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TO INFINITESIMAL DISTURBANCES.

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SPATIAL STABILITY OF PIPE POISEUILLE FLOW

TO

INFINITESIMAL DISTURBANCES

by

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DEPARTMENT OF MECHANICAL ENGINEERING

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# Carnegie-Mellon University

## CARNEGIE INSTITUTE OF TECHNOLOGY

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TITLE Spatial Stability of Pipe Poiseuille Flow to Infinitesimal

Disturbances

PRESENTED BY Vijay K. Garg

ACCEPTED BY THE DEPARTMENT OF Mechanical Engineering

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| | |
|---|---|
| a,b | exponents of r defined by equations (3.44) and (3.7) |
| $B_i$ | the $i^{th}$ complex constant defined by equations (3.23) |
| c | exponent of r defined by equations (3.8) and (3.44) |
| C | a closed contour in the complex k-plane |
| $C_i$ | the $i^{th}$ coefficient in the series expansion for the steady mean velocity (equation (3.22)) |
| $C_p$ | dimensionless phase velocity |
| $C_l$ | a closed contour in the complex determinant plane |
| d | exponent of r defined by equations (3.9) and (3.44) |
| det | determinant defined by equations (3.35) or (3.73) |
| D | the operator $\frac{d}{dr}$, $\frac{d}{dz}$ or $\frac{d}{dx}$ |
| e | the exponential function |
| $\underset{\sim}{e_r}, \underset{\sim}{e_z}, \underset{\sim}{e_\theta}$ | unit vectors in the radial, axial and circumferential direction respectively |
| f | complex function of a complex variable z; function of dependent and independent variables in a differential equation |
| $\bar{f}, \bar{g}$ | complex eigenfunctions defined by equations (3.37) |
| $F_i, G_i$ | the $i^{th}$ coefficients in the series expansions for $\bar{f}$ and $g$ (equations (3.44) and (3.56)) |
| h | step size for integration |
| j | $\sqrt{-1}$ |
| k | dimensionless complex wave number (variously called the eigenvalue or propagation constant) |
| $k^*$ | dimensional counterpart of k |

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| | |
|---|---|
| $k_i$ | the imaginary part of $k$ |
| $k_\ell$ | the $\ell^{th}$ value of $k$ (equation (1.2)) |
| $k_r$ | the real part of $k$ |
| $\ell$ | a positive integer |
| L | order of the step-by-step integration scheme |
| m | a positive integer corresponding either to the number of terms in the series expansion for $V_z$ or to the 'central' mode for an axisymmetric disturbance |
| M | number of eigenvalues enclosed by a certain region in the $k$-plane |
| Max | maximum value |
| n | the azimuthal mode number |
| N | number of zeros of $f(z)$ within a closed region |
| p | dimensionless perturbation pressure |
| $p^*$ | dimensional perturbation pressure |
| $\bar{p}$ | complex eigenfunction for perturbation pressure |
| $\hat{p}$ | complex quantity whose real part is p (equation (2.20)) |
| $\bar{p}^*$ | normalized value of $\bar{p}$ |
| $P_i$ | the $i^{th}$ independent solution for $\bar{p}$ |
| P | resultant pressure in the flow (see equation (2.2)); number of poles of $f(z)$ within a closed region C |
| $P_i$ | the $i^{th}$ coefficient in the series expansion for $\bar{p}$ (equations (3.9), (3.21), (3.44) and (3.56)) |
| $P_o^*$ | steady mean pressure (equation (2.2)) |
| q | integer for Gill's 'wall' mode for an axisymmetric disturbance |
| r | radial coordinate (synonymous with $r'$ from equation (2.13) onwards) |

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| | |
|---|---|
| $r'$ | dimensionless radial coordinate |
| $r_\ell$ | a radial distance such that $(r_s < r_\ell \le 1)$ |
| $r_m$ | radius at which the perturbation amplitude is maximum |
| $r_p$ | pipe radius |
| $r_s$ | radius up to which the series solution is carried |
| $R$ | Reynolds number |
| $t$ | time (synonymous with $t'$ from equation (2.14) onward) |
| $t'$ | dimensionless time |
| $U_i$ | the $i^{th}$ coefficient in the series expansion for $\bar{v}_z$ (equations (3.8), (3.20), (3.44) and (3.56)) |
| $v_r, v_z, v_\theta$ | dimensionless radial, axial and circumferential components of disturbance velocity |
| $\underset{\sim}{v}$ | disturbance velocity vector whose components are $v_r$, $v_z$ and $v_\theta$ |
| $v_r^*, v_z^*, v_\theta^*$ | dimensional counterparts of $v_r$, $v_z$, $v_\theta$ respectively |
| $\underset{\sim}{v^*}$ | vector whose components are $v_r^*$, $v_z^*$ and $v_\theta^*$ |
| $\bar{v}_r, \bar{v}_z, \bar{v}_\theta$ | complex eigenfunctions for $v_r$, $v_z$, $v_\theta$ respectively |
| $\hat{v}_r, \hat{v}_z, \hat{v}_\theta$ | complex quantities whose real parts are $v_r$, $v_z$, $v_\theta$ respectively (equations (2.17) through (2.19)) |
| $\underset{\sim}{\hat{v}}$ | vector whose components are $\hat{v}_r$, $\hat{v}_z$ and $\hat{v}_\theta$ |
| $v_{ri}, v_{zi}, v_{\theta i}$ | the $i^{th}$ independent solution for $\bar{v}_r$, $\bar{v}_z$, $\bar{v}_\theta$ |
| $\bar{v}_r^*, \bar{v}_z^*, \bar{v}_\theta^*$ | normalized values of $\bar{v}_r$, $\bar{v}_z$ and $\bar{v}_\theta$ respectively |
| $\underset{\sim}{v}_n$ | vector defined by equation (1.1) |
| $\underset{\sim}{v}_{n\ell}$ | vector defined by equation (1.2) |
| $\underset{\sim}{\bar{v}}_{n\ell}$ | vector defined by equation (1.3) |
| $V_r^*, V_z^*, V_\theta^*$ | radial, axial and circumferential components of the steady mean velocity (dimensional values) |

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$V_z$      dimensionless counterpart of $V_z^*$

$V_i$      the $i^{th}$ coefficient in the series expansion for $\bar{v}_r$ (equations (3.7) and (3.19))

$\underset{\sim}{V^*}$      velocity vector for the steady mean flow

$V_{z0}^*$      value of $V_z^*$ at the center of the pipe

$V_r, V_z, V_\theta$      radial, axial and circumferential components of the resultant flow velocity

$\underset{\sim}{V}$      resultant velocity vector

x      real part of the complex variable z; independent variable in a differential equation

X      real part of the complex function f(z)

y      imaginary part of the complex variable z; dependent variable in a differential equation

$y_c$      corrected value of y

$y_h$      value of y calculated with a step size of h

$y_{h/2}$      value of y calculated with a step size of h/2

Y      imaginary part of the complex function f(z)

z      axial coordinate (synonymous with z' from equation (2.13) onwards); complex variable as in Chapter 4

z'      dimensionless axial coordinate

$\alpha$      integer for the general exponent of r in a series

$\varepsilon$      error in step-by-step integration with a predictor-corrector method (equation (AI-4))

$\theta$      angular coordinate

$\lambda$      dimensionless wavelength

$\nu$      kinematic viscosity of the fluid

$\pi$      the constant 3.14159...

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$\rho$       density of the incompressible fluid

$\displaystyle\sum_{i=1}^{m}$       summation over i from 1 to m

$\phi$       phase angle for a complex quantity

$\omega$       dimensionless frequency

$\omega^*$       dimensional frequency

$\underset{\sim}{0}$       null vector

e.g.       for example

$|\ \ |$       absolute value

$Im(z)$       denotes imaginary part of the complex number z

$Re(z)$       denotes real part of the complex number z

# ABSTRACT

A theoretical study of the spatial (or convective) stability of Poiseuille flow in a rigid pipe to infinitesimal disturbances is presented in this dissertation. Both axisymmetric and non-axisymmetric disturbances were considered. The downstream propagation of the disturbance, that has a constant frequency and is imposed at a specified location in the fluid, is governed by four, coupled, linear, ordinary differential equations in terms of its velocity components and hydrodynamic pressure. The flow is considered to be convectively stable if the disturbance decays in the downstream direction.

For a wide range of Reynolds numbers, frequencies, and azimuthal mode numbers, the coupled differential equations were solved numerically for the complex eigenvalues, or wave numbers, each of which defines a mode of propagation. A series solution for small values of the pipe radius followed by step-by-step integration to the pipe wall was employed. In order to ascertain the number of eigenvalues within a closed region, an eigenvalue search technique was developed. The theoretical results predict the pipe Poiseuille flow to be stable to all infinitesimal, axisymmetric and non-axisymmetric disturbances up to Reynolds numbers of 10,000.

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Chapter I

INTRODUCTION

1.1  Literature Survey

The transition from laminar to turbulent flow of a viscous

fluid in a pipe was first experimentally observed by Osborne

Reynolds [1]* in 1883.  This phenomenon of transition, though in

engineering practice more important than that of the stability

of laminar motion, is yet to be clearly understood.  However, a

rich harvest of knowledge has been gained regarding the stability

of various fluid flows to small, though not necessarily

infinitesimal, disturbances - a prerequisite for understanding

the more complex phenomenon of transition.  Since a flow is

said to be stable if the effects of small disturbances do not

grow with respect to either time or downstream distance, it

should be pointed out that instability (or growth of disturbances)

does not necessarily lead to turbulent motion; it could lead to

another state of laminar motion, and in fact, this is often the

case [2].

Lord Rayleigh is generally credited to be the founder of

hydrodynamic stability theory but his efforts were largely

limited to inviscid fluids only.  Observing that the instability

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*Numbers in parentheses designate references listed at the
end of the dissertation.

generally occurs at large Reynolds numbers, Rayleigh conjectured
[3] that the mechanism of instability may be analyzed by first
neglecting the viscous force, and incorporating it later as a
stabilizing influence. This approach simplifies the analysis a
great deal since the viscous terms contain spatial derivatives
of the highest order; their neglect, therefore, reduces the order
of the stability equations. However, Rayleigh's own analysis
revealed that the inviscid plane Poiseuille flow is <u>not</u> unstable,
thus creating a paradox of sorts [4].

Lord Kelvin's work [5] for a related problem supported
Reynolds' idea [1] that the flow may be unstable for disturbances
of a certain magnitude, and stable for smaller disturbances. This
observation was further supported by Ekman [6] and others who
showed experimentally that for flow in a circular pipe, the
transition from laminar to turbulent flow can be delayed to very
high Reynolds numbers (40,000 and more) if sufficient care is
taken with the entry conditions.

The behavior of small disturbances to a flow can be analyzed
in two different ways. In the first of these, a disturbance is
assumed to be applied at an initial instant everywhere in the
fluid and is, in particular, periodic in the downstream direction.
The flow is then said to be "temporally" stable if the disturbance
decays with time and unstable if it grows. In the second approach,
it is assumed that a disturbance which is periodic in time is
imposed at a specified location in the fluid and propagates downstream.

If the disturbance decays in the downstream direction, the flow is considered to be "spatially" stable. Though the latter point of view is physically more realistic, almost all effort has been devoted to the former approach since the very inception of the stability theory. It is only in the last decade that the analysis of spatial stability has begun to draw some significant attention.

The temporal stability of Poiseuille flow in a pipe to axisymmetric disturbances was, perhaps, first analyzed theoretically by Sexl [7, 8]. His conclusions are, however, unreliable since he applied some artificial boundary conditions for mathematical simplicity. Later Pretsch [9] and Pekeris [10] found different sets of solutions for the case of a disturbance confined to a thin region either near the wall or near the center of the pipe. Taking account of both the limiting cases of Pretsch and Pekeris, Corcos and Sellars [11] reached a wrong conclusion that only a finite number of eigenfunctions exist for the pipe flow problem. Schensted [12] resolved the problem by showing conclusively that an infinite set of eigenfunctions exists in the case of axisymmetric disturbances and that the set is complete. A theoretical analysis of the temporal stability of pipe flow to non-axisymmetric disturbances has been recently carried out by Sadler [13]. All these studies indicate that the Poiseuille flow in a pipe is always temporally stable to infinitesimal, axisymmetric and non-axisymmetric disturbances.

The experimental work of Leite [14] also shows that the pipe flow is stable to axisymmetric disturbances. However, an experiment reported by Lessen et al. [15] indicates instability for the non-axisymmetric mode n = 1. In another experiment, Bhat [16] found a critical Reynolds number of 2140 for azimuthally periodic (or non-axisymmetric) disturbances. Since in these experiments, the disturbance imposed on the flow has a fixed frequency, it is imperative to study the problem theoretically in terms of the spatial stability analysis rather than the temporal analysis. The only efforts in this perspective were made by Gill [17], and Davey and Drazin [18] for axisymmetric disturbances applied to pipe flow. With their theoretical results in fairly good agreement with the experimental work of Leite [14], they concluded that the Poiseuille flow in a pipe is spatially stable to infinitesimal, axisymmetric disturbances. However, Gill's analytic solution was limited mainly to the particular cases of wall modes (r → 1) and central modes (r → 0) because of their simplifying nature. For a general r (0 ≤ r ≤ 1), his asymptotic expansion was too complicated to work with except for a very special case of $\omega R \rightarrow 0$.

Recently a theoretical investigation of the stability of finite, but small, axisymmetric disturbances applied to Poiseuille flow in a rigid pipe was carried out by Crowder [19]. Instead of the usual eigenvalue problem, he considered a specialized form of disturbance claimed to be similar to that generated by an

infinitesimally thin hollow cylinder placed concentrically within the pipe and oscillated axially with a constant frequency and amplitude. His conclusions are, however, questionable due to several reasons. Though Crowder clearly states that the flow is deemed stable if the disturbance decays with downstream axial distance, he refers to his data in such a way as to imply temporal, rather than spatial, instability. In fact, due to the paucity of data in his figures, it is questionable to conclude even temporal instability at Reynolds numbers of 10,000 and 100,000. Moreover from the rapidly growing, oscillatory nature of some of his results, it appears as if his solution had numerical instability due perhaps to a rather coarse grid size employed for the finite-difference solution of the disturbance equations.

The first part of this thesis extends the bounds of axisymmetric spatial stability analysis to a very wide range of frequency $\omega$ and Reynolds number $R$, and to disturbances anywhere in the range $0 \leq r \leq 1$. This significant extension of previous results [17] is accomplished by utilizing a numerical solution of the spatial stability equations -- a counterpart to the temporal analysis of Thomas for plane Poiseuille flow [20].

The rest of the present investigation deals for the first time with the spatial stability analysis of non-axisymmetric disturbances applied to pipe flow. It is believed that such an analysis may help to explain the experimentally observed instability [15, 16] of

Poiseuille flow in a pipe to non-axisymmetric disturbances - an observation heretofore unexplored by theory.

## 1.2 The Thesis Problem

As we have already mentioned, the purpose of this study is to analyze theoretically the behavior of small disturbances to viscous, incompressible Poiseuille flow in a circular pipe. Both axisymmetric and non-axisymmetric disturbances will be considered. The disturbance amplitude will be taken to be infinitesimal, leading to a set of linearized disturbance equations.

The corresponding mathematical problem can be expressed as follows: For time $t < 0$ we suppose that the Poiseuille flow is undisturbed, and that at $t = 0$, a sinusoidal disturbance is imposed at $z = 0$. This disturbance has a frequency $\omega$ and is an arbitrary function of the radius $r$ and angle $\theta$. For example, the $\theta$-dependence of the disturbance velocity vector $\underset{\sim}{v}$ can be expressed as a Fourier series in $\theta$ of the form

$$\underset{\sim}{v}(r,\theta,z,t) = Re[\underset{\sim}{\hat{v}}(r,\theta,z,t)] = Re[\sum_{n=0}^{\infty} \underset{\sim}{v}_n(r,z,t)e^{jn\theta}] \qquad (1.1)$$

For $n = 0$, we have an axisymmetric disturbance; for $n = 1$ the perturbations are constant along a simple helix; for $n = 2$, the perturbations are constant along two intertwined helices, and so on.

For the $(r,z,t)$ dependence, there are two possible ways of representing $\underset{\sim}{v}_n (r,z,t)$. One of them is for the temporal stability analysis and the other for the spatial analysis. We will only be concerned with the latter approach as the former has already been studied in detail. For the present investigation, each $\underset{\sim}{v}_n(r,z,t)$ can be expressed as a Fourier series of the form

$$\underset{\sim}{v}_n(r,z,t) = \sum_{\ell=1}^{\infty} \underset{\sim}{\overset{\sim}{v}}_{n\ell}(r,t)e^{k_\ell z} \tag{1.2}$$

where $k_\ell$ is the complex wave number to be found. Finally, the $t$-dependence of $\underset{\sim}{\overset{\sim}{v}}_{n\ell}(r,t)$ can be expressed as

$$\underset{\sim}{\overset{\sim}{v}}_{n\ell}(r,t) = \underset{\sim}{\overline{v}}_{n\ell}(r)e^{-j\omega t} \tag{1.3}$$

where $\omega$ is the real frequency of the disturbance. Thus the disturbance velocity vector is given by the double Fourier series as

$$\underset{\sim}{v}(r,\theta,z,t) = Re\left[ \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \underset{\sim}{\overline{v}}_{n\ell}(r) \exp(k_\ell z + jn\theta - j\omega t)\right] \tag{1.4}$$

Note that $\underset{\sim}{\hat{v}}(r,\theta,z,t)$, $\underset{\sim}{v}_n(r,z,t)$, $\underset{\sim}{\overset{\sim}{v}}_{n\ell}(r,t)$ and $\underset{\sim}{\overline{v}}_{n\ell}(r)$ are all complex vectors.

The decisive question to be investigated in this study is whether, for a specific Reynolds number, the disturbance of a given frequency $\omega$ grows or decays with the downstream distance

z when n takes on integer values 0,1,2,.... The flow is considered to be unstable when the disturbance grows with z and vice-versa. In terms of the complex wave number k, it implies stability for $k_r < 0$ and instability for $k_r > 0$.

In the case of temporal stability that has already been analyzed [13], the wave number k is considered real while the frequency $\omega$ is the complex eigenvalue sought for. The flow is then said to be temporally stable if $Im(\omega) < 0$. Another important difference between the two cases is that, because of the continuity equation (see equation (2.21)), the vector eigenfunction $\bar{\underset{\sim}{v}}(r)$ is equivalent to two scalar eigenfunctions for the temporal stability rather than three as in the case of spatial stability. The problem of convective or spatial stability is, therefore, much more complicated than that of temporal stability, which is quite likely the reason why it has been neglected in the past in favor of temporal studies.

Chapter 2

THEORETICAL ANALYSIS

## 2.1 Development of Stability Equations

The theory-of the stability of laminar flows decomposes the motion into a mean flow (whose stability constitutes the subject of the investigation) and into a disturbance superimposed on it. Thus in the resultant motion, the velocity vector is

$$\underset{\sim}{V} = \underset{\sim}{V}^* + \underset{\sim}{v}^* \qquad (2.1)$$

and the pressure is

$$P = P_o{}^* + p^* \qquad (2.2)$$

The principal assumptions for the analysis to follow are

i) Incompressible, Newtonian fluid with constant coefficient of viscosity

ii) Laminar flow conditions

iii) Steady mean flow

iv) No body forces

v) Infinitesimal disturbances

vi) Constant diameter pipe

vii) Rigid impermeable pipe wall

viii) No slip at the wall

ix) Boundedness at the pipe center

x) $\underset{\sim}{V}{}^* = V_z^*(r)\ \underset{\sim}{e}_z$, so that the mean flow has only an axial component of velocity and this component is a function of the radius of the pipe.
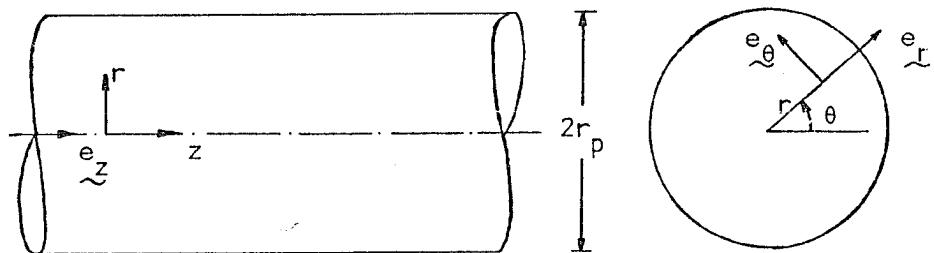


Figure 2.1   Cylindrical Coordinates for the Rigid Pipe

For such a fluid, the equations of motion in cylindrical coordinates are

Continuity

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0 \tag{2.3}$$

and momentum (Navier-Stokes equations)

$$\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right) + \frac{\partial (P/\rho)}{\partial r}$$

$$= \nu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_r}{\partial z^2}\right) \tag{2.4}$$

$$(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z}) + \frac{1}{r} \frac{\partial (P/\rho)}{\partial \theta}$$

$$= \nu (\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2}) \qquad (2.5)$$

$$(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r} \cdot \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z}) + \frac{\partial (P/\rho)}{\partial z}$$

$$= \nu (\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2}) \qquad (2.6)$$

The steady mean flow itself satisfies the equations

$$\frac{\partial (P_o^*/\rho)}{\partial r} = \frac{\partial (P_o^*/\rho)}{\partial \theta} = 0 \ ; \qquad \therefore \ P_o^*(r,\theta,z) = P_o^*(z)$$

$$(2.7)$$

$$\text{and} \quad \frac{\partial (P_o^*/\rho)}{\partial z} = \nu (\frac{\partial^2 V_z^*}{\partial r^2} + \frac{1}{r} \frac{\partial V_z^*}{\partial r})$$

so that the mean flow has the well-known Poiseuille velocity

distribution

$$V_z^*(r) = V_{zo}^* \{1 - (\frac{r}{r_p})^2\} \qquad (2.8)$$

$$V_r^* = V_\theta^* = 0$$

If we now substitute equations (2.1) and (2.2) in equations
(2.3) through (2.6), make use of equations (2.7), and neglect the
product terms of disturbance velocities with themselves and with
their spatial derivatives, we obtain the following set of
linearized hydrodynamic equations for disturbance velocity and
pressure

$$\frac{\partial v_r^*}{\partial r} + \frac{v_r^*}{r} + \frac{1}{r}\frac{\partial v_\theta^*}{\partial \theta} + \frac{\partial v_z^*}{\partial z} = 0 \tag{2.9}$$

$$\frac{\partial v_r^*}{\partial t} + V_z^*\frac{\partial v_r^*}{\partial z} + \frac{\partial(p^*/\rho)}{\partial r}$$

$$= \nu\left(\frac{\partial^2 v_r^*}{\partial r^2} + \frac{1}{r}\frac{\partial v_r^*}{\partial r} - \frac{v_r^*}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r^*}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta^*}{\partial \theta} + \frac{\partial^2 v_r^*}{\partial z^2}\right)$$

$$\tag{2.10}$$

$$\frac{\partial v_\theta^*}{\partial t} + V_z^*\frac{\partial v_\theta^*}{\partial z} + \frac{1}{r}\frac{\partial(p^*/\rho)}{\partial \theta}$$

$$= \nu\left(\frac{\partial^2 v_\theta^*}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta^*}{\partial r} - \frac{v_\theta^*}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta^*}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r^*}{\partial \theta} + \frac{\partial^2 v_\theta^*}{\partial z^2}\right)$$

$$\tag{2.11}$$

$$\frac{\partial v_z{}^*}{\partial t} + v_r{}^* \frac{\partial V_z{}^*}{\partial r} + V_z{}^* \frac{\partial v_z{}^*}{\partial z} + \frac{\partial (p^*/\rho)}{\partial z}$$

$$= \nu \left( \frac{\partial^2 v_z{}^*}{\partial r^2} + \frac{1}{r} \frac{\partial v_z{}^*}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_z{}^*}{\partial \theta^2} + \frac{\partial^2 v_z{}^*}{\partial z^2} \right) \qquad (2.12)$$

Equations (2.9) through (2.12) can be made non-dimensional by choosing as the respective units of length, velocity, and density the radius of the pipe $r_p$, the maximum velocity of the mean flow $V_{zo}{}^*$, and the density of the fluid $\rho$. Thus let

$$v_r = \frac{v_r{}^*}{V_{zo}{}^*} \qquad\qquad r' = \frac{r}{r_p} \qquad\qquad V_z = \frac{V_z{}^*}{V_{zo}{}^*}$$

$$v_\theta = \frac{v_\theta{}^*}{V_{zo}{}^*} \qquad\qquad z' = \frac{z}{r_p} \qquad\qquad p = \frac{p^*}{\rho V_{zo}{}^{*2}}$$

$$v_z = \frac{v_z{}^*}{V_{zo}{}^*} \qquad\qquad t' = \frac{t V_{zo}{}^*}{r_p} \qquad\qquad R = \frac{V_{zo}{}^* r_p}{\nu}$$

Making these substitutions in equations (2.9) through (2.12), dividing equation (2.9) by $V_{zo}{}^*/r_p$ and equations (2.10) through (2.12) by $V_{zo}{}^{*2}/r_p$, and dropping the primes from r,z,t for convenience, we get the following set of linear, partial differential equations for the fluctuation properties in non-dimensional form

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{2.13}$$

$$\frac{\partial v_r}{\partial t} + V_z \frac{\partial v_r}{\partial z} + \frac{\partial p}{\partial r} = \frac{1}{R}\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right.$$

$$\left. + \frac{\partial^2 v_r}{\partial z^2}\right) \tag{2.14}$$

$$\frac{\partial v_\theta}{\partial t} + V_z \frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{1}{R}\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right.$$

$$\left. + \frac{\partial^2 v_\theta}{\partial z^2}\right) \tag{2.15}$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + V_z \frac{\partial v_z}{\partial z} + \frac{\partial p}{\partial z} = \frac{1}{R}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

$$\tag{2.16}$$

Since the equations (2.13) through (2.16) are linear, the various Fourier components forming the arbitrary disturbances according to equation (1.4) are independent of one another. We can, therefore, consider only one Fourier component at a time. Thus, we hope to find solutions of the type

$$v_r(r,\theta,z,t) = Re[\hat{v}_r(r,\theta,z,t)] = Re[\overline{v}_r(r)\,\exp(kz + jn\theta - j\omega t)]$$

$$\tag{2.17}$$

$$v_\theta(r,\theta,z,t) = Re[\hat{v}_\theta(r,\theta,z,t)] = Re[\bar{v}_\theta(r) \exp(kz + jn\theta - j\omega t)]$$

$$(2.18)$$

$$v_z(r,\theta,z,t) = Re[\hat{v}_z(r,\theta,z,t)] = Re[\bar{v}_z(r) \exp(kz + jn\theta - j\omega t)]$$

$$(2.19)$$

$$p(r,\theta,z,t) = Re[\hat{p}(r,\theta,z,t)] = Re[\bar{p}(r) \exp(kz + jn\theta - j\omega t)]$$

$$(2.20)$$

where all variables are dimensionless, and where the superscript (^) indicates a preliminary complex solution which will lead to the real solution $v_r$, $v_\theta$, $v_z$, and p. The quantities $\bar{v}_r(r)$, $\bar{v}_\theta(r)$, $\bar{v}_z(r)$ and $\bar{p}(r)$ are all complex functions of r only. The four functions defined in equations (2.17) through (2.20) must obey the equations (2.13) through (2.16), and after operating with the derivatives in θ, z and t, and eliminating the exponential factors, we get after some rearrangement

$$(D + \frac{1}{r}) \bar{v}_r + \frac{jn}{r} \bar{v}_\theta + k\bar{v}_z = 0 \qquad (2.21)$$

$$[D^2 + \frac{1}{r} D - \{\frac{n^2 + 1}{r^2} - k^2 + R(kV_z - j\omega)\}] \bar{v}_r - j \frac{2n}{r^2} \bar{v}_\theta - RD\bar{p} = 0$$

$$(2.22)$$

$$\left[D^2 + \frac{1}{r}D - \left\{\frac{n^2+1}{r^2} - k^2 + R(kV_z - j\omega)\right\}\right]\bar{v}_\theta + j\frac{2n}{r^2}\bar{v}_r - \frac{jn}{r}R\bar{p} = 0$$

(2.23)

and $$\left[D^2 + \frac{1}{r}D - \left\{\frac{n^2}{r^2} - k^2 + R(kV_z - j\omega)\right\}\right]\bar{v}_z - R\bar{v}_r\,DV_z - Rk\bar{p} = 0$$

(2.24)

where D is the operator $\frac{d}{dr}$ , and, for Poiseuille flow in a pipe, $V_z$ is given by

$$V_z = (1 - r^2)$$

(2.25)

These equations no longer contain $\theta$, z and t as independent variables. For a given real, dimensionless frequency $\omega$, and a Reynolds number R, equations (2.21) through (2.24) must be solved for $\bar{v}_r(r)$, $\bar{v}_\theta(r)$, $\bar{v}_z(r)$ and $\bar{p}(r)$ with k and n as parameters. In fact, the dimensionless, complex wave number k is an eigenvalue and has to be determined first. This is done by applying the boundary conditions at the center and at the rigid wall of the pipe to solutions of the differential equations (equations (2.21) through (2.24)). It may be pointed out that dimensional counterparts of k and $\omega$ are

$$k^* = \frac{k}{r_p} \quad , \qquad \text{and} \qquad \omega^* = \frac{V_{zo}^*\,\omega}{r_p}$$

(2.26)

## 2.2  Boundary Conditions

### 2.2.1  At the Center of the Pipe (r = 0)

The physical restrictions require that no fluid velocity or pressure be unbounded or discontinuous at $r = 0$ (or at any other $r$ for that matter).  Since it is assumed that the velocity components and pressure vary as $\sin n\theta$ (or $\cos n\theta$), it is required that

$$\bar{v}_z(o) = \bar{p}(o) = 0 \quad ; \qquad n \neq 0 \quad , \tag{2.27.1}$$

otherwise $\bar{v}_z$ and $\bar{p}$ would be multi-valued at $r = 0$.  Figure 2.2 shows one of the many possible variations of $\bar{v}_z(r)$ and $\bar{p}(r)$; it is evident that for $n > 0$, $\bar{v}_z(o)$ and $\bar{p}(o)$ must vanish.  For $n = 0$, however, both $\bar{v}_z(o)$ and $\bar{p}(o)$ can have finite values.  Thus for axisymmetric disturbances

$$\bar{v}_z(o) = \text{finite}, \qquad \bar{p}(o) = \text{finite} \quad ; \qquad n = 0 \tag{2.27.2}$$
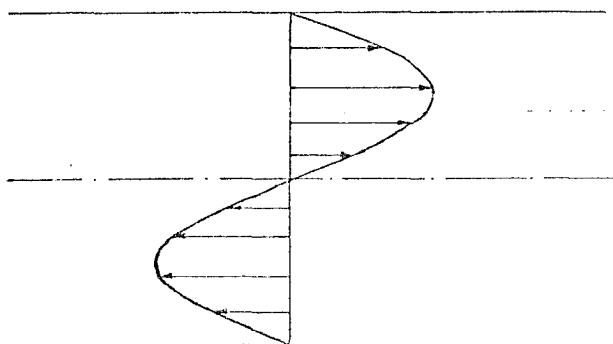


Figure 2.2  A Possible Mode Shape for the Eigenfunction $\bar{v}_z(r)$ or $\bar{p}(r)$; (n > 0)

For the eigenfunctions $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$, we must have [21]

$$\bar{v}_r(o) = \bar{v}_\theta(o) = 0 \quad ; \qquad n \neq 1 \quad , \tag{2.28.1}$$

$$\bar{v}_r(o) + j\bar{v}_\theta(o) = 0 \quad ; \qquad n = 1 \tag{2.28.2}$$

This can be seen from Figure 2.3 where streamlines are shown for various values of n.



$$n = 1 \qquad\qquad\qquad n = 2 \qquad\qquad\qquad n = 3$$
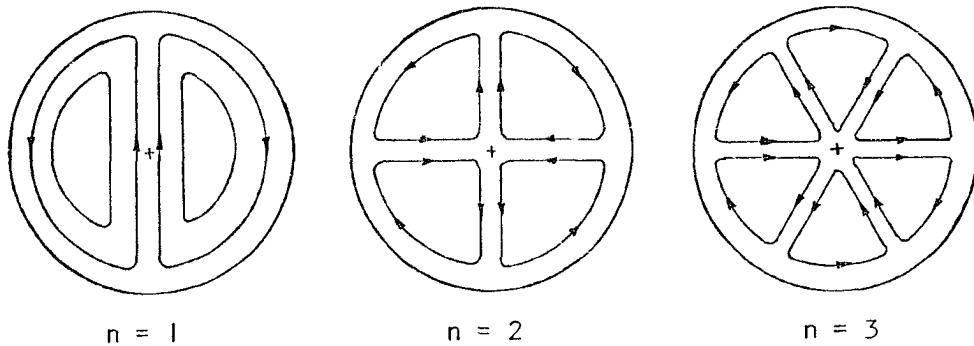
Figure 2.3  Streamlines of Flow for Various n

Thus except for n = 1, the streamlines of flow have to turn through a certain angle at r = 0 requiring, therefore, that $\bar{v}_r(o)$ and $\bar{v}_\theta(o)$ be individually zero.  For n = 1, the streamlines do not undergo such a change in direction at r = 0 and thus $\bar{v}_r(o)$ and $\bar{v}_\theta(o)$ can have finite values but must satisfy the equation (2.28.2) due to the continuity equation (equation (2.21)) and boundary condition

(equation (2.27.1)) for $\bar{v}_z(o)$. When n = 0, we will see that $\bar{v}_\theta(r)$ can be arbitrarily assumed to vanish for all r, thus requiring that $\bar{v}_r(o)$ be zero too.

### 2.2.2 At the Rigid Pipe Wall (r = 1)

The physical restrictions at r = 1 for the rigid, impermeable wall, assuming no slip condition, are

$$\bar{v}_r(1) = \bar{v}_\theta(1) = \bar{v}_z(1) = 0 \tag{2.29}$$

With these boundary conditions, the determination of the dimensionless, complex wave number k for a given dimensionless frequency ω and Reynolds number R with n as a parameter is an eigenvalue problem. Since the equations (2.21) through (2.24) contain second order derivatives in each velocity component but only first order derivative in the fluctuation pressure, the boundary conditions given above are adequate.

Chapter 3

SOLUTION OF THE DISTURBANCE EQUATIONS

Equations (2.21) through (2.24) together with boundary
conditions (equations (2.27) through (2.29)) completely govern the
propagation of an arbitrary, infinitesimal disturbance through an
incompressible, viscous, pipe flow. Their solution will determine
whether such a disturbance grows or decays with the downstream
distance z according as the real part of the complex wave number k
is positive or negative, for a given frequency $\omega$ and Reynolds
number R with n as a parameter. Though the basic technique
developed for solving the differential equation (equations (2.21)
through (2.24)) is the same for both axisymmetric and non-
axisymmetric disturbances, we will consider them separately for
the purpose of clarity.

3.1  Axisymmetric Disturbances (n = 0)

On substituting n = 0 in equations (2.21) through (2.24), we
find that the eigenfunction $\bar{v}_\theta(r)$ decouples itself completely from
others. Within the context of the stability theory, there is,
therefore, no production of $\bar{v}_\theta(r)$ and we can arbitrarily assume
that it vanishes under the effect of viscosity [22, page 229].
Thus for an axisymmetric disturbance, only two components of the
disturbance velocity - the radial and axial components - need be

considered. The set of four equations (equations (2.21) through (2.24)) reduces to one of three equations only, which are

$$(D + \frac{1}{r})\bar{v}_r + k \bar{v}_z = 0 \qquad (3.1)$$

$$[D^2 + \frac{1}{r} D - \{\frac{1}{r^2} - k^2 + R(kV_z - j\omega)\}] \bar{v}_r - R D\bar{p} = 0 \qquad (3.2)$$

$$[D^2 + \frac{1}{r} D - \{- k^2 + R(kV_z - j\omega)\}] \bar{v}_z - R\bar{v}_r DV_z - Rk\bar{p} = 0$$
$$(3.3)$$

The boundary conditions from Chapter 2, Section 2.2 are

$$\bar{v}_r(0) = 0, \ \bar{v}_z(0) = \text{finite}, \ \bar{p}(0) = \text{finite}$$

$$\bar{v}_r(1) = \bar{v}_z(1) = 0 \qquad (3.4)$$

This set of three linear differential equations can be reduced to a single fourth order differential equation in $\bar{v}_r(r)$. It may be pointed out here that the equation reported by Betchov and Criminale [22, page 229, equation (56.6)] is wrong. The correct fourth order differential equation for $\bar{v}_r(r)$ is

$$R(kV_z - j\omega)[D^2 + \frac{1}{r} D + (k^2 - \frac{1}{r^2})] \bar{v}_r - kRr \bar{v}_r D(\frac{DV_z}{r})$$

$$= [D^4 + \frac{2}{r} D^3 + (2k^2 - \frac{3}{r^2})D^2 + \frac{1}{r} (2k^2 + \frac{3}{r^2})D + (k^4 - \frac{2k^2}{r^2} - \frac{3}{r^4})] \bar{v}_r$$
$$(3.5)$$

The corresponding boundary conditions on $\bar{v}_r$ are obtained from equation (3.4) in conjunction with equation (3.1). They are

$$\bar{v}_r(0) = 0; \qquad D\bar{v}_r(0) = \text{finite}$$

$$\bar{v}_r(1) = 0; \qquad D\bar{v}_r(1) = 0$$

(3.6)

Though the number of dependent variables (or eigenfunctions) has been reduced from three to one, this fourth order differential equation is in itself too complicated to be solved analytically in a simple form.

An effort to solve these equations analytically was made by Gill [17] for real frequency $\omega$ and complex wave number $k$. However, his success was largely limited to the two particular cases of $r \to 0$ and $r \to 1$. For the general case of any $r$ away from either 0 or 1, his analytical solution was too complicated to work with except for the simple, and rather academic, case of $\omega R \to 0$. The application of his analytical analysis is, therefore, very much limited in practice. In order to cover a wider range of frequency and Reynolds number over the whole of the flow field $(0 \le r \le 1)$, it seemed desirable to attack the problem by direct numerical solution of the equations.

The boundary conditions in equation (3.4) require that the complex eigenfunctions $\bar{v}_z$ and $\bar{p}$ be finite at $r = 0$. We, therefore, encounter the problem of "starting" if we wish to solve this boundary value problem as an initial value problem. In fact, for that purpose, $D\bar{v}_r(0)$ and $D\bar{v}_z(0)$ are also unknown. For similar reasons, it is not possible to start backwards from $r = 1$. The following procedure was, therefore, developed for the problem at hand.

In short, the eigenfunctions are expanded as a power series in r near the pipe center. By taking an infinite number of terms in the series solution, it is theoretically possible to use the series expansion to r = I. In practice, however, the series solution is carried only up to a small but finite value of r (for example r = 0.I). The solution is then continued by any of the standard step-by-step integration techniques (Runge-Kutta or a predictor-corrector method) to the wall (r = I). The whole procedure is iterated until the boundary conditions at the wall are satisfied, thus resulting in a value for the complex wave number k.

### 3.1.1 Series Expansion

For the power series, let the eigenfunctions $\bar{v}_r(r)$, $\bar{v}_z(r)$ and $\bar{p}(r)$ be of the form

$$\bar{v}_r(r) = r^b(V_1 + V_2 r + V_3 r^2 + \ldots + V_\ell r^{\ell-1} + \ldots) \qquad (3.7)$$

$$\bar{v}_z(r) = r^c(U_1 + U_2 r + U_3 r^2 + \ldots + U_\ell r^{\ell-1} + \ldots) \qquad (3.8)$$

$$\bar{p}(r) = r^d(P_1 + P_2 r + P_3 r^2 + \ldots + P_\ell r^{\ell-1} + \ldots) \qquad (3.9)$$

where $V_1$, $U_1$, $P_1$, $V_2$, $U_2$, $P_2$,... are all complex constants. Though the steady stream velocity $V_z$ is given by equation (2.25) for Poiseuille flow in a pipe, let us develop the analysis for $V_z$ of the form

$$V_z(r) = C_1 + C_2 r + C_3 r^2 + \ldots + C_m r^{m-1} \qquad (3.10)$$

so that

$$DV_z(r) = C_2 + 2C_3 r + \ldots + (m-1)C_m r^{m-2} \qquad (3.11)$$

where $C_1$, $C_2$, ..., $C_m$ are all real constants. Substituting equations (3.7) through (3.11) into equations (3.1), (3.2) and (3.3), we get

From equation (3.1)

$$r^{b-1}[V_1(b+1)] + r^b[V_2(b+1+1)] + \ldots + r^{b+\ell-2}[V_\ell(b+\ell-1+1)] + \ldots$$

$$+ r^c(kU_1) + r^{c+1}(kU_2) + \ldots + r^{c+\ell-1}(kU_\ell) + \ldots = 0$$

$$(3.12)$$

From equation (3.2)

$$r^{b-2}[V_1\{b(b-1)+b-1\}] + r^{b-1}[V_2\{(b+1)b+(b+1)-1\}]$$

$$+ r^b[V_3\{(b+2)(b+1)+(b+2)-1\} + V_1\{k^2 - R(kC_1 - j\omega)\}]$$

$$+ r^{b+1}[V_4\{(b+3)(b+2)+(b+3)-1\} + V_2\{k^2 - R(kC_1 - j\omega)\} - V_1(kRC_2)]$$

$$+ \ldots$$

$$+ r^{b+\ell-3}[V_\ell\{(b+\ell-1)(b+\ell-2)+(b+\ell-1)-1\} +$$

$$V_{\ell-2}\{k^2 - R(kC_1 - j\omega)\} - kR\{\sum_{i=2}^{m} V_{\ell-i-1}C_i\}] + \ldots$$

$$- r^{d-1}\{dRP_1\} - r^d\{(d+1)RP_2\} - \ldots - r^{d+\ell-2}\{(d+\ell-1)RP_\ell\} - \ldots$$

$$= 0 \qquad (3.13)$$

And from equation (3.3)

$$r^{c-2}[U_1\{c(c-1)+c\}] + r^{c-1}[U_2\{(c+1)c+(c+1)\}]$$

$$+ r^c[U_3\{(c + 2)(c + 1) + (c + 2)\} + U_1\{k^2 - R(kC_1 - j\omega)\}]$$

$$+ r^{c+1}[U_4\{(c + 3)(c + 2) + (c + 3)\} + U_2\{k^2 - R(kC_1 - j\omega)\} - U_1(kRC_2)]$$

$$+ \ldots$$

$$+ r^{c+\ell-3}[U_\ell\{(c + \ell - 1)(c + \ell - 2) + (c + \ell - 1)\} +$$

$$U_{\ell-2}\{k^2 - R(kC_1 - j\omega)\} - kR \{\sum_{i=2}^{m} U_{\ell-i-1}C_i\}] + \ldots$$

$$- r^b V_1(RC_2) - r^{b+1}[V_1(2RC_3) + V_2(RC_2)] - \ldots - r^{b+\ell-1}[R \sum_{i=2}^{m} V_{\ell-i+2}C_i(i-1)]$$

$$- r^d(kRP_1) - r^{d+1}(kRP_2) - \ldots - r^{d+\ell-1}(kRP_\ell) - \ldots = 0$$

$$(3.14)$$

It should be pointed out that in equations (3.13) and (3.14) we have adopted the convention that any term inside the summation sign with either a zero or negative subscript is set to zero. To obtain the values of the exponents b, c and d, we note that, in each of equations (3.12) through (3.14), the coefficient of $r^\alpha$ must be zero for all $\alpha$. Then assuming that $V_1$, $U_1$ and $P_1$ are arbitrary non-zero constants (some or all of which may be independent), we get:

A. From equation (3.12),

either b - 1 = c or one of them is smaller than the other.

(i)  If (b - 1) is the smaller of the two, the term $r^{b-1}[V_1(b + 1)]$ is the leading term in the series arranged in ascending powers of r, so that

$$V_1(b + 1) = 0 \qquad \text{or} \quad b = -1$$

(Coefficients of remaining terms are also zero but these are irrelevant for the time being.)

(ii) If c is smaller than (b - 1), we get after reasoning in a way similar to that in (i), that

$$kU_1 = 0 \; ,$$

but this violates the assumption that $U_1$ be non-zero. Note that k is necessarily not zero since that implies a sustained wave of infinite wavelength.

(iii) If these two factors are equal but $V_1$ and $U_1$ are independent,

$$V_1 (b + 1) = 0 \qquad \text{or} \qquad b = -1 \; ,$$

and $kU_1 = 0$, but this is impossible.

(iv) In case the two factors (b - 1) and c are equal but $V_1$ and $U_1$ are dependent, nothing can be concluded regarding b since

$$V_1 (b + 1) + kU_1 = 0$$

Thus (i) through (iv) lead to the conclusion that either b = -1 or $U_1$ and $V_1$ are dependent.

B. From equation (3.13),

either b - 2 = d - 1 or one of them is smaller than the other.

(i)    If (b - 2) is smaller than (d - 1),

$$V_1 \{b(b - 1) + b - 1\} = 0 \quad \text{or} \quad b = \pm 1$$

(ii)　If $(d - 1)$ is smaller than $(b - 2)$,

$$RP_1(d) = 0 \quad \text{or} \quad d = 0$$

(iii) In case $(b - 2) = (d - 1)$, and $V_1$ and $P_1$ are independent, we get the same results as in (i) and (ii); however, if $V_1$ and $P_1$ are dependent, we are at a loss to evaluate b and d.

C.　From equation (3.14),

either $c - 2 = b = d$ or one or two of these factors is the smallest

(i)　　If $(c - 2)$ is the smallest,

$$U_1\{c(c - 1) + c\} = 0 \quad \text{or} \quad c = 0$$

(ii)　If b is the smallest,

$$-RV_1 C_2 = 0 \quad \text{or} \quad C_2 = 0 \text{ for } V_1 \neq 0$$

but $C_2$ is determined by the steady stream velocity $V_z$ (see equation (3.10)).

(iii) And if d is the smallest,

$$-kRP_1 = 0 \quad \text{or} \quad P_1 = 0$$

but this violates the assumption that $P_1 \neq 0$.

(iv)　Remarks similar to those above hold if two or all of these three factors are equal.

Looking over these conclusions, we find that

$$b = \pm 1$$
$$c = d = 0$$

The negative value for b is not possible due to boundedness of the solution at $r = 0$ (see equation (3.7)).

Thus    $b = 1$

and    $c = d = 0$

(3.15)

With this choice, all of the series for the eigenfunctions are bounded at $r = 0$. Also, the boundary conditions at $r = 0$ (equation (3.4)) are satisfied, that is

$$\bar{v}_r(0) = 0$$

$$\bar{v}_z(0) = U_1 \quad \text{(finite)}$$

$$\bar{p}(0) = P_1 \quad \text{(finite)}$$

The recurrence relations for the constants in the series expansions can now be determined. If we substitute equation (3.15) into equations (3.12) through (3.14), arrange the terms in ascending powers of $r$, and equate the coefficient for the $\ell^{th}$ term in each equation to zero, we find the following:

From equation (3.12),

$$r^{\ell-1}: \quad (\ell + 1)V_\ell + kU_\ell = 0 \tag{3.16}$$

From equation (3.13),

$$r^{\ell-2}: \quad (\ell^2 - 1)V_\ell + V_{\ell-2}\{k^2 - R(kC_1 - j\omega)\} - kR\{\sum_{i=2}^{m} V_{\ell-i-1}C_i\}$$

$$- (\ell - 1)RP_\ell = 0$$

(3.17)

And from equation (3.14),

$$r^{\ell-3}: \quad (\ell - 1)^2U_\ell + U_{\ell-2}\{k^2 - R(kC_1 - j\omega)\} - kR\{\sum_{i=2}^{m} U_{\ell-i-1}C_i\} - kRP_{\ell-2}$$

$$- R \left\{ \sum_{i=2}^{m} V_{\ell-i-1} C_i (i - 1) \right\} = 0 \tag{3.18}$$

where again, any term with either a zero or negative subscript is set to zero by convention.

While equations (3.'7) and (3.18) are identically true for $\ell = 1$, equation (3.16) indicates that $V_1$ and $U_1$ are not independent. For $\ell = 2$, equation (3.18) gives $U_2 = 0$. Then equation (3.16) gives $V_2 = 0$ and equation (3.17) requires that $P_2$ be zero. For $\ell = 4$, equations (3.16) through (3.18) give, on using the results for $\ell = 2$,

$$5V_4 + kU_4 = 0$$
$$15V_4 - kRC_2 V_1 - 3RP_4 = 0$$
$$9V_4 - kRC_2 U_1 - RC_2 V_1 = 0$$

If $C_2$ is arbitrarily selected to be zero, these equations give

$$V_4 = U_4 = P_4 = 0 .$$

It should be pointed out that for Poiseuille flow in a pipe, $C_2 = C_4 = C_5 = \ldots = C_m = 0$. Thus, in order to keep the following analysis simple but, at the same time, more general than that for Poiseuille flow alone, we will restrict the series expansion of $V_z$ (equation (3.10)) to the following condition

$$C_i = 0, \qquad\qquad i = 2,4,6,\ldots .$$

Then the non-zero terms inside the summation signs in equations (3.17) and (3.18) pertain only to the odd values of the index $i$.

If we now evaluate equations (3.16) through (3.18) for even values of the index $\ell$, we will find that all V, U, and P's with even subscripts are zero. Thus in the series expansion of the eigenfunctions (equations (3.7) through (3.9)), only odd subscripted V, U, and P's need to be related through the recurrence relations. However, in view of the computer application of this analysis, it will be worthwhile to modify the equations (3.7) through (3.10) as follows:

$$\bar{v}_r(r) = r(V_1 + V_2 r^2 + V_3 r^4 + \ldots + V_\ell r^{2(\ell-1)} + \ldots) \quad (3.19)$$

$$\bar{v}_z(r) = (U_1 + U_2 r^2 + U_3 r^4 + \ldots + U_\ell r^{2(\ell-1)} + \ldots) \quad (3.20)$$

$$\bar{p}(r) = (P_1 + P_2 r^2 + P_3 r^4 + \ldots + P_\ell r^{2(\ell-1)} + \ldots) \quad (3.21)$$

$$\text{and} \quad V_z(r) = [C_1 + C_2 r^2 + C_3 r^4 + \ldots + C_m r^{2(m-1)}] \quad (3.22)$$

Note that in equations (3.19) through (3.22), the subscripts on V, U, P, and C are numbered consecutively but within the parentheses, odd powers of r do not appear in view of the earlier conclusion.

Substitution of these series (equations (3.19) through (3.22)) into equations (3.1) through (3.3) yields equations similar to the equations (3.12) through (3.14). In each case, the $\ell^{th}$ smallest power of r has a coefficient given by the following:

From equation (3.1),

$$r^{2\ell-2}: \quad V_\ell[(2\ell - 1) + 1] + kU_\ell$$

From equation (3.2),

$$r^{2\ell-3}: \quad V_\ell[(2\ell - 1)(2\ell - 2) + (2\ell - 1) - 1]$$

$$+ \sum_{i=1}^{m} B_i V_{\ell-i} - RP_\ell(2\ell - 2)$$

And from equation (3.3),

$$r^{2\ell-4}: \quad U_\ell[(2\ell - 2)(2\ell - 3) + (2\ell - 2)]$$

$$+ \sum_{i=1}^{m} B_i U_{\ell-i} - 2R \sum_{i=2}^{m} (i - 1)C_i V_{\ell-i} - kRP_{\ell-1}$$

where  $B_1 = k^2 - R(kC_1 - j\omega)$

$$B_i = - kRC_i; \quad i = 2,3,\ldots \tag{3.23}$$

and  $\ell$ = a positive integer $(1,2,3,\ldots)$

Since these coefficients should be individually zero for all values

of $\ell$, we get the following recurrence relations for V, U, and P.

$$2\ell V_\ell + kU_\ell = 0 \tag{3.24}$$

$$[(2\ell - 1)^2 - 1]V_\ell + \sum_{i=1}^{m} B_i V_{\ell-i} - 2R(\ell - 1)P_\ell = 0 \tag{3.25}$$

$$4(\ell - 1)^2 U_\ell + \sum_{i=1}^{m} B_i U_{\ell-i} - 2R \sum_{i=2}^{m} (i - 1)C_i V_{\ell-i} - kRP_{\ell-1} = 0 \tag{3.26}$$

For clarity, we again repeat an earlier convention that in these

equations, any term with either a zero or negative subscript is

set to zero.

For $\ell = 1$, equations (3.25) and (3.26) are identically true,

and equation (3.24) shows a relation between $V_1$ and $U_1$. Thus $V_1$

and $U_1$ are dependent so that $P_1$ and only one of $U_1$ and $V_1$ are

independent. This is, in fact, what one may expect since only

two boundary conditions are to be satisfied at the rigid pipe wall. Let us take $U_1$ and $P_1$ as independent. Then from equation (3.24), $V_1$ is given by

$$V_1 = -\frac{kU_1}{2} \tag{3.27}$$

We can now find the higher subscripted V, U, and P's in terms of the independent constants $U_1$ and $P_1$.

Once $U_2$, $U_3$, etc. are known in terms of $U_1$ and $P_1$, calculation of $V_2$, $P_2$, $V_3$, $P_3$, etc. is very simple by use of equations (3.24) and (3.25). To find $U_2$, $U_3$, etc., we eliminate V between equations (3.24) and (3.26) to get

$$4(\ell - 1)^2 U_\ell + \sum_{i=1}^{m} B_i U_{\ell-i} + \frac{kR}{\ell} \sum_{i=2}^{m} (i - 1)C_i U_{\ell-i} - kRP_{\ell-1} = 0$$

On making use of equation (3.23) and after some rearrangement, we get

$$4(\ell - 1)^2 U_\ell = kRP_{\ell-1} - B_1 U_{\ell-1} - \frac{1}{\ell} \{ \sum_{i=2}^{m} B_i U_{\ell-i} (\ell - i + 1) \} \tag{3.28}$$

For convenience, we replace $\ell$ by $(\ell + 1)$ in equations (3.28), (3.24) and (3.25) to get the following recurrence relations for U, V and P

$$U_{\ell+1} = \frac{1}{4\ell^2} [kRP_\ell - B_1 U_\ell - \frac{1}{\ell + 1} \{ \sum_{i=2}^{m} (\ell + 2 - i)B_i U_{\ell+1-i} \}] \tag{3.29}$$

$$V_{\ell+1} = -\frac{1}{2(\ell + 1)} kU_{\ell+1} \tag{3.30}$$

$$P_{\ell+1} = \frac{1}{2R\ell} [\{(2\ell + 1)^2 - 1\} V_{\ell+1} + \sum_{i=1}^{m} B_i V_{\ell+1-i}] \tag{3.31}$$

where $V_1$ is given by equation (3.27), and where $\ell = 1, 2, 3, \ldots$.

With all coefficients in the series solutions (equations (3.19) through (3.21)) expressed in terms of $U_1$ and $P_1$, we can express any eigenfunction as a sum of <u>two</u> terms; for example, the radial velocity eigenfunction $\bar{v}_r(r)$ may be written as

$$\bar{v}_r(r) = v_{r1}(r)U_1 + v_{r2}(r)P_1 \qquad (3.32)$$

In order to understand this concept, $V_1, V_2, \ldots, V_\ell, \ldots$ (see equation (3.19)) must be expressed in terms of $U_1$ and $P_1$. Equation (3.27) gives $V_1$ directly in terms of $U_1$. To find $V_2$, we substitute $\ell = 1$ in equations (3.29) and (3.30) to get

$$U_2 = \frac{1}{4}\left[kRP_1 - B_1U_1 - \frac{1}{2}\sum_{i=2}^{m}(3-i)B_iU_{2-i}\right]$$

$$= \frac{1}{4}[kRP_1 - B_1U_1], \quad \text{(due to the convention for zero or negative subscripts)}$$

and $V_2 = -\frac{1}{4}kU_2$

$$= -\frac{1}{4}k\left[\frac{1}{4}\{kRP_1 - B_1U_1\}\right]$$

Thus, equation (3.19) gives

$$\bar{v}_r(r) = r\left[-\frac{k}{2}U_1 - \frac{1}{16}k(kRP_1 - B_1U_1)r^2 + \ldots\right]$$

$$= U_1\left[-\frac{1}{2}kr + \frac{1}{16}kB_1r^3 + \ldots\right] + P_1\left[-\frac{1}{16}k^2Rr^3 + \ldots\right]$$

And, in view of equation (3.32) above, we get

$$v_{r1}(r) = -\frac{1}{2}kr + \frac{1}{16}kB_1r^3 + \ldots$$

and $v_{r2}(r) = -\frac{1}{16}k^2Rr^3 + \ldots$

A similar expansion holds for the other eigenfunctions $\bar{v}_z(r)$ and $\bar{p}(r)$.

### 3.1.2 Step-by-Step Integration

As mentioned earlier, the series expansions for the eigenfunctions are carried only up to a small value of the radius r since, in actual practice, it is possible to sum only a _finite_ number of terms in a series. From this small value of r (e.g., r = 0.1), the solution is further continued by a step-by-step integration technique to the wall (r = 1).

It should be pointed out, however, that equations (3.1) through (3.3) are not directly useful for the step-by-step integration techniques due to the fact that equation (3.2) contains both the highest degree derivatives in the eigenfunctions $\bar{v}_r(r)$ and $\bar{p}(r)$. To circumvent this difficulty, we differentiate the continuity equation (equation (3.1)) with respect to r and solve for $D^2\bar{v}_r$. Then equation (3.2) can be solved for the unknown $D\bar{p}$ using the relation just obtained for $D^2\bar{v}_r$, and equation (3.3) directly gives $D^2\bar{v}_z$. Thus, the set of differential equations useful for step-by-step integration techniques is

$$D^2\bar{v}_r = \frac{1}{r^2}\bar{v}_r - \frac{1}{r}D\bar{v}_r - kD\bar{v}_z$$

$$D^2\bar{v}_z = R(k\bar{p} + \bar{v}_r DV_z) - \frac{1}{r}D\bar{v}_z + \{-k^2 + R(kV_z - j\omega)\}\bar{v}_z \quad (3.33)$$

and $\quad D\bar{p} = -\frac{1}{R}[kD\bar{v}_z + \{-k^2 + R(kV_z - j\omega)\}\bar{v}_r]$

The integration of these differential equations is carried out in terms of the <u>two</u> independent solutions of the eigenfunctions, that is, in terms of $v_{r1}(r)$ and $v_{r2}(r)$ for the eigenfunction $\bar{v}_r(r)$, and similarly for $\bar{v}_z(r)$ and $\bar{p}(r)$. Since the set of solutions $v_{r1}(r)$, $v_{z1}(r)$ and $p_1(r)$ is independent of the other set $v_{r2}(r)$, $v_{z2}(r)$ and $p_2(r)$, each set must satisfy the equations (3.33) independently. Thus the set of equations (3.33) is equivalent to <u>two</u> sets – one with a subscript 1 on $\bar{v}_r(r)$, $\bar{v}_z(r)$ and $\bar{p}(r)$ and the other with a subscript 2. Starting from the small value of r up to which the series solution is used, each of the two sets of equations (3.33) is solved independently of the other by a step-by-step integration technique to the pipe wall. This concept is diagrammatically represented in Figure 3.1 for the real parts of the independent solutions $v_{r1}(r)$ and $v_{r2}(r)$. It should be pointed out that $v_{r1}(r)$, $v_{r2}(r)$, etc. are, in general, complex functions. In Figure 3.1, the solid portion of the curve is obtained by series solution and the dashed portion by a step-by-step integration technique.

At this stage, it is pertinent to note that the boundary conditions at the center of the pipe (equations (3.4)) have already been taken care of by the series solution developed earlier. However, the boundary conditions at the rigid pipe wall are yet to be satisfied. It is <u>this</u> requirement that precludes all but a certain definite combination of the <u>two</u> independent solutions for every eigenfunction.
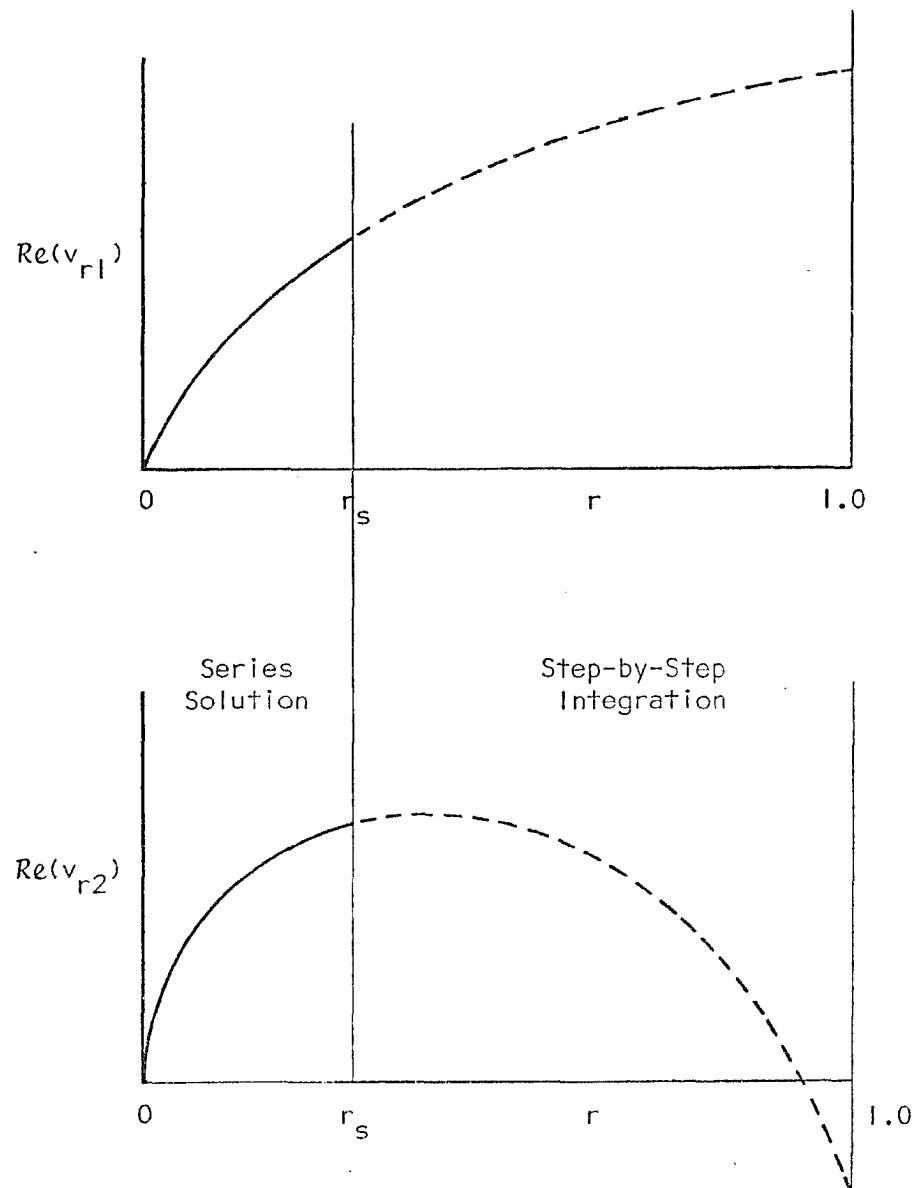
Figure 3.1  Diagrammatic Representation of $v_{r1}(r)$ and $v_{r2}(r)$

The boundary conditions at the rigid tube wall require that the disturbance velocity components be zero at $r = 1$, so that in view of equation (3.32), we have

$$
\begin{bmatrix} v_{r1}(1) & v_{r2}(1) \\ v_{z1}(1) & v_{z2}(1) \end{bmatrix} \begin{bmatrix} U_1 \\ P_1 \end{bmatrix} = \underset{\sim}{0}
\tag{3.34}
$$

If det stands for the determinant of the coefficient matrix, that is, if

$$
\det \equiv \begin{vmatrix} v_{r1}(1) & v_{r2}(1) \\ v_{z1}(1) & v_{z2}(1) \end{vmatrix} ,
\tag{3.35}
$$

it is required that det be zero for a non-trivial solution of the set of homogeneous equations (3.34). Since the determinant det is, in general, complex, we need to assure that the absolute value of det be zero, that is,

$$
|\det| = 0
\tag{3.36}
$$

Further, since the elements of the determinant in equation (3.35) are functions of the complex wave number $k$, frequency $\omega$, and Reynolds number $R$, only certain combinations of these parameters will allow equation (3.36) to be satisfied, thus yielding the eigenvalue $k$. Once an eigenvalue is found, the eigenfunctions $\bar{v}_r(r)$, $\bar{v}_z(r)$ and $\bar{p}(r)$ are easily calculated in terms of only one arbitrary constant, say $U_1$, since $P_1$ and $U_1$ are now related by equation (3.34) as follows

$$\frac{P_1}{U_1} = -\frac{v_{r1}(1)}{v_{r2}(1)} = -\frac{v_{z1}(1)}{v_{z2}(1)}$$

For convenience, we set $U_1 = (1.0 + j0.0)$, and find all the eigen-functions from equations of the form given in equation (3.32).

Before discussing the case of non-axisymmetric disturbances, let us elaborate on the reasons for analyzing it differently from the behavior of axisymmetric disturbances. As noted earlier, the circumferential component of disturbance velocity $\overline{v}_\theta(r)$ can be arbitrarily assumed to vanish for an axisymmetric disturbance, thus providing an obvious simplicity which is lacking in the case of a non-axisymmetric disturbance. A more important consequence of this simplicity is that for an axisymmetric disturbance, each eigenfunction can be represented as a combination of two linearly independent solutions which are linked by the boundary conditions at the rigid pipe wall. For a non-axisymmetric disturbance, it is natural to expect three such linearly independent solutions for each eigenfunction since all of the three disturbance velocity components must vanish at the rigid pipe wall. In contrast to a 2 x 2 determinant for the axisymmetric disturbance (equation (3.35)), we will, therefore, have a 3 x 3 determinant for the non-axisymmetric disturbance.

## 3.2 Non-axisymmetric Disturbances (n > 0)

The governing differential equations and boundary conditions for this case are given in equations (2.21) through (2.24), (2.27.1), (2.28) and (2.29). Since all of the eigenfunctions and their derivatives are not known at any one of the two end points (r = 0 or r = 1), a direct step-by-step integration of the differential equations is not possible. We, therefore, follow a procedure similar to the one developed above for axisymmetric disturbances. However, before we consider the power series expansion of the eigenfunctions, let us define the following combinations of $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$

$$\bar{f}(r) = \bar{v}_r(r) + j\,\bar{v}_\theta(r)$$
$$\text{and } \bar{g}(r) = \bar{v}_r(r) - j\,\bar{v}_\theta(r) \tag{3.37}$$

In view of the boundary condition for n = 1 (equation (2.28.2)), and differential equations (2.21) through (2.23), we will find that it is simpler to work with $\bar{f}(r)$ and $\bar{g}(r)$ instead of $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$. Once $\bar{f}(r)$ and $\bar{g}(r)$ are known, it is simple to calculate $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$ by the following relations

$$\bar{v}_r(r) = \frac{1}{2}\left[\bar{f}(r) + \bar{g}(r)\right]$$
$$\bar{v}_\theta(r) = \frac{1}{2j}\left[\bar{f}(r) - \bar{g}(r)\right] \tag{3.38}$$

To transform equations (2.21) through (2.24) in terms of $\bar{f}$, $\bar{g}$, $\bar{v}_z$ and $\bar{p}$, multiply equation (2.21) by 2, add and subtract j times equation (2.23) from equation (2.22), substitute equations

(3.37) and (3.38) wherever necessary, and rearrange a little to get the following

$$(D + \frac{n + 1}{r})\bar{f} + (D + \frac{1 - n}{r})\bar{g} + 2k\,\bar{v}_z = 0 \tag{3.39}$$

$$[D^2 + \frac{1}{r}D - \{\frac{(n + 1)^2}{r^2} - k^2 + R(kV_z - j\omega)\}]\bar{f} - R(D - \frac{n}{r})\bar{p} = 0 \tag{3.40}$$

$$[D^2 + \frac{1}{r}D - \{\frac{(n - 1)^2}{r^2} - k^2 + R(kV_z - j\omega)\}]\bar{g} - R(D + \frac{n}{r})\bar{p} = 0 \tag{3.41}$$

$$[D^2 + \frac{1}{r}D - \{\frac{n^2}{r^2} - k^2 + R(kV_z - j\omega)\}]\bar{v}_z - \frac{1}{2}R(\bar{f} + \bar{g})DV_z - Rk\bar{p} = 0 \tag{3.42}$$

The boundary conditions on $\bar{f}$, $\bar{g}$, $\bar{v}_z$ and $\bar{p}$ are (from Chapter 2, Section 2.2)

$$\bar{f}(0) = \bar{v}_r(0) + j\,\bar{v}_\theta(0) = 0\,; \quad n = 1, 2, 3, \ldots$$

$$\bar{g}(0) = \begin{Bmatrix} 0; & n = 2, 3, 4, \ldots \\ \text{finite}; & n = 1 \end{Bmatrix} \tag{3.43}$$

$$\bar{v}_z(0) = \bar{p}(0) = 0; \quad n = 1, 2, 3, \ldots$$

and $\bar{f}(1) = \bar{g}(1) = \bar{v}_z(1) = 0; \quad n = 1, 2, 3, \ldots$

### 3.2.1 Series Expansion

Let the power series expansion of $\bar{f}(r)$, $\bar{g}(r)$, $\bar{v}_z(r)$ and $\bar{p}(r)$ be of the form

$$\bar{f}(r) = r^a(F_1 + F_2 r + F_3 r^2 + \ldots + F_\ell r^{\ell - 1} + \ldots)$$

$$\bar{g}(r) = r^b(G_1 + G_2 r + G_3 r^2 + \ldots + G_\ell r^{\ell - 1} + \ldots)$$

$$\bar{v}_z(r) = r^c(U_1 + U_2 r + U_3 r^2 + \ldots + U_\ell r^{\ell-1} + \ldots) \qquad (3.44)$$

$$\bar{p}(r) = r^d(P_1 + P_2 r + P_3 r^2 + \ldots + P_\ell r^{\ell-1} + \ldots)$$

where all F, G, U and P's are complex constants.

In view of the analysis for axisymmetric disturbances, let the steady stream velocity $V_z$ be given by equation (3.22), where the constants $C_1, C_2, \ldots, C_m$ are all real.

Substituting equations (3.44) and (3.22) into equations (3.39) through (3.42), and using equation (3.23), we get from equation (3.39)

$$r^{a-1}[F_1(a + n + 1)] + r^a[F_2(a + 1 + n + 1)] + \ldots$$
$$+ r^{a+\ell-2}[F_\ell(a + \ell - 1 + n + 1)] + \ldots + r^{b-1}[G_1(b + 1 - n)]$$
$$+ r^b[G_2(b + 1 + 1 - n)] + \ldots + r^{b+\ell-2}[G_\ell(b + \ell - 1 + 1 - n)] + \ldots$$
$$+ r^c(2kU_1) + r^{c+1}(2kU_2) + \ldots + r^{c+\ell-1}(2kU_\ell) + \ldots = 0$$
$$(3.45)$$

From equation (3.40)

$$r^{a-2}[F_1\{a(a - 1) + a - (n + 1)^2\}] + r^{a-1}[F_2\{(a + 1)a + (a + 1) - (n + 1)^2\}]$$
$$+ r^a[F_3\{(a + 2)(a + 1) + (a + 2) - (n + 1)^2\} + F_1 B_1]$$
$$+ r^{a+1}[F_4\{(a + 3)(a + 2) + (a + 3) - (n + 1)^2\} + F_2 B_1]$$
$$+ r^{a+2}[F_5\{(a + 4)(a + 3) + (a + 4) - (n + 1)^2\} + F_3 B_1 + F_1 B_2]$$
$$+ \ldots$$
$$+ r^{a+\ell-3}[F_\ell\{(a + \ell - 1)(a + \ell - 2) + (a + \ell - 1) - (n + 1)^2\} + \sum_{i=1}^{m} F_{\ell-2i} B_i]$$
$$+ \ldots$$
$$- r^{d-1}[R(d - n)P_1] - r^d[R(d + 1 - n)P_2] - \ldots$$
$$- r^{d+\ell-2}[R(d + \ell - 1 - n)P_\ell] - \ldots = 0 \qquad (3.46)$$

From equation (3.41)

$$r^{b-2}[G_1\{b(b-1) + b - (n-1)^2\}] + r^{b-1}[G_2\{(b+1)b + (b+1) - (n-1)^2\}]$$

$$+ r^b[G_3\{(b+2)(b+1) + (b+2) - (n-1)^2\} + G_1B_1]$$

$$+ r^{b+1}[G_4\{(b+3)(b+2) + (b+3) - (n-1)^2\} + G_2B_1]$$

$$+ r^{b+2}[G_5\{(b+4)(b+3) + (b+4) - (n-1)^2\} + G_3B_1 + G_1B_2]$$

$$+ \ldots$$

$$+ r^{b+\ell-3}[G_\ell\{(b+\ell-1)(b+\ell-2) + (b+\ell-1) - (n-1)^2\} + \sum_{i=1}^{m} G_{\ell-2i}B_i]$$

$$+ \ldots$$

$$- r^{d-1}[R(d+n)P_1] - \ldots - r^{d+\ell-2}[R(d+\ell-1+n)P_\ell] - \ldots = 0$$

$$(3.47)$$

And from equation (3.42)

$$r^{c-2}[U_1\{c(c-1) + c - n^2\}] + r^{c-1}[U_2\{(c+1)c + (c+1) - n^2\}]$$

$$+ r^c[U_3\{(c+2)(c+1) + (c+2) - n^2\} + U_1B_1]$$

$$+ \ldots$$

$$+ r^{c+\ell-3}[U_\ell\{(c+\ell-1)(c+\ell-2) + (c+\ell-1) - n^2\} + \sum_{i=1}^{m} U_{\ell-2i}B_i]$$

$$+ \ldots$$

$$- r^{a+1}[F_1(RC_2)] - r^{a+2}[F_2(RC_2)] - r^{a+3}[F_3(RC_2) + F_1(2RC_3)] -$$

$$\ldots - r^{a+\ell}[R\{\sum_{i=2}^{m} (i-1)F_{\ell-2i+4}C_i\}] - \ldots$$

$$- r^{b+1}[G_1(RC_2)] - r^{b+2}[G_2(RC_2)] - r^{b+3}[G_3(RC_2) + G_1(2RC_3)] -$$

$$\ldots - r^{b+\ell}[R\{\sum_{i=2}^{m} (i-1)G_{\ell-2i+4}C_i\}] - \ldots$$

$$- r^d(RkP_1) - r^{d+1}(RkP_2) - \ldots - r^{d+\ell-1}(RkP_\ell) - \ldots = 0$$

$$(3.48)$$

where, again as for axisymmetric disturbances, any term with either a zero or negative subscript in equations (3.46) through (3.48) is set to zero.

To find relations for a, b, c and d, the coefficient of $r^\alpha$ is set to zero for all $\alpha$ in equations (3.45) through (3.48). Following a reasoning similar to that for axisymmetric disturbances (Section 3.1.1), and assuming that $F_1$, $G_1$, $U_1$ and $P_1$ are non-zero constants, we get

A. From equation (3.45),

either $(a - 1) = (b - 1) = c$ or one or two of these factors is the smallest.

(i) If $(a - 1)$ is the smallest,

$$F_1(a + n + 1) = 0 \quad \text{or} \quad a = -(n + 1)$$

(ii) If $(b - 1)$ is the smallest,

$$G_1(b + 1 - n) = 0 \quad \text{or} \quad b = (n - 1)$$

(iii) If c is the smallest,

$$2kU_1 = 0 \quad \text{or} \quad U_1 = 0 \ (\because k \neq 0),$$

but this violates the assumption that $U_1$ be non-zero.

(iv) If all or any two of these three factors are equal but $F_1$, $G_1$ and $U_1$ are independent, we get precisely the same conclusions as above; in case the corresponding $F_1$, $G_1$ and $U_1$ are dependent, no useful relation can be found for a and b.

B.  From equation (3.46),

either $(a - 2) = (d - 1)$ or one of them is smaller than the
other.

(i)    If $(a - 2)$ is the smaller of the two,

$F_1[a^2 - (n + 1)^2] = 0$   or   $a = \pm (n + 1)$

(ii)   If $(d - 1)$ is the smaller of the two,

$R(d - n)P_1 = 0$          or   $d = n$

(iii)  If these two factors are equal but $F_1$ and $P_1$ are inde-

pendent, we get the same results as in (i) and (ii);

however, if $F_1$ and $P_1$ are dependent, a and d cannot be

determined.

C.  From equation (3.47),

either $(b - 2) = (d - 1)$ or one of them is smaller than the
other.

(i)    If $(b - 2)$ is smaller than $(d - 1)$,

$G_1[b^2 - (n - 1)^2] = 0$   or   $b = \pm (n - 1)$

(ii)   If $(d - 1)$ is smaller than $(b - 2)$,

$R(d + n)P_1 = 0$          or   $d = -n$

(iii)  Remarks similar to those above for equation (3.46)

hold if both of $(b - 2)$ and $(d - 1)$ are equal.

D.  From equation (3.48),

either one or more of the four factors $(c - 2)$, $(a + 1)$,
$(b + 1)$ and d is the smallest exponent of r.

(i)    If $(c - 2)$ is the smallest,

$$U_1(c^2 - n^2) = 0 \qquad \text{or} \qquad c = \pm n$$

(ii)  If $(a + 1)$ is the smallest,

$$F_1(RC_2) = 0 \qquad \text{or} \qquad F_1 = 0 \text{ for } C_2 \neq 0$$

(iii) If $(b + 1)$ is the smallest,

$$G_1(RC_2) = 0 \qquad \text{or} \qquad G_1 = 0 \text{ for } C_2 \neq 0$$

(iv)  If $d$ is the smallest,

$$RkP_1 = 0 \qquad \text{or} \qquad P_1 = 0$$

The last three conclusions violate the assumption that $F_1$, $G_1$ and $P_1$ be non-zero.

(v)   If two or more of these factors are equal and if $F_1$, $G_1$, $U_1$ and $P_1$ are all independent, we get the same conclusions as in (i) through (iv); in case $U_1$ and one or more of $F_1$, $G_1$ and $P_1$ are dependent, nothing can be concluded about the value of $c$.

Looking over these conclusions, it is found that

$$a = \pm (n + 1)$$
$$b = \pm (n - 1)$$
$$c = \pm n$$
$$d = \pm n$$

Since $n$ is a positive integer, the negative values for $a$, $c$ and $d$ are not possible due to boundedness of the solution at $r = 0$. Also, if $b = -(n - 1)$, $\bar{g}(r)$ will be unbounded at $r = 0$ for $n > 1$. Thus, only the following values for $a$, $b$, $c$ and $d$ are retained

$$a = n + 1$$

$$b = n - 1 \qquad\qquad (3.49)$$

$$c = d = n$$

With this choice, the boundary conditions at $r = 0$ (equations (3.43)) are also satisfied, for example,

$$\bar{f}(0) = 0; \quad n = 1,2,3,\ldots$$

$$\bar{g}(0) = \left\{ \begin{array}{l} 0; \quad n = 2,3,4,\ldots \\ G_1 \text{(finite)}; \quad n = 1 \end{array} \right\}$$

and $\bar{v}_z(0) = \bar{p}(0) = 0; \quad n = 1,2,3,\ldots$

The recurrence relations for the constants in the series expansions can now be determined. To do so, substitute equation (3.49) into equations (3.45) through (3.48), arrange the terms in ascending powers of $r$, and equate the coefficient for the $\ell^{th}$ term in each equation to zero to get:

From equation (3.45),

$$r^{n+\ell-3}: \quad G_\ell(\ell - 1) + F_{\ell-2}(2n + \ell - 1) + 2kU_{\ell-2} = 0 \qquad (3.50)$$

From equation (3.46),

$$r^{n+\ell-2}: \quad F_\ell[(n+\ell)^2 - (n + 1)^2] + \sum_{i=1}^{m} F_{\ell-2i}B_i - R(\ell - 1)P_\ell = 0 \qquad (3.51)$$

From equation (3.47),

$$r^{n+\ell-4}: \quad G_\ell[(n + \ell - 2)^2 - (n - 1)^2] + \sum_{i=1}^{m} G_{\ell-2i}B_i - R(2n + \ell -3)P_{\ell-2} = 0 \qquad (3.52)$$

And from equation (3.48),

$$r^{n+\ell-3}: \quad U_\ell[(n + \ell - 1)^2 - n^2] + \sum_{i=1}^{m} U_{\ell-2i}B_i - R\{\sum_{i=2}^{m} C_i(i - 1)(F_{\ell-2i}$$

$$+ G_{\ell-2i+2})\} - kRP_{\ell-2} = 0 \tag{3.53}$$

where again, any term with either a zero or negative subscript is set to zero by convention.

Equations (3.50) through (3.53) are identically true for $\ell = 1$. For $\ell = 2$, equation (3.53) and any one of equations (3.50) and (3.52) give

$$U_2 = G_2 = 0 \tag{3.54}$$

Then from equations (3.50), (3.52) and (3.53) for $\ell = 4$, and from equation (3.51) for $\ell = 2$, we get on using equation (3.54)

$$3G_4 + (2n + 3)F_2 = 0$$

$$3G_4 - RP_2 = 0$$

$$3(2n + 3)U_4 - kRP_2 = 0$$

$$(2n + 3)F_2 - RP_2 = 0$$

or in matrix form,

$$\begin{bmatrix} 3 & (2n + 3) & 0 & 0 \\ 3 & 0 & -R & 0 \\ 0 & 0 & -kR & 3(2n + 3) \\ 0 & (2n + 3) & -R & 0 \end{bmatrix} \begin{bmatrix} G_4 \\ F_2 \\ P_2 \\ U_4 \end{bmatrix} = \underset{\sim}{0} \tag{3.55}$$

The determinant of the coefficient matrix $= -18R(2n + 3)^2$ which is not zero. Since the set of equations (3.55) is homogeneous, we conclude that

$$F_2 = P_2 = G_4 = U_4 = 0$$

In fact, if this process is continued, we find that <u>all</u> F, G, U and P's with <u>even</u> subscripts are zero since they satisfy the following set of homogeneous equations

$$
\begin{bmatrix}
(2\ell-1) & (2n+2\ell-1) & 0 & 0 \\
(2\ell-1) & 0 & -R & 0 \\
0 & 0 & -kR & (2\ell-1)(2n+2\ell-1) \\
0 & (2n+2\ell-1) & -R & 0
\end{bmatrix}
\begin{bmatrix}
G_{2\ell} \\
F_{2(\ell-1)} \\
P_{2(\ell-1)} \\
U_{2\ell}
\end{bmatrix}
= \underset{\sim}{0}
$$

where $\ell$ is a positive integer, and the determinant of the coefficient matrix $= -2R(2\ell - 1)^2(2n + 2\ell -1)^2$ which is <u>not</u> zero.

This conclusion is similar to the one obtained earlier for an axisymmetric disturbance, and in view of that analysis, we modify the subscript notation in equations (3.44) as follows:

$$\bar{f}(r) = r^{n+1}(F_1 + F_2 r^2 + F_3 r^4 + \ldots + F_\ell r^{2(\ell-1)} + \ldots)$$

$$\bar{g}(r) = r^{n-1}(G_1 + G_2 r^2 + G_3 r^4 + \ldots + G_\ell r^{2(\ell-1)} + \ldots)$$

$$\bar{v}_z(r) = r^n(U_1 + U_2 r^2 + U_3 r^4 + \ldots + U_\ell r^{2(\ell-1)} + \ldots)$$

$$\bar{p}(r) = r^n(P_1 + P_2 r^2 + P_3 r^4 + \ldots + P_\ell r^{2(\ell-1)} + \ldots)$$

(3.56)

Substitution of the series expansions (equations (3.56) and (3.22)) into equations (3.39) through (3.42) yields equations which are similar to equations (3.45) through (3.48). In each case, the $\ell^{\text{th}}$ smallest exponent of r has a coefficient given by the following:

From equation (3.39),

$$r^{n+2\ell-4}: \quad [G_\ell(n + 2\ell - 3 + 1 - n) + F_{\ell-1}(n + 2\ell - 3 + n + 1) + U_{\ell-1}(2k)]$$

From equation (3.40),

$$r^{n+2\ell-3}: \quad [F_\ell\{(n + 2\ell - 1)(n + 2\ell - 2) + (n + 2\ell - 1) - (n + 1)^2\}$$
$$+ \{\sum_{i=1}^{m} F_{\ell-i}B_i\} - R(n + 2\ell - 2 - n)P_\ell]$$

From equation (3.41),

$$r^{n+2\ell-5}: \quad [G_\ell\{(n + 2\ell - 3)(n + 2\ell - 4) + (n + 2\ell - 3) - (n - 1)^2\}$$
$$+ \{\sum_{i=1}^{m} G_{\ell-i}B_i\} - R(n + 2\ell - 4 + n)P_{\ell-1}]$$

And from equation (3.42),

$$r^{n+2\ell-4}: \quad [U_\ell\{(n + 2\ell - 2)(n + 2\ell - 3) + (n + 2\ell - 2) - n^2\}$$
$$+ \{\sum_{i=1}^{m} U_{\ell-i}B_i\} - kRP_{\ell-1} - R\{\sum_{i=2}^{m} (i - 1)C_i(F_{\ell-i} + G_{\ell+1-i})\}]$$

where $B_i$, $i = 1, 2, \ldots, m$ are given by equation (3.23).

Since these coefficients should be individually zero for all values of $\ell$, we get the following recurrence relations for F, G, U and P.

$$(\ell - 1)G_\ell + (n + \ell - 1)F_{\ell-1} + kU_{\ell-1} = 0 \qquad (3.57)$$

$$4(\ell - 1)(n + \ell)F_\ell + \{\sum_{i=1}^{m} F_{\ell-i}B_i\} - 2R(\ell - 1)P_\ell = 0 \qquad (3.58)$$

$$4(\ell - 1)(n + \ell - 2)G_\ell + \{\sum_{i=1}^{m} G_{\ell-i}B_i\} - 2R(n + \ell - 2)P_{\ell-1} = 0 \qquad (3.59)$$

$$4(\ell - 1)(n + \ell - 1)U_\ell + \{ \sum_{i=1}^{m} U_{\ell-i} B_i \} - R\{ \sum_{i=2}^{m} (i - 1)C_i (F_{\ell-i} + G_{\ell+1-i}) \}$$

$$- kRP_{\ell-1} = 0 \tag{3.60}$$

where any term having a factor with either a zero or negative

subscript is set to zero.

For $\ell = 1$, equations (3.57) through (3.60) are identically

true. For $\ell = 2$, equations (3.57) and (3.59) give

$$G_2 + (n + 1)F_1 + kU_1 = 0$$

$$4nG_2 + B_1 G_1 - 2nRP_1 = 0$$

Eliminating $G_2$ between these equations, we get

$$(n + 1)F_1 + kU_1 - \frac{B_1}{4n} G_1 + \frac{R}{2} P_1 = 0 \tag{3.61}$$

Equation (3.61) clearly shows that the first four constants

in the series expansions are not all independent. Since there is

only one relation between them (equation (3.61)), only three of

them are independent as was conjectured earlier in this chapter

at the end of Section 3.1.2. Let $G_1$, $U_1$ and $P_1$ be independent for

further analysis. Thus $F_1$ is given by

$$F_1 = \frac{1}{n + 1} (\frac{B_1}{4n} G_1 - \frac{R}{2} P_1 - kU_1) \tag{3.62}$$

For the determination of higher subscripted F, G, U and P's, we

note that equations (3.59) and (3.60) give $G_\ell$ and $U_\ell$ directly for

$\ell = 2,3,4,\ldots.$ Calculation of $F_\ell$ and $P_\ell$ ($\ell \geq 2$), however, is a

little involved since equation (3.58) contains both of them. To

circumvent this difficulty, we eliminate $F_\ell$ between equation (3.58) and that obtained from equation (3.57) after replacing $(\ell - 1)$ by $\ell$, to get

$$-4\ell \, G_{\ell+1} - 4kU_\ell + \frac{1}{(\ell - 1)} \sum_{i=1}^{m} F_{\ell-i} B_i - 2RP_\ell = 0 \qquad (3.63)$$

Also replacing $(\ell - 1)$ by $\ell$ in equation (3.59), we get after simplification

$$4\ell \, G_{\ell+1} + \frac{1}{(n + \ell - 1)} \sum_{i=1}^{m} G_{\ell+1-i} B_i - 2RP_\ell = 0 \qquad (3.64)$$

Adding equations (3.63) and (3.64), we get

$$- 4kU_\ell + \frac{1}{(\ell - 1)} \sum_{i=1}^{m} F_{\ell-i} B_i + \frac{1}{(n + \ell - 1)} \sum_{i=1}^{m} G_{\ell+1-i} B_i = 4RP_\ell \qquad (3.65)$$

For convenience, we replace $\ell$ by $(\ell + 1)$ in equations (3.59), (3.60), (3.65) and (3.58) to get the following recurrence relations for U, G, P and F

$$G_{\ell+1} = \frac{1}{2\ell} \left[ RP_\ell - \frac{1}{2(n + \ell - 1)} \sum_{i=1}^{m} G_{\ell+1-i} B_i \right] \qquad (3.66)$$

$$U_{\ell+1} = \frac{1}{4\ell(n + \ell)} \left[ kRP_\ell - \sum_{i=1}^{m} U_{\ell+1-i} B_i \right.$$
$$\left. + R \sum_{i=2}^{m} (i - 1)C_i (F_{\ell+1-i} + G_{\ell+2-i}) \right] \qquad (3.67)$$

$$P_{\ell+1} = \frac{1}{R} \left[ - kU_{\ell+1} + \frac{1}{4} \left\{ \frac{1}{\ell} \sum_{i=1}^{m} F_{\ell+1-i} B_i \right. \right.$$
$$\left. \left. + \frac{1}{(n + \ell)} \sum_{i=1}^{m} G_{\ell+2-i} B_i \right\} \right] \qquad (3.68)$$

and $F_{\ell+1} = \dfrac{1}{2(n + \ell + 1)} [RP_{\ell+1} - \dfrac{1}{2\ell} \sum\limits_{i=1}^{m} F_{\ell+1-i} B_i]$  (3.69)

with $F_1$ given by equation (3.62).

In that order, equations (3.66) through (3.69) give the coefficients for $\ell = 2,3,4,...$ in terms of $G_1$, $U_1$ and $P_1$. We can then express any eigenfunction as a sum of three terms; for example, the axial velocity eigenfunction $\bar{v}_z(r)$ may be written as

$$\bar{v}_z(r) = v_{z1}(r)G_1 + v_{z2}(r)U_1 + v_{z3}(r)P_1$$  (3.70)

It may be recalled that for an axisymmetric disturbance, there were only two such terms for an eigenfunction. The concept is, therefore, the same and further clarification is unnecessary in view of the explanation given earlier for the axisymmetric disturbance.

### 3.2.2  Step-by-Step Integration

The series expansions developed above are carried only up to a small value of the radius r. The solution is further continued by a step-by-step integration technique to the pipe wall in a manner analogous to that for axisymmetric disturbances. The only differences between this case and that of an axisymmetric disturbance are:

(i)  There are four eigenfunctions and, therefore, four differential equations for this case as compared to only three for the axisymmetric disturbance.

(ii) There are three independent solutions for each eigen-
function and three boundary conditions at the pipe
wall in place of only two for an axisymmetric
disturbance.

Conceptually, therefore, the problem is similar to that for
axisymmetric disturbances though it is more difficult to work out.
It should also be pointed out that the step-by-step integration
is carried directly in terms of the eigenfunctions $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$
rather than in terms of their combinations $\bar{f}(r)$ and $\bar{g}(r)$. This
is helpful since eventually it is desirable to have the eigen-
functions $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$ separately rather than in terms of $\bar{f}(r)$
and $\bar{g}(r)$. Since equation (2.22) contains both the highest
degree derivatives in the eigenfunctions $\bar{v}_r(r)$ and $\bar{p}(r)$, we have
to perform operations similar to those for an axisymmetric
disturbance in order to obtain the differential equations
suitable for step-by-step integration.

Thus, we differentiate the continuity equation (equation (2.21))
with respect to r and solve for $D^2\bar{v}_r$. Equations (2.22), (2.23) and
(2.24) are then solved for the unknowns $D\bar{p}$, $D^2\bar{v}_\theta$ and $D^2\bar{v}_z$
respectively to yield the following set

$$D^2\bar{v}_r = \frac{1}{r}(\frac{1}{r} - D)(\bar{v}_r + jn\,\bar{v}_\theta) - k\,D\bar{v}_z$$

$$D^2\bar{v}_\theta = j\frac{n}{r}(R\bar{p} - \frac{2}{r}\bar{v}_r) - \frac{1}{r}D\bar{v}_\theta + \{\frac{n^2+1}{r^2} - k^2 + R(kV_z - j\omega)\}\bar{v}_\theta$$

$$D^2\bar{v}_z = R(k\bar{p} + \bar{v}_r\,DV_z) - \frac{1}{r}D\bar{v}_z + \{\frac{n^2}{r^2} - k^2 + R(kV_z - j\omega)\}\bar{v}_z$$

$$(3.71)$$

and

$$D\bar{p} = \frac{1}{R} [(D^2 + \frac{1}{r} D)\bar{v}_r - j \frac{2n}{r^2} \bar{v}_\theta - (\frac{n^2 + 1}{r^2} - k^2 + R(kV_z - j\omega))\bar{v}_r]$$

On account of equation (3.70), the integration of these differential equations is carried out in terms of the _three_ independent solutions of the eigenfunctions, that is, in terms of $v_{z1}(r)$, $v_{z2}(r)$ and $v_{z3}(r)$ for the eigenfunction $\bar{v}_z(r)$, and similarly for $\bar{v}_r(r)$, $\bar{v}_\theta(r)$ and $\bar{p}(r)$. For further elaboration, the reader is referred to the discussion advanced earlier for an axisymmetric disturbance.

Having integrated the stability equations over the whole region $0 \leq r \leq 1$ such that boundary conditions at the center of the pipe are satisfied (by means of the series expansion), it is now required to satisfy the boundary conditions at the rigid pipe wall. These boundary conditions require that the disturbance velocity components be zero at $r = 1$, so that in view of equation (3.70), we must have

$$\begin{bmatrix} v_{r1}(1) & v_{r2}(1) & v_{r3}(1) \\ v_{\theta 1}(1) & v_{\theta 2}(1) & v_{\theta 3}(1) \\ v_{z1}(1) & v_{z2}(1) & v_{z3}(1) \end{bmatrix} \begin{bmatrix} G_1 \\ U_1 \\ P_1 \end{bmatrix} = \underset{\sim}{0} \qquad (3.72)$$

The determinant, whose absolute value should be zero, is, therefore, of order 3 as already speculated. Thus, the complex eigenvalue $k$ is again found by satisfying equation (3.36) where det is now given by the following equation

$$\det \equiv \begin{vmatrix} v_{r1}(1) & v_{r2}(1) & v_{r3}(1) \\ v_{\theta1}(1) & v_{\theta2}(1) & v_{\theta3}(1) \\ v_{z1}(1) & v_{z2}(1) & v_{z3}(1) \end{vmatrix} \qquad (3.73)$$

The ratios of $P_i$ and $U_i$ to $G_i$ are given by (see equation (3.72))

$$\frac{P_1}{G_1} = \frac{v_{\theta1}(1)\, v_{z2}(1) - v_{\theta2}(1)\, v_{z1}(1)}{v_{\theta2}(1)\, v_{z3}(1) - v_{\theta3}(1)\, v_{z2}(1)}$$

and $\qquad (3.74)$

$$\frac{U_i}{G_1} = \frac{v_{\theta3}(1)\, v_{z1}(1) - v_{\theta1}(1)\, v_{z3}(1)}{v_{\theta2}(1)\, v_{z3}(1) - v_{\theta3}(1)\, v_{z2}(1)}$$

so that by arbitrarily setting $G_1 = (1.0 + j0.0)$, the eigenfunctions can be found from equations of the form given in equation (3.70).

The fourth order Runge-Kutta method [23] and the Adams-Bashforth-Moulton predictor-corrector method [24] were used for the step-by-step integration of the stability equations. These methods are summarized in Appendix I.

Chapter 4

SEARCH FOR EIGENVALUES

Having developed the basic technique for solving the stability

equations for both axisymmetric and non-axisymmetric disturbances

applied to Poiseuille flow in a pipe, we are confronted with the

task of exploring the whole (or a significant part) of the complex

k-plane for possible eigenvalues. Unfortunately, this task is

generally not an easy one since equation (3.36), which determines

the eigenvalue k, is either transcendental or, as in our case,

some high order polynomial in k whose exact nature is unknown.

One of the methods employed for this purpose has been to use

directly one of the several numerical minimization schemes, such

as the Newton-Raphson iteration. This method has several

disadvantages. Since there are an infinite number of eigenvalues,

the most severe drawback of an iterative technique is its

tendency to bypass one eigenvalue and converge to another, so that

depending upon where the minimization procedure is started, some

important eigenvalues may be omitted. Thus, use of an iterative

technique alone leaves open the possibility that the least damped

mode may not be detected at all. In fact, it was observed during

the present investigation that unless the initial guess was close

to the true eigenvalue, the iteration technique used to improve

upon the value always slipped from the least stable mode to one

which was more stable, the problem becoming more critical as the

Reynolds number increases. Since this investigation is mainly concerned with locating possible instabilities of the Poiseuille flow in a pipe, it is evident that an iteration technique alone is highly inadequate.

A major advance in this context is reported in the work of Scarton [25] who developed a heuristic to extract all possible eigenvalues in a given region of the k plane. His technique, known as the 'Method of Eigenvalleys', is a highly improved form of the so-called grid methods that evaluate the determinant (equation (3.35) or (3.73)) for every mesh point in the region of interest. Conceptually, the method of eigenvalleys plots the absolute value of the determinant versus $(k_r, k_i)$. In fact, a three dimensional plot of the eigensurface [25, page 22] is not required. Instead, it is only sufficient to plot a well-labeled contour map of the eigensurface. The location of the eigenvalues is then provided by the minima of an eigenvalley.

The most severe disadvantage of any grid method including the method of eigenvalleys is the large amount of computer time used in evaluating the determinant at sufficient mesh points inside a given region of the k-plane. In addition, drawing a well-labeled contour map for the method of eigenvalleys is a highly complicated task even with the aid of a computer. It is desirable, therefore, to develop a method that not only ascertains the total number of eigenvalues in a certain region of the k-plane but is also simpler and less time consuming than the method of eigenvalleys. All

these criteria are very nicely satisfied by a method that stems directly from a result of the Cauchy integral theorem for determining the number of zeros of a complex function in any closed region of the argument in which the function is analytic.

For a function f(z), analytic except for poles in the interior of a closed curve C on the z-plane, and for f(z) and Df(z) continuous on C, the residue theorem gives [26, page 102, equation (41)]

$$N - P = \frac{1}{2\pi j} \int_C \frac{Df(z)}{f(z)} dz \qquad (4.1)$$

where N and P denote respectively the number of zeros and the number of poles of f(z) within the closed region C and where D is the operator $\frac{d}{dz}$. Since Df(z)/f(z) is the derivative of log[f(z)], to calculate the definite integral on the right-hand side of equation (4.1), it is sufficient to know only the variation of

$$\log|f(z)| + j \ \text{angle}[f(z)]$$

when the variable z describes the contour C in the positive sense, that is, in the counterclockwise sense. But $|f(z)|$ returns to its initial value, while the angle of f(z) increases by $2M\pi$, M being either zero or a positive or negative integer. Thus equation (4.1) reduces to

$$N - P = \frac{2M\pi j}{2\pi j} = M; \qquad (4.2)$$

that is, the difference (N - P) is equal to the integer quotient

obtained by dividing the variation of the angle of f(z) by $2\pi$ as

the variable z describes the boundary C in the positive sense.

The function f(z) can be expressed in terms of its real and

imaginary parts as follows

$$f(z) = X + jY$$

Then, as the point $z = x + jy$ describes the curve C in the positive

sense, the point whose coordinates are (X,Y) with respect to a

system of rectangular axes with the same orientation as the first

system, describes also a closed curve $C_1$, and it is only required

to count the number of revolutions which the radius vector joining

the origin of coordinates to the point (X,Y) has turned through in

one sense or the other. There is thus no need of plotting the

actual curve $C_1$. Instead, the integer M is given by the net

multiples of $2\pi$ by which the phase angle of f(z) changes in going

round the curve $C_1$. When the point z describes the curve C in the

positive sense, any counterclockwise movement of the phasor for

f(z) is considered positive while the clockwise movement is

considered negative.

In application to this problem, each element of the determinant

(equation (3.35) or (3.73)) is a function of the complex eigenvalue

k. In fact, the determinant can be considered as some high order

polynomial in k due to the recurrence relations for the coefficients

in the series solution of the stability equations. Therefore, the

function det(k) has no poles, that is, P ≡ 0 in equations (4.1)
and (4.2), and N = M. Thus, the problem of finding the number
of eigenvalues within a closed region of the k-plane is as simple
as counting the net multiples of $2\pi$ by which the phase angle of
the determinant (equation (3.35) or (3.73)) changes as k assumes
values on the closed contour in the k-plane.

The advantages of this simple method over the complicated
method of eigenvalleys are obvious. Firstly, in the present
method, it is only required to find the value of the determinant
for values of k on the boundary of the closed region to be investi-
gated. For the method of eigenvalleys, it may be recalled that
the determinant must be evaluated at all mesh points within and
on the boundary of the closed region in the k-plane. The saving
in computer time is, therefore, apparent. Secondly, no plotter
time is needed on the computer since a plot of the determinant
values or of the eigensurface [25] is not required for the present
method. Such a plot, however, is the very basis of the method of
eigenvalleys. The present method is, therefore, more efficient and
economical than the method of eigenvalleys.

While the present method is most useful in establishing the
number of eigenvalues inside a closed region of the k-plane, it
can also be used to provide a close approximation to the true
eigenvalue so that an iterative technique may converge to it.
If it so happens that the closed contour chosen on the k-plane
passes close to an eigenvalue within the contour at some location,

the phase angle of the determinant will change by about $180^{\circ}$ and

the magnitude of the determinant will show a dip. However,

depending on the closeness of the boundary to the eigenvalue, this

may or may not be the case. Nevertheless, we can always subdivide

the region investigated on the k-plane and thus isolate a

relatively small region which contains the eigenvalue. Thus, if

the net change in the phase angle of the determinant (equation

(3.35) or (3.73)) is $6\pi$ when k assumes values on the closed

contour XOAYX (Figure 4.1), it implies that there are 3 eigenvalues

inside the closed region XOAYX. In order to locate more closely
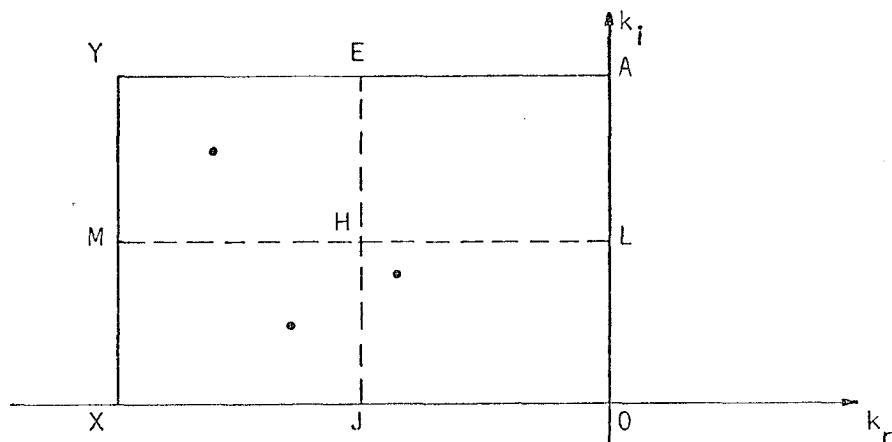


Figure 4.1. Some Eigenvalues in a Region of the k-Plane

where the three eigenvalues lie, we may divide the original region

into four smaller regions by evaluating the determinant along the

lines EJ and ML. Each of these four regions is then studied for

any possible eigenvalues using the values of the phase angle of

the determinant for values of k on respective boundaries. This

process can be continued until the region enclosing the eigenvalue

is small enough for an iterative technique to converge to the true

eigenvalue. Experience has shown that except for the least stable

mode, it is sufficient to know only two significant digits

correctly in the approximate eigenvalue. The secant method used

for the iterative minimization scheme then converges to the true

eigenvalue easily. For the least stable mode, it is generally

required to know the approximate eigenvalue to one more order of

accuracy. This requirement of starting with a good approximation

to the true eigenvalue is imperative for any iteration technique.

Even with the method of eigenvalleys [25], it is often necessary

to refine the eigensurface plot for the region enclosing the

eigenvalue so as to get a good enough approximation for convergence.

In order to ascertain theoretically the presence of unstable

modes for disturbances applied to pipe flow, it is only necessary

to determine whether there are any possible eigenvalues with

positive real parts. Primarily therefore, it is of interest to

investigate the first quadrant of the k-plane for possible eigen-

values. It should also be pointed out that for the above technique

to be valid, it is not at all required that the determinant be

zero or close to it. In fact, very large determinant values (of

the order of $10^{15}$) were often encountered while studying the

regions of interest on the k-plane. For meaningful results,

however, it is essential that the solution be numerically stable.

Later in Chapter 5, we will discuss a rather interesting observation

in this regard.

Chapter 5

RESULTS

The disturbance equations were solved using the technique

developed in Chapter 3 for both axisymmetric and non-axisymmetric

disturbances applied to Poiseuille flow in a rigid pipe. Thus,

in the series expansion for the steady mean flow velocity $V_z$

(equation (3.22)), only the first two constants $C_1$ and $C_2$ are non-

zero; in fact, $C_1 = -C_2 = 1.0$, and the higher C's are all zero.

Four values of the azimuthal mode number n (n = 0,1,2 and 3)

were tried with greater emphasis on n = 0 and n = 1. The dimension-

less frequency $\omega$ and Reynolds number R were varied so that the

product $\omega R$ covered a range from 200 to 30,000. In order to justify

the usefulness of the present technique, it is best to compare the

results obtained for the spatial stability of an axisymmetric

disturbance with those already available in the literature [14, 17].

5.1 Axisymmetric Disturbances (n = 0)

For an axisymmetric disturbance, it was relatively simple to

find some eigenvalues since Gill [17] has provided the equations for

obtaining a close estimate to the eigenvalues. In the present

notation, these equations [17, equations (4.5) and (4.6)] give

$$k_i R \simeq \omega R [1 + 4m(2\omega R)^{-1/2}]$$
(5.1)

and $k_r R \simeq -2m[4m + (2\omega R)^{1/2}]$ (5.2)

where m is an integer (m = 1,2,3...) corresponding to axisymmetric modes, and where it is necessary (for a good approximation) that $\omega$ be small compared with $R^{1/3}$.

It is clear from these equations that the governing parameter is the product $\omega R$ rather than the frequency $\omega$ and Reynolds number R independently. Though equations (5.1) and (5.2) give only approximate eigenvalues, it was found through numerical experimentation that so long as $\omega$ and R were varied in such a way as to keep the product $\omega R$ constant, the values of $k_r R$ and $k_i R$ remained almost constant. This provides an obvious facility by reducing the number of parameters from two to one. Thus, if the eigenvalue, say $k_1$, were known for $R_1 = 500$ and $\omega_1 = 1.0$, the eigenvalue $k_2$ for $R_2 = 5000$ and $\omega_2 = 0.1$ (so that $\omega_1 R_1 = \omega_2 R_2 = 500$) is approximately given by $k_1/10$ since $k_1 R_1 \simeq k_2 R_2$. It was found that on this basis, the first three decimal places in the values for $k_r$ and $k_i$ could be predicted correctly in most cases. This, in general, resulted in a better estimate for the eigenvalue than that provided by equations (5.1) and (5.2).

## 5.1.1 Stability of Flow

The real worth of the observation in the preceding paragraph lies in the fact that it is only necessary to investigate the stability of pipe Poiseuille flow for a wide enough range of the

product $\omega R$ rather than that of $\omega$ and $R$ independently. The effort required is, therefore, considerably reduced. Since the flow is unstable if the real part of $k$ is greater than zero, regions in the first quadrant of the k-plane were examined for any possible eigenvalues by means of the technique developed in Chapter 4. The product of $\omega$ and $R$ was varied from 200 to 10,000 for this part of the investigation. Some of the regions explored are shown in Figure 5.1. As k was varied to assume values on the boundary of these regions, it was found that the net change in the phase angle of the corresponding determinant (equation (3.35)) was zero. Thus no eigenvalue exists inside any of the regions in Figure 5.1 confirming, therefore, the earlier conclusions [14, 17, 27] that the Poiseuille flow in a pipe is spatially stable to all infinitesimal axisymmetric disturbances. All the possible eigenvalues for this problem are, therefore, such that their real parts are negative. This is also shown by the approximate equation (5.2).

In view of the observation that $\omega R$ is the governing factor for $k_r R$ and $k_i R$, it should be pointed out that if $R$ were increased and $\omega$ decreased indefinitely in order that the product $\omega R$ remains constant, the mode will become less and less stable since $k_r$ (and also $k_i$) will approach zero. For example, it was found that for a Reynolds number of 2000 and a frequency of 0.1, the least stable mode (that is one for which $|k_r|$ is minimum) has

$$k = (-0.0220485 + j0.1201397) \quad ,$$

so that the amplitude of this disturbance will not fall to half

its original value until a distance of about 16 pipe diameters

downstream of its source. If R is increased to 20,000 and $\omega$

decreased to 0.01 so that $\omega R = 200$ again, the approximate value

of $k_r$ will be -0.0022 so that for this disturbance amplitude to

fall to half its original value, a pipe length equivalent to about

160 diameters will be required. Moreover, it was found in con-

formity with Gill's observation [17] that as $\omega R$ decreases, both

$|k_r R|$ and $k_i R$ decrease. Gill also observed that for a zero

frequency disturbance (actually one for which $\omega R \rightarrow 0$), the

amplitude falls to half its original value at a distance of about

R/100 diameters from its source, that is at a distance of about

200 diameters for a Reynolds number of 20,000. Thus if an

infinitesimal axisymmetric disturbance of a very low frequency is

applied to the pipe Poiseuille flow at very high Reynolds numbers,

the mean flow will stay distorted for a long distance downstream.


## 5.1.2  Eigenvalues and Eigenfunctions

By varying the integer m in equation (5.1) and (5.2), approximate

eigenvalues were found for a wide range of $\omega R$, and then accurately

calculated using the method developed in Chapter 3. With m as a

parameter, Figures 5.2 and 5.3 show the variation of $k_r R$ and $k_i R$

with $\omega R$ as the independent variable. Eigenvalues were found for

more values of m than these figures indicate but for reasons of

clarity, they are not included here. Though $k_r R$ and $k_i R$ vary with different combinations of $\omega$ and $R$ such that $\omega R$ is constant, these variations are too small to be represented graphically on the scales used in Figures 5.2 and 5.3. By means of the eigenvalue-search-technique developed in Chapter 4, it was concluded that for a given $\omega R$, the least stable mode is the one given by $m = 1$. This is also the one with the largest wave length and phase velocity since the dimensionless wave length $\lambda$ is given by

$$\lambda = \frac{1}{k_i} \, , \qquad\qquad (5.3)$$

and the dimensionless phase velocity $C_p$ is given by

$$C_p = \frac{\omega}{k_i} \, . \qquad\qquad (5.4)$$

Although higher modes, that is, those with higher values of $m$ are more stable, Figure 5.3 shows that the corresponding wave length undergoes only a very slight decrease. Also, if the frequency is increased at a given Reynolds number, a given mode becomes more stable, and this time there is a marked decrease in wave length.

Figures 5.4 and 5.5 compare some results of the present investigation with those of Gill [17, Figure 1] for a Reynolds number of 4000 and a dimensionless frequency of 0.975 so that $\omega R = 3900$. The axial velocity eigenfunction $\bar{v}_z(r)$ is represented on these figures in terms of its amplitude and phase for two modes $m = 1$ and $m = 3$. The amplitude $|\bar{v}_z(r)|$ was normalized with respect

to its maximum value before plotting. Gill's results are shown by the dashed lines in these figures. For small r, the agreement is excellent and any small differences can be attributed to the inherent error involved in taking off values from a figure in a paper. There is an abrupt end to the dashed lines on these figures since Gill could not find the corresponding values beyond a certain radius due to his assumption that r is close to zero. This also accounts for the large error in his results for the phase angle of $\bar{V}_z(r)$ when m = 1 and r greater than about 0.4. The present technique is, of course, valid over the whole region $0 \leq r \leq 1$.

It may also be observed from Figure 5.5 that the mode m = 3 has a discontinuity of about 180° in the phase angle for $\bar{V}_z(r)$. Such phase discontinuities were in fact found experimentally by Leite [14, page 81] for several observed velocity distributions. Notice also that if a mode has such a phase relationship, the curve showing the radial variation of the amplitude (Figure 5.4) will have two maxima. This is yet another feature of many of the experimental curves of Leite [14].

Due to the inherent difficulty in finding the value for the phase $\phi$ of an eigenfunction outside the range $-180° \leq \phi \leq 180°$ on the computer, the eigenfunctions are represented here in terms of their real and imaginary parts. In fact, it is sufficient to exhibit only their normalized values. Since it is possible that the maximum magnitude of the real and imaginary parts of an eigenfunction may differ by several orders of magnitude, both

real and imaginary parts are normalized separately for a fair representation. Using the maximum magnitude norm on the interval $(0 \leq r \leq 1)$, we define a normalized eigenfunction for the radial velocity component by

$$\bar{v}_r^*(r) = \frac{Re[\bar{v}_r(r)]}{\underset{(0 \leq r \leq 1)}{Max} \left| Re[\bar{v}_r(r)] \right|} + j \frac{Im[\bar{v}_r(r)]}{\underset{(0 \leq r \leq 1)}{Max} \left| Im[\bar{v}_r(r)] \right|} \quad (5.5)$$

A similar transformation is carried out for $\bar{v}_z(r)$ and $\bar{p}(r)$ so that the maximum magnitudes of the real and imaginary parts of the normalized eigenfunctions are both unity.

Figure 5.6 shows the radial variation of the real and imaginary parts of the normalized eigenfunctions $\bar{v}_r^*(r)$ and $\bar{v}_z^*(r)$ for the least stable mode. The difference in the mode shape for the two combinations of $\omega$ and R is representative of the trend observed for other combinations also. The pressure eigenfunction is not shown in this figure since it was found to be almost uniform over the whole range $(0 \leq r \leq 1)$ for nearly all cases for the axisymmetric disturbance. Figure 5.7 shows the eigenfunctions for a more stable mode (m = 4). When compared with similar results for m = 1 in Figure 5.6, we find that the mode shape becomes more oscillatory as the mode becomes more stable. These figures clearly indicate that the boundary conditions at the center and wall of the pipe are satisfied.

It should be pointed out that once an eigenvalue was found, the eigenfunctions were calculated by properly combining the two independent solutions for them (equation (3.32)). This was accomplished by arbitrarily setting $U_I = 1.0 + j0.0$ (see Section 3.1.2). If, instead, $P_I$ is set equal to $(1.0 + j0.0)$, the real and imaginary parts of the eigenfunctions will, in general, have different radial variations from those in Figures 5.6 and 5.7. However, the more meaningful quantities like the magnitude and phase of the eigenfunctions are unchanged regardless of which of the two constants ($U_I$ or $P_I$) is set equal to $(1.0 + j0.0)$. Thus it is absolutely arbitrary, as it should, to set any one of $U_I$ and $P_I$ to 1.0. A similar comment is applicable to the eigen-functions shown in Figures 5.12 through 5.15 for the non-axisymmetric disturbance.

### 5.1.3  Gill's 'Wall' and 'Central' Modes

In his theoretical analysis for axisymmetric disturbances, Gill [17] referred to 'wall' and 'central' modes as those modes that correspond to a disturbance confined to a thin region either close to the pipe wall or close to the center of the pipe. In other words, the disturbance amplitude has a maxima lying close to the pipe center for central modes and close to the pipe wall for wall modes. The wall modes were designated by Gill with a symbol q and the central modes with a symbol m. The approximate equations

(5.1) and (5.2) are, in fact, for his central modes. Gill made this classification in order to simplify his asymptotic analysis. In the present investigation, however, such a classification is out of place on two grounds.

First of all, there is no need of making any simplifying assumptions that the disturbance be confined to a thin region in the pipe. Secondly, and this is more important, the central modes with high values of m are not distinguishable from the wall modes. Gill himself notes [17, page 160] that as m increases, the radius $r_m$ at which the disturbance amplitude is maximum increases, and that for m sufficiently large, $r_m$ tends to be of order unity which is actually the case for a wall mode. This observation was also confirmed during the present investigation. The results in Figure 5 of Gill's paper further emphasize this fact. It is unnecessary, therefore, to distinguish between wall and central modes.

### 5.1.4 Numerical Errors and Stability

It is an unfortunate fact that the values of the constants (V, U and P's in equations (3.19) through (3.21)) increase as more terms are retained in the series expansions of the eigenfunctions. Fortunately, however, the exponent of r also increases rapidly and since a term in the series consists of the product like $V_\ell r^{2(\ell - 1)}$, it was found that for r less than 1, the successive terms in the series decrease in magnitude. Then if $r_s$ represents the preassigned

value of r up to which the series solution was carried, the criteria

used for terminating the series was that for $r = r_s$ the ratio of the

last term retained to the partial sum up to a term preceding the

last one must be less than a preassigned epsilon. This ratio was

found for all the three series (equations (3.19) through (3.21))

and the maximum of these ratios was required to be less than epsilon.

In view of the accuracy requirements, a value of $10^{-12}$ or even lower

was used for epsilon in all cases. It was found that in general

about 11 to 15 terms were required in the series for a typical $r_s$

of 0.1; the higher number of terms required for a larger value of

the parameter $\omega R$. If at all, the problem due to the series solution

was critical only at very high values of $\omega R$ (over 10,000) but the

aforementioned value of epsilon could always be retained at the cost

of a relatively lower $r_s$ in order not to exceed a certain upper

limit for the number of terms required in the series.

Starting from $r = r_s$, the integration of the system equations

was accomplished by means of the fourth order Runge-Kutta method.

An outline of this method is given in Appendix I for easy reference.

Though it is generally argued [24, 28] that the Runge-Kutta method

requires twice as much work as the predictor-corrector for the same

accuracy, it is true only if the corrector is used once per step.

While checking some results by the predictor-corrector method[*], it

was found that the corrector must be iterated at least two or three

times for a certain accuracy. In such a situation, therefore, the

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[*]The particular predictor-corrector method used here is outlined
in Appendix I.

Runge-Kutta method seems preferable. In fact, Acton [24] casts serious doubts on iterating the corrector twice. Later in Section 5.2.3, we will have more reasons to agree with some of his other beliefs [24, page 134].

It is a well known fact [23, 24, 28] that the error involved in the Runge-Kutta method is not easily traceable. Collatz [23, page 71] has provided a rough guide for finding a reasonable length of the integration step but in application to this problem, it was found that his criteria was too stiff to be practicable. Ralston [28] has provided some error estimates for the Runge-Kutta method but they are good only for a single differential equation while we have three coupled differential equations for an axisymmetric disturbance. This error estimation problem also exists to some extent for the predictor-corrector method due to the coupled differential equations in our case. Moreover, the major problem with predictor-corrector integration schemes concerns stability rather than accuracy.

It seems that the best one can do to ensure accurate results [23, 24, 28] is to solve the problem twice, once for a step size of h and then for h/2. This rule of thumb was followed to check some results at random. Table 5.1 is representative of the accuracy of the present results. For a Reynolds number of 10,000 and a dimensionless frequency of 0.5, this table snows the values of k for the least stable mode for step sizes of 0.01, 0.005, 0.0025 and 0.001. A rather large number of decimal places is

Table 5.1   Values of k for various values of

the step size (R = 10,000, ω = 0.5, n = 0)

| Step Size | k |
|-----------|---|
| 0.0100 | (-0.02029414360 + j0.52153199809) |
| 0.0050 | (-0.02029412921 + j0.52153194376) |
| 0.0025 | (-0.02029412827 + j0.52153194041) |
| 0.0010 | (-0.02029412821 + j0.52153194019) |

carried in the values reported in this table in order to show

clearly the differences encountered by using smaller and smaller

step size.  As can be easily seen, the first seven decimal places

in the values of k are always identical.  According to Collatz [23],

if the calculations are performed with two step sizes of h and h/2,

the error in the calculation with the smaller step size can be

estimated by one $(2^L - 1)$-th part of the difference between the

results of the two calculations.  Thus the corrected value $y_c$ is

given by

$$y_c \simeq y_{h/2} - \frac{y_h - y_{h/2}}{2^L - 1}$$ 

(5.6)

where L is the order of the method used, and $y_h$ and $y_{h/2}$ are the

values of y calculated with step sizes of h and h/2 respectively.

With L = 4 for the fourth order Runge-Kutta method employed, the

values of k reported in Table 5.1 for step sizes of 0.005 and

0.0025 can be corrected. The results are shown in Table 5.2 with

the value of k for the step size of 0.001 included for easy

comparison. This table shows the last two values of k to be the

same up to the 11 places of decimals displayed here.


Table 5.2  Corrected Values of k from Table 5.1

Using equation (5.6)

| Step Size | Corrected Value of k |
|-----------|----------------------|
| 0.0050 | (-0.02029412825 + j0.52153194014) |
| 0.0025 | (-0.02029412821 + j0.52153194019) |
| 0.0010 | (-0.02029412821 + j0.52153194019)* |

*uncorrected as in Table 5.1


In short, this representative example illustrates that the error

involved in the present study was indeed very small.  The maximum

and minimum step sizes used were 0.01 and 0.001 with the latter

used only in cases of very high values of the parameter $\omega R$.  For

the axisymmetric disturbance no numerical instability was encountered

in any of the cases studied.  We did encounter this problem in a

few cases for the non-axisymmetric disturbance and will discuss

them later in Section 5.2.3.

## 5.2  Non-axisymmetric Disturbances ($n \neq 0$)

This being the first study of the spatial stability of a non-axisymmetric disturbance, the location of eigenvalues is completely unknown.  No formulae comparable to those in Equations (5.1) and (5.2) exist.  In such a case it is imperative to isolate a region on the k-plane that contains one or more eigenvalues by means of the eigenvalue search technique developed in Chapter 4.  Then by subdividing the region into smaller ones, a close enough approximation can be obtained for the iteration process to converge to the true eigenvalue.

### 5.2.1  Stability of Flow

It has already been stated that for the spatial instability of the flow, the real part of the complex wave number k must be positive, that is the corresponding eigenvalue must lie in the first quadrant of the k-plane.  The regions in this part of the k-plane were, therefore, explored for possible eigenvalues by means of the technique described in Chapter 4.  For this purpose, the azimuthal mode number n was taken to be 1 and the Reynolds number R and frequency $\omega$ were varied so that their product $\omega R$ covered a wide range from 200 to 10,000.  Figure 5.8 shows some of the regions examined.  Relatively small regions of the k-plane were examined for high values of $\omega R$ due to the unfortunate fact that a very small step size (0.001 or lower) must be used at high

values of ωR and kR in order to avoid numerical instability. It

was found that no eigenvalue exists inside the regions displayed

in Figure 5.8. This implies that up to a Reynolds number of 10,000,

the pipe Poiseuille flow is spatially stable to infinitesimal non-

axisymmetric disturbances for which n = 1. It will be shown in

the following section that the pipe flow is relatively more stable

to disturbances with n greater than 1. This is what one may also

expect on the basis that except for an axisymmetric disturbance,

n = 1 is the only disturbance that allows non-zero values for

$\bar{v}_r(o)$ and $\bar{v}_\theta(o)$ - (see equation (2.28.2)). It may, therefore, be

argued that it is less damped than those for which n = 2,3,....

Also, a recent theoretical analysis [13] has shown that the pipe

flow is temporally stable to non-axisymmetric disturbances with

n = 1.

## 5.2.2  Eigenvalues and Eigenfunctions

In view of the observation for an axisymmetric disturbance,

it was natural to investigate whether the frequency ω and Reynolds

number R could be combined into one parameter ωR. The many cases

studied with this objective in mind indicate that it holds as far

as one is looking for an approximation to some of the eigenvalues

in a certain region but not all as Figure 5.9 indicates. This

figure shows the number of eigenvalues present inside certain

regions of the second quadrant of the k-plane. Parts (a) and (b)

of this figure indicate the regions studied for $\omega R = 200$ such that the Reynolds number and frequency have a ten-fold variation. Since the region in part (b) is such that $k_r$ and $k_i$ can vary only one-tenth as much as for the region in part (a), these two regions must contain the same number of eigenvalues if $\omega R$ were to be the controlling factor for values of $k_r R$ and $k_i R$. It is clear, however, that such is not the case. The same is true of the regions in parts (c) and (d) for $\omega R = 500$; one contains 11 eigenvalues while the other has only 10. Since eigenvalues were isolated only for regions shown in parts (a) and (c) of this figure, it is not known which one(s) is(are) not present inside the regions in parts (b) and (d). Notice that such an investigation was not carried out for the axisymmetric disturbance.

Another observation made in this connection is that in some cases, the mode shapes have a marked difference for different combinations of $\omega$ and $R$ for which $\omega R$ is constant. Such was not the case for an axisymmetric disturbance; there the eigenfunctions were almost the same for all the cases studied with this purpose in mind.

It should be pointed out, however, that eigenvalues for the least stable mode were found for values of $R$ and $\omega$ corresponding to those in parts (b) and (d) of Figure 5.9, and the starting values for $k$ were taken from those for parts (a) and (c). Convergence to the true eigenvalue was rapid - only 7 or 8 iterations being necessary. In fact, this observation is

representative of the facility experienced in finding the least
stable mode for various values of $\omega$ and R. During these compu-
tations, it was found that convergence to the true eigenvalue was
very rapid if it so happened that the approximate value for k
could be taken on the basis that $\omega$R is the controlling parameter
for $k_r$R and $k_i$R.

For the least stable mode when n = 1, Figure 5.10 shows the
variation of $k_r$ and $k_i$ with R as the independent variable and $\omega$
as a parameter. It is clear from this figure that for a fixed
frequency, $|k_r|$ and $k_i$ both decrease as the Reynolds number increases;
that is, the mode becomes less stable and has a larger wave length
at higher Reynolds numbers. The effect of frequency at a fixed
Reynolds number is similar though relatively more significant for
the wave length. If the frequency is doubled at a fixed Reynolds
number, the wave length reduces by a factor of about 2 so that the
phase velocity (equation 5.4)) is nearly constant. These
conclusions are similar to those for the axisymmetric disturbance
as a careful study of Figures 5.2 and 5.3 will reveal. In Figure
5.10, the curves for $k_i$ seem to approach the zero value asymptotically.
Thus if the frequency is decreased and/or the Reynolds number is
increased indefinitely, the mode tends to approach a neutrally
stable state - an observation similar to the one for the axisymmetric
disturbance (Section 5.1.1).

A comparison of the eigenvalues for the least stable mode in
the case of four different values of the azimuthal mode number n

is provided by Figure 5.11. The results are presented in the form

of the variation of $k_r R$ and $k_i R$ with $\omega R$ as the independent variable.

Though the values of $k_r R$ and $k_i R$ do change with different combi-

nations of $\omega$ and R for which $\omega R$ is a constant, these changes are

too small to be represented graphically for the cases investigated.

The lower portion of this figure helps to prove an earlier statement

that n = 1 is the least stable of all modes including the one for

which n = 0. The possibility of such a behavior was speculated by

Betchov and Criminale [22, page 230] on the basis that in a pipe

flow, the production of vorticity (given by the term involving

$D(DV_z/r)$ in equation (3.5)) vanishes identically for n = 0 but not

for higher values of n. This implies, however, that n = 0 should

be the most stable mode of all while it is clear from Figure 5.11

that such is not the case. Since n = 0 allows for non-zero values

of $\bar{v}_z(o)$ and $\bar{p}(o)$ while for n = 2,3,..., all the eigenfunctions are

required to vanish at r = 0, it appears that the non-zero boundary

conditions at r = 0 for n = 0 have a more significant effect than

the non-zero production of vorticity for n = 2,3,.... Continuing

this argument further, both the non-zero boundary conditions at

r = 0 and the non-zero production of vorticity help to make n = 1

the least stable mode of all.

As far as $k_i$ or its reciprocal, the wave length, is concerned,

there is no such ambiguity; the value of $k_i R$ increases with higher

values of n at a constant $\omega R$. For the sake of clarity, the

variation of $k_i R$ has been shown on Figure 5.11 for only two values

of n. The curves for n = 1 and n = 2 lie in between those for n = 0
and n = 3. The effect of raising the azimuthal mode number n on the
wave length is very small at high values of ωR. It is interesting
to note that the curves drawn on Figure 5.11 are almost linear,
suggesting thereby that a simple power law can be used to relate
$k_r R$ and $k_i R$ with ωR. However, due to the lack of a simple relation
connecting the power law constants with the parameter n, it was
not pursued further.

In view of the earlier discussion for the axisymmetric
disturbance (Section 5.1.2), eigenfunctions are shown in Figures
5.12 through 5.15 in terms of their real and imaginary parts
normalized according to equation (5.5). The effect of changing
the azimuthal mode number n on the mode shape for the least stable
mode can be noticed by studying Figures 5.12 through 5.14. These
figures are for a Reynolds number of 5000 and a frequency of 0.1.
It is clear from these figures that as n varies from 1 to 3, some
of the eigenfunctions are affected more than the others. Due
perhaps to the non-zero values of $\overline{v}_r(o)$ and $\overline{v}_\theta(o)$ for n = 1, the
mode shape changes considerably as n varies from 1 to 2; the
change from 2 to 3 in the value of n being rather insignificant
for the mode shape.

For n = 1, a Reynolds number of 500 and a dimensionless
frequency of 1.0, Figure 5.15 shows the radial variation of the
eigenfunctions for the fourth least stable mode and one with the
smallest wave length and phase velocity in the rectangular region

marked by $(-1.5 \leq k_r \leq 5.0)$ and $(0 \leq k_i \leq 4.0)$. This modal

characterization was made possible by the eigenvalue search

technique described in Chapter 4. The conclusion, in general,

is the same as for an axisymmetric disturbance; the mode shape

becomes more oscillatory as the mode becomes more stable. For

different values of n, most of the eigenvalues for the least

stable modes are tabulated in Table 5.5. This table clearly

brings out the fact that kR is approximately governed by $\omega$R.

For n = 1, a frequency of 1.0 and Reynolds numbers of 200 and 500,

Table 5.6 shows all the eigenvalues present in the region of

the k-plane defined by $(-1.5 \leq k_r \leq 5.0)$ and $(0 \leq k_i \leq 4.0)$.


5.2.3  <u>Numerical Errors and Stability</u>

Due to the discussion already included in Section 5.1.4,

it seems appropriate to check the accuracy of our results by

comparing the values of k calculated on the basis of different

step sizes. Table 5.3 is a representative example of such values

of k for four step sizes. These results are for n = 1, a Reynolds

number of 10,000, and a frequency of 0.5. Therefore, except for n,

the parameters are the same as for the results in Table 5.1. It

is clear that the values of k in Table 5.3 are identical only up to

the fifth decimal place as compared to the seventh place for the

results in Table 5.1. Though still very small, the error involved

in the non-axisymmetric case is more than that for the axisymmetric

disturbance. This is to be expected due to the fact that for an axisymmetric disturbance there are only three eigenfunctions and two independent solutions for each eigenfunction as compared to four eigenfunctions and three independent solutions for each for a non-axisymmetric disturbance. The number of calculations required for the latter is, therefore, at least twice that for the former.

Table 5.3   Values of k for various values

of the step size (R = 10,000, $\omega$ = 0.5, n = 1)

| Step Size | k |
|-----------|---|
| 0.0100 | (−0.01722848619 + j0.53525172047) |
| 0.0050 | (−0.01722769846 + j0.53525112413) |
| 0.0025 | (−0.01722764735 + j0.53525108570) |
| 0.0010 | (−0.01722764397 + j0.53525108317) |

For step sizes of 0.005 and 0.0025, the values of k corrected by means of equation (5.6) are shown in Table 5.4 with the uncorrected value for the step size of 0.001 included for comparison. The last two values of k in this table differ only in the $11^{th}$ decimal place.

Table 5.4  Corrected Values of k from Table 5.3

Using equation (5.6)

| Step Size | Corrected Value of k |
|-----------|----------------------|
| 0.0050 | (-0.01722764595 + j0.53525108437) |
| 0.0025 | (-0.01722764394 + j0.53525108314) |
| 0.0010 | (-0.01722764397 + j0.53525108317) [*] |

[*] uncorrected as in Table 5.3

Numerical instability manifested itself only in very few cases.
In fact, only two such cases were encountered during the compu-
tation.  Once while converging to an eigenvalue for a non-axisymmetric
disturbance, the determinant (equation (3.73)) became 'exactly'
zero thus indicating the eigenvalue but the boundary conditions
at the wall were not satisfied.  The problem, however, disappeared
when the integration step size was reduced.

Another case was relatively more informative and requires
some discussion.  In order to investigate the number of eigenvalues,
if any, inside the region shown in part (d) of Figure 5.8, the
first attempt was made with a step size of 0.004.  According to
the eigenvalue search technique described in Chapter 4, it was
found that there should be one eigenvalue inside the region in
question.  After careful subdivisions of the region to get an
approximate eigenvalue, the final iterations to the true eigenvalue

were attempted with a step size of 0.002. Finding that successive
iterations always seemed to converge to some eigenvalue outside
the region of interest, the original contour (part (d) of Figure
5.8) was studied again by the eigenvalue search technique but
this time with a step size of 0.002; the result being that <u>no</u>
eigenvalue exists inside the closed region. Lower values of the
step size were tried to confirm that there really was no eigenvalue
present in the region of interest. It should also be pointed out,
however, that step sizes larger than 0.004 indicated wrongly the
presence of one eigenvalue. This case highlights the dangers
involved in taking a relatively big step size even while looking
for eigenvalues inside a closed region of the k-plane.

It is also of interest to note here that the predictor-
corrector method was used for the investigation under discussion,
and that the allowable error at each step was maintained constant
for all step sizes. Since a step size of 0.004 or larger gives
wrong results, it is evident that excessive iterations of the
corrector equation (in order to keep the error within a certain
limit) do not result in the 'true' solution of the differential
equation but only of the difference equation. Thus, as also
suggested by Acton [24, pages 133-134], iterations of the corrector
equation in the predictor-corrector method follow the law of
diminishing returns as they converge towards the iterated limit -
<u>not</u> towards the 'true' solution. It is, therefore, futile to

work harder and get wrong results with a larger step size. For more reliable results, a smaller step size must be used.

In view of the observation that the two cases that did result in numerical instabiiity could be resolved by taking a smaller step size, it appears that the round-off error was insignificant during the entire investigation. In any case, a choice of too conservative a step size must always be avoided since it not only requires more computation time but also involves the dangers of round-off error becoming significant.
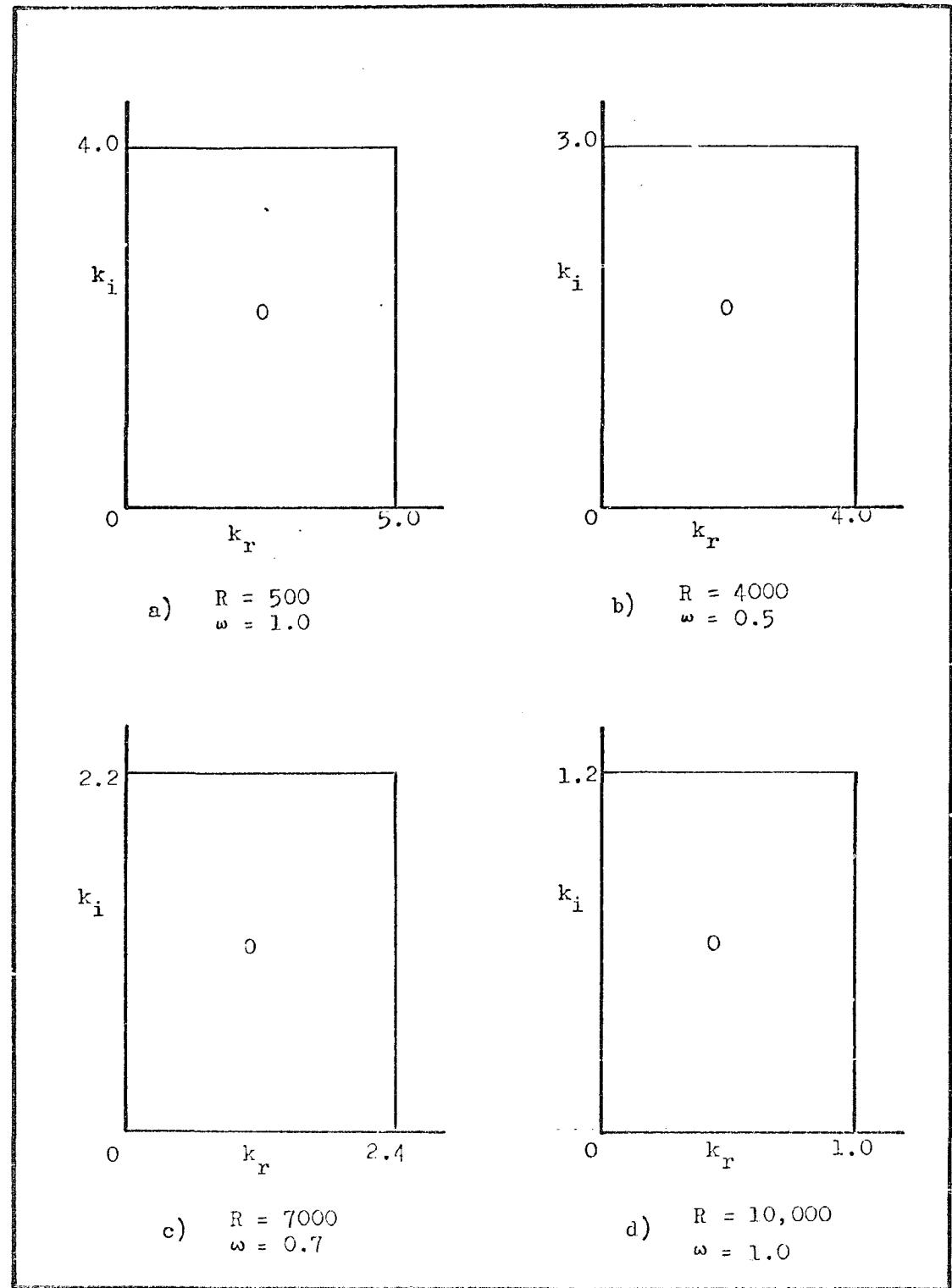
Figure 5.1. Regions on the k-plane with the number of Eigenvalues present inside the region (n = 0)
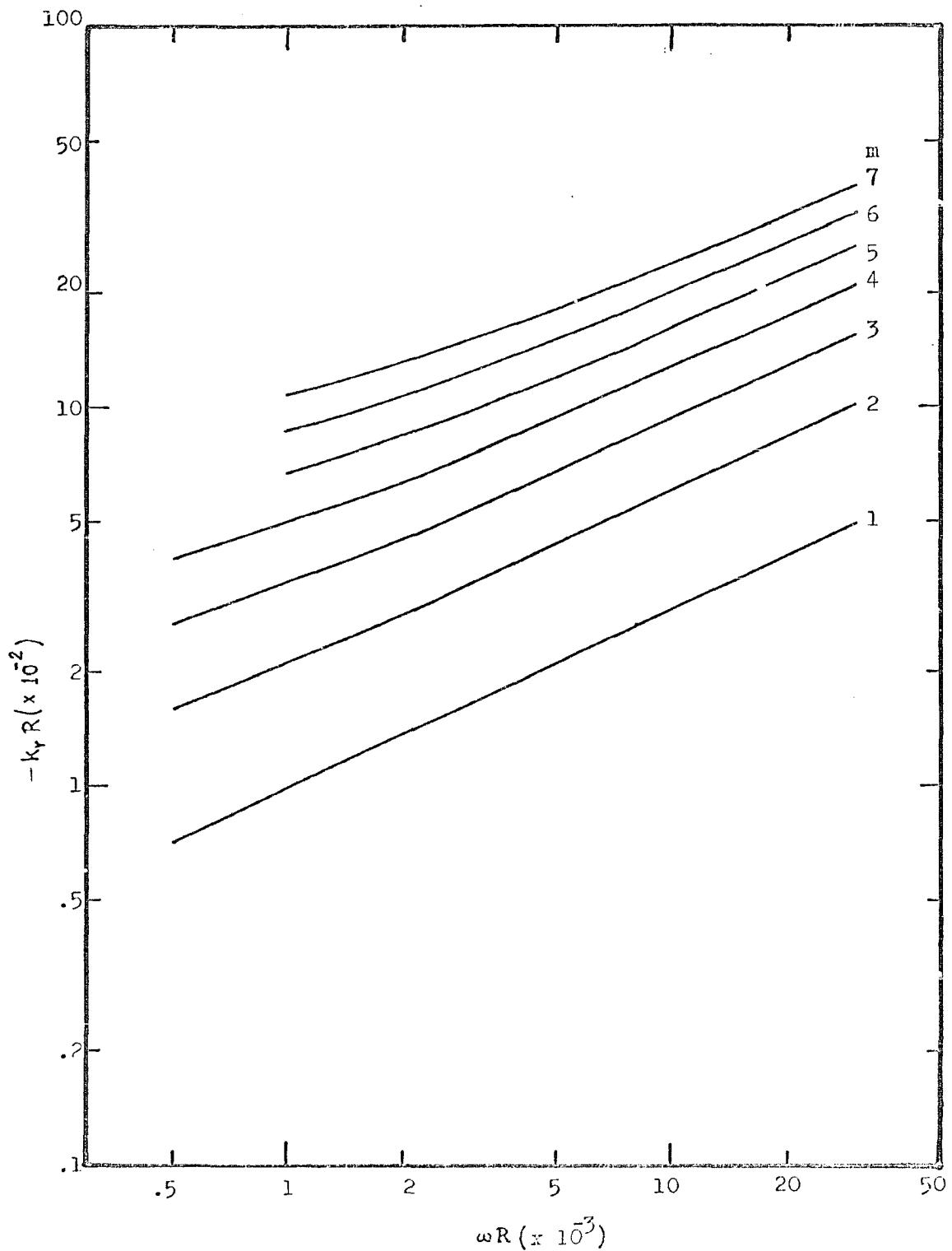
Figure 5.2. Variation of $k_r R$ with $\omega R$ for an Axisymmetric
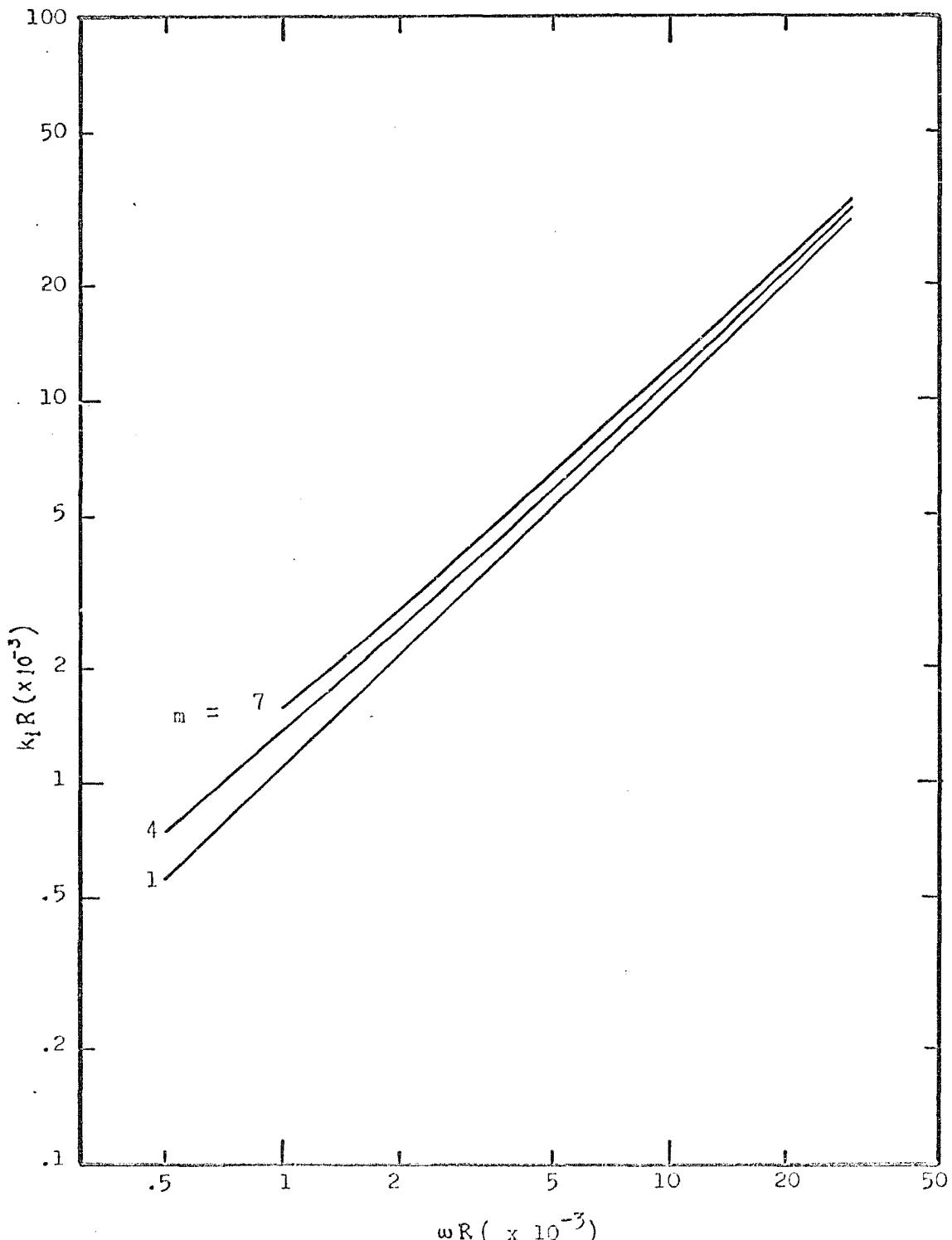
Disturbance

Figure 5.3. Variation of $k_iR$ with $\omega R$ for an Axisymmetric
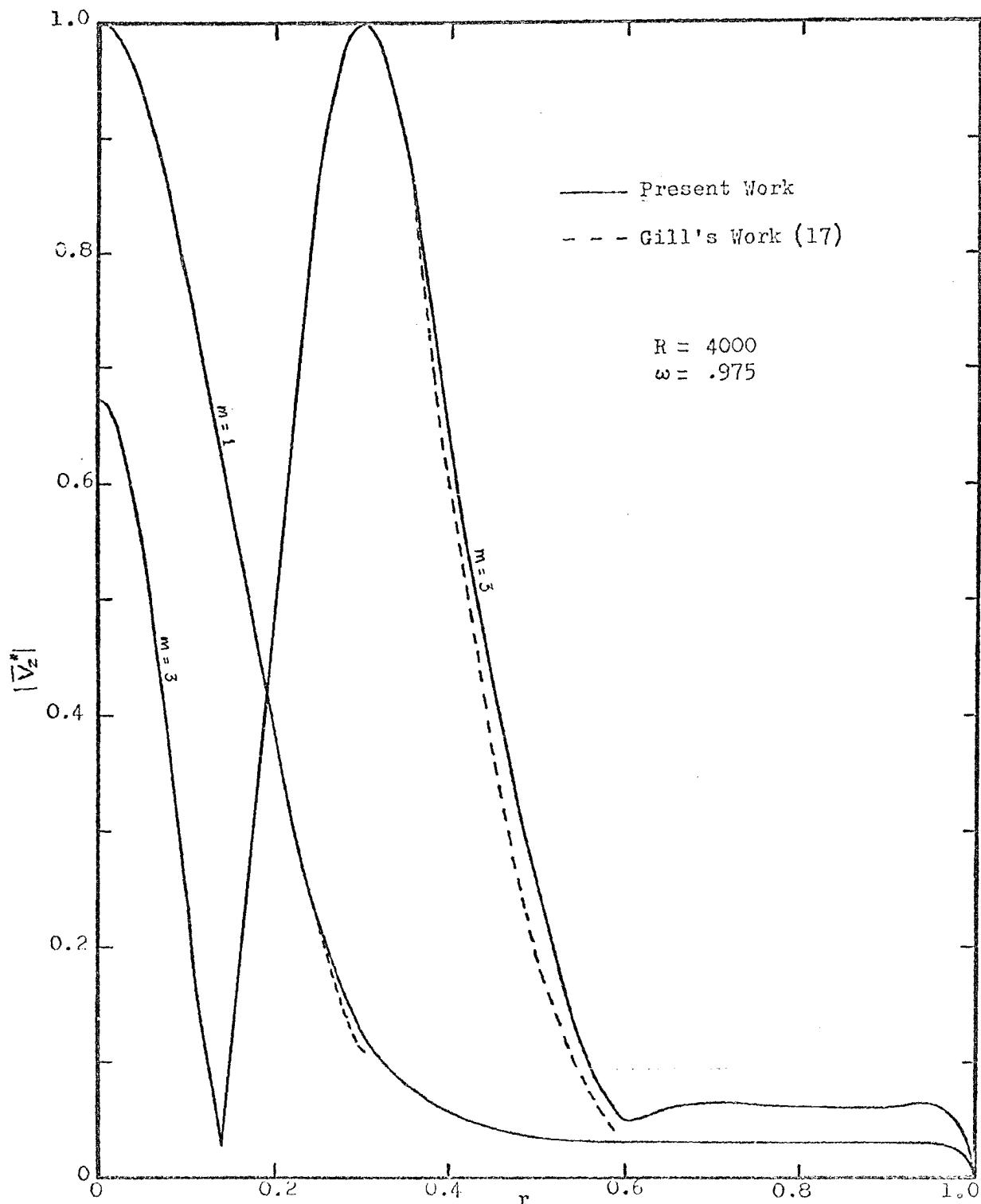
Disturbance

Figure 5.4. Comparison with Gill's Results (17, Figure 1) for an Axisymmetric Disturbance .
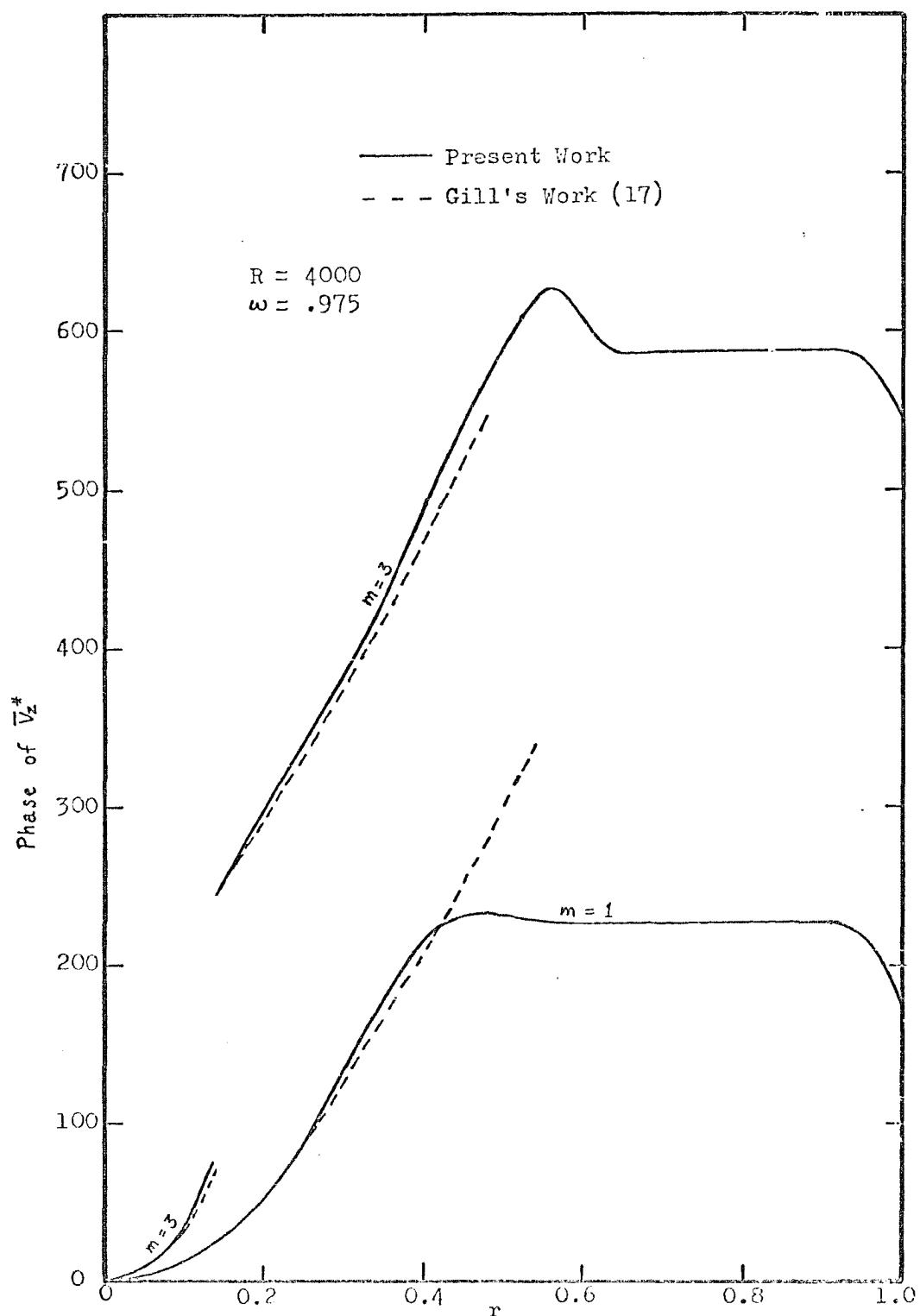
Figure 5.5.  Comparison with Gill's Results (17, Figure 1) for an Axisymmetric Disturbance

a) R = 2000, ω = 0.1

b) R = 10,000, ω = 0.2
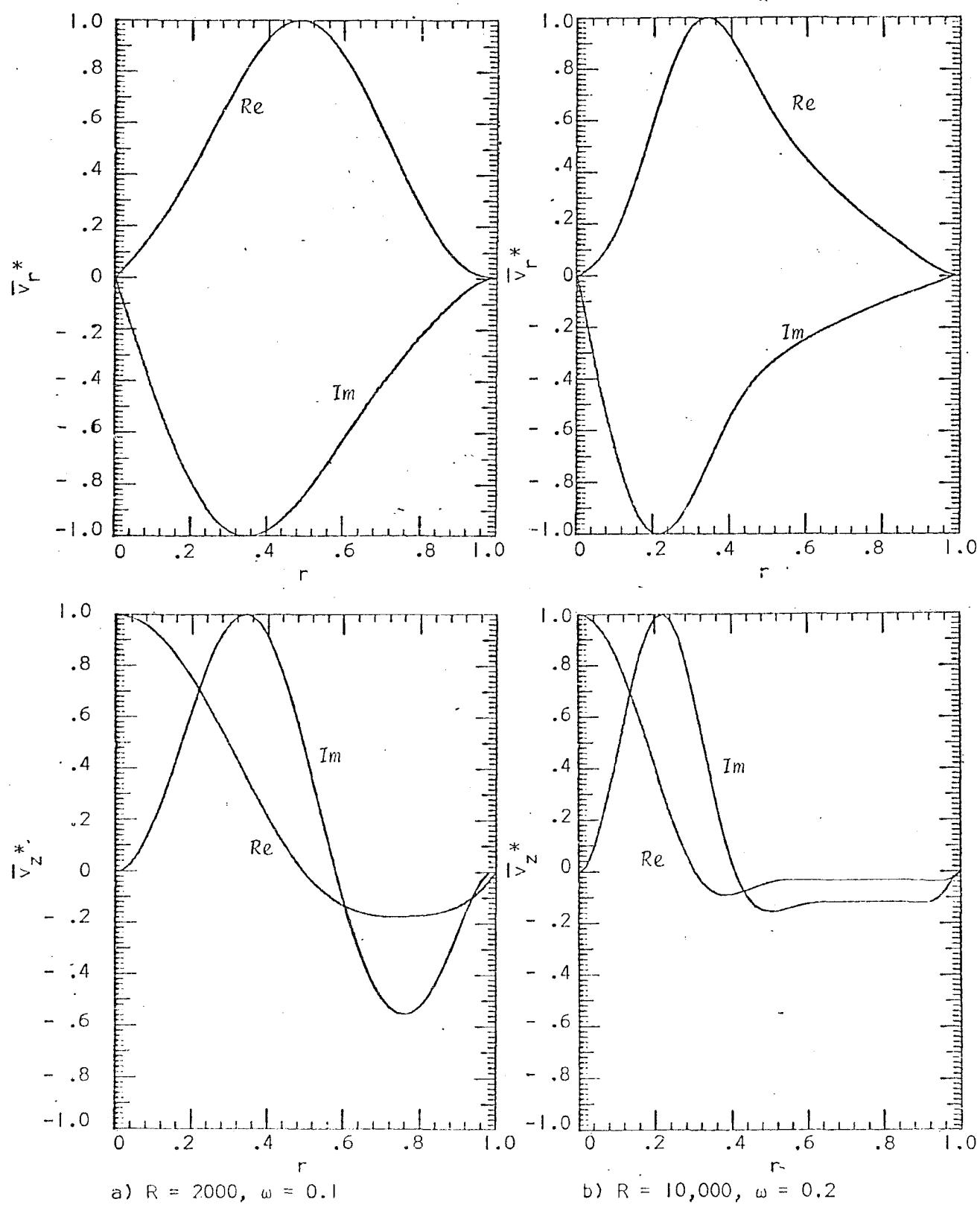
Figure 5.6   Eigenfunctions of the Least Stable Mode for an Axisymmetric Disturbance
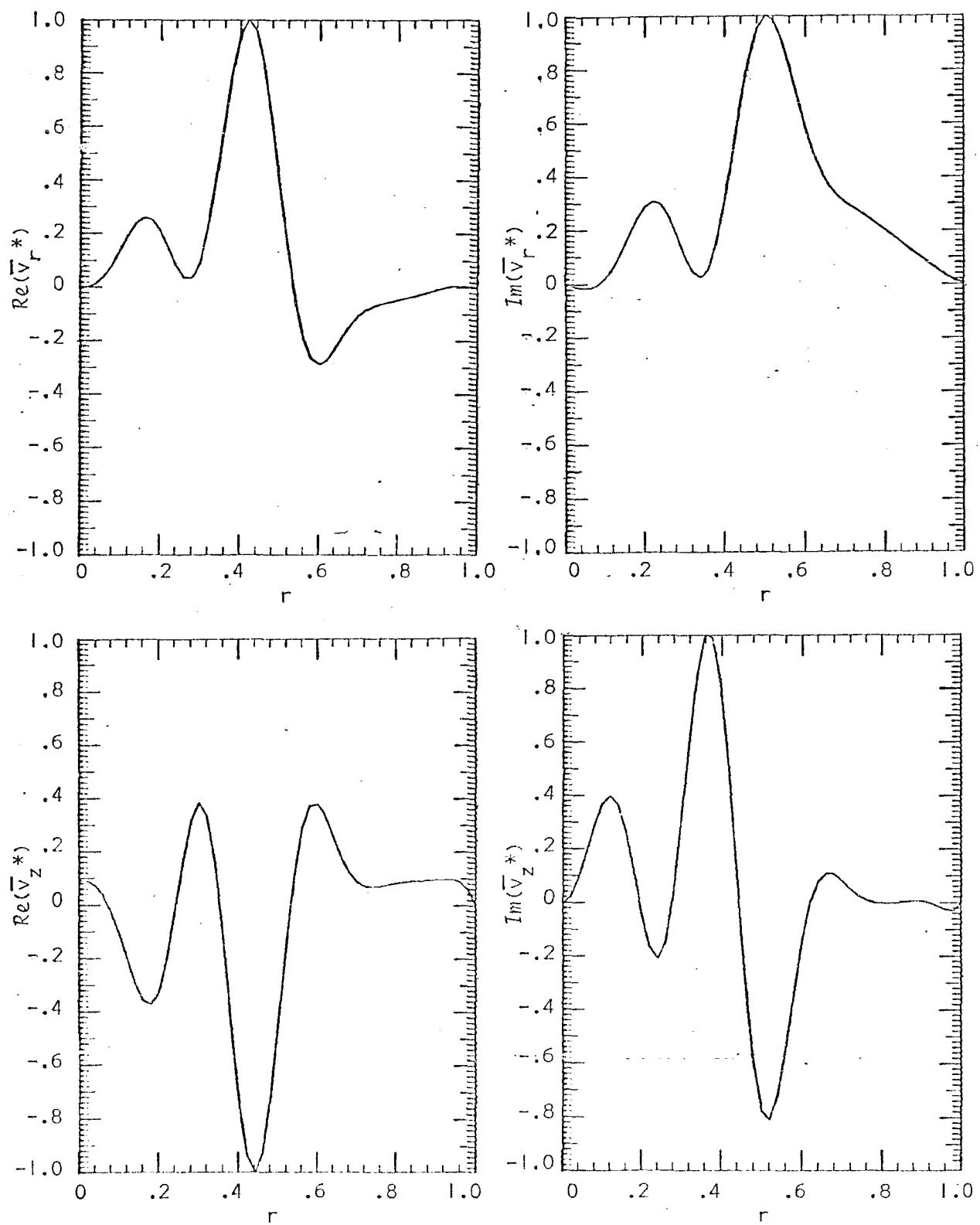
Figure 5.7  Eigenfunctions for an Axisymmetric Disturbance for m = 4
(R = 10,000, ω = 0.2)
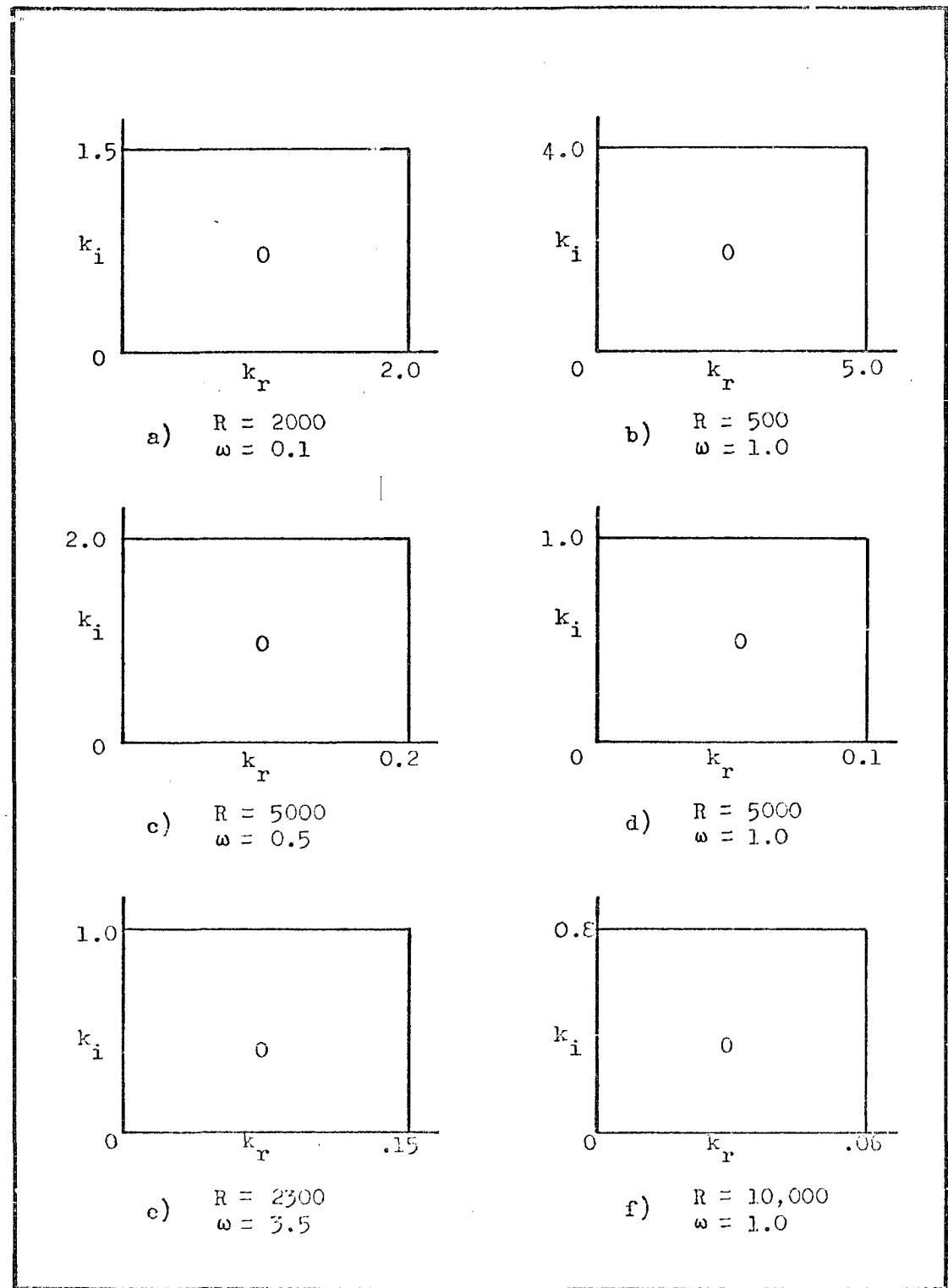
Figure 5.8. Number of Eigenvalues enclosed by some Regions of the k-plane for n = 1 .
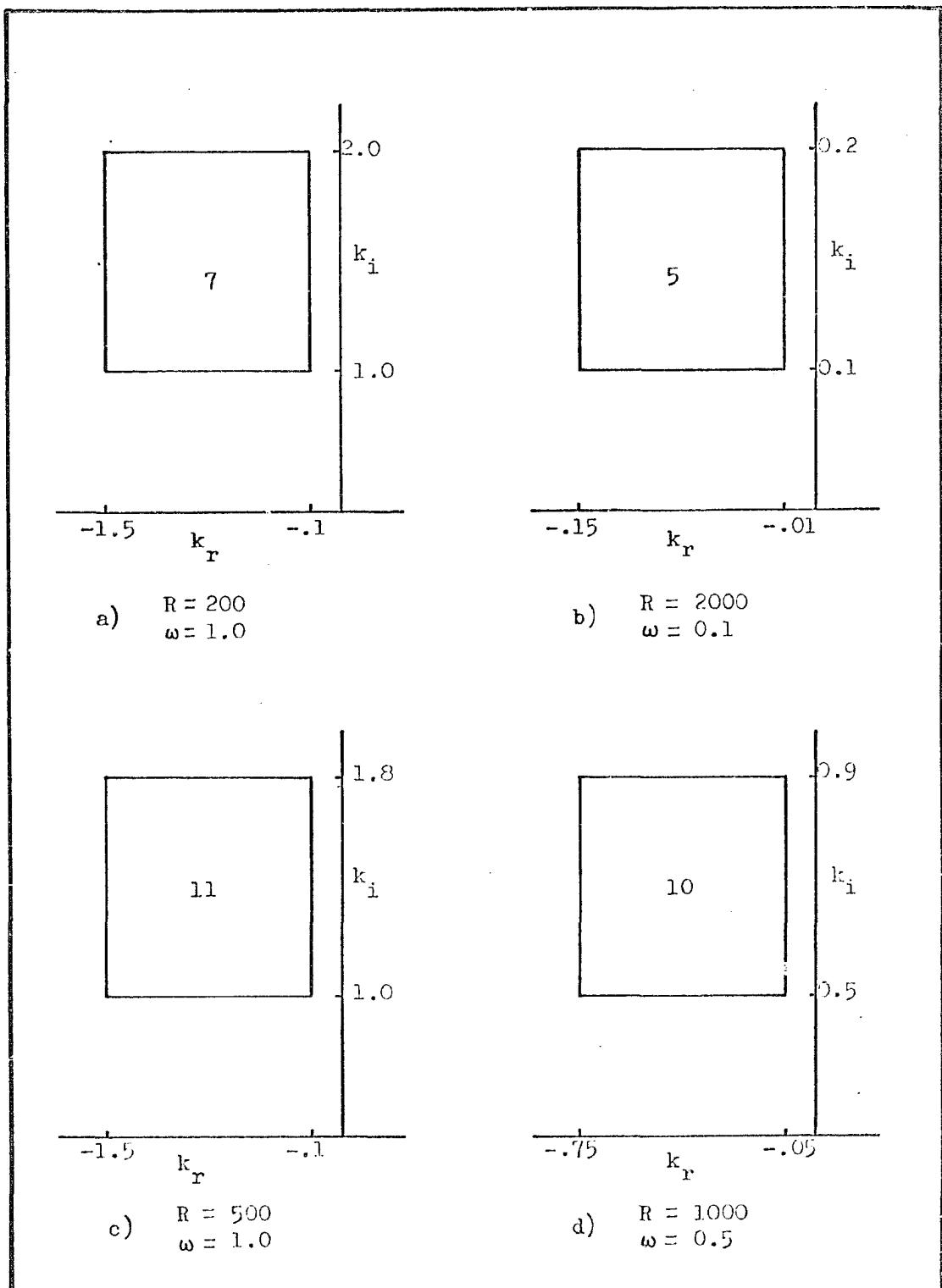
Figure 5.9. Number of Eigenvalues enclosed by some Regions of the k-plane for n = 1 .
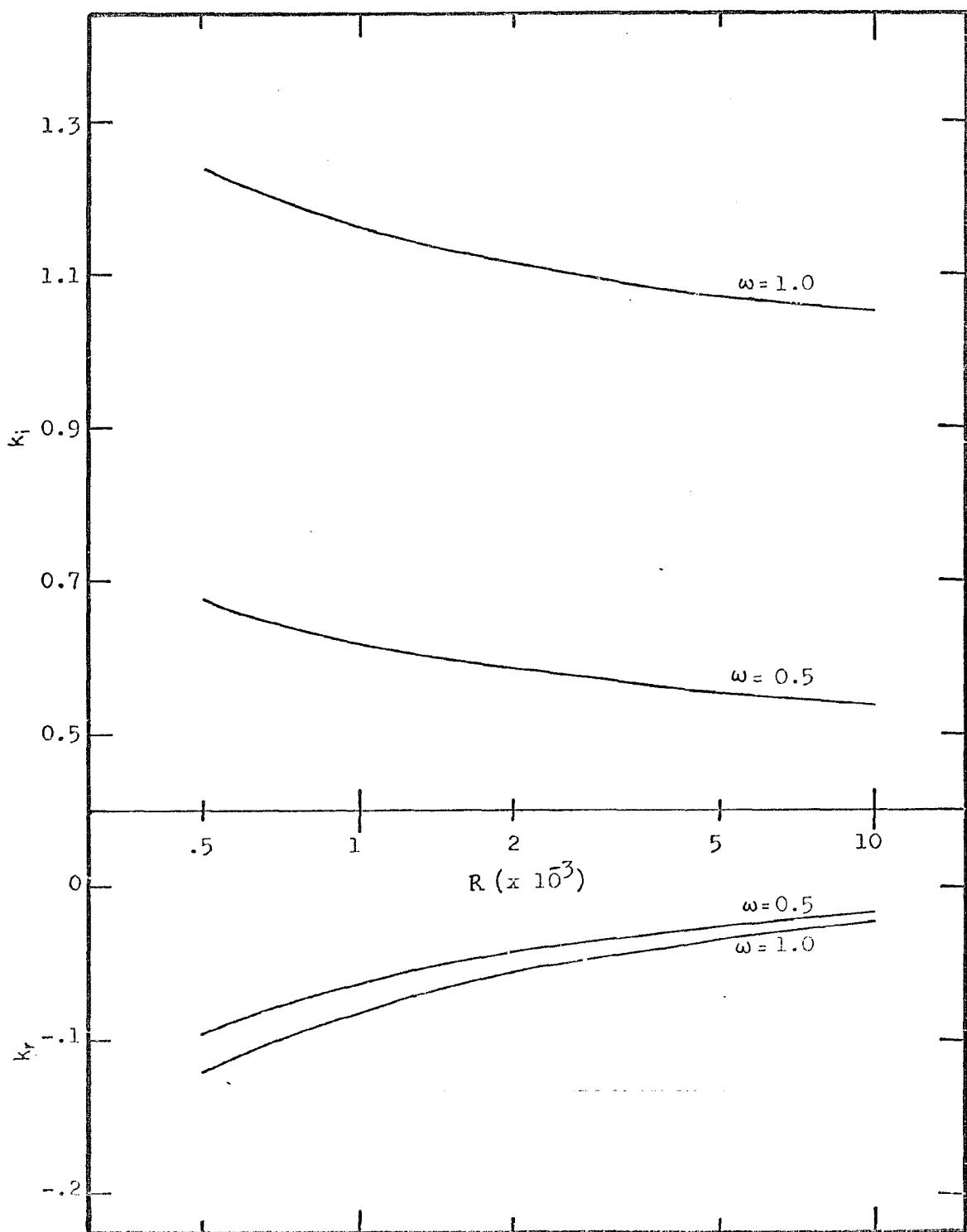
Figure 5.10.  Variation of $k_r$ and $k_i$ with R for the Least
Stable Mode for n = 1  .
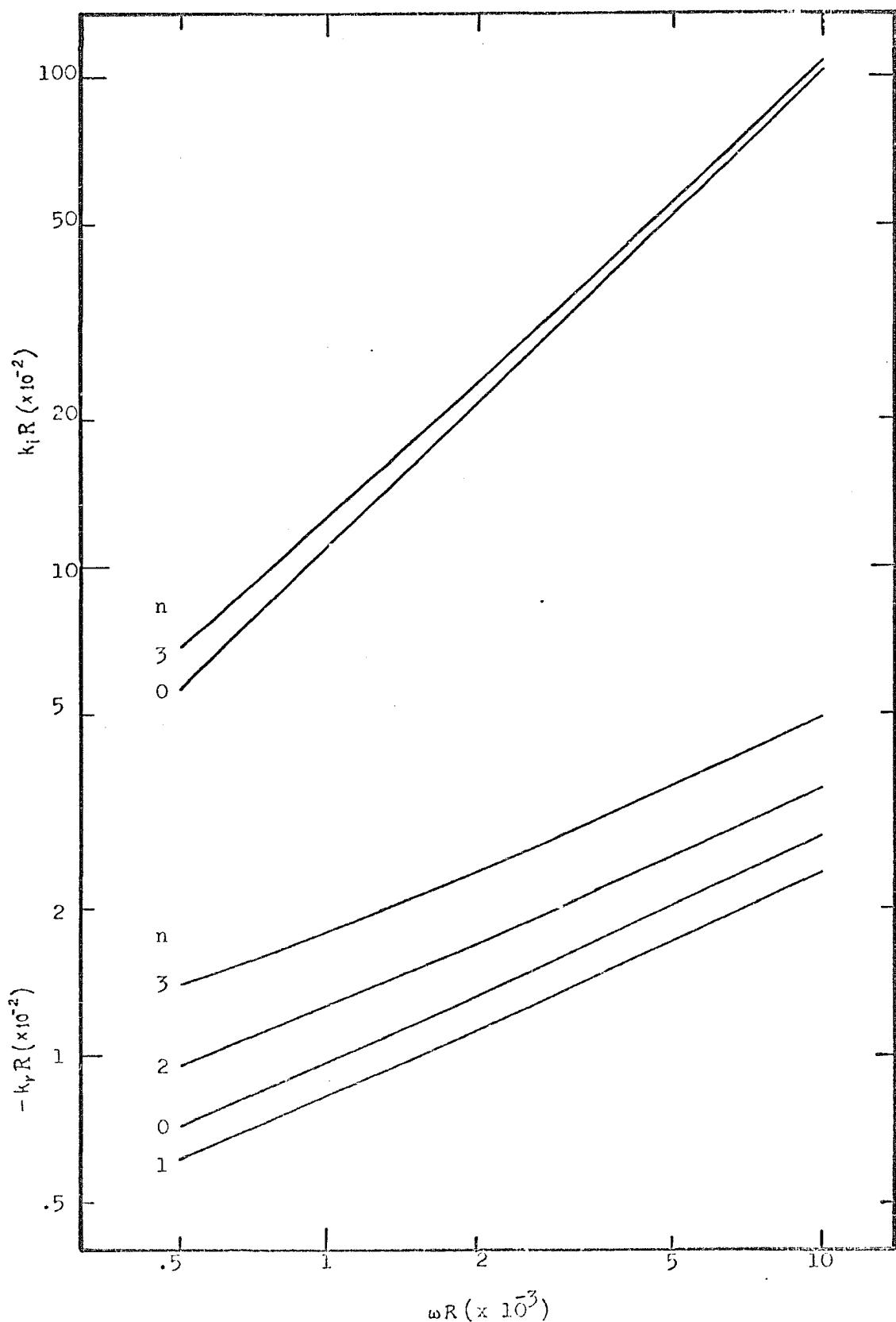
Figure 5.11. Variation of $k_r R$ and $k_i R$ with $\omega R$ for the Least Stable Mode for different values of n .
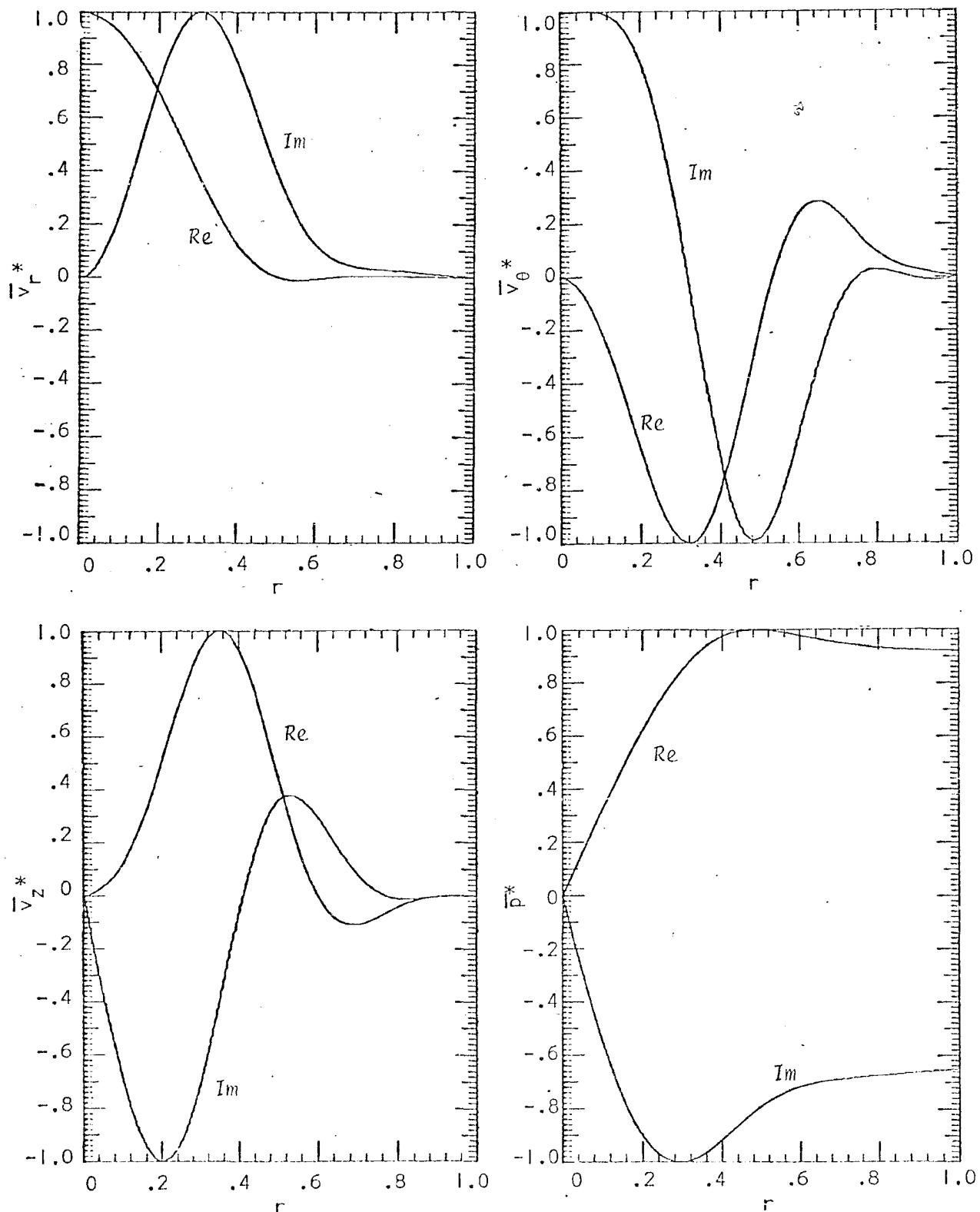
Figure 5.12  Eigenfunctions for the Least Stable Mode for n = 1
(R = 5000, ω = 0.1)
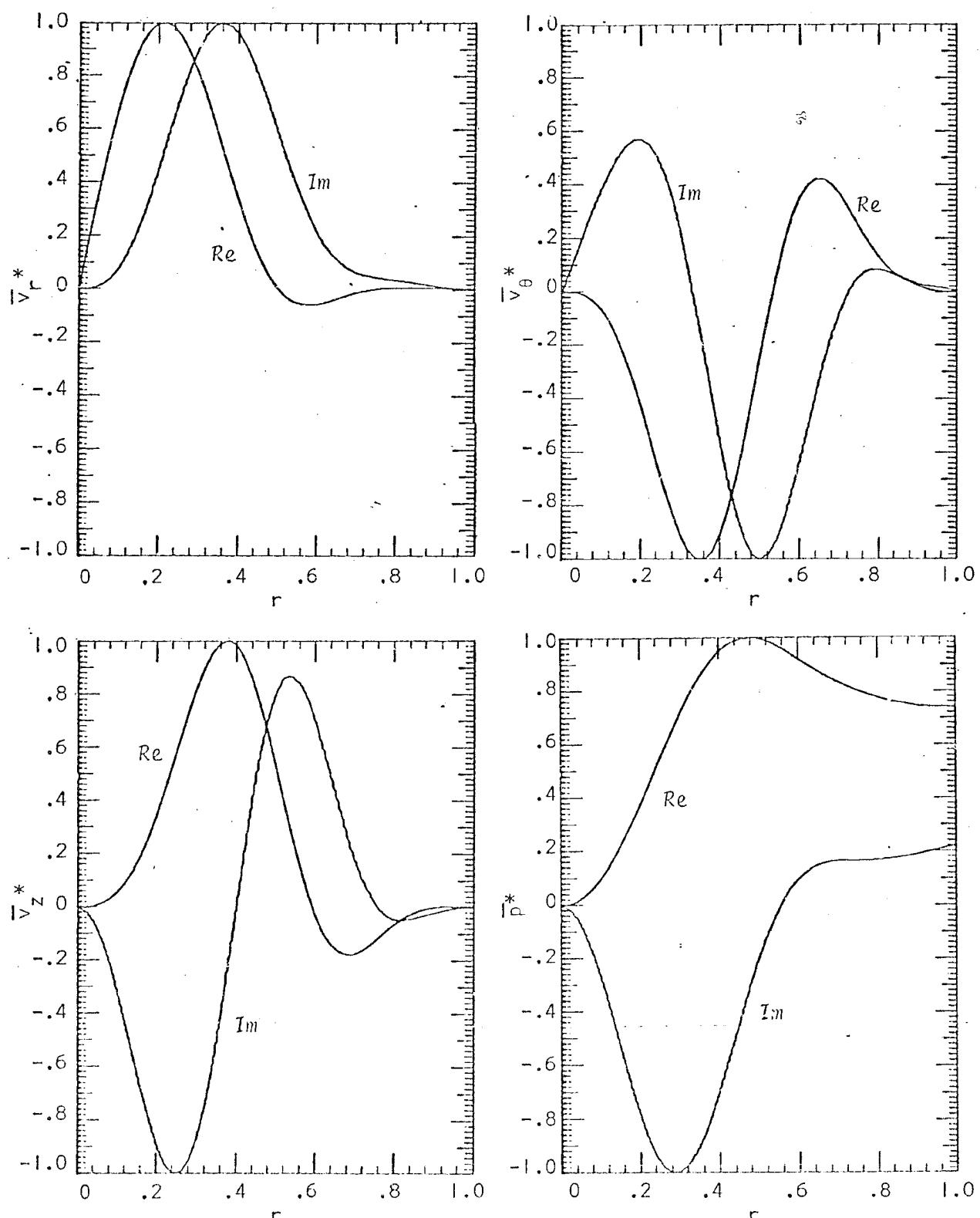
Figure 5.13  Eigenfunctions for the Least Stable Mode for n = 2
(R = 5000, ω = 0.1)
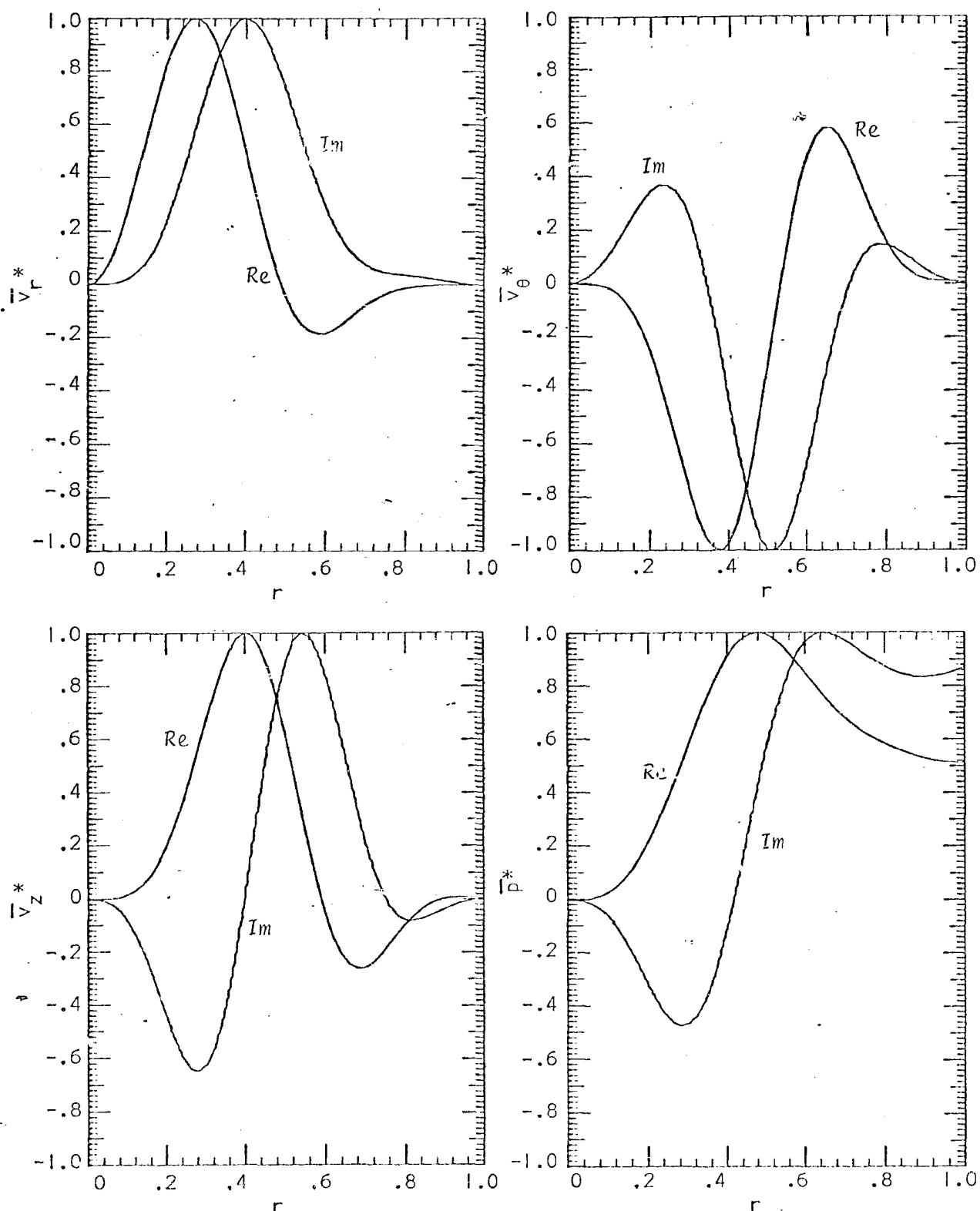
Figure 5.14  Eigenfunctions for the Least Stable Mode for n = 3
(R = 5000, ω = 0.1)
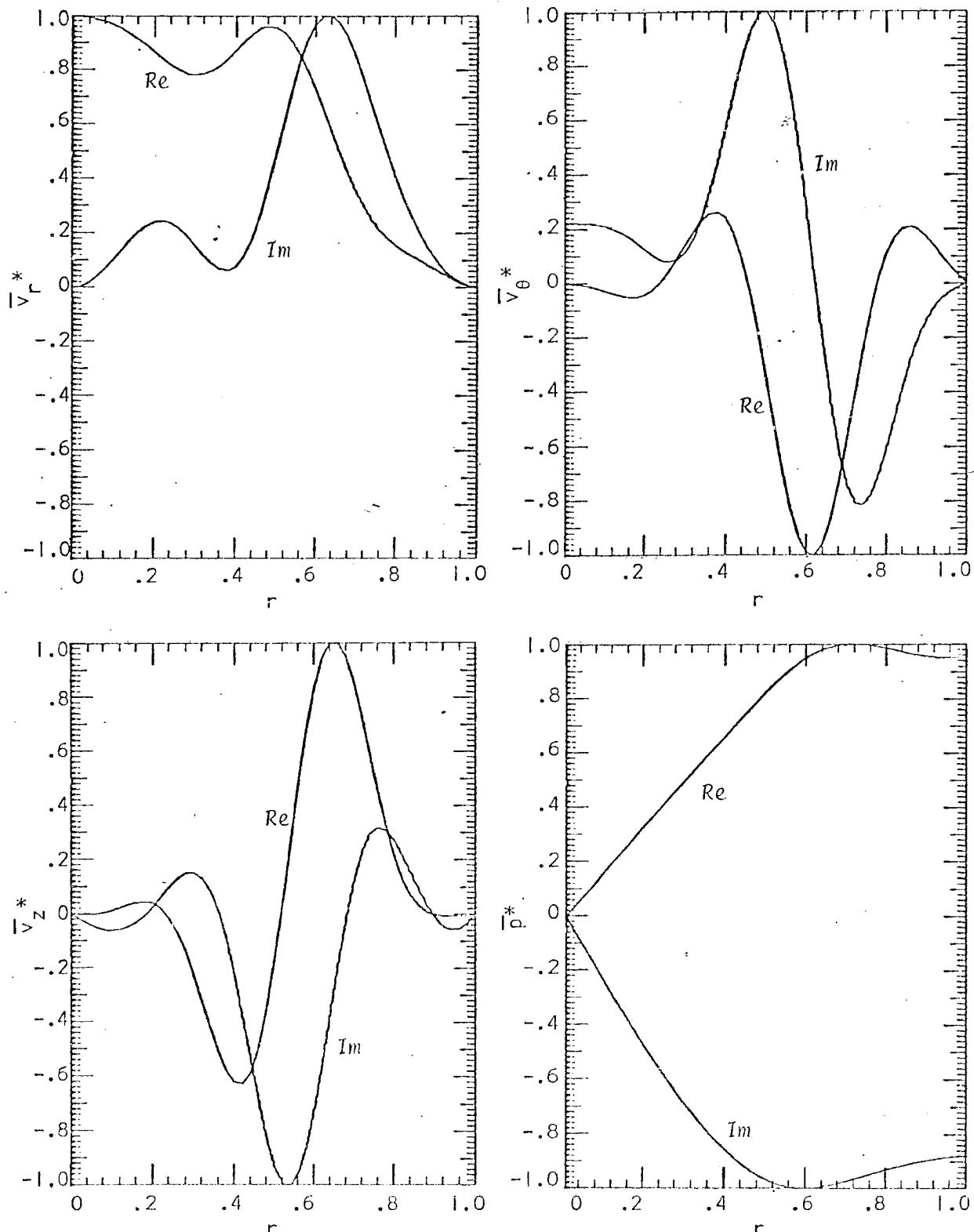
Figure 5.15   Eigenfunctions for the Fourth Least Stable Mode for n = 1
(R = 500, ω = 1.0)

## TABLE 5.5   EIGENVALUES FOR THE LEAST STABLE MODE

| n | R | $\omega$ | $k_r$ | $k_i$ | $\omega R$ | $k_r R$ | $k_i R$ |
|---|---|---|---|---|---|---|---|
| 0 | 200 | 1.00 | -.2306169 | 1.1974489 | 200 | -46.12 | 239.49 |
| 0 | 2000 | .10 | -.0220485 | .1201397 | 200 | -44.10 | 240.28 |
| 0 | 500 | 1.00 | -.1458821 | 1.1263924 | 500 | -72.94 | 563.20 |
| 0 | 1000 | .50 | -.0720160 | .5635374 | 500 | -72.02 | 563.54 |
| 0 | 5000 | .10 | -.0143442 | .1127291 | 500 | -71.72 | 563.65 |
| 0 | 2000 | .50 | -.0484902 | .5454682 | 1000 | -96.98 | 1090.94 |
| 0 | 2000 | 1.00 | -.0668253 | 1.0652629 | 2000 | -133.65 | 2130.53 |
| 0 | 10000 | .20 | -.0132537 | .2130853 | 2000 | -132.54 | 2130.85 |
| 0 | 5000 | .50 | -.0296603 | .5287794 | 2500 | -148.30 | 2643.90 |
| 0 | 5000 | 1.00 | -.0411276 | 1.0413187 | 5000 | -205.64 | 5206.59 |
| 0 | 10000 | .50 | -.0202941 | .5215319 | 5000 | -202.94 | 5215.32 |
| | | | | | | | |
| 1 | 200 | 1.00 | -.1927579 | 1.4080152 | 200 | -38.55 | 281.60 |
| 1 | 500 | .50 | -.0958004 | .6747065 | 250 | -47.90 | 337.35 |
| 1 | 500 | 1.00 | -.1209005 | 1.2355468 | 500 | -60.45 | 617.77 |
| 1 | 1000 | .50 | -.0623363 | .6179738 | 500 | -62.34 | 617.97 |
| 1 | 5000 | .10 | -.0125640 | .1235965 | 500 | -62.82 | 617.98 |
| 1 | 1000 | 1.00 | -.0818855 | 1.1626914 | 1000 | -81.89 | 1162.69 |
| 1 | 2000 | .50 | -.0416506 | .5814468 | 1000 | -83.30 | 1162.89 |
| 1 | 2000 | 1.00 | -.0559741 | 1.1130823 | 2000 | -111.95 | 2226.16 |
| 1 | 5000 | .50 | -.0250213 | .5504006 | 2500 | -125.11 | 2752.00 |
| 1 | 5000 | 1.00 | -.0342649 | 1.0704538 | 5000 | -171.32 | 5352.27 |
| 1 | 10000 | .50 | -.0172276 | .5352511 | 5000 | -172.28 | 5352.51 |
| 1 | 10000 | 1.00 | -.0238157 | 1.0494442 | 10000 | -238.16 | 10494.44 |
| | | | | | | | |
| 2 | 200 | 1.00 | -.3330430 | 1.5027363 | 200 | -66.61 | 300.55 |
| 2 | 2000 | .10 | -.0342364 | .1511539 | 200 | -68.47 | 302.31 |
| 2 | 1000 | .50 | -.0958640 | .6502761 | 500 | -95.86 | 650.28 |
| 2 | 5000 | .10 | -.0191875 | .1301243 | 500 | -95.94 | 650.62 |
| 2 | 2000 | .50 | -.0629062 | .6035592 | 1000 | -125.81 | 1207.12 |
| 2 | 2000 | 1.00 | -.0843488 | 1.1432984 | 2000 | -168.70 | 2286.60 |
| 2 | 5000 | .50 | -.0372121 | .5639535 | 2500 | -186.06 | 2819.77 |
| 2 | 5000 | 1.00 | -.0508443 | 1.0892089 | 5000 | -254.22 | 5446.04 |
| 2 | 10000 | 1.00 | -.0350749 | 1.0625678 | 10000 | -350.75 | 10625.68 |
| | | | | | | | |
| 3 | 2000 | .10 | -.0509292 | .1622041 | 200 | -101.86 | 324.41 |
| 3 | 1000 | .50 | -.1395855 | .6830121 | 500 | -139.59 | 683.01 |
| 3 | 5000 | .10 | -.0278959 | .1366924 | 500 | -139.48 | 683.46 |
| 3 | 1000 | 1.00 | -.1808304 | 1.2510464 | 1000 | -180.83 | 1251.05 |
| 3 | 2000 | 1.00 | -.1198719 | 1.1744056 | 2000 | -239.74 | 2348.81 |
| 3 | 5000 | 1.00 | -.0714672 | 1.1084881 | 5000 | -357.34 | 5542.44 |
| 3 | 10000 | 1.00 | -.0490261 | 1.0760490 | 10000 | -490.26 | 10760.49 |

TABLE 5.6    ALL EIGENVALUES INSIDE THE REGION

BOUNDED BY $-1.5 \leq k_r \leq 5.0$ and $0 \leq k_i \leq 4.0$

a)  R = 200 ,  $\omega$ = 1.0 ,  n = 1

| $k_r$ | $k_i$ |
|---|---|
| -.1927579 | 1.4080152 |
| -.3277873 | 1.5895616 |
| -.3965171 | 1.1164892 |
| -.7622516 | 1.2328901 |
| -.7645968 | 1.6558520 |
| -1.2432880 | 1.7950913 |
| -1.2521588 | 1.3054534 |

b)  R = 500 ,  $\omega$ = 1.0 ,  n = 1

| $k_r$ | $k_i$ |
|---|---|
| -.1209005 | 1.2355468 |
| -.2379789 | 1.0844600 |
| -.2935170 | 1.4025778 |
| -.3440662 | 1.7074794 |
| -.4322858 | 1.1800157 |
| -.5511817 | 1.5620836 |
| -.6615135 | 1.2722493 |
| -.8614424 | 1.6420581 |
| -.9292155 | 1.3561234 |
| -1.2188337 | 1.6920611 |
| -1.2410594 | 1.4212779 |

Chapter 6

CONCLUSIONS

It was observed that the two parameters, frequency $\omega$ and

Reynolds number R, could be combined into one, the product $\omega$R.

Thus, it was found that $\omega$R governs, in an approximate manner,

the values of kR.  This observation provided an obvious facility

of varying the product $\omega$R rather than $\omega$ and R separately.

The effect of Reynolds number, frequency, and the azimuthal

mode number n on the relative stability and wavelength of a mode

can be summarized as follows:

i) For a fixed value of $\omega$ and R, n = 1 is the least

stable mode of all.  In order of increasing stability,

the sequence for the values of n is 1,0,2,3,...

ii) For constant $\omega$ and R, the least stable mode (one for

which $|k_r|$ is minimum) has the largest wavelength

for n = 0.  The wavelength decreases only slightly

as n takes on higher integer values; the effect at

high values of $\omega$R being negligibly small.

iii) For a fixed n and $\omega$, the least stable mode becomes

less stable and has a larger wavelength as the

Reynolds number is increased.

iv) The effect of frequency for constant n and R is

similar to that in part (iii) above except that the

wavelength is almost doubled when the frequency is

reduced by a factor of 2.  Thus the modal phase

velocity remains almost constant.

v) For a fixed value of n, the least stable mode

tends to be neutrally stable, that is $k_r \to 0$

as the frequency of the disturbance is decreased

and the Reynolds number of flow is increased

indefinitely.

vi) For a constant n, $\omega$ and R, the mode shape becomes

more oscillatory as the mode becomes more stable,

that is as $|k_r|$ increases.

On the basis of theoretical results, it is concluded that up

to a Reynolds number of 10,000, the pipe Poiseuille flow is

spatially stable to both axisymmetric and non-axisymmetric

disturbances that have an infinitesimal amplitude.  For the

axisymmetric disturbance, the results are in agreement with

previous theoretical and experimental results [17, 14].  For the

convective stability of non-axisymmetric disturbances, no

theoretical results are available; this being the first such

study.  The experimentally observed instability [15, 16] for non-

axisymmetric disturbances is, however, in contradiction with the

present conclusion.  Bhat [16] experimentally found instability

for a Reynolds number of 2300 and a frequency of 3.5 (region in

part (e) of Figure 5.8).  Nevertheless, both Bhat [16] and Lessen

et al. [15] failed to specify the amplitude of the disturbance

that was employed.  In view of the experimental fact that Poiseuille

flow in a pipe is stable to infinitesimal axisymmetric disturbances but unstable to finite axisymmetric disturbances [27], it is felt that nonlinear effects of the disturbance, which have been neglected in the present investigation, must be considered in order to predict the experimental instability for non-axisymmetric disturbances.

Before concluding this study, it is appropriate to point out that the numerical technique developed here is highly versatile and can be easily extended to analyze the stability of a compressible fluid in an elastic tube.

## Appendix I

## STEP-BY-STEP INTEGRATION TECHNIQUES

The two methods used for the step-by-step integration of the coupled differential equations (equations (3.33) or (3.71)) are described below.

The fourth order Runge-Kutta method is best given in a tabular form [23]. Tables AI-I and AI-2 exhibit the appropriate formulae for first and second order differential equations.

### Table AI-I

Runge-Kutta Scheme for Differential Equations of the First Order; $Dy = f(x,y)$

| $x$ | $y$ | $K_\ell = h\, f(x,y)$ | Correction |
|---|---|---|---|
| $x_0$ | $y_0$ | $K_1$ | |
| $x_0 + \frac{1}{2} h$ | $y_0 + \frac{1}{2} K_1$ | $K_2$ | |
| $x_0 + \frac{1}{2} h$ | $y_0 + \frac{1}{2} K_2$ | $K_3$ | $K = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$ |
| $x_0 + h$ | $y_0 + K_3$ | $K_4$ | |
| $x_1 = x_0 + h$ | $y_1 = y_0 + K$ | | |

Table AI-2

Runge-Kutta Scheme for Differential Equations of the
Second Order; $D^2y = f(x,y,Dy)$

| $x$ | $y$ | $Dy$ | $K_{\ell} = \frac{h^2}{2} f(x,y,Dy)$ | Correction |
|---|---|---|---|---|
| $x_o$ | $y_o$ | $Dy_o$ | $K_1$ | |
| $x_o + \frac{1}{2} h$ | $y_o + \frac{1}{2} h\, Dy_o + \frac{1}{4} K_1$ | $Dy_o + K_1/h$ | $K_2$ | $K = \frac{1}{3}(K_1 + K_2 + K_3)$ |
| $x_o + \frac{1}{2} h$ | $y_o + \frac{1}{2} h\, Dy_o + \frac{1}{4} K_1$ | $Dy_o + K_2/h$ | $K_3$ | |
| $x_o + h$ | $y_o + h\, Dy_o + K_3$ | $Dy_o + 2K_3/h$ | $K_4$ | $K' = \frac{1}{3}(K_1 + 2K_2 + 2K_3 + K_4)$ |
| $x_1 = x_o + h$ | $y_1 = y_o + h\, Dy_o + K$ | $Dy_1 = Dy_o + K'/h$ | | |

In these tables, h is the step size, D is the operator $\frac{d}{dx}$, and $x_o$, $y_o$ and $Dy_o$ are the values known at the starting point of the integration step. The procedure is to calculate $K_1$ first and then go to the first column in the second row. After calculating $K_2$ by traversing the second row from left to right, we go to the first column in the third row, and so on till $K_4$ is calculated. The factors in the 'correction' column are then calculated before we go to the fifth row; $x_1$, $y_1$ and $Dy_1$ are the values at the end of the present step, that is they are the starting point values for the next integration step. The whole procedure is, therefore, repeated for successive steps.

Out of the many predictor-corrector methods [24], one that is relatively more stable and less complicated to use than the others is the fourth order Adams-Dashforth-Moulton system. For a first order differential equation

$$Dy = f(x,y) \; , \tag{A1-1}$$

the essential equations for this method are [24]

$$y_{1P} = y_o + \frac{h}{24} (55Dy_o - 59Dy_{-1} + 37Dy_{-2} - 9Dy_{-3}) \tag{A1-2}$$

and $y_{1C} = y_o + \frac{h}{24} (9Dy_1 + 19Dy_o - 5Dy_{-1} + Dy_{-2})$ (A1-3)

where  D - the operator $\frac{d}{dx}$

$y_{1P}$ - predicted value of y at the end of the integration step

$y_{1C}$ - corrected value of y at the end of the integration step

$y_o$, $Dy_o$ - values of y and Dy at the beginning of the step

h - step size

and where negative subscripts refer to the values of Dy at previous steps. This method, therefore, requires that values of Dy be known at three previous points in addition to the one at the starting point of the step.

The procedure is to find $y_{1P}$ from equation (AI-2) and use this value in equation (AI-I) to find (for the first time) a tentative value of the derivative $Dy_1$ at $x_1$. Equation (AI-3) is then used to calculate $y_{1C}$. Before proceeding further, it is advisable to check the error involved in $y_{1C}$. This error can be estimated by [24]

$$\varepsilon = \frac{|y_{1C} - y_{1P}|}{20} \qquad\qquad (AI-4)$$

If $\varepsilon$ is larger than a preassigned error bound, $y_{1C}$ is pumped into equation (AI-I) to get a new value for $Dy_1$ which yields yet another value for $y_{1C}$ from equation (AI-3). The error is again calculated between the two values of $y_{1C}$ and the process is repeated until $\varepsilon$ is less than the preassigned error limit. In fact, if more than 2 iterations of the corrector (equation (AI-3)) are required, it is advisable to cut the step size in half, and start forward again. Once the error is within the acceptable limit, $y_{1C}$ is pumped through the differential equation to get a value of the derivative $Dy_1$ at $x_1$ that is based on the current value of $y_{1C}$ since that is

going to be accepted as the correct value. The process is repeated for successive steps.

A second order differential equation can always be decomposed into two first order equations by defining $z \equiv Dy$. A separate treatment is, therefore, unnecessary for a second order differential equation. For the sake of clarity, the scheme used for step-by-step integration of the equations (3.71) is given below

i) Calculate $(\bar{v}_r, D\bar{v}_r, D^2\bar{v}_r)$, $(\bar{v}_\theta, D\bar{v}_\theta, D^2\bar{v}_\theta)$, $(\bar{v}_z, D\bar{v}_z, D^2\bar{v}_z)$ and $(\bar{p}, D\bar{p})$ at four values of $r$, namely $r_s$, $(r_s - h)$, $(r_s - 2h)$ and $(r_s - 3h)$ by means of the series solution. Let $\ell = 1$.

ii) Let $r_\ell = (r_s + \ell h)$. Calculate the predicted values $D\bar{v}_r$, $D\bar{v}_\theta$, $D\bar{v}_z$ and $\bar{p}$ at $r = r_\ell$ by means of equation (AI-2).

iii) Calculate the values of $\bar{v}_r$, $\bar{v}_\theta$ and $\bar{v}_z$ at $r = r_\ell$ by means of equation (AI-3).

iv) Using equations (3.71), calculate the values of $D^2\bar{v}_r$, $D^2\bar{v}_\theta$, $D^2\bar{v}_z$ and $D\bar{p}$ at $r = r_\ell$.

v) Recalculate $D\bar{v}_r$, $D\bar{v}_\theta$, $D\bar{v}_z$ and $\bar{p}$ at $r = r_\ell$ by means of equation (AI-3).

vi) Using equation (AI-4), calculate the errors involved in the most recent values of $D\bar{v}_r$, $D\bar{v}_\theta$, $D\bar{v}_z$ and $\bar{p}$ for $r = r_\ell$. If the errors are acceptable, proceed ahead; otherwise repeat steps (iii) to (vi) until the errors are acceptable.

vii) Set $\ell = \ell + 1$. Repeat steps (ii) to (vi) until

$r_\ell = 1$.

With the help of the above scheme, it should be easy to understand subroutine PREDCT listed in Appendix II.

Appendix II

COMPUTER PROGRAM


The numerical solution of the disturbance equations was

carried out by the application of Fortran V language to the

equations developed in Chapter 3.  In its final form, the computer

program consists of a main program and several subroutines.  In

view of the many differences between the analyses for axisymmetric

and non-axisymmetric disturbances, it is imperative that different

programs be used for the two cases.  The subroutines listed later

in this Appendix help to provide programs for the following:

i)  Determination of eigenvalues for a non-axisymmetric

   disturbance:

      The essential subroutines are SERIES, SECANT, EIGENF,

   NORMFN, NORMAB, MULDPC, DIVDPC, ABSDPC, and PLTMOD.  In

   addition, subroutines mentioned in either (a) or (b)

   below are required.

      a) When the Runge-Kutta method is used for the step-by

         step integration - subroutines EFSRRK and RUNGEK.

      b) When the predictor-corrector method is used for the

         step-by-step integration - subroutines EFSRPC and
         PREDCT.

ii) Determination of eigenvalues for an axisymmetric

   disturbance:

The subroutines required are ASERIS, AEFSRK, ARUNGE, ASECAN, AEIGNF, NORMFN, NORMAB, MULDPC, DIVDPC, ABSDPC, and PLTMOD.

In each case, the main program provides input of the necessary control constants, of some or all boundary conditions at $r = 0$, and of the known series constants. The comment cards included in the source listing explain clearly the inputs to the particular subroutine. In order to keep the dependence of each series for an eigenfunction on the two or three independent solutions, the necessary variables are subscripted by an index that assumes values 1 and 2 for axisymmetric disturbance, and values 1, 2 and 3 for a non-axisymmetric disturbance. More explanation can be found in subroutines SERIES and ASERIS.

It should be pointed out that for reasons of accuracy, it is imperative that variables be used in double precision form. Although the double precision complex arithmetic is not available on the Univac 1108 system it was created out of necessity. Thus, every double precision complex variable was expressed in terms of two double precision variables which represent the real and imaginary parts of the complex quantity.

```
C
            SUBROUTINE SERIES (RE,W,KR,KI,NU,AN,EPS,MNTS,RS,ANN,C,NTS,AXR,AXI)
C
C* THIS IS THE FIRST SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR A NON-AXI
C SYMMETRIC DISTURBANCE *
C
C FOR A PREASSIGNED VALUE OF THE RADIUS UPTO WHICH THE SERIES SOLUTION IS
C CARRIED, THIS SUBROUTINE EVALUATES THE TERMS IN THE SERIES EXPANSION OF
C THE EIGENFUNCTIONS (SEE SECTION 3.2.1). ALL THE SERIES ARE TERMINATED IF
C THE ABSOLUTE VALUE OF THE RATIO OF THE LAST TERM RETAINED TO THE PARTIAL
C SUM FOR ALL SERIES DOES NOT EXCEED A PREASSIGNED EPSILON ('EPS' HERE)
C
C VARIABLES TO BE SUPPLIED BY THE MAIN PROGRAMME ARE RE,W,KR,KI,NU,AN,EPS,
C MNTS,RS,ANN,C
C    RE - REYNOLDS NUMBER
C    W - DIMENSIONLESS FREQUENCY
C    KR,KI - REAL AND IMAGINARY PARTS OF THE DIMENSIONLESS WAVE NUMBER K
C    NU - NUMBER OF TERMS IN THE SERIES EXPANSION OF VZ (EQN. (3.22))
C    AN = N=1,2,3,...........
C    ANN = N+1
C    EPS - ALREADY EXPLAINED
C    MNTS - MAX. NUMBER OF TERMS ALLOWED IN THE SERIES EXPANSION, SHOULD NOT
C              EXCEED 20 FOR THE DIMENSIONS USED HERE
C    RS - THE VALUE OF R UPTO WHICH THE SERIES SOLUTION IS USED
C    C - ARRAY OF NU ELEMENTS FOR THE COEFFICIENTS IN THE SERIES EXPANSION
C        OF VZ (EQN. (3.22))
C    NTS - ACTUAL NUMBER OF TERMS USED IN THE SERIES EXPANSION FOR EIGEN-
C          FUNCTIONS. IT IS AN OUTPUT OF THE SUBROUTINE. (NTS.LE.MNTS)
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C BR(J),BI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF B'S(EQN. (3.23))
C
C GR(L,J),GI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF G'S. THE
C FIRST SUBSCRIPT INDICATES THE TERM NUMBER IN THE SERIES FOR THE EIGENFUNC
C TION G(R)(SEE EQN.(3.56)), AND THE SECOND SUBSCRIPT REFERS TO THE COEFFI-
C CIENT OF EITHER ONE OF THE THREE INDEPENDENT CONSTANTS (G1,U1,P1) FOR
C EQUATIONS OF THE FORM OF EQN.(3.70). FOR EXAMPLE, G3 CAN BE EXPRESSED AS
C        G3 = G31*G1 + G32*U1 + G33*P1
C    THEN  GR(3,1) = REAL PART OF G31
C          GR(3,2) = REAL PART OF G32
C    AND   GR(3,3) = REAL PART OF G33
C SIMILARLY FOR THE ARRAYS FOR FR,FI,UR,UI,PR,PI
C
C ACCORDING TO THE ABOVE SCHEME, THE FOLLOWING CONSTANTS ARE SET IN THE
C MAIN PROGRAMME
C    GR(1,2) = GI(1,2) = GR(1,3) = GI(1,3) = 0.0
C    UR(1,1) = UI(1,1) = UR(1,3) = UI(1,3) = 0.0
C    PR(1,1) = PI(1,1) = PR(1,2) = PI(1,2) = 0.0
C    GI(1,1) = UI(1,2) = PI(1,3) = 0.0
C    AND GR(1,1) = UR(1,2) = PR(1,3) =  SOME NON-ZERO CONSTANT
C
C ************************************************************************
C**** FROM NOW ONWARDS, THE SUBSCRIPT J WILL ALWAYS REFER TO ANY ONE OF THE
C THREE INDEPENDENT SOLUTIONS FOR THE EIGENFUNCTIONS. THUS, J = 1,2,3
C ************************************************************************
C
C GSR(J),GSI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE THREE INDE
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C PENDENT SUMS FOR THE EIGENFUNCTION G(R) EVALUATED AT R=RS.  SIMILARLY FOR
C FSR(J),FSI(J),USR(J),USI(J),PSR(J),PSI(J)
C
        DOUBLE PRECISION RE,W,KR,KI,AN,EPS,RS,ANN,BR(2),BI(2),AL,AI,AXR,AY
   11,SUR,SUI,SGR,SGI,SFR,SFI,FSR(3),FSI(3),GSR(3),GSI(3),USR(3),USI(3
   2),PSR(3),PSI(3),ARF(3),ARG(3),ARU(3),ARP(3),RP,ABSDPC,KSR,KSI
        DOUBLE PRECISION C(NU),GR(20,3),GI(20,3),FR(20,3),FI(20,3),UR(20,3
   3),UI(20,3),PR(20,3),PI(20,3)
        COMMON /AREA1/ GR,GI,FR,FI,UR,UI,PR,PI
        KSR=KR*KR-KI*KI
        KSI=2.0D0*KR*KI
        BR(1)=KSR-RE*KR*C(1)
        BI(1)=KSI+RE*(W-KI*C(1))
        DO 20 J=2,NU
        BR(J)=-KR*RE*C(J)
   20 BI(J)=-KI*RE*C(J)
C CALCULATING THE FIRST TERM IN THE SERIES FOR F(R)-SEE EQN.(3.62)
        DO 30 J=1,3
        CALL MULDPC (BR(1),BI(1),GR(1,J),GI(1,J),AXR,AXI)
        CALL MULDPC(KR,KI,UR(1,J),UI(1,J),KSR,KSI)
        FR(1,J)=(AXR/(4.D0*AN)-PR(1,J)*RE/2.D0-KSR)/ANN
   30 FI(1,J)=(AXI/(4.D0*AN)-PI(1,J)*RE/2.D0-KSI)/ANN
        DO 21 J=1,3
        FSR(J)=FR(1,J)
        FSI(J)=FI(1,J)
        GSR(J)=GR(1,J)
        GSI(J)=GI(1,J)
        USR(J)=UR(1,J)
        USI(J)=UI(1,J)
        PSR(J)=PR(1,J)
   21 PSI(J)=PI(1,J)
        L=0
   90 L=L+1
        AL=L
        DO 60 J=1,3
        SUR=0.0D0
        SUI=0.0D0
        SGR=0.0D0
        SGI=0.0D0
        SFR=0.0D0
        SFI=0.0D0
        DO 40 I=1,NU
        LL=L+1-I
        IF(LL.LE.0) GO TO 40
C SUR,SUI - REAL AND IMAGINARY PARTS OF THE FIRST SUMMATION IN EQN.(3.67)
        CALL MULDPC (BR(I),BI(I),UR(LL,J),UI(LL,J),AXR,AXI)
        SUR=SUR+AXR
        SUI=SUI+AXI
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SUMMATION IN EQN.(3.66)
        CALL MULDPC (BR(I),BI(I),GR(LL,J),GI(LL,J),AXR,AXI)
        SGR=SGR+AXR
        SGI=SGI+AXI
C SFR,SFI - REAL AND IMAGINARY PARTS OF THE SUMMATION IN EQN.(3.69)
        CALL MULDPC (BR(I),BI(I),FR(LL,J),FI(LL,J),AXR,AXI)
        SFR=SFR+AXR
        SFI=SFI+AXI
   40 CONTINUE
C CALCULATING THE G'S IN THE SERIES FOR G(R) - SEE EQN.(3.66)
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      GR(L+1,J)=(RE*PR(L,J)-SGR/(2.D0*(AN+AL-1.D0)))/(2.D0*AL)
      GI(L+1,J)=(RE*PI(L,J)-SGI/(2.D0*(AN+AL-1.D0)))/(2.D0*AL)
      SGR=0.0D0
      SGI=0.0D0
      DO 50 I=2,NU
      LL=L+1-I
      AI=I-1
      IF(LL.LE.0) GO TO 70
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SECOND SUMMATION IN EQN.(3.67)
      SGR= SGR+AI*RE*C(I)*FR(LL,J)
      SGI= SGI+AI*RE*C(I)*FI(LL,J)
  70  LL=LL+1
      IF(LL.LE.0) GO TO 50
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SECOND AND THIRD SUMMATIONS IN
C EQUATION (3.67)
      SGR= SGR+AI*RE*C(I)*GR(LL,J)
      SGI= SGI+AI*RE*C(I)*GI(LL,J)
  50  CONTINUE
C CALCULATING THE U'S IN THE SERIES FOR VZ(R) - SEE EQN.(3.67)
      CALL MULDPC (KR ,KI ,PR(L,J),PI(L,J),AXR,AXI)
      UR(L+1,J)=(AXR*RE-SUR+SGR)/(4.D0*AL*(AN+AL))
      UI(L+1,J)=(AXI*RE-SUI+SGI)/(4.D0*AL*(AN+AL))
      SGR=0.0D0
      SGI=0.0D0
      DO 80 I=1,NU
      LL=L+2-I
      IF(LL.LE.0) GO TO 80
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SECOND SUMMATION IN EQN.(3.68)
      CALL MULDPC (BR(I),BI(I),GR(LL,J),GI(LL,J),AXR,AXI)
      SGR=SGR+AXR
      SGI=SGI+AXI
  80  CONTINUE
C CALCULATING THE P'S IN THE SERIES FOR P(R) - SEE EQN.(3.68)
      CALL MULDPC(KR,KI,UR(L+1,J),UI(L+1,J),AXR,AXI)
      PR(L+1,J)=( -AXR      +(SFR/AL+SGR/(AL+AN))/4.D0)/RE
      PI(L+1,J)=( -AXI      +(SFI/AL+SGI/(AL+AN))/4.D0)/RE
C CALCULATING THE F'S IN THE SERIES FOR F(R) - SEE EQN.(3.69)
      FR(L+1,J)=(RE*PR(L+1,J)-SFR/(2.D0*AL))/(2.D0*(ANN+AL))
  60  FI(L+1,J)=(RE*PI(L+1,J)-SFI/(2.D0*AL))/(2.D0*(ANN+AL))
      IF(L.EQ.1) GO TO 90
C CALCULATING THE PARTIAL SUM OF THE SERIES FOR THE EIGENFUNCTIONS
C EVALUATED AT R=RS
      RP=RS**(2*(L-1))
      DO 35 J=1,3
      FSR(J)=FSR(J)+FR(L,J)*RP
      FSI(J)=FSI(J)+FI(L,J)*RP
      GSR(J)=GSR(J)+GR(L,J)*RP
      GSI(J)=GSI(J)+GI(L,J)*RP
      USR(J)=USR(J)+UR(L,J)*RP
      USI(J)=USI(J)+UI(L,J)*RP
      PSR(J)=PSR(J)+PR(L,J)*RP
  35  PSI(J)=PSI(J)+PI(L,J)*RP
      NTS=L+1
C FINDING THE ABSOLUTE VALUE OF THE RATIOS OF THE LAST TERMS IN THE SERIES
C (FOR THE THREE INDEPENDENT SOLUTIONS) TO THE PARTIAL SUM. ARF(J) TO ARP(
C ARE THE ARRAYS TO STORE THESE RATIOS
      RP=RS**(2*L)
      DO 25 J=1,3
```

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      IF(DABS(FSR(J)).LT.1.0D-70 .AND. DABS(FSI(J)).LT.1.0D-70) GO TO 61
      CALL DIVDPC (FR(NTS,J),FI(NTS,J),FSR(J),FSI(J),AXR,AXI)
      ARF(J)=ABSDPC(AXR,AXI)*RP
      GO TO 62
   61 ARF(J)=0.0D0
   62 IF(DABS(GSR(J)).LT.1.0D-70 .AND. DABS(GSI(J)).LT.1.0D-70) GO TO 63
      CALL DIVDPC (GR(NTS,J),GI(NTS,J),GSR(J),GSI(J),AXR,AXI)
      ARG(J)=ABSDPC(AXR,AXI)*RP
      GO TO 64
   63 ARG(J)=0.0D0
   64 IF(DABS(USR(J)).LT.1.0D-70 .AND. DABS(USI(J)).LT.1.0D-70) GO TO 66
      CALL DIVDPC (UR(NTS,J),UI(NTS,J),USR(J),USI(J),AXR,AXI)
      ARU(J)=ABSDPC(AXR,AXI)*RP   .
      GO TO 67
   66 ARU(J)=0.0D0
   67 IF(DABS(PSR(J)).LT.1.0D-70 .AND. DABS(PSI(J)).LT.1.0D-70) GO TO 68
      CALL DIVDPC (PR(NTS,J),PI(NTS,J),PSR(J),PSI(J),AXR,AXI)
      ARP(J)=ABSDPC(AXR,AXI)*RP
      GO TO 25
   68 ARP(J)=0.0D0
   25 CONTINUE
C AXR - MAXIMUM VALUE OF THE ABOVE RATIOS
      AXR=0.0D0
      DO 45 J=1,3
      AXR=DMAX1(AXR,ARF(J))
      AXR=DMAX1(AXR,ARG(J))
      AXR=DMAX1(AXR,ARU(J))
   45 AXR=DMAX1(AXR,ARP(J))
C AXI - LARGEST TERM (EITHER ITS REAL OR IMAGINARY PART) IN ALL THE SERIES
      AXI  =0.0D0
      DO 54 J=1,3
      AXI  =DMAX1(AXI   ,DABS(FR(NTS,J)))
      AXI  =DMAX1(AXI   ,DABS(FI(NTS,J)))
      AXI  =DMAX1(AXI   ,DABS(GR(NTS,J)))
      AXI  =DMAX1(AXI   ,DABS(GI(NTS,J)))
      AXI  =DMAX1(AXI   ,DABS(UR(NTS,J)))
      AXI  =DMAX1(AXI   ,DABS(UI(NTS,J)))
      AXI  =DMAX1(AXI   ,DABS(PR(NTS,J)))
   54 AXI  =DMAX1(AXI   ,DABS(PI(NTS,J)))
C ALL SERIES ARE TERMINATED IF EITHER AXR.LE.EPS OR IF AXI.GT.1.D70.  THE
C LATTER CRITERIA, THOUGH NEVER FOUND TO BE IMPORTANT DURING THE PRESENT
C INVESTIGATION, IS INCLUDED HERE TO AVOID VERY LARGE VALUES FOR THE TERMS
C IN THE SERIES
C
      IF(AXR.LE.EPS .OR. AXI.GT.1.0D70) RETURN
C
C AN ERROR MESSAGE IS PRINTED IF THE NUMBER OF TERMS REQUIRED IN THE SERIES
C FOR KERS EXCEED MNIS. IF EPS IS KEPT CONSTANT, RS SHOULD BE REDUCED OR
C MNTS AND CORRESPONDING DIMENSIONS INCREASED.
      IF( NTS .EQ.MNIS) GO TO 85
      GO TO 90
   85 WRITE(6,11) NTS,AXR,AXI
   11 FORMAT     (5X,9H EVEN FOR,I3,36H TERMS IN THE SERIES, THE MAX. RAT
     110 =,G10.4,33H  AND ABS. MAX. VALUE OF A TERM =,G10.4)
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE EFSRRK (HI,AN,AIM,MN,NTS,RS,NES)
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C
C* THIS IS THE SECOND SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR A NON-AXI
C SYMMETRIC DISTURBANCE *
C
C THIS SUBROUTINE CALCULATES THE EIGENFUNCTIONS ACCORDING TO THE SERIES
C EXPANSIONS (EQN.(3.56)) FOR (0.LE.R.LE.RS) AT STEPS OF HH. IT ALSO CAL-
C CULATES THE DERIVATES OF THE EIGENFUNCTIONS FOR DISTURBANCE VELOCITY
C COMPONENTS AT R=RS. THESE ARE REQUIRED LATER FOR STEP-BY-STEP INTEGRATION
C BY THE RUNGE-KUTTA METHOD.
C
C ***************************************************************************
C IF THE PREDICTOR-CORRECTOR METHOD IS USED FOR STEP-BY-STEP INTEGRATION,
C USE SUBROUTINE 'EFSRPC' IN PLACE OF THIS.
C ***************************************************************************
C
C THE INPUT VARIABLES NOT ALREADY EXPLAINED EARLIER ARE HH,NN,NES
C   HH - RADIAL INTERVAL AT WHICH THE EIGENFUNCTION VALUES ARE STORED FOR
C        PLOTTING IF AN EIGENVALUE IS FOUND. HH=0.02 HERE. IF A LOWER VALUE
C        IS DESIRED, RELEVANT DIMENSIONS MUST BE ACCORDINGLY INCREASED.
C   NN = AN = N = 1,2,3,.......
C   NES - NUMBER OF POINTS FOR WHICH EIGENFUNCTIONS FOUND BY THE SERIES
C        SOLUTION ARE STORED FOR PLOTTING LATER.   NES = 1 + RS/HH
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C VRR(L,J),VRI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE EIGEN-
C FUNCTION VR(R). THE FIRST SUBSCRIPT INDICATES THE RADIAL POSITION FOR
C WHICH THE EIGENFUNCTION VALUE IS STORED, AND THE SECOND SUBSCRIPT REFERS
C TO ANYONE OF THE THREE INDEPENDENT SOLUTIONS WHICH MAKE UP VR(R).
C SIMILAR IS THE CASE WITH ARRAYS FOR VZR,VZI,VTR,VTI,PRR,PRI
C
C DVRR(J),DVRI(J) - ARRAYS OF THREE ELEMENTS FOR THE THREE INDEPENDENT
C SOLUTIONS OF THE REAL AND IMAGINARY PARTS OF THE DERIVATIVE OF VR(R).
C FOR THE RUNGE-KUTTA METHOD, THEY ARE NEEDED ONLY AT R=RS.
C SIMILAR IS THE CASE FOR ARRAYS OF DVTR,DVTI,DVZR,DVZI
C
      DOUBLE PRECISION VRR(51,3),VRI(51,3),VTR(51,3),VTI(51,3),VZR(51,3)
     1,VZI(51,3),PRR(51,3),PRI(51,3),DVRR(3),DVRI(3),DVTR(3),DVTI(3),DVZ
     2R(3),DVZI(3),GSR,GSI,FSR,HH,R,RF,RG,RU,PP,RDG,AM,ANN,AF,AU,UI(20,3
     3),UR(20,3),FSI   ,RS,PR(20,3),PI(20,3), GR(20,3),GI(20,3),FR(20,3),
     4FI(20,3)
      COMMON /AREA1/ GR,GI,FR,FI,UR,UI,PR,PI
      COMMON /AREA2/ VRR,VRI,VTR,VTI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ DVRR,DVRI,DVTR,DVTI,DVZR,DVZI
C SETTING UP THE BOUNDARY CONDITIONS ON VR(R) AND VTHETA(R) AT R=0.  THE
C BOUNDARY CONDITIONS ON VZ(R) AND P(R) AT R=0 ARE SET IN THE MAIN PROGRAM
C SINCE THEY ARE ALL ZERO FOR (N.GE.1) .
C
      IF(NN.EQ.1) GO TO 17
      DO 16 J=1,3
      VRR(1,J)=0.0D0
      VRI(1,J)=0.0D0
      VTR(1,J)=0.0D0
   16 VTI(1,J)=0.0D0
      GO TO 14
   17 DO 18 J=1,3
      VRR(1,J)=GR(1,J)/2.0D0
      VRI(1,J)=GI(1,J)/2.0D0
```

```
         VTR(1,J)=-GI(1,J)/2.D0
   18    VTI(1,J)=GR(1,J)/2.D0
C  CALCULATING THE EIGENFUNCTIONS FOR (0.LE.R.LE.RS) AT INTERVALS OF HH BY
C  MEANS OF THE SERIES EXPANSIONS
   14    R=0.0D0
         DO 13 LL=2,NES
         R=R+HH
         RF=R**(NN+1)
         RU=R**NN
         IF(NN.EQ.1) GO TO 28
         RG=R**(NN-1)
         GO TO 37
   28    RG=1.0D0
   37    DO 13 J=1,3
C  INITIALIZING THE SUMMATION OF SERIES
         FSR   =FR(1,J)
         FSI   =FI(1,J)
         GSR   =GR(1,J)
         GSI   =GI(1,J)
         VZR(LL,J)=UR(1,J)
         VZI(LL,J)=UI(1,J)
         PRR(LL,J)=PR(1,J)
         PRI(LL,J)=PI(1,J)
C  SUMMING UP THE SERIES
         DO 36 L=2,NTS
         RP=R**(2*(L-1))
         FSR   =FSR   +FR(L,J)*RP
         FSI   =FSI   +FI(L,J)*RP
         GSR   =GSR   +GR(L,J)*RP
         GSI   =GSI   +GI(L,J)*RP
         VZR(LL,J)=VZR(LL,J)+UR(L,J)*RP
         VZI(LL,J)=VZI(LL,J)+UI(L,J)*RP
         PRR(LL,J)=PRR(LL,J)+PR(L,J)*RP
   36    PRI(LL,J)=PRI(LL,J)+PI(L,J)*RP
         VRR(LL,J)=(FSR   *RF+GSR   *RG)/2.D0
         VRI(LL,J)=(FSI   *RF+GSI   *RG)/2.D0
         VTR(LL,J)=(FSI   *RF-GSI   *RG)/2.D0
         VTI(LL,J)=(GSR   *RG-FSR   *RF)/2.D0
         VZR(LL,J)=VZR(LL,J)*RU
         VZI(LL,J)=VZI(LL,J)*RU
         PRR(LL,J)=PRR(LL,J)*RU
   13    PRI(LL,J)=PRI(LL,J)*RU
C  FINDING THE DERIVATIVES OF THE EIGENFUNCTIONS FOR DISTURBANCE VELOCITY
C  COMPONENTS AT R=RS USING THE SERIES SOLUTION.
         IF(NN.EQ.2) GO TO 33
         RDG=RS**(NN-2)
         GO TO 23
   33    RDG=1.0D0
   23    DO 19 J=1,3
         FSR=FR(1,J)*ANN
         FSI=FI(1,J)*ANN
         GSR=GR(1,J)*(AN-1.0D0)
         GSI=GI(1,J)*(AN-1.0D0)
         DVZR(J)=UR(1,J)*AN
         DVZI(J)=UI(1,J)*AN
         DO 24 L=2,NTS
         AF=NN+2*L-1
         AU=AF-1.0D0
```

```
C
            SUBROUTINE SERIES (RE,W,KR,KI,NU,AN,EPS,MNTS,RS,ANN,C,NTS,AXR,AXI)
C
C* THIS IS THE FIRST SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR A NON-AXI
C SYMMETRIC DISTURBANCE *
C
C FOR A PREASSIGNED VALUE OF THE RADIUS UPTO WHICH THE SERIES SOLUTION IS
C CARRIED, THIS SUBROUTINE EVALUATES THE TERMS IN THE SERIES EXPANSION OF
C THE EIGENFUNCTIONS (SEE SECTION 3.2.1). ALL THE SERIES ARE TERMINATED IF
C THE ABSOLUTE VALUE OF THE RATIO OF THE LAST TERM RETAINED TO THE PARTIAL
C SUM FOR ALL SERIES DOES NOT EXCEED A PREASSIGNED EPSILON ('EPS' HERE)
C
C VARIABLES TO BE SUPPLIED BY THE MAIN PROGRAMME ARE RE,W,KR,KI,NU,AN,EPS,
C MNTS,RS,ANN,C
C    RE - REYNOLDS NUMBER
C    W - DIMENSIONLESS FREQUENCY
C    KR,KI - REAL AND IMAGINARY PARTS OF THE DIMENSIONLESS WAVE NUMBER K
C    NU - NUMBER OF TERMS IN THE SERIES EXPANSION OF VZ (EQN. (3.22))
C    AN = N=1,2,3,............
C    ANN = N+1
C    EPS - ALREADY EXPLAINED
C    MNTS - MAX. NUMBER OF TERMS ALLOWED IN THE SERIES EXPANSION, SHOULD NOT
C            EXCEED 20 FOR THE DIMENSIONS USED HERE
C    RS - THE VALUE OF R UPTO WHICH THE SERIES SOLUTION IS USED
C    C - ARRAY OF NU ELEMENTS FOR THE COEFFICIENTS IN THE SERIES EXPANSION
C         OF VZ (EQN. (3.22))
C    NTS - ACTUAL NUMBER OF TERMS USED IN THE SERIES EXPANSION FOR EIGEN-
C            FUNCTIONS. IT IS AN OUTPUT OF THE SUBROUTINE. (NTS.LE.MNTS)
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C BR(J),BI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF B'S(EQN. (3.23))
C
C GR(L,J),GI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF G'S. THE
C FIRST SUBSCRIPT INDICATES THE TERM NUMBER IN THE SERIES FOR THE EIGENFUNC
C TION G(R)(SEE EQN.(3.56)), AND THE SECOND SUBSCRIPT REFERS TO THE COEFFI-
C CIENT OF EITHER ONE OF THE THREE INDEPENDENT CONSTANTS (G1,U1,P1) FOR
C EQUATIONS OF THE FORM OF EQN.(3.70), FOR EXAMPLE, G3 CAN BE EXPRESSED AS
C        G3 = G31*G1 + G32*U1 + G33*P1
C    THEN   GR(3,1) = REAL PART OF G31
C           GR(3,2) = REAL PART OF G32
C    AND    GR(3,3) = REAL PART OF G33
C SIMILARLY FOR THE ARRAYS FOR FR,FI,UR,UI,PR,PI
C
C ACCORDING TO THE ABOVE SCHEME, THE FOLLOWING CONSTANTS ARE SET IN THE
C MAIN PROGRAMME
C    GR(1,2) = GI(1,2) = GR(1,3) = GI(1,3) = 0.0
C    UR(1,1) = UI(1,1) = UR(1,3) = UI(1,3) = 0.0
C    PR(1,1) = PI(1,1) = PR(1,2) = PI(1,2) = 0.0
C    GI(1,1) = UI(1,2) = PI(1,3) = 0.0
C    AND GR(1,1) = UR(1,2) = PR(1,3) =  SOME NON-ZERO CONSTANT
C
C ****************************************************************
C**** FROM NOW ONWARDS, THE SUBSCRIPT J WILL ALWAYS REFER TO ANY ONE OF THE
C THREE INDEPENDENT SOLUTIONS FOR THE EIGENFUNCTIONS. THUS, J = 1,2,3
C ****************************************************************
C
C GSR(J),GSI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE THREE INDE
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C PENDENT SUMS FOR THE EIGENFUNCTION G(R) EVALUATED AT R=RS.   SIMILARLY FOR
C FSR(J),FSI(J),USR(J),USI(J),PSR(J),PSI(J)
C
      DOUBLE PRECISION RE,W,KR,KI,AN,EPS,RS,ANN,BR(2),BI(2),AL,AI,AXR,AY
     11,SUR,SUI,SGR,SGI,SFR,SFI,FSR(3),FSI(3),GSR(3),GSI(3),USR(3),USI(3
     2),PSR(3),PSI(3),ARF(3),ARG(3),ARU(3),ARP(3),RP,ABSDPC,KSR,KSI
      DOUBLE PRECISION C(NU),GR(20,3),GI(20,3),FR(20,3),FI(20,3),UR(20,3
     3),UI(20,3),PR(20,3),PI(20,3)
      COMMON /AREA1/ GR,GI,FR,FI,UR,UI,PR,PI
      KSR=KR*KR-KI*KI
      KSI=2.0D0*KR*KI
      BR(1)=KSR-RE*KR*C(1)
      BI(1)=KSI+RE*(W-KI*C(1))
      DO 20 J=2,NU
      BR(J)=-KR*RE*C(J)
   20 BI(J)=-KI*RE*C(J)
C CALCULATING THE FIRST TERM IN THE SERIES FOR F(R)-SEE EQN.(3.62)
      DO 30 J=1,3
      CALL MULDPC (BR(1),BI(1),GR(1,J),GI(1,J),AXR,AXI)
      CALL MULDPC(KR,KI,UR(1,J),UI(1,J),KSR,KSI)
      FR(1,J)=(AXR/(4.D0*AN)-PR(1,J)*RE/2.D0-KSR)/ANN
   30 FI(1,J)=(AXI/(4.D0*AN)-PI(1,J)*RE/2.D0-KSI)/ANN
      DO 21 J=1,3
      FSR(J)=FR(1,J)
      FSI(J)=FI(1,J)
      GSR(J)=GR(1,J)
      GSI(J)=GI(1,J)
      USR(J)=UR(1,J)
      USI(J)=UI(1,J)
      PSR(J)=PR(1,J)
   21 PSI(J)=PI(1,J)
      L=0
   90 L=L+1
      AL=L
      DO 60 J=1,3
      SUR=0.0D0
      SUI=0.0D0
      SGR=0.0D0
      SGI=0.0D0
      SFR=0.0D0
      SFI=0.0D0
      DO 40 I=1,NU
      LL=L+1-I
      IF(LL.LE.0) GO TO 40
C SUR,SUI - REAL AND IMAGINARY PARTS OF THE FIRST SUMMATION IN EQN.(3.67)
      CALL MULDPC (BR(I),BI(I),UR(LL,J),UI(LL,J),AXR,AXI)
      SUR=SUR+AXR
      SUI=SUI+AXI
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SUMMATION IN EQN.(3.66)
      CALL MULDPC (BR(I),BI(I),GR(LL,J),GI(LL,J),AXR,AXI)
      SGR=SGR+AXR
      SGI=SGI+AXI
C SFR,SFI - REAL AND IMAGINARY PARTS OF THE SUMMATION IN EQN.(3.69)
      CALL MULDPC (BR(I),BI(I),FR(LL,J),FI(LL,J),AXR,AXI)
      SFR=SFR+AXR
      SFI=SFI+AXI
   40 CONTINUE
C CALCULATING THE G'S IN THE SERIES FOR G(R) - SEE EQN.(3.66)
```

```
      GR(L+1,J)=(RE*PR(L,J)-SGR/(2.D0*(AN+AL-1.D0)))/(2.D0*AL)
      GI(L+1,J)=(RE*PI(L,J)-SGI/(2.D0*(AN+AL-1.D0)))/(2.D0*AL)
      SGR=0.0D0
      SGI=0.0D0
      DO 50 I=2,NU
      LL=L+1-I
      AI=I-1
      IF(LL.LE.0) GO TO 70
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SECOND SUMMATION IN EQN.(3.67)
      SGR= SGR+AI*RE*C(I)*FR(LL,J)
      SGI= SGI+AI*RE*C(I)*FI(LL,J)
   70 LL=LL+1
      IF(LL.LE.0) GO TO 50
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SECOND AND THIRD SUMMATIONS IN
C EQUATION (3.67)
      SGR= SGR+AI*RE*C(I)*GR(LL,J)
      SGI= SGI+AI*RE*C(I)*GI(LL,J)
   50 CONTINUE
C CALCULATING THE U'S IN THE SERIES FOR VZ(R) - SEE EQN.(3.67)
      CALL MULDPC (KR ,KI ,PR(L,J),PI(L,J),AXR,AXI)
      UR(L+1,J)=(AXR*RE-SUR+SGR)/(4.D0*AL*(AN+AL))
      UI(L+1,J)=(AXI*RE-SUI+SGI)/(4.D0*AL*(AN+AL))
      SGR=0.0D0
      SGI=0.0D0
      DO 80 I=1,NU
      LL=L+2-I
      IF(LL.LE.0) GO TO 80
C SGR,SGI - REAL AND IMAGINARY PARTS OF THE SECOND SUMMATION IN EQN.(3.68)
      CALL MULDPC (BR(I),BI(I),GR(LL,J),GI(LL,J),AXR,AXI)
      SGR=SGR+AXR
      SGI=SGI+AXI
   80 CONTINUE
C CALCULATING THE P'S IN THE SERIES FOR P(R) - SEE EQN.(3.68)
      CALL MULDPC(KR,KI,UR(L+1,J),UI(L+1,J),AXR,AXI)
      PR(L+1,J)=( -AXR      +(SFR/AL+SGR/(AL+AN))/4.D0)/RE
      PI(L+1,J)=( -AXI      +(SFI/AL+SGI/(AL+AN))/4.D0)/RE
C CALCULATING THE F'S IN THE SERIES FOR F(R) - SEE EQN.(3.69)
      FR(L+1,J)=(RE*PR(L+1,J)-SFR/(2.D0*AL))/(2.D0*(ANN+AL))
   60 FI(L+1,J)=(RE*PI(L+1,J)-SFI/(2.D0*AL))/(2.D0*(ANN+AL))
      IF(L.EQ.1) GO TO 90
C CALCULATING THE PARTIAL SUM OF THE SERIES FOR THE EIGENFUNCTIONS
C EVALUATED AT R=RS
      RP=RS**(2*(L-1))
      DO 35 J=1,3
      FSR(J)=FSR(J)+FR(L,J)*RP
      FSI(J)=FSI(J)+FI(L,J)*RP
      GSR(J)=GSR(J)+GR(L,J)*RP
      GSI(J)=GSI(J)+GI(L,J)*RP
      USR(J)=USR(J)+UR(L,J)*RP
      USI(J)=USI(J)+UI(L,J)*RP
      PSR(J)=PSR(J)+PR(L,J)*RP
   35 PSI(J)=PSI(J)+PI(L,J)*RP
      NTS=L+1
C FINDING THE ABSOLUTE VALUE OF THE RATIOS OF THE LAST TERMS IN THE SERIES
C (FOR THE THREE INDEPENDENT SOLUTIONS) TO THE PARTIAL SUM. ARF(J) TO ARP(J)
C ARE THE ARRAYS TO STORE THESE RATIOS
      RP=RS**(2*L)
      DO 25 J=1,3
```

```
       IF(DABS(FSR(J)).LT.1.0D-70 .AND. DABS(FSI(J)).LT.1.0D-70) GO TO 61
       CALL DIVDPC (FR(NTS,J),FI(NTS,J),FSR(J),FSI(J),AXR,AXI)
       ARF(J)=ABSDPC(AXR,AXI)*RP
       GO TO 62
   61  ARF(J)=0.0D0
   62  IF(DABS(GSR(J)).LT.1.0D-70 .AND. DABS(GSI(J)).LT.1.0D-70) GO TO 63
       CALL DIVDPC (GR(NTS,J),GI(NTS,J),GSR(J),GSI(J),AXR,AXI)
       ARG(J)=ABSDPC(AXR,AXI)*RP
       GO TO 64
   63  ARG(J)=0.0D0
   64  IF(DABS(USR(J)).LT.1.0D-70 .AND. DABS(USI(J)).LT.1.0D-70) GO TO 66
       CALL DIVDPC (UR(NTS,J),UI(NTS,J),USR(J),USI(J),AXR,AXI)
       ARU(J)=ABSDPC(AXR,AXI)*RP
       GO TO 67
   66  ARU(J)=0.0D0
   67  IF(DABS(PSR(J)).LT.1.0D-70 .AND. DABS(PSI(J)).LT.1.0D-70) GO TO 68
       CALL DIVDPC (PR(NTS,J),PI(NTS,J),PSR(J),PSI(J),AXR,AXI)
       ARP(J)=ABSDPC(AXR,AXI)*RP
       GO TO 25
   68  ARP(J)=0.0D0
   25  CONTINUE
C  AXR - MAXIMUM VALUE OF THE ABOVE RATIOS
       AXR=0.0D0
       DO 45 J=1,3
       AXR=DMAX1(AXR,ARF(J))
       AXR=DMAX1(AXR,ARG(J))
       AXR=DMAX1(AXR,ARU(J))
   45  AXR=DMAX1(AXR,ARP(J))
C  AXI - LARGEST TERM (EITHER ITS REAL OR IMAGINARY PART) IN ALL THE SERIES
       AXI =0.0D0
       DO 54 J=1,3
       AXI =DMAX1(AXI    ,DABS(FR(NTS,J)))
       AXI =DMAX1(AXI    ,DABS(FI(NTS,J)))
       AXI =DMAX1(AXI    ,DABS(GR(NTS,J)))
       AXI =DMAX1(AXI    ,DABS(GI(NTS,J)))
       AXI =DMAX1(AXI    ,DABS(UR(NTS,J)))
       AXI =DMAX1(AXI    ,DABS(UI(NTS,J)))
       AXI =DMAX1(AXI    ,DABS(PR(NTS,J)))
   54  AXI =DMAX1(AXI    ,DABS(PI(NTS,J)))
C  ALL SERIES ARE TERMINATED IF EITHER AXR.LE.EPS OR IF AXI.GT.1.D70.  THE
C  LATTER CRITERIA, THOUGH NEVER FOUND TO BE IMPORTANT DURING THE PRESENT
C  INVESTIGATION, IS INCLUDED HERE TO AVOID VERY LARGE VALUES FOR THE TERMS
C  IN THE SERIES
C
       IF(AXR.LE.EPS .OR. AXI.GT.1.0D70) RETURN
C
C  AN ERROR MESSAGE IS PRINTED IF THE NUMBER OF TERMS REQUIRED IN THE SERIES
C  FOR R=RS EXCEED MNIS. IF EPS IS KEPT CONSTANT, RS SHOULD BE REDUCED OR
C  MNTS AND CORRESPONDING DIMENSIONS INCREASED.
       IF( NTS .EQ.MNIS) GO TO 85
       GO TO 90
   85  WRITE(6,11) NTS,AXR,AXI
   11  FORMAT     (5X,9H EVEN FOR,I3,38H TERMS IN THE SERIES, THE MAX. RAT
      110 =,G10.4,33H  AND ABS. MAX. VALUE OF A TERM =,G10.4)
       RETURN
       END
C  ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
       SUBROUTINE EFSKRK (HN,AN,AIM,HN,NTS,RS,MES)
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C
C* THIS IS THE SECOND SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR A NON-AXI
C SYMMETRIC DISTURBANCE *
C
C THIS SUBROUTINE CALCULATES THE EIGENFUNCTIONS ACCORDING TO THE SERIES
C EXPANSIONS (EQN.(3.56)) FOR (0.LE.R.LE.RS) AT STEPS OF HH. IT ALSO CAL-
C CULATES THE DERIVATES OF THE EIGENFUNCTIONS FOR DISTURBANCE VELOCITY
C COMPONENTS AT R=RS. THESE ARE REQUIRED LATER FOR STEP-BY-STEP INTEGRATION
C BY THE RUNGE-KUTTA METHOD.
C
C **********************************************************************
C IF THE PREDICTOR-CORRECTOR METHOD IS USED FOR STEP-BY-STEP INTEGRATION,
C USE SUBROUTINE 'EFSRPC' IN PLACE OF THIS.
C **********************************************************************
C
C THE INPUT VARIABLES NOT ALREADY EXPLAINED EARLIER ARE HH,NN,NES
C    HH - RADIAL INTERVAL AT WHICH THE EIGENFUNCTION VALUES ARE STORED FOR
C         PLOTTING IF AN EIGENVALUE IS FOUND. HH=0.02 HERE. IF A LOWER VALUE
C         IS DESIRED, RELEVANT DIMENSIONS MUST BE ACCORDINGLY INCREASED.
C    NN = AN = N = 1,2,3,........
C    NES - NUMBER OF POINTS FOR WHICH EIGENFUNCTIONS FOUND BY THE SERIES
C          SOLUTION ARE STORED FOR PLOTTING LATER.  NES = 1 + RS/HH
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C VRR(L,J),VRI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE EIGEN-
C FUNCTION VR(R). THE FIRST SUBSCRIPT INDICATES THE RADIAL POSITION FOR
C WHICH THE EIGENFUNCTION VALUE IS STORED, AND THE SECOND SUBSCRIPT REFERS
C TO ANYONE OF THE THREE INDEPENDENT SOLUTIONS WHICH MAKE UP VR(R).
C SIMILAR IS THE CASE WITH ARRAYS FOR VZR,VZI,VTR,VTI,PRR,PRI
C
C DVRR(J),DVRI(J) - ARRAYS OF THREE ELEMENTS FOR THE THREE INDEPENDENT
C SOLUTIONS OF THE REAL AND IMAGINARY PARTS OF THE DERIVATIVE OF VR(R).
C FOR THE RUNGE-KUTTA METHOD, THEY ARE NEEDED ONLY AT R=RS.
C SIMILAR IS THE CASE FOR ARRAYS OF DVTR,DVTI,DVZR,DVZI
C
         DOUBLE PRECISION VRR(51,3),VRI(51,3),VTR(51,3),VTI(51,3),VZR(51,3)
        1,VZI(51,3),PRR(51,3),PRI(51,3),DVRR(3),DVRI(3),DVTR(3),DVTI(3),DVZ
        2R(3),DVZI(3),GSR,GSI,FSR,HH,R,RF,RG,RU,PP,RDG,AM,ANN,AF,AU,UI(20,3
        3),UR(20,3),FSI  ,RS,PR(20,3),PI(20,3), GR(20,3),GT(20,3),FR(20,3),
        4FI(20,3)
         COMMON /AREA1/ GR,GI,FR,FI,UR,UI,PR,PI
         COMMON /AREA2/ VRR,VRI,VTR,VTI,VZR,VZI,PRR,PRI
         COMMON /AREA3/ DVRR,DVRI,DVTR,DVTI,DVZR,DVZI
C SETTING UP THE BOUNDARY CONDITIONS ON VR(R) AND VTHETA(R) AT R=0.  THE
C BOUNDARY CONDITIONS ON VZ(R) AND P(R) AT R=0 ARE SET IN THE MAIN PROGRAM
C SINCE THEY ARE ALL ZERO FOR (N.GE.1) .
C
         IF(NN.EQ.1) GO TO 17
         DO 16 J=1,3
         VRR(1,J)=0.0D0
         VRI(1,J)=0.0D0
         VTR(1,J)=0.0D0
   16    VTI(1,J)=0.0D0
         GO TO 14
   17    DO 18 J=1,3
         VRR(1,J)=GR(1,J)/2.D0
         VRI(1,J)=GI(1,J)/2.D0
```

```
      VTR(1,J)=-GI(1,J)/2.D0
   18 VT1(1,J)=GR(1,J)/2.D0
C CALCULATING THE EIGENFUNCTIONS FOR (0.LE.R.LE.RS) AT INTERVALS OF HH BY
C MEANS OF THE SERIES EXPANSIONS
   14 R=0.0D0
      DO 13 LL=2,NES
      R=R+HH
      RF=R**(NN+1)
      RU=R**NN
      IF(NN.EQ.1) GO TO 28
      RG=R**(NN-1)
      GO TO 37
   28 RG=1.0D0
   37 DO 13 J=1,3
C INITIALIZING THE SUMMATION OF SERIES
      FSR    =FR(1,J)
      FSI    =FI(1,J)
      GSR    =GR(1,J)
      GSI    =GI(1,J)
      VZR(LL,J)=UR(1,J)
      VZI(LL,J)=UI(1,J)
      PRR(LL,J)=PR(1,J)
      PRI(LL,J)=PI(1,J)
C SUMMING UP THE SERIES
      DO 36 L=2,NTS
      RP=R**(2*(L-1))
      FSR    =FSR    +FR(L,J)*RP
      FSI    =FSI    +FI(L,J)*RP
      GSR    =GSR    +GR(L,J)*RP
      GSI    =GSI    +GI(L,J)*RP
      VZR(LL,J)=VZR(LL,J)+UR(L,J)*RP
      VZI(LL,J)=VZI(LL,J)+UI(L,J)*RP
      PRR(LL,J)=PRR(LL,J)+PR(L,J)*RP
   36 PRI(LL,J)=PRI(LL,J)+PI(L,J)*RP
      VRR(LL,J)=(FSR    *RF+GSR    *RG)/2.D0
      VRI(LL,J)=(FSI    *RF+GSI    *RG)/2.D0
      VTR(LL,J)=(FSI    *RF-GSI    *RG)/2.D0
      VTI(LL,J)=(GSR    *RG-FSR    *RF)/2.D0
      VZR(LL,J)=VZR(LL,J)*RU
      VZI(LL,J)=VZI(LL,J)*RU
      PRR(LL,J)=PRR(LL,J)*RU
   13 PRI(LL,J)=PRI(LL,J)*RU
C FINDING THE DERIVATIVES OF THE EIGENFUNCTIONS FOR DISTURBANCE VELOCITY
C COMPONENTS AT R=RS USING THE SERIES SOLUTION.
      IF(NN.EQ.2) GO TO 33
      RDG=RS**(NN-2)
      GO TO 23
   33 RDG=1.0D0
   23 DO 19 J=1,3
      FSR=FR(1,J)*ANN
      FSI=FI(1,J)*ANN
      GSR=GR(1,J)*(ANN-1.0D0)
      GSI=GI(1,J)*(ANN-1.0D0)
      DVZR(J)=UR(1,J)*AN
      DVZI(J)=UI(1,J)*AN
      DO 24 L=2,NTS
      AF=AN+2*L-1
      AU=AF-1.0D0
```

```
        RP=RS**(2*(L-1))
         FSR= FSR+FR(L,J)*RP*AF
         FSI= FSI+FI(L,J)*RP*AF
         GSR= GSR+GR(L,J)*RP*(AU-1.D0)
         GSI= GSI+GI(L,J)*RP*(AU-1.D0)
        DVZR(J)=DVZR(J)+UR(L,J)*RP*AU
   24   DVZI(J)=DVZI(J)+UI(L,J)*RP*AU
         DVRR(    J)=( FSR*RU+ GSR*RDG)/2.D0
         DVRI(    J)=( FSI*RU+ GSI*RDG)/2.D0
         DVTR(    J)=( FSI*RU- GSI*RDG)/2.D0
         DVTI(    J)=( GSR*RDG- FSR*RU)/2.D0
        DVZR(J)=DVZR(J)*RG
   19   DVZI(J)=DVZI(J)*RG
        RETURN
        END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
        SUBROUTINE RUNGEK(RE,W,KR,KI,LS,AN,RS,NU,C,H,NES,NPE,NG,ND)
C
C* THIS IS THE THIRD SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR A NON-AXI
C SYMMETRIC DISTURBANCE *
C
C STARTING FROM R=RS (UPTO WHICH THE SERIES SOLUTION WAS USED), THIS SUBROU
C TINE INTEGRATES THE STABILITY EQUATIONS (3.71) IN A STEP-BY-STEP SCHEME
C BY THE USE OF FOURTH ORDER RUNGE-KUTTA METHOD.
C
C THE INPUT VARIABLES NOT ALREADY CLARIFIED ARE LS,H,NPE,NG,ND
C   LS = 1 + RS/H
C   H - THE STEP SIZE IN RADIAL DIRECTION FOR STEP-BY-STEP INTEGRATION SO
C         THAT (H.LE.HH)
C   NPE - NUMBER OF POINTS FOR WHICH THE EIGENFUNCTION VALUES ARE STORED
C          FOR PLOTTING LATER ON. NPE = 1 + 1./HH . IF NPE IS TO EXCEED 51,
C          RELEVANT DIMENSIONS MUST BE INCREASED.
C   NG = (ND-1)/(NPE-1) = HH/H
C   ND = 1 + 1./H
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C EVRR(J) TO EPRI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE THREE
C INDEPENDENT SOLUTIONS OF THE EIGENFUNCTIONS AT THE STARTING POINT OF
C EVERY FORWARD STEP
C
C CVRR TO CPRI - REAL AND IMAGINARY PARTS OF THE EIGENFUNCTION VALUES
C PERTAINING TO THOSE FOR THE Y COLUMN IN TABLES AI-1 AND AI-2 (APPENDIX I)
C
C CDVRR TO CDVZI - REAL AND IMAGINARY PARTS OF THE VALUES PERTAINING TO
C THOSE FOR THE DY COLUMN IN TABLE AI-2 (APPENDIX I)
C
C VZM - THE STEDY MEAN FLOW VELOCITY (EQN.(3.22))
C DVZM - DERIVATIVE OF VZM WITH RESPECT TO R
C FVRR TO FPRI - REAL AND IMAGINARY PARTS OF THE LEFT HAND SIDES OF
C EQUATIONS (3.71)
C
C VRRK1 TO PRIK4 - REAL AND IMAGINARY PARTS OF THE FOUR CONSTANTS K1 TO K4
C FOR THE RUNGE-KUTTA METHOD (SEE APPENDIX I, TABLES AI-1 AND AI-2)
C
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DOUBLE PRECISION KR,KI
        DIMENSION EPRI(3),VRR(51,3),VRI(51,3),VTR(51,3),VTI(51,3),VZR(51,3
```

```
   1),VZI(51,3),PRR(51,3),PRI(51,3),DVRR(3),DVRI(3),DVTR(3),DVTI(3),DV
   22R(3),DVZI(3),EVRR(3),EVRI(3),EVTR(3),EVTI(3),EVZR(3),EVZI(3),EPRR
   3(3),C(NU)
      COMMON /AREA2/ VRR,VRI,VTR,VTI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ DVRR,DVRI,DVTR,DVTI,DVZR,DVZI
      K=NES
      MUL=1
      DO 76 J=1,3
      EVRR(J)=VRR(NES,J)
      EVRI(J)=VRI(NES,J)
      EVTR(J)=VTR(NES,J)
      EVTI(J)=VTI(NES,J)
      EVZR(J)=VZR(NES,J)
      EVZI(J)=VZI(NES,J)
      EPRR(J)=PRR(NES,J)
   76 EPRI(J)=PRI(NES,J)
      RR=RS
      HS=H*H/2.D0
      M=LS+1
      DO 42 I=M,ND
      DO 72 J=1,3
      R=RR
      MRK=1
C INITIALIZING THE Y AND DY COLUMNS OF TABLES AI-1 AND AI-2 (APPENDIX I)
      CVRR=EVRR(J)
      CVRI=EVRI(J)
      CVTR=EVTR(J)
      CVTI=EVTI(J)
      CVZR=EVZR(J)
      CVZI=EVZI(J)
      CPRR=EPRR(J)
      CPRI=EPRI(J)
      CDVRR=DVRR(J)
      CDVRI=DVRI(J)
      CDVTR=DVTR(J)
      CDVTI=DVTI(J)
      CDVZR=DVZR(J)
      CDVZI=DVZI(J)
   51 SR=1.D0/(R*R)
C CALCULATING VZ AND DVZ FOR THE STEADY MEAN FLOW
      VZM=C(1)
      DVZM=0.0D0
      DO 44 L=2,NU
      AR=2*(L-1)
      VZM=VZM+C(L)*(R**(2*(L-1)))
   44 DVZM=DVZM+AR*C(L)*(R**(2*L-3))
C CALCULATING THE LEFT HAND SIDES OF EQUATIONS (3.71) WRITTEN IN THEIR REAL
C AND IMAGINARY PARTS
C
      AR=(AN*AN+1.D0)*SR+KR*(RE*VZM-KR)+KI*KI
      AI=KI*(RE*VZM-2.D0*KR)-RE*W
   74 CALL MULUPC (AR,AI,CVTR,CVTI,AXR,AXI)
      FVTR=2.D0*AN*SR*CVRI+AXR-(AN*RE*CPRI+CDVTR)/R
      FVTI=-2.D0*AN*SR*CVRR+AXI+(AN*RE*CPRR-CDVTI)/R
      CALL MULUPC (AR,AI,CVZR,CVZI,AXR,AXI)
      CALL MULUPC (KR,KI,CPRR,CPRI,BXR,BXI)
      FVZR=RE*(DVZM*CVRR+BXR)-CDVZR/R-SR*CVZR+AYR
      FVZI=RE*(DVZM*CVRI+BXI)-CDVZI/R-SR*CVZI+AYI
```

```
      CALL MULDPC (KR,KI,CDVZR,CDVZI,AXR,AXI)
      FVRR=SR*(CVRR-AN*CVTI)-(CDVRR-AM*CDVTI)/R-AXR
      FVRI=SR*(CVRI+AN*CVTR)-(CDVRI+AN*CDVTR)/R-AXI
      CALL MULDPC (AR,AI,CVRR,CVRI,BXR,BXI)
      FPRR=(SR*CVRR-AXR-BXR+AN*(CDVTI/R+SR*CVTI))/RE
      FPRI=(SR*CVRI-AXI-BXI-AN*(CDVTR/R+SR*CVTR))/RE
      GO TO(55,59,65,71), MRK
C CALCULATING K1'S AND SECOND ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
   55 VRRK1=HS*FVRR
      VRIK1=HS*FVRI
      VTRK1=HS*FVTR
      VTIK1=HS*FVTI
      VZRK1=HS*FVZR
      VZIK1=HS*FVZI
      PRRK1=H*FPRR
      PRIK1=H*FPRI
      R=RR+H/2.D0
      CVRR=EVRR(J)+DVRR(J)*H/2.D0+VRRK1/4.D0
      CVRI=EVRI(J)+DVRI(J)*H/2.D0+VRIK1/4.D0
      CVTR=EVTR(J)+DVTR(J)*H/2.D0+VTRK1/4.D0
      CVTI=EVTI(J)+DVTI(J)*H/2.D0+VTIK1/4.D0
      CVZR=EVZR(J)+DVZR(J)*H/2.D0+VZRK1/4.D0
      CVZI=EVZI(J)+DVZI(J)*H/2.D0+VZIK1/4.D0
      CPRR=EPRR(J)+PRRK1/2.D0
      CPRI=EPRI(J)+PRIK1/2.D0
      CDVRR=DVRR(J)+VRRK1/H
      CDVRI=DVRI(J)+VRIK1/H
      CDVTR=DVTR(J)+VTRK1/H
      CDVTI=DVTI(J)+VTIK1/H
      CDVZR=DVZR(J)+VZRK1/H
      CDVZI=DVZI(J)+VZIK1/H
      MRK=2
      GO TO 51
C CALCULATING K2'S AND THIRD ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
   59 VRRK2=HS*FVRR
      VRIK2=HS*FVRI
      VTRK2=HS*FVTR
      VTIK2=HS*FVTI
      VZRK2=HS*FVZR
      VZIK2=HS*FVZI
      PRRK2=H*FPRR
      PRIK2=H*FPRI
      CPRR=EPRR(J)+PRRK2/2.D0
      CPRI=EPRI(J)+PRIK2/2.D0
      CDVRR=DVRR(J)+VRRK2/H
      CDVRI=DVRI(J)+VRIK2/H
      CDVTR=DVTR(J)+VTRK2/H
      CDVTI=DVTI(J)+VTIK2/H
      CDVZR=DVZR(J)+VZRK2/H
      CDVZI=DVZI(J)+VZIK2/H
      MRK=3
      GO TO 74
C CALCULATING K3'S AND FOURTH ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
   65 VRRK3=HS*FVRR
      VRIK3=HS*FVRI
```

```
      VTRK3=HS*FVTR
      VTIK3=HS*FVTI
      VZRK3=HS*FVZR
      VZIK3=HS*FVZI
      PRRK3=H*FPRR
      PRIK3=H*FPRI
      R=RR+H
      CVRR=EVRR(J)+DVRR(J)*H+VRRK3
      CVRI=EVRI(J)+DVRI(J)*H+VRIK3
      CVTR=EVTR(J)+DVTR(J)*H+VTRK3
      CVTI=EVTI(J)+DVTI(J)*H+VTIK3
      CVZR=EVZR(J)+DVZR(J)*H+VZRK3
      CVZI=EVZI(J)+DVZI(J)*H+VZIK3
      CPRR=EPRR(J)+PRRK3
      CPRI=EPRI(J)+PRIK3
      CDVRR=DVRR(J)+2.D0*VRRK3/H
      CDVRI=DVRI(J)+2.D0*VRIK3/H
      CDVTR=DVTR(J)+2.D0*VTRK3/H
      CDVTI=DVTI(J)+2.D0*VTIK3/H
      CDVZR=DVZR(J)+2.D0*VZRK3/H
      CDVZI=DVZI(J)+2.D0*VZIK3/H
      MRK=4
      GO TO 51
C CALCULATING K4'S AND FIFTH ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
   71 VRRK4=HS*FVRR
      VRIK4=HS*FVRI
      VTRK4=HS*FVTR
      VTIK4=HS*FVTI
      VZRK4=HS*FVZR
      VZIK4=HS*FVZI
      PRRK4=H*FPRR
      PRIK4=H*FPRI
      EVRR(J)=EVRR(J)+DVRR(J)*H+(VRRK1+VRRK2+VRRK3)/3.D0
      EVRI(J)=EVRI(J)+DVRI(J)*H+(VRIK1+VRIK2+VRIK3)/3.D0
      EVTR(J)=EVTR(J)+DVTR(J)*H+(VTRK1+VTRK2+VTRK3)/3.D0
      EVTI(J)=EVTI(J)+DVTI(J)*H+(VTIK1+VTIK2+VTIK3)/3.D0
      EVZR(J)=EVZR(J)+DVZR(J)*H+(VZRK1+VZRK2+VZRK3)/3.D0
      EVZI(J)=EVZI(J)+DVZI(J)*H+(VZIK1+VZIK2+VZIK3)/3.D0
      EPRR(J)=EPRR(J)+(PRRK1+2.D0*PRRK2+2.D0*PRRK3+PRRK4)/6.D0
      EPRI(J)=EPRI(J)+(PRIK1+2.D0*PRIK2+2.D0*PRIK3+PRIK4)/6.D0
      DVRR(J)=DVRR(J)+(VRRK1+2.D0*VRRK2+2.D0*VRRK3+VRRK4)/(3.D0*H)
      DVRI(J)=DVRI(J)+(VRIK1+2.D0*VRIK2+2.D0*VRIK3+VRIK4)/(3.D0*H)
      DVTR(J)=DVTR(J)+(VTRK1+2.D0*VTRK2+2.D0*VTRK3+VTRK4)/(3.D0*H)
      DVTI(J)=DVTI(J)+(VTIK1+2.D0*VTIK2+2.D0*VTIK3+VTIK4)/(3.D0*H)
      DVZR(J)=DVZR(J)+(VZRK1+2.D0*VZRK2+2.D0*VZRK3+VZRK4)/(3.D0*H)
   72 DVZI(J)=DVZI(J)+(VZIK1+2.D0*VZIK2+2.D0*VZIK3+VZIK4)/(3.D0*H)
      RR=R
C THE NEXT ARITHMETIC IF STATEMENT IMPLIES THAT IF R IS EQUAL TO A MULTIPLE
C OF HH, EIGENFUNCTION VALUES ARE TO BE STORED IN PROPER ARRAYS BEFORE
C INTEGRATION IS CONTINUED FURTHER.
      IF(R-LS-MUL*HG) 42,57,42
   57 MUL=MUL+1
      K=K+1
      DO 49 J=1,5
      VRR(K,J)=EVRR(J)
      VRI(K,J)=EVRI(J)
      VTR(K,J)=EVTR(J)
```

```
      VTI(K,J)=EVTI(J)
      VZR(K,J)=EVZR(J)
      VZI(K,J)=EVZI(J)
      PRR(K,J)=EPRR(J)
   49 PRI(K,J)=EPRI(J)
   42 CONTINUE
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE SECANT (N,*,RE,W,KR,KI,E,E1,E2,NT,H,NG,NN,RS)
C
C* THIS IS THE FOURTH AND LAST SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR
C A NON-AXISYMMETRIC DISTURBANCE *
C
C HAVING INTEGRATED THE STABILITY EQUATIONS OVER THE WHOLE DOMAIN
C (0.LE.R.LE.1) FOR AN APPROXIMATE EIGENVALUE, THIS SUBROUTINE EMPLOYS THE
C SECANT METHOD IN ORDER TO CONVERGE TO THE TRUE EIGENVALUE.
C
C IF THE PREDICTOR-CORRECTOR METHOD IS USED AND IF IT IS DESIRED TO PUNCH
C 'EPL' ON CARDS (SEE SUBROUTINE 'PREDCT' FOR THE MEANING OF EPL), CARRY
C EPL IN THE ARGUMENT LIST, SPECIFY IT AS A DOUBLE PRECISION VARIABLE,
C INSERT EPL BETWEEN 'RS' AND 'ADET' IN THE STATEMENT PUNCH 71, AND CHANGE
C 6X TO G6.1 IN FORMAT NO. 71. ALSO CHANGE ''DATA TE/'RK'/'' TO ''DATA TE
C /'PC'/'' FOR PURPOSES OF CLARITY. EXCEPT FOR THESE CHANGES, THE SUBROU-
C TINE 'SECANT' IS GOOD REGARDLESS OF THE METHOD USED FOR STEP-BY-STEP
C INTEGRATION FOR A NON-AXISYMMETRIC DISTURBANCE
C
C THE INPUT VARIABLES NOT ALREADY CLARIFIED ARE N,*,E,E1,E2,NT
C   N = NPE (SEE SUBROUTINE 'RUNGEK')
C   * - FOR IMPLICATION OF THE 'MULTIPLE RETURN' FACILITY PROVIDED ON
C        THE UNIVAC 1108 MACHINE
C   E - THE MAXIMUM VALUE OF THE DETERMINANT WHICH IS CONSIDERED CLOSE
C        ENOUGH TO 'ZERO' FOR K TO BE A TRUE EIGENVALUE.
C   E1,E2 - MULTIPLIERS FOR KR AND KI FOR THE NEXT GUESS OF THE EIGENVALUE.
C            THESE ARE ESSENTIAL FOR THE SECANT METHOD.
C   NT - ITERATION NUMBER FOR THE SECANT METHOD. THE MAIN PROGRAMME MUST
C        SET NT = 0
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C DR,DI - REAL AND IMAGINARY PARTS OF THE DETERMINANT (EQN.(3.73))
C ADET - ABSOLUTE VALUE OF THE DETERMINANT. ADET = DSQRT(DR*DR+DI*DI)
C
C URR(L) TO PEI(L) - ARRAYS FOR THE COMPLETE EIGENFUNCTIONS ONCE A TRUE
C EIGENVALUE IS FOUND. THESE ARRAYS HAVE VALUES FOR THE LEFT HAND SIDES OF
C EQUATIONS OF THE FORM GIVEN IN EQN.(3.70)
C
C TE - AN ALPHANUMERIC CHARACTER WHOSE VALUE IS 'RK' IF RUNGE-KUTTA METHOD
C IS USED, AND 'PC' IF A PREDICTOR-CORRECTOR METHOD IS USED FOR THE STEP-
C BY-STEP INTEGRATION
C
      DOUBLE PRECISION RE,W,KR,KI,E,DR,DI,DRP,DIP,KR1,KT1,KR2,KT2,CR(2),
     1CI(2),ADET,ABSDPC,H,VEC(6),URR(51),URI(51),UTR(51),UTI(51),UZR(51)
     2,UZI(51),PER(51),PEI(51),VRR(51,3),VRI(51,3),VTR(51,3),VTI(51,3),V
     3ZR(51,3),VZI(51,3),PRR(51,3),PRI(51,3),RS
      DIMENSION KM(4)
      COMMON /AREA2/ VRR,VRI,VTR,VTI,VZR,VZI,PRR,PRI
      DATA TE/'RK'/
```

```
      NT=NT+1
C CALCULATING THE DETERMINANT BY EXPANDING ON THE ELEMENTS OF THE FIRST
C ROW (SEE EQUATION (3.73))
      CALL MULDPC (VTR(N,2),VTI(N,2),VZR(N,3),VZI(N,3),CR(1),CI(1))
      CALL MULDPC (VIR(N,3),VTI(N,3),VZR(N,2),VZI(N,2),CR(2),CI(2))
      VEC(1)=CR(1)-CR(2)
      VEC(2)=CI(1)-CI(2)
C VEC(L) - ARRAY CONTAINING THE SIX QUANTITIES ON THE RIGHT HAND SIDES
C OF EQUATION (3.74)
      CALL MULDPC (VEC(1),VEC(2),VRR(N,1),VRI(N,1),DR,DT)
      CALL MULDPC (VIR(N,3),VTI(N,3),VZR(N,1),VZI(N,1),CR(1),CI(1))
      CALL MULDPC (VTR(N,1),VTI(N,1),VZR(N,3),VZI(N,3),CR(2),CI(2))
      VEC(3)=CR(1)-CR(2)
      VEC(4)=CI(1)-CI(2)
      CALL MULDPC (VEC(3),VEC(4),VRR(N,2),VRI(N,2),CR(1),CI(1))
      DR=DR+CR(1)
      DI=DI+CI(1)
      CALL MULDPC (VTR(N,1),VTI(N,1),VZR(N,2),VZI(N,2),CR(1),CI(1))
      CALL MULDPC (VIR(N,2),VTI(N,2),VZR(N,1),VZI(N,1),CR(2),CI(2))
      VEC(5)=CR(1)-CR(2)
      VEC(6)=CI(1)-CI(2)
      CALL MULDPC (VEC(5),VEC(6),VRR(N,3),VRI(N,3),CR(1),CI(1))
      DR=DR+CR(1)
      DI=DI+CI(1)
      ADET=ABSDPC(DR,DI)
      WRITE(6,52) H,RE,W,KR,KI,DR,DI,ADET,NT
   52 FORMAT(1H0,F9.4,F10.1,F8.3,2D24.14,3G14.5,I5)
C FOR K TO BE AN EIGENVALUE, THE ABSOLUTE VALUE OF THE DETERMINANT MUST
C BE .LE. A PREASSIGNED EPSILON ('E' HERE)
C
      IF(ADET.LE.E) GO TO 81
C
C GIVE UP IF THE NUMBER OF ITERATIONS EXCEED 20
C
      IF(NT.GT.20) RETURN
C
C THE SECANT METHOD FOR ITERATING TO THE EIGENVALUE. THIS METHOD KEEPS
C THE TWO MOST RECENTLY COMPUTED POINTS AT EACH ITERATION AND JOINS THEM
C BY A STRAIGHT LINE TO GET THE NEW ESTIMATE FOR THE ROOT
      IF(NT.GT.1) GO TO 53
      DRP=DR
      DIP=DI
      KR1=KR
      KI1=KI
      KR=E1*KR
      KI=E2*KI
C THE 'RETURN 2' STATEMENT SENDS CONTROL IN THE MAIN PROGRAMME TO CALL
C THE SUBROUTINE 'SERIES' AGAIN FOR TRIAL WITH THE NEW ESTIMATE OF K
      RETURN 2
   53 CALL MULDPC (KR,KI,DRP,DIP,CR(1),CI(1))
      CALL MULDPC (KR1,KI1,DR,DI,CR(2),CI(2))
      CR(1)=CR(1)-CR(2)
      CI(1)=CI(1)-CI(2)
      CR(2)=DRP-DR
      CI(2)=DIP-DI
      CALL DIVDPC (CR(1),CI(1),CR(2),CI(2),KR2,KI2)
      DRP=DR
      DIP=DI
```

```
          KR1=KR
          KI1=KI
          KR=KR2
          KI=KI2
          RETURN 2
C HAVING DETERMINED AN EIGENVALUE, THE RATIOS OF EQN.(3.74) ARE CALCULATED
       81 CALL DIVDPC(VEC(3),VEC(4),VEC(1),VEC(2),CR(1),CI(1))
          CALL DIVDPC(VEC(5),VEC(6),VEC(1),VEC(2),CR(2),CI(2))
          PUNCH 71, NN,H,RE,W,KR,KI,RS,ADEI,TE
       71 FORMAT(I1,F5.4,F7.1,F5.3,G22.16,G21.16,F3.2,6X,G8.3,A2)
C THE FOLLOWING CALLS ARE FOR FILLING UP THE ARRAYS URR(L) TO PEI(L), AND
C FOR PLOTTING THE NORMALIZED EIGENFUNCTIONS
          CALL EIGENF (VRR,VRI,URR,URI,CR,CI,N)
          CALL EIGENF (VTR,VTI,UTR,UTI,CR,CI,N)
          CALL EIGENF (VZR,VZI,UZR,UZI,CR,CI,N)
          CALL EIGENF (PRR,PRI,PER,PEI,CR,CI,N)
          CALL NORMFN (URR,RE,W,KR,KI,H,N,NG,1)
          CALL NORMFN (URI,RE,W,KR,KI,H,N,NG,2)
          CALL NORMAB (URR,URI,RE,W,KR,KI,H,N,NG,VEC,RM,1)
          CALL NORMFN (UTR,RE,W,KR,KI,H,N,NG,3)
          CALL NORMFN (UTI,RE,W,KR,KI,H,N,NG,4)
          CALL NORMAB (UTR,UTI,RE,W,KR,KI,H,N,NG,VEC,RM,2)
          CALL NORMFN (UZR,RE,W,KR,KI,H,N,NG,5)
          CALL NORMFN (UZI,RE,W,KR,KI,H,N,NG,6)
          CALL NORMAB (UZR,UZI,RE,W,KR,KI,H,N,NG,VEC,RM,3)
          CALL NORMFN (PER,RE,W,KR,KI,H,N,NG,7)
          CALL NORMFN (PEI,RE,W,KR,KI,H,N,NG,8)
          CALL NORMAB (PER,PEI,RE,W,KR,KI,H,N,NG,VEC,RM,4)
C PUNCHING OUT THE MAXIMA OF THE ABSOLUTE VALUES OF THE FOUR EIGENFUNCTIONS
C AND THE RADII AT WHICH THEY OCCUR.
          PUNCH 82, (VEC(J),RM(J),J=1,4)
       82 FORMAT(4(G15.10,F4.2))
          RETURN
          END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
          SUBROUTINE EIGENF (AR,AI,BR,BI,CR,CI,N)
C
C FOR DETERMINING THE NET EIGENFUNCTIONS BY PROPERLY COMBINING THE THREE
C INDEPENDENT SOLUTIONS FOR THEM (EQUATION (3.70))
C
C CR(L),CI(L) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE RATIOS IN
C EQUATION (3.74)
C BR(L),BI(L) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE EIGENFUNCTION
C
C AR(L,J),AI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE THREE
C INDEPENDENT SOLUTIONS OF AN EIGENFUNCTION
C
          DOUBLE PRECISION AR(51,3),AI(51,3),BR(51),BI(51),CR(2),CI(2),AX,BX
          DO 82 L=1,N
          BR(L)=AR(L,1)
          BI(L)=AI(L,1)
          DO 82 J=2,3
          CALL MULDPC(AR(L,J),AI(L,J),CR(J-1),CI(J-1),AX,BX)
          BR(L)=BR(L)+AX
       82 BI(L)=BI(L)+BX
          RETURN
          END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

```
        SUBROUTINE NORMEN (A,RE,W,KR,KI,H,N,I,K)
C
C FOR NORMALIZING A GIVEN FUNCTION WITH RESPECT TO ITS ABSOLUTE MAXIMUM
C VALUE WITHIN THE WHOLE DOMAIN. IT ALSO CALLS FOR PLOTTING THE
C NORMALIZED VALUE.
C
C A(L) - ARRAY OF FUNCTION VALUES IN THE DOMAIN
C FM - THE ABSOLUTE MAXIMUM VALUE OF THE GIVEN FUNCTION (REAL OR IMAGINARY
C PART OF AN EIGENFUNCTION HERE)
C RM - THE VALUE OF THE INDEPENDENT VARIABLE (R HERE) AT WHICH 'FM' OCCURS
C
        DOUBLE PRECISION A(N),B,RE,W,KR,KI,H,FM
        DIMENSION V(8),F(51)
        DATA V/'VRR','VRI','VTR','VTI','VZR','VZI','PRR','PRI'/
        C=1
C FINDING FM AND RM
        FM=0.0D0
        DO 30 L=1,N
        B=DABS(A(L))
        IF(B.LT.FM) GO TO 30
        FM=B
        NR=L
   30 CONTINUE
        B=NR-1
        RM=H*B+C
        WRITE(6,20) H,RE,W,KR,KI,V(K),FM,RM
   20 FORMAT(1H1, 3HH =,F6.4,3X,4HRE =,F7.1,3X,3HW =,F6.3,3X,4HKR =,G15.
        18,3X,4HKI =,G14.8,4X,19HABS. MAX. VALUE OF ,A3,2H =,G10.4,3X,6HAT
        2R =,F5.2,/)
C CALLING THE PLOT ROUTINE FOR PLOTTING THE NORMALIZED FUNCTION VALUES
        DO 40 L=1,N
        B=L-1
        R=H*B+C
        F(L)=A(L)/FM
        CALL PLTMOD (F(L),R,1.0,-1.0)
   40 CONTINUE
C PUNCHING OUT THE NORMALIZED FUNCTION VALUES, FM AND RM
        PUNCH 50, F,FM,RM
   50 FORMAT(5(10F8.5/),F8.5,G15.10,F4.2)
        IF(K.GT.6) RETURN
C
C THIS STATEMENT HELPS TO SHOW THE DEGREE OF FULFILMENT OF THE BOUNDARY
C CONDITIONS AT R=1 FOR THE VALUE OF K TAKEN AS THE EIGENVALUE. IF THERE
C ARE ANY NUMERICAL INSTABILITIES EITHER DUE TO A LARGE STEP SIZE OR DUE
C TO TOO MUCH ROUND-OFF ERROR (FOUND TO BE INSIGNIFICANT DURING THE PRESENT
C INVESTIGATION ), THE BOUNDARY CONDITIONS AT R=1 WILL NOT BE SATISFIED
C EVEN IF THE DETERMINANT VALUE IS 'ZERO'.
C
        WRITE(6,10) V(K),F(N)
   10 FORMAT(//,10X,24HTHE NORMALIZED VALUE OF ,A3,52H WHICH SHOULD BE E
        1XACTLY ZERO AT THE RIGID WALL IS =,G11.5)
        RETURN
        END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
        SUBROUTINE NORMAB (AR,AI,RE,W,KR,KI,H,N,I,B,RM,K)
C
C FOR NORMALIZING THE ABSOLUTE VALUE OF A GIVEN COMPLEX FUNCTION (AN
C EIGENFUNCTION HERE), AND CALLING FOR ITS PLOT
```

```
C
C B(K) - ELEMENTS OF THIS ARRAY CONTAIN THE MAX. VALUE OF THE ABSOLUTE
C VALUE OF THE EIGENFUNCTIONS. STORED FOR PUNCHING THEM ON ONE CARD IN
C SUBROUTINE 'SECANT' OR 'ASECAN'
C
C RM(K) - ARRAY FOR THE VALUE OF RADII AT WHICH B(K) OCCUR. STORED FOR
C THE SAME REASON AS B(K).
C AB(L) - ARRAY FOR THE ABSOLUTE VALUES OF THE COMPLEX FUNCTION
C
      DOUBLE PRECISION AR(N),AI(N),AB(51),RE,W,KR,KI,H,P(4),ABSDPC
      DIMENSION V(4),RM(4)
      DATA V/'VR','VT','VZ','PR'/
      C=I
C FINDING THE ABSOLUTE VALUE OF THE COMPLEX FUNCTION AND ITS MAXIMA
      B(K)=0.0D0
      DO 30 L=1,N
      AB(L)=ABSDPC(AR(L),AI(L))
      IF(AB(L).LT.B(K)) GO TO 30
      B(K)=AB(L)
      NR=L
   30 CONTINUE
      D=NR-1
      RM(K)=H*D*C
      WRITE(6,20) H,RE,W,KR,KI,V(K),B(K),RM(K)
   20 FORMAT(1H1, 3HH =,F6.4,3X,4HRE =,F7.1,3X,3HW =,F6.3,3X,4HKR =,G15.
     18,3X,4HKI =,G14.6,4X,19HMAX. VALUE OF ABS. ,A2,2H =,G10.4,3X,6HAT
     2R =,F5.2,/)
C CALLING THE PLOT ROUTINE FOR PLOTTING THE NORMALIZED ABSOLUTE VALUES
      DO 40 L=1,N
      D=L-1
      R=H*D*C
      FN=AB(L)/B(K)
      CALL PLTMOD (FN,R,1.0,-1.0)
   40 CONTINUE
      RETURN
      END
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE MULDPC (A,B,C,D,E,F)
C
C FOR COMPUTING THE COMPLEX PRODUCT  (A+IB)(C+ID) = E+IF .  IT IS ALSO
C CAPABLE OF COMPUTING WITHOUT CONFUSION MULDPC (A,B,A,B,A,B) WHICH
C IMPLIES THAT (A+IB)**2 = (IS REPLACED BY) = A+IB
C
      DOUBLE PRECISION A,B,C,D,E,F,AA,BB,CC,DD
C
C THE NEXT FOUR STATEMENTS ARE REQUIRED TO GET AROUND A SYSTEM BUG
      AA=A
      BB=B
      CC=C
      DD=D
      E=AA*CC-BB*DD
      F=AA*DD+BB*CC
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE DIVDPC (A,B,C,D,E,F)
C
C FOR COMPUTING THE COMPLEX DIVISION  (A+IB)/(C+ID) = E+IF .  IT IS ALSO
```

```
C CAPABLE OF COMPUTING WITHOUT CONFUSION (A+IB)/(C+ID) = (IS REPLACED BY)
C = (A+IB) OR (C+ID)
C
      DOUBLE PRECISION A,B,C,D,E,F,AA,BB,CC,DD,G
C
C THE NEXT FOUR STATEMENTS ARE REQUIRED TO GET AROUND A SYSTEM BUG
      AA=A
      BB=B
      CC=C
      DD=D
      G=CC*CC+DD*DD
      E=(AA*CC+BB*DD)/G
      F=(BB*CC-AA*DD)/G
      RETURN
      END
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DOUBLE PRECISION FUNCTION ABSDPC(A,B)
C
C FOR COMPUTING THE ABSOLUTE VALUE OF A DOUBLE PRECISION COMPLEX VARIABLE
C
      DOUBLE PRECISION A,B
      ABSDPC=DSQRT(A*A+B*B)
      RETURN
      END
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE PLTMOD (FRI,R,C,D)
C
C THIS SUBROUTINE PLOTS ON THE PRINTER PAPER A FUNCTION EVALUATED AT
C EQUIDISTANT VALUES OF THE INDEPENDENT VARIABLE. IT IS A SLIGHTLY MODIFIED
C VERSION OF THE PLOT ROUTINE WRITTEN BY DR. R. W. HORNBECK, MECHANICAL
C ENGINEERING DEPARTMENT, CARNEGIE-MELLON UNIVERSITY.
C
C FRI - VALUE OF THE FUNCTION TO BE PLOTTED
C R - VALUE OF THE INDEPENDENT VARIABLE
C C - MAXIMUM VALUE OF THE FUNCTION IN THE WHOLE DOMAIN
C D - MINIMUM VALUE OF THE FUNCTION IN THE WHOLE DOMAIN
C
C A POINT CORRESPONDING TO THE FUNCTION VALUE IS PLOTTED EACH TIME THE
C SUBROUTINE IS CALLED. AS THE PRINTER PAPER ADVANCES AT A CONSTANT RATE,
C THIS SUBROUTINE IS USEFUL ONLY FOR PLOTTING A FUNCTION KNOWN AT EQUIDIS-
C TANT VALUES OF THE INDEPENDENT VARIABLE.
C
      DIMENSION A(102),E(102)
      DATA A/' ',',','.',49*' ','+',49*' ',',','.'/,B/'*'/
      DATA E/' ',',','.',49*' ','+',49*' ',',','.'/
      I=(FRI-D)*100./(C-D)+2.
      A(I)=B
      WRITE(6,100) A,FRI,R
100   FORMAT(102A1,4X,6HF(R) =,F8.5,2X,3HR =,F4.2)
      DO 101 I=1,102
101   A(I)=E(I)
      RETURN
      END
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE EFSRPC (HR,AN,AIN,NN,ITS,RS,MES,H,NPC)
C
C* THIS IS THE SECOND SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR A NON-
C AXISYMMETRIC DISTURBANCE *
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C
C THIS SUBROUTINE CALCULATES THE EIGENFUNCTIONS ACCORDING TO THE SERIES
C EXPANSIONS (EQN.(3.56)) FOR (0.LE.R.LE.RS) AT STEPS OF HH. IT ALSO CAL-
C CULATES THE EIGENFUNCTIONS AND THEIR FIRST AND SECOND DERIVATIVES AT
C FOUR VALUES OF R NAMELY RS, (RS-H), (RS-2*H), AND (RS-3*H).  THESE ARE
C REQUIRED LATER FOR STEP-BY-STEP INTEGRATION BY THE FOURTH ORDER PREDICTOR
C -CORRECTOR METHOD.
C
C ****************************************************************
C IF THE RUNGE-KUTTA METHOD IS USED FOR STEP-BY-STEP INTEGRATION, USE
C SUBROUTINE 'EFSRRK' IN PLACE OF THIS.
C ****************************************************************
C
C THE INPUT VARIABLES NOT ALREADY EXPLAINED EARLIER ARE HH,NN,NES,NPC,H
C   HH - RADIAL INTERVAL AT WHICH THE EIGENFUNCTION VALUES ARE STORED FOR
C        PLOTTING IF AN EIGENVALUE IS FOUND. HH=0.02 HERE. IF A LOWER VALUE
C        IS DESIRED, RELEVANT DIMENSIONS MUST BE ACCORDINGLY INCREASED.
C   NN = AN = N = 1,2,3.........
C   NES - NUMBER OF POINTS FOR WHICH EIGENFUNCITONS FOUND BY THE SERIES
C         SOLUTION ARE STORED FOR PLOTTING LATER.  NES = 1 + RS/HH
C   H - THE STEP SIZE IN RADIAL DIRECTION FOR STEP-BY-STEP INTEGRATION SO
C       THAT (H.LE.HH)
C   NPC - THE ORDER OF THE PREDICTOR-CORRECTOR METHOD USED (NPC=4 HERE)
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C VRR(L,J),VRI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE EIGEN-
C FUNCTION VR(R). THE FIRST SUBSCRIPT INDICATES THE RADIAL POSITION FOR
C WHICH THE EIGENFUNCTION VALUE IS STORED, AND THE SECOND SUBSCRIPT REFERS
C TO ANYONE OF THE THREE INDEPENDENT SOLUTIONS WHICH MAKE UP VR(R).
C SIMILAR IS THE CASE WITH ARRAYS FOR VZR,VZI,VTR,VTI,PPR,PRI
C
C CVRR(L,J) TO CDDVZI(L,J) - 22 ARRAYS TO STORE THE VALUES OF THE EIGEN-
C FUNCTIONS AND THEIR FIRST AND SECOND DERIVATIVES FOR THE FOUR VALUES OF
C R NAMELY RS, (RS-H), (RS-2*H), RS-3*H). THE SIZE OF EACH OF THESE ARRAYS
C MUST BE (NPC+1,3). THESE ARE ALL NEEDED LATER FOR THE PREDICTOR-
C CORRECTOR METHOD.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION GR(20,3),VRR(51,3),VRI(51,3),VTR(51,3),VTI(51,3),VZR(51,
     13),VZI(51,3),PRR(51,3),PRI(51,3),CVRR(5,3),CVRI(5,3),CVTR(5,3),CVT
     2I(5,3),CVZR(5,3),CVZI(5,3),CDVRR(5,3),CDVRI(5,3),CDVTR(5,3),CDVTI(
     35,3),CDVZR(5,3),CDVZI(5,3),CDPRR(5,3),CDPRI(5,3),CDDVRR(5,3),CDDVR
     4I(5,3),CDDVTR(5,3),CDDVTI(5,3),CDDVZR(5,3),CDDVZI(5,3),CPRR(5,3),C
     5PRI(5,3),GI(20,3),FR(20,3),FI(20,3),UR(20,3),UI(20,3),PR(20,3),PI(
     620,3)
      COMMON /AREA1/ GR,GI,FR,FI,UR,UI,PR,PI
      COMMON /AREA2/ VRR,VRI,VTR,VTI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ CVRR,CVRI,CVTR,CVTI,CVZR,CVZI,CPRR,CPRI,CDVRR,CDVRI
     1,CDVTR,CDVTI,CDVZR,CDVZI,CDPRR,CDPRI,CDDVRR,CDDVRT,CDDVTR,CDDVTI,C
     2DDVZR,CDDVZI
C SETTING UP THE BOUNDARY CONDITIONS ON VR(R) AND VTHETA(R) AT R=0.  THE
C BOUNDARY CONDITIONS ON VZ(R) AND P(R) AT R=0 ARE SET IN THE MAIN PROGRAM
C SINCE THEY ARE ALL ZERO FOR (N.GE.1) .
C
      IF(NN.EQ.1) GO TO 17
      DO 16 J=1,3
      VRR(1,J)=0.0D0
```

```
      VRI(1,J)=0.0D0
      VTR(1,J)=0.0D0
   16 VTI(1,J)=0.0D0
      GO TO 14
   17 DO 18 J=1,3
      VRR(1,J)=GR(1,J)/2.D0
      VRI(1,J)=GI(1,J)/2.D0
      VTR(1,J)=-GI(1,J)/2.D0
   18 VTI(1,J)=GR(1,J)/2.D0
C FINDING THE EIGENFUNCTIONS AND THEIR FIRST AND SECOND DERIVATIVES AT
C THE FOUR VALUES OF R USING THE SERIES SOLUTION
   14 R=NPC
      R=RS-R*H
      DO 19 LL=1,NPC
      R=R+H
      RF=R**(NN+1)
      RU=R**NN
      IF(NN.EQ.1) GO TO 22
      RG=R**(NN-1)
      GO TO 32
   22 RG=1.0D0
      RDG=1.0D0/R
      RDDG=RDG/R
      GO TO 23
   32 IF(NN.EQ.2) GO TO 33
      RDG=R**(NN-2)
      GO TO 34
   33 RDG=1.0D0
      RDDG=RDG/R
      GO TO 23
   34 IF(NN.EQ.3) GO TO 36
      RDDG=R**(NN-3)
      GO TO 23
   36 RDDG=1.0D0
   23 DO 19 J=1,3
C INITIALISING THE SERIES FOR EIGENFUNCTIONS, THEIR FIRST AND
C SECOND DERIVATIVES
      FSR    =FR(1,J)
      FSI    =FI(1,J)
      GSR    =GR(1,J)
      GSI    =GI(1,J)
      CVZR(LL,J)=UR(1,J)
      CVZI(LL,J)=UI(1,J)
      CPRR(LL,J)=PR(1,J)
      CPRI(LL,J)=PI(1,J)
      DFSR=FR(1,J)*ANN
      DFSI=FI(1,J)*ANN
      DGSR=GR(1,J)*(AN-1.0D0)
      DGSI=GI(1,J)*(AN-1.0D0)
      CDVZR(LL,J)=UR(1,J)*AN
      CDVZI(LL,J)=UI(1,J)*AN
      CDPRR(LL,J)=PR(1,J)*AN
      CDPRI(LL,J)=PI(1,J)*AN
      DDFSR=FR(1,J)*ANN*ANN
      DDFSI=FI(1,J)*ANN*ANN
      DDGSR=GR(1,J)*(AN-2.0D0)*(AN-1.0D0)
      DDGSI=GI(1,J)*(AN-2.0D0)*(AN-1.0D0)
      CDDVZR(LL,J)=UR(1,J)*AN*(AN-1.D0)
```

```
      CDDVZI(LL,J)=UI(1,J)*AN*(AN-1.D0)
      DO 24 L=2,NTS
      AF=NN+2*L-1
      AU=AF-1.0D0
      AG=AU-1.0D0
      AP=AG-1.0D0
      RP=R**(2*(L-1))
C SUMMING UP THE SERIES
      FSR     =FSR     +FR(L,J)*RP
      FSI     =FSI     +FI(L,J)*RP
      GSR     =GSR     +GR(L,J)*RP
      GSI     =GSI     +GI(L,J)*RP
      CVZR(LL,J)=CVZR(LL,J)+UR(L,J)*RP
      CVZI(LL,J)=CVZI(LL,J)+UI(L,J)*RP
      CPRR(LL,J)=CPRR(LL,J)+PR(L,J)*RP
      CPRI(LL,J)=CPRI(LL,J)+PI(L,J)*RP
      DFSR=DFSR+FR(L,J)*RP*AF
      DFSI=DFSI+FI(L,J)*RP*AF
      DGSR=DGSR+GR(L,J)*RP*AG
      DGSI=DGSI+GI(L,J)*RP*AG
      CDVZR(LL,J)=CDVZR(LL,J)+UP(L,J)*RP*AU
      CDVZI(LL,J)=CDVZI(LL,J)+UI(L,J)*RP*AU
      CDPRR(LL,J)=CDPRR(LL,J)+PR(L,J)*RP*AU
      CDPRI(LL,J)=CDPRI(LL,J)+PI(L,J)*RP*AU
      DDFSR=DDFSR+FR(L,J)*RP*AF*AU
      DDFSI=DDFSI+FI(L,J)*RP*AF*AU
      DDGSR=DDGSR+GR(L,J)*RP*AG*AP
      DDGSI=DDGSI+GI(L,J)*RP*AG*AP
      CDDVZR(LL,J)=CDDVZR(LL,J)+UR(L,J)*RP*AU*AG
   24 CDDVZI(LL,J)=CDDVZI(LL,J)+UI(L,J)*RP*AU*AG
      CVRR(LL,J)=(FSR     *RF+GSR     *RG)/2.D0
      CVRI(LL,J)=(FSI     *RF+GSI     *RG)/2.D0
      CVTR(LL,J)=(FSI     *RF-GSI     *RG)/2.D0
      CVTI(LL,J)=(GSR     *RG-FSR     *RF)/2.D0
      CVZR(LL,J)=CVZR(LL,J)*RU
      CVZI(LL,J)=CVZI(LL,J)*RU
      CPRR(LL,J)=CPRR(LL,J)*RU
      CPRI(LL,J)=CPRI(LL,J)*RU
      CDVRR(LL,J)=(DFSR*RU+DGSR*RDG)/2.D0
      CDVRI(LL,J)=(DFSI*RU+DGSI*RDG)/2.D0
      CDVTR(LL,J)=(DFSI*RU-DGSI*RDG)/2.D0
      CDVTI(LL,J)=(DGSR*RDG-DFSR*RU)/2.D0
      CDVZR(LL,J)=CDVZR(LL,J)*RG
      CDVZI(LL,J)=CDVZI(LL,J)*RG
      CDPRR(LL,J)=CDPRR(LL,J)*RG
      CDPRI(LL,J)=CDPRI(LL,J)*RG
      CDDVRR(LL,J)=(DDFSR*RG+DDGSR*RDDG)/2.D0
      CDDVRI(LL,J)=(DDFSI*RG+DDGSI*RDDG)/2.D0
      CDDVTR(LL,J)=(DDFSI*RG-DDGSI*RDDG)/2.D0
      CDDVTI(LL,J)=(DDGSR*RDDG-DDFSR*RG)/2.D0
      CDDVZR(LL,J)=CDDVZR(LL,J)*RDG
   19 CDDVZI(LL,J)=CDDVZI(LL,J)*RDG
C CALCULATING THE EIGENFUNCTIONS FOR (0.LE.R.LE.RS) AT INTERVALS OF HH BY
C MEANS OF THE SERIES EXPANSIONS
      R=0.0D0
      DO 15 LL=2,NES
      R=R+HH
      RF=R**(NN+1)
```

```
      RU=R**NN
      IF(NN.EQ.1) GO TO 28
      RG=R**(NN-1)
      GO TO 37
   28 RG=1.0D0
   37 DO 13 J=1,3
C INITIALIZING THE SUMMATION OF SERIES
      FSR    =FR(1,J)
      FSI    =FI(1,J)
      GSR    =GR(1,J)
      GSI    =GI(1,J)
      VZR(LL,J)=UR(1,J)
      VZI(LL,J)=UI(1,J)
      PRR(LL,J)=PR(1,J)
      PRI(LL,J)=PI(1,J)
C SUMMING UP THE SERIES
      DO 38 L=2,NTS
      RP=R**(2*(L-1))
      FSR    =FSR    +FR(L,J)*RP
      FSI    =FSI    +FI(L,J)*RP
      GSR    =GSR    +GR(L,J)*RP
      GSI    =GSI    +GI(L,J)*RP
      VZR(LL,J)=VZR(LL,J)+UR(L,J)*RP
      VZI(LL,J)=VZI(LL,J)+UI(L,J)*RP
      PRR(LL,J)=PRR(LL,J)+PR(L,J)*RP
   38 PRI(LL,J)=PRI(LL,J)+PI(L,J)*RP
      VRR(LL,J)=(FSR    *RF+GSR    *RG)/2.D0
      VRI(LL,J)=(FSI    *RF+GSI    *RG)/2.D0
      VTR(LL,J)=(FSI    *RF-GSI    *RG)/2.D0
      VTI(LL,J)=(GSR    *RG-FSR    *RF)/2.D0
      VZR(LL,J)=VZR(LL,J)*RU
      VZI(LL,J)=VZI(LL,J)*RU
      PRR(LL,J)=PRR(LL,J)*RU
   13 PRI(LL,J)=PRI(LL,J)*RU
      RETURN
      END
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE PREDCT (RE,W,KR,KI,ND,NG,AM,RS,NU,C,EPL,H,LS,NPC,NPE,NE
     1S)
C
C* THIS IS THE THIRD SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR A NON-
C AXISYMMETRIC DISTURBANCE *
C
C STARTING FROM R=RS (UPTO WHICH THE SERIES SOLUTION WAS USED), THIS SUBROU
C TINE INTEGRATES THE STABILITY EQUATIONS (3.71) IN A STEP-BY-STEP SCHEME
C BY THE USE OF FOURTH ORDER ADAMS-BASHFORTH-MOULTON PREDICTOR-
C CORRECTOR METHOD. SEE APPENDIX I FOR MORE CLARIFICATION.
C
C THE INPUT VARIABLES NOT ALREADY CLARIFIED ARE LS,EPL,NPE,NG,ND
C   LS = 1 + RS/H
C   EPL - ALLOWABLE ERROR BOUND PER STEP (SEE EQN.(AI-4) AND RELATED
C         DISCUSSION)
C   NPE - NUMBER OF POINTS FOR WHICH THE EIGENFUNCTION VALUES ARE STORED
C         FOR PLOTTING LATER ON. NPE = 1 + 1./HH . IF NPE IS TO EXCEED 51,
C         RELEVANT DIMENSIONS MUST BE INCREASED.
C   NG = (ND-1)/(NPE-1) = HH/H
C   ND = 1 + 1./H
C
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C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C C1 TO C7 - COEFFICIENTS IN THE PREDICTOR-CORRECTOR EQUATIONS (AI-2)
C AND (AI-3)
C CVRR(L,J) TO CDDVZI(L,J) - FOR THESE 22 ARRAYS REFER TO SUBROUTINE EFSRPC
C
C ADVRR TO APRI - PREDICTED VALUES I.E. VALUES OF THE LEFT HAND SIDES
C OF EQUATIONS (AI-2) FOR EIGENFUNCTIONS AND/OR THEIR DERIVATIVES
C
C VZM - THE STEDY MEAN FLOW VELOCITY (EQN.(3.22))
C DVZM - DERIVATIVE OF VZM WITH RESPECT TO R
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION KR,KI
      DIMENSION C(NU), VRR(51,3),VRI(51,3),VTR(51,3),VTT(51,3),VZR(51,3)
     1,VZI(51,3),PRR(51,3),PRI(51,3),CVRR(5,3),CVRI(5,3),CVTR(5,3),CVTI(
     25,3),CVZR(5,3),CVZI(5,3),CDVRR(5,3),CDVRI(5,3),CDVTR(5,3),CDVTI(5,
     33),CDVZR(5,3),CDVZI(5,3),CDPRR(5,3),CDPRI(5,3),CDDVRR(5,3),CDDVRI(
     45,3),CDDVTR(5,3),CDDVTI(5,3),CDDVZR(5,3),CDDVZI(5,3),CPRR(5,3),CPR
     5I(5,3)
      COMMON /AREA2/ VRR,VRI,VTR,VTT,VZR,VZI,PRR,PRI
      COMMON /AREA3/ CVRR,CVRI,CVTR,CVTI,CVZR,CVZI,CPRR,CPRI,CDVRR,CDVRI
     1,CDVTR,CDVTI,CDVZR,CDVZI,CDPRR,CDPRI,CDDVRR,CDDVRI,CDDVTR,CDDVTI,C
     2DDVZR,CDDVZI
      K=NES
      MUL=1
      N=NPC+1
      C1=55.D0*H/24.D0
      C2=59.D0*H/24.D0
      C3=37.D0*H/24.D0
      C4= 9.D0*H/24.D0
      C5=19.D0*H/24.D0
      C6=C1/11.D0
      C7=C5/5.D0
      M=LS+1
      R=RS
      DO 42 L=M,ND
      R=R+H
      SR=1.D0/(R*R)
C CALCULATING VZ AND DVZ FOR THE STEADY MEAN FLOW
      VZM=C(1)
      DVZM=0.0D0
      DO 44 J=2,NU
      AR=2*(J-1)
      VZM=VZM+C(J)*(R**(2*(J-1)))
   44 DVZM=DVZM+AR*C(J)*(R**(2*J-3))
      AR=(AR*AR+1.D0)*SR+KR*(RE*VZM-KR)+KI*KI
      AI=KI*(RE*VZM-2.D0*KR)-RE*W
      DO 47 J=1,3
      NM=1
      NIT=0
C USING THE PREDICTOR EQUATION (AI-2)
C
      ADVRR=CDVRR(NPC,J)+C1*CDDVRR(NPC,J)-C2*CDDVRR(N-2,J)+C3*CDDVRR(N-3
     1,J)-C4*CDDVRR(N-4,J)
      ADVRI=CDVRI(NPC,J)+C1*CDDVRI(NPC,J)-C2*CDDVRI(N-2,J)+C3*CDDVRI(N-3
     1,J)-C4*CDDVRI(N-4,J)
      ADVTR=CDVTR(NPC,J)+C1*CDDVTR(NPC,J)-C2*CDDVTR(N-2,J)+C3*CDDVTR(N-3
```

```
     1,J)-C4*CDDVTR(N-4,J)
      ADVTI=CDVTI(NPC,J)+C1*CDDVTI(NPC,J)-C2*CDDVTI(N-2,J)+C3*CDDVTI(N-3
     1,J)-C4*CDDVTI(N-4,J)
      ADVZR=CDVZR(NPC,J)+C1*CDDVZR(NPC,J)-C2*CDDVZR(N-2,J)+C3*CDDVZR(N-3
     1,J)-C4*CDDVZR(N-4,J)
      ADVZI=CDVZI(NPC,J)+C1*CDDVZI(NPC,J)-C2*CDDVZI(N-2,J)+C3*CDDVZI(N-3
     1,J)-C4*CDDVZI(N-4,J)
      APRR=CPRR(NPC,J)+C1*CDPRR(NPC,J)-C2*CDPRR(N-2,J)+C3*CDPRR(N-3,J)-C
     14*CDPRR(N-4,J)
      APRI=CPRI(NPC,J)+C1*CDPRI(NPC,J)-C2*CDPRI(N-2,J)+C3*CDPRI(N-3,J)-C
     14*CDPRI(N-4,J)
C USING THE CORRECTOR EQUATION (AI-3) FOR VELOCITY EIGENFUNCTIONS ONLY
C SINCE THEY INVOLVE SECOND ORDER DIFFERENTIAL EQUATIONS
C
   43 CVRR(N,J)=CVRR(NPC,J)+C4*ADVRR+C5*CDVRR(NPC,J)-C6*CDVRR(N-2,J)+C7*
     1CDVRR(N-3,J)
      CVRI(N,J)=CVRI(NPC,J)+C4*ADVRI+C5*CDVRI(NPC,J)-C6*CDVRI(N-2,J)+C7*
     1CDVRI(N-3,J)
      CVTR(N,J)=CVTR(NPC,J)+C4*ADVTR+C5*CDVTR(NPC,J)-C6*CDVTR(N-2,J)+C7*
     1CDVTR(N-3,J)
      CVTI(N,J)=CVTI(NPC,J)+C4*ADVTI+C5*CDVTI(NPC,J)-C6*CDVTI(N-2,J)+C7*
     1CDVTI(N-3,J)
      CVZR(N,J)=CVZR(NPC,J)+C4*ADVZR+C5*CDVZR(NPC,J)-C6*CDVZR(N-2,J)+C7*
     1CDVZR(N-3,J)
      CVZI(N,J)=CVZI(NPC,J)+C4*ADVZI+C5*CDVZI(NPC,J)-C6*CDVZI(N-2,J)+C7*
     1CDVZI(N-3,J)
C CALCULATING THE LEFT HAND SIDES OF EQUATIONS (3.71) WRITTEN IN THEIR REAL
C AND IMAGINARY PARTS
C
      CALL MULDPC (KR,KI,ADVZR,ADVZI,AXR,AXI)
      CDDVRR(N,J)=SR*(CVRR(N,J)-AN*CVTI(N,J))-(ADVRR-AN*ADVTI)/R-AXR
      CDDVRI(N,J)=SR*(CVRI(N,J)+AN*CVTR(N,J))-(ADVRI+AN*ADVTR)/R-AXI
      CALL MULDPC (AR,AI,CVTR(N,J),CVTI(N,J),AXR,AXI)
      CDDVTR(N,J)=-(AN*RE*APRI+ADVTR)/R+2.D0*AN*SR*CVRI(N,J)+AXR
      CDDVTI(N,J)= (AN*RE*APRR-ADVTI)/R-2.D0*AN*SR*CVRR(N,J)+AXI
      CALL MULDPC (AR,AI,CVZR(N,J),CVZI(N,J),AXR,AXI)
      CALL MULDPC (KR,KI,APRR,APRI,BXR,BXI)
      CDDVZR(N,J)=RE*(UVZM*CVRR(N,J)+BXR)-ADVZR/R-SR*CVZR(N,J)+AXR
      CDDVZI(N,J)=RE*(UVZM*CVRI(N,J)+BXI)-ADVZI/R-SR*CVZI(N,J)+AXI
      CALL MULDPC (AR,AI,CVRR(N,J),CVRI(N,J),AXR,AXI)
      CDPRR(N,J)=(CDDVRR(N,J)+ADVRR/R+2.D0*AN*SR*CVTI(N,J)-AXR)/RE
      CDPRI(N,J)=(CDDVRI(N,J)+ADVRI/R-2.D0*AN*SR*CVTR(N,J)-AXI)/RE
C
C USING THE CORRECTOR EQUATION (AI-3)
C
      CDVRR(N,J)=CDVRR(NPC,J)+C4*CDDVRR(N,J)+C5*CDDVRR(NPC,J)-C6*CDDVRR(
     1N-2,J)+C7*CDDVRR(N-3,J)
      CDVRI(N,J)=CDVRI(NPC,J)+C4*CDDVRI(N,J)+C5*CDDVRI(NPC,J)-C6*CDDVRI(
     1N-2,J)+C7*CDDVRI(N-3,J)
      CDVTR(N,J)=CDVTR(NPC,J)+C4*CDDVTR(N,J)+C5*CDDVTR(NPC,J)-C6*CDDVTR(
     1N-2,J)+C7*CDDVTR(N-3,J)
      CDVTI(N,J)=CDVTI(NPC,J)+C4*CDDVTI(N,J)+C5*CDDVTI(NPC,J)-C6*CDDVTI(
     1N-2,J)+C7*CDDVTI(N-3,J)
      CDVZR(N,J)=CDVZR(NPC,J)+C4*CDDVZR(N,J)+C5*CDDVZR(NPC,J)-C6*CDDVZR(
     1N-2,J)+C7*CDDVZR(N-3,J)
      CDVZI(N,J)=CDVZI(NPC,J)+C4*CDDVZI(N,J)+C5*CDDVZI(NPC,J)-C6*CDDVZI(
     1N-2,J)+C7*CDDVZI(N-3,J)
      CPRR(N,J)=CPRR(NPC,J)+C4*CDPRR(N,J)+C5*CDPRR(NPC,J)-C6*CDPRR(N-2,J
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```
      1)+C7*CDPRR(N-3,J)
       CPRI(N,J)=CPRI(NPC,J)+C4*CDPRI(N,J)+C5*CDPRI(NPC,J)-C6*CDPRI(N-2,J
      1)+C7*CDPRI(N-3,J)
       IF(NM.EQ.2) GO TO 39
       NIT=NIT+1
C FINDING THE MAXIMA OF THE ERRORS INVOLVED FOR ALL EIGENFUNCTIONS USING
C EQUATION (AI-4)
       BXR=ADVRR-CDVRR(N,J)
       BXI=ADVRI-CDVRI(N,J)
       AXR=ABSDPC(BXR,BXI)
       BXR=ADVTR-CDVTR(N,J)
       BXI=ADVTI-CDVTI(N,J)
       AXI=ABSDPC(BXR,BXI)
       AXR=DMAX1(AXR,AXI)
       BXR=ADVZR-CDVZR(N,J)
       BXI=ADVZI-CDVZI(N,J)
       AXI=ABSDPC(BXR,BXI)
       AXR=DMAX1(AXR,AXI)
       BXR=APRR-CPRR(N,J)
       BXI=APRI-CPRI(N,J)
       AXI=ABSDPC(BXR,BXI)
C AXR - MAXIMUM ERROR
       AXR=DMAX1(AXR,AXI)/2D.D0
       ADVRR=CDVRR(N,J)
       ADVRI=CDVRI(N,J)
       ADVTR=CDVTR(N,J)
       ADVTI=CDVTI(N,J)
       ADVZR=CDVZR(N,J)
       ADVZI=CDVZI(N,J)
       APRR=CPRR(N,J)
       APRI=CPRI(N,J)
       IF(AXR.LE.EPL) GO TO 41
C A WARNING IS ISSUED IF THE MAX. ERROR IS NOT WITHIN THE ERROR BOUND
C (I.E. EPL) EVEN AFTER 5 ITERATIONS OF THE CORRECTOR.
       IF(NIT-5 ) 43,46,46
   41 NM=2
       GO TO 43
   46 WRITE(6,48) NIT,AXR,L,J
   48 FORMAT   (10X,39H THE STEP SIZE IS TOO LARGE, EVEN AFTER,I3,29H I
      1TERATIONS, THE MAX. ERROR =,D11.4,I8,I4)
       NM=2
       GO TO 43
C SHIFTING THE ELEMENTS IN THE 22 ARRAYS BEFORE GOING TO THE NEXT
C INTEGRATION STEP.
   39 DO 47 LL=1,NPC
       CVRR(LL,J)=CVRR(LL+1,J)
       CVRI(LL,J)=CVRI(LL+1,J)
       CVTR(LL,J)=CVTR(LL+1,J)
       CVTI(LL,J)=CVTI(LL+1,J)
       CVZR(LL,J)=CVZR(LL+1,J)
       CVZI(LL,J)=CVZI(LL+1,J)
       CPRR(LL,J)=CPRR(LL+1,J)
       CPRI(LL,J)=CPRI(LL+1,J)
       CDVRR(LL,J)=CDVRR(LL+1,J)
       CDVRI(LL,J)=CDVRI(LL+1,J)
       CDVTR(LL,J)=CDVTR(LL+1,J)
       CDVTI(LL,J)=CDVTI(LL+1,J)
       CDVZR(LL,J)=CDVZR(LL+1,J)
```

```
      CDVZI(LL,J)=CDVZI(LL+1,J)
      CDPRR(LL,J)=CDPRR(LL+1,J)
      CDPRI(LL,J)=CDPRI(LL+1,J)
      CDDVRR(LL,J)=CDDVRR(LL+1,J)
      CDDVRI(LL,J)=CDDVRI(LL+1,J)
      CDDVTR(LL,J)=CDDVTR(LL+1,J)
      CDDVTI(LL,J)=CDDVTI(LL+1,J)
      CDDVZR(LL,J)=CDDVZR(LL+1,J)
   47 CDDVZI(LL,J)=CDDVZI(LL+1,J)
C THE NEXT ARITHMETIC IF STATEMENT IMPLIES THAT IF R IS EQUAL TO A MULTIPLE
C OF HH, EIGENFUNCTION VALUES ARE TO BE STORED IN PROPER ARRAYS BEFORE
C INTEGRATION IS CONTINUED FURTHER.
      IF(L-LS-MUL*NG) 42,57,42
   57 MUL=MUL+1
      K=K+1
      DO 49 J=1,3
      VRR(K,J)=CVRR(NPC,J)
      VRI(K,J)=CVRI(NPC,J)
      VTI(K,J)=CVTI(NPC,J)
      VTR(K,J)=CVTR(NPC,J)
      VZR(K,J)=CVZR(NPC,J)
      VZI(K,J)=CVZI(NPC,J)
      PRR(K,J)=CPRR(NPC,J)
   49 PRI(K,J)=CPRI(NPC,J)
   42 CONTINUE
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE ASERIS (RE,W,KR,KI,NU,RS,EPS,MNTS,C,NTS,AXR,AXI)
C
C* THIS IS THE FIRST SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR AN AXISYM
C METRIC DISTURBANCE *
C
C FOR A PREASSIGNED VALUE OF THE RADIUS UPTO WHICH THE SERIES SOLUTION IS
C CARRIED, THIS SUBROUTINE EVALUATES THE TERMS IN THE SERIES EXPANSION OF
C THE EIGENFUNCTIONS (SEE SECTION 3.1.1). ALL THE SERIES ARE TERMINATED IF
C THE ABSOLUTE VALUE OF THE RATIO OF THE LAST TERM RETAINED TO THE PARTIAL
C SUM FOR ALL SERIES DOES NOT EXCEED A PREASSIGNED EPSILON ('EPS' HERE)
C
C VARIABLES TO BE SUPPLIED BY THE MAIN PROGRAMME ARE RE,W,KR,KI,NU,RS,
C EPS,MNTS,C
C    RE - REYNOLDS NUMBER
C    W - DIMENSIONLESS FREQUENCY
C    KR,KI - REAL AND IMAGINARY PARTS OF THE DIMENSIONLESS WAVE NUMBER K
C    NU - NUMBER OF TERMS IN THE SERIES EXPANSION OF VZ (EQN. (3.22))
C    RS - THE VALUE OF R UPTO WHICH THE SERIES SOLUTION IS USED
C    EPS - ALREADY EXPLAINED
C    MNTS - MAX. NUMBER OF TERMS ALLOWED IN THE SERIES EXPANSION, SHOULD NOT
C           EXCEED 20 FOR THE DIMENSIONS USED HERE
C    C - ARRAY OF NU ELEMENTS FOR THE COEFFICIENTS IN THE SERIES EXPANSION
C        OF VZ (EQN. (3.22))
C    NTS - ACTUAL NUMBER OF TERMS USED IN THE SERIES EXPANSION FOR EIGEN-
C          FUNCTIONS. IT IS AN OUTPUT OF THE SUBROUTINE. (NTS.LE.MNTS)
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C BR(J),BI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF B'S(EQN. (3.23))
C
```

```
C VR(L,J),VI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF V'S. THE
C FIRST SUBSCRIPT INDICATES THE TERM NUMBER IN THE SERIES FOR THE EIGENFUNC
C TION VR(R)(SEE EQN.(3.19)), AND THE SECOND SUBSCRIPT REFERS TO THE
C COEFFICIENT OF EITHER ONE OF THE TWO INDEPENDENT CONSTANTS (U1,P1) FOR
C EQUATIONS OF THE FORM OF EQN.(3.32), FOR EXAMPLE, V3 CAN BE EXPRESSED AS
C       V3 = V31*U1 + V32*P1
C   THEN  VR(3,1) = REAL PART OF V31
C   AND   VR(3,2) = REAL PART OF V32
C SIMILARLY FOR THE ARRAYS FOR UR,UI,PR,PI
C
C ACCORDING TO THE ABOVE SCHEME, THE FOLLOWING CONSTANTS ARE SET IN THE
C MAIN PROGRAMME (ALSO SEE EQN.(3.27))
C   VR(1,2) = VI(1,2) = UR(1,2) = UI(1,2) = 0.0
C   PR(1,1) = PI(1,1) = UI(1,1) = PI(1,2) = 0.0
C   AND UR(1,1) = PR(1,2) = SOME NON-ZERO CONSTANT
C
C ***********************************************************************
C**** FROM NOW ONWARDS, THE SUBSCRIPT J WILL ALWAYS REFER TO ANY ONE OF THE
C   TWO INDEPENDENT SOLUTIONS FOR THE EIGENFUNCTIONS. THUS, J = 1,2
C ***********************************************************************
C
C VSR(J),VSI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE TWO INDE-
C PENDENT SUMS FOR THE EIGENFUNCTION VR(R) EVALUATED AT R=RS. SIMILARLY
C FOR USR(J),USI(J),PSR(J),PSI(J)
C
      DOUBLE PRECISION RE,W,KR,KI,AL,EPS,RS,ALL,BR(2),BI(2),AXR,AXI,SUR,
     1SUI,SVR,SVI,VSR(2),VSI(2),USR(2),USI(2),PSR(2),PSI(2),ARV(2),ARU(2
     2),ARP(2),RP,ABSDPC,KSR,KSI,C(NU),VR(20,2),VI(20,2),UR(20,2),UI(20,
     32),PR(20,2),PI(20,2)
      COMMON /AREA1/ VR,VI,UR,UI,PR,PI
      KSR=KR*KR-KI*KI
      KSI=2.0D0*KR*KI
      BR(1)=KSR-RE*KR*C(1)
      BI(1)=KSI+RE*(W-KI*C(1))
      DO 20 J=2,NU
      BR(J)=-KR*RE*C(J)
   20 BI(J)=-KI*RE*C(J)
C CALCULATING THE FIRST TERM IN THE SERIES FOR VR(R)-SEE EQN.(3.27)
      CALL MULDPC (KR,KI,UR(1,1),UI(1,1),AXR,AXI)
      VR(1,1)=-AXR/2.D0
      VI(1,1)=-AXI/2.D0
      DO 21 J=1,2
      VSR(J)=VR(1,J)
      VSI(J)=VI(1,J)
      USR(J)=UR(1,J)
      USI(J)=UI(1,J)
      PSR(J)=PR(1,J)
   21 PSI(J)=PI(1,J)
      L=0
   90 L=L+1
      AL=L
      DO 60 J=1,2
      SUR=0.0D0
      SUI=0.0D0
      SVR=0.0D0
      SVI=0.0D0
      DO 50 I=2,NU
      LL=L+1-I
```

```
        IF(LL.LE.0) GO TO 50
C SUR,SUI - REAL AND IMAGINARY PARTS OF THE SUMMATION IN EQN.(3.29)
        CALL MULDPC (BR(I),BI(I),UR(LL,J),UI(LL,J),AXR,AXI)
        ALL =LL+1
        SUR=SUR+AXR*ALL
        SUI=SUI+AXI*ALL
   50 CONTINUE
C CALCULATING THE U'S IN THE SERIES FOR VZ(R) - SEE EQN.(3.29)
        CALL MULDPC (BR(1),BI(1),UR(L ,J),UI(L ,J),AXR,AXI)
        CALL MULDPC (KR ,KI ,PR(L,J),PI(L,J),KSR,KSI)
        UR(L+1,J)=(RE*KSR-AXR-SUR/(AL+1.D0))/(4.D0*AL*AL)
        UI(L+1,J)=(RE*KSI-AXI-SUI/(AL+1.D0))/(4.D0*AL*AL)
C CALCULATING THE V'S IN THE SERIES FOR VR(R) - SEE EQN.(3.30)
        CALL MULDPC (KR,KI,UR(L+1,J),UI(L+1,J),AXR,AXI)
        VR(L+1,J)=-AXR/(2.D0*(AL+1.D0))
        VI(L+1,J)=-AXI/(2.D0*(AL+1.D0))
        DO 40 I=1,NU
        LL=L+1-I
        IF(LL.LE.0) GO TO 40
C SVR,SVI - REAL AND IMAGINARY PARTS OF THE SUMMATION IN EQN.(3.31)
        CALL MULDPC (BR(I),BI(I),VR(LL,J),VI(LL,J),AXR,AXI)
        SVR=SVR+AXR
        SVI=SVI+AXI
   40 CONTINUE
C CALCULATING THE P'S IN THE SERIES FOR P(R) - SEE EQN.(3.31)
        PR(L+1,J)=(AXR+((2.D0*AL+1.D0)**2-1.D0)*VR(L+1,J))/(2.D0*RE*AL)
   60 PI(L+1,J)=(AXI+((2.D0*AL+1.D0)**2-1.D0)*VI(L+1,J))/(2.D0*RE*AL)
        IF(L.EQ.1) GO TO 90
C CALCULATING THE PARTIAL SUM OF THE SERIES FOR THE EIGENFUNCTIONS
C EVALUATED AT R=RS
        RP=RS**(2*(L-1))
        DO 35 J=1,2
        VSR(J)=VSR(J)+VR(L,J)*RP
        VSI(J)=VSI(J)+VI(L,J)*RP
        USR(J)=USR(J)+UR(L,J)*RP
        USI(J)=USI(J)+UI(L,J)*RP
        PSR(J)=PSR(J)+PR(L,J)*RP
   35 PSI(J)=PSI(J)+PI(L,J)*RP
        NTS=L+1
C FINDING THE ABSOLUTE VALUE OF THE RATIOS OF THE LAST TERMS IN THE SERIES
C (FOR THE TWO INDEPENDENT SOLUTIONS) TO THE PARTIAL SUM. ARV(J) TO ARP(J)
C ARE THE ARRAYS TO STORE THESE RATIOS
        RP=RS**(2*L)
        DO 25 J=1,2
        IF(DABS(VSR(J)).LT.1.0D-70 .AND. DABS(VSI(J)).LT.1.0D-70) GO TO 61
        CALL DIVDPC (VR(NTS,J),VI(NTS,J),VSR(J),VSI(J),AXR,AXI)
        ARV(J)=ABSDPC(AXR,AXI)*RP
        GO TO 64
   61 ARV(J)=0.0D0
   64 IF(DABS(USR(J)).LT.1.0D-70 .AND. DABS(USI(J)).LT.1.0D-70) GO TO 66
        CALL DIVDPC (UR(NTS,J),UI(NTS,J),USR(J),USI(J),AXR,AXI)
        ARU(J)=ABSDPC(AXR,AXI)*RP
        GO TO 67
   66 ARU(J)=0.0D0
   67 IF(DABS(PSR(J)).LT.1.0D-70 .AND. DABS(PSI(J)).LT.1.0D-70) GO TO 68
        CALL DIVDPC (PR(NTS,J),PI(NTS,J),PSR(J),PSI(J),AXR,AXI)
        ARP(J)=ABSDPC(AXR,AXI)*RP
        GO TO 25
```

```
   68 ARP(J)=0.0D0
   25 CONTINUE
C AXR - MAXIMUM VALUE OF THE ABOVE RATIOS
      AXR=0.0D0
      DO 45 J=1,2
      AXR=DMAX1(AXR,ARV(J))
      AXR=DMAX1(AXR,ARU(J))
   45 AXR=DMAX1(AXR,ARP(J))
C AX1 - LARGEST TERM (EITHER ITS REAL OR IMAGINARY PART) IN ALL THE SERIES
      AX1 =0.0D0
      DO 54 J=1,2
      AX1 =DMAX1(AX1    ,DABS(VR(NTS,J)))
      AX1 =DMAX1(AX1    ,DABS(VI(NTS,J)))
      AX1 =DMAX1(AX1    ,DABS(UR(NTS,J)))
      AX1 =DMAX1(AX1    ,DABS(UI(NTS,J)))
      AX1 =DMAX1(AX1    ,DABS(PR(NTS,J)))
   54 AX1 =DMAX1(AX1    ,DABS(PI(NTS,J)))
C ALL SERIES ARE TERMINATED IF EITHER AXR.LE.EPS OR IF AX1.GT.1.D70.   THE
C LATTER CRITERIA, THOUGH NEVER FOUND TO BE IMPORTANT DURING THE PRESENT
C INVESTIGATION, IS INCLUDED HERE TO AVOID VERY LARGE VALUES FOR THE TERMS
C IN THE SERIES
C
      IF(AXR.LE.EPS .OR. AX1.GT.1.0D70) RETURN
C
C AN ERROR MESSAGE IS PRINTED IF THE NUMBER OF TERMS REQUIRED IN THE SERIES
C FOR R=RS EXCEED MNTS. IF EPS IS KEPT CONSTANT, RS SHOULD BE REDUCED OR
C MNTS AND CORRESPONDING DIMENSIONS INCREASED.
      IF( NTS .EQ.MNTS) GO TO 85
      GO TO 90
   85 WRITE(6,11) NTS,AXR,AX1
   11 FORMAT    (5X,9H EVEN FOR,I3,38H TERMS IN THE SERIES, THE MAX. RAT
     110 =,G10.4,33H  AND ABS. MAX. VALUE OF A TERM =,G10.4)
      RETURN
      END
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE AEFSRK (HH,NES,NTS,RS)
C
C* THIS IS THE SECOND SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR AN AXISYM
C METRIC DISTURBANCE *
C
C THIS SUBROUTINE CALCULATES THE EIGENFUNCTIONS ACCORDING TO THE SERIES
C EXPANSIONS (EQN.(3.19) THROUGH (3.21)) FOR (0.LE.R.LE.RS) AT STEPS OF HH.
C IT ALSO CALCULATES THE DERIVATES OF THE EIGENFUNCTIONS FOR DISTURBANCE
C VELOCITY COMPONENTS AT R=RS. THESE ARE REQUIRED LATER FOR STEP-BY-STEP
C INTEGRATION BY THE RUNGE-KUTTA METHOD.
C
C THE INPUT VARIABLES NOT ALREADY EXPLAINED EARLIER ARE HH,NES
C   HH - RADIAL INTERVAL AT WHICH THE EIGENFUNCTION VALUES ARE STORED FOR
C        PLOTTING IF AN EIGENVALUE IS FOUND. HH=0.02 HERE. IF A LOWER VALUE
C        IS DESIRED, RELEVANT DIMENSIONS MUST BE ACCORDINGLY INCREASED.
C   NES - NUMBER OF POINTS FOR WHICH EIGENFUNCTIONS FOUND BY THE SERIES
C         SOLUTION ARE STORED FOR PLOTTING LATER.   NES = 1 + RS/HH
C
C THE BOUNDARY CONDITIONS AT R=0 ARE ALSO SET IN THE MAIN PROGRAMME.
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C VRR(L,J),VRI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE EIGEN-
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C FUNCTION VR(R). THE FIRST SUBSCRIPT INDICATES THE RADIAL POSITION FOR
C WHICH THE EIGENFUNCTION VALUE IS STORED, AND THE SECOND SUBSCRIPT REFERS
C TO ANYONE OF THE TWO INDEPENDENT SOLUTIONS WHICH MAKE UP VR(R).
C SIMILAR IS THE CASE WITH ARRAYS FOR VZR,VZI,PRR,PRI
C
C DVRR(J),DVRI(J) - ARRAYS OF TWO ELEMENTS FOR THE TWO INDEPENDENT
C SOLUTIONS OF THE REAL AND IMAGINARY PARTS OF THE DERIVATIVE OF VR(R).
C FOR THE RUNGE-KUTTA METHOD, THEY ARE NEEDED ONLY AT R=RS.
C SIMILAR IS THE CASE FOR ARRAYS OF DVZR,DVZI
C
      DOUBLE PRECISION VRR(51,2),VRI(51,2),VZR(51,2),VZI(51,2),PRR(51,2)
     1,PRI(51,2),DVRR(2),DVRI(2),DVZR(2),DVZI(2), PI(20,2) ,R,HH,RS,RP,A
     2V,RU,VR(20,2),VI(20,2),UR(20,2),UI(20,2),PR(20,2)
      COMMON /AREA1/ VR,VI,UR,UI,PR,PI
      COMMON /AREA2/ VRR,VRI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ DVRR,DVRI,DVZR,DVZI
C CALCULATING THE EIGENFUNCTIONS FOR (0.LE.R.LE.RS) AT INTERVALS OF HH BY
C MEANS OF THE SERIES EXPANSIONS
      R=0.0D0
      DO 13 LL=2,NES
      R=R+HH
      DO 13 J=1,2
C INITIALIZING THE SUMMATION OF SERIES
      VRR(LL,J)=VR(1,J)
      VRI(LL,J)=VI(1,J)
      VZR(LL,J)=UR(1,J)
      VZI(LL,J)=UI(1,J)
      PRR(LL,J)=PR(1,J)
      PRI(LL,J)=PI(1,J)
C SUMMING UP THE SERIES
      DO 30 L=2,NTS
      RP=R**(2*(L-1))
      VRR(LL,J)=VRR(LL,J)+VR(L,J)*RP
      VRI(LL,J)=VRI(LL,J)+VI(L,J)*RP
      VZR(LL,J)=VZR(LL,J)+UR(L,J)*RP
      VZI(LL,J)=VZI(LL,J)+UI(L,J)*RP
      PRR(LL,J)=PRR(LL,J)+PR(L,J)*RP
   30 PRI(LL,J)=PRI(LL,J)+PI(L,J)*RP
      VRR(LL,J)=VRR(LL,J)*R
   13 VRI(LL,J)=VRI(LL,J)*R
C FINDING THE DERIVATIVES OF THE EIGENFUNCTIONS FOR DISTURBANCE VELOCITY
C COMPONENTS AT R=RS USING THE SERIES SOLUTION.
      DO 24 J=1,2
      DVRR(J)=VR(1,J)
      DVRI(J)=VI(1,J)
      DVZR(J)=0.0D0
      DVZI(J)=0.0D0
      DO 24 L=2,NTS
      AV=2*L-1
      RP=RS**(2*(L-1))
      RU=RS**(2*L-3)
      DVRR(J)=DVRR(J)+VR(L,J)*RP*AV
      DVRI(J)=DVRI(J)+VI(L,J)*RP*AV
      DVZR(J)=DVZR(J)+UR(L,J)*RU*(AV-1.D0)
   24 DVZI(J)=DVZI(J)+UI(L,J)*RU*(AV-1.D0)
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

```
      SUBROUTINE ARUNGE(RE,W,KR,KI,LS,NPE,RS,NU,C,H,NES,NG,ND)
C
C* THIS IS THE THIRD SUBROUTINE CALLED BY THE MAIN PROGRAM4E FOR AN AXISYM
C METRIC DISTURBANCE *
C
C STARTING FROM R=RS (UPTO WHICH THE SERIES SOLUTION WAS USED), THIS SUBROU
C TINE INTEGRATES THE STABILITY EQUATIONS (3.35) IN A STEP-BY-STEP SCHEME
C BY THE USE OF FOURTH ORDER RUNGE-KUTTA METHOD.
C
C THE INPUT VARIABLES NOT ALREADY CLARIFIED ARE LS,H,NPE,NG,ND
C   LS = 1 + RS/H
C   H - THE STEP SIZE IN RADIAL DIRECTION FOR STEP-BY-STEP INTEGRATION SO
C       THAT (H.LE.HH)
C   NPE - NUMBER OF POINTS FOR WHICH THE EIGENFUNCTION VALUES ARE STORED
C         FOR PLOTTING LATER ON. NPE = 1 + 1./HH . IF NPE IS TO EXCEED 51,
C         RELEVANT DIMENSIONS MUST BE INCREASED.
C   NG = (ND-1)/(NPE-1) = HH/H
C   ND   1 + 1./H
C
C ADDIT     L IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C EVRR(J,   . EPRI(J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE TWO
C INDEPENDENT SOLUTIONS OF THE EIGENFUNCTIONS AT THE STARTING POINT OF
C EVERY FORWARD STEP
C
C CVRR TO CPRI - REAL AND IMAGINARY PARTS OF THE EIGENFUNCTION VALUES
C PERTAINING TO THOSE FOR THE Y COLUMN IN TABLES AI-1 AND AI-2 (APPENDIX I)
C
C CDVRR TO CDVZI - REAL AND IMAGINARY PARTS OF THE VALUES PERTAINING TO
C THOSE FOR THE DY COLUMN IN TABLE AI-2 (APPENDIX I)
C
C VZM - THE STEDY MEAN FLOW VELOCITY (EQN.(3.22))
C DVZM - DERIVATIVE OF VZM WITH RESPECT TO R
C FVRR TO FPRI - REAL AND IMAGINARY PARTS OF THE LEFT HAND SIDES OF
C EQUATIONS (3.35)
C
C VRRK1 TO PRIK4 - REAL AND IMAGINARY PARTS OF THE FOUR CONSTANTS K1 TO K4
C FOR THE RUNGE-KUTTA METHOD (SEE APPENDIX I, TABLES AI-1 AND AI-2)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION KR,KI
      DIMENSION  C(NU),VRR(51,2),VRI(51,2),VZR(51,2),VZI(51,2),PRR(51,2)
     1,PRI(51,2),DVRR(2),DVRI(2),DVZR(2),DVZI(2),FVRR(2),FVRI(2),FVZR(2)
     2,FVZI(2),EPRR(2),EPRI(2)
      COMMON /AREA2/ VRR,VRI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ DVRR,DVRI,DVZR,DVZI
      K=NES
      MUL=1
      DO 76 J=1,2
      EVRR(J)=VRR(NES,J)
      EVRI(J)=VRI(NES,J)
      EVZR(J)=VZR(NES,J)
      EVZI(J)=VZI(NES,J)
      EPRR(J)=PRR(NES,J)
   76 EPRI(J)=PRI(NES,J)
      RR=RS
      HS=HH/2.00
      M=LS+1
```

```
      DO 42 I=M,ND
      DO 72 J=1,2
      R=RR
      MRK=1
C INITIALIZING THE Y AND DY COLUMNS OF TABLES AI-1 AND AI-2 (APPENDIX I)
      CVRR=EVRR(J)
      CVRI=EVRI(J)
      CVZR=EVZR(J)
      CVZI=EVZI(J)
      CPRR=EPRR(J)
      CPRI=EPRI(J)
      CDVRR=DVRR(J)
      CDVRI=DVRI(J)
      CDVZR=DVZR(J)
      CDVZI=DVZI(J)
   51 SR=1.D0/(R*R)
C CALCULATING VZ AND DVZ FOR THE STEADY MEAN FLOW
      VZM=C(1)
      DVZM=0.0D0
      DO 44 L=2,NU
      AR=2*(L-1)
      VZM=VZM+C(L)*(R**(2*(L-1)))
   44 DVZM=DVZM+AR*C(L)*(R**(2*L-3))
C CALCULATING THE LEFT HAND SIDES OF EQUATIONS (3.71) WRITTEN IN THEIR REAL
C AND IMAGINARY PARTS
C
      AR=KR*(RE*VZM-KR)+KI*KI
      AI=KI*(RE*VZM-2.D0*KR)-RE*W
   74 CALL MULDPC (AR,AI,CVZR,CVZI,AXR,AXI)
      CALL MULDPC (KR,KI,CPRR,CPRI,BXR,BXI)
      FVZR=RE*(DVZM*CVRR+BXR)-CDVZR/R+AXR
      FVZI=RE*(DVZM*CVRI+BXI)-CDVZI/R+AXI
      CALL MULDPC (KR,KI,CDVZR,CDVZI,AXR,AXI)
      FVRR=SR*CVRR-CDVRR/R-AXR
      FVRI=SR*CVRI-CDVRI/R-AXI
      CALL MULDPC (AR,AI,CVRR,CVRI,BXR,BXI)
      FPRR=-(AXR+BXR)/RE
      FPRI=-(AXI+BXI)/RE
      GO TO(55,59,65,71), MRK
C CALCULATING K1'S AND SECOND ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
   55 VRRK1=HS*FVRR
      VRIK1=HS*FVRI
      VZRK1=HS*FVZR
      VZIK1=HS*FVZI
      PRRK1=H*FPRR
      PRIK1=H*FPRI
      R=RR+H/2.D0
      CVRR=EVRR(J)+DVRR(J)*H/2.D0+VRRK1/4.D0
      CVRI=EVRI(J)+DVRI(J)*H/2.D0+VRIK1/4.D0
      CVZR=EVZR(J)+DVZR(J)*H/2.D0+VZRK1/4.D0
      CVZI=EVZI(J)+DVZI(J)*H/2.D0+VZIK1/4.D0
      CPRR=EPRR(J)+PRRK1/2.D0
      CPRI=EPRI(J)+PRIK1/2.D0
      CDVRR=DVRR(J)+VRRK1/H
      CDVRI=DVRI(J)+VRIK1/H
      CDVZR=DVZR(J)+VZRK1/H
      CDVZI=DVZI(J)+VZIK1/H
```

```
          MRK=2
          GO TO 51
C CALCULATING K2'S AND THIRD ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
     59 VRRK2=HS*FVRR
          VRIK2=HS*FVRI
          VZRK2=HS*FVZR
          VZIK2=HS*FVZI
          PRRK2=H*FPRR
          PRIK2=H*FPRI
          CPRR=EPRR(J)+PRRK2/2.DU
          CPRI=EPRI(J)+PRIK2/2.DU
          CDVRR=DVRR(J)+VRRK2/H
          CDVRI=DVRI(J)+VRIK2/H
          CDVZR=DVZR(J)+VZRK2/H
          CDVZI=DVZI(J)+VZIK2/H
          MRK=3
          GO TO 74
C CALCULATING K3'S AND FOURTH ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
     65 VRRK3=HS*FVRR
          VRIK3=HS*FVRI
          VZRK3=HS*FVZR
          VZIK3=HS*FVZI
          PRRK3=H*FPRR
          PRIK3=H*FPRI
          R=RR+H
          CVRR=EVRR(J)+DVRR(J)*H+VRRK3
          CVRI=EVRI(J)+DVRI(J)*H+VRIK3
          CVZR=EVZR(J)+DVZR(J)*H+VZRK3
          CVZI=EVZI(J)+DVZI(J)*H+VZIK3
          CPRR=EPRR(J)+PRRK3
          CPRI=EPRI(J)+PRIK3
          CDVRR=DVRR(J)+2.DO+VRRK3/H
          CDVRI=DVRI(J)+2.DO*VRIK3/H
          CDVZR=DVZR(J)+2.DO*VZRK3/H
          CDVZI=DVZI(J)+2.DO*VZIK3/H
          MRK=4
          GO TO 51
C CALCULATING K4'S AND FIFTH ROW OF X,Y AND DY COLUMNS IN TABLES AI-1 AND
C AI-2 (APPENDIX I)
     71 VRRK4=HS*FVRR
          VRIK4=HS*FVRI
          VZRK4=HS*FVZR
          VZIK4=HS*FVZI
          PRRK4=H*FPRR
          PRIK4=H*FPRI
          EVRR(J)=EVRR(J)+DVRR(J)*H+(VRRK1+VRRK2+VRRK3)/3.DO
          EVRI(J)=EVRI(J)+DVRI(J)*H+(VRIK1+VRIK2+VRIK3)/3.DO
          EVZR(J)=EVZR(J)+DVZR(J)*H+(VZRK1+VZRK2+VZRK3)/3.DO
          EVZI(J)=EVZI(J)+DVZI(J)*H+(VZIK1+VZIK2+VZIK3)/3.DO
          EPRR(J)=EPRR(J)+(PRRK1+2.DO*PRRK2+2.DO*PRRK3+PRRK4)/6.DO
          EPRI(J)=EPRI(J)+(PRIK1+2.DO*PRIK2+2.DO*PRIK3+PRIK4)/6.DO
          DVRR(J)=DVRR(J)+(VRRK1+2.DO*VRRK2+2.DO*VRRK3+VRRK4)/(3.DO*H)
          DVRI(J)=DVRI(J)+(VRIK1+2.DO*VRIK2+2.DO*VRIK3+VRIK4)/(3.DO*H)
          DVZR(J)=DVZR(J)+(VZRK1+2.DO*VZRK2+2.DO*VZRK3+VZRK4)/(3.DO*H)
     72 DVZI(J)=DVZI(J)+(VZIK1+2.DO*VZIK2+2.DO*VZIK3+VZIK4)/(3.DO*H)
          RR=R
```

```
C THE NEXT ARITHMETIC IF STATEMENT IMPLIES THAT IF R IS EQUAL TO A MULTIPLE
C OF HH, EIGENFUNCTION VALUES ARE TO BE STORED IN PROPER ARRAYS BEFORE
C INTEGRATION IS CONTINUED FURTHER.
      IF(I-LS-MUL*NG) 42,57,42
   57 MUL=MUL+1
      K=K+1
      DO 49 J=1,2
      VRR(K,J)=EVRR(J)
      VRI(K,J)=EVRI(J)
      VZR(K,J)=EVZR(J)
      VZI(K,J)=EVZI(J)
      PRR(K,J)=EPRR(J)
   49 PRI(K,J)=EPRI(J)
   42 CONTINUE
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE ASECAN (N,*,RE,W,KR,KI,E,E1,E2,NT,H,NG,NN,RS)
C
C* THIS IS THE FOURTH AND LAST SUBROUTINE CALLED BY THE MAIN PROGRAMME FOR
C AN AXISYMMETRIC DISTURBANCE *
C
C HAVING INTEGRATED THE STABILITY EQUATIONS OVER THE WHOLE DOMAIN
C (0.LE.R.LE.1) FOR AN APPROXIMATE EIGENVALUE, THIS SUBROUTINE EMPLOYS THE
C SECANT METHOD IN ORDER TO CONVERGE TO THE TRUE EIGENVALUE.
C
C THE INPUT VARIABLES NOT ALREADY CLARIFIED ARE N,*,E,E1,E2,NT
C   N = NPE (SEE SUBROUTINE 'ARUNGE')
C   * - FOR IMPLICATION OF THE 'MULTIPLE RETURN' FACILITY PROVIDED ON
C          THE UNIVAC 1108 MACHINE
C   E - THE MAXIMUM VALUE OF THE DETERMINANT WHICH IS CONSIDERED CLOSE
C          ENOUGH TO 'ZERO' FOR K TO BE A TRUE EIGENVALUE.
C   E1,E2 - MULTIPLIERS FOR KR AND KI FOR THE NEXT GUESS OF THE EIGENVALUE.
C          THESE ARE ESSENTIAL FOR THE SECANT METHOD.
C   NT - ITERATION NUMBER FOR THE SECANT METHOD. THE MAIN PROGRAMME MUST
C          SET NT = 0
C
C ADDITIONAL IMPORTANT VARIABLES USED INSIDE THE SUBROUTINE ARE
C
C DR,DI - REAL AND IMAGINARY PARTS OF THE DETERMINANT (EQN.(3.35))
C ADET - ABSOLUTE VALUE OF THE DETERMINANT. ADET = DSQRT(DR*DR+DI*DI)
C
C URR(L) TO PEI(L) - ARRAYS FOR THE COMPLETE EIGENFUNCTIONS ONCE A TRUE
C EIGENVALUE IS FOUND. THESE ARRAYS HAVE VALUES FOR THE LEFT HAND SIDES OF
C EQUATIONS OF THE FORM GIVEN IN EQN.(3.32)
C
      DOUBLE PRECISION RE,W,KR,KI,E,DR,DI,URP,DTP,KR1,KT1,KR2,KT2,CR(2),
     1CI(2),ADET,ADSDPC,H,VEC(4),URR(51),URI(51),UZR(51),UZI(51),PER(51)
     2,PEI(51),VRR(51,2),VRI(51,2),VZR(51,2),VZI(51,2),PRR(51,2),PRI(51,
     32),RS
      DIMENSION RM(4)
      COMMON /APEA2/ VRR,VRI,VZR,VZI,PRR,PRI
      DATA TE/'RR'/
      NT=NT+1
C CALCULATING THE DETERMINANT (SEE EQN.(3.35))
      CALL MULDPC (VRR(N,1),VRI(N,1),VZR(N,2),VZI(N,2),CR(1),CI(1))
      CALL MULDPC (VRR(N,2),VRI(N,2),VZR(N,1),VZI(N,1),CR(2),CI(2))
      DR      =CR(1)-CR(2)
```

```
      DI    =CI(1)-CI(2)
      ADET=ABSDPC(DR,DI)
      WRITE(6,52) H,RE,W,KR,KI,DR,DI,ADET,NT
   52 FORMAT(1H0,F9.4,F10.1,F8.3,2G24.14,3G14.5,I5)
C FOR K TO BE AN EIGENVALUE, THE ABSOLUTE VALUE OF THE DETERMINANT MUST
C BE .LE. A PREASSIGNED EPSILON ('E' HERE)
C
      IF(ADET.LE.E) GO TO 61
C
C GIVE UP IF THE NUMBER OF ITERATIONS EXCEED 20
C
      IF(NT.GT.20) RETURN
C
C THE SECANT METHOD FOR ITERATING TO THE EIGENVALUE. THIS METHOD KEEPS
C THE TWO MOST RECENTLY COMPUTED POINTS AT EACH ITERATION AND JOINS THEM
C BY A STRAIGHT LINE TO GET THE NEW ESTIMATE FOR THE ROOT
      IF(NT.GT.1) GO TO 53
      DRP=DR
      DIP=DI
      KR1=KR
      KI1=KI
      KR=E1*KR
      KI=E2*KI
C THE 'RETURN 2' STATEMENT SENDS CONTROL IN THE MAIN PROGRAMME TO CALL
C THE SUBROUTINE 'ASERIS' AGAIN FOR TRIAL WITH THE NEW ESTIMATE OF K
      RETURN 2
   53 CALL MULDPC (KR,KI,DRP,DIP,CR(1),CI(1))
      CALL MULDPC (KR1,KI1,DR,DI,CR(2),CI(2))
      CR(1)=CR(1)-CR(2)
      CI(1)=CI(1)-CI(2)
      CR(2)=DRP-DR
      CI(2)=DIP-DI
      CALL DIVDPC (CR(1),CI(1),CR(2),CI(2),KR2,KI2)
      DRP=DR
      DIP=DI
      KR1=KR
      KI1=KI
      KR=KR2
      KI=KI2
      RETURN 2
C THE FOLLOWING IS DONE ONLY IF AN EIGENVALUE IS FOUND
   61 CALL DIVDPC(VRR(N,1),VRI(N,1),VRR(N,2),VRI(N,2),DRP,DTP)
      PUNCH 71, NN,H,RE,W,KR,KI,KS,ADET,TE
   71 FORMAT(I1,F5.4,F7.1,F5.3,G22.16,G21.16,F5.2,6X,G8.3,A2)
C THE FOLLOWING CALLS ARE FOR FILLING UP THE ARRAYS URR(L) TO PEI(L), AND
C FOR PLOTTING THE NORMALIZED EIGENFUNCTIONS
      CALL ALIGNF (VRR,VRI,URR,URI,DRP,DIP,N)
      CALL ALIGNF (VZR,VZI,UZR,UZI,DRP,DIP,N)
      CALL ALIGNF (PRR,PRI,PER,PEI,DRP,DIP,N)
      CALL NORMFN (URR,RE,W,KR,KI,H,N,NG,1)
      CALL NORMFN (URI,RE,W,KR,KI,H,N,NG,2)
      CALL NORMAB (URR,URI,PE,W,KR,KI,H,N,NG,VEC,RM,1)
      CALL NORMFN (UZR,RE,W,KR,KI,H,N,NG,5)
      CALL NORMFN (UZI,RE,W,KR,KI,H,N,NG,6)
      CALL NORMAB (UZR,UZI,PE,W,KR,KI,H,N,NG,VEC,RM,3)
      CALL NORMFN (PER,PE,W,KR,KI,H,N,NG,7)
      CALL NORMFN (PEI,RE,W,KR,KI,H,N,NG,8)
      CALL NORMAB (PER,PEI,PE,W,KR,KI,H,N,NG,VEC,RM,4)
```

```
C PUNCHING OUT THE MAXIMA OF THE ABSOLUTE VALUES OF THE THREE EIGEN-
C FUNCTIONS AND THE RADII AT WHICH THEY OCCUR.
      PUNCH 82, VEC(1),RM(1),(VEC(J),RM(J),J=3,4)
   82 FORMAT(3(G15.10,F4.2))
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE AEIGNF (AR,AI,BR,BI,CR,CI,N)
C
C FOR DETERMINING THE NET EIGENFUNCTIONS BY PROPERLY COMBINING THE TWO
C INDEPENDENT SOLUTIONS FOR THEM (EQUATION (3.32))
C
C AR(L,J),AI(L,J) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE THREE
C INDEPENDENT SOLUTIONS OF AN EIGENFUNCTION
C
C BR(L),BI(L) - ARRAYS FOR THE REAL AND IMAGINARY PARTS OF THE EIGENFUNCTION
C
      DOUBLE PRECISION AR(51,2),AI(51,2),BR(51),BI(51),CR,CI,AX,BX
      DO 82 L=1,N
      BR(L)=AR(L,1)
      BI(L)=AI(L,1)
      CALL MULDPC(AR(L,2),AI(L,2),CR,CI,AX,BX)
      BR(L)=BR(L)-AX
   82 BI(L)=BI(L)-BX
      RETURN
      END
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C THE ** MAIN ** PROGRAM FOR A NON-AXISYMMETRIC DISTURBANCE WHEN THE
C  RUNGE-KUTTA METHOD IS USED FOR THE STEP-BY-STEP INTEGRATION
C
      DOUBLE PRECISION VRR(51,3),VRI(51,3),VTR(51,3),VTI(51,3),VZR(51,3)
     1,VZI(51,3),PRR(51,3),PRI(51,3),DVRR(3),DVRI(3),DVTR(3),DVTI(3),DVZ
     2R(3),DVZI(3),AXI,E,RE,W,KR,KI,AN,EPS,RS,H,ANN,C(2),GR(20,3),GI(20,
     33),FR(20,3),FI(20,3),UR(20,3),UI(20,3),PR(20,3),PI(20,3),HH,AXR
      COMMON /AREA1/ GR,GI,FR,FI,UR,UI,PR,PI
      COMMON /AREA2/ VRR,VRI,VTR,VTI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ DVRR,DVRI,DVTR,DVTI,DVZR,DVZI
      DATA NU,NPE/2,51/
      READ(5,15) (GI(1,J),UI(1,J),PI(1,J),J=1,3),(GR(1,J),PR(1,J-1),J=2,
     13),UR(1,1),UR(1,3),(VZR(1,J),VZI(1,J),PRR(1,J),PRI(1,J),J=1,3)
   15 FORMAT(27G2.0)
      READ(5,10) (C(J),J=1,NU),EPS,MNTS,KN
   10 FORMAT(3G7.1,2I3)
      HH=NPE-1
      HH=1.D0/HH
C KN - NUMBER OF EIGENVALUES DESIRED
      DO 18 II=1,KN
      READ(5,56) RE,W,KR,KI,RS,E,E1,E2,NES,LS,NG,ND,NN
   56 FORMAT(6G7.1,2F6.0,3I3,I4,I2)
      WRITE(6,78) NN,C,RS,EPS,E,E1,E2,MNTS,NPE,NES,LS,NG,ND
   78 FORMAT(1H1,20X,38HMETHOD USED - FOURTH ORDER RUNGE-KUTTA,10X,3HN =
     1,I2///,5G11.2,2F9.3,5I6,I8)
      GR(1,1)=1.D0/RE
      UR(1,2)=GR(1,1)
      PR(1,3)=GR(1,1)
      AN=NN
      ANN=NN+1
      H=1.D-1
```

```
      H=1.D0/H
      NT=0
   96 CALL      SERIES (RE,W,KR,KI,NU,AN,EPS,MNTS,RS,ANN,C,NTS,AXR,AXI)
      CALL      EFSRRK (HH,AN,ANN,NN,NTS,RS,NES)
      CALL      RUNGEK(RE,W,KR,KI,LS,AN,RS,NU,C,H,NES,NPE,NG,ND)
      CALL    SECANT (NPE,$96,RE,W,KR,KI,E,E1,E2,NT,H,NG,NN,RS)
      WRITE(6,26) NTS,AXR,AXI
   26 FORMAT (//,10X,32H NUMBER OF TERMS IN THE SERIES =,I3,23H  WHEN TH
     1E MAX. RATIO =,G10.4,33H  AND ABS. MAX. VALUE OF A TERM =,G10.4)
   18 CONTINUE
      STOP
      END
C THE FIRST DATA CARD (FOR FORMAT NO. 15) TOTAL NO. OF DATA CARDS = KN+2
0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C THE ** MAIN ** PROGRAM FOR A NON-AXISYMMETRIC DISTURBANCE WHEN THE
C PREDICTOR-CORRECTOR METHOD IS USED FOR THE STEP-BY-STEP INTEGRATION
C
      DOUBLE PRECISION VRR(51,3),VRI(51,3),VTR(51,3),VTI(51,3),VZR(51,3)
     1,VZI(51,3),PRR(51,3),PRI(51,3),CVRR(5,3),CVRI(5,3),CVTR(5,3),CVTI(
     25,3),CVZR(5,3),CVZI(5,3),CDVRR(5,3),CDVRI(5,3),CDVTR(5,3),CDVTI(5,
     33),CDVZR(5,3),CDVZI(5,3),CDPRR(5,3),CDPRI(5,3),CDDVRR(5,3),CDDVRI(
     45,3),CDDVTR(5,3),CDDVTI(5,3),CDDVZR(5,3),CDDVZI(5,3),CPRR(5,3),CPR
     51(5,3),AXI,AXR,HH,RE,W,KR,KI,AN,EPS,RS,H,ANN,C(2),GR(20,3),GI(20,3
     6),FR(20,3),FI(20,3),UR(20,3),UI(20,3),PR(20,3),PI(20,3),EPL,E
      COMMON /AREA1/ GR,GI,FR,FI,UR,UI,PR,PI
      COMMON /AREA2/ VRR,VRI,VTR,VTI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ CVRR,CVRI,CVTR,CVTI,CVZR,CVZI,CPRR,CPRI,CDVRR,CDVRI
     1,CDVTR,CDVTI,CDVZR,CDVZI,CDPRR,CDPRI,CDDVRR,CDDVRI,CDDVTR,CDDVTI,C
     2DDVZR,CDDVZI
      DATA NU,NPE,NPC/2,51,4/
      READ(5,15) (GI(1,J),UI(1,J),PI(1,J),J=1,3),(GR(1,J),PR(1,J-1),J=2,
     13),UR(1,1),UR(1,3),(VZR(1,J),VZI(1,J),PRR(1,J),PRI(1,J),J=1,3)
   15 FORMAT(27G2.0)
      READ(5,10) (C(J),J=1,NU),EPS,EPL,MNTS,KN
   10 FORMAT(4G7.1,2I3)
      HH=NPE-1
      HH=1.D0/HH
C KN - NUMBER OF EIGENVALUES DESIRED
      DO 18 II=1,KN
      READ(5,56) RE,W,KR,KI,RS,E,E1,E2,NES,LS,NG,ND,NN
   56 FORMAT(6G7.1,2F6.0,3I3,I4,I2)
      WRITE(6,78) NN,C,RS,EPS,EPL,E,E1,E2,MNTS,NPE,NPC,LS,NG,NES,ND
   78 FORMAT(1H1,20X,45HMETHOD USED - ADAMS ET AL PREDICTOR-CORRECTOR,10
     1X,3HN =,I2,///,6G11.2,2F9.3,6I6,I8)
      GR(1,1)=1.D0/RE
      UR(1,2)=GR(1,1)
      PR(1,3)=GR(1,1)
      AN=NN
      ANN=NN+1
      H=ND-1
      H=1.D0/H
      NT=0
   96 CALL      SERIES (RE,W,KR,KI,NU,AN,EPS,MNTS,RS,ANN,C,NTS,AXR,AXI)
      CALL      EFSRPC (HH,AN,ANN,NN,NTS,RS,NES,H,NPC)
      CALL PREDCT(RE,W,KR,KI,ND,NG,AN,RS,NU,C,EPL,H,LS,NPC,NPE,NES)
      CALL    SECANT (NPE,$96,RE,W,KR,KI,E,E1,E2,NT,H,NG,NN,RS,EPL)
      WRITE(6,26) NTS,AXR,AXI
```

```
   26 FORMAT (//,10X,32H NUMBER OF TERMS IN THE SERIES =,I3,23H  WHEN TH
     1E MAX. RATIO =,G10.4,33H  AND ABS. MAX. VALUE OF A TERM =,G10.4)
   18 CONTINUE
      STOP
      END
C THE FIRST DATA CARD (FOR FORMAT NO. 15) TOTAL NO. OF DATA CARDS = KN+2
0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C THE ** MAIN ** PROGRAM FOR AN AXISYMMETRIC DISTURBANCE
C
      DOUBLE PRECISION VRR(51,2),VRI(51,2),VZR(51,2),VZI(51,2),PRR(51,2)
     1,PRI(51,2),DVRR(2),DVRI(2),DVZR(2),DVZI(2),RE,W,KR,KI,AXR,AXI,H,HH
     2,RS,EPS,E,PR(20,2),PI(20,2),C(2),VR(20,2),VI(20,2),UR(20,2),UI(20,
     32)
      COMMON /AREA1/ VR,VI,UR,UI,PR,PI
      COMMON /AREA2/ VRR,VRI,VZR,VZI,PRR,PRI
      COMMON /AREA3/ DVRR,DVRI,DVZR,DVZI
      DATA NU,NPE/2,51/
      READ(5,15) VR(1,2),VI(1,2),(UR(1,J),UI(1,J),PR(1,J),PI(1,J),J=1,2)
     1,(VRR(1,J),VRI(1,J),VZR(1,J),VZI(1,J),PRR(1,J),PRI(1,J),J=1,2)
   15 FORMAT(22G3.0)
      READ(5,10) (C(J),J=1,NU),EPS,MNTS,KN
   10 FORMAT(3G7.1,2I3)
      HH=NPE-1
      HH=1.D0/HH
C KN - NUMBER OF EIGENVALUES DESIRED
      DO 18 II=1,KN
      READ(5,56) RE,W,KR,KI,RS,E,E1,E2,NES,LS,NG,ND,NN
   56 FORMAT(6G7.1,2F6.0,3I3,I4,I2)
      WRITE(6,78) NN,C,RS,EPS,E,E1,E2,MNTS,NPE,NES,LS,NG,ND
   78 FORMAT(1H1,20X,38HMETHOD USED - FOURTH ORDER RUNGE-KUTTA,10X,3HN =
     1,I2,///,5G11.2,2F9.3,5I6,I8)
      H=ND-1
      H=1.D0/H
      NT=0
      AXI=II
      AXR=DSQRT(2.*W*RE)
      KR=-2.*AXI*(AXR+4.*AXI)/RE
      KI=W*(1.+4.*AXI/AXR)
   96 CALL      ASERIS (RE,W,KR,KI,NU,RS,EPS,MNTS,C,NTS,AXR,AXI)
      CALL      AEFSRK (HH,NES,NTS,RS)
      CALL      ARUNGE(RE,W,KR,KI,LS,NPE,RS,NU,C,H,NES,NG,ND)
      CALL   ASECAN (NPE,$96,RE,W,KR,KI,E,E1,E2,NT,H,NG,NN,RS)
      WRITE(6,26) NTS,AXR,AXI
   26 FORMAT (//,10X,32H NUMBER OF TERMS IN THE SERIES =,I3,23H  WHEN TH
     1E MAX. RATIO =,G10.4,33H  AND ABS. MAX. VALUE OF A TERM =,G10.4)
   18 CONTINUE
      STOP
      END
C THE FIRST DATA CARD (FOR FORMAT NO. 15) TOTAL NO. OF DATA CARDS = KN+2
0.00.01.00.00.00.00.00.01.00.00.00.01.00.00.00.00.00.00.00.01.00.0
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