.0076e-11
system_10	RK4 (TP12)	0.020	4.5011e-12	10.000	5.4561e-12
system_10	RK4 (TP13)	0.040	4.3721e-13	10.000	5.1514e-13
system_10	RK4 (TP14)	0.060	7.1054e-14	10.000	7.272e-14
system_10	LA3 (TP8)	0.000	1.9603e-08	10.000	2.3468e-08
system_10	LA3 (TP9)	0.010	2.1781e-09	10.000	2.5435e-09
system_10	LA3 (TP10)	0.010	2.278e-10	10.000	2.705e-10
system_10	LA3 (TP11)	0.020	2.4233e-11	10.000	2.8082e-11
system_10	LA3 (TP12)	0.040	2.478e-12	10.000	2.9076e-12
system_10	LA3 (TP13)	0.060	2.7711e-13	10.000	3.1497e-13
system_10	LA3 (TP14)	0.090	6.8168e-14	10.000	6.5503e-14
system_10	LC3 (TP8)	0.000	2.6295e-08	10.000	3.4923e-08
system_10	LC3 (TP9)	0.010	3.0011e-09	10.000	3.521e-09
system_10	LC3 (TP10)	0.010	2.8753e-10	10.000	3.508e-10
system_10	LC3 (TP11)	0.020	2.8342e-11	10.000	3.4456e-11
system_10	LC3 (TP12)	0.030	2.7964e-12	10.000	3.3326e-12
system_10	LC3 (TP13)	0.050	2.9554e-13	10.000	3.4062e-13
system_10	LC3 (TP14)	0.070	6.0396e-14	10.000	5.9508e-14
system_10	Riot (05, 1e-11)	0m0.148s	3.2904e-11	10.000	4.4509e-11
system_10	Riot (10, 1e-11)	0m0.154s	2.276e-14	10.000	2.4266e-12
system_10	Riot (15, 1e-11)	0m0.235s	2.1427e-14	10.000	2.0872e-14
system_10	Valencia-IVP (0.00025)	0m1.280s	0.00015473	10.000	0.0022794
system_10	Valencia-IVP (0.0025)	0m0.111s	0.0015521	10.000	0.022876
system_10	Valencia-IVP (0.025)	0m0.014s	0.016012	10.000	0.23397
system_10	VNODE-LP (15, 1e-14,1e-14)	0m0.007s	1.6653e-15	10.000	1.4988e-15
system_10	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	1.2212e-15	10.000	1.1102e-15
system_10	VNODE-LP (25, 1e-14,1e-14)	0m0.004s	9.992e-16	10.000	1.1102e-15

Table 3.8: Simulation results of Problem 11

Problems	Methods	c5t	c5w	c6t	c6w
system_11	TAYLOR4 (TP8)	0.260	1.5364e-07	10.000	0.00011249
system_11	TAYLOR4 (TP9)	0.380	1.6536e-08	10.000	0.0001409
system_11	TAYLOR4 (TP10)	0.600	1.6928e-09	10.000	6.8266e-05
system_11	TAYLOR4 (TP11)	0.950	1.7436e-10	10.000	7.4563e-06
system_11	TAYLOR4 (TP12)	1.490	1.8469e-11	10.000	7.9824e-07
system_11	TAYLOR4 (TP13)	2.280	2.9283e-12	10.000	1.0116e-07
system_11	TAYLOR4 (TP14)	3.610	1.9837e-12	10.000	3.6623e-08
system_11	RK4 (TP8)	0.160	1.4924e-07	10.000	9.9104e-05
system_11	RK4 (TP9)	0.240	1.6173e-08	10.000	0.00010979
system_11	RK4 (TP10)	0.370	1.6512e-09	10.000	3.7122e-05
system_11	RK4 (TP11)	0.560	1.6831e-10	10.000	4.4121e-06
system_11	RK4 (TP12)	0.910	1.7229e-11	10.000	4.5013e-07
system_11	RK4 (TP13)	1.390	2.037e-12	10.000	5.0184e-08
system_11	RK4 (TP14)	2.130	6.8701e-13	10.000	1.2723e-08
system_11	LA3 (TP8)	0.150	1.3016e-07	10.000	0.00027567
system_11	LA3 (TP9)	0.210	1.3811e-08	10.000	5.0329e-05
system_11	LA3 (TP10)	0.320	1.5537e-09	10.000	3.3377e-05
system_11	LA3 (TP11)	0.500	1.6718e-10	10.000	4.3944e-06
system_11	LA3 (TP12)	0.790	1.5877e-11	10.000	4.2412e-07
system_11	LA3 (TP13)	1.240	1.8541e-12	10.000	4.6319e-08
system_11	LA3 (TP14)	1.890	5.9908e-13	10.000	1.1111e-08
system_11	LC3 (TP8)	0.140	1.2294e-07	10.000	0.00022257
system_11	LC3 (TP9)	0.200	1.2053e-08	10.000	5.6171e-05
system_11	LC3 (TP10)	0.310	1.1696e-09	10.000	4.3e-05
system_11	LC3 (TP11)	0.470	1.1365e-10	10.000	4.8341e-06
system_11	LC3 (TP12)	0.740	1.1288e-11	10.000	4.7542e-07
system_11	LC3 (TP13)	1.160	1.3585e-12	10.000	5.0885e-08
system_11	LC3 (TP14)	1.770	5.1648e-13	10.000	1.1353e-08
system_11	Riot (05, 1e-11)	0m0.593s	3.3225e-10	10.000	3.6967e-08
system_11	Riot (10, 1e-11)	0m0.299s	6.505e-12	10.000	3.2633e-09
system_11	Riot (15, 1e-11)	0m0.436s	3.5971e-14	10.000	5.0365e-10
system_11	Valencia-IVP (0.00025)	0m1.732s	0.011564	4.825	986.14
system_11	Valencia-IVP (0.0025)	0m0.252s	0.11774	2.902	1.5629
system_11	Valencia-IVP (0.025)	0m0.094s	1.5234	1.050	1.7124
system_11	VNODE-LP (15, 1e-14,1e-14)	0m0.015s	1.3101e-14	10.000	2.7778e-12
system_11	VNODE-LP (20, 1e-14,1e-14)	0m0.013s	9.1038e-15	10.000	1.9398e-12
system_11	VNODE-LP (25, 1e-14,1e-14)	0m0.011s	6.8834e-15	10.000	2.2919e-12

Table 3.9: Simulation results of Problem 13

Problems	Methods	c5t	c5w	c6t	c6w
system_13	TAYLOR4 (TP8)	0.100	6.0623e-08	10.000	1.1392e-05
system_13	TAYLOR4 (TP9)	0.160	6.3074e-09	10.000	6.6022e-06
system_13	TAYLOR4 (TP10)	0.260	6.6362e-10	10.000	6.3809e-06
system_13	TAYLOR4 (TP11)	0.410	6.9288e-11	10.000	5.969e-06
system_13	TAYLOR4 (TP12)	0.630	8.6562e-12	10.000	5.8669e-06
system_13	TAYLOR4 (TP13)	0.990	3.336e-12	10.000	9.4036e-06
system_13	TAYLOR4 (TP14)	1.570	4.281e-12	10.000	2.4348e-05
system_13	RK4 (TP8)	0.070	7.7716e-08	10.000	1.1601e-05
system_13	RK4 (TP9)	0.120	8.0154e-09	10.000	2.9548e-06
system_13	RK4 (TP10)	0.180	8.5062e-10	10.000	3.2373e-06
system_13	RK4 (TP11)	0.290	8.8824e-11	10.000	4.3262e-06
system_13	RK4 (TP12)	0.440	9.7406e-12	10.000	5.0541e-06
system_13	RK4 (TP13)	0.690	1.9238e-12	10.000	4.0228e-06
system_13	RK4 (TP14)	1.100	1.6866e-12	10.000	1.052e-05
system_13	LA3 (TP8)	0.060	5.6343e-08	10.000	2.5172e-05
system_13	LA3 (TP9)	0.090	6.0874e-09	10.000	1.0084e-05
system_13	LA3 (TP10)	0.140	6.5448e-10	10.000	5.8655e-06
system_13	LA3 (TP11)	0.220	6.8319e-11	10.000	5.7753e-06
system_13	LA3 (TP12)	0.350	7.3896e-12	10.000	4.6608e-06
system_13	LA3 (TP13)	0.530	1.4424e-12	10.000	3.0252e-06
system_13	LA3 (TP14)	0.830	1.2559e-12	10.000	3.7585e-06
system_13	LC3 (TP8)	0.060	5.7775e-08	10.000	3.7157e-05
system_13	LC3 (TP9)	0.100	6.2167e-09	10.000	1.62e-05
system_13	LC3 (TP10)	0.150	6.5544e-10	10.000	7.0966e-06
system_13	LC3 (TP11)	0.250	6.8894e-11	10.000	7.2423e-06
system_13	LC3 (TP12)	0.390	7.4376e-12	10.000	6.7877e-06
system_13	LC3 (TP13)	0.590	1.5206e-12	10.000	1.347e-05
system_13	LC3 (TP14)	0.920	1.3589e-12	10.000	9.5534e-06
system_13	Riot (05, 1e-11)	0m0.182s	2.3274e-10	10.000	2.2851e-09
system_13	Riot (10, 1e-11)	0m0.119s	3.5083e-14	10.000	1.236e-10
system_13	Riot (15, 1e-11)	0m0.153s	1.1813e-13	10.000	5.4101e-12
system_13	Valencia-IVP (0.00025)	0m1.141s	0.0044966	7.088	999.86
system_13	Valencia-IVP (0.0025)	0m0.099s	0.045269	5.923	999.03
system_13	Valencia-IVP (0.025)	0m0.017s	0.48459	4.650	990.84
system_13	VNODE-LP (15, 1e-14,1e-14)	0m0.004s	6.2172e-15	10.000	2.4802e-13
system_13	VNODE-LP (20, 1e-14,1e-14)	0m0.005s	3.9968e-15	10.000	2.3404e-13
system_13	VNODE-LP (25, 1e-14,1e-14)	0m0.005s	1.7764e-15	10.000	1.1502e-13

Table 3.10: Simulation results of Problem 14

Problems	Methods	c5t	c5w	c6t	c6w
system_14	TAYLOR4 (TP8)	0.180	6.7792e-07	10.000	3.6732e+06
system_14	TAYLOR4 (TP9)	0.260	6.9365e-08	10.000	3.7168e+05
system_14	TAYLOR4 (TP10)	0.420	6.9965e-09	10.000	37470
system_14	TAYLOR4 (TP11)	0.640	7.1965e-10	10.000	3831
system_14	TAYLOR4 (TP12)	1.000	9.1987e-11	10.000	487.39
system_14	TAYLOR4 (TP13)	1.570	4.0941e-11	10.000	212.59
system_14	TAYLOR4 (TP14)	2.460	5.42e-11	10.000	280.91
system_14	RK4 (TP8)	0.140	8.8443e-07	10.000	4.8078e+06
system_14	RK4 (TP9)	0.210	9.0238e-08	10.000	4.8664e+05
system_14	RK4 (TP10)	0.330	9.1356e-09	10.000	49032
system_14	RK4 (TP11)	0.520	9.2979e-10	10.000	4954.2
system_14	RK4 (TP12)	0.830	1.0077e-10	10.000	536.16
system_14	RK4 (TP13)	1.250	2.2155e-11	10.000	116.14
system_14	RK4 (TP14)	1.980	2.1288e-11	10.000	110.34
system_14	LA3 (TP8)	0.110	6.5762e-07	10.000	3.6344e+06
system_14	LA3 (TP9)	0.160	6.8229e-08	10.000	3.6887e+05
system_14	LA3 (TP10)	0.250	6.9439e-09	10.000	37284
system_14	LA3 (TP11)	0.390	7.0554e-10	10.000	3768.7
system_14	LA3 (TP12)	0.630	7.6625e-11	10.000	407.83
system_14	LA3 (TP13)	0.960	1.6641e-11	10.000	87.117
system_14	LA3 (TP14)	1.500	1.5774e-11	10.000	81.805
system_14	LC3 (TP8)	0.120	6.6269e-07	10.000	3.6549e+06
system_14	LC3 (TP9)	0.180	6.8267e-08	10.000	3.7023e+05
system_14	LC3 (TP10)	0.280	7.0143e-09	10.000	37343
system_14	LC3 (TP11)	0.440	7.0725e-10	10.000	3774.1
system_14	LC3 (TP12)	0.700	7.7222e-11	10.000	410.63
system_14	LC3 (TP13)	1.150	1.7465e-11	10.000	91.328
system_14	LC3 (TP14)	1.660	1.7025e-11	10.000	88.352
system_14	Riot (03, 1e-11)	0m2.181s	1.0466e-05	-0.000	1.0466e-05
system_14	Riot (04, 1e-11)	0m1.239s	2.1448e-08	-0.000	2.1448e-08
system_14	Riot (05, 1e-11)	0m0.348s	7.1298e-09	8.208	2.2565e+261
system_14	Riot (06, 1e-11)	0m0.194s	2.2129e-09	-0.000	2.2129e-09
system_14	Riot (10, 1e-11)	0m0.126s	4.0075e-12	1.000	4.0075e-12
system_14	Riot (15, 1e-11)	0m0.175s	1.2037e-11	10.000	1.5302e+136
system_14	Valencia-IVP (0.00025)	0m1.778s	0.090273	3.670	999.58
system_14	Valencia-IVP (0.0025)	0m0.165s	0.90282	2.973	998.44
system_14	Valencia-IVP (0.025)	0m0.021s	9.1235	2.275	967.86
system_14	VNODE-LP (15, 1e-14,1e-14)	0m0.008s	1.9185e-13	10.000	1.0508
system_14	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	2.2737e-13	10.000	1.25
system_14	VNODE-LP (25, 1e-14,1e-14)	0m0.005s	9.2371e-14	10.000	0.48828

Table 3.11: Simulation results of Problem 15

Problems	Methods	c5t	c5w	c6t	c6w
system_15	TAYLOR4 (TP8)	0.110	0.9093	10.000	0.91298
system_15	TAYLOR4 (TP9)	0.160	0.9093	10.000	0.91296
system_15	TAYLOR4 (TP10)	0.250	0.9093	10.000	0.91296
system_15	TAYLOR4 (TP11)	0.410	0.9093	10.000	0.91295
system_15	TAYLOR4 (TP12)	0.650	0.9093	10.000	0.91297
system_15	TAYLOR4 (TP13)	1.030	0.9093	10.000	0.91297
system_15	TAYLOR4 (TP14)	1.590	0.9093	10.000	0.91296
system_15	RK4 (TP8)	0.070	0.9093	10.000	0.91299
system_15	RK4 (TP9)	0.110	0.9093	10.000	0.91296
system_15	RK4 (TP10)	0.180	0.9093	10.000	0.91295
system_15	RK4 (TP11)	0.280	0.9093	10.000	0.91296
system_15	RK4 (TP12)	0.450	0.9093	10.000	0.91295
system_15	RK4 (TP13)	0.710	0.9093	10.000	0.91296
system_15	RK4 (TP14)	1.090	0.9093	10.000	0.91295
system_15	LA3 (TP8)	0.060	1.004	10.000	41.485
system_15	LA3 (TP9)	0.090	0.96902	10.000	25.255
system_15	LA3 (TP10)	0.140	0.94981	10.000	9.715
system_15	LA3 (TP11)	0.220	0.93481	10.000	6.4485
system_15	LA3 (TP12)	0.350	0.926	10.000	3.4445
system_15	LA3 (TP13)	0.550	0.92025	10.000	1.6699
system_15	LA3 (TP14)	0.870	0.91549	10.000	2.3746
system_15	LC3 (TP8)	0.060	1.0058	10.000	63.011
system_15	LC3 (TP9)	0.100	0.97512	10.000	22.843
system_15	LC3 (TP10)	0.160	0.95246	10.000	16.319
system_15	LC3 (TP11)	0.240	0.93554	10.000	8.0286
system_15	LC3 (TP12)	0.460	0.92607	10.000	4.1775
system_15	LC3 (TP13)	0.620	0.92054	10.000	1.9364
system_15	LC3 (TP14)	0.970	0.91552	10.000	1.4643
system_15	Riot (05, 1e-11)	0m0.360s	0.92101	10.000	0.91295
system_15	Riot (10, 1e-11)	0m0.155s	0.93965	10.000	0.91295
system_15	Riot (15, 1e-11)	0m0.202s	0.93965	10.000	0.91295
system_15	Valencia-IVP (0.00025)	0m0.976s	3.6323	3.799	999.63
system_15	Valencia-IVP (0.0025)	0m0.088s	3.6817	3.785	999.37
system_15	Valencia-IVP (0.025)	0m0.014s	4.2116	3.650	997.82
system_15	VNODE-LP (15, 1e-14,1e-14)	0m0.004s	0.9093	10.000	8.3669
system_15	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	0.9093	10.000	8.3669
system_15	VNODE-LP (25, 1e-14,1e-14)	0m0.003s	0.9093	10.000	8.3669

Table 3.12: Simulation results of Problem 16

Problems	Methods	c5t	c5w	c6t	c6w
system_16	TAYLOR4 (TP8)	0.190	5.0338	10.000	2.6716e+12
system_16	TAYLOR4 (TP9)	0.290	5.0338	10.000	2.6716e+12
system_16	TAYLOR4 (TP10)	0.430	5.0338	10.000	2.6716e+12
system_16	TAYLOR4 (TP11)	0.660	5.0338	10.000	2.6716e+12
system_16	TAYLOR4 (TP12)	1.040	5.0338	10.000	2.6716e+12
system_16	TAYLOR4 (TP13)	1.620	5.0338	10.000	2.6716e+12
system_16	TAYLOR4 (TP14)	2.530	5.0338	10.000	2.6716e+12
system_16	RK4 (TP8)	0.140	5.0338	10.000	2.6716e+12
system_16	RK4 (TP9)	0.210	5.0338	10.000	2.6716e+12
system_16	RK4 (TP10)	0.330	5.0338	10.000	2.6716e+12
system_16	RK4 (TP11)	0.530	5.0338	10.000	2.6716e+12
system_16	RK4 (TP12)	0.840	5.0338	10.000	2.6716e+12
system_16	RK4 (TP13)	1.270	5.0338	10.000	2.6716e+12
system_16	RK4 (TP14)	1.960	5.0338	10.000	2.6716e+12
system_16	LA3 (TP8)	0.110	5.0368	10.000	2.6879e+12
system_16	LA3 (TP9)	0.170	5.035	10.000	2.678e+12
system_16	LA3 (TP10)	0.250	5.0343	10.000	2.6742e+12
system_16	LA3 (TP11)	0.410	5.034	10.000	2.6726e+12
system_16	LA3 (TP12)	0.670	5.0339	10.000	2.672e+12
system_16	LA3 (TP13)	1.030	5.0339	10.000	2.6718e+12
system_16	LA3 (TP14)	1.570	5.0338	10.000	2.6717e+12
system_16	LC3 (TP8)	0.120	5.0391	10.000	2.7006e+12
system_16	LC3 (TP9)	0.190	5.0359	10.000	2.6828e+12
system_16	LC3 (TP10)	0.280	5.0347	10.000	2.676e+12
system_16	LC3 (TP11)	0.450	5.0342	10.000	2.6734e+12
system_16	LC3 (TP12)	0.720	5.034	10.000	2.6723e+12
system_16	LC3 (TP13)	1.140	5.0339	10.000	2.6719e+12
system_16	LC3 (TP14)	1.740	5.0339	10.000	2.6717e+12
system_16	Riot (05, 1e-11)	0m0.607s	5.0338	-0.000	3.4e+150
system_16	Riot (10, 1e-11)	0m0.160s	5.0338	-0.000	3.3409e+248
system_16	Riot (15, 1e-11)	0m0.204s	5.0338	-0.000	1.3096e+136
system_16	Valencia-IVP (0.00025)	0m1.641s	5.1241	2.748	999.74
system_16	Valencia-IVP (0.0025)	0m0.155s	5.9373	2.635	999.64
system_16	Valencia-IVP (0.025)	0m0.022s	14.218	2.200	938.36
system_16	VNODE-LP (15, 1e-14,1e-14)	0m0.004s	5.0338	10.000	2.6716e+12
system_16	VNODE-LP (20, 1e-14,1e-14)	0m0.004s	5.0338	10.000	2.6716e+12
system_16	VNODE-LP (25, 1e-14,1e-14)	0m0.005s	5.0338	10.000	2.6716e+12

Table 3.13: Simulation results of Problem 17

Problems	Methods	c5t	c5w	c6t	c6w
system_17	TAYLOR4 (TP8)	0.020	2.5429e-08	10.000	2.3333e-08
system_17	TAYLOR4 (TP9)	0.030	2.695e-09	10.000	2.4776e-09
system_17	TAYLOR4 (TP10)	0.050	2.7876e-10	10.000	2.6014e-10
system_17	TAYLOR4 (TP11)	0.080	2.859e-11	10.000	2.673e-11
system_17	TAYLOR4 (TP12)	0.130	3.0154e-12	10.000	2.7828e-12
system_17	TAYLOR4 (TP13)	0.200	4.6429e-13	10.000	3.6043e-13
system_17	TAYLOR4 (TP14)	0.000	0	0.000	0
system_17	RK4 (TP8)	0.010	5.9725e-08	10.000	5.6092e-08
system_17	RK4 (TP9)	0.010	6.7171e-09	10.000	6.2806e-09
system_17	RK4 (TP10)	0.010	6.4465e-10	10.000	6.282e-10
system_17	RK4 (TP11)	0.020	5.8932e-11	10.000	5.8241e-11
system_17	RK4 (TP12)	0.040	5.3604e-12	10.000	5.1803e-12
system_17	RK4 (TP13)	0.060	5.1581e-13	10.000	4.8617e-13
system_17	RK4 (TP14)	0.090	8.5709e-14	10.000	6.3449e-14
system_17	LA3 (TP8)	0.010	2.395e-08	10.000	2.1498e-08
system_17	LA3 (TP9)	0.010	2.5485e-09	10.000	2.4479e-09
system_17	LA3 (TP10)	0.020	2.7709e-10	10.000	2.569e-10
system_17	LA3 (TP11)	0.030	2.8204e-11	10.000	2.6542e-11
system_17	LA3 (TP12)	0.050	2.9106e-12	10.000	2.7096e-12
system_17	LA3 (TP13)	0.080	3.2618e-13	10.000	2.916e-13
system_17	LA3 (TP14)	0.130	8.2823e-14	10.000	5.429e-14
system_17	LC3 (TP8)	0.010	3.2526e-08	10.000	3.1401e-08
system_17	LC3 (TP9)	0.010	3.4509e-09	10.000	3.3385e-09
system_17	LC3 (TP10)	0.020	3.6045e-10	10.000	3.4087e-10
system_17	LC3 (TP11)	0.030	3.4278e-11	10.000	3.2206e-11
system_17	LC3 (TP12)	0.040	3.2934e-12	10.000	3.1542e-12
system_17	LC3 (TP13)	0.070	3.4661e-13	10.000	3.1558e-13
system_17	LC3 (TP14)	0.110	7.2831e-14	10.000	5.0293e-14
system_17	Riot (05, 1e-11)	0m0.209s	4.0267e-11	-0.000	4.3024e-11
system_17	Riot (10, 1e-11)	0m0.153s	3.8114e-13	-0.000	4.3851e-12
system_17	Riot (15, 1e-11)	0m0.249s	1.7208e-14	-0.000	2.2093e-14
system_17	Valencia-IVP (0.00025)	0m1.248s	0.00062591	10.000	0.012037
system_17	Valencia-IVP (0.0025)	0m0.108s	0.0062999	10.000	0.12039
system_17	Valencia-IVP (0.025)	0m0.015s	0.06731	9.275	1.1674
system_17	VNODE-LP (15, 1e-14,1e-14)	0m0.007s	2.1094e-15	10.000	1.0825e-15
system_17	VNODE-LP (20, 1e-14,1e-14)	0m0.009s	1.1102e-15	10.000	9.1593e-16
system_17	VNODE-LP (25, 1e-14,1e-14)	0m0.010s	1.2212e-15	10.000	5.8287e-16

Table 3.14: Simulation results of Problem 18

Problems	Methods	c5t	c5w	c6t	c6w
system_18	TAYLOR4 (TP8)	0.080	2.3166	1.247	80.315
system_18	TAYLOR4 (TP9)	0.120	2.2033	1.271	63.866
system_18	TAYLOR4 (TP10)	0.190	2.136	1.286	50.824
system_18	TAYLOR4 (TP11)	0.280	2.0957	1.296	40.262
system_18	TAYLOR4 (TP12)	0.440	2.0711	1.302	31.806
system_18	TAYLOR4 (TP13)	0.700	2.0558	1.305	25.183
system_18	TAYLOR4 (TP14)	0.000	1	0.000	1
system_18	RK4 (TP8)	0.040	2.032	1.315	92.9
system_18	RK4 (TP9)	0.060	2.031	1.315	73.775
system_18	RK4 (TP10)	0.090	2.0305	1.315	58.317
system_18	RK4 (TP11)	0.140	2.0303	1.315	46.315
system_18	RK4 (TP12)	0.210	2.0303	1.314	36.66
system_18	RK4 (TP13)	0.330	2.0302	1.313	29.062
system_18	RK4 (TP14)	0.520	2.0302	1.312	22.972
system_18	LA3 (TP8)	0.040	2.634	1.188	103.56
system_18	LA3 (TP9)	0.050	2.3653	1.232	82.448
system_18	LA3 (TP10)	0.080	2.2265	1.262	64.848
system_18	LA3 (TP11)	0.130	2.1482	1.281	51.565
system_18	LA3 (TP12)	0.180	2.1026	1.293	40.939
system_18	LA3 (TP13)	0.280	2.0752	1.300	32.465
system_18	LA3 (TP14)	0.450	2.0583	1.304	25.656
system_18	LC3 (TP8)	0.040	3.3388	1.118	99.411
system_18	LC3 (TP9)	0.060	2.6504	1.185	79.498
system_18	LC3 (TP10)	0.090	2.3694	1.230	63.574
system_18	LC3 (TP11)	0.140	2.227	1.261	50.594
system_18	LC3 (TP12)	0.200	2.1486	1.280	39.994
system_18	LC3 (TP13)	0.310	2.1029	1.292	31.772
system_18	LC3 (TP14)	0.490	2.0753	1.299	25.11
system_18	Riot (05, 1e-11)	0m3.154s	0.89498	-0.000	5.6525
system_18	Riot (10, 1e-11)	0m12.527s	0.7695	-0.000	13.258
system_18	Riot (15, 1e-11)	0m46.473s	0.76476	-0.000	12.845
system_18	Valencia-IVP (0.00025)	0m3.609s	2.5351	1.309	62.299
system_18	Valencia-IVP (0.0025)	0m0.385s	2.4744	0.983	2.4744
system_18	Valencia-IVP (0.025)	0m0.046s	2.1873	0.875	2.1873
system_18	VNODE-LP (15, 1e-14,1e-14)	0m0.008s	1.952	1.352	106.72
system_18	VNODE-LP (20, 1e-14,1e-14)	0m0.013s	4.4163	1.079	154.57
system_18	VNODE-LP (25, 1e-14,1e-14)	0m0.032s	189.75	0.944	189.75

Table 3.15: Simulation results of Problem 19

Problems	Methods	c5t	c5w	c6t	c6w
system_19	TAYLOR4 (TP8)	0.060	0.66694	10.000	0.18508
system_19	TAYLOR4 (TP9)	0.090	0.65131	10.000	0.15696
system_19	TAYLOR4 (TP10)	0.130	0.64145	10.000	0.14286
system_19	TAYLOR4 (TP11)	0.210	0.63542	10.000	0.13516
system_19	TAYLOR4 (TP12)	0.330	0.63167	10.000	0.13071
system_19	TAYLOR4 (TP13)	0.520	0.62932	10.000	0.12803
system_19	TAYLOR4 (TP14)	0.000	1	0.000	1
system_19	RK4 (TP8)	0.030	0.62552	10.000	0.12388
system_19	RK4 (TP9)	0.040	0.62541	10.000	0.12377
system_19	RK4 (TP10)	0.070	0.62536	10.000	0.12372
system_19	RK4 (TP11)	0.100	0.62534	10.000	0.1237
system_19	RK4 (TP12)	0.150	0.62533	10.000	0.12369
system_19	RK4 (TP13)	0.240	0.62533	10.000	0.12369
system_19	RK4 (TP14)	0.380	0.62533	10.000	0.12369
system_19	LA3 (TP8)	0.030	0.67072	10.000	0.19253
system_19	LA3 (TP9)	0.040	0.65354	10.000	0.15985
system_19	LA3 (TP10)	0.060	0.64288	10.000	0.14432
system_19	LA3 (TP11)	0.090	0.63625	10.000	0.13591
system_19	LA3 (TP12)	0.130	0.63216	10.000	0.13112
system_19	LA3 (TP13)	0.210	0.62963	10.000	0.12827
system_19	LA3 (TP14)	0.330	0.62803	10.000	0.12654
system_19	LC3 (TP8)	0.030	0.69287	10.000	0.25335
system_19	LC3 (TP9)	0.040	0.66627	10.000	0.18198
system_19	LC3 (TP10)	0.060	0.65057	10.000	0.15488
system_19	LC3 (TP11)	0.090	0.6409	10.000	0.14156
system_19	LC3 (TP12)	0.140	0.63504	10.000	0.13436
system_19	LC3 (TP13)	0.220	0.63142	10.000	0.13021
system_19	LC3 (TP14)	0.350	0.62915	10.000	0.12771
system_19	Riot (05, 1e-11)	0m3.192s	0.44827	-0.000	0.13094
system_19	Riot (10, 1e-11)	0m12.762s	0.44389	-0.000	0.057421
system_19	Riot (15, 1e-11)	0m40.498s	0.44387	-0.000	0.055362
system_19	Valencia-IVP (0.00025)	0m2.772s	2.8979	1.191	3.7768
system_19	Valencia-IVP (0.0025)	0m0.287s	2.9052	1.175	3.694
system_19	Valencia-IVP (0.025)	0m0.041s	2.9872	1.300	5.8585
system_19	VNODE-LP (15, 1e-14,1e-14)	0m0.008s	0.88761	6.361	151.77
system_19	VNODE-LP (20, 1e-14,1e-14)	0m0.010s	0.98714	3.815	218.19
system_19	VNODE-LP (25, 1e-14,1e-14)	0m0.008s	1.1388	2.597	270.43

Table 3.16: Simulation results of Problem 20

Problems	Methods	c5t	c5w	c6t	c6w
system_20	TAYLOR4 (TP8)	0.040	0.0052454	10.000	5.7321e-09
system_20	TAYLOR4 (TP9)	0.060	0.0052389	10.000	5.9775e-10
system_20	TAYLOR4 (TP10)	0.100	0.005235	10.000	1.3097e-10
system_20	TAYLOR4 (TP11)	0.160	0.0052325	10.000	7.6695e-11
system_20	TAYLOR4 (TP12)	0.000	0.2	0.000	0.2
system_20	TAYLOR4 (TP13)	0.000	0.2	0.000	0.2
system_20	TAYLOR4 (TP14)	0.000	0.2	0.000	0.2
system_20	RK4 (TP8)	0.010	0.0052285	10.000	9.8518e-09
system_20	RK4 (TP9)	0.020	0.0052284	10.000	1.2709e-09
system_20	RK4 (TP10)	0.040	0.0052284	10.000	1.5888e-10
system_20	RK4 (TP11)	0.060	0.0052284	10.000	8.1081e-11
system_20	RK4 (TP12)	0.100	0.0052284	10.000	6.8557e-11
system_20	RK4 (TP13)	0.000	0.2	0.000	0.2
system_20	RK4 (TP14)	0.000	0.2	0.000	0.2
system_20	LA3 (TP8)	0.010	0.0052955	10.000	2.5286e-07
system_20	LA3 (TP9)	0.030	0.0052591	10.000	8.833e-09
system_20	LA3 (TP10)	0.040	0.0052431	10.000	8.3868e-10
system_20	LA3 (TP11)	0.060	0.0052358	10.000	1.9991e-10
system_20	LA3 (TP12)	0.100	0.0052323	10.000	1.02e-10
system_20	LA3 (TP13)	0.000	0.2	0.000	0.2
system_20	LA3 (TP14)	0.000	0.2	0.000	0.2
system_20	LC3 (TP8)	0.010	0.0053599	10.000	9.8946e-07
system_20	LC3 (TP9)	0.020	0.0052888	10.000	5.6014e-08
system_20	LC3 (TP10)	0.030	0.005257	10.000	4.6691e-09
system_20	LC3 (TP11)	0.050	0.0052427	10.000	2.7076e-10
system_20	LC3 (TP12)	0.090	0.0052359	10.000	1.1279e-10
system_20	LC3 (TP13)	0.140	0.0052325	10.000	8.2115e-11
system_20	LC3 (TP14)	0.210	0.0052308	10.000	7.2424e-11
system_20	Riot (05, 1e-11)	0m2.343s	0.0051337	-0.000	6.9818e-11
system_20	Riot (10, 1e-11)	0m0.506s	0.0051337	-0.000	6.6049e-11
system_20	Riot (15, 1e-11)	0m1.011s	0.0051337	-0.000	6.6032e-11
system_20	Valencia-IVP (0.00025)	0m2.020s	5.7609	1.371	895.46
system_20	Valencia-IVP (0.0025)	0m0.244s	6.1709	1.123	8.035
system_20	Valencia-IVP (0.025)	0m0.030s	7.1228	0.750	7.1228
system_20	VNODE-LP (15, 1e-14,1e-14)	0m0.003s	0.0053622	10.000	6.9172e-11
system_20	VNODE-LP (20, 1e-14,1e-14)	0m0.005s	0.0053887	10.000	6.957e-11
system_20	VNODE-LP (25, 1e-14,1e-14)	0m0.007s	0.0054356	10.000	7.0287e-11

Table 3.17: Simulation results of Problem 21

Problems	Methods	c5t	c5w	c6t	c6w
system_21	TAYLOR4 (TP8)	0.030	3.0733e-08	10.000	6.8721e-09
system_21	TAYLOR4 (TP9)	0.050	3.2389e-09	10.000	1.1268e-09
system_21	TAYLOR4 (TP10)	0.070	3.3001e-10	10.000	9.6522e-11
system_21	TAYLOR4 (TP11)	0.110	3.3614e-11	10.000	8.5003e-12
system_21	TAYLOR4 (TP12)	0.000	0	0.000	0
system_21	TAYLOR4 (TP13)	0.000	0	0.000	0
system_21	TAYLOR4 (TP14)	0.000	0	0.000	0
system_21	RK4 (TP8)	0.010	3.3937e-08	10.000	7.4364e-09
system_21	RK4 (TP9)	0.020	3.4224e-09	10.000	1.0865e-09
system_21	RK4 (TP10)	0.030	3.4031e-10	10.000	7.6861e-11
system_21	RK4 (TP11)	0.050	3.39e-11	10.000	1.1213e-11
system_21	RK4 (TP12)	0.090	3.4204e-12	10.000	1.3034e-12
system_21	RK4 (TP13)	0.000	0	0.000	0
system_21	RK4 (TP14)	0.000	0	0.000	0
system_21	LA3 (TP8)	0.010	2.6881e-08	8.634	3.8833e-08
system_21	LA3 (TP9)	0.020	2.8558e-09	10.000	1.9854e-09
system_21	LA3 (TP10)	0.030	2.9342e-10	10.000	1.4172e-10
system_21	LA3 (TP11)	0.060	2.9966e-11	10.000	1.0167e-11
system_21	LA3 (TP12)	0.090	3.0833e-12	10.000	8.5887e-13
system_21	LA3 (TP13)	0.140	3.908e-13	10.000	9.9032e-14
system_21	LA3 (TP14)	0.000	0	0.000	0
system_21	LC3 (TP8)	0.010	3.0304e-08	10.000	5.0799e-07
system_21	LC3 (TP9)	0.020	2.7984e-09	10.000	3.9342e-08
system_21	LC3 (TP10)	0.030	2.6206e-10	10.000	2.426e-10
system_21	LC3 (TP11)	0.050	2.5378e-11	10.000	1.213e-11
system_21	LC3 (TP12)	0.070	2.458e-12	10.000	1.243e-12
system_21	LC3 (TP13)	0.120	3.082e-13	10.000	1.0303e-13
system_21	LC3 (TP14)	0.190	1.39e-13	10.000	1.6875e-14
system_21	Riot (05, 1e-11)	0m0.346s	4.0035e-11	-0.000	2.075e-12
system_21	Riot (10, 1e-11)	0m0.168s	4.4511e-12	-0.000	7.0832e-14
system_21	Riot (15, 1e-11)	0m0.211s	2.1094e-14	-0.000	2.1094e-14
system_21	Valencia-IVP (0.00025)	0m1.174s	0.073251	3.678	900.35
system_21	Valencia-IVP (0.0025)	0m0.095s	0.74627	2.210	6.0933
system_21	Valencia-IVP (0.025)	0m0.032s	6.312	0.975	6.312
system_21	VNODE-LP (15, 1e-14,1e-14)	0m0.008s	3.9968e-15	10.000	1.1102e-15
system_21	VNODE-LP (20, 1e-14,1e-14)	0m0.007s	2.8866e-15	10.000	1.1102e-15
system_21	VNODE-LP (25, 1e-14,1e-14)	0m0.006s	1.9984e-15	10.000	1.1102e-15

Table 3.18: Simulation results of Problem 22

Problems	Methods	c5t	c5w	c6t	c6w
system_22	TAYLOR4 (TP8)	0.060	1.3818	10.000	1.3831
system_22	TAYLOR4 (TP9)	0.080	1.3818	10.000	1.3831
system_22	TAYLOR4 (TP10)	0.120	1.3818	10.000	1.3831
system_22	TAYLOR4 (TP11)	0.170	1.3818	10.000	1.3831
system_22	TAYLOR4 (TP12)	0.270	1.3818	10.000	1.3831
system_22	TAYLOR4 (TP13)	0.440	1.3818	10.000	1.3831
system_22	TAYLOR4 (TP14)	0.690	1.3818	10.000	1.3831
system_22	RK4 (TP8)	0.040	1.3818	10.000	1.3831
system_22	RK4 (TP9)	0.060	1.3818	10.000	1.3831
system_22	RK4 (TP10)	0.090	1.3818	10.000	1.3831
system_22	RK4 (TP11)	0.130	1.3818	10.000	1.3831
system_22	RK4 (TP12)	0.210	1.3818	10.000	1.3831
system_22	RK4 (TP13)	0.330	1.3818	10.000	1.3831
system_22	RK4 (TP14)	0.520	1.3818	10.000	1.3831
system_22	LA3 (TP8)	0.030	1.4465	10.000	5.1497
system_22	LA3 (TP9)	0.040	1.4248	10.000	3.046
system_22	LA3 (TP10)	0.070	1.4096	10.000	3.5315
system_22	LA3 (TP11)	0.100	1.4	10.000	2.4605
system_22	LA3 (TP12)	0.170	1.3935	10.000	2.5072
system_22	LA3 (TP13)	0.250	1.3891	10.000	1.8036
system_22	LA3 (TP14)	0.410	1.3865	10.000	1.5151
system_22	LC3 (TP8)	0.040	1.4501	10.000	4.8497
system_22	LC3 (TP9)	0.050	1.427	10.000	4.1688
system_22	LC3 (TP10)	0.080	1.4116	10.000	2.9464
system_22	LC3 (TP11)	0.110	1.4004	10.000	3.0065
system_22	LC3 (TP12)	0.180	1.394	10.000	2.0322
system_22	LC3 (TP13)	0.290	1.3895	10.000	1.7565
system_22	LC3 (TP14)	0.450	1.3867	10.000	1.7305
system_22	Riot (05, 1e-11)	0m0.215s	1.3818	-0.000	1.3831
system_22	Riot (10, 1e-11)	0m0.147s	1.3818	-0.000	1.3831
system_22	Riot (15, 1e-11)	0m0.192s	1.3818	-0.000	1.3831
system_22	Valencia-IVP (0.00025)	0m0.980s	2.7189	6.907	999.97
system_22	Valencia-IVP (0.0025)	0m0.090s	2.724	6.897	999.51
system_22	Valencia-IVP (0.025)	0m0.014s	2.7767	6.800	990.15
system_22	VNODE-LP (15, 1e-14,1e-14)	0m0.003s	1.3818	10.000	25.373
system_22	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	1.3818	10.000	25.373
system_22	VNODE-LP (25, 1e-14,1e-14)	0m0.005s	1.3818	10.000	25.373

Table 3.19: Simulation results of Problem 23

Problems	Methods	c5t	c5w	c6t	c6w
system_23	TAYLOR4 (TP8)	0.050	1.5913e-08	10.000	1.6814e-06
system_23	TAYLOR4 (TP9)	0.080	1.7046e-09	10.000	2.6103e-07
system_23	TAYLOR4 (TP10)	0.120	1.8306e-10	10.000	1.6375e-07
system_23	TAYLOR4 (TP11)	0.180	1.903e-11	10.000	1.6848e-07
system_23	TAYLOR4 (TP12)	0.280	2.212e-12	10.000	2.0574e-08
system_23	TAYLOR4 (TP13)	0.450	6.5636e-13	10.000	6.3359e-09
system_23	TAYLOR4 (TP14)	0.710	7.6073e-13	10.000	7.5749e-09
system_23	RK4 (TP8)	0.040	1.9834e-08	10.000	7.863e-07
system_23	RK4 (TP9)	0.050	2.2172e-09	10.000	2.5607e-07
system_23	RK4 (TP10)	0.080	2.3651e-10	10.000	8.9245e-08
system_23	RK4 (TP11)	0.130	2.4555e-11	10.000	1.3865e-07
system_23	RK4 (TP12)	0.200	2.6081e-12	10.000	2.2231e-08
system_23	RK4 (TP13)	0.310	4.2721e-13	10.000	3.913e-09
system_23	RK4 (TP14)	0.490	3.0509e-13	10.000	2.9406e-09
system_23	LA3 (TP8)	0.030	1.5086e-08	10.000	1.2796e-06
system_23	LA3 (TP9)	0.040	1.6451e-09	10.000	3.7812e-07
system_23	LA3 (TP10)	0.060	1.7698e-10	10.000	1.8245e-07
system_23	LA3 (TP11)	0.100	1.8517e-11	10.000	1.1868e-07
system_23	LA3 (TP12)	0.150	1.9926e-12	10.000	1.811e-08
system_23	LA3 (TP13)	0.240	3.233e-13	10.000	3.106e-09
system_23	LA3 (TP14)	0.370	2.256e-13	10.000	2.248e-09
system_23	LC3 (TP8)	0.030	1.5774e-08	10.000	1.8135e-06
system_23	LC3 (TP9)	0.050	1.7152e-09	10.000	2.948e-07
system_23	LC3 (TP10)	0.070	1.7917e-10	10.000	3.0775e-07
system_23	LC3 (TP11)	0.110	1.8552e-11	10.000	1.3905e-07
system_23	LC3 (TP12)	0.170	1.9949e-12	10.000	1.8803e-08
system_23	LC3 (TP13)	0.270	3.3529e-13	10.000	3.2553e-09
system_23	LC3 (TP14)	0.420	2.4336e-13	10.000	2.448e-09
system_23	Riot (05, 1e-11)	0m0.102s	5.6269e-11	-0.000	7.3491e-10
system_23	Riot (10, 1e-11)	0m0.114s	2.7978e-14	-0.000	4.2883e-11
system_23	Riot (15, 1e-11)	0m0.139s	4.1966e-14	-0.000	1.0757e-12
system_23	Valencia-IVP (0.00025)	0m1.130s	0.00046233	10.000	5.0012
system_23	Valencia-IVP (0.0025)	0m0.095s	0.0046322	10.000	50.642
system_23	Valencia-IVP (0.025)	0m0.014s	0.047235	10.000	574.87
system_23	VNODE-LP (15, 1e-14,1e-14)	0m0.003s	1.9984e-15	10.000	8.3933e-14
system_23	VNODE-LP (20, 1e-14,1e-14)	0m0.004s	9.992e-16	10.000	7.5051e-14
system_23	VNODE-LP (25, 1e-14,1e-14)	0m0.006s	6.6613e-16	10.000	4.7073e-14

Table 3.20: Simulation results of Problem 24

Problems	Methods	c5t	c5w	c6t	c6w
system_24	TAYLOR4 (TP8)	0.060	1.9324	10.000	14317
system_24	TAYLOR4 (TP9)	0.080	1.9324	10.000	14317
system_24	TAYLOR4 (TP10)	0.110	1.9324	10.000	14317
system_24	TAYLOR4 (TP11)	0.170	1.9324	10.000	14317
system_24	TAYLOR4 (TP12)	0.270	1.9324	10.000	14317
system_24	TAYLOR4 (TP13)	0.440	1.9324	10.000	14317
system_24	TAYLOR4 (TP14)	0.670	1.9324	10.000	14317
system_24	RK4 (TP8)	0.040	1.9324	10.000	14317
system_24	RK4 (TP9)	0.060	1.9324	10.000	14317
system_24	RK4 (TP10)	0.080	1.9324	10.000	14317
system_24	RK4 (TP11)	0.130	1.9324	10.000	14317
system_24	RK4 (TP12)	0.200	1.9324	10.000	14317
system_24	RK4 (TP13)	0.330	1.9324	10.000	14317
system_24	RK4 (TP14)	0.510	1.9324	10.000	14317
system_24	LA3 (TP8)	0.030	1.9328	10.000	14347
system_24	LA3 (TP9)	0.050	1.9326	10.000	14329
system_24	LA3 (TP10)	0.070	1.9325	10.000	14322
system_24	LA3 (TP11)	0.100	1.9325	10.000	14319
system_24	LA3 (TP12)	0.160	1.9324	10.000	14318
system_24	LA3 (TP13)	0.250	1.9324	10.000	14318
system_24	LA3 (TP14)	0.400	1.9324	10.000	14317
system_24	LC3 (TP8)	0.040	1.9331	10.000	14371
system_24	LC3 (TP9)	0.050	1.9327	10.000	14338
system_24	LC3 (TP10)	0.070	1.9325	10.000	14325
system_24	LC3 (TP11)	0.110	1.9325	10.000	14320
system_24	LC3 (TP12)	0.180	1.9325	10.000	14318
system_24	LC3 (TP13)	0.320	1.9324	10.000	14318
system_24	LC3 (TP14)	0.470	1.9324	10.000	14317
system_24	Riot (05, 1e-11)	0m0.222s	1.9324	-0.000	21721
system_24	Riot (10, 1e-11)	0m0.148s	1.9324	-0.000	21718
system_24	Riot (15, 1e-11)	0m0.193s	1.9324	-0.000	21703
system_24	Valencia-IVP (0.00025)	0m1.214s	1.9329	7.337	999.94
system_24	Valencia-IVP (0.0025)	0m0.114s	1.9368	7.320	998.62
system_24	Valencia-IVP (0.025)	0m0.014s	1.977	7.175	998.34
system_24	VNODE-LP (15, 1e-14,1e-14)	0m0.004s	1.9324	10.000	14317
system_24	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	1.9324	10.000	14317
system_24	VNODE-LP (25, 1e-14,1e-14)	0m0.002s	1.9324	10.000	14317

Table 3.21: Simulation results of Problem 25

Problems	Methods	c5t	c5w	c6t	c6w
system_25	TAYLOR4 (TP8)	0.040	1.3772e-08	10.000	0.00016615
system_25	TAYLOR4 (TP9)	0.070	1.5056e-09	10.000	1.7744e-05
system_25	TAYLOR4 (TP10)	0.110	1.6064e-10	10.000	1.8561e-06
system_25	TAYLOR4 (TP11)	0.160	1.6748e-11	10.000	1.9164e-07
system_25	TAYLOR4 (TP12)	0.260	1.8416e-12	10.000	2.0722e-08
system_25	TAYLOR4 (TP13)	0.410	4.1234e-13	10.000	4.1866e-09
system_25	TAYLOR4 (TP14)	0.650	4.0101e-13	10.000	3.7742e-09
system_25	RK4 (TP8)	0.030	1.7587e-08	10.000	0.00021528
system_25	RK4 (TP9)	0.050	1.9552e-09	10.000	2.3157e-05
system_25	RK4 (TP10)	0.070	2.0936e-10	10.000	2.4276e-06
system_25	RK4 (TP11)	0.110	2.1817e-11	10.000	2.501e-07
system_25	RK4 (TP12)	0.180	2.26e-12	10.000	2.5778e-08
system_25	RK4 (TP13)	0.280	3.1397e-13	10.000	3.3969e-09
system_25	RK4 (TP14)	0.450	1.6809e-13	10.000	1.6121e-09
system_25	LA3 (TP8)	0.030	1.2714e-08	10.000	0.00016167
system_25	LA3 (TP9)	0.040	1.4237e-09	10.000	1.7264e-05
system_25	LA3 (TP10)	0.060	1.537e-10	10.000	1.7994e-06
system_25	LA3 (TP11)	0.090	1.61e-11	10.000	1.8681e-07
system_25	LA3 (TP12)	0.140	1.714e-12	10.000	1.9526e-08
system_25	LA3 (TP13)	0.210	2.3848e-13	10.000	2.5806e-09
system_25	LA3 (TP14)	0.340	1.2601e-13	10.000	1.2041e-09
system_25	LC3 (TP8)	0.030	1.2989e-08	10.000	0.00016663
system_25	LC3 (TP9)	0.040	1.473e-09	10.000	1.7777e-05
system_25	LC3 (TP10)	0.060	1.562e-10	10.000	1.8255e-06
system_25	LC3 (TP11)	0.100	1.6272e-11	10.000	1.878e-07
system_25	LC3 (TP12)	0.150	1.7181e-12	10.000	1.9601e-08
system_25	LC3 (TP13)	0.240	2.4358e-13	10.000	2.6186e-09
system_25	LC3 (TP14)	0.380	1.3412e-13	10.000	1.2862e-09
system_25	Riot (05, 1e-11)	0m0.104s	5.7086e-11	-0.000	0.0013639
system_25	Riot (10, 1e-11)	0m0.109s	3.7192e-15	-0.000	3.7192e-15
system_25	Riot (15, 1e-11)	0m0.089s	0	-0.000	5.7732e-15
system_25	Valencia-IVP (0.00025)	0m1.087s	0.00029389	10.000	2.7571
system_25	Valencia-IVP (0.0025)	0m0.093s	0.0029465	10.000	27.915
system_25	Valencia-IVP (0.025)	0m0.015s	0.030251	10.000	316.61
system_25	VNODE-LP (15, 1e-14,1e-14)	0m0.004s	9.992e-16	10.000	8.9433e-12
system_25	VNODE-LP (20, 1e-14,1e-14)	0m0.004s	8.8818e-16	10.000	7.9496e-12
system_25	VNODE-LP (25, 1e-14,1e-14)	0m0.004s	8.3267e-16	10.000	6.2134e-12

Table 3.22: Simulation results of Problem 26

Problems	Methods	c5t	c5w	c6t	c6w
system_26	TAYLOR4 (TP8)	0.190	1.2981	10.000	0.00023241
system_26	TAYLOR4 (TP9)	0.280	1.2981	10.000	0.00022813
system_26	TAYLOR4 (TP10)	0.410	1.2981	10.000	0.00022749
system_26	TAYLOR4 (TP11)	0.610	1.2981	10.000	0.000229
system_26	TAYLOR4 (TP12)	0.940	1.2981	10.000	0.00022964
system_26	TAYLOR4 (TP13)	1.480	1.2981	10.000	0.00022949
system_26	TAYLOR4 (TP14)	2.360	1.2981	10.000	0.00022948
system_26	RK4 (TP8)	0.130	1.2981	10.000	0.00023297
system_26	RK4 (TP9)	0.190	1.2981	10.000	0.00022751
system_26	RK4 (TP10)	0.300	1.2981	10.000	0.00022782
system_26	RK4 (TP11)	0.470	1.2981	10.000	0.00022747
system_26	RK4 (TP12)	0.730	1.2981	10.000	0.00022897
system_26	RK4 (TP13)	1.150	1.2981	10.000	0.00022902
system_26	RK4 (TP14)	1.800	1.2981	10.000	0.00022939
system_26	LA3 (TP8)	0.110	1.7614	10.000	47.327
system_26	LA3 (TP9)	0.160	1.6099	10.000	5.1636
system_26	LA3 (TP10)	0.240	1.5196	10.000	3.3388
system_26	LA3 (TP11)	0.380	1.4341	10.000	1.4904
system_26	LA3 (TP12)	0.580	1.394	10.000	0.77395
system_26	LA3 (TP13)	0.910	1.3539	10.000	0.27453
system_26	LA3 (TP14)	1.440	1.3298	10.000	0.09762
system_26	LC3 (TP8)	0.110	1.7962	10.000	57.572
system_26	LC3 (TP9)	0.170	1.6345	10.000	4.2967
system_26	LC3 (TP10)	0.260	1.544	10.000	3.6973
system_26	LC3 (TP11)	0.420	1.4629	10.000	3.522
system_26	LC3 (TP12)	0.630	1.3763	10.000	0.85036
system_26	LC3 (TP13)	1.000	1.3639	10.000	1.2933
system_26	LC3 (TP14)	1.600	1.3283	10.000	0.13024
system_26	Riot (05, 1e-11)	0m0.592s	1.2981	-0.000	0.00023441
system_26	Riot (10, 1e-11)	0m0.217s	1.2981	-0.000	0.00022716
system_26	Riot (15, 1e-11)	0m0.302s	1.2981	-0.000	0.00022731
system_26	Valencia-IVP (0.00025)	0m1.817s	277.25	1.238	999.84
system_26	Valencia-IVP (0.0025)	0m0.156s	287.15	1.230	996.77
system_26	Valencia-IVP (0.025)	0m0.022s	421.64	1.125	867.43
system_26	VNODE-LP (15, 1e-14,1e-14)	0m0.007s	1.2981	10.000	6.8883
system_26	VNODE-LP (20, 1e-14,1e-14)	0m0.008s	1.2981	10.000	6.8883
system_26	VNODE-LP (25, 1e-14,1e-14)	0m0.007s	1.2981	10.000	6.8883

Table 3.23: Simulation results of Problem 27

Problems	Methods	c5t	c5w	c6t	c6w
system_27	TAYLOR4 (TP8)	0.150	9.9382e-08	10.000	9.9453e-07
system_27	TAYLOR4 (TP9)	0.210	1.0984e-08	10.000	2.4889e-07
system_27	TAYLOR4 (TP10)	0.320	1.1848e-09	10.000	3.0464e-07
system_27	TAYLOR4 (TP11)	0.500	1.2016e-10	10.000	2.585e-07
system_27	TAYLOR4 (TP12)	0.790	1.366e-11	10.000	2.1613e-07
system_27	TAYLOR4 (TP13)	1.220	3.6535e-12	10.000	3.5367e-07
system_27	TAYLOR4 (TP14)	1.900	3.9741e-12	10.000	1.2891e-06
system_27	RK4 (TP8)	0.110	1.2932e-07	10.000	1.3248e-06
system_27	RK4 (TP9)	0.160	1.4163e-08	10.000	2.0043e-07
system_27	RK4 (TP10)	0.250	1.5098e-09	10.000	1.5059e-07
system_27	RK4 (TP11)	0.410	1.5404e-10	10.000	1.5609e-07
system_27	RK4 (TP12)	0.640	1.6365e-11	10.000	1.7223e-07
system_27	RK4 (TP13)	0.970	2.5304e-12	10.000	1.8389e-07
system_27	RK4 (TP14)	1.540	1.6289e-12	10.000	1.7725e-07
system_27	LA3 (TP8)	0.090	1.077e-07	10.000	0.0021861
system_27	LA3 (TP9)	0.130	1.1278e-08	10.000	0.00013224
system_27	LA3 (TP10)	0.200	1.1811e-09	10.000	3.2675e-05
system_27	LA3 (TP11)	0.310	1.2139e-10	10.000	3.5598e-06
system_27	LA3 (TP12)	0.490	1.2736e-11	10.000	1.2922e-06
system_27	LA3 (TP13)	0.760	1.9278e-12	10.000	3.4537e-07
system_27	LA3 (TP14)	1.230	1.2119e-12	10.000	3.651e-07
system_27	LC3 (TP8)	0.100	1.1371e-07	10.000	0.0045486
system_27	LC3 (TP9)	0.150	1.1741e-08	10.000	0.0004441
system_27	LC3 (TP10)	0.220	1.2156e-09	10.000	4.9058e-05
system_27	LC3 (TP11)	0.350	1.2311e-10	10.000	4.7915e-06
system_27	LC3 (TP12)	0.540	1.297e-11	10.000	1.3287e-06
system_27	LC3 (TP13)	0.840	1.9971e-12	10.000	4.0411e-07
system_27	LC3 (TP14)	1.290	1.3069e-12	10.000	1.3527e-06
system_27	Riot (05, 1e-11)	0m0.256s	1.8868e-10	-0.000	2.7813e+09
system_27	Riot (10, 1e-11)	0m0.164s	1.199e-14	-0.000	3.4514e-08
system_27	Riot (15, 1e-11)	0m0.230s	8.793e-14	-0.000	1.8045e-12
system_27	Valencia-IVP (0.00025)	0m1.391s	0.1407	2.649	999.19
system_27	Valencia-IVP (0.0025)	0m0.126s	1.4595	2.205	988.39
system_27	Valencia-IVP (0.025)	0m0.021s	21.761	1.650	925.46
system_27	VNODE-LP (15, 1e-14,1e-14)	0m0.006s	9.992e-15	10.000	9.4229e-14
system_27	VNODE-LP (20, 1e-14,1e-14)	0m0.005s	5.9952e-15	10.000	5.4546e-14
system_27	VNODE-LP (25, 1e-14,1e-14)	0m0.004s	5.9952e-15	10.000	3.6599e-14

Table 3.24: Simulation results of Problem 28

Problems	Methods	c5t	c5w	c6t	c6w
system_28	TAYLOR4 (TP8)	4.130	20.932	0.283	20.932
system_28	TAYLOR4 (TP9)	5.960	17.968	0.282	17.968
system_28	TAYLOR4 (TP10)	8.860	15.338	0.280	15.338
system_28	TAYLOR4 (TP11)	12.540	13.052	0.282	13.052
system_28	TAYLOR4 (TP12)	18.350	11.079	0.280	11.079
system_28	TAYLOR4 (TP13)	27.080	9.385	0.277	9.385
system_28	TAYLOR4 (TP14)	38.040	7.9094	0.275	7.9094
system_28	RK4 (TP8)	4.160	23.152	0.284	23.152
system_28	RK4 (TP9)	3.300	19.86	0.284	19.86
system_28	RK4 (TP10)	4.790	17.013	0.285	17.013
system_28	RK4 (TP11)	6.890	14.506	0.283	14.506
system_28	RK4 (TP12)	9.590	12.365	0.279	12.365
system_28	RK4 (TP13)	13.920	10.475	0.278	10.475
system_28	RK4 (TP14)	20.070	8.8608	0.276	8.8608
system_28	LA3 (TP8)	1.960	24.682	0.274	24.682
system_28	LA3 (TP9)	2.960	21.217	0.276	21.217
system_28	LA3 (TP10)	4.250	18.246	0.276	18.246
system_28	LA3 (TP11)	6.350	15.618	0.278	15.618
system_28	LA3 (TP12)	8.760	13.281	0.278	13.281
system_28	LA3 (TP13)	12.720	11.269	0.277	11.269
system_28	LA3 (TP14)	18.360	9.5482	0.276	9.5482
system_28	LC3 (TP8)	1.870	24.835	0.265	24.835
system_28	LC3 (TP9)	2.830	21.005	0.270	21.005
system_28	LC3 (TP10)	4.250	17.966	0.273	17.966
system_28	LC3 (TP11)	6.320	15.404	0.276	15.404
system_28	LC3 (TP12)	8.960	13.092	0.276	13.092
system_28	LC3 (TP13)	13.090	11.123	0.277	11.123
system_28	LC3 (TP14)	19.040	9.4039	0.275	9.4039
system_28	Riot (05, 1e-11)	0m29.200s	0	-0.000	4.2446
system_28	Riot (10, 1e-11)	18m44.691s	0	-0.000	4.0786
system_28	Riot (15, 1e-11)	210m1.595s	0	-0.000	4.5904
system_28	Valencia-IVP (0.00025)	0m2.126s	1.1713	0.162	1.1713
system_28	Valencia-IVP (0.0025)	0m0.733s	3.1672	0.395	3.1672
system_28	Valencia-IVP (0.025)	0m0.027s	0.95755	0.075	0.95755
system_28	VNODE-LP (15, 1e-14,1e-14)	0m0.309s	18.119	0.155	18.119
system_28	VNODE-LP (20, 1e-14,1e-14)	0m0.299s	22.402	0.140	22.402
system_28	VNODE-LP (25, 1e-14,1e-14)	0m0.301s	25.252	0.128	25.252

Table 3.25: Simulation results of Problem 29

Problems	Methods	c5t	c5w	c6t	c6w
system_29	TAYLOR4 (TP8)	0.560	3.5522e-07	10.000	3.7763e-07
system_29	TAYLOR4 (TP9)	0.830	3.6736e-08	10.000	3.9348e-08
system_29	TAYLOR4 (TP10)	1.250	3.7576e-09	10.000	4.0256e-09
system_29	TAYLOR4 (TP11)	1.990	3.7343e-10	10.000	4.03e-10
system_29	TAYLOR4 (TP12)	3.120	3.7579e-11	10.000	4.0723e-11
system_29	TAYLOR4 (TP13)	5.370	4.5068e-12	10.000	4.8452e-12
system_29	TAYLOR4 (TP14)	7.360	1.6607e-12	10.000	1.7164e-12
system_29	RK4 (TP8)	0.340	5.0539e-07	10.000	5.8051e-07
system_29	RK4 (TP9)	0.460	5.8113e-08	10.000	6.5063e-08
system_29	RK4 (TP10)	0.710	5.7374e-09	10.000	6.4068e-09
system_29	RK4 (TP11)	1.000	6.2862e-10	10.000	6.6449e-10
system_29	RK4 (TP12)	1.540	6.1003e-11	10.000	6.3473e-11
system_29	RK4 (TP13)	2.420	5.9718e-12	10.000	6.1513e-12
system_29	RK4 (TP14)	3.700	8.6475e-13	10.000	8.7147e-13
system_29	LA3 (TP8)	0.340	2.3268e-07	10.000	3.1546e-07
system_29	LA3 (TP9)	0.470	2.4512e-08	10.000	3.3829e-08
system_29	LA3 (TP10)	0.710	2.3962e-09	10.000	3.3821e-09
system_29	LA3 (TP11)	1.060	2.281e-10	10.000	3.2909e-10
system_29	LA3 (TP12)	1.660	2.1874e-11	10.000	3.1581e-11
system_29	LA3 (TP13)	2.600	2.3522e-12	10.000	3.2499e-12
system_29	LA3 (TP14)	4.030	5.4412e-13	10.000	6.1251e-13
system_29	LC3 (TP8)	0.340	2.9012e-07	10.000	4.1042e-07
system_29	LC3 (TP9)	0.460	2.9152e-08	10.000	4.3464e-08
system_29	LC3 (TP10)	0.680	2.7738e-09	10.000	4.2533e-09
system_29	LC3 (TP11)	1.000	2.5993e-10	10.000	3.9511e-10
system_29	LC3 (TP12)	1.560	2.3858e-11	10.000	3.5866e-11
system_29	LC3 (TP13)	2.480	2.4346e-12	10.000	3.4715e-12
system_29	LC3 (TP14)	3.730	5.4146e-13	10.000	6.1394e-13
system_29	Riot (05, 1e-11)	0m1.818s	3.2308e-10	-0.000	5.7962e-09
system_29	Riot (10, 1e-11)	0m1.333s	6.1563e-12	-0.000	1.0335e-10
system_29	Riot (15, 1e-11)	0m2.386s	9.6034e-15	-0.000	9.6034e-15
system_29	Valencia-IVP (0.00025)	0m3.140s	0.001153	10.000	0.057922
system_29	Valencia-IVP (0.0025)	0m0.516s	0.01199	6.265	0.2962
system_29	Valencia-IVP (0.025)	0m0.226s	0.17131	1.200	0.2357
system_29	VNODE-LP (15, 1e-14,1e-14)	0m0.080s	1.8485e-14	10.000	1.5952e-14
system_29	VNODE-LP (20, 1e-14,1e-14)	0m0.099s	1.199e-14	10.000	1.1606e-14
system_29	VNODE-LP (25, 1e-14,1e-14)	0m0.107s	9.4924e-15	10.000	8.9239e-15

Table 3.26: Simulation results of Problem 30

Problems	Methods	c5t	c5w	c6t	c6w
system_30	TAYLOR4 (TP8)	16.140	80.278	0.297	80.278
system_30	TAYLOR4 (TP9)	24.390	63.718	0.298	63.718
system_30	TAYLOR4 (TP10)	36.140	50.476	0.299	50.476
system_30	TAYLOR4 (TP11)	53.200	40.071	0.299	40.071
system_30	TAYLOR4 (TP12)	77.710	31.765	0.299	31.765
system_30	TAYLOR4 (TP13)	113.360	25.075	0.297	25.075
system_30	TAYLOR4 (TP14)	159.450	19.754	0.296	19.754
system_30	RK4 (TP8)	9.090	92.703	0.302	92.703
system_30	RK4 (TP9)	13.060	73.155	0.302	73.155
system_30	RK4 (TP10)	19.520	58.308	0.303	58.308
system_30	RK4 (TP11)	28.190	46.348	0.300	46.348
system_30	RK4 (TP12)	40.790	36.606	0.300	36.606
system_30	RK4 (TP13)	60.340	28.938	0.299	28.938
system_30	RK4 (TP14)	87.210	22.85	0.298	22.85
system_30	LA3 (TP8)	7.670	103.64	0.290	103.64
system_30	LA3 (TP9)	11.260	81.407	0.293	81.407
system_30	LA3 (TP10)	16.760	65.057	0.296	65.057
system_30	LA3 (TP11)	24.890	51.523	0.298	51.523
system_30	LA3 (TP12)	36.280	40.863	0.298	40.863
system_30	LA3 (TP13)	55.410	32.308	0.298	32.308
system_30	LA3 (TP14)	82.570	25.57	0.297	25.57
system_30	LC3 (TP8)	7.590	101.33	0.282	101.33
system_30	LC3 (TP9)	11.350	79.317	0.288	79.317
system_30	LC3 (TP10)	17.050	63.817	0.293	63.817
system_30	LC3 (TP11)	25.100	50.599	0.296	50.599
system_30	LC3 (TP12)	36.770	40.033	0.296	40.033
system_30	LC3 (TP13)	55.480	31.626	0.297	31.626
system_30	LC3 (TP14)	78.840	25.014	0.297	25.014
system_30	Riot				
system_30	Valencia-IVP (0.00025)	0m13.555s	57.455	0.332	57.455
system_30	Valencia-IVP (0.0025)	0m0.494s	4.4295	0.245	4.4295
system_30	Valencia-IVP (0.025)	0m0.108s	3.7929	0.200	3.7929
system_30	VNODE-LP (15, 1e-14,1e-14)	0m0.194s	105.32	0.259	105.32
system_30	VNODE-LP (20, 1e-14,1e-14)	0m0.186s	146.87	0.237	146.87
system_30	VNODE-LP (25, 1e-14,1e-14)	0m0.187s	188.72	0.220	188.72

Table 3.27: Simulation results of Problem 31

Problems	Methods	c5t	c5w	c6t	c6w
system_31	TAYLOR4 (TP8)	2.230	1.2578e-07	10.000	5.8598e-05
system_31	TAYLOR4 (TP9)	3.470	1.3976e-08	10.000	5.2745e-05
system_31	TAYLOR4 (TP10)	5.330	1.5217e-09	10.000	1.2559e-05
system_31	TAYLOR4 (TP11)	8.330	1.5944e-10	10.000	1.4182e-06
system_31	TAYLOR4 (TP12)	13.050	1.6868e-11	10.000	1.5655e-07
system_31	TAYLOR4 (TP13)	20.400	2.2326e-12	10.000	2.1442e-08
system_31	TAYLOR4 (TP14)	31.840	1.028e-12	10.000	1.025e-08
system_31	RK4 (TP8)	1.290	1.0486e-07	10.000	4.4015e-05
system_31	RK4 (TP9)	1.850	1.2451e-08	10.000	4.0884e-05
system_31	RK4 (TP10)	2.940	1.3932e-09	10.000	1.3356e-05
system_31	RK4 (TP11)	4.390	1.5137e-10	10.000	1.457e-06
system_31	RK4 (TP12)	6.870	1.5606e-11	10.000	1.5259e-07
system_31	RK4 (TP13)	10.920	1.6551e-12	10.000	1.6821e-08
system_31	RK4 (TP14)	16.650	3.2484e-13	10.000	3.6533e-09
system_31	LA3 (TP8)	1.250	7.1058e-08	10.000	8.3149e-05
system_31	LA3 (TP9)	1.780	7.6279e-09	10.000	4.4857e-05
system_31	LA3 (TP10)	2.670	8.5419e-10	10.000	1.2882e-05
system_31	LA3 (TP11)	4.270	9.3902e-11	10.000	1.3788e-06
system_31	LA3 (TP12)	6.580	9.7906e-12	10.000	1.4299e-07
system_31	LA3 (TP13)	10.140	1.0789e-12	10.000	1.5696e-08
system_31	LA3 (TP14)	16.130	2.7699e-13	10.000	3.3895e-09
system_31	LC3 (TP8)	1.090	1.7432e-07	10.000	4.6058e-05
system_31	LC3 (TP9)	1.490	1.9061e-08	10.000	4.6763e-05
system_31	LC3 (TP10)	2.220	1.9137e-09	10.000	1.9889e-05
system_31	LC3 (TP11)	3.450	1.8493e-10	10.000	1.9633e-06
system_31	LC3 (TP12)	5.250	1.7977e-11	10.000	1.9146e-07
system_31	LC3 (TP13)	8.310	1.821e-12	10.000	1.9509e-08
system_31	LC3 (TP14)	12.850	3.1761e-13	10.000	3.486e-09
system_31	Riot (05, 1e-11)	0m8.552s	1.3195e-10	-0.000	3.7849e-08
system_31	Riot (10, 1e-11)	0m4.423s	4.2645e-12	-0.000	5.8043e-09
system_31	Riot (15, 1e-11)	0m4.983s	1.8874e-15	-0.000	1.2535e-10
system_31	Valencia-IVP (0.00025)	0m55.912s	0.0020183	4.793	1.5566
system_31	Valencia-IVP (0.0025)	0m4.192s	0.020632	3.252	1.8903
system_31	Valencia-IVP (0.025)	0m0.399s	0.25275	1.800	1.0445
system_31	VNODE-LP (15, 1e-14,1e-14)	0m0.160s	9.26e-15	10.000	1.3792e-13
system_31	VNODE-LP (20, 1e-14,1e-14)	0m0.181s	4.9093e-15	10.000	9.2898e-14
system_31	VNODE-LP (25, 1e-14,1e-14)	0m0.205s	4.0697e-15	10.000	7.63e-14

Table 3.28: Simulation results of Problem 32

Problems	Methods	c5t	c5w	c6t	c6w
system_32	TAYLOR4 (TP8)	0.110	1.0131e-07	10.000	1.092e-06
system_32	TAYLOR4 (TP9)	0.150	1.1876e-08	10.000	1.2968e-07
system_32	TAYLOR4 (TP10)	0.220	1.5521e-09	10.000	1.3318e-08
system_32	TAYLOR4 (TP11)	0.330	1.5477e-10	10.000	1.4345e-09
system_32	TAYLOR4 (TP12)	0.520	1.9744e-11	10.000	1.544e-10
system_32	TAYLOR4 (TP13)	0.900	2.9185e-12	10.000	1.8198e-11
system_32	TAYLOR4 (TP14)	1.280	1.549e-12	10.000	4.4076e-12
system_32	RK4 (TP8)	0.050	1.1569e-07	10.000	1.6157e-06
system_32	RK4 (TP9)	0.070	1.2497e-08	10.000	2.1179e-07
system_32	RK4 (TP10)	0.110	1.4826e-09	10.000	2.7584e-08
system_32	RK4 (TP11)	0.170	1.5983e-10	10.000	3.5086e-09
system_32	RK4 (TP12)	0.250	1.6631e-11	10.000	4.1913e-10
system_32	RK4 (TP13)	0.400	1.9895e-12	10.000	5.1035e-11
system_32	RK4 (TP14)	0.630	4.9205e-13	10.000	6.4764e-12
system_32	LA3 (TP8)	0.060	4.8721e-08	10.000	9.5298e-07
system_32	LA3 (TP9)	0.080	5.2775e-09	10.000	9.6145e-08
system_32	LA3 (TP10)	0.130	5.5251e-10	10.000	1.0333e-08
system_32	LA3 (TP11)	0.180	5.6823e-11	10.000	1.1945e-09
system_32	LA3 (TP12)	0.290	5.9162e-12	10.000	1.2948e-10
system_32	LA3 (TP13)	0.460	8.0025e-13	10.000	1.4912e-11
system_32	LA3 (TP14)	0.710	4.0501e-13	10.000	2.1458e-12
system_32	LC3 (TP8)	0.060	8.8322e-08	10.000	1.0067e-06
system_32	LC3 (TP9)	0.080	1.0516e-08	10.000	1.0107e-07
system_32	LC3 (TP10)	0.110	1.3559e-09	10.000	1.2095e-08
system_32	LC3 (TP11)	0.180	1.5019e-10	10.000	1.3537e-09
system_32	LC3 (TP12)	0.250	1.6706e-11	10.000	1.5879e-10
system_32	LC3 (TP13)	0.390	2.0792e-12	10.000	1.7959e-11
system_32	LC3 (TP14)	0.610	4.7784e-13	10.000	2.528e-12
system_32	Riot (05, 1e-11)	0m2.160s	8.7466e-11	-0.000	2.9713e-10
system_32	Riot (10, 1e-11)	0m0.781s	1.2124e-13	-0.000	4.0483e-11
system_32	Riot (15, 1e-11)	0m0.815s	1.3411e-13	-0.000	1.8493e-11
system_32	Valencia-IVP (0.00025)	1m35.630s	0.00026492	10.000	0.28978
system_32	Valencia-IVP (0.0025)	0m2.151s	0.0026499	10.000	2.9143
system_32	Valencia-IVP (0.025)	0m0.272s	0.026604	10.000	31.409
system_32	VNODE-LP (15, 1e-14,1e-14)	0m0.039s	9.77e-15	10.000	6.3727e-14
system_32	VNODE-LP (20, 1e-14,1e-14)	0m0.044s	8.8818e-15	10.000	7.3386e-14
system_32	VNODE-LP (25, 1e-14,1e-14)	0m0.040s	7.9936e-15	10.000	3.586e-14

Table 3.29: Simulation results of Problem 33

Problems	Methods	c5t	c5w	c6t	c6w
system_33	TAYLOR4 (TP8)	0.110	0.81192	10.000	0.20314
system_33	TAYLOR4 (TP9)	0.150	0.81192	10.000	0.20314
system_33	TAYLOR4 (TP10)	0.230	0.81192	10.000	0.20314
system_33	TAYLOR4 (TP11)	0.360	0.81192	10.000	0.20314
system_33	TAYLOR4 (TP12)	0.530	0.81192	10.000	0.20314
system_33	TAYLOR4 (TP13)	0.860	0.81192	10.000	0.20314
system_33	TAYLOR4 (TP14)	1.330	0.81192	10.000	0.20314
system_33	RK4 (TP8)	0.060	0.81192	10.000	0.20315
system_33	RK4 (TP9)	0.080	0.81192	10.000	0.20314
system_33	RK4 (TP10)	0.110	0.81192	10.000	0.20314
system_33	RK4 (TP11)	0.170	0.81192	10.000	0.20314
system_33	RK4 (TP12)	0.270	0.81192	10.000	0.20314
system_33	RK4 (TP13)	0.410	0.81192	10.000	0.20314
system_33	RK4 (TP14)	0.640	0.81192	10.000	0.20314
system_33	LA3 (TP8)	0.060	0.81202	10.000	0.20448
system_33	LA3 (TP9)	0.080	0.81197	10.000	0.20373
system_33	LA3 (TP10)	0.130	0.81194	10.000	0.20338
system_33	LA3 (TP11)	0.200	0.81193	10.000	0.20324
system_33	LA3 (TP12)	0.300	0.81193	10.000	0.20318
system_33	LA3 (TP13)	0.470	0.81192	10.000	0.20316
system_33	LA3 (TP14)	0.740	0.81192	10.000	0.20315
system_33	LC3 (TP8)	0.060	0.81211	10.000	0.20522
system_33	LC3 (TP9)	0.080	0.812	10.000	0.204
system_33	LC3 (TP10)	0.120	0.81196	10.000	0.20349
system_33	LC3 (TP11)	0.170	0.81194	10.000	0.20329
system_33	LC3 (TP12)	0.260	0.81193	10.000	0.2032
system_33	LC3 (TP13)	0.400	0.81192	10.000	0.20317
system_33	LC3 (TP14)	0.620	0.81192	10.000	0.20315
system_33	Riot (05, 1e-11)	0m3.466s	0.81192	-0.000	0.20314
system_33	Riot (10, 1e-11)	0m0.842s	0.81192	-0.000	0.20314
system_33	Riot (15, 1e-11)	0m0.886s	0.81192	-0.000	0.20314
system_33	Valencia-IVP (0.00025)	1m30.726s	0.8123	10.000	243.87
system_33	Valencia-IVP (0.0025)	0m1.521s	0.81566	10.000	249.32
system_33	Valencia-IVP (0.025)	0m0.257s	0.85019	10.000	309.55
system_33	VNODE-LP (15, 1e-14,1e-14)	0m0.041s	0.81192	10.000	0.20314
system_33	VNODE-LP (20, 1e-14,1e-14)	0m0.042s	0.81192	10.000	0.20314
system_33	VNODE-LP (25, 1e-14,1e-14)	0m0.039s	0.81192	10.000	0.20314

Table 3.30: Simulation results of Problem 34

Problems	Methods	c5t	c5w	c6t	c6w
system_34	TAYLOR4 (TP8)	0.010	4.7235e-09	10.000	2.9591e-07
system_34	TAYLOR4 (TP9)	0.010	7.8377e-10	10.000	3.0655e-08
system_34	TAYLOR4 (TP10)	0.020	1.0829e-10	10.000	3.5486e-09
system_34	TAYLOR4 (TP11)	0.020	1.2753e-11	10.000	4.2103e-10
system_34	TAYLOR4 (TP12)	0.030	1.3936e-12	10.000	4.4459e-11
system_34	TAYLOR4 (TP13)	0.040	2.1538e-13	10.000	7.7094e-12
system_34	TAYLOR4 (TP14)	0.070	1.2879e-13	10.000	5.361e-12
system_34	RK4 (TP8)	0.010	1.4271e-09	10.000	7.7418e-08
system_34	RK4 (TP9)	0.010	2.0589e-10	10.000	8.1568e-09
system_34	RK4 (TP10)	0.020	2.1419e-11	10.000	8.3585e-10
system_34	RK4 (TP11)	0.020	2.3852e-12	10.000	8.5283e-11
system_34	RK4 (TP12)	0.030	2.78e-13	10.000	9.3454e-12
system_34	RK4 (TP13)	0.040	6.2172e-14	10.000	1.9824e-12
system_34	RK4 (TP14)	0.060	5.5955e-14	10.000	1.8456e-12
system_34	LA3 (TP8)	0.010	8.073e-11	10.000	1.7684e-07
system_34	LA3 (TP9)	0.010	6.0024e-11	10.000	2.3486e-08
system_34	LA3 (TP10)	0.010	5.8509e-12	10.000	2.36e-09
system_34	LA3 (TP11)	0.020	5.4801e-13	10.000	2.6067e-10
system_34	LA3 (TP12)	0.020	7.7716e-14	10.000	2.9845e-11
system_34	LA3 (TP13)	0.030	3.0642e-14	10.000	4.0146e-12
system_34	LA3 (TP14)	0.050	3.6859e-14	10.000	1.5241e-12
system_34	LC3 (TP8)	0.010	4.5581e-10	10.000	1.7673e-07
system_34	LC3 (TP9)	0.010	1.0584e-10	10.000	1.8314e-08
system_34	LC3 (TP10)	0.020	1.5158e-11	10.000	2.4613e-09
system_34	LC3 (TP11)	0.020	1.7217e-12	10.000	2.8973e-10
system_34	LC3 (TP12)	0.010	1.9762e-13	10.000	2.8706e-11
system_34	LC3 (TP13)	0.040	4.7074e-14	10.000	4.4125e-12
system_34	LC3 (TP14)	0.060	4.4409e-14	10.000	1.6751e-12
system_34	Riot (05, 1e-11)	0m0.304s	1.3289e-12	-0.000	1.8114e-10
system_34	Riot (10, 1e-11)	0m0.241s	5.7954e-14	-0.000	3.439e-12
system_34	Riot (15, 1e-11)	0m0.268s	6.9944e-14	-0.000	6.0574e-13
system_34	Valencia-IVP (0.00025)	0m42.641s	1.6439e-05	10.000	0.0004796
system_34	Valencia-IVP (0.0025)	0m1.277s	0.00016439	10.000	0.0047963
system_34	Valencia-IVP (0.025)	0m0.165s	0.001644	10.000	0.047992
system_34	VNODE-LP (15, 1e-14,1e-14)	0m0.008s	8.8818e-16	10.000	3.5527e-14
system_34	VNODE-LP (20, 1e-14,1e-14)	0m0.010s	8.8818e-16	10.000	3.6415e-14
system_34	VNODE-LP (25, 1e-14,1e-14)	0m0.009s	8.8818e-16	10.000	2.931e-14

Table 3.31: Simulation results of Problem 35

Problems	Methods	c5t	c5w	c6t	c6w
system_35	TAYLOR4 (TP8)	0.010	0.94449	10.000	4.7953
system_35	TAYLOR4 (TP9)	0.020	0.94449	10.000	4.795
system_35	TAYLOR4 (TP10)	0.030	0.94449	10.000	4.7949
system_35	TAYLOR4 (TP11)	0.030	0.94449	10.000	4.7949
system_35	TAYLOR4 (TP12)	0.050	0.94449	10.000	4.7948
system_35	TAYLOR4 (TP13)	0.070	0.94449	10.000	4.7948
system_35	TAYLOR4 (TP14)	0.090	0.94449	10.000	4.7948
system_35	RK4 (TP8)	0.010	0.94449	10.000	4.7948
system_35	RK4 (TP9)	0.020	0.94449	10.000	4.7948
system_35	RK4 (TP10)	0.020	0.94449	10.000	4.7948
system_35	RK4 (TP11)	0.020	0.94449	10.000	4.7948
system_35	RK4 (TP12)	0.040	0.94449	10.000	4.7948
system_35	RK4 (TP13)	0.050	0.94449	10.000	4.7948
system_35	RK4 (TP14)	0.070	0.94449	10.000	4.7948
system_35	LA3 (TP8)	0.010	0.94461	10.000	4.8179
system_35	LA3 (TP9)	0.010	0.9446	10.000	4.809
system_35	LA3 (TP10)	0.010	0.94456	10.000	4.8035
system_35	LA3 (TP11)	0.020	0.94454	10.000	4.8002
system_35	LA3 (TP12)	0.030	0.94452	10.000	4.7982
system_35	LA3 (TP13)	0.040	0.94451	10.000	4.7969
system_35	LA3 (TP14)	0.050	0.9445	10.000	4.7962
system_35	LC3 (TP8)	0.010	0.94473	10.000	4.8375
system_35	LC3 (TP9)	0.010	0.94466	10.000	4.8213
system_35	LC3 (TP10)	0.020	0.94461	10.000	4.8111
system_35	LC3 (TP11)	0.020	0.94457	10.000	4.805
system_35	LC3 (TP12)	0.030	0.94454	10.000	4.8012
system_35	LC3 (TP13)	0.030	0.94452	10.000	4.7988
system_35	LC3 (TP14)	0.060	0.94451	10.000	4.7973
system_35	Riot (05, 1e-11)	0m26.070s	0.93958	-0.000	4.3033
system_35	Riot (10, 1e-11)	0m21.763s	0.93958	-0.000	4.3033
system_35	Riot (15, 1e-11)	0m1.415s	0.93958	-0.000	4.3033
system_35	Valencia-IVP (0.00025)	0m46.038s	0.93957	10.000	4.2038
system_35	Valencia-IVP (0.0025)	0m1.842s	0.93976	10.000	4.2101
system_35	Valencia-IVP (0.025)	0m0.161s	0.94163	10.000	4.2741
system_35	VNODE-LP (15, 1e-14,1e-14)	0m0.010s	0.94965	10.000	5.8441
system_35	VNODE-LP (20, 1e-14,1e-14)	0m0.008s	0.94965	10.000	5.8753
system_35	VNODE-LP (25, 1e-14,1e-14)	0m0.011s	0.94965	10.000	5.8753

Table 3.32: Simulation results of Problem 36

Problems	Methods	c5t	c5w	c6t	c6w
system_36	TAYLOR4 (TP8)	0.390	3.8302e-07	10.000	3.3818e-06
system_36	TAYLOR4 (TP9)	0.480	5.0076e-08	10.000	4.2414e-07
system_36	TAYLOR4 (TP10)	0.600	6.2302e-09	10.000	5.0646e-08
system_36	TAYLOR4 (TP11)	0.860	7.2726e-10	10.000	5.7903e-09
system_36	TAYLOR4 (TP12)	1.210	8.1487e-11	10.000	6.4198e-10
system_36	TAYLOR4 (TP13)	1.790	2.7196e-11	10.000	1.2778e-10
system_36	TAYLOR4 (TP14)	2.760	1.3596e-11	10.000	5.2596e-11
system_36	RK4 (TP8)	0.300	2.8762e-07	10.000	2.7084e-06
system_36	RK4 (TP9)	0.310	4.616e-08	10.000	3.7416e-07
system_36	RK4 (TP10)	0.380	6.6357e-09	10.000	4.8865e-08
system_36	RK4 (TP11)	0.480	8.5565e-10	10.000	5.9451e-09
system_36	RK4 (TP12)	0.630	1.0328e-10	10.000	6.8487e-10
system_36	RK4 (TP13)	0.880	1.2269e-11	10.000	7.8007e-11
system_36	RK4 (TP14)	1.240	1.7035e-12	10.000	1.1177e-11
system_36	LA3 (TP8)	0.300	2.2169e-07	10.000	2.0499e-06
system_36	LA3 (TP9)	0.330	3.3496e-08	10.000	2.7089e-07
system_36	LA3 (TP10)	0.390	4.619e-09	10.000	3.451e-08
system_36	LA3 (TP11)	0.510	5.7408e-10	10.000	4.0876e-09
system_36	LA3 (TP12)	0.650	6.7628e-11	10.000	4.6189e-10
system_36	LA3 (TP13)	0.910	7.7591e-12	10.000	5.2355e-11
system_36	LA3 (TP14)	1.420	1.231e-12	10.000	8.5425e-12
system_36	LC3 (TP8)	0.290	2.2911e-07	10.000	2.0717e-06
system_36	LC3 (TP9)	0.330	3.4984e-08	10.000	2.7398e-07
system_36	LC3 (TP10)	0.390	4.9685e-09	10.000	3.4966e-08
system_36	LC3 (TP11)	0.500	6.4688e-10	10.000	4.22e-09
system_36	LC3 (TP12)	0.660	7.7153e-11	10.000	4.8028e-10
system_36	LC3 (TP13)	0.890	9.0523e-12	10.000	5.493e-11
system_36	LC3 (TP14)	1.320	1.3678e-12	10.000	8.8445e-12
system_36	Riot (05, 1e-11)	0m1.095s	3.8821e-11	-0.000	2.6445e-10
system_36	Riot (10, 1e-11)	0m0.857s	2.176e-13	-0.000	4.5475e-12
system_36	Riot (15, 1e-11)	0m1.818s	3.1442e-13	-0.000	1.2212e-12
system_36	Valencia-IVP (0.00025)	1m34.728s	8.8326e-05	10.000	0.00054692
system_36	Valencia-IVP (0.0025)	0m1.368s	0.00088326	10.000	0.0054692
system_36	Valencia-IVP (0.025)	0m0.178s	0.0088326	10.000	0.054692
system_36	VNODE-LP (15, 1e-14,1e-14)	0m0.014s	1.3323e-14	10.000	9.4147e-14
system_36	VNODE-LP (20, 1e-14,1e-14)	0m0.014s	1.1546e-14	10.000	8.0824e-14
system_36	VNODE-LP (25, 1e-14,1e-14)	0m0.014s	7.9936e-15	10.000	5.9508e-14

Table 3.33: Simulation results of Problem 37

Problems	Methods	c5t	c5w	c6t	c6w
system_37	TAYLOR4 (TP8)	0.460	0.36452	10.000	1.0893
system_37	TAYLOR4 (TP9)	0.570	0.3637	10.000	1.0799
system_37	TAYLOR4 (TP10)	0.800	0.36315	10.000	1.0739
system_37	TAYLOR4 (TP11)	1.140	0.36279	10.000	1.0699
system_37	TAYLOR4 (TP12)	1.690	0.36255	10.000	1.0674
system_37	TAYLOR4 (TP13)	2.530	0.3624	10.000	1.0658
system_37	TAYLOR4 (TP14)	3.930	0.3623	10.000	1.0648
system_37	RK4 (TP8)	0.290	0.36214	10.000	1.063
system_37	RK4 (TP9)	0.360	0.36214	10.000	1.0631
system_37	RK4 (TP10)	0.370	0.36214	10.000	1.0631
system_37	RK4 (TP11)	0.560	0.36214	10.000	1.0631
system_37	RK4 (TP12)	0.780	0.36214	10.000	1.0631
system_37	RK4 (TP13)	1.090	0.36214	10.000	1.0631
system_37	RK4 (TP14)	1.630	0.36214	10.000	1.0631
system_37	LA3 (TP8)	0.320	0.36125	10.000	1.0578
system_37	LA3 (TP9)	0.300	0.36152	10.000	1.0594
system_37	LA3 (TP10)	0.430	0.36172	10.000	1.0606
system_37	LA3 (TP11)	0.590	0.36186	10.000	1.0615
system_37	LA3 (TP12)	0.800	0.36196	10.000	1.062
system_37	LA3 (TP13)	1.150	0.36202	10.000	1.0624
system_37	LA3 (TP14)	1.740	0.36207	10.000	1.0627
system_37	LC3 (TP8)	0.310	0.36119	10.000	1.0583
system_37	LC3 (TP9)	0.360	0.36148	10.000	1.0598
system_37	LC3 (TP10)	0.450	0.36169	10.000	1.0609
system_37	LC3 (TP11)	0.590	0.36184	10.000	1.0616
system_37	LC3 (TP12)	0.780	0.36195	10.000	1.0621
system_37	LC3 (TP13)	1.200	0.36201	10.000	1.0625
system_37	LC3 (TP14)	1.710	0.36206	10.000	1.0627
system_37	Riot (05, 1e-11)	1m11.410s	0.25904	-0.000	0.51435
system_37	Riot (10, 1e-11)	0m5.525s	0.25904	-0.000	0.51435
system_37	Riot (15, 1e-11)	0m20.456s	0.25904	-0.000	0.51435
system_37	Valencia-IVP (0.00025)	1m26.397s	0.25956	10.000	0.51575
system_37	Valencia-IVP (0.0025)	0m1.774s	0.26021	10.000	0.52027
system_37	Valencia-IVP (0.025)	0m0.170s	0.26796	10.000	0.56814
system_37	VNODE-LP (15, 1e-14,1e-14)	0m0.012s	0.26197	10.000	0.53714
system_37	VNODE-LP (20, 1e-14,1e-14)	0m0.014s	0.26206	10.000	0.53773
system_37	VNODE-LP (25, 1e-14,1e-14)	0m0.015s	0.26225	10.000	0.53846

Table 3.34: Simulation results of Problem 38

Problems	Methods	c5t	c5w	c6t	c6w
system_38	TAYLOR4 (TP8)	0.170	2.7922e-08	10.000	7.4431e-08
system_38	TAYLOR4 (TP9)	0.220	3.2672e-09	10.000	7.942e-09
system_38	TAYLOR4 (TP10)	0.270	4.1428e-10	10.000	8.3748e-10
system_38	TAYLOR4 (TP11)	0.400	4.6597e-11	10.000	8.5737e-11
system_38	TAYLOR4 (TP12)	0.560	5.4421e-12	10.000	8.7429e-12
system_38	TAYLOR4 (TP13)	0.870	6.8723e-13	10.000	9.297e-13
system_38	TAYLOR4 (TP14)	1.320	2.669e-13	10.000	1.6706e-13
system_38	RK4 (TP8)	0.120	2.1332e-08	10.000	4.498e-08
system_38	RK4 (TP9)	0.160	2.19e-09	10.000	4.8124e-09
system_38	RK4 (TP10)	0.220	2.2353e-10	10.000	4.9613e-10
system_38	RK4 (TP11)	0.300	2.2414e-11	10.000	5.1359e-11
system_38	RK4 (TP12)	0.430	2.2782e-12	10.000	5.2231e-12
system_38	RK4 (TP13)	0.620	2.7422e-13	10.000	5.4531e-13
system_38	RK4 (TP14)	0.930	9.8588e-14	10.000	8.3211e-14
system_38	LA3 (TP8)	0.090	2.3226e-08	10.000	5.5003e-08
system_38	LA3 (TP9)	0.120	2.0602e-09	10.000	5.7806e-09
system_38	LA3 (TP10)	0.160	1.8549e-10	10.000	5.8628e-10
system_38	LA3 (TP11)	0.220	1.7422e-11	10.000	6.0634e-11
system_38	LA3 (TP12)	0.280	1.6807e-12	10.000	6.1279e-12
system_38	LA3 (TP13)	0.420	1.9051e-13	10.000	6.3527e-13
system_38	LA3 (TP14)	0.600	6.2839e-14	10.000	8.3517e-14
system_38	LC3 (TP8)	0.100	3.1915e-08	10.000	3.6288e-08
system_38	LC3 (TP9)	0.080	2.9665e-09	10.000	3.5504e-09
system_38	LC3 (TP10)	0.180	3.4742e-10	10.000	3.6358e-10
system_38	LC3 (TP11)	0.240	3.4815e-11	10.000	3.6803e-11
system_38	LC3 (TP12)	0.310	3.511e-12	10.000	3.7114e-12
system_38	LC3 (TP13)	0.480	3.8325e-13	10.000	3.8405e-13
system_38	LC3 (TP14)	0.690	8.8596e-14	10.000	6.0923e-14
system_38	Riot (05, 1e-11)	0m1.119s	8.3338e-11	-0.000	3.9802e-10
system_38	Riot (10, 1e-11)	0m0.599s	3.0975e-14	-0.000	2.307e-11
system_38	Riot (15, 1e-11)	0m0.755s	4.4409e-15	-0.000	4.7198e-14
system_38	Valencia-IVP (0.00025)	1m10.629s	0.00053855	9.927	935.08
system_38	Valencia-IVP (0.0025)	0m4.512s	0.0054036	7.390	83.458
system_38	Valencia-IVP (0.025)	0m0.436s	0.055881	4.675	9.5271
system_38	VNODE-LP (15, 1e-14,1e-14)	0m0.027s	2.3315e-15	10.000	1.7986e-14
system_38	VNODE-LP (20, 1e-14,1e-14)	0m0.023s	1.4433e-15	10.000	1.2323e-14
system_38	VNODE-LP (25, 1e-14,1e-14)	0m0.026s	1.4155e-15	10.000	1.1435e-14

Table 3.35: Simulation results of Problem 39

Problems	Methods	c5t	c5w	c6t	c6w
system_39	TAYLOR4 (TP8)	0.180	0.098956	10.000	0.0040241
system_39	TAYLOR4 (TP9)	0.250	0.098918	10.000	0.0040059
system_39	TAYLOR4 (TP10)	0.350	0.098895	10.000	0.0039945
system_39	TAYLOR4 (TP11)	0.500	0.098881	10.000	0.0039875
system_39	TAYLOR4 (TP12)	0.760	0.098872	10.000	0.0039831
system_39	TAYLOR4 (TP13)	1.150	0.098866	10.000	0.0039803
system_39	TAYLOR4 (TP14)	1.730	0.098863	10.000	0.0039785
system_39	RK4 (TP8)	0.110	0.098857	10.000	0.0039757
system_39	RK4 (TP9)	0.170	0.098857	10.000	0.0039756
system_39	RK4 (TP10)	0.230	0.098857	10.000	0.0039756
system_39	RK4 (TP11)	0.310	0.098856	10.000	0.0039756
system_39	RK4 (TP12)	0.480	0.098856	10.000	0.0039756
system_39	RK4 (TP13)	0.690	0.098856	10.000	0.0039756
system_39	RK4 (TP14)	1.060	0.098856	10.000	0.0039756
system_39	LA3 (TP8)	0.100	0.10458	10.000	0.0045642
system_39	LA3 (TP9)	0.130	0.10263	10.000	0.0043234
system_39	LA3 (TP10)	0.180	0.10129	10.000	0.0041864
system_39	LA3 (TP11)	0.240	0.1004	10.000	0.0041045
system_39	LA3 (TP12)	0.330	0.099845	10.000	0.0040554
system_39	LA3 (TP13)	0.470	0.099481	10.000	0.0040253
system_39	LA3 (TP14)	0.690	0.09925	10.000	0.0040067
system_39	LC3 (TP8)	0.110	0.10484	10.000	0.0046338
system_39	LC3 (TP9)	0.150	0.10273	10.000	0.0043531
system_39	LC3 (TP10)	0.200	0.10128	10.000	0.0042001
system_39	LC3 (TP11)	0.270	0.1004	10.000	0.004112
system_39	LC3 (TP12)	0.380	0.099832	10.000	0.0040597
system_39	LC3 (TP13)	0.560	0.099472	10.000	0.0040278
system_39	LC3 (TP14)	0.820	0.099244	10.000	0.0040082
system_39	Riot (05, 1e-11)	0m3.777s	0.09197	-0.000	1.135e-05
system_39	Riot (10, 1e-11)	6m32.012s	0.09682	-0.000	0.24626
system_39	Riot (15, 1e-11)	13m4.722s	0.09682	-0.000	0.24626
system_39	Valencia-IVP (0.00025)	0m23.487s	0.67999	2.515	881.5
system_39	Valencia-IVP (0.0025)	0m1.379s	0.68374	2.303	6.9672
system_39	Valencia-IVP (0.025)	0m0.247s	0.73359	2.275	9.8884
system_39	VNODE-LP (15, 1e-14,1e-14)	0m0.028s	0.10211	10.000	0.29379
system_39	VNODE-LP (20, 1e-14,1e-14)	0m0.028s	0.10278	10.000	0.30109
system_39	VNODE-LP (25, 1e-14,1e-14)	0m0.025s	0.10322	10.000	0.3087

Table 3.36: Simulation results of Problem 40

Problems	Methods	c5t	c5w	c6t	c6w
system_40	TAYLOR4 (TP8)	25.590	1.5611e-06	9.522	0.62151
system_40	TAYLOR4 (TP9)	35.300	1.886e-07	10.000	0.24996
system_40	TAYLOR4 (TP10)	49.740	2.2374e-08	10.000	0.029197
system_40	TAYLOR4 (TP11)	74.140	2.6015e-09	10.000	0.0094394
system_40	TAYLOR4 (TP12)	113.610	2.9517e-10	10.000	0.0011596
system_40	TAYLOR4 (TP13)	173.010	3.3894e-11	10.000	0.0001425
system_40	TAYLOR4 (TP14)	266.180	5.7838e-12	10.000	2.3248e-05
system_40	RK4 (TP8)	18.480	1.0075e-06	10.000	0.54239
system_40	RK4 (TP9)	21.920	1.3601e-07	10.000	0.1267
system_40	RK4 (TP10)	30.550	1.5332e-08	10.000	0.023281
system_40	RK4 (TP11)	43.010	1.6852e-09	10.000	0.0039165
system_40	RK4 (TP12)	64.720	1.8093e-10	10.000	0.00066257
system_40	RK4 (TP13)	101.010	1.9397e-11	10.000	7.4275e-05
system_40	RK4 (TP14)	151.690	2.5571e-12	10.000	9.829e-06
system_40	LA3 (TP8)	16.790	1.1266e-06	10.000	0.60409
system_40	LA3 (TP9)	21.760	1.3529e-07	10.000	0.15636
system_40	LA3 (TP10)	29.580	1.6075e-08	10.000	0.026108
system_40	LA3 (TP11)	41.390	1.8938e-09	10.000	0.0063105
system_40	LA3 (TP12)	60.530	2.1847e-10	10.000	0.00083967
system_40	LA3 (TP13)	91.840	2.4841e-11	10.000	0.00010056
system_40	LA3 (TP14)	136.740	3.209e-12	10.000	1.2899e-05
system_40	LC3 (TP8)	16.910	1.2305e-06	9.878	0.61908
system_40	LC3 (TP9)	21.020	1.5037e-07	10.000	0.16173
system_40	LC3 (TP10)	28.780	1.8244e-08	10.000	0.033471
system_40	LC3 (TP11)	39.800	2.1564e-09	10.000	0.0072601
system_40	LC3 (TP12)	58.290	2.5061e-10	10.000	0.0009416
system_40	LC3 (TP13)	88.440	2.8377e-11	10.000	0.00011561
system_40	LC3 (TP14)	133.470	3.5767e-12	10.000	1.4508e-05
system_40	Riot (05, 1e-11)	0m26.087s	1.9465e-10	0.000	0
system_40	Riot (10, 1e-11)	11m50.212s	5.0149e-12	0.000	0
system_40	Riot (15, 1e-11)	60m12.975s	7.1054e-15	0.000	0
system_40	Valencia-IVP (0.00025)	0m12.132s	0.0010009	0.000	0
system_40	Valencia-IVP (0.0025)	0m0.366s	0.010036	0.000	0
system_40	Valencia-IVP (0.025)	0m0.035s	0.10322	0.000	0
system_40	VNODE-LP (15, 1e-14,1e-14)	0m0.046s	3.8192e-14	3.000	2.9702e-10
system_40	VNODE-LP (20, 1e-14,1e-14)	0m3.850s	2.7978e-14	0.000	0
system_40	VNODE-LP (25, 1e-14,1e-14)	0m4.400s	2.1316e-14	0.000	0

Table 3.37: Simulation results of Problem 41

Problems	Methods	c5t	c5w	c6t	c6w
system_41	TAYLOR4 (TP8)	38.240	0.23341	2.193	0.81586
system_41	TAYLOR4 (TP9)	56.240	0.23356	2.144	0.84571
system_41	TAYLOR4 (TP10)	82.870	0.23312	2.150	0.844
system_41	TAYLOR4 (TP11)	129.130	0.23292	2.133	0.86159
system_41	TAYLOR4 (TP12)	203.650	0.23286	2.139	0.85592
system_41	TAYLOR4 (TP13)	317.250	0.2329	2.131	0.85896
system_41	TAYLOR4 (TP14)	501.260	0.23288	2.155	0.84068
system_41	RK4 (TP8)	24.250	0.23001	2.229	0.79366
system_41	RK4 (TP9)	28.080	0.22991	2.223	0.79461
system_41	RK4 (TP10)	40.330	0.22957	2.211	0.80802
system_41	RK4 (TP11)	59.630	0.23089	2.190	0.8191
system_41	RK4 (TP12)	90.700	0.23209	2.165	0.83117
system_41	RK4 (TP13)	144.140	0.23228	2.156	0.83803
system_41	RK4 (TP14)	224.750	0.23232	2.149	0.84697
system_41	LA3 (TP8)	22.090	0.23611	2.198	0.81753
system_41	LA3 (TP9)	27.740	0.23533	2.193	0.81736
system_41	LA3 (TP10)	40.040	0.23467	2.157	0.83501
system_41	LA3 (TP11)	59.630	0.23386	2.143	0.84664
system_41	LA3 (TP12)	90.130	0.23336	2.160	0.83456
system_41	LA3 (TP13)	139.080	0.23337	2.149	0.84272
system_41	LA3 (TP14)	223.950	0.2331	2.150	0.84385
system_41	LC3 (TP8)	20.800	0.23786	2.181	0.821
system_41	LC3 (TP9)	29.050	0.23605	2.187	0.8203
system_41	LC3 (TP10)	40.420	0.23551	2.158	0.83568
system_41	LC3 (TP11)	59.040	0.23434	2.140	0.84825
system_41	LC3 (TP12)	91.580	0.23364	2.156	0.84324
system_41	LC3 (TP13)	142.150	0.23351	2.141	0.85052
system_41	LC3 (TP14)	220.420	0.23324	2.140	0.85526
system_41	Riot (05, 1e-11)	4m0.951s	0.22004	0.000	0
system_41	Riot (10, 1e-11)	81m51.368s	0.22004	0.000	0
system_41	Riot (15, 1e-11)	305m35.205s	0.22004	0.000	0
system_41	Valencia-IVP (0.00025)	0m10.623s	0.3966	0.000	0
system_41	Valencia-IVP (0.0025)	0m0.275s	0.4067	0.000	0
system_41	Valencia-IVP (0.025)	0m0.029s	0.51161	0.000	0
system_41	VNODE-LP (15, 1e-14,1e-14)	0m0.056s	0.24701	2.251	1.0915
system_41	VNODE-LP (20, 1e-14,1e-14)	0m0.061s	0.24758	2.240	1.1135
system_41	VNODE-LP (25, 1e-14,1e-14)	0m0.068s	0.24797	2.231	1.1282

Table 3.38: Simulation results of Problem 42

Problems	Methods	c5t	c5w	c6t	c6w
system_42	TAYLOR4 (TP8)	0.670	6.5182e-08	10.000	0.0001853
system_42	TAYLOR4 (TP9)	0.970	6.9772e-09	10.000	7.5231e-05
system_42	TAYLOR4 (TP10)	1.510	7.267e-10	10.000	2.3754e-05
system_42	TAYLOR4 (TP11)	2.340	7.4017e-11	10.000	3.93e-06
system_42	TAYLOR4 (TP12)	3.720	7.7236e-12	10.000	4.2675e-07
system_42	TAYLOR4 (TP13)	5.810	1.1897e-12	10.000	6.9992e-08
system_42	TAYLOR4 (TP14)	9.150	7.8271e-13	10.000	4.9014e-08
system_42	RK4 (TP8)	0.420	4.9849e-08	10.000	6.6443e-05
system_42	RK4 (TP9)	0.600	5.3644e-09	10.000	4.1734e-05
system_42	RK4 (TP10)	0.880	5.6878e-10	10.000	2.3717e-05
system_42	RK4 (TP11)	1.360	5.7543e-11	10.000	3.2772e-06
system_42	RK4 (TP12)	2.140	5.9515e-12	10.000	3.4427e-07
system_42	RK4 (TP13)	3.380	6.9655e-13	10.000	4.1534e-08
system_42	RK4 (TP14)	5.250	2.3448e-13	10.000	1.4537e-08
system_42	LA3 (TP8)	0.380	5.4075e-08	10.000	0.00017238
system_42	LA3 (TP9)	0.530	5.9316e-09	10.000	4.4174e-05
system_42	LA3 (TP10)	0.800	6.3172e-10	10.000	2.4504e-05
system_42	LA3 (TP11)	1.180	6.4227e-11	10.000	3.3231e-06
system_42	LA3 (TP12)	1.840	6.5794e-12	10.000	3.4972e-07
system_42	LA3 (TP13)	2.940	7.5517e-13	10.000	4.1783e-08
system_42	LA3 (TP14)	4.530	2.1938e-13	10.000	1.3757e-08
system_42	LC3 (TP8)	0.360	5.3813e-08	10.000	0.00013458
system_42	LC3 (TP9)	0.480	5.0378e-09	10.000	7.047e-05
system_42	LC3 (TP10)	0.700	4.7183e-10	10.000	2.1268e-05
system_42	LC3 (TP11)	1.030	4.4507e-11	10.000	2.4301e-06
system_42	LC3 (TP12)	1.600	4.3485e-12	10.000	2.4349e-07
system_42	LC3 (TP13)	2.530	4.9805e-13	10.000	2.9361e-08
system_42	LC3 (TP14)	3.990	1.7741e-13	10.000	1.1287e-08
system_42	Riot (05, 1e-11)	0m0.410s	1.4272e-10	-0.000	2.2876e-08
system_42	Riot (10, 1e-11)	0m0.197s	4.0634e-14	-0.000	1.0613e-09
system_42	Riot (15, 1e-11)	0m0.264s	1.8874e-15	-0.000	1.1936e-09
system_42	Valencia-IVP (0.00025)	0m4.192s	0.00030347	9.119	981.67
system_42	Valencia-IVP (0.0025)	0m0.741s	0.0030419	7.175	270.69
system_42	Valencia-IVP (0.025)	0m0.118s	0.031193	5.000	19.406
system_42	VNODE-LP (15, 1e-14,1e-14)	0m0.010s	5.5511e-15	10.000	3.5123e-12
system_42	VNODE-LP (20, 1e-14,1e-14)	0m0.007s	3.7748e-15	10.000	2.3554e-12
system_42	VNODE-LP (25, 1e-14,1e-14)	0m0.010s	3.6637e-15	10.000	2.6627e-12

Table 3.39: Simulation results of Problem 43

Problems	Methods	c5t	c5w	c6t	c6w
system_43	TAYLOR4 (TP8)	0.950	0.40394	3.329	6930.6
system_43	TAYLOR4 (TP9)	1.460	0.40494	3.308	4411.4
system_43	TAYLOR4 (TP10)	2.430	0.40469	3.306	2799.3
system_43	TAYLOR4 (TP11)	3.590	0.4051	3.299	1762.2
system_43	TAYLOR4 (TP12)	5.760	0.4055	3.289	1112.5
system_43	TAYLOR4 (TP13)	9.260	0.40555	3.287	701.1
system_43	TAYLOR4 (TP14)	14.500	0.40561	3.280	442.54
system_43	RK4 (TP8)	0.530	0.40361	3.338	8726.8
system_43	RK4 (TP9)	0.800	0.40434	3.333	5561.3
system_43	RK4 (TP10)	1.200	0.40498	3.316	3507.6
system_43	RK4 (TP11)	1.850	0.40521	3.301	2218.8
system_43	RK4 (TP12)	2.930	0.40515	3.288	1405.5
system_43	RK4 (TP13)	4.610	0.4055	3.282	888.14
system_43	RK4 (TP14)	7.290	0.40557	3.279	560.43
system_43	LA3 (TP8)	0.490	0.41849	3.263	11235
system_43	LA3 (TP9)	0.680	0.41437	3.282	7043.4
system_43	LA3 (TP10)	1.030	0.41131	3.280	4481.6
system_43	LA3 (TP11)	1.590	0.40936	3.277	2831.3
system_43	LA3 (TP12)	2.470	0.40803	3.276	1786.1
system_43	LA3 (TP13)	3.930	0.40718	3.283	1129.3
system_43	LA3 (TP14)	6.240	0.40664	3.276	713.89
system_43	LC3 (TP8)	0.460	0.42315	3.230	10495
system_43	LC3 (TP9)	0.660	0.41741	3.254	6704.5
system_43	LC3 (TP10)	0.990	0.41374	3.259	4299.6
system_43	LC3 (TP11)	1.500	0.41082	3.267	2691.3
system_43	LC3 (TP12)	2.310	0.40895	3.271	1711.2
system_43	LC3 (TP13)	3.680	0.40774	3.271	1079.2
system_43	LC3 (TP14)	5.800	0.40699	3.272	680.82
system_43	Riot (05, 1e-11)	0m57.400s	0.36095	0.000	0
system_43	Riot (10, 1e-11)	42m34.441s	0.36736	0.000	0
system_43	Riot (15, 1e-11)	335m18.382s	0.36736	0.000	0
system_43	Valencia-IVP (0.00025)	0m4.077s	0.63512	2.885	954.65
system_43	Valencia-IVP (0.0025)	0m0.463s	0.63944	2.860	300.15
system_43	Valencia-IVP (0.025)	0m0.116s	0.68415	2.650	29.66
system_43	VNODE-LP (15, 1e-14,1e-14)	0m0.010s	0.55406	2.715	13888
system_43	VNODE-LP (20, 1e-14,1e-14)	0m0.011s	0.55889	2.580	29046
system_43	VNODE-LP (25, 1e-14,1e-14)	0m0.009s	0.52831	2.438	43755

Table 3.40: Simulation results of Problem 44

Problems	Methods	c5t	c5w	c6t	c6w
system_44	TAYLOR4 (TP8)	99.250	1.6729e-08	10.000	5.1709e-07
system_44	TAYLOR4 (TP9)	147.970	1.7394e-09	10.000	7.0242e-08
system_44	TAYLOR4 (TP10)	226.180	1.7919e-10	10.000	9.4333e-09
system_44	TAYLOR4 (TP11)	350.780	1.8123e-11	10.000	1.2442e-09
system_44	TAYLOR4 (TP12)	555.550	1.82e-12	10.000	1.6044e-10
system_44	TAYLOR4 (TP13)	878.990	1.9962e-13	10.000	2.0348e-11
system_44	TAYLOR4 (TP14)	1374.220	1.9529e-13	10.000	2.7485e-12
system_44	RK4 (TP8)	87.650	2.171e-08	10.000	6.5343e-07
system_44	RK4 (TP9)	130.540	2.2593e-09	10.000	8.9434e-08
system_44	RK4 (TP10)	200.110	2.3279e-10	10.000	1.2006e-08
system_44	RK4 (TP11)	312.950	2.3621e-11	10.000	1.5783e-09
system_44	RK4 (TP12)	493.220	2.3793e-12	10.000	2.0472e-10
system_44	RK4 (TP13)	783.320	2.4301e-13	10.000	2.5788e-11
system_44	RK4 (TP14)	1230.930	7.9048e-14	10.000	3.2618e-12
system_44	LA3 (TP8)	71.940	1.6981e-08	10.000	5.8303e-07
system_44	LA3 (TP9)	98.920	1.7519e-09	10.000	6.9277e-08
system_44	LA3 (TP10)	149.090	1.7923e-10	10.000	8.6749e-09
system_44	LA3 (TP11)	233.660	1.8224e-11	10.000	1.1029e-09
system_44	LA3 (TP12)	361.000	1.8307e-12	10.000	1.3978e-10
system_44	LA3 (TP13)	9.390	0	0.000	0
system_44	LA3 (TP14)	13.960	0	0.000	0
system_44	LC3 (TP8)	75.260	1.7199e-08	10.000	6.3209e-07
system_44	LC3 (TP9)	106.040	1.7313e-09	10.000	7.4196e-08
system_44	LC3 (TP10)	163.720	1.801e-10	10.000	9.1825e-09
system_44	LC3 (TP11)	248.310	1.8118e-11	10.000	1.157e-09
system_44	LC3 (TP12)	9.380	0	0.000	0
system_44	LC3 (TP13)	9.390	0	0.000	0
system_44	LC3 (TP14)	13.940	0	0.000	0
system_44	Riot				
system_44	Valencia-IVP (0.00025)	0m17.732s	0.00067987	8.555	999.95
system_44	Valencia-IVP (0.0025)	0m1.845s	0.0068261	7.338	997.5
system_44	Valencia-IVP (0.025)	0m0.222s	0.071092	6.000	977.47
system_44	VNODE-LP (15, 1e-14,1e-14)	0m0.026s	8.3267e-16	10.000	1.0658e-14
system_44	VNODE-LP (20, 1e-14,1e-14)	0m0.019s	4.996e-16	10.000	5.5511e-15
system_44	VNODE-LP (25, 1e-14,1e-14)	0m0.014s	1.9429e-16	10.000	3.9968e-15

Table 3.41: Simulation results of Problem 45

Problems	Methods	c5t	c5w	c6t	c6w
system_45	TAYLOR4 (TP8)	109.900	0.36788	10.000	0.1126
system_45	TAYLOR4 (TP9)	162.030	0.36788	10.000	0.1126
system_45	TAYLOR4 (TP10)	247.300	0.36788	10.000	0.1126
system_45	TAYLOR4 (TP11)	385.000	0.36788	10.000	0.1126
system_45	TAYLOR4 (TP12)	610.920	0.36788	10.000	0.1126
system_45	TAYLOR4 (TP13)	978.780	0.36788	10.000	0.1126
system_45	TAYLOR4 (TP14)	1538.830	0.36788	10.000	0.1126
system_45	RK4 (TP8)	95.190	0.36788	10.000	0.1126
system_45	RK4 (TP9)	141.530	0.36788	10.000	0.1126
system_45	RK4 (TP10)	218.660	0.36788	10.000	0.1126
system_45	RK4 (TP11)	338.220	0.36788	10.000	0.1126
system_45	RK4 (TP12)	538.980	0.36788	10.000	0.1126
system_45	RK4 (TP13)	853.390	0.36788	10.000	0.1126
system_45	RK4 (TP14)	1335.820	0.36788	10.000	0.1126
system_45	LA3 (TP8)	75.870	0.38756	10.000	0.18747
system_45	LA3 (TP9)	106.810	0.38077	10.000	0.15599
system_45	LA3 (TP10)	164.780	0.37601	10.000	0.13858
system_45	LA3 (TP11)	253.600	0.37304	10.000	0.12843
system_45	LA3 (TP12)	392.330	0.37115	10.000	0.12237
system_45	LA3 (TP13)	628.270	0.36994	10.000	0.11868
system_45	LA3 (TP14)	990.970	0.36917	10.000	0.1164
system_45	LC3 (TP8)	79.310	0.38884	10.000	0.192
system_45	LC3 (TP9)	114.090	0.38141	10.000	0.15824
system_45	LC3 (TP10)	175.810	0.37636	10.000	0.13975
system_45	LC3 (TP11)	272.780	0.37326	10.000	0.1291
system_45	LC3 (TP12)	431.510	0.37127	10.000	0.12276
system_45	LC3 (TP13)	676.980	0.37001	10.000	0.11892
system_45	LC3 (TP14)	1073.260	0.36923	10.000	0.11655
system_45	Riot				
system_45	Valencia-IVP (0.00025)	0m17.383s	2.72	4.274	999.57
system_45	Valencia-IVP (0.0025)	0m1.838s	2.7353	4.263	997.63
system_45	Valencia-IVP (0.025)	0m0.222s	2.8947	4.150	973.41
system_45	VNODE-LP (15, 1e-14,1e-14)	0m0.024s	0.36788	10.000	0.66718
system_45	VNODE-LP (20, 1e-14,1e-14)	0m0.020s	0.36788	10.000	0.66718
system_45	VNODE-LP (25, 1e-14,1e-14)	0m0.013s	0.36788	10.000	0.66718

Table 3.42: Simulation results of Problem 46

Problems	Methods	c5t	c5w	c6t	c6w
system_46	TAYLOR4 (TP8)	206.640	2.8129e-07	10.000	4.6794e-08
system_46	TAYLOR4 (TP9)	295.550	3.9758e-08	10.000	6.1744e-09
system_46	TAYLOR4 (TP10)	442.500	5.6043e-09	10.000	9.4693e-10
system_46	TAYLOR4 (TP11)	659.730	7.9915e-10	10.000	1.3713e-10
system_46	TAYLOR4 (TP12)	1006.260	1.1174e-10	10.000	2.0165e-11
system_46	TAYLOR4 (TP13)	1520.690	1.557e-11	10.000	2.9375e-12
system_46	TAYLOR4 (TP14)	2311.850	2.162e-12	10.000	4.3671e-13
system_46	RK4 (TP8)	181.660	3.5581e-07	10.000	6.0115e-08
system_46	RK4 (TP9)	266.450	5.0501e-08	10.000	8.4781e-09
system_46	RK4 (TP10)	398.290	7.1016e-09	10.000	1.1435e-09
system_46	RK4 (TP11)	595.700	9.957e-10	10.000	1.7022e-10
system_46	RK4 (TP12)	900.870	1.3983e-10	10.000	2.492e-11
system_46	RK4 (TP13)	1372.050	1.9572e-11	10.000	3.7196e-12
system_46	RK4 (TP14)	2083.170	2.7271e-12	10.000	5.4328e-13
system_46	LA3 (TP8)	142.750	3.029e-07	10.000	8.59e-08
system_46	LA3 (TP9)	204.050	3.6926e-08	10.000	7.3781e-09
system_46	LA3 (TP10)	301.230	4.8977e-09	10.000	9.0584e-10
system_46	LA3 (TP11)	455.320	6.5218e-10	10.000	1.2048e-10
system_46	LA3 (TP12)	683.080	9.0125e-11	10.000	1.6853e-11
system_46	LA3 (TP13)	1035.690	1.2352e-11	10.000	2.3525e-12
system_46	LA3 (TP14)	18.270	0	0.000	0
system_46	LC3 (TP8)	149.890	3.2956e-07	10.000	1.0752e-07
system_46	LC3 (TP9)	219.390	4.0029e-08	10.000	8.8952e-09
system_46	LC3 (TP10)	323.850	5.1723e-09	10.000	1.0649e-09
system_46	LC3 (TP11)	490.370	6.9923e-10	10.000	1.2867e-10
system_46	LC3 (TP12)	737.050	9.5419e-11	10.000	1.7671e-11
system_46	LC3 (TP13)	1110.750	1.3104e-11	10.000	2.5332e-12
system_46	LC3 (TP14)	18.420	0	0.000	0
system_46	Riot				
system_46	Valencia-IVP (0.00025)	0m19.620s	0.90083	1.613	998.27
system_46	Valencia-IVP (0.0025)	0m2.097s	10.696	1.383	994.33
system_46	Valencia-IVP (0.025)	0m0.280s	717.1	1.000	717.1
system_46	VNODE-LP (15, 1e-14,1e-14)	0m0.112s	2.9109e-15	10.000	8.7708e-14
system_46	VNODE-LP (20, 1e-14,1e-14)	0m0.064s	1.5613e-15	10.000	3.9968e-14
system_46	VNODE-LP (25, 1e-14,1e-14)	0m0.040s	8.3267e-16	10.000	2.4092e-14

Table 3.43: Simulation results of Problem 47

Problems	Methods	c5t	c5w	c6t	c6w
system_47	TAYLOR4 (TP8)	251.600	0.073576	10.000	9.138e-06
system_47	TAYLOR4 (TP9)	370.930	0.073576	10.000	9.0857e-06
system_47	TAYLOR4 (TP10)	574.250	0.073576	10.000	9.08e-06
system_47	TAYLOR4 (TP11)	896.240	0.073576	10.000	9.08e-06
system_47	TAYLOR4 (TP12)	1411.760	0.073576	10.000	9.08e-06
system_47	TAYLOR4 (TP13)	2235.520	0.073576	10.000	9.08e-06
system_47	TAYLOR4 (TP14)	3574.470	0.073576	10.000	9.08e-06
system_47	RK4 (TP8)	214.830	0.073576	10.000	9.1474e-06
system_47	RK4 (TP9)	329.930	0.073576	10.000	9.0882e-06
system_47	RK4 (TP10)	514.490	0.073576	10.000	9.08e-06
system_47	RK4 (TP11)	804.210	0.073576	10.000	9.08e-06
system_47	RK4 (TP12)	1261.520	0.073576	10.000	9.08e-06
system_47	RK4 (TP13)	1985.340	0.073576	10.000	9.08e-06
system_47	RK4 (TP14)	3126.690	0.073576	10.000	9.08e-06
system_47	LA3 (TP8)	172.260	0.073587	10.000	5.1859e-05
system_47	LA3 (TP9)	253.280	0.073581	10.000	2.9554e-05
system_47	LA3 (TP10)	387.100	0.073578	10.000	1.9645e-05
system_47	LA3 (TP11)	606.460	0.073577	10.000	1.4898e-05
system_47	LA3 (TP12)	939.230	0.073576	10.000	1.2458e-05
system_47	LA3 (TP13)	1476.800	0.073576	10.000	1.1107e-05
system_47	LA3 (TP14)	2318.910	0.073576	10.000	1.0316e-05
system_47	LC3 (TP8)	183.610	0.073588	10.000	5.7105e-05
system_47	LC3 (TP9)	275.560	0.073581	10.000	3.2142e-05
system_47	LC3 (TP10)	417.670	0.073578	10.000	2.056e-05
system_47	LC3 (TP11)	655.720	0.073577	10.000	1.5314e-05
system_47	LC3 (TP12)	1024.290	0.073576	10.000	1.2655e-05
system_47	LC3 (TP13)	1599.140	0.073576	10.000	1.1206e-05
system_47	LC3 (TP14)	2513.770	0.073576	10.000	1.0371e-05
system_47	Riot				
system_47	Valencia-IVP (0.00025)	0m19.696s	43.149	1.244	998.7
system_47	Valencia-IVP (0.0025)	0m2.122s	62.436	1.215	989.09
system_47	Valencia-IVP (0.025)	0m0.270s	832.17	0.975	832.17
system_47	VNODE-LP (15, 1e-14,1e-14)	0m0.112s	0.073576	10.000	0.19992
system_47	VNODE-LP (20, 1e-14,1e-14)	0m0.063s	0.073576	10.000	0.19992
system_47	VNODE-LP (25, 1e-14,1e-14)	0m0.038s	0.073576	10.000	0.19992

Table 3.44: Simulation results of Problem 48

Problems	Methods	c5t	c5w	c6t	c6w
system_48	TAYLOR4 (TP8)	270.510	3.5098e-08	10.000	7.1884e-07
system_48	TAYLOR4 (TP9)	411.140	3.7378e-09	10.000	1.0437e-07
system_48	TAYLOR4 (TP10)	636.760	3.9037e-10	10.000	1.5031e-08
system_48	TAYLOR4 (TP11)	1003.860	3.9284e-11	10.000	2.1706e-09
system_48	TAYLOR4 (TP12)	1586.020	4.0629e-12	10.000	3.1246e-10
system_48	TAYLOR4 (TP13)	2484.510	4.9355e-13	10.000	4.4848e-11
system_48	TAYLOR4 (TP14)	3908.730	1.9457e-13	10.000	6.4941e-12
system_48	RK4 (TP8)	244.580	4.4892e-08	10.000	9.0045e-07
system_48	RK4 (TP9)	372.300	4.8038e-09	10.000	1.306e-07
system_48	RK4 (TP10)	583.480	5.112e-10	10.000	1.8909e-08
system_48	RK4 (TP11)	921.360	5.1871e-11	10.000	2.7231e-09
system_48	RK4 (TP12)	1449.170	5.2396e-12	10.000	3.9235e-10
system_48	RK4 (TP13)	2274.660	5.6177e-13	10.000	5.6309e-11
system_48	RK4 (TP14)	3598.580	1.0719e-13	10.000	8.1017e-12
system_48	LA3 (TP8)	192.150	3.5165e-08	10.000	5.7435e-06
system_48	LA3 (TP9)	282.410	3.6459e-09	10.000	3.0749e-07
system_48	LA3 (TP10)	432.900	3.8153e-10	10.000	2.5881e-08
system_48	LA3 (TP11)	688.310	3.9668e-11	10.000	2.7242e-09
system_48	LA3 (TP12)	1069.590	4.036e-12	10.000	3.2499e-10
system_48	LA3 (TP13)	24.680	0	0.000	0
system_48	LA3 (TP14)	32.600	0	0.000	0
system_48	LC3 (TP8)	206.350	3.7145e-08	10.000	9.4115e-06
system_48	LC3 (TP9)	303.850	3.7256e-09	10.000	3.9047e-07
system_48	LC3 (TP10)	468.430	3.9032e-10	10.000	2.9785e-08
system_48	LC3 (TP11)	738.850	3.9451e-11	10.000	3.0202e-09
system_48	LC3 (TP12)	1151.230	4.015e-12	10.000	3.5442e-10
system_48	LC3 (TP13)	24.670	0	0.000	0
system_48	LC3 (TP14)	32.170	0	0.000	0
system_48	Riot				
system_48	Valencia-IVP (0.00025)	0m24.122s	0.004682	4.352	999.44
system_48	Valencia-IVP (0.0025)	0m2.676s	0.047669	3.725	994.41
system_48	Valencia-IVP (0.025)	0m0.311s	0.57528	2.950	913.46
system_48	VNODE-LP (15, 1e-14,1e-14)	0m0.041s	8.0491e-16	10.000	9.194e-16
system_48	VNODE-LP (20, 1e-14,1e-14)	0m0.029s	7.2164e-16	10.000	3.4001e-16
system_48	VNODE-LP (25, 1e-14,1e-14)	0m0.023s	3.0531e-16	10.000	2.498e-16

Table 3.45: Simulation results of Problem 49

Problems	Methods	c5t	c5w	c6t	c6w
system_49	TAYLOR4 (TP8)	309.470	0.10763	10.000	0.011897
system_49	TAYLOR4 (TP9)	471.270	0.10763	10.000	0.011895
system_49	TAYLOR4 (TP10)	733.000	0.10763	10.000	0.011895
system_49	TAYLOR4 (TP11)	1166.480	0.10763	10.000	0.011895
system_49	TAYLOR4 (TP12)	1859.450	0.10763	10.000	0.011895
system_49	TAYLOR4 (TP13)	2949.570	0.10763	10.000	0.011895
system_49	TAYLOR4 (TP14)	4695.330	0.10763	10.000	0.011895
system_49	RK4 (TP8)	272.790	0.10763	10.000	0.011897
system_49	RK4 (TP9)	422.230	0.10763	10.000	0.011895
system_49	RK4 (TP10)	655.010	0.10763	10.000	0.011895
system_49	RK4 (TP11)	1042.530	0.10763	10.000	0.011895
system_49	RK4 (TP12)	1639.250	0.10763	10.000	0.011895
system_49	RK4 (TP13)	2579.950	0.10763	10.000	0.011895
system_49	RK4 (TP14)	4085.610	0.10763	10.000	0.011895
system_49	LA3 (TP8)	212.960	0.11444	10.000	0.059049
system_49	LA3 (TP9)	317.320	0.11195	10.000	0.033987
system_49	LA3 (TP10)	489.220	0.11035	10.000	0.02343
system_49	LA3 (TP11)	773.440	0.10933	10.000	0.018369
system_49	LA3 (TP12)	1212.350	0.1087	10.000	0.015688
system_49	LA3 (TP13)	1913.060	0.1083	10.000	0.01418
system_49	LA3 (TP14)	3027.240	0.10805	10.000	0.013295
system_49	LC3 (TP8)	227.850	0.11525	10.000	0.063765
system_49	LC3 (TP9)	339.800	0.11229	10.000	0.035622
system_49	LC3 (TP10)	526.690	0.11049	10.000	0.024098
system_49	LC3 (TP11)	839.800	0.10941	10.000	0.018691
system_49	LC3 (TP12)	1309.320	0.10874	10.000	0.015859
system_49	LC3 (TP13)	2073.580	0.10833	10.000	0.014276
system_49	LC3 (TP14)	3273.800	0.10807	10.000	0.013352
system_49	Riot				
system_49	Valencia-IVP (0.00025)	0m24.032s	5.8874	2.488	999.56
system_49	Valencia-IVP (0.0025)	0m2.571s	5.9852	2.475	998.38
system_49	Valencia-IVP (0.025)	0m0.314s	7.1174	2.350	997.96
system_49	VNODE-LP (15, 1e-14,1e-14)	0m0.044s	0.10763	10.000	0.011895
system_49	VNODE-LP (20, 1e-14,1e-14)	0m0.030s	0.10763	10.000	0.011895
system_49	VNODE-LP (25, 1e-14,1e-14)	0m0.021s	0.10763	10.000	0.011895

Table 3.46: Simulation results of Problem 56

Problems	Methods	c5t	c5w	c6t	c6w
system_56	TAYLOR4 (TP8)	1.210	1.2465e-05	10.000	8.3213e-05
system_56	TAYLOR4 (TP9)	1.470	4.9579e-06	10.000	4.0366e-05
system_56	TAYLOR4 (TP10)	2.060	2.875e-06	10.000	3.0631e-05
system_56	TAYLOR4 (TP11)	2.980	9.1925e-07	10.000	3.1768e-05
system_56	TAYLOR4 (TP12)	4.320	2.9164e-07	10.000	0.00013262
system_56	TAYLOR4 (TP13)	6.340	8.4087e-08	10.000	0.00010804
system_56	TAYLOR4 (TP14)	9.610	1.159e-08	10.000	3.8782e-05
system_56	RK4 (TP8)	0.590	2.9768e-07	10.000	1.6874e-05
system_56	RK4 (TP9)	0.800	4.3149e-08	10.000	1.0541e-05
system_56	RK4 (TP10)	1.040	6.4105e-09	10.000	8.784e-06
system_56	RK4 (TP11)	1.550	8.693e-10	10.000	2.2891e-06
system_56	RK4 (TP12)	2.300	1.1902e-10	10.000	3.4672e-07
system_56	RK4 (TP13)	3.410	1.572e-11	10.000	4.8711e-08
system_56	RK4 (TP14)	5.200	2.0466e-12	10.000	6.8005e-09
system_56	LA3 (TP8)	0.590	2.3105e-07	10.000	2.6021e-05
system_56	LA3 (TP9)	0.790	3.7592e-08	10.000	1.2067e-05
system_56	LA3 (TP10)	1.090	5.8336e-09	10.000	9.2705e-06
system_56	LA3 (TP11)	1.520	8.9354e-10	10.000	2.7425e-06
system_56	LA3 (TP12)	2.210	1.3327e-10	10.000	4.3035e-07
system_56	LA3 (TP13)	3.110	1.9496e-11	10.000	6.3061e-08
system_56	LA3 (TP14)	4.550	2.8287e-12	10.000	9.2639e-09
system_56	LC3 (TP8)	0.600	2.2727e-07	10.000	2.0461e-05
system_56	LC3 (TP9)	0.780	3.6407e-08	10.000	1.6216e-05
system_56	LC3 (TP10)	1.090	5.4528e-09	10.000	9.3004e-06
system_56	LC3 (TP11)	1.570	7.8127e-10	10.000	2.411e-06
system_56	LC3 (TP12)	2.350	1.062e-10	10.000	3.3882e-07
system_56	LC3 (TP13)	3.370	1.3902e-11	10.000	4.5448e-08
system_56	LC3 (TP14)	5.080	1.8026e-12	10.000	6.1851e-09
system_56	Riot (02, 1e-11)	0m2.480s	2.643e-07	-0.000	0.001449
system_56	Riot (05, 1e-11)	0m0.300s	6.8263e-11	-0.000	2.0833e-07
system_56	Riot (10, 1e-11)	0m0.259s	1.0353e-12	-0.000	1.1906e-09
system_56	Riot (15, 1e-11)	0m0.375s	4.563e-14	-0.000	6.2571e-12
system_56	Valencia-IVP (0.00025)	0m1.982s	0.00019354	10.000	4.7911
system_56	Valencia-IVP (0.0025)	0m0.184s	0.0019484	10.000	48.755
system_56	Valencia-IVP (0.025)	0m0.026s	0.020834	10.000	582.16
system_56	VNODE-LP (15, 1e-14,1e-14)	0m0.015s	4.6629e-15	10.000	6.9611e-14
system_56	VNODE-LP (20, 1e-14,1e-14)	0m0.017s	3.5527e-15	10.000	5.948e-14
system_56	VNODE-LP (25, 1e-14,1e-14)	0m0.019s	2.7756e-15	10.000	3.9801e-14

Table 3.47: Simulation results of Problem 57

Problems	Methods	c5t	c5w	c6t	c6w
system_57	TAYLOR4 (TP8)	1.060	0.0067999	10.000	0.0034968
system_57	TAYLOR4 (TP9)	1.420	0.0067971	10.000	0.0033828
system_57	TAYLOR4 (TP10)	2.050	0.0067966	10.000	0.003375
system_57	TAYLOR4 (TP11)	2.970	0.0067958	10.000	0.0033756
system_57	TAYLOR4 (TP12)	4.230	0.0067957	10.000	0.0034279
system_57	TAYLOR4 (TP13)	6.310	0.0067956	10.000	0.0034369
system_57	TAYLOR4 (TP14)	9.570	0.0067956	10.000	0.0033881
system_57	RK4 (TP8)	0.580	0.0067958	10.000	0.0033643
system_57	RK4 (TP9)	0.780	0.0067956	10.000	0.0033621
system_57	RK4 (TP10)	1.060	0.0067956	10.000	0.0033584
system_57	RK4 (TP11)	1.530	0.0067956	10.000	0.0033523
system_57	RK4 (TP12)	2.410	0.0067956	10.000	0.0033504
system_57	RK4 (TP13)	3.320	0.0067956	10.000	0.0033501
system_57	RK4 (TP14)	5.180	0.0067956	10.000	0.00335
system_57	LA3 (TP8)	0.580	0.0069796	10.000	0.013207
system_57	LA3 (TP9)	0.770	0.0069339	10.000	0.0083059
system_57	LA3 (TP10)	1.060	0.0068985	10.000	0.0086576
system_57	LA3 (TP11)	1.480	0.0068688	10.000	0.0060924
system_57	LA3 (TP12)	2.160	0.0068461	10.000	0.0054237
system_57	LA3 (TP13)	3.030	0.0068304	10.000	0.0045981
system_57	LA3 (TP14)	4.490	0.0068194	10.000	0.004046
system_57	LC3 (TP8)	0.590	0.0070117	10.000	0.014465
system_57	LC3 (TP9)	0.770	0.0069595	10.000	0.0095625
system_57	LC3 (TP10)	1.070	0.0069121	10.000	0.010461
system_57	LC3 (TP11)	1.520	0.0068748	10.000	0.0065855
system_57	LC3 (TP12)	2.280	0.0068492	10.000	0.0056602
system_57	LC3 (TP13)	3.300	0.0068311	10.000	0.0047481
system_57	LC3 (TP14)	5.050	0.0068188	10.000	0.004001
system_57	Riot (05, 1e-11)	0m0.342s	0.013481	-0.000	33.434
system_57	Riot (10, 1e-11)	0m0.308s	0.012937	-0.000	4.2549
system_57	Riot (15, 1e-11)	0m0.517s	0.012937	-0.000	1.078
system_57	Valencia-IVP (0.00025)	0m1.863s	0.015962	10.000	288.91
system_57	Valencia-IVP (0.0025)	0m0.180s	0.017692	10.000	337.44
system_57	Valencia-IVP (0.025)	0m0.024s	0.035905	10.000	921.84
system_57	VNODE-LP (15, 1e-14,1e-14)	0m0.015s	0.0067956	10.000	0.054773
system_57	VNODE-LP (20, 1e-14,1e-14)	0m0.018s	0.0067956	10.000	0.054773
system_57	VNODE-LP (25, 1e-14,1e-14)	0m0.018s	0.0067956	10.000	0.054773

Table 3.48: Simulation results of Problem 58

Problems	Methods	c5t	c5w	c6t	c6w
system_58	TAYLOR4 (TP8)	0.180	7.7141e-08	10.000	3.4549e-05
system_58	TAYLOR4 (TP9)	0.260	8.0285e-09	10.000	2.1271e-05
system_58	TAYLOR4 (TP10)	0.410	8.2963e-10	10.000	9.8007e-06
system_58	TAYLOR4 (TP11)	0.640	8.5321e-11	10.000	3.2405e-06
system_58	TAYLOR4 (TP12)	1.000	8.848e-12	10.000	3.459e-07
system_58	TAYLOR4 (TP13)	1.550	1.6449e-12	10.000	5.4546e-08
system_58	TAYLOR4 (TP14)	2.730	1.3616e-12	10.000	3.6384e-08
system_58	RK4 (TP8)	0.130	1.1435e-07	10.000	3.0612e-05
system_58	RK4 (TP9)	0.190	1.5557e-08	10.000	1.7471e-05
system_58	RK4 (TP10)	0.280	1.8288e-09	10.000	8.2663e-06
system_58	RK4 (TP11)	0.440	1.9948e-10	10.000	4.5881e-06
system_58	RK4 (TP12)	0.680	2.3118e-11	10.000	5.8802e-07
system_58	RK4 (TP13)	1.030	2.7485e-12	10.000	7.0181e-08
system_58	RK4 (TP14)	1.600	6.4215e-13	10.000	1.6374e-08
system_58	LA3 (TP8)	0.100	7.7582e-08	10.000	3.862e-05
system_58	LA3 (TP9)	0.150	8.4071e-09	10.000	1.8014e-05
system_58	LA3 (TP10)	0.230	1e-09	10.000	5.9569e-06
system_58	LA3 (TP11)	0.350	1.1954e-10	10.000	4.7884e-06
system_58	LA3 (TP12)	0.550	1.4015e-11	10.000	5.7684e-07
system_58	LA3 (TP13)	0.840	1.6018e-12	10.000	6.6192e-08
system_58	LA3 (TP14)	1.300	4.8228e-13	10.000	1.4357e-08
system_58	LC3 (TP8)	0.110	7.6055e-08	10.000	3.2019e-05
system_58	LC3 (TP9)	0.160	7.0379e-09	10.000	1.7252e-05
system_58	LC3 (TP10)	0.240	7.073e-10	10.000	8.826e-06
system_58	LC3 (TP11)	0.370	6.9519e-11	10.000	2.7407e-06
system_58	LC3 (TP12)	0.570	7.0854e-12	10.000	2.6685e-07
system_58	LC3 (TP13)	0.880	9.0639e-13	10.000	3.0817e-08
system_58	LC3 (TP14)	1.360	4.0279e-13	10.000	1.0658e-08
system_58	Riot (05, 1e-11)	0m0.386s	6.7986e-11	-0.000	1.8892e-06
system_58	Riot (10, 1e-11)	0m0.225s	7.1609e-13	-0.000	3.3649e-08
system_58	Riot (15, 1e-11)	0m0.310s	2.1094e-14	-0.000	7.9267e-07
system_58	Valencia-IVP (0.00025)	0m1.907s	0.0032029	4.129	968.02
system_58	Valencia-IVP (0.0025)	0m0.289s	0.032453	3.468	825.8
system_58	Valencia-IVP (0.025)	0m0.063s	0.36874	2.325	2.7348
system_58	VNODE-LP (15, 1e-14,1e-14)	0m0.007s	9.992e-15	10.000	5.2854e-13
system_58	VNODE-LP (20, 1e-14,1e-14)	0m0.007s	5.9952e-15	10.000	3.5797e-13
system_58	VNODE-LP (25, 1e-14,1e-14)	0m0.008s	5.107e-15	10.000	2.6821e-13

Table 3.49: Simulation results of Problem 59

Problems	Methods	c5t	c5w	c6t	c6w
system_59	TAYLOR4 (TP8)	0.320	0.61888	2.332	409.62
system_59	TAYLOR4 (TP9)	0.500	0.61763	2.342	627.45
system_59	TAYLOR4 (TP10)	0.780	0.62016	2.343	947.95
system_59	TAYLOR4 (TP11)	1.220	0.61634	2.353	1393.3
system_59	TAYLOR4 (TP12)	1.960	0.61604	2.356	2026.9
system_59	TAYLOR4 (TP13)	3.070	0.61584	2.358	2938.4
system_59	TAYLOR4 (TP14)	4.880	0.61601	2.356	1820.2
system_59	RK4 (TP8)	0.190	0.61554	2.340	309.16
system_59	RK4 (TP9)	0.290	0.61552	2.346	482.14
system_59	RK4 (TP10)	0.460	0.61552	2.351	729.25
system_59	RK4 (TP11)	0.700	0.61571	2.354	1074.3
system_59	RK4 (TP12)	1.150	0.61887	2.351	1577.8
system_59	RK4 (TP13)	1.740	0.61551	2.358	2300
system_59	RK4 (TP14)	2.740	0.61551	2.359	2656.2
system_59	LA3 (TP8)	0.170	0.62868	2.301	256.97
system_59	LA3 (TP9)	0.240	0.6239	2.322	405.24
system_59	LA3 (TP10)	0.370	0.62081	2.335	610.27
system_59	LA3 (TP11)	0.580	0.61885	2.344	914.53
system_59	LA3 (TP12)	0.890	0.61796	2.350	1348.2
system_59	LA3 (TP13)	1.410	0.61684	2.354	1965
system_59	LA3 (TP14)	2.230	0.61635	2.357	2848.1
system_59	LC3 (TP8)	0.170	0.63191	2.285	262.91
system_59	LC3 (TP9)	0.260	0.6259	2.312	421.05
system_59	LC3 (TP10)	0.390	0.62194	2.329	642.8
system_59	LC3 (TP11)	0.610	0.61954	2.341	952.94
system_59	LC3 (TP12)	1.050	0.62126	2.343	1402.2
system_59	LC3 (TP13)	1.490	0.61753	2.352	2043.1
system_59	LC3 (TP14)	2.350	0.61652	2.356	2965.4
system_59	Riot (05, 1e-11)	0m7.354s	0	0.000	0
system_59	Riot (10, 1e-11)	6m33.869s	0.58244	0.000	0
system_59	Riot (15, 1e-11)	53m34.326s	0.58244	0.000	0
system_59	Valencia-IVP (0.00025)	0m3.563s	1.4356	1.733	990.17
system_59	Valencia-IVP (0.0025)	0m0.469s	1.5086	1.698	818.5
system_59	Valencia-IVP (0.025)	0m0.100s	3.003	1.300	23.135
system_59	VNODE-LP (15, 1e-14,1e-14)	0m0.013s	1.4378	1.641	3.7929e+05
system_59	VNODE-LP (20, 1e-14,1e-14)	0m0.013s	1.8859	1.527	1.176e+06
system_59	VNODE-LP (25, 1e-14,1e-14)	0m0.011s	2.2062	1.455	1.9992e+06

Table 3.50: Simulation results of Problem 60

Problems	Methods	c5t	c5w	c6t	c6w
system_60	TAYLOR4 (TP8)	0.980	3.6313e-08	10.000	2.0172e-05
system_60	TAYLOR4 (TP9)	1.510	3.91e-09	10.000	1.0452e-05
system_60	TAYLOR4 (TP10)	2.370	4.1545e-10	10.000	2.449e-06
system_60	TAYLOR4 (TP11)	3.760	4.2142e-11	10.000	2.5501e-07
system_60	TAYLOR4 (TP12)	5.900	4.5062e-12	10.000	2.7482e-08
system_60	TAYLOR4 (TP13)	9.510	8.6153e-13	10.000	4.9282e-09
system_60	TAYLOR4 (TP14)	15.000	7.2609e-13	10.000	3.8885e-09
system_60	RK4 (TP8)	0.560	5.1365e-08	10.000	1.0619e-05
system_60	RK4 (TP9)	0.850	5.3363e-09	10.000	6.8093e-06
system_60	RK4 (TP10)	1.280	5.4613e-10	10.000	3.8676e-06
system_60	RK4 (TP11)	2.460	5.5102e-11	10.000	4.6275e-07
system_60	RK4 (TP12)	3.240	5.5893e-12	10.000	4.885e-08
system_60	RK4 (TP13)	5.110	6.1329e-13	10.000	5.556e-09
system_60	RK4 (TP14)	7.970	2.0783e-13	10.000	1.3849e-09
system_60	LA3 (TP8)	0.550	5.2924e-08	10.000	1.3089e-05
system_60	LA3 (TP9)	0.840	5.5683e-09	10.000	1.2154e-05
system_60	LA3 (TP10)	1.230	5.7092e-10	10.000	2.8252e-06
system_60	LA3 (TP11)	2.120	5.8239e-11	10.000	3.2878e-07
system_60	LA3 (TP12)	3.040	5.9095e-12	10.000	3.4149e-08
system_60	LA3 (TP13)	4.770	6.4371e-13	10.000	4.0022e-09
system_60	LA3 (TP14)	7.540	1.9762e-13	10.000	1.2146e-09
system_60	LC3 (TP8)	0.580	3.2515e-08	10.000	1.2785e-05
system_60	LC3 (TP9)	0.850	3.4269e-09	10.000	1.1738e-05
system_60	LC3 (TP10)	1.330	3.5506e-10	10.000	2.5568e-06
system_60	LC3 (TP11)	2.090	3.6346e-11	10.000	2.6495e-07
system_60	LC3 (TP12)	3.310	3.6904e-12	10.000	2.7388e-08
system_60	LC3 (TP13)	5.160	4.6496e-13	10.000	3.3249e-09
system_60	LC3 (TP14)	8.120	2.0606e-13	10.000	1.1795e-09
system_60	Riot (05, 1e-11)	0m0.401s	1.0846e-10	-0.000	4.1356e-07
system_60	Riot (10, 1e-11)	0m0.208s	1.3138e-12	-0.000	1.4383e-08
system_60	Riot (15, 1e-11)	0m0.293s	2.3981e-14	-0.000	1.4009e-09
system_60	Valencia-IVP (0.00025)	0m2.208s	0.0012113	10.000	21.282
system_60	Valencia-IVP (0.0025)	0m0.282s	0.012152	8.033	944.65
system_60	Valencia-IVP (0.025)	0m0.049s	0.12493	5.225	615.14
system_60	VNODE-LP (15, 1e-14,1e-14)	0m0.015s	6.3283e-15	10.000	1.8436e-12
system_60	VNODE-LP (20, 1e-14,1e-14)	0m0.013s	5.9952e-15	10.000	2.2619e-12
system_60	VNODE-LP (25, 1e-14,1e-14)	0m0.013s	3.9968e-15	10.000	1.127e-12

Table 3.51: Simulation results of Problem 61

Problems	Methods	c5t	c5w	c6t	c6w
system_61	TAYLOR4 (TP8)	1.180	0.0054407	10.000	8.3228
system_61	TAYLOR4 (TP9)	1.830	0.0053657	10.000	15.453
system_61	TAYLOR4 (TP10)	2.860	0.0054044	10.000	14.813
system_61	TAYLOR4 (TP11)	4.600	0.0054132	10.000	320.66
system_61	TAYLOR4 (TP12)	7.210	0.0054145	10.000	11.811
system_61	TAYLOR4 (TP13)	11.320	0.0053453	10.000	313.11
system_61	TAYLOR4 (TP14)	18.070	0.0054383	10.000	725.35
system_61	RK4 (TP8)	0.620	0.0052643	10.000	3.3273
system_61	RK4 (TP9)	0.920	0.0052883	10.000	12.87
system_61	RK4 (TP10)	1.420	0.0054387	10.000	8.7345
system_61	RK4 (TP11)	2.330	0.0053817	10.000	14.179
system_61	RK4 (TP12)	3.520	0.0053906	10.000	90.053
system_61	RK4 (TP13)	5.520	0.0054731	10.000	40.322
system_61	RK4 (TP14)	8.640	0.0054604	10.000	52.883
system_61	LA3 (TP8)	0.600	0.0053293	10.000	11.103
system_61	LA3 (TP9)	0.890	0.0053306	10.000	15.398
system_61	LA3 (TP10)	1.340	0.0053562	10.000	10.597
system_61	LA3 (TP11)	2.080	0.0054059	10.000	24.382
system_61	LA3 (TP12)	3.320	0.0054158	10.000	22.094
system_61	LA3 (TP13)	5.200	0.0054598	9.972	75209
system_61	LA3 (TP14)	8.110	0.0054368	10.000	84.479
system_61	LC3 (TP8)	0.620	0.0053219	10.000	15.264
system_61	LC3 (TP9)	0.930	0.0053593	10.000	13.911
system_61	LC3 (TP10)	1.430	0.005359	10.000	12.418
system_61	LC3 (TP11)	2.270	0.0054463	10.000	63.773
system_61	LC3 (TP12)	3.630	0.0054206	10.000	26.739
system_61	LC3 (TP13)	5.640	0.0054502	9.731	81583
system_61	LC3 (TP14)	8.820	0.0054423	10.000	44.156
system_61	Riot (05, 1e-11)	0m29.113s	0.016523	0.000	0
system_61	Riot (10, 1e-11)	2m2.447s	0.016523	0.000	0
system_61	Riot (15, 1e-11)	9m16.121s	0.016523	0.000	0
system_61	Valencia-IVP (0.00025)	0m2.193s	0.0070886	7.850	995.84
system_61	Valencia-IVP (0.0025)	0m0.314s	0.018078	7.098	938.56
system_61	Valencia-IVP (0.025)	0m0.049s	0.13117	5.150	535.8
system_61	VNODE-LP (15, 1e-14,1e-14)	0m0.015s	0.0064256	9.464	1.0425e+08
system_61	VNODE-LP (20, 1e-14,1e-14)	0m0.011s	0.007766	9.213	4.7889e+08
system_61	VNODE-LP (25, 1e-14,1e-14)	0m0.012s	0.0087521	9.173	1.0624e+09

Table 3.52: Simulation results of Problem 62

Problems	Methods	c5t	c5w	c6t	c6w
system_62	TAYLOR4 (TP8)	0.020	1.2802e-09	10.000	1.0457e-06
system_62	TAYLOR4 (TP9)	0.020	1.2802e-09	10.000	1.4972e-07
system_62	TAYLOR4 (TP10)	0.020	9.382e-10	10.000	1.6447e-08
system_62	TAYLOR4 (TP11)	0.030	1.4039e-10	10.000	1.7013e-09
system_62	TAYLOR4 (TP12)	0.030	1.5092e-11	10.000	1.7605e-10
system_62	TAYLOR4 (TP13)	0.050	2.1814e-12	10.000	2.251e-11
system_62	TAYLOR4 (TP14)	0.070	9.8055e-13	10.000	9.5852e-12
system_62	RK4 (TP8)	0.020	7.5438e-11	10.000	6.0009e-07
system_62	RK4 (TP9)	0.020	7.5438e-11	10.000	2.0981e-07
system_62	RK4 (TP10)	0.020	7.5438e-11	10.000	2.6419e-08
system_62	RK4 (TP11)	0.020	7.5438e-11	10.000	2.6952e-09
system_62	RK4 (TP12)	0.020	1.1987e-11	10.000	2.6037e-10
system_62	RK4 (TP13)	0.020	1.7266e-12	10.000	2.6858e-11
system_62	RK4 (TP14)	0.040	4.0501e-13	10.000	4.7393e-12
system_62	LA3 (TP8)	0.020	2.0744e-10	10.000	3.7513e-07
system_62	LA3 (TP9)	0.020	2.0744e-10	10.000	6.8085e-08
system_62	LA3 (TP10)	0.020	2.0744e-10	10.000	8.2075e-09
system_62	LA3 (TP11)	0.020	8.3048e-11	10.000	9.0913e-10
system_62	LA3 (TP12)	0.020	2.4023e-11	10.000	1.1102e-10
system_62	LA3 (TP13)	0.030	3.3396e-12	10.000	1.3493e-11
system_62	LA3 (TP14)	0.040	6.0396e-13	10.000	3.4959e-12
system_62	LC3 (TP8)	0.020	8.3944e-11	10.000	2.7272e-07
system_62	LC3 (TP9)	0.020	8.3944e-11	10.000	1.0016e-07
system_62	LC3 (TP10)	0.020	8.3944e-11	10.000	1.1054e-08
system_62	LC3 (TP11)	0.020	7.875e-11	10.000	1.1258e-09
system_62	LC3 (TP12)	0.020	1.1283e-11	10.000	1.1067e-10
system_62	LC3 (TP13)	0.030	1.5774e-12	10.000	1.2335e-11
system_62	LC3 (TP14)	0.040	3.8369e-13	10.000	3.304e-12
system_62	Riot (05, 1e-11)	0m0.096s	7.887e-13	-0.000	3.9957e-11
system_62	Riot (10, 1e-11)	0m0.116s	7.9226e-13	-0.000	2.2027e-13
system_62	Riot (15, 1e-11)	0m0.139s	9.3081e-13	-0.000	5.0093e-13
system_62	Valencia-IVP (0.00025)	0m1.501s	8e-06	10.000	9.0701e-05
system_62	Valencia-IVP (0.0025)	0m0.135s	8.0004e-05	10.000	0.00090724
system_62	Valencia-IVP (0.025)	0m0.017s	0.00080027	10.000	0.0090954
system_62	VNODE-LP (15, 1e-14,1e-14)	0m0.006s	1.0658e-14	10.000	1.0303e-13
system_62	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	1.0658e-14	10.000	1.1013e-13
system_62	VNODE-LP (25, 1e-14,1e-14)	0m0.005s	1.0658e-14	10.000	1.1013e-13

Table 3.53: Simulation results of Problem 63

Problems	Methods	c5t	c5w	c6t	c6w
system_63	TAYLOR4 (TP8)	0.050	0.91819	10.000	4.9513
system_63	TAYLOR4 (TP9)	0.060	0.91498	10.000	4.7543
system_63	TAYLOR4 (TP10)	0.090	0.9127	10.000	4.6373
system_63	TAYLOR4 (TP11)	0.130	0.9111	10.000	4.5661
system_63	TAYLOR4 (TP12)	0.200	0.91002	10.000	4.5221
system_63	TAYLOR4 (TP13)	0.320	0.90932	10.000	4.4947
system_63	TAYLOR4 (TP14)	0.500	0.90886	10.000	4.4776
system_63	RK4 (TP8)	0.030	0.90814	10.000	4.452
system_63	RK4 (TP9)	0.030	0.9081	10.000	4.4501
system_63	RK4 (TP10)	0.050	0.90808	10.000	4.4493
system_63	RK4 (TP11)	0.070	0.90808	10.000	4.449
system_63	RK4 (TP12)	0.110	0.90807	10.000	4.4489
system_63	RK4 (TP13)	0.160	0.90807	10.000	4.4488
system_63	RK4 (TP14)	0.250	0.90807	10.000	4.4488
system_63	LA3 (TP8)	0.030	0.94854	10.000	5.196
system_63	LA3 (TP9)	0.030	0.93622	10.000	4.9019
system_63	LA3 (TP10)	0.040	0.927	10.000	4.7292
system_63	LA3 (TP11)	0.060	0.9205	10.000	4.6236
system_63	LA3 (TP12)	0.090	0.91616	10.000	4.5583
system_63	LA3 (TP13)	0.140	0.91325	10.000	4.5176
system_63	LA3 (TP14)	0.220	0.91136	10.000	4.492
system_63	LC3 (TP8)	0.030	0.95645	10.000	5.5166
system_63	LC3 (TP9)	0.030	0.94129	10.000	5.0708
system_63	LC3 (TP10)	0.050	0.9303	10.000	4.8246
system_63	LC3 (TP11)	0.060	0.92279	10.000	4.6803
system_63	LC3 (TP12)	0.100	0.91757	10.000	4.5928
system_63	LC3 (TP13)	0.140	0.91414	10.000	4.5388
system_63	LC3 (TP14)	0.230	0.91192	10.000	4.5052
system_63	Riot (05, 1e-11)	0m0.226s	6.1391e-12	-0.000	2.1793e-10
system_63	Riot (10, 1e-11)	0m0.219s	6.1391e-12	-0.000	8.3134e-13
system_63	Riot (15, 1e-11)	0m0.222s	3.6238e-13	-0.000	3.979e-13
system_63	Valencia-IVP (0.00025)	0m3.804s	1.4207	4.983	939.4
system_63	Valencia-IVP (0.0025)	0m0.416s	1.4208	4.960	184.88
system_63	Valencia-IVP (0.025)	0m0.067s	1.4224	3.675	6.8657
system_63	VNODE-LP (15, 1e-14,1e-14)	0m0.006s	1.1898	5.765	12397
system_63	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	1.1582	4.716	24367
system_63	VNODE-LP (25, 1e-14,1e-14)	0m0.004s	1.161	4.394	39403

Table 3.54: Simulation results of Problem 64

Problems	Methods	c5t	c5w	c6t	c6w
system_64	TAYLOR4 (TP8)	0.390	1.4114e-06	10.000	0.00068375
system_64	TAYLOR4 (TP9)	0.390	1.2487e-06	10.000	0.00015597
system_64	TAYLOR4 (TP10)	0.430	4.3621e-07	10.000	4.6461e-05
system_64	TAYLOR4 (TP11)	0.560	1.2507e-07	10.000	1.1767e-05
system_64	TAYLOR4 (TP12)	0.730	3.3494e-08	10.000	2.9977e-06
system_64	TAYLOR4 (TP13)	0.980	8.5987e-09	10.000	7.7019e-07
system_64	TAYLOR4 (TP14)	1.400	2.3403e-09	10.000	1.9712e-07
system_64	RK4 (TP8)	0.330	4.21e-11	10.000	4.7828e-07
system_64	RK4 (TP9)	0.330	4.21e-11	10.000	4.7336e-08
system_64	RK4 (TP10)	0.330	3.0917e-11	10.000	4.8889e-09
system_64	RK4 (TP11)	0.400	4.2763e-12	10.000	5.0111e-10
system_64	RK4 (TP12)	0.470	4.4072e-13	10.000	5.0163e-11
system_64	RK4 (TP13)	0.610	4.842e-14	10.000	5.218e-12
system_64	RK4 (TP14)	0.840	5.6413e-15	10.000	7.4518e-13
system_64	LA3 (TP8)	0.330	4.4317e-11	10.000	3.2208e-07
system_64	LA3 (TP9)	0.330	4.4317e-11	10.000	3.4219e-08
system_64	LA3 (TP10)	0.360	2.6981e-11	10.000	3.3887e-09
system_64	LA3 (TP11)	0.400	3.1899e-12	10.000	3.4486e-10
system_64	LA3 (TP12)	0.470	3.5908e-13	10.000	3.4774e-11
system_64	LA3 (TP13)	0.640	3.9885e-14	10.000	3.6753e-12
system_64	LA3 (TP14)	0.890	4.7254e-15	10.000	6.1373e-13
system_64	LC3 (TP8)	0.330	4.1986e-11	10.000	3.5873e-07
system_64	LC3 (TP9)	0.330	4.1986e-11	10.000	3.7852e-08
system_64	LC3 (TP10)	0.370	2.5936e-11	10.000	3.8917e-09
system_64	LC3 (TP11)	0.400	3.0693e-12	10.000	3.9943e-10
system_64	LC3 (TP12)	0.470	3.4297e-13	10.000	4.0659e-11
system_64	LC3 (TP13)	0.650	3.849e-14	10.000	4.2375e-12
system_64	LC3 (TP14)	0.880	4.5519e-15	10.000	6.6169e-13
system_64	Riot (05, 1e-11)	0m0.136s	3.194e-14	-0.000	1.1558e-10
system_64	Riot (10, 1e-11)	0m0.253s	5.4123e-16	-0.000	1.35e-13
system_64	Riot (15, 1e-11)	0m0.252s	5.4123e-16	-0.000	6.9278e-14
system_64	Valencia-IVP (0.00025)	0m1.721s	1.0417e-05	10.000	0.00016797
system_64	Valencia-IVP (0.0025)	0m0.165s	0.00010417	10.000	0.0016797
system_64	Valencia-IVP (0.025)	0m0.019s	0.0010417	10.000	0.016797
system_64	VNODE-LP (15, 1e-14,1e-14)	0m0.004s	6.245e-17	10.000	9.77e-15
system_64	VNODE-LP (20, 1e-14,1e-14)	0m0.005s	6.9389e-17	10.000	1.199e-14
system_64	VNODE-LP (25, 1e-14,1e-14)	0m0.004s	6.9389e-17	10.000	1.0658e-14

Table 3.55: Simulation results of Problem 65

Problems	Methods	c5t	c5w	c6t	c6w
system_65	TAYLOR4 (TP8)	0.410	0.25212	10.000	2.7137
system_65	TAYLOR4 (TP9)	0.410	0.25212	10.000	2.7126
system_65	TAYLOR4 (TP10)	0.500	0.25212	10.000	2.7121
system_65	TAYLOR4 (TP11)	0.590	0.25211	10.000	2.7118
system_65	TAYLOR4 (TP12)	0.760	0.25211	10.000	2.7116
system_65	TAYLOR4 (TP13)	1.070	0.25211	10.000	2.7115
system_65	TAYLOR4 (TP14)	1.570	0.25211	10.000	2.7114
system_65	RK4 (TP8)	0.340	0.25211	10.000	2.7113
system_65	RK4 (TP9)	0.330	0.25211	10.000	2.7113
system_65	RK4 (TP10)	0.370	0.25211	10.000	2.7113
system_65	RK4 (TP11)	0.400	0.25211	10.000	2.7113
system_65	RK4 (TP12)	0.470	0.25211	10.000	2.7113
system_65	RK4 (TP13)	0.650	0.25211	10.000	2.7113
system_65	RK4 (TP14)	0.860	0.25211	10.000	2.7113
system_65	LA3 (TP8)	0.330	0.25211	10.000	2.7134
system_65	LA3 (TP9)	0.330	0.25211	10.000	2.7127
system_65	LA3 (TP10)	0.370	0.25211	10.000	2.7122
system_65	LA3 (TP11)	0.400	0.25211	10.000	2.7119
system_65	LA3 (TP12)	0.500	0.25211	10.000	2.7117
system_65	LA3 (TP13)	0.690	0.25211	10.000	2.7115
system_65	LA3 (TP14)	0.940	0.25211	10.000	2.7115
system_65	LC3 (TP8)	0.330	0.25211	10.000	2.7145
system_65	LC3 (TP9)	0.330	0.25211	10.000	2.7133
system_65	LC3 (TP10)	0.370	0.25211	10.000	2.7126
system_65	LC3 (TP11)	0.400	0.25211	10.000	2.7121
system_65	LC3 (TP12)	0.500	0.25211	10.000	2.7118
system_65	LC3 (TP13)	0.650	0.25211	10.000	2.7117
system_65	LC3 (TP14)	0.900	0.25211	10.000	2.7115
system_65	Riot (05, 1e-11)	0m5.669s	0.25147	-0.000	2.6697
system_65	Riot (10, 1e-11)	0m1.551s	0.25147	-0.000	2.6698
system_65	Riot (15, 1e-11)	0m5.042s	0.25147	-0.000	2.6698
system_65	Valencia-IVP (0.00025)	0m1.576s	0.25147	10.000	2.6699
system_65	Valencia-IVP (0.0025)	0m0.146s	0.25147	10.000	2.6716
system_65	Valencia-IVP (0.025)	0m0.021s	0.25177	10.000	2.6883
system_65	VNODE-LP (15, 1e-14,1e-14)	0m0.006s	0.25278	10.000	2.7636
system_65	VNODE-LP (20, 1e-14,1e-14)	0m0.006s	0.25278	10.000	2.7636
system_65	VNODE-LP (25, 1e-14,1e-14)	0m0.005s	0.25278	10.000	2.7636

Table 3.56: Simulation results of Problem 71

Problems	Methods	c5t	c5w	c6t	c6w
system_71	TAYLOR4 (TP8)	0.410	0.34183	0.723	0.34183
system_71	TAYLOR4 (TP9)	0.610	0.34398	0.723	0.34398
system_71	TAYLOR4 (TP10)	0.930	0.34513	0.723	0.34513
system_71	TAYLOR4 (TP11)	1.530	0.34637	0.723	0.34637
system_71	TAYLOR4 (TP12)	2.420	0.34685	0.723	0.34685
system_71	TAYLOR4 (TP13)	3.600	0.34733	0.723	0.34733
system_71	TAYLOR4 (TP14)	5.500	0.34747	0.723	0.34747
system_71	RK4 (TP8)	0.410	0.34107	0.710	0.34107
system_71	RK4 (TP9)	0.800	0.34517	0.718	0.34517
system_71	RK4 (TP10)	0.970	0.34595	0.719	0.34595
system_71	RK4 (TP11)	0.610	0.34721	0.721	0.34721
system_71	RK4 (TP12)	0.940	0.34711	0.721	0.34711
system_71	RK4 (TP13)	1.510	0.34743	0.722	0.34743
system_71	RK4 (TP14)	2.290	0.34757	0.722	0.34757
system_71	LA3 (TP8)	0.320	0.34419	0.714	0.34419
system_71	LA3 (TP9)	0.270	0.34689	0.720	0.34689
system_71	LA3 (TP10)	0.390	0.34737	0.721	0.34737
system_71	LA3 (TP11)	0.570	0.34704	0.721	0.34704
system_71	LA3 (TP12)	0.900	0.34744	0.722	0.34744
system_71	LA3 (TP13)	1.440	0.34753	0.722	0.34753
system_71	LA3 (TP14)	2.220	0.34779	0.722	0.34779
system_71	LC3 (TP8)	0.310	0.34572	0.715	0.34572
system_71	LC3 (TP9)	0.270	0.34545	0.715	0.34545
system_71	LC3 (TP10)	0.400	0.34696	0.719	0.34696
system_71	LC3 (TP11)	0.600	0.3477	0.721	0.3477
system_71	LC3 (TP12)	0.940	0.34745	0.721	0.34745
system_71	LC3 (TP13)	1.490	0.34765	0.722	0.34765
system_71	LC3 (TP14)	2.300	0.34772	0.722	0.34772
system_71	Riot				
system_71	Valencia-IVP (0.00025)	0m9.028s	0	0.000	0
system_71	Valencia-IVP (0.0025)	0m0.112s	0	0.000	0
system_71	Valencia-IVP (0.025)	0m0.007s	0	0.000	0
system_71	VNODE-LP (15, 1e-14,1e-14)	0m0.028s	0.093606	1.088	0.078438
system_71	VNODE-LP (20, 1e-14,1e-14)	0m0.035s	0.094651	1.085	0.080607
system_71	VNODE-LP (25, 1e-14,1e-14)	0m0.034s	0.095228	1.083	0.081672

Table 3.57: Simulation results of Problem 72

Problems	Methods	c5t	c5w	c6t	c6w
system_72	TAYLOR4 (TP8)	0.110	1.888e-08	10.000	1.5233e-07
system_72	TAYLOR4 (TP9)	0.160	1.991e-09	10.000	8.6791e-08
system_72	TAYLOR4 (TP10)	0.250	2.0536e-10	10.000	9.4023e-08
system_72	TAYLOR4 (TP11)	0.410	2.076e-11	10.000	1.1399e-08
system_72	TAYLOR4 (TP12)	0.640	2.1992e-12	10.000	1.6071e-09
system_72	TAYLOR4 (TP13)	0.990	3.4417e-13	10.000	2.5738e-10
system_72	TAYLOR4 (TP14)	1.550	2.3176e-13	10.000	6.5053e-11
system_72	RK4 (TP8)	0.090	2.476e-08	10.000	1.6617e-07
system_72	RK4 (TP9)	0.140	2.538e-09	10.000	4.0938e-08
system_72	RK4 (TP10)	0.200	2.6417e-10	10.000	5.5074e-08
system_72	RK4 (TP11)	0.330	2.6895e-11	10.000	3.9127e-09
system_72	RK4 (TP12)	0.520	2.7741e-12	10.000	7.941e-10
system_72	RK4 (TP13)	0.800	3.2496e-13	10.000	1.5842e-10
system_72	RK4 (TP14)	1.230	1.0836e-13	10.000	3.334e-11
system_72	LA3 (TP8)	0.070	1.8797e-08	10.000	1.0346e-06
system_72	LA3 (TP9)	0.110	1.9786e-09	10.000	1.3939e-07
system_72	LA3 (TP10)	0.160	2.0676e-10	10.000	9.6798e-08
system_72	LA3 (TP11)	0.250	2.1116e-11	10.000	1.2856e-08
system_72	LA3 (TP12)	0.750	2.126e-12	10.000	1.7966e-09
system_72	LA3 (TP13)	0.620	2.4913e-13	10.000	2.5436e-10
system_72	LA3 (TP14)	0.950	8.1712e-14	10.000	4.3266e-11
system_72	LC3 (TP8)	0.080	1.9008e-08	10.000	3.6404e-06
system_72	LC3 (TP9)	0.120	2.0101e-09	10.000	2.7038e-07
system_72	LC3 (TP10)	0.180	2.0895e-10	10.000	1.3971e-07
system_72	LC3 (TP11)	0.280	2.1388e-11	10.000	2.2361e-08
system_72	LC3 (TP12)	0.430	2.1335e-12	10.000	2.1033e-09
system_72	LC3 (TP13)	0.690	2.5269e-13	10.000	2.8453e-10
system_72	LC3 (TP14)	1.040	8.632e-14	10.000	4.7709e-11
system_72	Riot (05, 1e-11)	0m1.648s	6.8875e-11	-0.000	0.0018269
system_72	Riot (10, 1e-11)	0m1.461s	4.1078e-15	-0.000	7.1333e-13
system_72	Riot (15, 1e-11)	0m1.542s	1.4155e-15	-0.000	9.9245e-15
system_72	Valencia-IVP (0.00025)	1m10.076s	0.011379	4.194	999.68
system_72	Valencia-IVP (0.0025)	0m0.692s	0.11581	3.530	992.01
system_72	Valencia-IVP (0.025)	0m0.061s	1.3941	2.750	956.94
system_72	VNODE-LP (15, 1e-14,1e-14)	0m0.014s	9.1593e-16	10.000	1.9629e-16
system_72	VNODE-LP (20, 1e-14,1e-14)	0m0.010s	9.1593e-16	10.000	1.4984e-16
system_72	VNODE-LP (25, 1e-14,1e-14)	0m0.010s	3.8858e-16	10.000	7.7839e-17

Table 3.58: Simulation results of Problem 73

Problems	Methods	c5t	c5w	c6t	c6w
system_73	TAYLOR4 (TP8)	0.200	0.64903	10.000	0.00011543
system_73	TAYLOR4 (TP9)	0.280	0.64903	10.000	0.00011391
system_73	TAYLOR4 (TP10)	0.410	0.64903	10.000	0.00011378
system_73	TAYLOR4 (TP11)	0.620	0.64903	10.000	0.00011407
system_73	TAYLOR4 (TP12)	1.730	0.64903	10.000	0.00011593
system_73	TAYLOR4 (TP13)	1.560	0.64903	10.000	0.00011625
system_73	TAYLOR4 (TP14)	2.420	0.64903	10.000	0.00011618
system_73	RK4 (TP8)	0.130	0.64903	10.000	0.00011568
system_73	RK4 (TP9)	0.200	0.64903	10.000	0.0001142
system_73	RK4 (TP10)	0.310	0.64903	10.000	0.00011396
system_73	RK4 (TP11)	0.480	0.64903	10.000	0.00011377
system_73	RK4 (TP12)	0.760	0.64903	10.000	0.00011457
system_73	RK4 (TP13)	1.200	0.64903	10.000	0.00011523
system_73	RK4 (TP14)	1.850	0.64903	10.000	0.00011523
system_73	LA3 (TP8)	0.110	0.90804	10.000	44.669
system_73	LA3 (TP9)	0.160	0.84683	10.000	11.972
system_73	LA3 (TP10)	0.240	0.74939	10.000	1.4182
system_73	LA3 (TP11)	0.380	0.72469	10.000	2.9375
system_73	LA3 (TP12)	0.600	0.6851	10.000	0.3029
system_73	LA3 (TP13)	0.980	0.67844	10.000	0.28271
system_73	LA3 (TP14)	1.500	0.66826	10.000	0.1337
system_73	LC3 (TP8)	0.120	0.95006	10.000	41.555
system_73	LC3 (TP9)	0.170	0.85351	10.000	4.46
system_73	LC3 (TP10)	0.270	0.78623	10.000	3.7844
system_73	LC3 (TP11)	0.420	0.72942	10.000	1.3191
system_73	LC3 (TP12)	0.660	0.69002	10.000	0.35275
system_73	LC3 (TP13)	1.060	0.67725	10.000	0.3206
system_73	LC3 (TP14)	1.660	0.66841	10.000	0.061096
system_73	Riot (05, 1e-11)	0m1.815s	0.64903	-0.000	0.00011995
system_73	Riot (10, 1e-11)	0m2.136s	0.64903	-0.000	0.0001136
system_73	Riot (15, 1e-11)	0m3.216s	0.64903	-0.000	0.00011366
system_73	Valencia-IVP (0.00025)	1m1.164s	138.84	1.367	999.11
system_73	Valencia-IVP (0.0025)	0m0.278s	145.77	1.355	994.94
system_73	Valencia-IVP (0.025)	0m0.029s	243.46	1.225	891.88
system_73	VNODE-LP (15, 1e-14,1e-14)	0m0.024s	0.64903	10.000	3.4442
system_73	VNODE-LP (20, 1e-14,1e-14)	0m0.015s	0.64903	10.000	3.4442
system_73	VNODE-LP (25, 1e-14,1e-14)	0m0.016s	0.64903	10.000	3.4442

Table 3.59: Simulation results of Problem 74

Problems	Methods	c5t	c5w	c6t	c6w
system_74	TAYLOR4 (TP8)	0.250	430.91	0.785	430.91
system_74	TAYLOR4 (TP9)	0.350	652.41	0.785	652.41
system_74	TAYLOR4 (TP10)	0.490	283.48	0.785	283.48
system_74	TAYLOR4 (TP11)	0.670	559.65	0.785	559.65
system_74	TAYLOR4 (TP12)	0.890	576.25	0.785	576.25
system_74	TAYLOR4 (TP13)	1.290	234.53	0.785	234.53
system_74	TAYLOR4 (TP14)	0.000	0	0.000	0
system_74	RK4 (TP8)	0.130	624.24	0.785	624.24
system_74	RK4 (TP9)	0.160	57.925	0.785	57.925
system_74	RK4 (TP10)	0.220	330.01	0.785	330.01
system_74	RK4 (TP11)	0.310	268.64	0.785	268.64
system_74	RK4 (TP12)	0.440	44.208	0.785	44.208
system_74	RK4 (TP13)	0.620	267.16	0.785	267.16
system_74	RK4 (TP14)	0.860	74.118	0.785	74.118
system_74	LA3 (TP8)	0.130	76.095	0.785	76.095
system_74	LA3 (TP9)	0.190	45.448	0.785	45.448
system_74	LA3 (TP10)	0.260	62.95	0.785	62.95
system_74	LA3 (TP11)	0.370	64.448	0.785	64.448
system_74	LA3 (TP12)	0.530	527.45	0.785	527.45
system_74	LA3 (TP13)	0.730	21.878	0.785	21.878
system_74	LA3 (TP14)	1.040	266.61	0.785	266.61
system_74	LC3 (TP8)	0.100	90.528	0.785	90.528
system_74	LC3 (TP9)	0.130	61.895	0.785	61.895
system_74	LC3 (TP10)	0.170	79.971	0.785	79.971
system_74	LC3 (TP11)	0.240	104.2	0.785	104.2
system_74	LC3 (TP12)	0.320	8.7342	0.785	8.7342
system_74	LC3 (TP13)	0.450	205.63	0.785	205.63
system_74	LC3 (TP14)	0.640	258.77	0.785	258.77
system_74	Riot (05, 1e-11)	0m0.791s	0	0.000	0
system_74	Riot (10, 1e-11)	0m0.430s	0	0.000	0
system_74	Riot (15, 1e-11)	0m0.613s	0	0.000	0
system_74	Valencia-IVP (0.00025)	0m9.104s	668.07	0.783	668.07
system_74	Valencia-IVP (0.0025)	0m0.165s	60.454	0.765	60.454
system_74	Valencia-IVP (0.025)	0m0.014s	5.325	0.650	5.325
system_74	VNODE-LP (15, 1e-14,1e-14)	0m0.014s	4992.7	0.015	4992.7
system_74	VNODE-LP (20, 1e-14,1e-14)	0m0.023s	2.2247e-07	0.785	2.2247e-07
system_74	VNODE-LP (25, 1e-14,1e-14)	0m0.010s	16182	0.001	16182

3.1.3 Discussion

Firstly, we count the number of problem for which each method (for each order and each precision) is first in term of solution diameter, second or last. This account is done for the simulation at 1 second and at 10 seconds. The results are summarized in the table 3.60. Of course, we are aware that the results are biased by the number of methods we have. Nevertheless, this table allows us to consider that Valencia and Riot are not valid competitors.

Table 3.60: Number of times a method produced the sharpest enclosure or the second sharpest enclosure.

Method	c5w (1st)	c5w (2nd)	c5w (last)	c6w (1st)	c6w (2nd)	c6w (last)
RK	103	35	8	58	39	8
Vnode-LP	70	28	9	44	27	8
Riot	36	11	0	24	12	2
Valencia	3	3	49	3	2	49

After this reduction of competitors, only the best results for our three order-4 Runge-Kutta methods, and for Vnode are kept for comparison. We present in the spider graph 3.1, respectively 3.2, the normalized results (divided by the median and multiplied by 10) for each problem for a simulation at 1 second, respectively at 10 seconds. The median used to normalize the results is computed with all the methods: Taylor4, RK4, LA3, LC3 Riot, Valencia and Vnode (for all precision and all order).

Remark: for the graph 3.1, we truncate the results at 25 for the clarity. It leads to the truncation of LC3 result for problem 44, initially at 178, the result is set at 25. In the same manner, the results are also truncated at 50 for the graph 3.2, fifteen times for Vnode, one time for LC3 and one time for RK4.

We can easily see on spider graph 3.1 that the Runge-Kutta methods are more stable, by describing a circle while Vnode results are more in a star shape. Moreover, the implicit methods (LA3 and LC3) provide better results than the explicit RK4 in a majority of problems. This fact is even more clear on the graph 3.2. On this latter graph, we can also see that Vnode fails many times while at least one of our Runge-Kutta methods performs a good simulation for all the considered problems. Finally, if Vnode are the best on many problems, by our stability and our better results for some problems, we can conclude that our tool is a good competitor for Vnode. The last remark but not the least, it is important to remember that we have currently only methods of order 4, when Vnode can use a Taylor at order 25 !

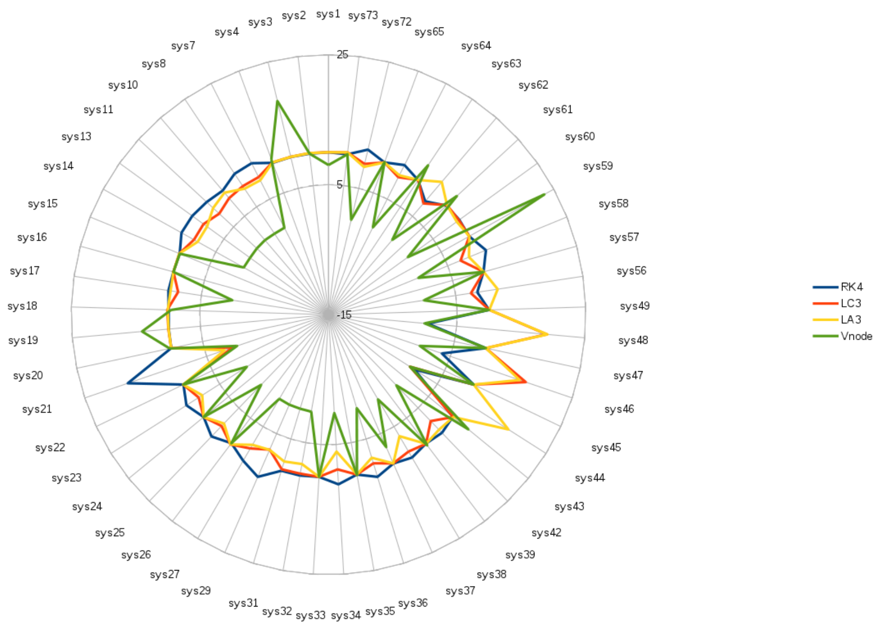


Figure 3.1: Results gathered in spider graph for a simulation of 1 second, for the methods: RK4, LC3, LA3 and Vnode

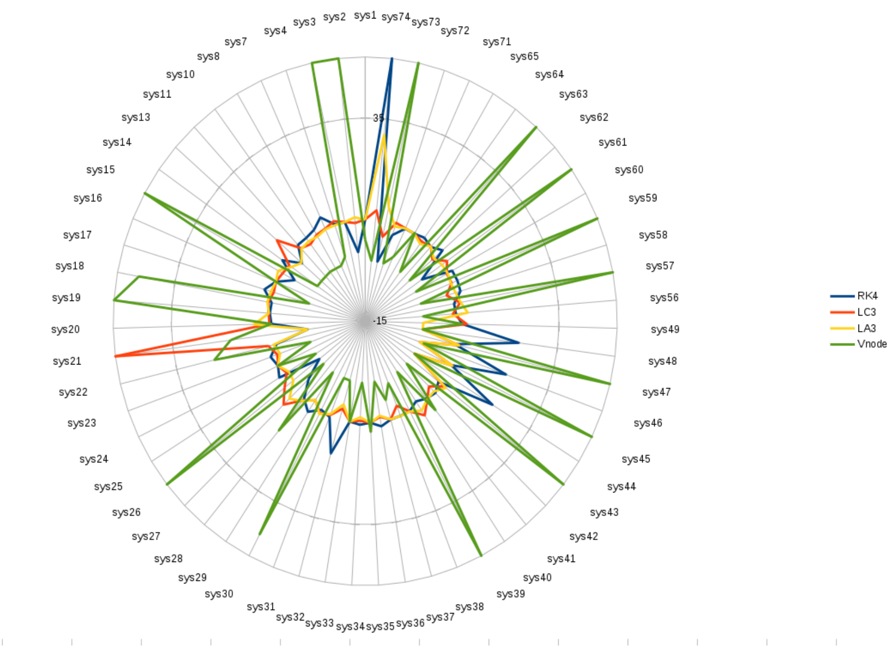


Figure 3.2: Results gathered in spider graph for a simulation of 10 seconds, for the methods: RK4, LC3, LA3 and Vnode

3.2 Detest benchmark

3.2.1 Disclaimer

This section reports the results of the solution of various problems coming from the DETEST benchmark. For each problem, different validated methods of Runge-Kutta of order 4 are applied among: the classical formula of Runge-Kutta (explicit), the Lobatto-3a formula (implicit) and the Lobatto-3c formula (implicit). Moreover, an homemade version of Taylor series, limited to order 5 and using affine arithmetic, is also applied on each problem.

For each problem, we report the following metrics:

- c5t: user time taken to simulate the problem for 1 second.
- c5w: the final diameter of the solution (infinity norm is used).
- c6t: the time to breakdown the method with a maximal limit of 10 seconds.
- c6w: the diameter of the solution a the breakdown time.

3.2.2 Results

Table 3.61: Simulation results of Problem ns_A1

Problems	Methods	c5t	c5w	c6t	c6w
ns_A1	TAYLOR4 (TP4)	0.030	9.146e-06	10.000	6.3861e-06
ns_A1	TAYLOR4 (TP6)	0.030	5.0222e-07	2.000	9.7332e-07
ns_A1	TAYLOR4 (TP8)	0.060	6.0636e-09	2.000	5.7233e-08
ns_A1	TAYLOR4 (TP10)	0.120	6.3146e-11	2.000	6.7023e-10
ns_A1	TAYLOR4 (TP12)	0.300	7.1687e-13	10.000	5.5133e-12
ns_A1	TAYLOR4 (TP14)	0.020	9.146e-06	10.000	6.3861e-06
ns_A1	RK4 (TP4)	0.010	9.146e-06	10.000	6.2632e-06
ns_A1	RK4 (TP6)	0.020	7.1338e-07	2.000	1.236e-06
ns_A1	RK4 (TP8)	0.030	7.4993e-09	2.000	4.3775e-08
ns_A1	RK4 (TP10)	0.060	8.4251e-11	2.000	6.7118e-10
ns_A1	RK4 (TP12)	0.160	8.8185e-13	10.000	7.5966e-12
ns_A1	RK4 (TP14)	0.010	9.146e-06	10.000	6.2632e-06
ns_A1	LA3 (TP4)	0.010	1.531e-06	10.000	7.6554e-06
ns_A1	LA3 (TP6)	0.020	4.0741e-07	2.000	8.1525e-07
ns_A1	LA3 (TP8)	0.020	5.4981e-09	2.000	4.1256e-08
ns_A1	LA3 (TP10)	0.050	6.1542e-11	2.000	5.8395e-10
ns_A1	LA3 (TP12)	0.130	6.7724e-13	10.000	5.3249e-12
ns_A1	LA3 (TP14)	0.010	1.531e-06	10.000	7.6554e-06
ns_A1	LC3 (TP4)	0.010	2.3003e-06	10.000	7.8708e-05
ns_A1	LC3 (TP6)	0.020	3.8815e-07	2.000	1.1053e-06
ns_A1	LC3 (TP8)	0.030	5.8283e-09	2.000	4.7752e-08
ns_A1	LC3 (TP10)	0.060	6.1916e-11	2.000	6.2382e-10
ns_A1	LC3 (TP12)	0.140	6.7468e-13	10.000	5.3717e-12
ns_A1	LC3 (TP14)	0.020	2.3003e-06	10.000	7.8708e-05

Table 3.62: Simulation results of Problem ns_A2

Problems	Methods	c5t	c5w	c6t	c6w
ns_A2	TAYLOR4 (TP4)	0.040	8.4667e-05	10.000	8.3982e-05
ns_A2	TAYLOR4 (TP6)	0.050	1.5064e-06	2.000	3.1707e-06
ns_A2	TAYLOR4 (TP8)	0.080	2.1535e-08	2.000	1.4215e-07
ns_A2	TAYLOR4 (TP10)	0.180	2.4119e-10	2.000	1.6527e-09
ns_A2	TAYLOR4 (TP12)	0.440	2.6019e-12	10.000	1.3628e-11
ns_A2	TAYLOR4 (TP14)	0.040	8.4667e-05	10.000	8.3982e-05
ns_A2	RK4 (TP4)	0.020	1.6376e-05	10.000	4.3081e-05
ns_A2	RK4 (TP6)	0.030	2.3535e-06	2.000	4.883e-06
ns_A2	RK4 (TP8)	0.040	4.9213e-08	2.000	2.9883e-07
ns_A2	RK4 (TP10)	0.060	5.2365e-10	2.000	3.509e-09
ns_A2	RK4 (TP12)	0.140	4.6034e-12	10.000	2.7551e-11
ns_A2	RK4 (TP14)	0.020	1.6376e-05	10.000	4.3081e-05
ns_A2	LA3 (TP4)	0.020	9.2675e-06	10.000	2.3649e-05
ns_A2	LA3 (TP6)	0.030	1.1342e-06	2.000	3.1789e-06
ns_A2	LA3 (TP8)	0.040	1.9817e-08	2.000	1.116e-07
ns_A2	LA3 (TP10)	0.080	2.3168e-10	2.000	1.5203e-09
ns_A2	LA3 (TP12)	0.200	2.5466e-12	10.000	1.3165e-11
ns_A2	LA3 (TP14)	0.020	9.2675e-06	10.000	2.3649e-05
ns_A2	LC3 (TP4)	0.020	7.5652e-06	10.000	2.6357e-05
ns_A2	LC3 (TP6)	0.030	1.4833e-06	2.000	3.5055e-06
ns_A2	LC3 (TP8)	0.040	2.6328e-08	2.000	1.5975e-07
ns_A2	LC3 (TP10)	0.070	2.9172e-10	2.000	1.9148e-09
ns_A2	LC3 (TP12)	0.150	2.828e-12	10.000	1.5799e-11
ns_A2	LC3 (TP14)	0.020	7.5652e-06	10.000	2.6357e-05

Table 3.63: Simulation results of Problem ns_A3

Problems	Methods	c5t	c5w	c6t	c6w
ns_A3	TAYLOR4 (TP4)	0.050	0.00043573	10.000	0.0041836
ns_A3	TAYLOR4 (TP6)	0.060	8.3465e-06	2.000	1.6766e-05
ns_A3	TAYLOR4 (TP8)	0.110	1.0131e-07	2.000	2.4257e-07
ns_A3	TAYLOR4 (TP10)	0.220	1.5521e-09	2.000	3.3297e-09
ns_A3	TAYLOR4 (TP12)	0.550	1.9743e-11	10.000	1.544e-10
ns_A3	TAYLOR4 (TP14)	0.050	0.00043573	10.000	0.0041836
ns_A3	RK4 (TP4)	0.030	0.00014736	10.000	0.004336
ns_A3	RK4 (TP6)	0.040	8.2963e-06	2.000	2.5968e-05
ns_A3	RK4 (TP8)	0.050	1.1569e-07	2.000	5.2775e-07
ns_A3	RK4 (TP10)	0.120	1.4826e-09	2.000	1.0182e-08
ns_A3	RK4 (TP12)	0.260	1.6631e-11	10.000	4.1913e-10
ns_A3	RK4 (TP14)	0.030	0.00014736	10.000	0.004336
ns_A3	LA3 (TP4)	0.030	7.0869e-05	10.000	0.0049959
ns_A3	LA3 (TP6)	0.040	4.0701e-06	2.000	9.8443e-06
ns_A3	LA3 (TP8)	0.060	4.8721e-08	2.000	1.7833e-07
ns_A3	LA3 (TP10)	0.130	5.5251e-10	2.000	2.0766e-09
ns_A3	LA3 (TP12)	0.310	5.917e-12	10.000	1.2948e-10
ns_A3	LA3 (TP14)	0.030	7.0869e-05	10.000	0.0049959
ns_A3	LC3 (TP4)	0.030	8.4934e-05	10.000	0.0056809
ns_A3	LC3 (TP6)	0.030	8.2958e-06	2.000	1.9432e-05
ns_A3	LC3 (TP8)	0.060	8.8322e-08	2.000	2.3749e-07
ns_A3	LC3 (TP10)	0.110	1.3559e-09	2.000	3.5262e-09
ns_A3	LC3 (TP12)	0.250	1.6706e-11	10.000	1.5879e-10
ns_A3	LC3 (TP14)	0.030	8.4934e-05	10.000	0.0056809

Table 3.64: Simulation results of Problem ns_A4

Problems	Methods	c5t	c5w	c6t	c6w
ns_A4	TAYLOR4 (TP4)	0.030	4.7235e-09	10.000	5.5164e-05
ns_A4	TAYLOR4 (TP6)	0.030	4.7235e-09	10.000	2.3393e-05
ns_A4	TAYLOR4 (TP8)	0.030	4.7235e-09	10.000	3.347e-07
ns_A4	TAYLOR4 (TP10)	0.040	1.0831e-10	10.000	4.3438e-09
ns_A4	TAYLOR4 (TP12)	0.080	1.394e-12	10.000	5.5834e-11
ns_A4	TAYLOR4 (TP14)	0.030	4.7235e-09	10.000	5.5164e-05
ns_A4	RK4 (TP4)	0.020	1.9557e-09	10.000	9.2536e-06
ns_A4	RK4 (TP6)	0.020	1.9557e-09	10.000	7.5759e-06
ns_A4	RK4 (TP8)	0.020	1.4271e-09	10.000	1.0095e-07
ns_A4	RK4 (TP10)	0.030	2.1419e-11	10.000	1.0543e-09
ns_A4	RK4 (TP12)	0.060	2.9976e-13	10.000	1.1743e-11
ns_A4	RK4 (TP14)	0.020	1.9557e-09	10.000	9.2536e-06
ns_A4	LA3 (TP4)	0.020	8.073e-11	10.000	3.1954e-06
ns_A4	LA3 (TP6)	0.020	8.073e-11	10.000	3.1954e-06
ns_A4	LA3 (TP8)	0.020	8.073e-11	10.000	2.7486e-07
ns_A4	LA3 (TP10)	0.020	5.8513e-12	10.000	3.7748e-09
ns_A4	LA3 (TP12)	0.040	7.7716e-14	10.000	4.1114e-11
ns_A4	LA3 (TP14)	0.020	8.073e-11	10.000	3.1954e-06
ns_A4	LC3 (TP4)	0.020	4.5581e-10	10.000	4.0216e-06
ns_A4	LC3 (TP6)	0.020	4.5581e-10	10.000	4.0216e-06
ns_A4	LC3 (TP8)	0.020	4.5581e-10	10.000	2.0247e-07
ns_A4	LC3 (TP10)	0.030	1.517e-11	10.000	3.328e-09
ns_A4	LC3 (TP12)	0.050	2.0917e-13	10.000	4.2366e-11
ns_A4	LC3 (TP14)	0.020	4.5581e-10	10.000	4.0216e-06

Table 3.65: Simulation results of Problem ns_A5

Problems	Methods	c5t	c5w	c6t	c6w
ns_A5	TAYLOR4 (TP4)	0.070	3.7194e-05	10.000	0.0019729
ns_A5	TAYLOR4 (TP6)	0.070	3.7194e-05	10.000	0.00023788
ns_A5	TAYLOR4 (TP8)	0.090	3.5949e-05	10.000	0.00011626
ns_A5	TAYLOR4 (TP10)	0.140	3.5909e-05	10.000	0.00011476
ns_A5	TAYLOR4 (TP12)	0.310	3.5909e-05	10.000	0.00011475
ns_A5	TAYLOR4 (TP14)	0.070	3.7194e-05	10.000	0.0019729
ns_A5	RK4 (TP4)	0.060	1.9565e-05	10.000	0.00017718
ns_A5	RK4 (TP6)	0.050	1.9565e-05	10.000	0.00013889
ns_A5	RK4 (TP8)	0.060	1.9565e-05	10.000	6.419e-05
ns_A5	RK4 (TP10)	0.070	1.9502e-05	10.000	6.2343e-05
ns_A5	RK4 (TP12)	0.120	1.9501e-05	10.000	6.2318e-05
ns_A5	RK4 (TP14)	0.060	1.9565e-05	10.000	0.00017718
ns_A5	LA3 (TP4)	0.060	1.2266e-05	10.000	0.0001074
ns_A5	LA3 (TP6)	0.060	1.2266e-05	10.000	8.3944e-05
ns_A5	LA3 (TP8)	0.060	1.2241e-05	10.000	4.0144e-05
ns_A5	LA3 (TP10)	0.080	1.2208e-05	10.000	3.9021e-05
ns_A5	LA3 (TP12)	0.150	1.2208e-05	10.000	3.9011e-05
ns_A5	LA3 (TP14)	0.060	1.2266e-05	10.000	0.0001074
ns_A5	LC3 (TP4)	0.060	3.0021e-06	10.000	8.1197e-05
ns_A5	LC3 (TP6)	0.060	3.0021e-06	10.000	5.8567e-05
ns_A5	LC3 (TP8)	0.060	2.9819e-06	10.000	1.0499e-05
ns_A5	LC3 (TP10)	0.080	2.9478e-06	10.000	9.4283e-06
ns_A5	LC3 (TP12)	0.140	2.9474e-06	10.000	9.4188e-06
ns_A5	LC3 (TP14)	0.060	3.0021e-06	10.000	8.1197e-05

Table 3.66: Simulation results of Problem ns_B1

Problems	Methods	c5t	c5w	c6t	c6w
ns_B1	TAYLOR4 (TP4)	0.120	0.00041929	10.000	0.16516
ns_B1	TAYLOR4 (TP6)	0.180	5.8337e-06	2.000	8.016e-05
ns_B1	TAYLOR4 (TP8)	0.270	1.5364e-07	2.000	3.7536e-05
ns_B1	TAYLOR4 (TP10)	0.610	1.6928e-09	2.000	1.2351e-05
ns_B1	TAYLOR4 (TP12)	1.500	1.847e-11	10.000	7.9824e-07
ns_B1	TAYLOR4 (TP14)	0.120	0.00041929	10.000	0.16516
ns_B1	RK4 (TP4)	0.060	0.00054791	10.000	0.093055
ns_B1	RK4 (TP6)	0.090	7.7186e-06	2.000	7.0418e-05
ns_B1	RK4 (TP8)	0.160	1.4924e-07	2.000	8.8254e-06
ns_B1	RK4 (TP10)	0.370	1.6512e-09	2.000	3.6011e-06
ns_B1	RK4 (TP12)	1.260	1.7231e-11	10.000	4.5013e-07
ns_B1	RK4 (TP14)	0.070	0.00054791	10.000	0.093055
ns_B1	LA3 (TP4)	0.060	0.00052296	10.000	0.7639
ns_B1	LA3 (TP6)	0.090	5.981e-06	2.000	6.7454e-05
ns_B1	LA3 (TP8)	0.150	1.3016e-07	2.000	3.6223e-05
ns_B1	LA3 (TP10)	0.380	1.5537e-09	2.000	1.2472e-05
ns_B1	LA3 (TP12)	0.820	1.5877e-11	10.000	4.2412e-07
ns_B1	LA3 (TP14)	0.060	0.00052296	10.000	0.7639
ns_B1	LC3 (TP4)	0.070	0.00074279	10.000	7.958
ns_B1	LC3 (TP6)	0.080	8.5157e-06	2.000	0.00010335
ns_B1	LC3 (TP8)	0.160	1.2294e-07	2.000	3.5055e-05
ns_B1	LC3 (TP10)	0.330	1.1696e-09	2.000	5.7342e-06
ns_B1	LC3 (TP12)	0.770	1.1289e-11	10.000	4.7543e-07
ns_B1	LC3 (TP14)	0.070	0.00074279	10.000	7.958

Table 3.67: Simulation results of Problem ns_B2

Problems	Methods	c5t	c5w	c6t	c6w
ns_B2	TAYLOR4 (TP4)	0.310	8.7614e-05	10.000	0.00010474
ns_B2	TAYLOR4 (TP6)	0.540	1.0578e-06	2.000	7.0248e-06
ns_B2	TAYLOR4 (TP8)	1.030	2.3614e-08	2.000	4.1597e-06
ns_B2	TAYLOR4 (TP10)	2.470	2.5418e-10	2.000	4.92e-08
ns_B2	TAYLOR4 (TP12)	6.170	2.8764e-12	10.000	2.2351e-10
ns_B2	TAYLOR4 (TP14)	0.330	8.7614e-05	10.000	0.00010474
ns_B2	RK4 (TP4)	0.200	9.85e-05	10.000	0.00014666
ns_B2	RK4 (TP6)	0.320	1.4878e-06	2.000	5.1326e-06
ns_B2	RK4 (TP8)	0.630	2.8479e-08	2.000	2.8268e-06
ns_B2	RK4 (TP10)	1.510	3.244e-10	2.000	5.9749e-08
ns_B2	RK4 (TP12)	3.710	3.4948e-12	10.000	1.4806e-10
ns_B2	RK4 (TP14)	0.200	9.85e-05	10.000	0.00014666
ns_B2	LA3 (TP4)	0.210	0.00011841	10.000	0.049815
ns_B2	LA3 (TP6)	0.270	1.0755e-06	2.000	8.8929e-06
ns_B2	LA3 (TP8)	0.490	2.1817e-08	2.000	3.4971e-06
ns_B2	LA3 (TP10)	1.110	2.4909e-10	2.000	4.7146e-08
ns_B2	LA3 (TP12)	2.830	2.6863e-12	10.000	1.7301e-10
ns_B2	LA3 (TP14)	0.200	0.00011841	10.000	0.049815
ns_B2	LC3 (TP4)	0.210	0.00011385	10.000	0.11981
ns_B2	LC3 (TP6)	0.290	1.2619e-06	2.000	1.129e-05
ns_B2	LC3 (TP8)	0.540	2.2956e-08	2.000	4.0489e-06
ns_B2	LC3 (TP10)	1.240	2.5586e-10	2.000	4.9842e-08
ns_B2	LC3 (TP12)	3.060	2.7098e-12	10.000	2.22e-10
ns_B2	LC3 (TP14)	0.200	0.00011385	10.000	0.11981

Table 3.68: Simulation results of Problem ns_B3

Problems	Methods	c5t	c5w	c6t	c6w
ns_B3	TAYLOR4 (TP4)	0.240	0.00012496	10.000	0.00010291
ns_B3	TAYLOR4 (TP6)	0.330	2.3385e-06	2.000	5.5216e-06
ns_B3	TAYLOR4 (TP8)	0.640	2.7922e-08	2.000	2.5563e-07
ns_B3	TAYLOR4 (TP10)	1.400	4.1428e-10	2.000	4.1314e-09
ns_B3	TAYLOR4 (TP12)	3.370	5.4439e-12	10.000	8.7434e-12
ns_B3	TAYLOR4 (TP14)	0.230	0.00012496	10.000	0.00010291
ns_B3	RK4 (TP4)	0.140	0.00015668	10.000	3.2946e-05
ns_B3	RK4 (TP6)	0.200	1.8062e-06	2.000	3.3727e-06
ns_B3	RK4 (TP8)	0.380	2.132e-08	2.000	1.8169e-07
ns_B3	RK4 (TP10)	0.830	2.2347e-10	2.000	1.9818e-09
ns_B3	RK4 (TP12)	2.050	2.2799e-12	10.000	5.204e-12
ns_B3	RK4 (TP14)	0.140	0.00015668	10.000	3.2946e-05
ns_B3	LA3 (TP4)	0.140	4.8032e-05	10.000	4.5514e-05
ns_B3	LA3 (TP6)	0.170	2.1365e-06	2.000	4.9673e-06
ns_B3	LA3 (TP8)	0.260	2.3226e-08	2.000	2.0047e-07
ns_B3	LA3 (TP10)	0.540	1.8545e-10	2.000	1.8394e-09
ns_B3	LA3 (TP12)	1.220	1.6824e-12	10.000	6.1119e-12
ns_B3	LA3 (TP14)	0.150	4.8032e-05	10.000	4.5514e-05
ns_B3	LC3 (TP4)	0.150	7.7124e-05	10.000	8.204e-05
ns_B3	LC3 (TP6)	0.210	1.6073e-06	2.000	3.8007e-06
ns_B3	LC3 (TP8)	0.290	3.1901e-08	2.000	2.6442e-07
ns_B3	LC3 (TP10)	0.590	3.4737e-10	2.000	3.3251e-09
ns_B3	LC3 (TP12)	1.410	3.5121e-12	10.000	3.7007e-12
ns_B3	LC3 (TP14)	0.140	7.7124e-05	10.000	8.204e-05

Table 3.69: Simulation results of Problem ns_B4

Problems	Methods	c5t	c5w	c6t	c6w
ns_B4	TAYLOR4 (TP4)	17.290	0.0016863	4.340	0.90148
ns_B4	TAYLOR4 (TP6)	19.640	8.0895e-05	7.751	1.2422
ns_B4	TAYLOR4 (TP8)	27.690	1.5611e-06	9.522	0.62151
ns_B4	TAYLOR4 (TP10)	54.410	2.2374e-08	10.000	0.029197
ns_B4	TAYLOR4 (TP12)	123.820	2.9517e-10	10.000	0.0011596
ns_B4	TAYLOR4 (TP14)	17.280	0.0016863	4.340	0.90148
ns_B4	RK4 (TP4)	16.010	0.00053691	5.537	1.439
ns_B4	RK4 (TP6)	17.720	2.7416e-05	8.560	0.69971
ns_B4	RK4 (TP8)	19.930	1.0075e-06	10.000	0.54239
ns_B4	RK4 (TP10)	33.190	1.5332e-08	10.000	0.023281
ns_B4	RK4 (TP12)	70.370	1.8093e-10	10.000	0.00066257
ns_B4	RK4 (TP14)	16.010	0.00053691	5.537	1.439
ns_B4	LA3 (TP4)	15.970	0.00044803	3.505	0.031706
ns_B4	LA3 (TP6)	17.740	2.2201e-05	8.281	0.86537
ns_B4	LA3 (TP8)	17.980	1.1266e-06	10.000	0.60409
ns_B4	LA3 (TP10)	31.530	1.6075e-08	10.000	0.026108
ns_B4	LA3 (TP12)	65.940	2.1847e-10	10.000	0.00083967
ns_B4	LA3 (TP14)	15.940	0.00044803	3.505	0.031706
ns_B4	LC3 (TP4)	14.510	0.00097031	4.842	1.322
ns_B4	LC3 (TP6)	17.500	2.2588e-05	8.261	0.88373
ns_B4	LC3 (TP8)	18.120	1.2305e-06	9.878	0.61908
ns_B4	LC3 (TP10)	31.120	1.8244e-08	10.000	0.033471
ns_B4	LC3 (TP12)	63.970	2.5061e-10	10.000	0.0009416
ns_B4	LC3 (TP14)	14.450	0.00097031	4.842	1.322

Table 3.70: Simulation results of Problem ns_B5

Problems	Methods	c5t	c5w	c6t	c6w
ns_B5	TAYLOR4 (TP4)	0.340	0.00024281	10.000	0.023851
ns_B5	TAYLOR4 (TP6)	0.420	4.2469e-06	10.000	0.00038581
ns_B5	TAYLOR4 (TP8)	0.730	6.5182e-08	10.000	0.00039017
ns_B5	TAYLOR4 (TP10)	1.880	7.267e-10	10.000	2.2839e-05
ns_B5	TAYLOR4 (TP12)	4.040	7.7236e-12	10.000	4.2675e-07
ns_B5	TAYLOR4 (TP14)	0.340	0.00024281	10.000	0.023851
ns_B5	RK4 (TP4)	0.230	0.00012717	10.000	0.018828
ns_B5	RK4 (TP6)	0.280	3.3117e-06	10.000	0.00050419
ns_B5	RK4 (TP8)	0.710	4.9849e-08	10.000	0.00028774
ns_B5	RK4 (TP10)	0.960	5.6878e-10	10.000	2.2317e-05
ns_B5	RK4 (TP12)	2.320	5.9515e-12	10.000	3.4427e-07
ns_B5	RK4 (TP14)	0.230	0.00012717	10.000	0.018828
ns_B5	LA3 (TP4)	0.230	4.6884e-05	10.000	0.085944
ns_B5	LA3 (TP6)	0.250	3.5205e-06	10.000	0.00074212
ns_B5	LA3 (TP8)	0.410	5.4075e-08	10.000	0.00049104
ns_B5	LA3 (TP10)	0.860	6.3172e-10	10.000	2.4014e-05
ns_B5	LA3 (TP12)	1.930	6.5794e-12	10.000	3.4972e-07
ns_B5	LA3 (TP14)	0.230	4.6884e-05	10.000	0.085944
ns_B5	LC3 (TP4)	0.230	4.1633e-05	10.000	0.099077
ns_B5	LC3 (TP6)	0.260	3.8362e-06	10.000	0.0014268
ns_B5	LC3 (TP8)	0.390	5.3813e-08	10.000	0.00028027
ns_B5	LC3 (TP10)	0.770	4.7183e-10	10.000	2.0975e-05
ns_B5	LC3 (TP12)	1.730	4.3485e-12	10.000	2.4349e-07
ns_B5	LC3 (TP14)	0.230	4.1633e-05	10.000	0.099077

Table 3.71: Simulation results of Problem ns_D1

Problems	Methods	c5t	c5w	c6t	c6w
ns_D1	TAYLOR4 (TP4)	25.810	0.006207	5.396	2.925
ns_D1	TAYLOR4 (TP6)	34.860	0.0034041	9.153	1.8021
ns_D1	TAYLOR4 (TP8)	53.420	0.0033342	10.000	1.3536
ns_D1	TAYLOR4 (TP10)	99.050	0.0033352	8.847	2.0949
ns_D1	TAYLOR4 (TP12)	188.630	0.0016658	8.083	2.7048
ns_D1	TAYLOR4 (TP14)	25.820	0.006207	5.396	2.925
ns_D1	RK4 (TP4)	20.920	0.0028112	6.395	1.7957
ns_D1	RK4 (TP6)	31.130	0.0016874	10.000	0.38124
ns_D1	RK4 (TP8)	44.450	0.0016637	10.000	0.30351
ns_D1	RK4 (TP10)	46.830	0.0016633	10.000	0.21176
ns_D1	RK4 (TP12)	90.540	0.00083237	9.009	1.8083
ns_D1	RK4 (TP14)	20.910	0.0028112	6.395	1.7957
ns_D1	LA3 (TP4)	18.400	0.0022911	3.265	0.023256
ns_D1	LA3 (TP6)	24.620	0.00073054	9.243	1.7605
ns_D1	LA3 (TP8)	32.510	0.0006961	10.000	0.5364
ns_D1	LA3 (TP10)	48.120	0.00069491	10.000	0.532
ns_D1	LA3 (TP12)	91.680	0.0006949	8.836	1.8679
ns_D1	LA3 (TP14)	18.470	0.0022911	3.265	0.023256
ns_D1	LC3 (TP4)	18.290	0.0019492	3.291	0.023226
ns_D1	LC3 (TP6)	24.570	0.00026326	9.563	1.9057
ns_D1	LC3 (TP8)	30.500	0.00022948	10.000	0.19742
ns_D1	LC3 (TP10)	48.300	0.00022838	10.000	0.72164
ns_D1	LC3 (TP12)	95.330	9.4802e-05	9.079	1.8038
ns_D1	LC3 (TP14)	18.230	0.0019492	3.291	0.023226

Table 3.72: Simulation results of Problem ns_E1

Problems	Methods	c5t	c5w	c6t	c6w
ns_E1	TAYLOR4 (TP4)	0.570	1.4351	2.719	18.977
ns_E1	TAYLOR4 (TP6)	0.930	0.42517	2.000	0.00010071
ns_E1	TAYLOR4 (TP8)	1.450	0.20741	2.000	8.381e-06
ns_E1	TAYLOR4 (TP10)	3.080	0.13387	2.000	1.0032e-07
ns_E1	TAYLOR4 (TP12)	7.210	0.050696	10.000	22.567
ns_E1	TAYLOR4 (TP14)	0.570	1.4351	2.719	18.977
ns_E1	RK4 (TP4)	0.420	0.012668	10.000	1.2214
ns_E1	RK4 (TP6)	0.500	0.017561	2.000	6.1085e-05
ns_E1	RK4 (TP8)	0.700	0.031314	2.000	5.143e-06
ns_E1	RK4 (TP10)	1.140	0.030614	2.000	6.6196e-08
ns_E1	RK4 (TP12)	2.540	0.031647	10.000	0.91288
ns_E1	RK4 (TP14)	0.410	0.012668	10.000	1.2214
ns_E1	LA3 (TP4)	0.410	0.013204	10.000	0.34595
ns_E1	LA3 (TP6)	0.500	0.010426	2.000	8.1548e-05
ns_E1	LA3 (TP8)	0.690	0.013066	2.000	3.5913e-06
ns_E1	LA3 (TP10)	1.110	0.015096	2.000	5.1198e-08
ns_E1	LA3 (TP12)	2.420	0.011244	10.000	0.33604
ns_E1	LA3 (TP14)	0.420	0.013204	10.000	0.34595
ns_E1	LC3 (TP4)	0.410	0.0095702	10.000	0.26912
ns_E1	LC3 (TP6)	0.500	0.01023	2.000	8.7855e-05
ns_E1	LC3 (TP8)	0.720	0.010676	2.000	3.8404e-06
ns_E1	LC3 (TP10)	1.080	0.0095686	2.000	4.2571e-08
ns_E1	LC3 (TP12)	2.050	0.0091033	10.000	0.22942
ns_E1	LC3 (TP14)	0.410	0.0095702	10.000	0.26912

Table 3.73: Simulation results of Problem ns_E2

Problems	Methods	c5t	c5w	c6t	c6w
ns_E2	TAYLOR4 (TP4)	0.080	0.00040596	10.000	0.015868
ns_E2	TAYLOR4 (TP6)	0.120	5.6232e-06	2.000	0.00010071
ns_E2	TAYLOR4 (TP8)	0.180	7.7141e-08	2.000	8.381e-06
ns_E2	TAYLOR4 (TP10)	0.410	8.2963e-10	2.000	1.0032e-07
ns_E2	TAYLOR4 (TP12)	1.010	8.848e-12	10.000	3.459e-07
ns_E2	TAYLOR4 (TP14)	0.080	0.00040596	10.000	0.015868
ns_E2	RK4 (TP4)	0.050	0.0003214	10.000	0.015258
ns_E2	RK4 (TP6)	0.070	7.4223e-06	2.000	6.1085e-05
ns_E2	RK4 (TP8)	0.130	1.1435e-07	2.000	5.143e-06
ns_E2	RK4 (TP10)	0.280	1.8288e-09	2.000	6.6196e-08
ns_E2	RK4 (TP12)	0.670	2.3118e-11	10.000	5.8802e-07
ns_E2	RK4 (TP14)	0.050	0.0003214	10.000	0.015258
ns_E2	LA3 (TP4)	0.050	0.00025275	10.000	0.066185
ns_E2	LA3 (TP6)	0.060	3.8427e-06	2.000	8.1548e-05
ns_E2	LA3 (TP8)	0.100	7.7582e-08	2.000	3.5913e-06
ns_E2	LA3 (TP10)	0.230	1e-09	2.000	5.1198e-08
ns_E2	LA3 (TP12)	0.550	1.4015e-11	10.000	5.7683e-07
ns_E2	LA3 (TP14)	0.050	0.00025275	10.000	0.066185
ns_E2	LC3 (TP4)	0.050	0.00027974	10.000	0.10799
ns_E2	LC3 (TP6)	0.060	3.9986e-06	2.000	8.7855e-05
ns_E2	LC3 (TP8)	0.110	7.6055e-08	2.000	3.8404e-06
ns_E2	LC3 (TP10)	0.230	7.073e-10	2.000	4.2571e-08
ns_E2	LC3 (TP12)	0.580	7.085e-12	10.000	2.6685e-07
ns_E2	LC3 (TP14)	0.060	0.00027974	10.000	0.10799

Table 3.74: Simulation results of Problem ns_E3

Problems	Methods	c5t	c5w	c6t	c6w
ns_E3	TAYLOR4 (TP4)	0.300	0.0001603	10.000	0.015971
ns_E3	TAYLOR4 (TP6)	0.500	2.0937e-06	2.000	9.0741e-06
ns_E3	TAYLOR4 (TP8)	0.990	3.6313e-08	2.000	2.1037e-07
ns_E3	TAYLOR4 (TP10)	2.430	4.1545e-10	2.000	2.2064e-09
ns_E3	TAYLOR4 (TP12)	5.990	4.5057e-12	10.000	2.7482e-08
ns_E3	TAYLOR4 (TP14)	0.300	0.0001603	10.000	0.015971
ns_E3	RK4 (TP4)	0.230	0.00012793	10.000	0.020574
ns_E3	RK4 (TP6)	0.320	2.9744e-06	2.000	1.12e-05
ns_E3	RK4 (TP8)	0.570	5.1365e-08	2.000	3.5177e-07
ns_E3	RK4 (TP10)	1.270	5.4612e-10	2.000	4.1754e-09
ns_E3	RK4 (TP12)	3.160	5.5893e-12	10.000	4.885e-08
ns_E3	RK4 (TP14)	0.240	0.00012793	10.000	0.020574
ns_E3	LA3 (TP4)	0.190	0.00026818	10.000	0.036244
ns_E3	LA3 (TP6)	0.340	2.8738e-06	2.000	1.0633e-05
ns_E3	LA3 (TP8)	0.540	5.2924e-08	2.000	2.7291e-07
ns_E3	LA3 (TP10)	1.230	5.7092e-10	2.000	2.9675e-09
ns_E3	LA3 (TP12)	3.000	5.9095e-12	10.000	3.4149e-08
ns_E3	LA3 (TP14)	0.190	0.00026818	10.000	0.036244
ns_E3	LC3 (TP4)	0.210	0.00014667	10.000	0.047597
ns_E3	LC3 (TP6)	0.300	1.9905e-06	2.000	9.1701e-06
ns_E3	LC3 (TP8)	0.560	3.2515e-08	2.000	2.1352e-07
ns_E3	LC3 (TP10)	1.300	3.5506e-10	2.000	2.2933e-09
ns_E3	LC3 (TP12)	3.230	3.6904e-12	10.000	2.7388e-08
ns_E3	LC3 (TP14)	0.210	0.00014667	10.000	0.047597

Table 3.75: Simulation results of Problem ns_E4

Problems	Methods	c5t	c5w	c6t	c6w
ns_E4	TAYLOR4 (TP4)	0.040	1.2137e-09	10.000	3.702e-06
ns_E4	TAYLOR4 (TP6)	0.040	1.2137e-09	2.000	7.75e-08
ns_E4	TAYLOR4 (TP8)	0.040	1.2137e-09	2.000	7.75e-08
ns_E4	TAYLOR4 (TP10)	0.040	9.0002e-10	2.000	2.5812e-09
ns_E4	TAYLOR4 (TP12)	0.060	1.4914e-11	10.000	1.7366e-10
ns_E4	TAYLOR4 (TP14)	0.040	1.2137e-09	10.000	3.702e-06
ns_E4	RK4 (TP4)	0.020	7.5673e-11	10.000	7.2614e-07
ns_E4	RK4 (TP6)	0.020	7.5673e-11	2.000	3.0567e-09
ns_E4	RK4 (TP8)	0.020	7.5673e-11	2.000	3.0567e-09
ns_E4	RK4 (TP10)	0.020	7.5673e-11	2.000	1.7044e-09
ns_E4	RK4 (TP12)	0.030	1.1987e-11	10.000	2.519e-10
ns_E4	RK4 (TP14)	0.020	7.5673e-11	10.000	7.2614e-07
ns_E4	LA3 (TP4)	0.020	2.0709e-10	10.000	3.5712e-07
ns_E4	LA3 (TP6)	0.020	2.0709e-10	2.000	1.1984e-08
ns_E4	LA3 (TP8)	0.020	2.0709e-10	2.000	1.1984e-08
ns_E4	LA3 (TP10)	0.020	2.0709e-10	2.000	2.0098e-09
ns_E4	LA3 (TP12)	0.030	2.4286e-11	10.000	1.1126e-10
ns_E4	LA3 (TP14)	0.020	2.0709e-10	10.000	3.5712e-07
ns_E4	LC3 (TP4)	0.020	8.4192e-11	10.000	3.6162e-07
ns_E4	LC3 (TP6)	0.020	8.4192e-11	2.000	6.1039e-09
ns_E4	LC3 (TP8)	0.020	8.4192e-11	2.000	6.1039e-09
ns_E4	LC3 (TP10)	0.020	8.4192e-11	2.000	2.083e-09
ns_E4	LC3 (TP12)	0.030	1.1283e-11	10.000	1.1097e-10
ns_E4	LC3 (TP14)	0.020	8.4192e-11	10.000	3.6162e-07

3.2.3 Discussion

In the past tables the methods are highlighted in blue for the best one and in grey for the second one, at one second and at ten seconds of simulation. We can easily conclude from these results that the Runge-Kutta methods are more efficient than the Taylor approach, and moreover, that the implicit ones are often better than the explicit RK4 method.

3.3 Other problems

In this section, we present few well-known problems. For each one, the results provided by the logger and an image obtained with our 3D-plotter are given. These results are listed in order to prove that our tool is able to simulate some difficult problems.

3.3.1 Affine-uncertain

The problem is the following:

Initial states: $y_0 = ([0.8, 1.2]; [0.8, 1.2]; [0.8, 1.2]; [0.8, 1.2]; [0.8, 1.2])$

$$\text{Some interval parameters: } \left| \begin{array}{l} a1 = [-1.1, -0.9] \\ a2 = [-4.1, -3.9] \\ a3 = [-3.1, -2.9] \\ \quad a4 = [0.9, 1.1] \\ a5 = [-2.1, -1.9] \end{array} \right.$$

$$\text{The differential system: } \dot{y} = \begin{cases} a1 * y[0] + a2 * y[1] + [-0.1, 0.1] \\ -a2 * y[0] + a1 * y[1] + y[2] + [-0.1, 0.1] \\ a3 * y[2] + a4 * y[3] + [-0.1, 0.1] \\ -a4 * y[2] + a3 * y[3] + [-0.1, 0.1] \\ a5 * y[4] + [-0.1, 0.1] \end{cases}$$

```
Solution at t=2.000000 :
([-2.39334, 2.05433] ; [-2.19454, 2.48459] ; [-0.0626994, 0.0651442] ;
[-0.0672006, 0.0606297] ; [-0.0418624, 0.0784937])
Diameter : (4.44767 ; 4.67913 ; 0.127844 ; 0.12783 ; 0.120356)
Rejected picard :2
Accepted picard :673
Step min :0.00174779
Step max :0.00313099
Truncature error max :2.06329e-12
```

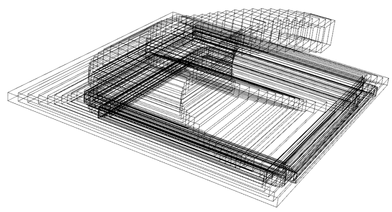


Figure 3.3: Simulation of the affine uncertain system

3.3.2 circle

The problem is the following:

Initial states: $y_0 = ([0, 0.1]; [0.95, 1.05])$

The differential system: $\dot{y} = \begin{cases} -y[1] \\ y[0] \end{cases}$

```
Solution at t=100.000000 :
([0.476077, 0.622885] ; [0.763549, 0.910451])
Diameter : (0.146808 ; 0.146902)
Rejected picard :0
Accepted picard :1400
Step min :0.01
Step max :0.0730046
Truncature error max :1.60132e-08
```

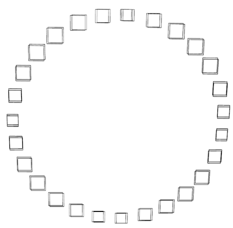


Figure 3.4: Simulation of the circle system

3.3.3 Lambert: linear problem (p213)

The problem is the following:

Initial states: $y_0 = (2, 3, 0)$

The differential system:

$$\dot{y} = \begin{cases} -2.0 * y[0] + y[1] + 2.0 * sin(y[2]) \\ 998.0 * y[0] - 999.0 * y[1] + 999.0 * (cos(y[2]) - sin(y[2])) \\ 1.0 \end{cases}$$

```
Solution at t=10.000000 :
([-0.544487, -0.543374] ; [-0.839843, -0.838119] ; [10, 10])
Diameter : (0.00111361 ; 0.00172438 ; 0)
Rejected picard :3
Accepted picard :26873
Step min :0.000204486
Step max :0.00040387
Truncature error max :7.99348e-06
```

3.3.4 Lambert: non linear and stiff problem (p223)

The problem is the following:

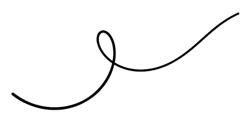


Figure 3.5: Simulation of the Lambert linear system

Initial states: $y_0 = (1, e^{-1}, 1)$
The differential system:

$$\dot{y} = \begin{cases} 1/y[0] - y[1] * exp(y[2] * y[2])/(y[2] * y[2]) - y[2] \\ 1/y[1] - exp(y[2] * y[2]) - 2 * y[2] * exp(-y[2] * y[2]) \\ 1 \end{cases}$$

```
Solution at t=1.500000 :
([0.399984, 0.400016] ; [0.00193044, 0.00193047] ; [2.5, 2.5])
Diameter : (3.26431e-05 ; 3.27931e-08 ; 9.19886e-12)
Rejected picard :1
Accepted picard :21060
Step min :4.79902e-06
Step max :0.0152239
Truncature error max :2.70897e-08
```

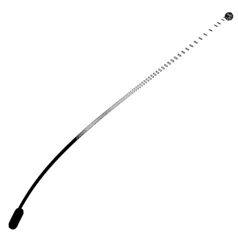


Figure 3.6: Simulation of the Lambert stiff system

3.3.5 Lorentz

The problem is the following:
Initial states: $y_0 = (15, 15, 36)$

$$\text{Some parameters: } \left| \begin{aligned} sigma &= 10 \\ rho &= 15 \\ beta &= 8/3 \end{aligned} \right.$$

$$\text{The differential system: } \dot{y} = \begin{cases} sigma * (y[1] - y[0]) \\ y[0] * (rho - y[2]) - y[1] \\ y[0] * y[1] - beta * y[2] \end{cases}$$

```
Solution at t=4.000000 :
([-4.89107, -4.60361] ; [-0.207979, 0.199418] ; [28.8958, 29.2391])
Diameter : (0.287467 ; 0.407397 ; 0.343226)
Rejected picard :5
Accepted picard :7419
Step min :0.0003125
Step max :0.000924129
Truncature error max :2.52501e-13
```

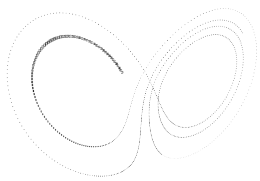


Figure 3.7: Simulation of the Lorentz system

3.3.6 oil-reservoir

The problem is the following:

Initial states: $y_0 = (10, 0)$

One parameter: $\left| stiffness = 0.001 \text{ } or \text{ } 0.0001 \right.$

$$\text{The differential system: } \dot{y} = \begin{cases} y[1] \\ y[1] * y[1] - 3.0/(stiffness + y[0] * y[0]) \end{cases}$$

```
Solution at t=50.000000 :
([-8.27752, -8.27751] ; [-0.224547, -0.224547])
Diameter : (6.2308e-06 ; 1.56923e-07)
Rejected picard :3
Accepted picard :71076
Step min :1e-06
Step max :0.016677
Truncature error max :6.57475e-11
```

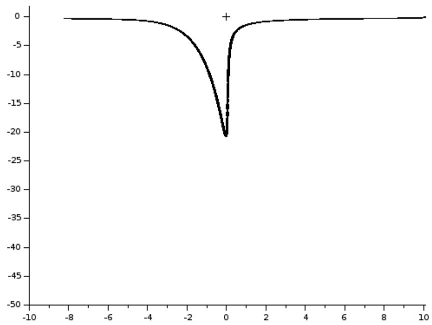


Figure 3.8: Simulation of the oil-reservoir system (stiffness=$1e-03$)

```
Solution at t=50.000000 :
([-8.56149, -8.56146] ; [-0.216578, -0.216577])
Diameter : (2.91622e-05 ; 6.85792e-07)
Rejected picard :2
Accepted picard :73200
Step min :1e-06
Step max :0.0166808
Truncature error max :2.2382e-08
```

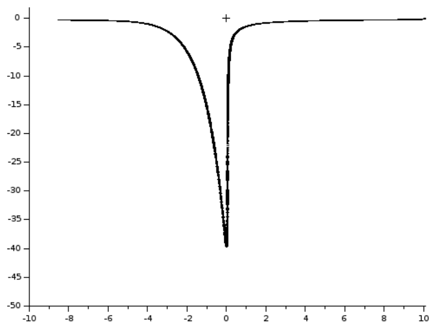


Figure 3.9: Simulation of the oil-reservoir system (stiffness=$1e-04$)

3.3.7 vanderpol

The problem is the following:
Initial states: $y_0 = (2, 0)$
One parameter: $\left| mu = 1.0 \; or \; 2.0 \right.$
The differential system: $\dot{y} = \begin{cases} y[1] \\ mu * (1.0 - y[0] * y[0]) * y[1] - y[0]) \end{cases}$

```
 Solution at t=50.000000 :
([-2.03535, -1.97923] ; [0.0419892, 0.0988844])
Diameter : (0.0561216 ; 0.0568952)
Rejected picard :1
Accepted picard :6789
Step min :0.00140294
Step max :0.012461
Truncature error max :2.53534e-12
```

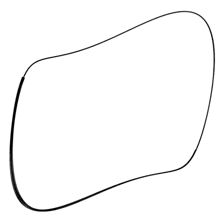


Figure 3.10: Simulation of the vanderpol system ($\mu = 1$)

```
Solution at t=40.000000 :
([1.0493, 1.49018] ; [-0.879307, -0.404504])
Diameter : (0.440879 ; 0.474803)
Rejected picard :2
Accepted picard :7163
Step min :0.00217604
Step max :0.0106292
Truncature error max :3.89696e-12
```

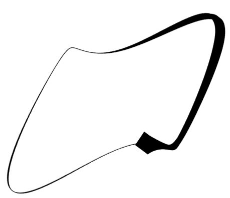


Figure 3.11: Simulation of the vanderpol system ($\mu = 2$)

3.3.8 volterra

The problem is the following:

Initial states: $y_0 = (1.0; 3.0)$, with a potentially added uncertainty $[-0.01, 0.01]$.

The differential system: $\dot{y} = \begin{cases} 2.0 * y[0] * (1.0 - y[1]) \\ -y[1] * (1.0 - y[0]) \end{cases}$

```
 Solution at t=5.488138 :
([1, 1] ; [3, 3])
Diameter : (2.25385e-10 ; 3.90135e-10)
Rejected picard :3
Accepted picard :1642
Step min :0.00115543
Step max :0.00830212
Truncature error max :3.11242e-14
```

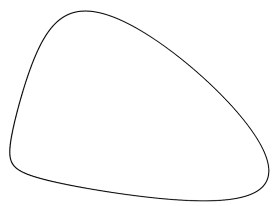


Figure 3.12: Simulation of the volterra system

```
Solution at t=5.488138 :
([0.919632, 1.08037] ; [2.92806, 3.07194])
Diameter : (0.160737 ; 0.14388)
Rejected picard :3
Accepted picard :1702
Step min :0.000912773
Step max :0.00819498
Truncature error max :3.13532e-14
```

3.3.9 orbit

The problem is the following:

Initial states: $y_0 = (0.994; 0; 0; -2.00158510637908252240537862224)$

Some parameters: $\left| \begin{array}{l} mu = 0.012277471 \\ mu_h = 1.0 - mu \end{array} \right.$

The differential system:

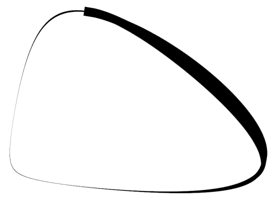


Figure 3.13: Simulation of the volterra system with uncertainties

$$\dot{y} = \begin{cases} y[2] \\ y[3] \\ y[0] + 2.0 * y[3] - mu_h * (y[0] + mu)/(sqrt((y[0] + mu) * (y[0] + mu) + \\ (y[1]) * (y[1])) * ((y[0] + mu) * (y[0] + mu) + (y[1]) * (y[1]))) \\ -mu * (y[0] - mu_h)/(sqrt((y[0] - mu_h) * (y[0] - mu_h) + \\ (y[1]) * (y[1])) * ((y[0] - mu_h) * (y[0] - mu_h) + (y[1]) * (y[1]))) \\ y[1] - 2.0 * y[2] - mu_h * y[1]/(sqrt((y[0] + mu) * (y[0] + mu) + \\ (y[1] * y[1])) * ((y[0] + mu) * (y[0] + mu) + (y[1] * y[1]))) \\ -mu * y[1]/(sqrt((y[0] - mu_h) * (y[0] - mu_h) + \\ (y[1] * y[1])) * ((y[0] - mu_h) * (y[0] - mu_h) + (y[1] * y[1]))) \end{cases}$$

```
Solution at t=5.000000 :
([-0.00635623, 0.0513502] ; [0.826695, 0.905428] ;
[-0.188858, -0.0486894] ; [-0.496981, -0.351559])
Diameter : (0.0577065 ; 0.0787324 ; 0.140169 ; 0.145422)
Rejected picard :0
Accepted picard :45489
Step min :1e-07
Step max :0.000310353
Truncature error max :2.18118e-12
```

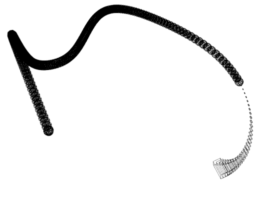


Figure 3.14: Simulation of the orbit system

3.3.10 Rossler

The problem is the following:

Initial states: $y_0 = (0; -10.3; 0.03)$

Some parameters: $\left| \begin{array}{l} a = 0.2 \\ b = 0.2 \\ c = 5.7 \end{array} \right.$

The differential system: $\dot{y} = \begin{cases} -(y[1] + y[2]) \\ y[0] + a * y[1] \\ b + y[2] * (y[0] - c) \end{cases}$

```
Solution at t=50.000000 :
([10.1496, 11.4172] ; [-7.78271, -5.69522] ; [0.0544862, 0.0971181])
Diameter : (1.26763 ; 2.08749 ; 0.0426319)
Rejected picard :6
Accepted picard :7123
Step min :0.001
Step max :0.0132739
Truncature error max :1.3313e-11
```

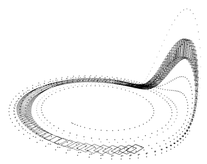


Figure 3.15: Simulation of the Rossler system

3.3.11 Discussion

In this section, we perform some testes on other problems, coming from the literature. It is not provided to compare with the other tools, but just to prove that we can model and perform some classical problems.

Chapter 4

Conclusion

To conclude, we present in this report our tool for the validated simulation of ordinary differential equations. It is based on the Runge-Kutta methods, in explicit and implicit form. By using the affine arithmetic, it is able to counteract the wrapping effect with a more simple approach than other approach, such as QR factorization for example. The results presented in this report prove that our tool is equivalent to the best software currently available (Vnode). It is important to notice that our tool is really steady, providing good results for all the benchmark with at least one of its Runge-Kutta method. And last but not the least, we present in this report only the methods at order four, when Vnode can use Taylor series at order twenty-five. Our approach is then validated and is strongly promising by using higher order.

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