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The Equilibrium and Stability of the Gaseous Component of the Galaxy

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ABSTRACT

The distribution of gas satisfying both the hydrostatic equilibrium and Poisson conditions with distance above the galactic plane is derived and compared with Schmidt's observations of the gas density at the galactic tangential points. The presence of simple one-dimensional equipartition magnetic and cosmic-ray components is introduced explicitly in the gas hydrostatic equilibrium equation. The gas distribution $\rho_g(z)$ is calculated for two limiting cases: (i) the total mass density at the galactic plane is equal to the sum of the observed gas and star densities in the solar neighborhood ($0.089\ M_\odot/pc^3$), and (ii) the z component of the galactic gravitational acceleration K_z is supplied by Oort, in which case we implicitly assume that the total mass density at the galactic plane equals $0.15\ M_\odot/pc^3$. Using recently observed values for the magnetic and cosmic-ray pressures, the z component of the turbulent gas velocity in the solar neighborhood is established.

A theoretical expression is derived for the half-thickness of the equilibrium disk of galactic gas. The half-thickness is found to be directly proportional to Q, where Q^2 equals the square of the z component of the turbulent gas velocity plus the magnetic pressure divided by the gas density plus the cosmic-ray pressure divided by the gas density, and inversely proportional to the square root of the total mass density at the galactic plane. Using the Innanen galactic

mass model, the half-thickness is computed as a function of distance from the galactic center and compared to the observations of McGee and Milton. The observed increase in half-thickness at distances beyond the solar distance R_o is reproduced theoretically. Finally, a galactic mass model is derived, using the observed half-width of the gas layer and an appropriate value for the z component of the turbulent gas velocity independent of R. Comparison of the two mass models in the region $4 \text{ kpc} \leq R \leq R_o$ indicates that Q may systematically increase with decreasing R.

A time-independent, linear, plane- and axially-symmetric perturbation analysis is performed on a plane-parallel layer of non-magnetic, non-rotating gas. The gas layer is itself immersed in a plane-stratified isothermal slab of stars which supply a self-consistent gravitational field. Only the gaseous component is perturbed. Expressions are derived for the perturbed gas potential and perturbed gas density that satisfy both the Poisson and hydrostatic equilibrium equations. The equation governing the size of the perturbations is found to be analogous to the one-dimensional time-independent Schrodinger equation for a particle bound by a potential well, and with similar boundary conditions. The perturbation radius is computed as a function of the parameter Q and compared with the Jeans' and Ledoux radii. Isodensity contours of the marginally unstable state are constructed. Large flattened objects with masses in the range 10^6 - 10^7 $M_\odot$ and radii in the symmetry plane characteristically 1-2 kpc result. Both the size and flattening increase as the star density at the galactic plane increases.

Finally, the stability of a self-gravitating isothermal gas disk threaded by a one-dimensional equipartition magnetic field is considered with respect to waves with motions in the $\vec{B}_e - \vec{g}$ plane and perpendicular to it. In the latter case, the magnetic field modifies the minimum length necessary to produce gravitational instability by the factor $\left(\frac{c^2 + a^2/2}{c^2}\right)^{\frac{1}{2}}$, where c is the isothermal sound speed and a is the Alfvén velocity.

I. INTRODUCTION

In recent years observations of interstellar absorption lines, interstellar extinction, and 21-cm radiation have aided in determining the distribution of the gaseous component in our galaxy. This paper discusses, among other topics, two rather recent observational findings relating to the equilibrium state of the interstellar gas disk: (i) the run of gas density ρ_g with distance above and below the galactic plane (Schmidt 1956), and (ii) the rapid increase in half-thickness of the gas layer at distances from the galactic center greater than the solar distance (McGee and Milton 1964) and its relatively uniform thickness at distances between R_o and about 4 kpc (Schmidt 1956; McGee and Milton 1964). To compute the equilibrium state of the gas disk, the Poisson equation and the hydrostatic equilibrium equations for the gaseous and stellar components of a two-fluid, infinitely-extended, plane-parallel mixture are used. Each component is assumed to be isothermal but each has its own characteristic temperature. The analysis in addition includes one-dimensional magnetic and cosmic-ray components, each constrained to vary with ρ_g according to equipartition assumptions.

Next, the equilibrium disk of non-magnetic gas is subject to time-independent, linear, plane- and axially-symmetric perturbations, the stellar distribution remaining unaffected. The perturbations are applied to the gas hydrostatic and Poisson equations in an effort to find the marginally unstable state. Because the analysis is time-independent, we are unable to follow the evolution of the perturbation. The marginally unstable configuration is determined, however, and is found to consist of flattened clouds of gas immersed in a plane-parallel slab of stars. The gas perturbations are found to have masses of 10^6 - 10^7 $M_\odot$ and radii of 1-2 kpc.

Finally, a similar perturbation analysis is performed on a magnetic self-gravitating isothermal gas disk immersed in a static isothermal star disk. Propagation along $\vec{B}_e$ leads to a system of equations with no obvious solution. Propagation across $\vec{B}_e$ leads to a single simple equation from which the minimum length needed to induce gravitational instability can be computed as a function of c, the isothermal sound speed of the gas, and B_e.

II. THE EQUILIBRIUM DISTRIBUTION OF GAS ABOVE THE GALACTIC PLANE

The initial equilibrium configuration of the galactic gaseous, stellar, magnetic, and cosmic-ray components may be pictured as follows. A plane-stratified (in z) infinitely-extended (in r) non-rotating gas slab is immersed in a star slab of similar description, the gas and star densities attaining maximum values at the mid-plane $z = 0$. The weight of the gas layer confines a one-dimensional magnetic field and cosmic-ray gas, the direction of the field assumed parallel to the plane $z = 0$. Early optical polarization measurements (Hiltner 1956) gave support to such a simple magnetic topology. However, more recent optical polarization measurements (Mathewson 1968; Mathewson and Nicholls 1968) on large numbers of stars indicate the presence of a helical component with a pitch angle of $7°$.

The Poisson and hydrostatic equilibrium equations expressed in cylindrical geometry may be written as follows:

$$\frac{d^2\varphi}{dz^2} = -4\pi G\ (\rho_g + \rho_*) \tag{1}$$

$$\frac{d}{dz}\ (P_g + B^2/8\pi + P_{c-r}) = \rho_g \frac{d\varphi}{dz} \tag{2}$$

and

$$\frac{dP_*}{dz} = \rho_* \frac{d\varphi}{dz}\quad , \tag{3}$$

where ρ_g and ρ_* are the gas and star densities, P_g and P_* are the gas and star pressures, and φ is the total gravitational potential, contributed to by both gas and stars. The two terms containing derivatives with respect to r in the expression for ∇^2 are omitted. Their contribution is less than 10% of the total at the sun's distance from the galactic center. The equation of state relating the pressure and density of the isothermal gaseous component can be written in the following manner

$$P_g = \langle v_{tz}^2 \rangle\ \rho_g \tag{4}$$

where $\langle v_{tz}^2 \rangle$ is the z component of the combined turbulent and thermal gas velocities, each squared. Following Parker (1966, 1969) we assume that $\langle v_{tz}^2 \rangle$ and the ratios of magnetic and cosmic-ray pressures to gas pressure are independent of z, allowing us to write that

$$B^2(z)/\rho_g(z) = B_o^2/\rho_{go}$$

and (5)

$$P_{c-r}(z)/\rho_g(z) = P_{c-ro}/\rho_{go}$$

where B_o, P_{c-ro}, and ρ_{go} are the values of B, P_{c-r}, and ρ_g evaluated at the galactic plane (z = 0). With the aid of equations (5), equation (4), and a similar equation for the stellar component

$$P_* = \langle v_{*z}^2 \rangle \rho_* \quad , \tag{6}$$

the two hydrostatic equilibrium equations assume the form

$$Q^2 \frac{d}{dz} \ln \rho_g = \frac{d\varphi}{dz} \tag{7}$$

$$\langle v_{*z}^2 \rangle \frac{d}{dz} \ln \rho_* = \frac{d\varphi}{dz} \tag{8}$$

where Q^2 denotes the quantity $[\langle v_{tz}^2 \rangle + B_o^2/8\pi\rho_{go} + P_{c-ro}/\rho_{go}]$. We have here explicitly assumed that both the gaseous and stellar components of the two-fluid mixture are isothermal, i.e. Q^2 and $\langle v_{*z}^2 \rangle$ are independent of z (but are unequal).

The Poisson equation and the stellar hydrostatic equilibrium equation need not be called upon explicitly to determine the distribution $\rho_g(z)/\rho_{go}$ if the gravitational acceleration $\frac{d\varphi}{dz}$ is known as a function of z. Oort (1960) determined this acceleration by analyzing star counts and radial velocities of K giants. The density of all matter found by Oort in the solar neighborhood is 0.15 $M_\odot/pc^3$, significantly greater than the combined observed density of stars and gas which is .089 $M_\odot/pc^3$ (Luyten 1968; Weaver 1970). More recently Woolley and Stewart (1967), noting that the velocity dispersion of A-type stars increases near the galactic pole with increasing magnitude, concluded that the density of all matter in the solar vicinity is 0.11 $M_\odot/pc^3$, considerably closer to the observed density than was found by Oort.

The question of the true density in the solar neighborhood and hence of the component of the gravitational acceleration perpendicular to the galactic plane is, in the author's opinion, not yet fully settled. We choose therefore to follow two separate courses in calculating $\rho_g(z)$, keeping in mind that the 'correct' approach probably lies somewhere between the following two limiting cases. (i) The Poisson equation and the two hydrostatic equilibrium equations are solved simultaneously for $\rho_g(z)$, assuming that the total mass density at the galactic plane is represented by the sum of the observed gas and star densities in the solar neighborhood (0.089 $M_\odot/pc^3$). Since the bulk of the stellar mass in the solar vicinity resides in the main sequence G, K, and M stars (Oort 1960) and since each type has nearly the same velocity dispersion of 18 km/sec in the z direction (Woolley 1952; Wehlau 1957), a unique value of 18 km/sec for $(\langle v_{*z}^2 \rangle)^{\frac{1}{2}}$ will be assumed for all stars. (ii) The gas hydrostatic equilibrium equation is solved for $\rho_g(z)$ using Oort's values of the z component of the galactic gravitational acceleration K_z at the sun's distance from the galactic center. Using these K_z we implicitly assume that $\rho_{go} + \rho_{*o} = 0.15\ M_{\odot}/pc^3$, where ρ_{go} and ρ_{*o} are the values of the gas and star densities at the galactic plane. This procedure is more accurate than Parker's (1966, 1969) work along similar lines because the simplifying approximation $K_z(z) = \langle K_z \rangle$ = constant is not employed.

We proceed with (i) by equating the left-hand sides of equations (7) and (8) and integrating, with the result that

$$\rho_* = \rho_{*o}\ (\rho_g/\rho_{go})^{Q^2/\langle v_{*z}^2 \rangle} . \tag{9}$$

Combining this with the Poisson equation then gives

$$\frac{d^2\varphi}{dz^2} = -4\pi G \left[\rho_g + \rho_{*o}\ (\rho_g/\rho_{go})^{Q^2/\langle v_{*z}^2 \rangle}\right] . \tag{10}$$

The gas hydrostatic equilibrium equation is differentiated and combined with equation (10):

$$\frac{d^2}{dz^2} \ln \rho_g/\rho_{go} = \frac{-4\pi G}{Q^2} \left[\rho_g + \rho_{*o}\ (\rho_g/\rho_{go})^{Q^2/\langle v_{*z}^2 \rangle}\right] . \tag{11}$$

Equation (11) may be reduced to a dimensionless form by defining four dimensionless parameters

$$\alpha = \rho_g(z)/\rho_{go}$$

$$\beta = z/H_g \text{ where } H_g = Q/(8\pi G\rho_{go})^{\frac{1}{2}} \tag{12}$$

$$\gamma = \rho_{*o}/\rho_{go}$$

and

$$\delta = Q^2/\langle v_{*z}^2\rangle$$

in which case equation (11) becomes

$$\frac{d^2 \ln\alpha}{d\beta^2} = -\tfrac{1}{2}\ [\alpha + \gamma\alpha^{\delta}] \quad . \tag{13}$$

Equation (13) is solved numerically for $\alpha(\beta)$ with three values of δ and hence of Q. Q is chosen to be 5.0, 7.7, and 10.0 km/sec; $(\langle v_{*z}^2\rangle)^{\frac{1}{2}}$ is chosen to be 18 km/sec. The observed gas and star densities ρ_{go} = 1 H atom/cm^3 = 0.025 $M_\odot/pc^3$ (Weaver 1970) and ρ_{*o} = 0.064 $M_\odot/pc^3$ (Luyten 1968) are used. The resultant curves of $\rho_g(z)/\rho_{go}$ are presented in Figure 1 together with a smooth curve through Schmidt's (1956) observed data, the latter corrected by the change in galactic distance scale R_o(new)/R_o(old) = 10.0 kpc/8.2kpc = 1.22.

When z < 200 pc the solution with Q = 7.7 km/sec agrees closely with Schmidt's observations. When z > 200 pc the observed curve lies above the theoretical solution, the discrepancy increasing with increasing z. Some possible causes of this discrepancy will be discussed shortly.

If Q is composed of turbulent, thermal, magnetic and cosmic-ray components, i.e.

$$Q^2 = \langle v_{tz}^2 \rangle + B_o^2/8\pi\rho_{go} + P_{c\text{-}ro}/\rho_{go} \tag{14}$$

$(\langle v_{tz}^2 \rangle)^{\frac{1}{2}}$ can then be determined by employing the Q found from Figure 1 and the observed values $B_o^2/8\pi$, $P_{c\text{-}ro}$, and ρ_{go}. Using a cosmic-ray pressure of 0.50×10^{-12} dynes/cm^2 (Parker 1969), a magnetic pressure of 0.37×10^{-12} dynes/cm^2 (Parker 1969; Verschuur 1969) corresponding to a field strength of 3.5×10^{-6} gauss, and $\rho_{go} = 0.025\ M_\odot/pc^3$, we find that $(\langle v_{tz}^2 \rangle)^{\frac{1}{2}} = 2.66$ km/sec.

Recent evidence from RR Lyrae star counts near the galactic center (Plaut 1970) indicates that R_o may be significantly less than 10 kpc. If we choose R_o to be 8.2 kpc and go through the same procedure as above, $(\langle v_{tz}^2 \rangle)^{\frac{1}{2}}$ turns out to be 0 km/sec, i.e. there is no room for gas turbulence. This result can clearly be ruled out by 21-cm line profile observations.

The second approach consists in solving the gas hydrostatic equilibrium equation using Oort's K_z. We recall equation (7)

$$Q^2 \frac{d}{dz} \ln\rho_g/\rho_{go} = \frac{d\varphi}{dz} = K_z \ . \tag{15}$$

Integrating between the limits $z = 0$ and $z = z$ we can solve for ρ_g/ρ_{go}:

$$\rho_g(z)/\rho_{go} = e^{\frac{1}{Q^2}\int_0^z K_z d_z} \quad . \tag{16}$$

Notice that neither ρ_{*o} nor $\langle v_{*z}^2\rangle$ appear here. Rather they are contained implicitly in K_z. The computed distribution $\rho_g(z)/\rho_{go}$ is shown in Figure 2 together with a smooth curve through Schmidt's data, once again corrected for the change in galactic scale length. $Q = 9.84$ km/sec gives the closest fit to the observed curve for $z < 200$ pc. This solution follows closely the simultaneous solution of the Poisson equation and the two hydrostatic equilibrium equations computed above with $Q = 7.7$ km/sec, as a comparison of Figures 1 and 2 shows. The higher Q arising from the use of Oort's K_z is directly related to the higher density of all matter found by Oort (0.15 $M_\odot/pc^3$ vs. 0.089 $M_\odot/pc^3$) at the galactic plane. Using $Q = 9.84$ km/sec with the previously quoted values for $B_o^2/8\pi$, P_{c-ro}, and ρ_{go}, we find that $(\langle v_{tz}^2\rangle)^{\frac{1}{2}} = 6.68$ km/sec. We should emphasize once again that $(\langle v_{tz}^2\rangle)^{\frac{1}{2}}$ is a function of the value chosen for R_o, since a change in R_o is reflected in the observed half-thickness of the gas layer. A decrease in R_o causes a corresponding decrease in half-thickness, and a smaller gas turbulent velocity is necessary to support the gas layer against gravity. Specifically, if $R_o = 8.2$ kpc rather than 10.0 kpc, $(\langle v_{tz}^2\rangle)^{\frac{1}{2}}$ becomes 3.44 km/sec.

Let us comment briefly here as to the outcome of approaches (i) and (ii). Only approach (ii) using Oort's K_z (implicitly assuming that

$\rho_{*o} + \rho_{go} = 0.15\ M_{\odot}/pc^3$) and $R_o = 10$ kpc yields $(\langle v_{tz}^2 \rangle)^{\frac{1}{2}}$ in the neighborhood of 7 km/sec as observed by Westerhout (1956) and others. All other assumptions result in turbulent gas velocities at least a factor of two smaller. Indeed, we may use this result as evidence that the Oort limit is substantially correct and that $R_o = 10$ kpc.

The theoretical distribution $\rho_g(z)/\rho_{go}$ falls off too rapidly with distance above the galactic plane greater than 200 pc for both approaches. Three possible explanations are (a) the gas hydrostatic equilibrium equation fails to hold when $z > 200$ pc, (b) the assumptions that $B^2(z)/\rho_g(z)\langle v_{tz}^2 \rangle$ and $P_{c-r}(z)/\rho_g(z)\langle v_{tz}^2 \rangle$ are independent of z break down when $z > 200$ pc, and (c) the observations at $z > 200$ pc are incorrect.

III. THE HALF-THICKNESS OF THE EQUILIBRIUM GAS DISK

Schmidt (1956) observed the thickness of the neutral hydrogen layer at several tangential points of the galaxy and found that between about 3 kpc from the galactic center and R_o, the half-thickness of the layer is surprisingly constant, with a value of 220 pc (268 pc on the new galactic distance scale with $R_o = 10$ kpc). More recently, Kerr (1969) has found a constant half-thickness of 200 pc over the region 4 kpc $\leq R \leq R_o$. Westerhout's (1956) isodensity contours of neutral hydrogen indicate that the half-thickness of the gas layer increases substantially with increasing distance from the galactic center beyond the sun's position R_o. Recently, McGee and Milton (1964) confirmed the relatively constant half-thickness at distances between 3 kpc and R_o (on the old distance scale). Beyond R_o they found that the half-thickness increases rapidly, reaching almost 1400 pc 13 kpc from the galactic center. We shall attempt to explain these observations from a theoretical point of view.

We recall equation (13) with the substitution $y = \ln \alpha$:

$$\frac{d^2y}{d\beta^2} = -\tfrac{1}{2}\,[e^y + \gamma e^{\delta y}]\ . \tag{17}$$

$\delta = Q^2/\langle v_{*z}^2\rangle$ is typically on the order of 0.1 to 0.2 and can be set equal to zero with little loss of accuracy. Equation (17) therefore becomes

$$\frac{d^2y}{d\beta^2} = -\tfrac{1}{2}\,[e^y + \gamma]\ . \tag{18}$$

The homogeneous equation

$$\frac{d^2y}{d\beta^2} = 0 \tag{19}$$

has the solution $y = A + B\beta$. But since $y(\beta = 0) = \frac{dy}{d\beta}(\beta = 0) = 0$, $A = B = 0$ and thus $y = 0$. To find the particular solution to the inhomogeneous equation we integrate equation (18) twice between the limits $\beta = 0$ and $\beta = \beta$, and find that

$$y = -\tfrac{1}{2}\,e^{\beta} - \tfrac{1}{4}\gamma\beta^2 + \tfrac{1}{2}\beta + \tfrac{1}{2}\ . \tag{20}$$

But $y = \ln\alpha$, hence

$$\alpha = e^{-\frac{1}{2}[e^{\beta} + \frac{1}{2}\gamma\beta^2 - \beta - 1]} . \tag{21}$$

Expanding e^{β} and keeping terms only to β^2, equation (21) becomes

$$\alpha \approx e^{-\frac{\beta^2}{4}[1 + \gamma]} . \tag{22}$$

Setting $\alpha = \rho_g(z)/\rho_{go} = \frac{1}{2}$, equation (22) is solved for $\beta_{\frac{1}{2}}$:

$$\beta_{\frac{1}{2}} \approx \left[\frac{2.773}{1 + \gamma}\right]^{\frac{1}{2}} . \tag{23}$$

The half-thickness, i.e. the distance between the points above and below the galactic plane where the gas density is exactly one-half its value at the plane, expressed in centimeters, can be written $z_{\frac{1}{2}} \approx 2\beta_{\frac{1}{2}}Hg$, i.e.

$$z_{\frac{1}{2}} \approx \left[\frac{1.386Q^2}{\pi G}\right]^{\frac{1}{2}} \left[\frac{1}{\rho_{*o} + \rho_{go}}\right]^{\frac{1}{2}} . \tag{24}$$

Equation (24) reveals that the half-thickness is proportional to the square root of Q^2, which we recall is equal to the combined gas turbulence squared plus magnetic pressure/ρ_g plus cosmic-ray pressure/ρ_g, all quantities evaluated at the galactic plane, and is inversely proportional to the square root of the total mass density at the galactic plane.

Taking $z_{\frac{1}{2}}$ as known from the observations of McGee and Milton, we may use equation (24) (i) to compare the observed and predicted distributions of $z_{\frac{1}{2}}(R)$, using a galactic mass model for $\rho_{To}(R) = \rho_{go}(R) + \rho_{*o}(R)$ and choosing that value of Q which gives the best fit to the observations at $R = R_o$ (where Q is assumed to be independent of distance from the galactic center), (ii) to derive a galactic mass model, assuming an appropriate value for Q independent of R, and (iii) to derive Q(R) on the basis of a galactic mass model. Schmidt (1956) has briefly explored (iii) for $R < R_o$ using his galactic mass model. We shall pursue (i) and (ii).

To pursue (i) the quantity $\rho_{To} = \rho_{*o} + \rho_{go}$ (the total density of matter at the galactic plane) must be known as a function of distance from the galactic center. Interstellar obscuration by dust severely limits our knowledge of $\rho_{*o}(R)$ at distances beyond about 1 kpc from the sun. Theoretically determined densities $\rho_{To}(R)$ must therefore be used. The Innanen galactic mass model with R_o = 10.0 kpc and $V_\odot$ = 252 km/sec is selected (Table 1). Q is chosen to be 7.25, 10.25, and 13.25 km/sec. The three curves for $z_{\frac{1}{2}}(R)$ that result are compared with the observations in Figure 3.

At distances less than about 8 kpc from the galactic center, all three computed curves fall below the mean curve through the observed points, suggesting either that a larger value is required for Q or that the total mass density at the galactic plane ρ_{To} has been overestimated. The curve with Q = 10.25 km/sec exhibits the best agreement with the observed half-thickness of 268 pc at the sun's position (assuming $R_o = 10$ kpc). This relatively high value of Q is in part due to the high mass density found by Innanen in the solar neighborhood ($0.149\ M_\odot/pc^3$) and to the approximations employed in deriving equation (24). All three curves exhibit the sharp rise in $z_{\frac{1}{2}}(R)$ beyond R_o found by McGee and Milton. We interpret, therefore, the increase in half-thickness of the gas layer to be due to the decrease in ρ_{To} at distances beyond R_o. In addition, because of the sharp rise in the computed half-width $z_{\frac{1}{2}}(R)$ in the outer galactic regions, there is the suggestion that Q decreases systematically with increasing distance over the range $R > 14$ kpc.

We have implicitly assumed here that the plane of maximum gas density is perfectly flat. 21-cm observations (Kerr 1957; Gum, Kerr, and Westerhout 1960) indicate, however, that a significant warpage occurs in the layer of interstellar hydrogen. In the region $R < R_o$, the points of maximum hydrogen density define a plane to within 50 pc. However, for $R > R_o$, the layer curves upward on one side of the galaxy and downward on the opposite side, the deviation from the principal plane reaching 1 kpc in the very outer rim of the galaxy. Theoretical discussions concerning the physical mechanism responsible for the distortion include interaction of the galactic halo with intergalactic gas (Kahn and Woltjer 1959), interaction of the galactic gas disk with the Magellanic Clouds at or near their present distance (Avner and King 1967), and interaction with the Magellanic Clouds during a close passage (Habing and Visser 1966; Hunter and Toomre 1969). If the increase in half-thickness is causally related to the warpage, our derivation may not be applicable.

The question naturally arises as to whether the increase in half-thickness of the gas layer is observed in galaxies other than our own. Low resolution 21-cm studies of extragalactic systems (Roberts 1969) at present offer no answer to this question. We can only hope to use the interstellar dust of the external galaxy as a gas tracer. Further, the system must be observed almost exactly edge on so as to minimize projection effects which confuse the issue. One favorable such galaxy, NGC 5866, has been observed by Burbidge and Burbidge (1960). Its dust layer, which extends very far from the center of the galaxy, remains remarkably flat in the region $R < R_1$. For $R > R_1$, both the half-thickness and warpage increase noticeably with increasing R.

To pursue (ii), i.e. to construct a galactic mass model, we are faced with the problem of choosing appropriate values of Q(R). What makes this problem exceptionally difficult is that Q can change if either of the three quantities $\langle v_{tz}^2 \rangle$, B_o^2/ρ_{go}, or P_{c-ro}/ρ_{go} vary with R. Because we quite obviously lack this information, Q is taken to be independent of R for purposes of simplicity, and is determined by the constraints that $\rho_{To} = 0.15\ M_\odot/pc^3$ and $z_{\frac{1}{2}} = 268$ pc in the solar vicinity. Under these restrictions, Q becomes 10.25 km/sec.

The resultant galactic mass model is presented in Table 1 along with Innanen's galactic mass model for purposes of comparison. That ρ_{To} is independent of R over the range 4 kpc $\leq R \leq$ 10 kpc is a result of the constancy of the observed half-width and the assumed constancy of Q. Our mass densities are systematically less than those found by

Innanen over this region, the disagreement becoming more substantial as R decreases. The agreement at $R = R_o$ is forced by our choice of Q. The derived mass density is proportional to Q^2. Uncertainties of a factor of at least 1.5 percent in Q introduce uncertainties greater than a factor of two in the mass model. To this must be added the uncertainties in the observed values of $z_{\frac{1}{2}}(R)$. The importance of our mass model at $R < R_o$ rests not in the exact numerical values derived for $\rho_{To}(R)$ (most certainly incorrect) but in demonstrating that present-day galactic mass models (e.g. Innanen's) cannot be reproduced by simply assuming that the gas and star disks obey hydrostatic and Poisson considerations and that $z_{\frac{1}{2}}$ and Q remain constant over the range 4 kpc $\le R \le$ 10 kpc. If the mass densities quoted by Innanen over this range are even near correct, it is clear that we must part with one of our assumptions. The restriction $Q(R)$ = constant appears to be the weakest link in the argument. Indeed, if Q is determined by the condition that $z_{\frac{1}{2}} = z_{\frac{1}{2}}$ observed and $\rho_{To} = \rho_{To\ \text{Innanen}}$, it becomes 24.8 km/sec at R = 4 kpc, significantly greater than its value at $R = R_o$. The suggested increase in $Q(R)$ with decreasing R may be due to larger gas turbulent velocities, possibly caused by an increase in the rate of star formation in regions near the galactic center (Schmidt 1970).

IV. THE STABILITY OF A SELF-GRAVITATING NON-MAGNETIC GAS DISK

The best-known perturbation analysis with sought for applications in galactic astronomy was carried out by Jeans (1928), and is appropriately called the Jeans' instability problem. Jeans considered as the initial equilibrium state an infinite static medium with a uniform density and gravitational potential field throughout. The last assumption is, however, inconsistent with a uniform non-zero density. It is this inconsistency which has prompted others to reinvestigate Jeans' analysis with self-consistent equilibrium density and potential distributions.

Jeans applied perturbations of the form

$$\begin{aligned} P &= P_o + \Delta P \\ \rho &= \rho_o + \Delta\rho \\ u &= 0 + \Delta u \end{aligned} \tag{25}$$

where P_o and ρ_o are the equilibrium quantities, to the hydrostatic equilibrium, continuity, Poisson, and heat equations, and after linearization of the resulting equations found the condition that

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 - 4\pi G\rho_o\right)\Delta\rho = 0 \tag{26}$$

where c_s is the isothermal sound speed of the medium. Jeans further assumed that $\Delta\rho$ takes the form

$$\Delta\rho = \Delta\rho_o \, e^{i(\vec{k} \cdot \vec{r} - \sigma t)} \tag{27}$$

in which case equation (26) readily yields that

$$(k^2 c_s^2 - \sigma^2 - 4\pi G\rho_o) = 0 \; . \tag{28}$$

If σ^2 is taken to be zero, the condition for neutral equilibrium, and if we recognize that the wave number $k = 2\pi/\lambda$, we find that

$$\lambda_J = c_s \; (\pi/\rho_o G)^{\frac{1}{2}} \; . \tag{29}$$

Equations (27), (28), and (29) tell us that all disturbances of dimension greater than λ_J will lead to gravitational instability. Conversely, all disturbances of length less than λ_J are gravitationally stable.

The first modification to Jeans' analysis of any real importance to galactic astronomy was undertaken by Ledoux (1951), who considered the more realistic initial equilibrium state of a plane-parallel non-rotating slab of self-gravitating isothermal gas (no stars). The gas density at the plane of symmetry assumes its maximum value of ρ_o and decreases with distance from the plane according to the well-known formula

$$\rho(z)/\rho_o \propto \operatorname{sech}^2 \; (z/H) \tag{30}$$

(Schmid - Burgk 1967) where H is the scale height of the gas layer in the z direction. With these self-consistent initial conditions, Ledoux proceeds to perform time-dependent, linear, plane- and axially-symmetric perturbations on the appropriate equations (cited above), and finds that the gas layer is just marginally stable against sinusoidal perturbations in the density, pressure, and velocity. The perturbations have infinitesimal amplitudes, and their lengths are found to be given by the formula

$$\lambda_L = c_s(2\pi/G\rho_o)^{\frac{1}{2}} . \tag{31}$$

This is precisely the same restriction as was found by Jeans, except that ρ is replaced by $\rho_o/2$.

The next improvement was by Schmid-Burgk (1967), who considers essentially the same equilibrium state as Ledoux (no stars). The amplitudes of the perturbations are not, however, restricted to be infinitesimal. In the limit of small amplitudes, the solution tends to that of Ledoux (equation 31), while in the limit of large amplitudes, the gas divides into a series of separate parallel cylinders spaced by $\lambda_L = c_s(2\pi/G\rho_o)^{\frac{1}{2}}$.

Toomre (1964) has investigated the gravitational stability of a large flattened system of stars in an initial equilibrium state where self-gravity balances the centrifugal forces due to rotation. He concludes that the system will form large-scale condensations unless large turbulent velocities are present. If the system does not possess these large random velocities, it will be unstable to disturbances that may approach the scale of the system in overall dimension.

Goldreich and Lynden-Bell (1965) consider the gravitational stability of plane-stratified, self-gravitating, uniformly-rotating disks of gas. They find that (i) pressure effects tend to stabilize short wavelength disturbances while rotation tends to stabilize long wavelength disturbances and (ii) when the quantity $\pi G\bar{\rho}/4\Omega^2$ is greater than about 1.0 (the exact number depending on the value of the gas polytropic index), disturbances of dimension several times the gas layer thickness become unstable.

Finally, Lin and Shu (1964) have discussed a density-wave theory of galactic spirals, neglecting the dispersion of star velocities and gas turbulence. In a later paper, Lin and Shu (1966) include the effects of the dispersion of stellar velocities and gas turbulence. More recently, Lin, Yuan, and Shu (1969) have included the presence of a magnetic field and its effect on the gas. The overall aim of the density-wave theory is to explain (a) the apparently large time scales involved in spiral structure in the face of differential galactic rotation and (b) the large scale dimension of spiral structure. The theory calculates the response of an infinitesimally thin sheet of gas and stars to a resultant spiral gravitational field.

With these ideas in mind, let us proceed with the perturbation analysis. We wish to subject a plane-stratified, non-rotating, non-magnetic distribution of gas to linear, time-independent, plane- and axially-symmetric perturbations. The plane of symmetry is the mid-plane $z = 0$, where the gas and star densities take on their maximum values.

The axis of symmetry is perpendicular to the symmetry plane and arbitrarily positioned. The gas is immersed in the gravitational field of a plane-parallel distribution of stars. Both components of the two-fluid mixture are isothermal but each has a different sound speed. We should stress that only the gaseous component is perturbed; the stars merely supply a self-consistent gravitational field which adds to the gravitational field of the gas. In addition, by restricting the perturbations to being time-independent, the marginally unstable state can be found if it exists, but we are unable to follow the initial equilibrium state as it evolves to the new equilibrium. It is hoped that the inclusion of the stellar component will lend the analysis an increased sense of reality as concerns its application to the Galaxy.

The time-independent perturbations may be written as follows:

$$\rho_g(r,z) = \rho_{eg}(z) + \Delta\rho_g(r,z) \qquad \text{a}$$

$$P_g(r,z) = P_{eg}(z) + \Delta P_g(r,z) = \langle v_{tz}^2 \rangle \, \rho_g(r,z) \qquad (32) \quad \text{b}$$

and

$$\varphi_g(r,z) + \varphi_*(r,z) = \varphi_{eg}(z) + \varphi_{e*}(z) + \Delta\varphi_g(r,z) \,. \qquad \text{c}$$

The subscript e refers to the equilibrium quantities, functions of z only (the distance from the symmetry plane), while Δ refers to the perturbed quantities, functions of both r (the distance from the symmetry axis) and z. Since only the gaseous component is perturbed, we can write that

$$\varphi_* (r,z) = \varphi_{e*} (z) \tag{33}$$

so that equation (32) c becomes

$$\varphi_g (r,z) = \varphi_{eg} (z) + \Delta\varphi_g (r,z) \ . \qquad (32)\ c'$$

Equations (32) a and c′ and (33) are substituted into the gas hydrostatic equilibrium equation

$$\frac{Q^2}{\rho_g} \vec{\nabla} \rho_g = \vec{\nabla} (\varphi_g + \varphi_*) \tag{34}$$

and if terms only to first order in $\Delta\rho_g$ and $\Delta\varphi_g$ are kept, we are left with the equation

$$\frac{1}{\rho_{eg}} \vec{\nabla}\rho_{eg} + \frac{1}{\rho_{eg}} \vec{\nabla} (\Delta\rho_g) - \frac{\Delta\rho_g}{\rho_{eg}^2} \vec{\nabla}\rho_{eg} = \frac{1}{Q^2} \vec{\nabla} (\varphi_{eg} + \varphi_{e*}) + \frac{1}{Q^2} \vec{\nabla}(\Delta\varphi_g). \tag{35}$$

The gas hydrostatic equilibrium equation for the initial equilibrium layer of gas and stars is easily written as

$$\frac{1}{\rho_{eg}} \vec{\nabla} \rho_{eg} = \frac{1}{Q^2} \vec{\nabla} (\varphi_{eg} + \varphi_{e*}) \ . \tag{36}$$

Equation (35) thus simplifies to

$$\frac{1}{\rho_{eg}} \vec{\nabla} (\Delta\rho_g) - \frac{\Delta\rho_g}{\rho_{eg}^2} \vec{\nabla}\rho_{eg} = \frac{1}{Q^2} \vec{\nabla} (\Delta\varphi_g) \ . \tag{37}$$

Next we define a new variable $\epsilon(r,z)$:

$$\epsilon(r,z) = \frac{\Delta\rho_g(r,z)}{\rho_{eg}(z)} \quad . \tag{38}$$

Equation (37) can now be written in terms of ϵ:

$$\vec{\nabla}\epsilon = \frac{1}{Q^2}\vec{\nabla}\,(\Delta\varphi_g) \quad . \tag{39}$$

Integrating equation (39) then gives

$$\epsilon = \frac{1}{Q^2}\,\Delta\varphi_g + \text{constant} \quad . \tag{40}$$

Operating on equation (40) with the Laplacian ∇^2 yields

$$\nabla^2\epsilon = \frac{1}{Q^2}\,\nabla^2\,(\Delta\varphi_g) = -\frac{1}{Q^2}\,4\pi G\Delta\rho_g = -\frac{1}{2H_g^2}\,\rho_{eg}/\rho_{ego}\;\epsilon \tag{41}$$

where we have introduced for the first time the Poisson equation

$$\nabla^2(\Delta\varphi_g) = -4\pi G\Delta\rho_g \tag{42}$$

and where it can be shown that

$$H_g = \left(\frac{Q^2}{8\pi G\rho_{ego}}\right)^{\frac{1}{2}} \tag{43}$$

is a scale height for the gas in the absence of stars. Equation (41) may be solved by the method of separation of variables, in which case ϵ is written as

$$\varepsilon(r,z) = X(z)\, Y(r) \; . \tag{44}$$

After some simplification, equation (41) becomes

$$-\frac{[Y''(r) + \frac{1}{r} Y'(r)]}{Y(r)} = k^2 = \frac{\left[X''(z) + \frac{1}{2H_g^2} \rho_{eg}/\rho_{ego} X(z)\right]}{X(z)} \; . \tag{45}$$

The separation constant k^2 arises because the left-hand side of equation (45) is a function of r only while the right-hand side is a function of z only.

The differential equation for Y(r)

$$Y''(r) + \frac{1}{r} Y'(r) + k^2 Y(r) = 0 \tag{46}$$

has the solution

$$Y(r) = J_o(kr) \tag{47}$$

where J_o is the zero order Bessel function. The other linearly-independent solution to equation (46) $\rightarrow \infty$ as $r \rightarrow 0$, and is therefore not considered further. The differential equation for $X(z) = X_k(z)$ is

$$X_k''(z) + \left[\frac{1}{2H_g^2} \rho_{eg}(z)/\rho_{ego} - k^2\right] X_k(z) = 0 \; , \tag{48}$$

with the boundary conditions

$$X_k'(0) = 0 \qquad \text{a}$$

$$\lim_{z \to \pm\infty} X_k(z) = 0 \; . \qquad \text{b} \tag{49}$$

In addition, $X_k(0)$ is set equal to 1 for purposes of normalization.

The first of equations (49) arises since $\Delta\rho_g$ is an even function of z. The second comes from the requirement that the ratio $\Delta\rho_g/\rho_{eg}$ tend to zero as $|z| \to \infty$.

The general solution to equation (41) can be expressed as an integral over the product $X_k(z)\ J_o(kr)$ with the appropriate expansion amplitudes A(k):

$$\epsilon(r,z) = \int_o^\infty A(k)\ X_k(z)\ J_o(kr)\ dk\ , \tag{50}$$

where $J_o(kr)$ is the solution to equation (46) and $X_k(z)$ is the solution to equation (48) subject to the boundary conditions expressed by equations (49) a and b. Equations (40) and (50) may now be combined to give

$$\Delta\varphi_g(r,z) = \langle v_{tz}^2\rangle \int_o^\infty A(k)\ X_k(z) J_o(kr) dk - \langle v_{tz}^2\rangle \text{ constant.} \tag{51}$$

From the theory of differential equations, the solution to Poisson's equation for the perturbed gas potential is just the sum of the particular solution of the inhomogeneous equation and the general solution of the homogeneous equation. Equation (51) with the constant = 0 corresponds to the former. The general solution to the homogeneous equation is equivalent to determining the solution to Laplace's equation

$$\nabla^2 \Delta\varphi_{gh} = 0, \tag{52}$$

the subscript h reminding us that we are seeking the general solution to the homogeneous equation.

Expanding $\Delta\varphi_{gh}$ in terms of $J_o(kr)$, we have that

$$\Delta\varphi_{gh}(r,z) = \int_o^\infty s_k(z)\ J_o(kr)\ dk\ . \tag{53}$$

In cylindrical coordinates, equation (52) assumes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Delta\varphi_{gh}\right) + \frac{\partial^2}{\partial z^2}\Delta\varphi_{gh} = 0\ . \tag{54}$$

When combined with the expansion from equation (53), the second term becomes simply

$$\frac{\partial^2}{\partial z^2}\Delta\varphi_{gh} = \int_o^\infty s_k''(z)\ J_o(kr)\ dk\ . \tag{55}$$

Let us turn our attention now to the first term of equation (54). We find after some differentiation and algebraic manipulation that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Delta\varphi_{gh}\right) = \int_o^\infty ks_k(z)\left[\frac{J_o'(kr)}{r} + k\ J_o''\ (kr)\right] dk\ . \tag{56}$$

Addition of equations (55) and (56) then leads to the desired equation:

$$0 = \nabla^2\Delta\varphi_{gh} = \int_o^\infty \left\{s_k''(z)\ J_o(kr) + ks_k(z)\left[\frac{J_o'(kr)}{r} + kJ_o''\ (kr)\right]\right\} dk. \tag{57}$$

We can make use of the well-known relations between the zeroth and first order Bessel functions J_o and J_1 and their derivatives

$$J_o'(kr) = - J_1(kr)$$
$$J_o''(kr) = - J_o(kr) + \frac{1}{kr} J_1(kr) \tag{58}$$

to simplify equation (57):

$$0 = \int_0^\infty \left[s_k''(z) \; J_o(kr) - k^2 \; s_k(z) \; J_o(kr) \right] dk \; . \tag{59}$$

Since the J_o functions are orthogonal, we can write that

$$s_k''(z) - k^2 \; s_k(z) = 0 \; . \tag{60}$$

The solution to this equation is just

$$s_k(z) = w_{1ko} \; e^{-k|z|} + w_{2ko} \; e^{+k|z|} \; . \tag{61}$$

The w_{2ko} must vanish if the boundary condition $\lim_{z \to \pm\infty} \Delta\varphi_g = 0$ is to be satisfied.

We have now obtained the functional form of the coefficients $s_k(z)$ in terms of which $\Delta\varphi_{gh}$ has been expanded. Substitution of equation (61) into equation (53) then easily gives

$$\Delta\varphi_{gh}(r,z) = \int_0^\infty w_{1ko} \; J_o(kr) \; e^{-k|z|} \; dk \; . \tag{62}$$

It should be noted that equation (62) satisfies the boundary condition that the perturbed gas potential be symmetric with respect to the plane

$z = 0$. Addition of equation (62), the general solution of the Laplace equation, to equation (51) is then the general solution of the Poisson equation for the perturbed gas potential, satisfying, of course, the gas hydrostatic equilibrium equation:

$$\Delta\varphi_g(r,z) = \int_0^\infty J_o(kr)\left[A(k)\ X_k(z) + w_{1ko}\ e^{-k|z|}\right] dk\ . \tag{63}$$

For the sake of completeness an expression for the perturbed gas density satisfying the Poisson and hydrostatic equilibrium conditions can be obtained easily from equations (38) and (50):

$$\Delta\rho_g(r,z) = \rho_{eg}(z) \int_0^\infty A(k)\ X_k(z)\ J_o(kr)\ dk \tag{64}$$

where again $J_o(kr)$ is the solution to equation (46) and $X_k(z)$ is the solution to equation (48).

Let us now attempt to derive $\Delta\varphi_g(r,z)$ by yet another approach, beginning with the solution to the Poisson equation for the perturbed gas potential expressed in cylindrical coordinates:

$$\Delta\varphi_g(r_o,\theta_o,z_o) = G\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^\infty \frac{1}{R}\ \Delta\rho_g\ (r,\theta,z)\ r dr d\theta dz\ . \tag{65}$$

(r_o,θ_o,z_o) and (r,θ,z) are any two points, the distance between them being R. It can be shown that

$$\frac{1}{R} = \sum_{m=o}^{\infty} \epsilon_m \cos[m(\theta-\theta_o)] \int_o^{\infty} J_m(kr)\ J_m(kr_o)\ e^{-k|z-z_o|}\ dk\ . \tag{66}$$

Equations (65) and (66) are readily combined to give

$$\Delta\varphi_g(r_o, \theta_o, z_o) = \tag{67}$$

$$G\int_o^{\infty} rdr \int_{-\infty}^{\infty} dz \sum_{m=o}^{\infty} \epsilon_m \int^{\infty} J_m(kr) J_m(kr_o)\,\Delta\rho_g(r,z) e^{-k|z-z_o|}\, dk \int_o^{2\pi} \cos[m(\theta-\theta_o)] d\theta.$$

Noting that

$$\int_o^{2\pi} \cos[m(\theta-\theta_o)]\ d\theta = 2\pi\delta_{mo} \tag{68}$$

where δ_{mo} is the Kronecker delta symbol, equation (67) simplifies to

$$\Delta\varphi_g(r_o, z_o) = 2\pi G \int_o^{\infty} rdr \int_{-\infty}^{\infty} dz\ \ \Delta\rho_g(r,z) \int_o^{\infty} dk J_o(kr)\ J_o(kr_o)\ e^{-k|z-z_o|}. \tag{69}$$

The perturbed gas density $\Delta\rho_g(r,z)$ may be expanded in terms of the zero order Bessel function $J_o(k'r)$ and the appropriate expansion amplitudes $T_{k'}(z)$:

$$\Delta\rho_g(r,z) = \int_o^{\infty} T_{k'}(z)\ J_o(k'r)\ dk' \tag{70}$$

in which case equation (69) becomes

$$\Delta\varphi_g(r_o, z_o) = 2\pi G \int_{-\infty}^{\infty} dz \int_o^{\infty} dk \int_o^{\infty} dk' J_o(kr_o) e^{-k|z-z_o|}\ T_{k'}(z) \int_o^{\infty} rdr J_o(kr) J_o(k'r). \tag{71}$$

It is noted that the integral over r is just

$$\int_0^\infty r dr \; J_o(kr) \; J_o(k'r) = \frac{1}{k} \, \delta(k'-k) \; . \tag{72}$$

Equation (71) then easily simplifies to

$$\Delta\varphi_g(r_o,z_o) = 2\pi G \int_{-\infty}^{\infty} dz \int_0^\infty \frac{dk}{k} \, J_o(kr_o) \; e^{-k|z-z_o|} \, T_k(z) \; . \tag{73}$$

We proceed by noting that two expressions for $\Delta\rho_g(r,z)$ have so far been obtained, and represented by equations (64) and (70). If they are equated we find that

$$T_k(z) = A(k) \; \rho_{eg}(z) \; X_k(z) \; , \tag{74}$$

which upon substitution into equation (73) results in

$$\Delta\varphi_g(r_o,z_o) = 2\pi G \int_0^\infty dk A(k) \frac{J_o(kr_o)}{k} \int_{-\infty}^{\infty} dz e^{-k|z-z_o|} \; \rho_{eg}(z) \; X_k(z) \; . \tag{75}$$

Equation (48) can be recalled to supply an expression for the quantity $\rho_{eg}(z) \; X_k(z)$ appearing in equation (75):

$$\rho_{eg}(z) \; X_k(z) = 2\rho_{go} H_g^2 \left[X_k(z) \; k^2 - X_k''(z) \right] . \tag{76}$$

It is this condition that, when substituted into equation (75), constrains $\Delta\varphi_g(r_o,z_o)$ to satisfy implicitly hydrostatic equilibrium:

$$\Delta\varphi_g(r_o,z_o) = 4\pi G H_g^2 \rho_{go}\int_0^\infty dk A(k)\, \frac{J_o(kr_o)}{k}\int_{-\infty}^\infty dz e^{-k|z-z_o|}\left[k^2 X_k(z) - X_k''(z)\right]. \tag{77}$$

To proceed further, it becomes necessary to evaluate the integral over z. This is most easily done by splitting the range into two separate parts:

$$\int_{-\infty}^\infty e^{-k|z-z_o|}\left[k^2 X_k(z) - X_k''(z)\right]dz = -\int_{-\infty}^{z_o} X_k''(z)\, e^{+k(z-z_o)}dz$$

$$+k^2\int_{-\infty}^{z_o} X_k(z)\, e^{+k(z-z_o)}dz - \int_{z_o}^\infty X_k''\, e^{-k(z-z_o)}dz + k^2\int_{z_o}^\infty X_k(z)\, e^{-k(z-z_o)}dz. \tag{78}$$

For notational simplicity, the four integrals appearing on the right-hand side of equation (78) are labeled A, B, C, and D, respectively. Integral A will be derived in detail via the method of integration by parts. Only the result of a similar integration will be stated for integral C, since the same procedure is used in its derivation. Integrals B and D need not be manipulated further. We let

$$u = e^{+k(z-z_o)} \qquad du = k\, e^{+k(z-z_o)}$$

$$dv = X_k''(z)\, dz \qquad v = X_k'(z) \tag{79}$$

so that integral A becomes

$$A = -X_k'(z)\, e^{+k(z-z_o)}\Big|_{-\infty}^{z_o} + k\int_{-\infty}^{z_o} X_k'(z)\, e^{+k(z-z_o)}dz\ . \tag{80}$$

The last integral in equation (80) can similarly be evaluated by the method of integration by parts, letting

$$u = e^{+k(z-z_o)} \qquad du = k\, e^{+k(z-z_o)}$$
$$dv = X_k'(z)\, dz \qquad v = X_k(z) \tag{81}$$

and thus

$$\int_{-\infty}^{z_o} X_k'(z)\, e^{+k(z-z_o)} dz = X_k(z)\, e^{+k(z-z_o)}\Big|_{-\infty}^{z_o} - k\int_{-\infty}^{z_o} X_k(z) e^{+k(z-z_o)} dz. \tag{82}$$

Combining equations (80) and (82) then leads to

$$A = -X_k'(z)\, e^{+k(z-z_o)}\Big|_{-\infty}^{z_o} + kX_k(z) e^{+k(z-z_o)}\Big|_{-\infty}^{z_o} - k^2\int_{-\infty}^{z_o} X_k(z) e^{+k(z-z_o)} dz. \tag{83}$$

After a similar set of operations, integral C becomes

$$C = -X_k'(z) e^{-k(z-z_o)}\Big|_{z_o}^{\infty} - kX_k(z) e^{-k(z-z_o)}\Big|_{z_o}^{\infty} - k^2\int_{z_o}^{\infty} X_k(z) e^{-k(z-z_o)} dz. \tag{84}$$

Integral B

$$B = k^2 \int_{-\infty}^{z_o} X_k(z)\, e^{+k(z-z_o)} dz \tag{85}$$

and integral D

$$D = k^2 \int_{z_o}^{\infty} X_k(z)\, e^{-k(z-z_o)}\, dz \tag{86}$$

need not be simplified further. Equations (83) through (86) are added, with the result that

$$\frac{1}{k}\int_{-\infty}^{\infty} e^{-k(z-z_o)}\left[X_k(z)k^2 - X_k''(z)\right]dz = A+B+C+D = -X_k'(z)\ e^{+k(z-z_o)}\Big|_{-\infty}^{z_o}$$

$$+kX_k(z)\ e^{+k(z-z_o)}\Big|_{-\infty}^{z_o} - X_k'(z)\ e^{-k(z-z_o)}\Big|_{z_o}^{\infty} - kX_k(z)e^{-k(z-z_o)}\Big|_{z_o}^{\infty}. \quad (87)$$

In order to evaluate these four terms in the limit that $z \to +\infty$ or $-\infty$, we recall the differential equation for $X_k(z)$:

$$X_k''(z) + \left[\frac{1}{2H_g^2}\ \rho_{eg}(z)/\rho_{go} - k^2\right] X_k(z) = 0\ . \quad (48)$$

In the limit that $z \to \pm\infty$, equation (48) becomes

$$X_k''(z) = k^2 X_k(z) \quad (88)$$

since $\lim_{z\to\pm\infty}\left[\frac{1}{2H_g^2}\ \rho_{eg}(z)/\rho_{go}\right] = 0$. The solution to equation (88) can easily be written

$$\begin{aligned} z > 0:\ \lim_{z\to+\infty} X_k(z) &= \alpha_k^{(+)}\ e^{-kz} + \beta_k^{(+)}\ e^{+kz} \\ z < 0:\ \lim_{z\to-\infty} X_k(z) &= \alpha_k^{(-)}\ e^{-kz} + \beta_k^{(-)}\ e^{+kz}\ . \end{aligned} \quad (89)$$

Referring back to equation (64) for the perturbed gas density

$$\Delta\rho_g(r,z)/\rho_{eg}(z) = \int_0^{\infty} A(k)\ X_k(z)\ J_o(kr)\ dk \quad (64)$$

it is clear that unless $\lim_{z\to\pm\infty} X_k(z) = 0$, the ratio $\Delta\rho_g(r,z)/\rho_{eg}(z)$ will not $\to 0$ as $|z| \to \infty$. This condition requires therefore that $\beta_k^{(+)} = \alpha_k^{(-)} = 0$. Symmetry of $\Delta\rho_g(r,z)$ about the plane $z = 0$ results in the further restriction that $\alpha_k^{(+)} = \beta_k^{(-)}$. Equations (89) therefore become

$$\begin{aligned} z > 0:&\quad \lim_{z\to+\infty} X_k(z) = \gamma e^{-kz} \\ z < 0:&\quad \lim_{z\to-\infty} X_k(z) = \gamma e^{+kz} \end{aligned} \tag{90}$$

where $\gamma = \alpha_k^{(+)} = \beta_k^{(-)}$. In addition, it can easily be verified that

$$\begin{aligned} z > 0:&\quad \lim_{z\to+\infty} X_k'(z) = -\gamma k e^{-kz} \\ z < 0:&\quad \lim_{z\to-\infty} X_k'(z) = \gamma k e^{+kz} \,. \end{aligned} \tag{91}$$

We can now evaluate each term in equation (87) at its appropriate infinite limit. For example, consider the limit of the first term in equation (87):

$$\lim_{z\to-\infty} \left[- X_k'(z)\; e^{+k(z-z_o)}\right] = \lim_{z\to-\infty} \left[-\gamma k e^{-kz_o} e^{+2kz}\right] = 0. \tag{92}$$

Similarly, for the other three limits appearing in equation (87) we have that

$$\lim_{z\to-\infty}\left[kX_k(z)\ e^{+k(z-z_o)}\right] = 0$$

$$\lim_{z\to+\infty}\left[-\ X_k'(z)\ e^{-k(z-z_o)}\right] = 0 \qquad (93)$$

and

$$\lim_{z\to+\infty}\left[-kX_k(z)\ e^{-k(z-z_o)}\right] = 0\ .$$

We are now able to evaluate the integral appearing in equation (77), the expression for the perturbed gas potential, as

$$\int_{-\infty}^{\infty} e^{-k|z-z_o|}\left[k^2X_k(z) - X_k''(z)\right]dz =$$

$$-X_k'(z_o) + kX_k(z_o) + X_k'(z_o) + kX_k(z_o) = 2kX_k(z_o)\ . \qquad (94)$$

Equation (77) therefore simplifies to

$$\Delta\varphi_{g2}(r,z) = 8\pi GH_g^2\rho_{go}\int_0^{\infty} A(k)\ X_k(z)\ J_o(rk)\ dk \qquad (95)$$

where the arbitrary coordinates (r_o,z_o) have been replaced by (r,z). This is, however, just the particular solution to the Poisson equation for which we have derived another expression above in the way of equation (51) (with the constant = 0):

$$\Delta\varphi_{g1}(r,z) = \langle v_{tz}^2\rangle \int_0^{\infty} A(k)\ X_k(z)\ J_o(kr)\ dk\ . \qquad (51)$$

Subtracting equation (95) from (51), we find that

$$\Delta\varphi_{g1}(r,z) - \Delta\varphi_{g2}(r,z) = [Q^2 - 8\pi G H_g^2 \rho_{ego}] \int_0^\infty A(k) X_k(z) J_o(kr)dk. \qquad (96)$$

H_g has previously been defined above by equation (43)

$$H_g^2 = Q^2/8\pi G\rho_{ego}$$

and therefore the bracketed term in equation (96) vanishes, as of course it must, since both $\Delta\varphi_{g1}$ and $\Delta\varphi_{g2}$ satisfy the Poisson and gas hydrostatic equilibrium equations. Indeed, the preceding derivation gives us confidence that the original expression for the general solution to the Poisson equation, equation (63)

$$\Delta\varphi_g(r,z) = \int_0^\infty J_o(kr) \left[A(k)\ X_k(z) + w_{1ko} e^{-k|z|}\right] dk \qquad (63)$$

is correct. $X_k(z)$ is of course obtained from the solution to equation (48) subject to the boundary conditions imposed by equations (49) a and b.

It is appropriate at this time to consider more closely the nature of equations (48) and (49) a and b

$$X''(z) + \left[\frac{1}{2H_g^2}\ \rho_{eg}(z)/\rho_{ego} - k^2\right] X_k(z) = 0 \qquad (48)$$

$$X_k'(o) = 0 \qquad \text{a}$$

$$(49)$$

$$\lim_{z\to\pm\infty} X_k(z) = 0 \qquad \text{b}$$

as regards the nature of the marginally unstable state. Equations (48) and (49) define what is known as the Sturm-Liouville problem from the theory of differential equations. The Sturm-Liouville problem is basically that of determining (a) the relationship of the separation constant k to the function $X_k(z)$ and (b) the influence on k of the boundary conditions imposed on $X_k(z)$. Stated more simply, we are searching for those values of the separation constant k that lead to functions $X_k(z)$ which satisfy both equation (48) and the boundary conditions imposed by equations (49).

The similarity of our problem to that of a particle of total energy E under the influence of a one-dimensional potential well of depth V(z) is obvious once we realize that the spatial component of the particle wave function $\psi(z)$ satisfies the one-dimensional time-independent Schrodinger equation

$$\frac{d^2\psi}{dz^2} + \frac{2m}{\hbar^2}[E - V(z)]\,\psi = 0 \tag{97}$$

subject to the restriction that $\int_{-\infty}^{\infty} |\psi^2|\, dz$ remain finite, i.e. that

$$\lim_{z\to\pm\infty} \psi(z) = 0 \ . \tag{98}$$

Equation (49) b is analogous to the square integrability condition imposed by equation (98), while equation (48), which defines $X_k(z)$, the z part of $\epsilon(r,z) = \int_0^{\infty} A(k)\ X_k(z)\ J_0(kr)\ dk = \Delta\rho_g(r,z)/\rho_{eg}(z)$, is analogous to equation (97) which defines the particle wave function. Rewriting equation (48) in a slightly different way

$$X_k''(z) + \frac{2m}{\hbar^2}\left[\left(\frac{-\hbar^2}{2m}k^2\right) - \left(\frac{-\hbar^2}{2m}\frac{1}{2H_g^2}\rho_{eg}(z)/\rho_{ego}\right)\right]X_k(z) = 0 \qquad (48')$$

brings out the analogy even more closely, with the equivalent potential energy and particle energy given by

$$V(z) = \frac{-\hbar^2}{2m}\frac{1}{2H_g^2}\rho_{eg}(z)/\rho_{ego} \qquad \text{a}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (99)

$$E = \frac{-\hbar^2}{2m}k^2 \quad . \qquad \text{b}$$

Since (a) $\frac{1}{2H_g^2}\rho_{eg}(z)/\rho_{ego} > 0$ for all z and attains a maximum value at $z = 0$ and (b) $k^2 > 0$, we conclude that formally the problem of solving for those values of the separation constant k which yield $X_k(z)$ satisfying equations (48) and the boundary conditions (49) a and b is equivalent to finding the 'allowed' energies of the particle in a potential well of depth $\frac{\hbar^2}{2m}\frac{1}{2H_g^2}$ and form $V(z) = \frac{-\hbar^2}{2m}\frac{1}{2H_g^2}\rho_{eg}(z)/\rho_{ego}$. It can easily be shown that for an isothermal plane-parallel layer of gas, $\rho_{eg}(z)/\rho_{ego} = C_1 \operatorname{sech}^2 z/C_2$, where C_1 and C_2 are constants. However, for the problem at hand, specifically an isothermal plane layer of gas immersed in an isothermal plane layer of stars, it was shown above that no simple analytical expression is obtained for the gas distribution $\rho_{eg}(z)/\rho_{ego}$. Instead, equation (11)

$$\frac{d^2 \ln}{dz^2} \rho_{eg}/\rho_{ego} = \frac{-4\pi G}{Q^2}\left[\rho_{eg} + \rho_{*o}(\rho_{eg}/\rho_{ego})^{Q^2/\langle v_{*z}^2\rangle}\right] \qquad (11)$$

must be used in conjunction with equations (48) and (49).

It will be instructive if we briefly review (Powell and Crasemann 1961) the solution to the time-independent Schrodinger equation for an electron of negative total energy subject to a one-dimensional rectangular potential well defined by

$$V(z) = \begin{cases} 0 & (z < 0) \\ -V_o & (0 < z < a) \\ 0 & (a < z) \end{cases} \quad . \qquad (100)$$

The unnormalized wave function $\psi(z)$ can be shown to be

$$\psi(z) = \begin{cases} e^{kz} & (z < 0) \\ Ae^{i\sigma z} + Be^{-i\sigma z} & (0 < z < a) \\ Ce^{-kz} & (a < z) \end{cases} \quad . \qquad (101)$$

The continuity conditions at $z = 0$ and $z = a$ are found to be

$$\begin{aligned} 1 &= A + B \\ k &= i\sigma(A - B) \\ Ce^{-ka} &= Ae^{i\sigma a} + Be^{-i\sigma a} \\ -kCe^{-ka} &= i\sigma(Ae^{i\sigma a} - Be^{-i\sigma a}) \quad . \end{aligned} \qquad (102)$$

Since the three constants A, B, and C must satisfy four equations, it is clear that only certain discrete values of the electron energy E will be allowed. Another way of viewing this is that only certain discrete values of E give wave functions in the range $0 < z < a$ which fit smoothly with decaying exponentials over the range $z < 0$ and $a < z$. We proceed by eliminating A, B, and C from equations (102), with the result that

$$2 \cot(\sigma a) = \frac{\sigma}{k} - \frac{k}{\sigma} \, . \tag{103}$$

Equation (103) can be solved by introducing the quantities

$$\gamma = \sqrt{\frac{2mV_o a^2}{\hbar^2}}$$

and

$$\alpha = \frac{\sigma a}{\gamma} = \sqrt{1 + E/V_o} \tag{104}$$

in which case we have that

$$\gamma a = (n-1)\pi + 2 \cos^{-1}\alpha \quad (n = 1,2,\ldots) \, . \tag{105}$$

Powell and Crasemann go on to show that the number of allowed discrete states is the greatest integer contained in the quantity $(\gamma/\pi + 1)$, which increases as V_o and a increase. If we approximate the distribution $\rho_{eg}(z)/\rho_{ego}$ from equation (11) by a rectangular distribution with

$$V_o = \frac{1}{2H_g^2} \frac{\hbar^2}{2m}$$

and

$$a = 2H_g \tag{106}$$

it is found that

$$\gamma = \sqrt{\frac{a^2}{2H_g^2}} = \sqrt{2}$$

in which case $(\gamma/\pi + 1) = 1.45$. This tells us to expect in the neighborhood of one or two allowed discrete values of the separation constant k to result from a solution of equations (11) and (48) subject to equations (49).

In analogy to the case of the rectangular potential well, k assumes only discrete values since an oscillatory function $X_k(z)$ in the interval $|-k^2| < |-\frac{1}{2H_g^2}\rho_{eg}(z)/\rho_{ego}|$ must be joined smoothly to decaying exponentials at two points defined by the condition $|-k^2| = |-\frac{1}{2H_g^2}\rho_{eg}(z)/\rho_{ego}|$. Clearly, only certain values of k will allow such a smooth superposition of functions. In addition, increasing the depth of the potential well by decreasing H_g^2 (and thus increasing ρ_{ego} or decreasing Q^2) will result in more eigenvalues k being found, and an increase in absolute magnitude of k for those eigenvalues already found to exist. ρ_{*o} and $\langle v_{*z}^2\rangle$ have no effect on the central depth of the potential well. They will, however, affect the width, in the sense that increasing ρ_{*o} and decreasing $\langle v_{*z}^2\rangle$ decreases the width which thereby decreases the number of eigenvalues and causes the remaining values of k to decrease in absolute magnitude. In summation then, the depth of the effective potential well is determined only by ρ_{ego} and Q^2, while the half-width or shape is determined by ρ_{ego}, Q^2, ρ_{*o}, and

$\langle v_{*z}^2 \rangle$. It is at this point then that the difference between our analysis and that of Ledoux becomes critical, for by assigning ρ_{*o} a non-zero value, we change the value of k. Since k will later be shown to be related to the radius of the perturbation in the plane z = 0, we should expect to find a different value for this radius than does Ledoux.

It seems appropriate now to digress slightly and discuss the numerical approach employed in finding the discrete values of the separation constant k. It has been noted previously that the solution to equation (48) in the limit $z \to +\infty$ is just

$$X_k(z) = \alpha_k^{(+)} e^{-kz} + \beta_k^{(+)} e^{+kz} . \tag{107}$$

We wish to find the restriction on k if $\beta_k^{(+)}$ is to vanish. This can most easily be accomplished by differentiating equation (107)

$$X_k'(z) = - \alpha_k^{(+)} k e^{-kz} + \beta_k^{(+)} k e^{+kz} \tag{108}$$

and eliminating $\alpha_k^{(+)}$ between equations (107) and (108), with the result that in the limit $z \to +\infty$,

$$X_k'(z) + kX_k(z) = 2k\beta_k^{(+)} e^{+kz} . \tag{109}$$

Since $\beta_k^{(+)}$ must vanish if the boundary condition $\lim_{z\to+\infty} X_k(z) = 0$ is satisfied, we have that

$$X_k'(z) + kX_k(z) = 0 . \tag{110}$$

To derive the eigenvalues k from the restriction dictated by equation (110), a sufficiently large value is considered for z, denoted by the symbol z_c, for which equation (110) holds with good accuracy. Typically, $z_c > 500$ pc. Next equation (48), the second order differential equation for $X_k(z)$, is solved numerically with the aid of equation (11), the equation for the gas distribution $\rho_{eg}(z)/\rho_{ego}$, allowing us to tabulate $X_k(z_c)$ and $X_k'(z_c)$. The quantity $P = X_k(z_c) + kX_k'(z_c)$ is then calculated at each of four equally spaced points covering the entire range over which k varies. The eigenvalue k lies in the interval over which P changes sign since we are seeking only those values of k for which P = 0. The eigenvalue or eigenvalues can be found to any desired accuracy by using this procedure. (A simple calculation shows that the answer converges most rapidly if the number of subintervals into which the previous interval has been divided is equal to three.)

Now that a numerical procedure has been developed to determine the eigenvalues k, we must determine how they relate to the size of the resulting 'perturbations'. It will be recalled that equation (47)

$$Y(r) = J_o(kr) \tag{47}$$

expresses the solution to the second order differential equation for $Y(r)$, where $\epsilon(r,z) = Y(r)\, X_k(z)$. From equation (47) we note that k must have units of reciprocal length, and that for a given value of k, the first zero in $Y(r)$ occurs when

$$r_1 = \alpha_1/k_1 \tag{111}$$

where α_1 is the first zero of the zero order Bessel function J_o and k_1 is the smallest eigenvalue found by the numerical methods enumerated above. More generally, if more than one eigenvalue k is found, we have that

$$r_n = \alpha_n/k_n \tag{112}$$

where α_n is correspondingly the nth zero of J_o. Focusing on equation (111) we observe that k_1 determines the radius of the perturbation in the galactic plane z = 0 since

$$\epsilon(r_1 = \alpha_1/k_1, z = 0) = X_k(0)J_o(\alpha_1) = \Delta\rho_g(\alpha_1/k_1, 0)/\rho_{eg}(0) = 0.$$

When equations (11) and (48) are solved simultaneously subject to the boundary conditions imposed by equations (49) a and b, just one eigenvalue k is found, consistent with the results found above for a rectangular potential well of equal depth and width $2H_g$. Of course, ρ_{ego}, ρ_{*o}, $(\langle v_{*z}^2\rangle)^{\frac{1}{2}}$, and Q must be specified. We have chosen $\rho_{ego} = 0.025\ M_\odot/pc^3$, $\rho_{*o} = 0.064 M_\odot/pc^3$, and $(\langle v_{*z}^2\rangle)^{\frac{1}{2}} = 18$ km/sec. Q is allowed to vary from 0 to 20 km/sec so that the eigenvalue k is determined as a function of Q. Equations (11), (48), and (111) enable us to find the perturbation radius r_1 as a function of Q. The results are plotted in Figure 4. The perturbation radius increases smoothly from 200 pc at Q = 1 km/sec to 1 kpc at 6 km/sec to 3 kpc at 18 km/sec. Over the range of interest to galactic astronomers, 6 km/sec $\leq Q \leq$ 12 km/sec, the radius varies between 1 and 2 kpc.

We can derive the isodensity contours of the marginally unstable state by recalling the perturbation equation for the gas density

$$\rho_g(r,z) = \rho_{eg}(z) + \Delta\rho_g(r,z) \ . \tag{32 a}$$

Since only one eigenvalue was found for our particular choice of ρ_{ego}, ρ_{*o}, and $\langle v_{*z}^2 \rangle$, the integral over k in the expression for the perturbed gas density (equation 64) reduces to just one term:

$$\Delta\rho_g(r,z) = A(k)\rho_{eg}(z)X_k(z)J_o(kr) \ . \tag{113}$$

Equations (32) a and (113) may be combined to give

$$\frac{\rho_g(r,z)}{\rho_{eg}(0)} = \frac{\rho_{eg}(z)}{\rho_{eg}(0)} \left[1 + A(k)X_k(z)J_o(kr)\right] \ . \tag{114}$$

At the center of the perturbation where $r = z = 0$, $X_k(z) = 1$, $J_o(kr) = 1$, and $\rho_{eg}(z)/\rho_{eg}(0) = 1$. Equation (114) evaluated at $r = z = 0$ therefore becomes

$$\frac{\rho_g(0,0)}{\rho_{eg}(0)} = 1 + A(k) \ . \tag{115}$$

Since the perturbation analysis is linear in nature, we must restrict the expansion amplitude $A(k)$ to lie in the range $0 \leq A(k) \leq 1.0$, i.e. $1.0 \leq \rho_g(0,0)/\rho_{eg}(0) \leq 2.0$. Making use of equation (11) for the quantity $\rho_{eg}(z)/\rho_{eg}(0)$, equation (48) for $X_k(z)$, and numerical tables for $J_0(kr)$, equation (114) may be used to construct an (r,z) grid of values for the quantity $\rho_g(r,z)/\rho_{eg}(0)$ once $A(k)$ is chosen, enabling us to construct contours of equal values of $\rho_g(r,z)/\rho_{eg}(0)$. Notice also that from equation (113), $\Delta\rho_g(r,z)$ vanishes on cylinders perpendicular to the plane $z = 0$ and of radius $r_1 = \alpha_1/k_1$. Choosing $A(k) = 0.5$, Figure 5 displays the three isodensity contours $\rho_g(r,z)/\rho_{eg}(0) = 1.3$, 0.7, and 0.3 constructed from equation (114). Choosing $A(k) = 1.0$, Figure 6 displays the three isodensity contours $\rho_g(r,z)/\rho_{eg}(0) = 1.7$, 1.0, and 0.3. The perturbations clearly are flattened objects.

We might ask what effect a change in ρ_{*o} has on the size and shape of the gas clouds. To answer this question we recall that increasing ρ_{*o} causes k to decrease. In addition, equation (111) tells us that $r_1 = \alpha_1/k_1$. r_1 therefore tends to increase as ρ_{*o} increases. Also r_1 tends to decrease as ρ_{*o} decreases. However, increasing ρ_{*o} decreases the half-thickness of the perturbation, causing it therefore to become flatter. By similar reasoning we find that increasing ρ_{ego} causes k to increase, and thus r_1 to decrease (analogous to the result of Jeans and Ledoux).

It will be instructive to compare the radius of our perturbations using $\rho_{ego} = 0.025\ M_{\odot}/pc^3$, $\rho_{*o} = 0.064\ M_{\odot}/pc^3$, and $(\langle v_{*z}^2\rangle)^{\frac{1}{2}} = 18$ km/sec with the Jeans' radius (which equals $\lambda_J/2$) and the Ledoux radius $(\lambda_L/2)$, each calculated with $\rho_{ego} = 0.025\ M_{\odot}/pc^3$. Q will be left as a free parameter, taking on the values 1, 5, 10, 15, and 20 km/sec.

The results are presented in Table 2, and are consistent with the discussion of the previous paragraph. The addition of a stellar component clearly increases the perturbation radius.

Before checking with the observations in an attempt to identify our 1-2 kpc gas perturbations with astronomical objects, we should stress that perturbations of the star layer were completely neglected. From a gravitational viewpoint, attention was focused only on perturbations of a second order quantity, the gaseous component of the galactic disk. It seems reasonable that gross galactic features such as spiral structure are related to perturbations in first order galactic quantities, specifically the stellar component. This seems all the more clear since the recent work of Toomre (1964) and of Lin and Shu (1964) on star-driven disturbances. We might, however, hope that our analysis would shed some light on the structure of the gaseous component within the spiral arms.

Continuing, equations (11) and (48) may be put into a dimensionless form, in which case they become

$$\frac{d^2}{d\beta^2} \ln\alpha = -\tfrac{1}{2}\left[\alpha + \gamma\alpha^{\delta}\right] \tag{116}$$

and

$$\frac{d^2}{d\beta^2} X_k = -\left[\alpha/2 - c_k^2\right] X_k \,, \tag{117}$$

where

$$k = C_k/H_g$$

and where α, β, γ, δ, and H_g have been defined above by equations (12).

The radius of the perturbation in the galactic plane (or equivalently the first node in J_o) can be written

$$r_1 = \left[2.4048/c_1\right] H_g \ . \tag{118}$$

But since

$$H_g = \left(Q^2/8\pi G\rho_{ego}\right)^{\frac{1}{2}},$$

r_1 depends on the ratio $Q/\rho_{ego}^{\frac{1}{2}}$, precisely the functional dependence found by Jeans and Ledoux. If Q is independent of distance from the galactic center beyond R_o, r_1 would be expected to increase as ρ_{go} decreases with increasing distance from the galactic center beyond R_o. Westerhout (1956) finds that

$$\frac{\rho_{ego}(R = 11 \text{ kpc})}{\rho_{ego}(R = 15 \text{ kpc})} \approx 4.7$$

from which it follows that

$$\frac{r_1(R = 15 \text{ kpc})}{r_1(R = 11 \text{ kpc})} \approx 2.2$$

due to a change in ρ_{ego} alone.

Finally, we note that the mass of an individual perturbation may be expressed by an integral of the density over a cylinder perpendicular to the galactic plane, of infinite height, and of radius $r_1 = \alpha_1/k_1$:

$$M = 2\pi \int_{-\infty}^{+\infty} \int_{-r_1}^{+r_1} \rho_g(r,z) r dr dz \ . \tag{119}$$

Equation (119) may be evaluated numerically with the aid of equation (114). Taking $Q = 10.0$ km/sec and values quoted previously for ρ_{ego}, ρ_{*o}, and $(\langle v_{*z}^2 \rangle)^{\frac{1}{2}}$, we find that $M = 5 \times 10^6\ M_\odot$.

Finally, we note in passing that McGee and Milton (1964) believe they have found that the principal elements of spiral arms are "clouds of enormous size" and mass $10^7\ M_\odot$. In addition they claim that "the size of the clouds apparantly increases markedly with distance from the galactic center greater than the solar distance (R_o)." These findings, if correct, do not contradict our results.

V. THE STABILITY OF A SELF-GRAVITATING MAGNETIC GAS DISK

In section IV we considered the stability of a non-magnetic, self-gravitating, isothermal gas disk in the presence of a static isothermal star disk. We wish now to discuss the stability of a magnetic, self-gravitating, isothermal gas disk of infinite conductivity immersed in a static isothermal star disk.

Parker (1966) has studied the stability of a gas disk (no stars), magnetic field $B_e(z)\hat{e}_y$, and cosmic-ray gas with respect to waves with motions in the $\vec{B}_e - \vec{g}$ plane. The gravitational acceleration $\vec{g}$ was assumed to be independent of z. A time-dependent linear perturbation analysis revealed that the interstellar gas-field system is subject to a Raleigh-Taylor type instability, driven by the magnetic field and cosmic-ray gas. The gas coagulates into clouds suspended in the magnetic field. The minimum e-folding time is found to be $\tau_{min} = 1.2 \times 10^7$ years. Parker neglected, however, the self-gravitation of the gas. We should point out that Parker included the relationship $\Delta p = \gamma p_o \Delta\rho/\rho_o$ with arbitrary γ in his discussion.

In this section we shall consider both motions in the $\vec{B}_e - \vec{g}$ plane and perpendicular to it, and will include the effect of self-gravitation of the gas. Only the isothermal case (γ=1) will be studied.

a) Motions in the $\vec{B}_e - \vec{g}$ Plane

The relationship between the vectors $\vec{g}$, $\vec{k}$, and $\vec{B}_e$ and the xyz coordinate axis is shown in Figure 7. The basic equations to be considered are the continuity, momentum, hydromagnetic, Poisson, and heat equation. However, the latter simplifies to

$$p_g = c^2 \rho_g$$

in the isothermal approximation (γ=1), where c is the isothermal sound speed of the gas. The equations may be written as follows:

$$\frac{\partial}{\partial t}\rho_g + \vec{v}\cdot\nabla\rho_g + \rho_g\nabla\cdot\vec{v} = 0 \tag{120}$$

$$\rho_g\frac{d}{dt}\vec{v} + \nabla P_g - \frac{1}{4\pi}(\nabla\times\vec{B})\times\vec{B} + \rho_g\nabla\varphi = 0 \tag{121}$$

$$\frac{\partial}{\partial t}\vec{B} - \nabla\times(\vec{v}\times\vec{B}) = 0 \tag{122}$$

and

$$\nabla^2\varphi_g - 4\pi G\rho_g = 0\ . \tag{123}$$

Introducing the perturbations

$$\rho_g = \rho_{eg} + \Delta\rho_g \qquad \text{a}$$

$$P_g = P_{eg} + \Delta P_g = c^2\rho_g \qquad \text{b}$$

$$\varphi_g = \varphi_{eg} + \Delta\varphi_g \qquad (124) \qquad \text{c}$$

$$\varphi = \varphi_e + \Delta\varphi_g = \varphi_{eg} + \varphi_{e*} + \Delta\varphi_g \qquad \text{d}$$

$$\vec{B} = \vec{B}_e + \Delta\vec{B} \qquad \text{e}$$

and retaining terms only up to first order in the perturbed quantities, equations (120) - (123) become

$$\frac{\partial}{\partial t}\Delta\rho_g + \rho_{eg}\frac{\partial}{\partial y}v_y + \left(\rho_{eg}\frac{\partial}{\partial z} + \frac{d}{dz}\rho_{eg}\right)v_z = 0 \tag{125}$$

$$c^2\frac{\partial}{\partial x}\Delta\rho_g + \frac{B_e}{4\pi}\frac{\partial}{\partial x}\Delta B_y - \frac{B_e}{4\pi}\frac{\partial}{\partial y}\Delta B_x + \rho_{eg}\frac{\partial}{\partial x}\Delta\varphi_g = 0 \tag{126}$$

$$c^2\frac{\partial}{\partial y}\Delta\rho_g + \rho_{eg}\frac{\partial}{\partial t}v_y - \frac{1}{4\pi}\Delta B_z\frac{d}{dz}B_e + \rho_{eg}\frac{\partial}{\partial y}\Delta\varphi_g = 0 \tag{127}$$

$$c^2\frac{\partial}{\partial z}\Delta\rho_g + \rho_{eg}\frac{\partial}{\partial t}v_z + \Delta\rho_g\frac{d}{dz}\varphi_e + \frac{B_e}{4\pi}\frac{\partial}{\partial z}\Delta B_y + \frac{1}{4\pi}\Delta B_y\frac{d}{dz}B_e - \frac{B_e}{4\pi}\frac{\partial}{\partial y}\Delta B_z$$
$$+\rho_{eg}\frac{\partial}{\partial z}\Delta\varphi_g = 0 \tag{128}$$

$$\frac{\partial}{\partial t}\Delta B_x - B_e\frac{\partial}{\partial y}v_x = 0 \tag{129}$$

$$\frac{\partial}{\partial t}\Delta B_y + (B_e\frac{\partial}{\partial z} + \frac{d}{dz}B_e)v_z = 0 \tag{130}$$

$$\frac{\partial}{\partial t}\Delta B_z - B_e\frac{\partial}{\partial y}v_z = 0 \tag{131}$$

$$4\pi G\Delta\rho_g - \nabla^2\Delta\varphi_g = 0\ . \tag{132}$$

Equations (126), (127), and (128) are, respectively, the x, y, and z components of the momentum equation (121); equations (129), (130), and (131) are, respectively, the x, y, and z components of the hydromagnetic equation (122). Since we restrict the velocity to lie in the y - z plane, $v_x = 0$ and from equation (129) it follows that $\Delta B_x = 0$.

The coefficients of the system of equations (125) - (132) are all independent of t, x, and y, allowing us to Fourier analyze in these variables ($\partial/\partial t \to n$, $\partial/\partial x \to ik_x$, $\partial/\partial y \to ik_y$). But since $\vec{k} = k_y\hat{e}_y + k_z\hat{e}_z$, $k_x = 0$, and equations (125) - (132) simplify to

$$n\Delta\rho_g + \rho_{eg}ik_yv_y + (\rho_{eg}\frac{\partial}{\partial z} + \frac{d}{dz}\rho_{eg})v_z = 0 \tag{133}$$

$$c^2 ik_y \Delta\rho_g + n\rho_{eg} v_y - \frac{1}{4\pi}\Delta B_z \frac{d}{dz} B_e + \rho_{eg} ik_y \Delta\varphi_g = 0 \tag{134}$$

$$c^2 \frac{\partial}{\partial z}\Delta\rho_g + n\rho_{eg} v_z + \Delta\rho_g \frac{d}{dz}\varphi_e + \frac{B_e}{4\pi}\frac{\partial}{\partial z}\Delta B_y + \frac{1}{4\pi}\Delta B_y \frac{d}{dz} B_e - \frac{B_e}{4\pi} ik_y \Delta B_z$$

$$+ \rho_{eg}\frac{\partial}{\partial z}\Delta\varphi_g = 0 \tag{135}$$

$$n\Delta B_y + \left(B_e \frac{\partial}{\partial z} + \frac{d}{dz} B_e\right) v_z = 0 \tag{136}$$

$$n\Delta B_z - B_e ik_y v_z = 0 \tag{137}$$

$$4\pi G\Delta\rho_g - (\partial^2/\partial z^2 - k_y^2)\Delta\varphi_g = 0\ . \tag{138}$$

If we write that

$$\frac{d}{dz}\rho_{eg} = f(z)\rho_{eg} \tag{139}$$

then it follows that

$$\frac{d}{dz} B_e = \tfrac{1}{2} f B_e \tag{140}$$

since B_e^2/ρ_{eg} is independent of z. Equation (139) is consistent with the presence of a stellar component. Proceeding,

$$\frac{d}{dz}\varphi_e = -g_e = \frac{-1}{\rho_{eg}}\frac{d}{dz}(P_{eg} + B_e^2/8\pi) = \frac{-1}{\rho_{eg}}(c^2 + a^2/2)\frac{d}{dz}\rho_{eg}$$

$$= \frac{-c^2}{\rho_{eg}}(1 + \alpha)\frac{d}{dz}\rho_{eg} = -c^2 f(1 + \alpha) \tag{141}$$

where

$$\alpha = \tfrac{1}{2}a^2/c^2 = B_e^2/8\pi c^2 \rho_{eg}, \tag{142}$$

$$\frac{1}{B_e}\frac{\partial}{\partial z}\Delta B_y = \frac{\partial}{\partial z}(\Delta B_y/B_e) - \Delta B_y \frac{d}{dz}\frac{1}{B_e} = (\partial/\partial z + f/2)\frac{\Delta B_y}{B_e}, \tag{143}$$

and

$$\frac{1}{\rho_{eg}}\frac{\partial}{\partial z}\Delta\rho_g = \frac{\partial}{\partial z}(\Delta\rho_g/\rho_{eg}) - \Delta\rho_g \frac{\partial}{\partial z}\frac{1}{\rho_{eg}} = (\partial/\partial z + f)\frac{\Delta\rho_g}{\rho_{eg}} . \tag{144}$$

With the aid of equations (139) - (144) the system (133) - (138) becomes

$$\frac{n\Delta\rho_g}{\rho_{eg}} + ik_y v_y + (\partial/\partial z + f)v_z = 0 \tag{145}$$

$$ik_y\frac{\Delta\rho_g}{\rho_{eg}} + \frac{n}{c^2}v_y - \frac{f\alpha\Delta B_z}{B_e} + \frac{ik_y}{c^2}\Delta\varphi_g = 0 \tag{146}$$

$$(\partial/\partial z - f\alpha)\frac{\Delta\rho_g}{\rho_{eg}} + \frac{n}{c^2}v_z + 2\alpha(\partial/\partial z + \tfrac{1}{2}f)\frac{\Delta B_y}{B_e} + \frac{f\alpha\Delta B_y}{B_e} - 2\alpha ik_y\frac{\Delta B_z}{B_e}$$

$$+ \frac{\partial}{\partial z}\frac{\Delta\varphi_g}{c^2} = 0 \tag{147}$$

$$(\partial/\partial z + \tfrac{1}{2}f)v_z + \frac{n\Delta B_y}{B_e} = 0 \tag{148}$$

$$ik_y v_z - \frac{n\Delta B_z}{B_e} = 0 \tag{149}$$

$$\frac{4\pi G}{c^2}\frac{\Delta\rho_g}{\rho_{eg}} - (\partial^2/\partial z^2 - k_y^2)\frac{\Delta\varphi_g}{c^2} = 0 \ . \tag{150}$$

The usual way to proceed is to Fourier analyze in the remaining variable z and to set the determinant of the coefficients of the system (146) - (151) equal to zero so as to avoid the trivial solution for $\Delta\rho_g$, v_y, v_z, ΔB_y, ΔB_z, and $\Delta\varphi_g$. A dispersion relation between n and k_y is then obtained, each solution called a mode. However, not all of the coefficients in this system are independent of z, rendering Fourier analysis in z a useless exercise, for the dispersion relation would contain unknown integrals over $\Delta\varphi_g$ and other of the variables. We will proceed by attempting to derive from the system (145) - (150) a single differential equation in z for $\Delta\varphi_g$. With the appropriate boundary conditions, such an equation would implicitly contain the desired dispersion relation.

If we make the substitutions

$$\epsilon = \Delta\rho_g/\rho_{eg} \qquad \text{a}$$

$$\delta = \Delta B_y/B_e \qquad \text{b}$$

$$\gamma = \Delta B_z/B_e \qquad \text{c} \tag{151}$$

$$\psi = \Delta\varphi_g/c^2 \qquad \text{d}$$

and, for purposes of simplicity, set $n = 0$ which corresponds to the case of marginal instability, equations (145) - (150) simplify to

$$ik_y v_y + (\partial/\partial z + f)v_z = 0 \tag{152}$$

$$ik_y \epsilon - f\alpha\gamma + ik_y \psi = 0 \tag{153}$$

$$(\partial/\partial z - f\alpha)\epsilon + 2\alpha(\partial/\partial z + f)\delta - 2ik_y\alpha\gamma + \frac{\partial}{\partial z}\psi = 0 \tag{154}$$

$$(\partial/\partial z + \tfrac{1}{2}f)v_z = 0 \tag{155}$$

$$ik_y v_z = 0 \tag{156}$$

$$\frac{4\pi G}{c^2}\epsilon - (\partial^2/\partial z^2 - k_y^2)\psi = 0\ . \tag{157}$$

Equations (152) and (156) imply that $v_y = v_z = 0$, not a surprising result since all velocities must vanish in the marginally unstable state.

It appears at first glance that we have three equations (153), (154), and (157) in the four unknowns ϵ, δ, γ, and ψ. δ and γ are not, however, independent of one another. In fact, from equations (136) and (137) we have that

$$\delta = \frac{-f}{ik_y}(\partial/\partial z + 1)\gamma \tag{158}$$

and equations (153), (154), and (157) thus become

$$ik_y \epsilon - f\alpha\gamma + ik_y \psi = 0 \tag{159}$$

$$(\partial/\partial z - f\alpha)\epsilon - \frac{2f\alpha}{ik_y}\left[\partial^2\gamma/\partial z^2 + (1 + f)\partial\gamma/\partial z + (f - k_y^2/f)\gamma\right] + \frac{\partial\psi}{\partial z} = 0 \tag{160}$$

$$\frac{4\pi G\epsilon}{c^2} - (\partial^2/\partial z^2 - k_y^2)\psi = 0 \ . \tag{161}$$

We have been unsuccessful in trying to eliminate ϵ and γ in an attempt to obtain a single differential equation for ψ, and so will terminate our discussion of the marginally unstable mode with motions in the $\vec{B}_e - \vec{g}$ plane.

b) Motions in a Plane Perpendicular to the $\vec{B}_e - \vec{g}$ Plane

The case $\vec{k} \perp \vec{B}_e$ for an initially uniform non-self-gravitating gas is particularly interesting. The longitudinal mode with $v_x = 0$, v_y finite (i.e. $\vec{v} \perp \vec{B}_e$) propagates at the phase velocity $v_p = (B_o^2/4\pi\rho_o + c^2)^{\frac{1}{2}}$ and is appropriately called the magnetosonic mode. The magnetic restoring forces are not due to curvature of the field lines as is true of the Alfvén mode, but to pressure gradients. In addition, the instability criterion (n real and positive) is $k < (c^2/c^2+a^2)^{\frac{1}{2}} k_J$ where a is the Alfvén velocity and where k_J is the Jeans' wave number defined by

$$k_J = (4\pi G\rho_o)^{\frac{1}{2}}/c \ . \tag{162}$$

Expressed in terms of length, $\lambda > \left[(c^2 + a^2)/c^2\right]^{\frac{1}{2}} \lambda_J$ where λ_J is the Jeans' length quoted above. Since the bracketed quantity is greater than one, the presence of a magnetic field renders gravitational instability more difficult.

With these remarks in mind, it seems appropriate to redefine the problem at hand. We seek to determine how a simple one-dimensional magnetic field (along x) affects the stability criteria of waves propagating perpendicular to $\vec{B}_e$ (along y) in an isothermal self-gravitating gas slab of infinite conductivity immersed in a static star slab. The geometrical relationship between $\vec{g}$, $\vec{k}$, and $\vec{B}_e$ is shown in Figure 8.

The basic equations may be written as follows:

$$\frac{\partial}{\partial t}\rho_g + \vec{v}\cdot\nabla\rho_g + \rho_g\nabla\cdot\vec{v} = 0 \tag{163}$$

$$\rho_g\frac{d}{dt}\vec{v} + \nabla P_g - \frac{1}{4\pi}\vec{B}\cdot\nabla\vec{B} + \frac{1}{8\pi}\nabla B^2 + \rho_g\nabla\varphi = 0 \tag{164}$$

$$\frac{\partial}{\partial t}\vec{B} - \nabla\times(\vec{v}\times\vec{B}) = 0 \tag{165}$$

and

$$\nabla^2\varphi_g - 4\pi G\rho_g = 0\ . \tag{166}$$

If we apply the perturbations

$$\rho_g = \rho_{eg} + \Delta\rho_g \qquad \text{a}$$

$$P_g = P_{eg} + \Delta P_g = c^2\rho_g \qquad (167) \qquad \text{b}$$

$$\varphi_g = \varphi_{eg} + \Delta\varphi_g \qquad \text{c}$$

$$\varphi = \varphi_e + \Delta\varphi_g = \varphi_{eg} + \varphi_{e*} + \Delta\varphi_g \qquad \text{d}$$

$$\vec{B} = \vec{B}_e + \Delta\vec{B} \qquad \text{e}$$

and retain terms only up to first order in the perturbed quantities, equations (163) - (166) become

$$\frac{\partial}{\partial t}\Delta\rho_g + \vec{v}\cdot\nabla\rho_{eg} + \rho_{eg}\nabla\cdot\vec{v} = 0 \tag{168}$$

$$\nabla\varphi_e\Delta\rho_g + \nabla\Delta P_g + \rho_{eg}\frac{\partial}{\partial t}\vec{v} + \frac{1}{4\pi}\nabla(\vec{B}_e\cdot\Delta\vec{B}) + \rho_{eg}\nabla\Delta\varphi_g = 0 \tag{169}$$

$$\frac{\partial}{\partial t}\Delta\vec{B} + \vec{v}\cdot\nabla\vec{B}_e + \vec{B}_e(\nabla\cdot\vec{v}) - (\vec{B}_e\cdot\nabla)\vec{v} = 0 \tag{170}$$

$$4\pi G\Delta\rho_g - \nabla^2\Delta\varphi_g = 0 \tag{171}$$

where we have introduced a well-known vector identity into the hydromagnetic equation (165).

The coefficients of the system of equations (168) - (171) are all independent of t, x, and y, allowing us to Fourier analyze in these variables ($\partial/\partial t \to n$, $\partial/\partial x \to ik_x$, $\partial/\partial y \to ik_y$). But since $\vec{k} = k_y\hat{e}_y + k_z\hat{e}_z$ for the case under study, $k_x = 0$, so that equations (168) - (171) simplify to

$$n\Delta\rho_g + \rho_{eg}ik_yv_y + (\rho_{eg}\partial/\partial z + d/dz\,\rho_{eg})v_z = 0 \tag{172}$$

$$ik_yc^2\Delta\rho_g + n\rho_{eg}v_y + ik_y\frac{B_e}{4\pi}\Delta B_x + \rho_{eg}ik_y\Delta\varphi_g = 0 \tag{173}$$

$$c^2\frac{\partial}{\partial z}\Delta\rho_g + \frac{d}{dz}\varphi_e\Delta\rho_g + n\rho_{eg}v_z + \frac{B_e}{4\pi}\frac{\partial}{\partial z}\Delta B_x + \frac{1}{4\pi}\Delta B_x\frac{d}{dz}B_e + \rho_{eg}\frac{\partial}{\partial z}\Delta\varphi_g = 0 \quad (174)$$

$$B_e ik_y v_y + (B_e\partial/\partial z + d/dz B_e)v_z + n\Delta B_x = 0 \quad (175)$$

$$-4\pi G\Delta\rho_g + (\partial^2/\partial z^2 - k_y^2)\Delta\varphi_g = 0\ . \quad (176)$$

Equations (173) and (174) are the y and z components of equation (169), the equation expressing the conservation of linear momentum, while equation (175) is the x component of the hydromagnetic equation (170). The y and z components of equation (170) need not be written explicitly since they merely express the fact that $\Delta B_y = \Delta B_z = 0$.

If we write that

$$\frac{d}{dz}\rho_{eg} = f(z)\rho_{eg} \quad (177)$$

as above, then since B_e^2/ρ_{eg} is independent of z, it follows that

$$\frac{d}{dz}B_e = \tfrac{1}{2}fB_e\ . \quad (178)$$

Equation (177) is consistent with the presence of a stellar component. In addition, we recall that

$$\frac{d}{dz}\varphi_e = -c^2 f(1 + \alpha) \quad (179)$$

where

$$\alpha = \tfrac{1}{2}a^2/c^2 = B_e^2/8\pi c^2\rho_{eg}\ . \quad (180)$$

Equations (177) - (180) may be used to simplify the perturbed equations:

$$n\frac{\Delta\rho_g}{\rho_{eg}} + ik_y v_y + (\partial/\partial z + f)v_z = 0 \tag{181}$$

$$ik_y c^2 \frac{\Delta\rho_g}{\rho_{eg}} + nv_y + ik_y \frac{B_e^2}{4\pi\rho_{eg}} \frac{\Delta B_x}{B_e} + ik_y \Delta\varphi_g = 0 \tag{182}$$

$$c^2 \frac{1}{\rho_{eg}}\frac{\partial}{\partial z}\Delta\rho_g - c^2 f(1+\alpha)\frac{\Delta\rho_g}{\rho_{eg}} + nv_z + \frac{B_e^2}{8\pi\rho_{eg}}\frac{1}{B_e}\frac{\partial}{\partial z}\Delta B_x + \frac{fB_e^2}{8\pi\rho_{eg}}\frac{\Delta B_x}{B_e}$$

$$+ \rho_{eg}\frac{\partial}{\partial z}\Delta\varphi_g = 0 \tag{183}$$

$$ik_y v_y + \left(\partial/\partial z + \frac{1}{B_e}\frac{d}{dz}B_e\right)v_z + n\frac{\Delta B_x}{B_e} = 0 \tag{184}$$

$$-4\pi G\Delta\rho_g + (\partial^2/\partial z^2 - k_y^2)\Delta\varphi_g = 0 \;. \tag{185}$$

If we recall that

$$\frac{1}{B_e}\frac{\partial}{\partial z}\Delta B_x = (\partial/\partial z + \tfrac{1}{2}f)\frac{\Delta B_x}{B_e} \tag{186}$$

and

$$\frac{1}{\rho_{eg}}\frac{\partial}{\partial z}\Delta\rho_g = (\partial/\partial z + f)\frac{\Delta\rho_g}{\rho_{eg}} \;, \tag{187}$$

and let

$$\epsilon = \Delta\rho_g/\rho_{eg} \qquad a$$

$$\delta = \Delta B_x/B_e \qquad (188) \qquad b$$

and

$$\psi = \Delta\varphi_g/c^2 \qquad c$$

the system of equations (181) - (185) reduces to

$$n\epsilon + ik_y v_y + (\partial/\partial z + f)v_z = 0 \tag{189}$$

$$\epsilon + \frac{n}{ik_y c^2} v_y + 2\alpha\delta + \psi = 0 \tag{190}$$

$$(\partial/\partial z - f\alpha)\epsilon + \frac{n}{c^2} v_z + 2\alpha(\partial/\partial z + f)\delta + \partial/\partial z\psi = 0 \tag{191}$$

$$ik_y v_y + (\partial/\partial z + \tfrac{1}{2}f)v_z + n\delta = 0 \tag{192}$$

$$\frac{-4\pi G\rho_{eg}}{c^2}\epsilon + (\partial^2/\partial z^2 - k_y^2)\psi = 0\ . \tag{193}$$

As was true in Section a) above, the coefficients are not all independent of z, rendering Fourier analysis in z a useless exercise. Thus, a simple analytic dispersion relation $n(k_y)$ cannot be derived from the system (189) - (193) by setting the determinant of the coefficients equal to zero. We may proceed, however, as before by attempting to obtain a single differential equation in ψ, which together with the proper boundary conditions gives the desired dispersion relation.

Proceeding with these remarks in mind, we differentiate equation

(190) and subtract equation (191), with the result that

$$\frac{n}{c^2}\left(\frac{1}{ik_y}\frac{\partial}{\partial z}v_y - v_z\right) + f\alpha(\epsilon - 2\delta) = 0 \;. \tag{194}$$

If we set $n = 0$, the condition for marginal instability, it follows from equation (194) that $\epsilon = 2\delta$ and from equations (189) and (192) that $v_y = v_z = 0$. This is not an unexpected result since all velocities vanish in the marginally unstable state. Proceeding with the analysis with $n = 0$, equation (190) assumes the form

$$\epsilon(1 + \alpha) + \psi = 0$$

and thus

$$\epsilon = -\frac{\psi}{1 + \alpha} \;. \tag{195}$$

The perturbed Poisson equation (193) thus becomes

$$\frac{\partial^2\psi}{\partial z^2} + \left[\frac{4\pi G\rho_{eg}}{c^2(1+\alpha)} - k_y^2\right]\psi = 0 \;. \tag{196}$$

If we recall from above that

$$H_g^2 = c^2/8\pi G\rho_{ego} \;, \tag{43}$$

equation (196) becomes

$$\frac{\partial^2\psi}{\partial z^2} + \left[\frac{1}{2H_g^2(1+\alpha)}\frac{\rho_{eg}(z)}{\rho_{ego}} - k_y^2\right]\psi = 0 \tag{197}$$

with the boundary conditions

$$\frac{\partial}{\partial z}\,\psi(z=0) = 0 \tag{198}$$

and

$$\lim_{|z|\to\infty} \psi = 0 \; . \tag{199}$$

It should be stressed that equation (197) was derived for the case of marginal instability ($n = 0$), and therefore as an obvious check on its validity we require that it reduce to equation (48) in the limit that $B_e \to 0$ ($\alpha \to 0$). This requirement is indeed satisfied. Equation (48), it may be recalled, expresses the condition for the marginal instability of a non-magnetic, self-gravitating, isothermal, plane-stratified gas disk in the presence of static star disk. In fact, equation (197) differs from equation (48) only by the factor $1 + \alpha = 1 + B_e^2/8\pi c^2 \rho_{eg}$ appearing in the second term. Since $\alpha > 1$, the depth of the equivalent potential well is reduced, and from the discussion above, the eigenvalue k_y decreases in absolute magnitude. Thus, the 'perturbation radius' in the symmetry plane (proportional to $\frac{1}{k_y}$) is increased. This is entirely reasonable from a physical point of view; the presence of a magnetic field enhances the difficulty of gravitational instability to result from a given disturbance.

To actually calculate the perturbation radius, we merely equate $c^2(1+\alpha)$ with Q^2 defined above. Thus, for example, if $c = 10$ km/sec, $B_{eo} = 3.5 \times 10^{-6}$ gauss, $\rho_{ego} = 0.025\ M_\odot/pc^3$, and $\rho_{*o} = 0.064\ M_\odot/pc^3$, α becomes 0.222, and $[c^2(1+\alpha)]^{\frac{1}{2}} = 11.1$ km/sec. However, from

Figure 4 r_1(10 km/sec) = 1.65 kpc while r_1(11.1 km/sec) = 1.87 kpc. The presence of a one-dimensional magnetic field of strength 3.5 x 10^{-6} gauss (Verschuur 1969) therefore increases the minimum size of a disturbance (propagating $\perp \vec{B}_e$) necessary to induce gravitational instability from 1.65 to 1.87 kpc when the isothermal sound speed of the gas is 10 km/sec and ρ_{ego} and ρ_{*o} assume the values quoted above.

In conclusion, if λ_k is the length for which a non-magnetic, self-gravitating, isothermal gas layer becomes marginally unstable, the introduction of a simple one-dimensional equipartition magnetic field modifies λ_k by the factor $(1 + \alpha)^{\frac{1}{2}} = (c^2 + \frac{a^2}{2} / c^2)^{\frac{1}{2}}$, where a is the Alfvén velocity = $B_{eo}/(4\pi\rho_{ego})^{\frac{1}{2}}$. This result is surprisingly similar to the modification induced by the presence of a one-dimensional equipartition magnetic field in an initially uniform, self-gravitating gas disk, when the disturbance propagates across $\vec{B}_e$. Finally, if $\vec{B}_e$ is essentially parallel to the direction of spiral arms as the observations indicate, the marginally unstable perturbations are expected to be ellipsoidal rather than spheroidal in shape.

VI. SUMMARY

The distribution of gas density with distance above the galactic plane $\rho_g(z)$ is computed in Section II, on the assumption that the gas and star disks obey hydrostatic and Poisson considerations. Comparing the theoretical gas distribution with Schmidt's observations of $\rho_g(z)$

at the galactic tangential points enables us to derive the z component of the turbulent gas velocity. Arguments are presented which (i) support the Oort mass limit (0.15 $M_{\odot}/pc^3$ in the solar vicinity) and (ii) indicate that $R_o = 10$ kpc.

The same theoretical considerations are employed in Section III to derive an expression for the half-thickness of the galactic gas disk. Employing the Innanen galactic mass model and the observations of McGee and Milton relating to the half-width of the neutral hydrogen layer in the region 4 kpc $\leq R \leq$ 16 kpc, it seems likely that (i) Q (defined above) is a decreasing function of distance from the galactic center and (ii) the increase in half-thickness in the range $R > R_o$ is due to a decrease in gas plus star density at the galactic plane.

In Section IV a non-magnetic isothermal gas disk in the presence of an isothermal star disk is subjected to time-independent axially-symmetric perturbations. Flattened objects centered on the symmetry plane result, with radii characteristically 1-2 kpc. The 21-cm observations of McGee and Milton do not contradict these findings.

Finally, a gas disk threaded by a simple one-dimensional equipartition magnetic field is subject to similar perturbations. The magnetic field tends to inhibit gravitational instability, in the sense that the minimum length necessary to induce gravitational instability for propagation across $\vec{B}_e$ is increased by the factor $(c^2 + \frac{a^2}{2}/c^2)^{\frac{1}{2}}$, where c is the isothermal sound speed of the gas and a is the Alfvén velocity.

APPENDIX

MOTIONS IN THE $\vec{B}_e$ - $\vec{g}$ PLANE: THE USE OF A VECTOR POTENTIAL

When $\vec{k}$ and $\vec{B}_e$ are constrained to lie along the y axis (Figure 7) and $\vec{v}$ lies in the y-z plane ($v_x = 0$), Parker (1966) has found the vector potential $\vec{A}$ to be a useful quantity, where

$$\vec{B} = \nabla \times \vec{A} . \tag{A1}$$

Written in this way, the requirement $\vec{\nabla} \cdot \vec{B} = 0$ is automatically satisfied since the divergence of the curl of any vector vanishes identically. The hydromagnetic equation thus becomes

$$\nabla \times \left[\frac{\partial}{\partial t} \vec{A} - (\vec{v} \times \vec{B})\right] = 0 \tag{A2}$$

from which it follows that

$$\frac{\partial}{\partial t} \vec{A} = \vec{v} \times \vec{B} + \nabla\psi . \tag{A3}$$

ψ may be set equal to zero, it being an arbitrary scalar potential. $\vec{A}$ is a useful quantity because we are dealing with two-dimensional motion (v_y and v_z) so that $\vec{A}$ is constant along magnetic lines of force.

Equation (A3) may be simplified by the introduction of a well-known vector identity:

$$\vec{v}\times\vec{B} = \vec{v}\times(\nabla\times\vec{A}) = -\vec{v}\cdot\nabla\vec{A} + \nabla(\vec{v}\cdot\vec{A}) - \vec{A}\cdot\nabla\vec{v} - \vec{A} \times (\nabla \times \vec{v}) . \tag{A4}$$

$\vec{v} \cdot \vec{A}$ vanishes since $\vec{v}$ is constrained to the y-z plane and $\vec{A}$ has only an x component. Similarly, both $\vec{A} \cdot \nabla\vec{v}$ and $\vec{A} \times (\nabla \times \vec{v})$ vanish, so that we are left with

$$\frac{\partial}{\partial t} \vec{A} + \vec{v} \cdot \nabla\vec{A} = 0 \qquad \text{(A5)}$$

as the hydromagnetic equation. From this equation it follows that

$$\frac{d}{dt} \vec{A} = 0 , \qquad \text{(A6)}$$

i.e. $\vec{A}$ is a constant of the motion.

In terms of $\vec{A}$, the continuity, momentum, hydromagnetic, and Poisson equations may be written as follows:

$$\frac{\partial}{\partial t} \rho_g + \vec{v} \cdot \nabla\rho_g + \rho_g \nabla \cdot \vec{v} = 0 \qquad \text{(A7)}$$

$$\rho_g \frac{d\vec{v}}{dt} + \nabla P_g + \frac{1}{4\pi} \nabla^2 A(\nabla A) + \rho_g \nabla\varphi = 0 \qquad \text{(A8)}$$

$$\frac{\partial}{\partial t} \vec{A} - \vec{v} \times \vec{B} = 0 \qquad \text{(A9)}$$

$$\nabla^2\varphi_g - 4\pi G\rho_g = 0 \qquad \text{(A10)}$$

where $\vec{A} = A\hat{e}_x$ and

$$-\frac{1}{4\pi} (\nabla\times\vec{B})\times\vec{B} = -\frac{1}{4\pi} [(\nabla\times\nabla\times\vec{A}) \times \vec{B}] = \frac{1}{4\pi} (\nabla^2 A) \times \vec{B} = \frac{1}{4\pi} \nabla^2 A(\nabla A) .$$

Introducing the perturbations

$$\rho_g = \rho_{eg} + \Delta\rho_g \qquad \text{a}$$

$$P_g = P_{eg} + \Delta P_g = c^2 \rho_g \qquad \text{b}$$

$$\varphi_g = \varphi_{eg} + \Delta\varphi_g \qquad \text{(A11)} \qquad \text{c}$$

$$\varphi = \varphi_e + \Delta\varphi_g \qquad \text{d}$$

$$\overline{A} = \vec{A}_e + \Delta\vec{A} \qquad \text{e}$$

equations (A7) - (A10) become

$$\frac{\partial}{\partial t} \Delta\rho_g + \rho_{eg} \frac{\partial}{\partial y} v_y + \left(\rho_{eg} \frac{\partial}{\partial z} + \frac{d}{dz} \rho_{eg}\right) v_z = 0 \qquad \text{(A12)}$$

$$c^2 \nabla\Delta\rho_g + \rho_{eg} \frac{\partial}{\partial t} \vec{v} + \Delta\rho_g \nabla\varphi_e + \frac{1}{4\pi} [(\nabla^2 A_e)(\nabla\Delta A) + (\nabla^2 \Delta A)(\nabla A_e)] + \rho_{eg} \nabla\Delta\varphi_g = 0 \qquad \text{(A13)}$$

$$\frac{\partial}{\partial t} \Delta\vec{A} - \vec{v} \times \vec{B}_e = 0 \qquad \text{(A14)}$$

$$4\pi G \Delta\rho_g - \nabla^2 \Delta\varphi_g = 0 \ . \qquad \text{(A15)}$$

The nonvanishing component of $\vec{v} \times \vec{B}_e$ is just $-B_e v_z \hat{e}_x$. In addition, we may write as before that

$$\frac{d}{dz}\,\rho_{eg} = f(z)\rho_{eg}$$

and

$$\frac{d}{dz}\,B_e = \tfrac{1}{2}fB_e \quad .$$

Since

$$\vec{B}_e = \nabla \times \vec{A}_e = \frac{d}{dz}\,A_e\,\hat{e}_y$$

it follows that

$$B_e = \frac{d}{dz}\,A_e \tag{A16}$$

and

$$\nabla^2 A_e = \frac{d^2}{dz^2}\,A_e = \frac{d}{dz}\,B_e = \tfrac{1}{2}fB_e \ . \tag{A17}$$

Thus, equations (A12) - (A15) become

$$n\Delta\rho_g + \rho_{eg}ik_y v_y + \left(\rho_{eg}\frac{\partial}{\partial z} + \frac{d}{dz}\,\rho_{eg}\right) v_z = 0 \tag{A18}$$

$$c^2 ik_y\Delta\rho_g + n\rho_{eg}v_y + \frac{ik_y fB_e}{8\pi}\Delta A + \rho_{eg}\,ik_y\,\Delta\varphi_y = 0 \tag{A19}$$

$$c^2\,\frac{\partial}{\partial z}\,\Delta\rho_g + n\rho_{eg}v_z - c^2 f(1+\alpha)\Delta\rho_g + \frac{B_e}{4\pi}\left[\tfrac{1}{2}f\,\frac{\partial}{\partial z}\,\Delta A + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\Delta A\right] +$$

$$\rho_{eg}\,\frac{\partial}{\partial z}\,\Delta\varphi_g = 0 \tag{A20}$$

$$n\Delta A + v_z B_e = 0 \tag{A21}$$

$$4\pi G\Delta\rho_g - \left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\Delta\varphi_g = 0 \tag{A22}$$

where we have recalled equation (141) for $d\varphi_e/dz$ and have Fourier analyzed in t, x, and y. Equations (A19) and (A20) are, respectively, the y and z components of the momentum equation.

Removing v_z from the system by means of equation (A21) we find that

$$n\frac{\Delta\rho_g}{\rho_{eg}} + ik_y v_y - \frac{n}{B_e}\left(\frac{\partial}{\partial z} + f\right)\Delta A = 0 \tag{A23}$$

$$ik_y\frac{\Delta\rho_g}{\rho_{eg}} + \frac{n}{c^2}v_y + \frac{ik_y f\alpha}{B_e}\Delta A + ik_y\frac{\Delta\varphi_g}{c^2} = 0 \tag{A24}$$

$$\left(\frac{\partial}{\partial z} + f\right)\frac{\Delta\rho_g}{\rho_{eg}} - f(1+\alpha)\frac{\Delta\rho_g}{\rho_{eg}} - \frac{n^2}{c^2 B_e}\Delta A + 2\alpha\left[\tfrac{1}{2}f\frac{\partial}{\partial z} + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\right]\Delta A$$

$$+ \frac{\partial}{\partial z}\frac{\Delta\varphi_g}{c^2} = 0 \tag{A25}$$

$$\frac{4\pi G}{c^2}\frac{\Delta\rho_g}{\rho_{eg}} - \frac{1}{\rho_{eg}}\left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\frac{\Delta\varphi_g}{c^2} = 0 \tag{A26}$$

where we have made use of equation (144) for $\frac{1}{\rho_{eg}}\frac{\partial}{\partial z}\Delta\rho_g$. If we define the new variables

$$\xi = B_e \frac{\Delta\rho_g}{\rho_{eg}} \tag{A27}$$

$$\eta = B_e v_y \tag{A28}$$

and
$$\psi = \frac{\Delta\rho_g}{c^2} \;, \tag{A29}$$

equations (A23) - (A26) become

$$n\xi + ik_y\eta - n(\frac{\partial}{\partial z} + f)\ \Delta A = 0 \tag{A30}$$

$$ik_y\xi + \frac{n}{c^2}\eta + ik_y f\alpha\Delta A + ik_y B_e\psi = 0 \tag{A31}$$

$$\left[\frac{\partial}{\partial z} - f(\alpha + \tfrac{1}{2})\right]\xi + \left[-\left(2\alpha k_y^2 + \frac{n^2}{c^2}\right) + \alpha f\,\frac{\partial}{\partial z} + 2\alpha\,\frac{\partial^2}{\partial z^2}\right]\Delta A + \frac{\partial}{\partial z}\,\psi = 0 \tag{A32}$$

$$\frac{4\pi G}{c^2}\,\xi - \frac{B_e}{\rho_{eg}}\left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\psi = 0\ . \tag{A33}$$

If, for the sake of simplicity, we restrict ourselves to the marginally unstable state ($n=0$), it follows from equation (A30) that $\eta = 0$ (i.e. $v_y = 0$), and from equation (A21) that $v_z = 0$, and equations (A30) - (A33) simplify to

$$ik_y\xi + ik_y f\alpha\Delta A + ik_y B_e\psi = 0 \tag{A34}$$

$$\left[\frac{\partial}{\partial z} - f(\alpha+\tfrac{1}{2})\right]\xi + \left[-2\alpha k_y^2 + \alpha f \frac{\partial}{\partial z} + 2\alpha \frac{\partial^2}{\partial z^2}\right]\Delta A + \frac{\partial}{\partial z}\psi = 0 \tag{A35}$$

$$\frac{4\pi G}{c^2}\xi - \frac{B_e}{\rho_{eg}}\left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\psi = 0\ . \tag{A36}$$

We have found equations (A34) - (A36) no easier to solve than the system (159) - (161), and will terminate the search for the marginally unstable mode with motions in the $\vec{B}_e$ - $\vec{g}$ plane.

TABLE 1

Galactic Mass Models

R(kpc)	ρ_{To} Innanen ($M_\odot/pc^3$)	ρ_{To} Kellman ($M_\odot/pc^3$)
4.0	0.872	0.149
6.0	0.470	0.149
8.2	0.269	0.149
10.0	0.149	0.149
12.0	0.062	0.0251
14.0	0.006	0.010
16.0		0.0056

TABLE 2

Q (km/sec)	r_1 (Jeans) (kpc)	r_1 (Ledoux) (kpc)	r_1 (Kellman) (kpc)
1	0.086	0.122	0.166
5	0.430	0.608	0.830
10	0.86	1.216	1.66
15	1.29	1.824	2.50
20	1.72	2.432	3.33

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FIGURE CAPTIONS

Figure 1. The distribution $\rho_g(z)/\rho_{go}$ (——) for three values of Q, obtained by solving the Poisson and two hydrostatic equilibrium equations. A smooth curve through Schmidt's observed data (----) is also shown.

Figure 2. The distribution $\rho_g(z)/\rho_{go}$ (——) obtained by solving the gas hydrostatic equilibrium equation with Oort's values of K_z. A smooth curve through Schmidt's observed data (----) is also shown.

Figure 3. The theoretically determined half-thickness of the gas layer (——) for three values of Q. A mean curve through the observations of McGee and Milton (----) is shown.

Figure 4. The dimension of the gas perturbations in the symmetry plane (z = 0) as a function of the z component of the turbulent gas velocity.

Figure 5. The isodensity contours $\rho_g(r,z)/\rho_{eg}(o)$ = 1.3, 0.7, and 0.3 for a gas perturbation with A(k) = 0.5.

Figure 6. The isodensity contours $\rho_g(r,z)/\rho_{eg}(o)$ = 1.7, 1.0, and 0.3 for a gas perturbation with A(k) = 1.0.

Figure 7. The relationship between $\vec{g}$, $\vec{k}$, and $\vec{B}_e$ when $\vec{k} \parallel \vec{B}$.

Figure 8. The relationship between $\vec{g}$, $\vec{k}$, and $\vec{B}_e$ when $\vec{k} \perp \vec{B}$.

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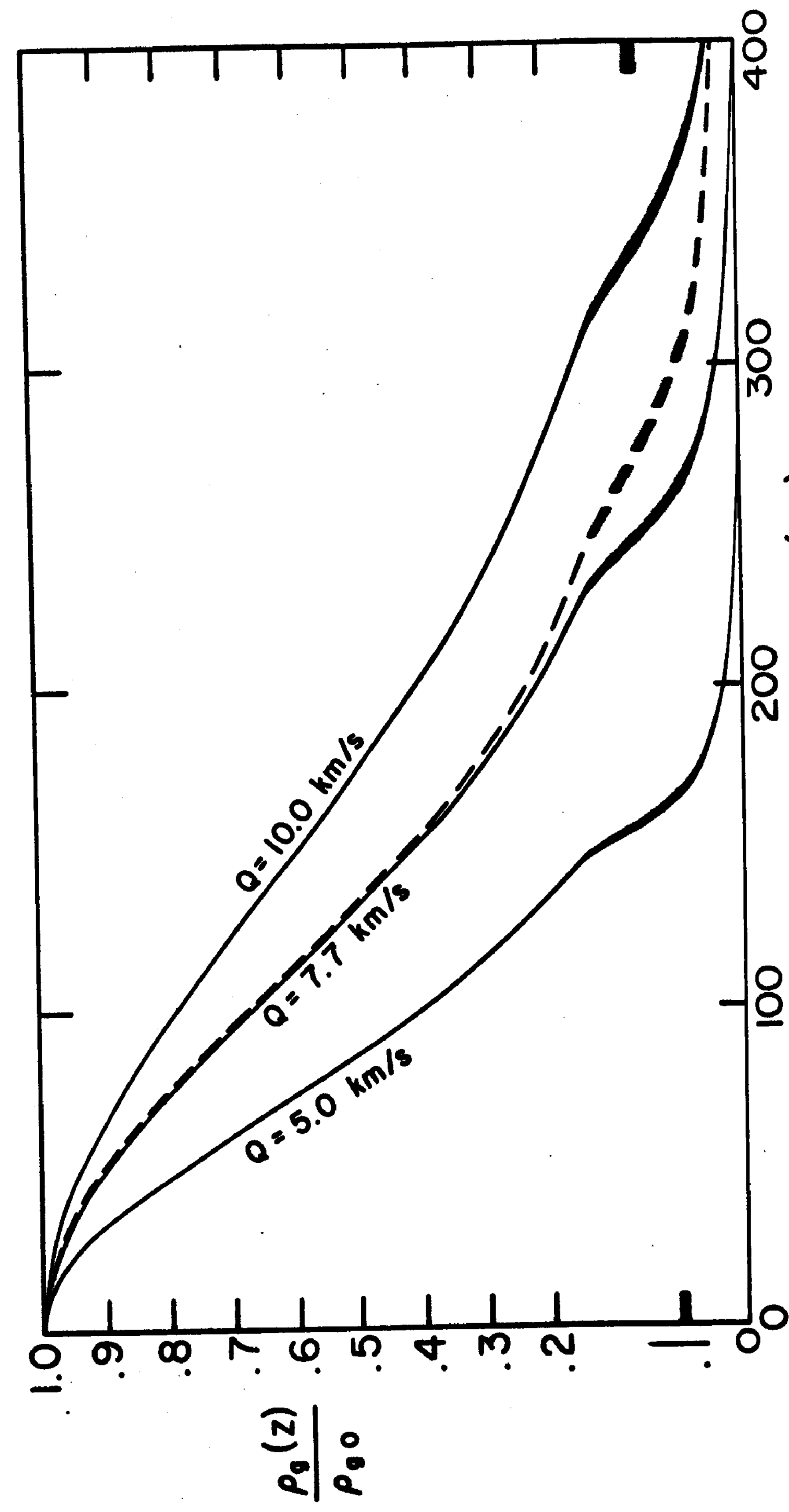


$\frac{\rho_g(z)}{\rho_{g0}}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
100
200
300
400
z(pc)
Q = 10.0 km/s
Q = 7.7 km/s
Q = 5.0 km/s

Figure 1

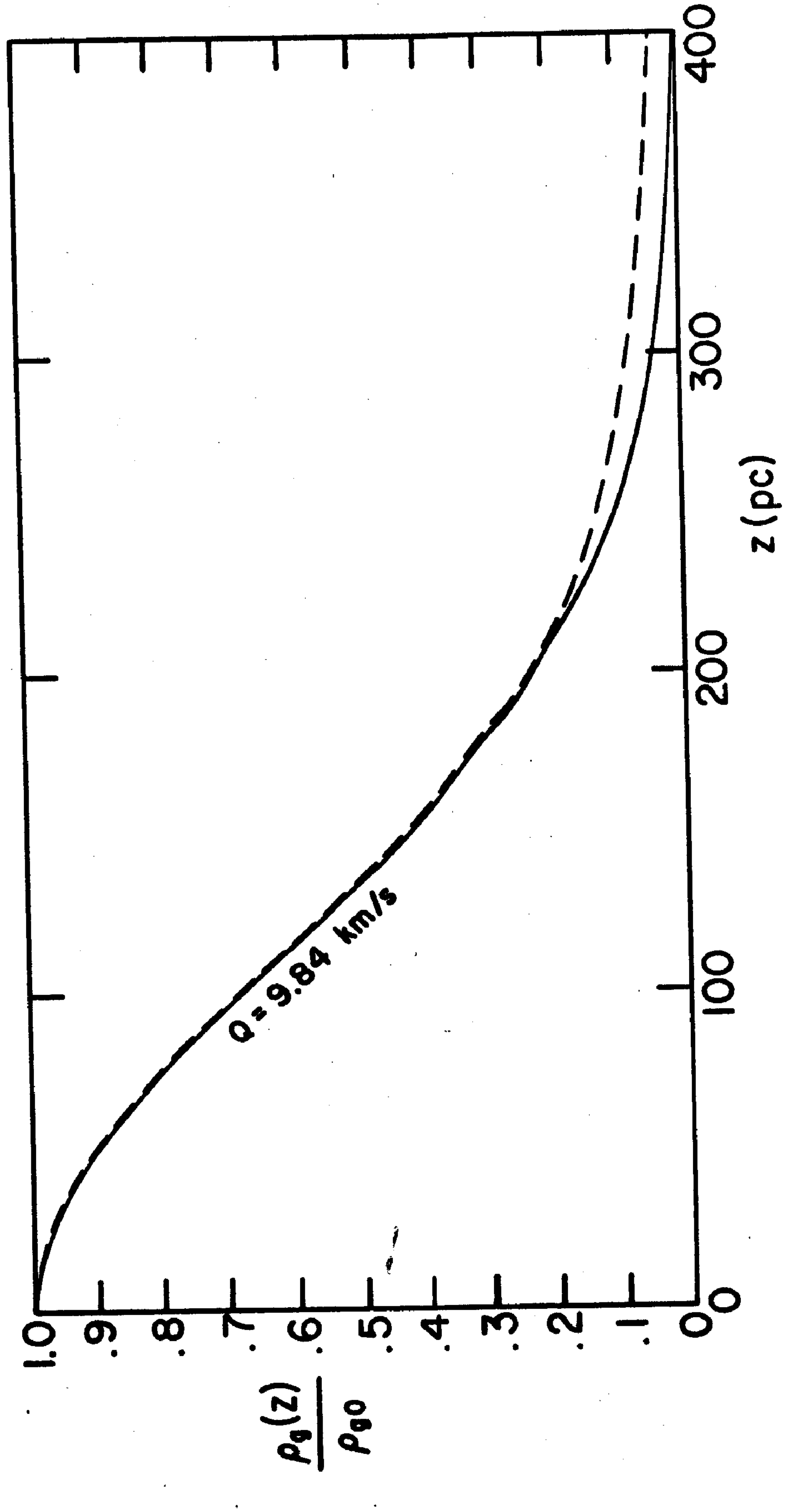


Figure 2

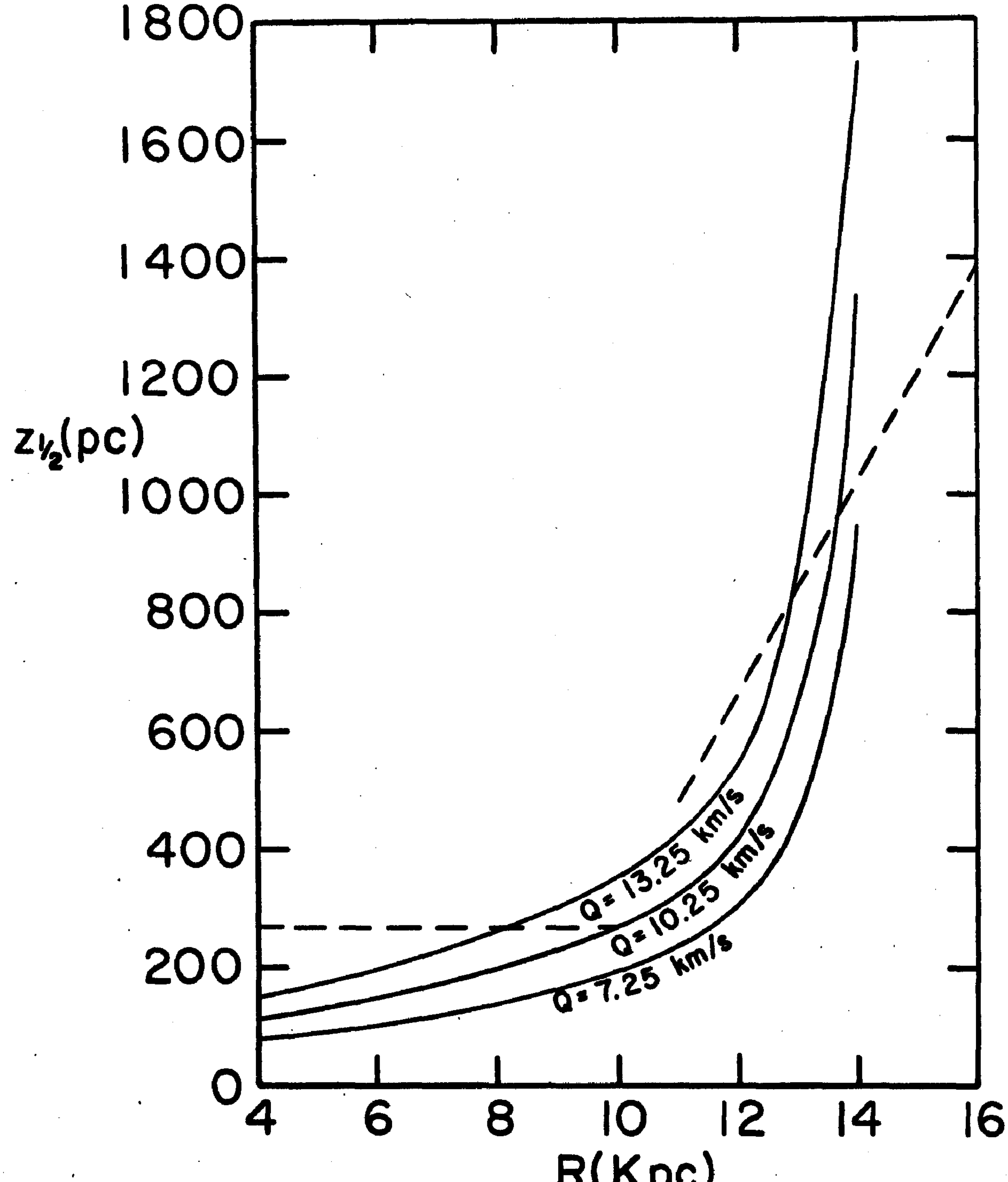

1800
1600
1400
1200
1000
800
600
400
200
0
z½(pc)
4
6
8
10
12
14
16
R(Kpc)
Q= 13.25 km/s
Q= 10.25 km/s
Q= 7.25 km/s

Figure 3

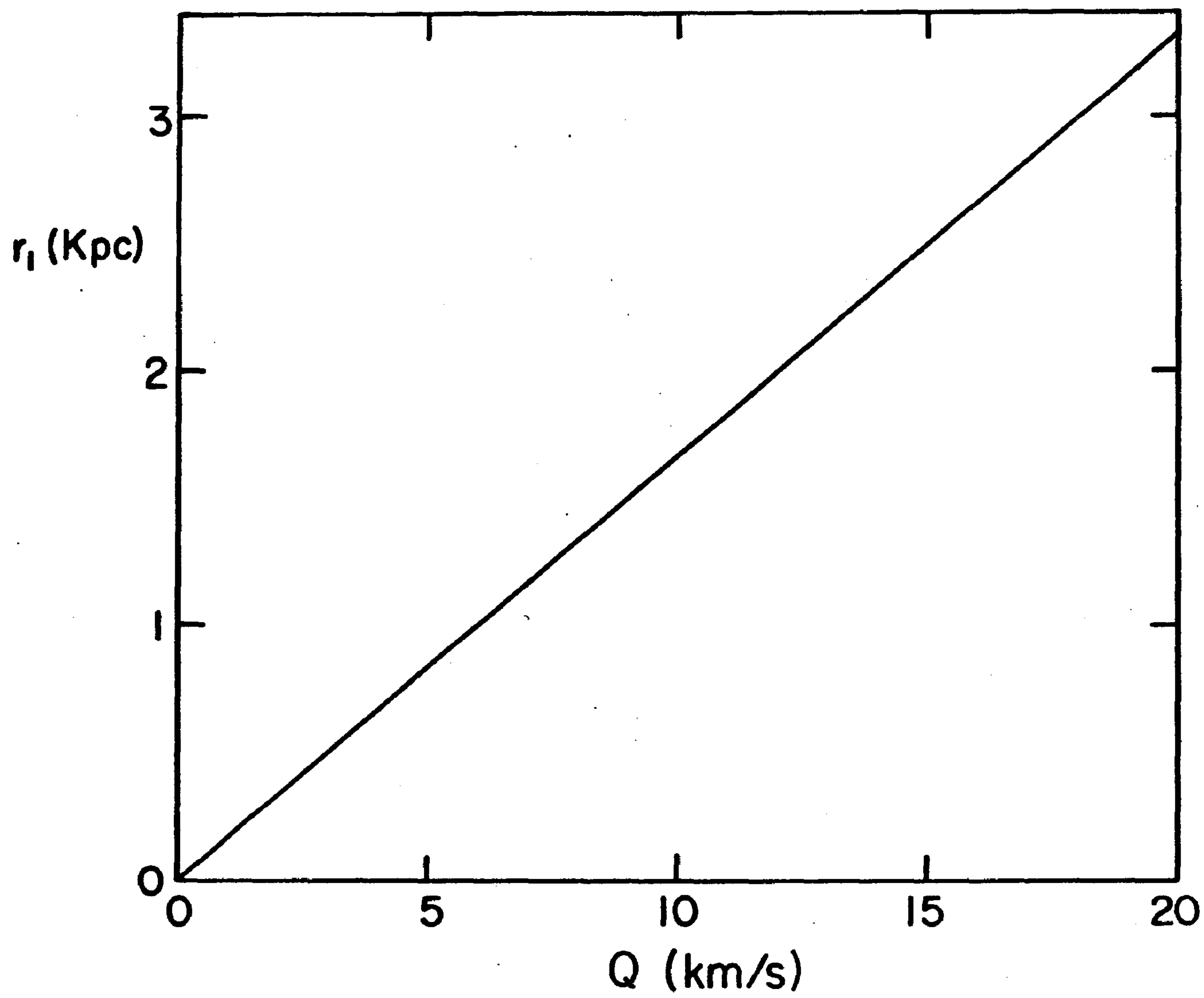


Figure 4

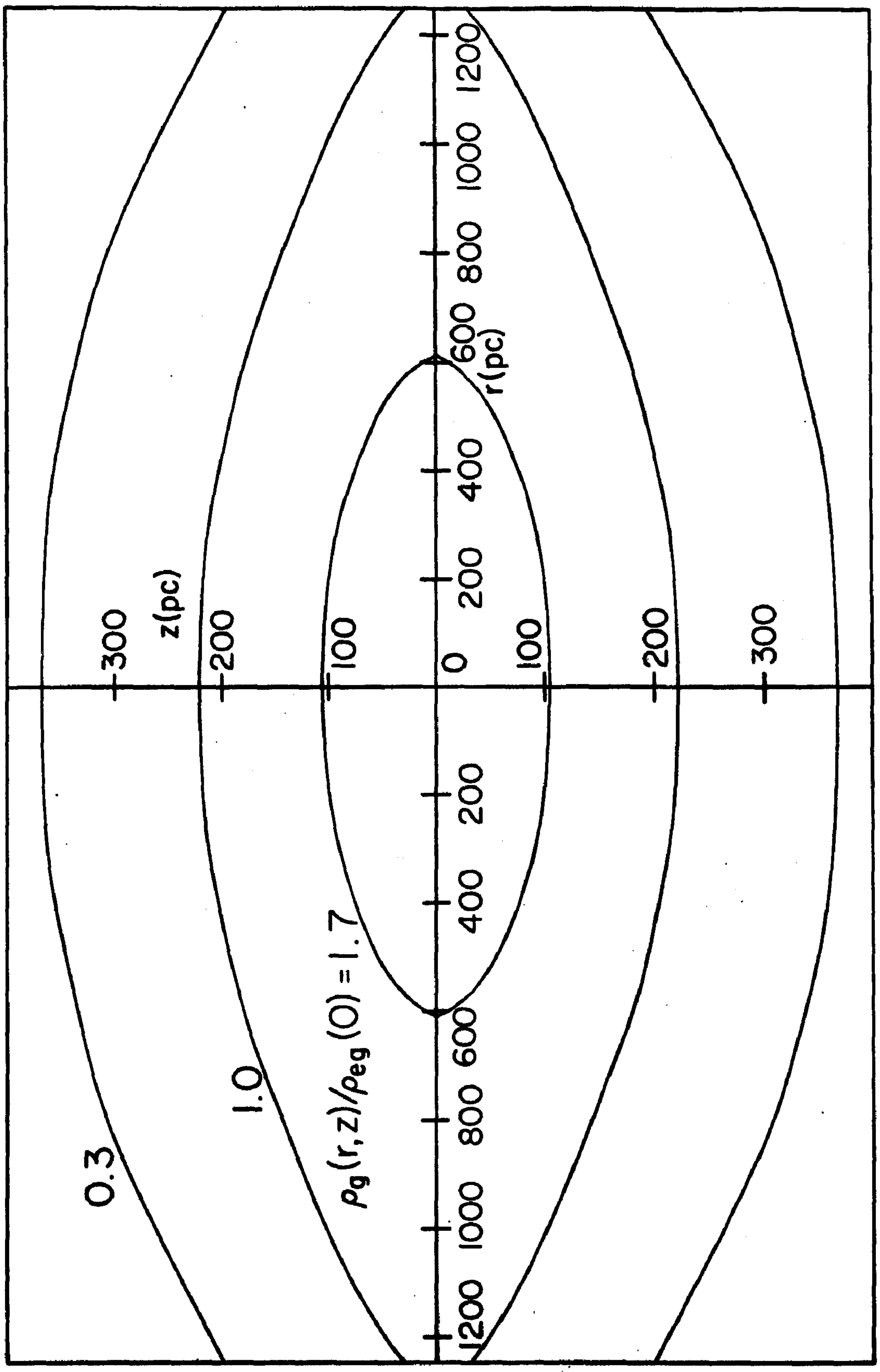
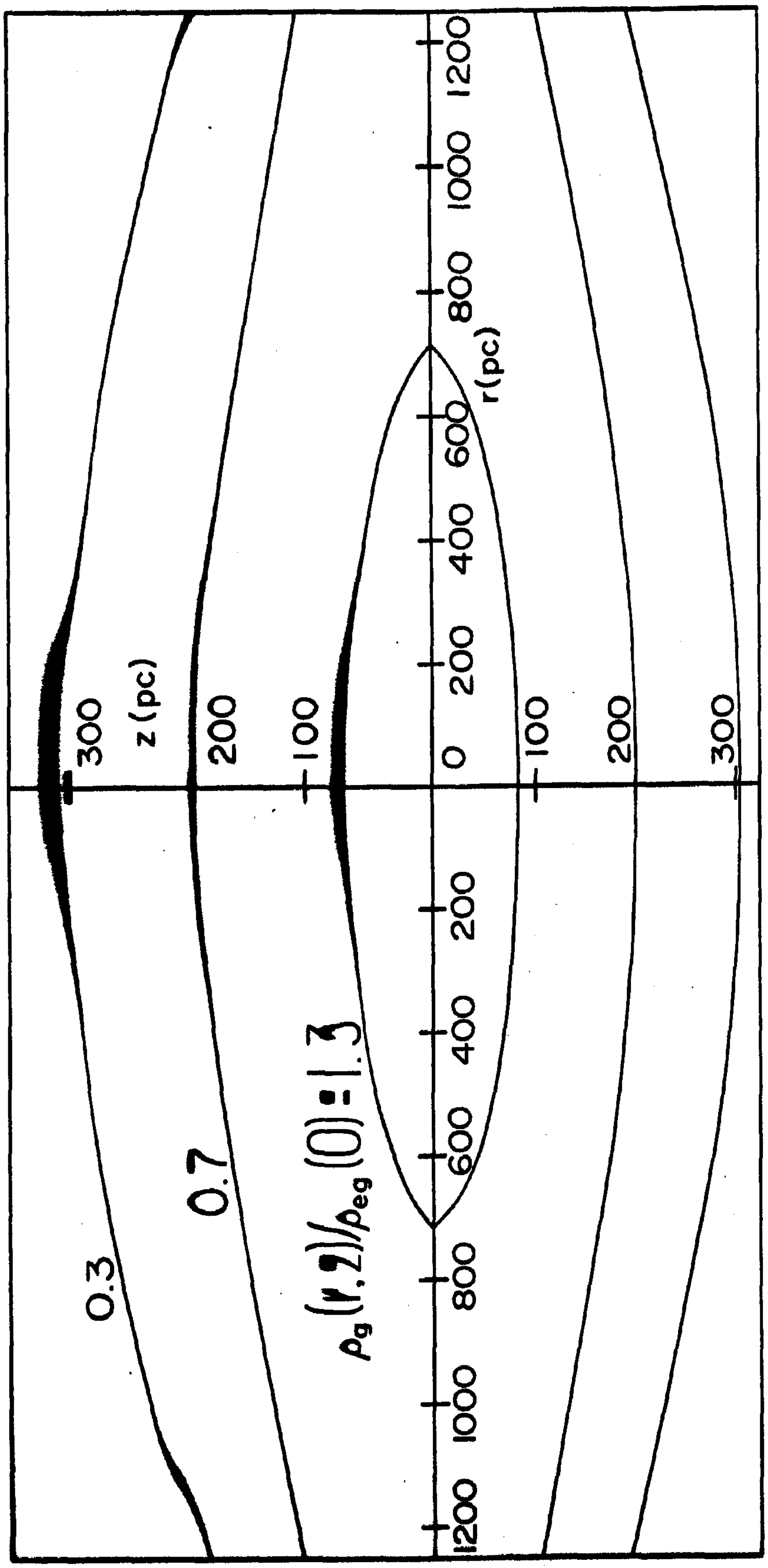

z(pc)
r(pc)
300
200
100
0
100
200
300
200
400
600
800
1000
1200
0.3
1.0
$\rho_g(r,z)/\rho_{eg}(0) = 1.7$

Figure 5



0.3
0.7
$\rho_g(r,z)/\rho_{eg}(0) = 1.3$
z (pc)
r (pc)
300
200
100
0
100
200
300
1200
1000
800
600
400
200
200
400
600
800
1000
1200

Figure 6

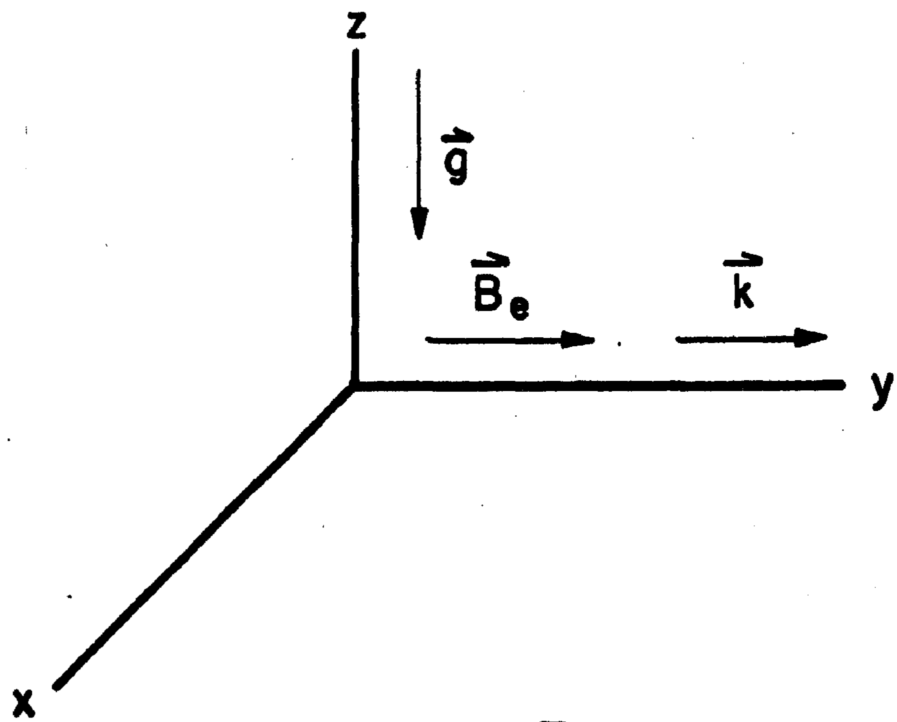


Figure 7

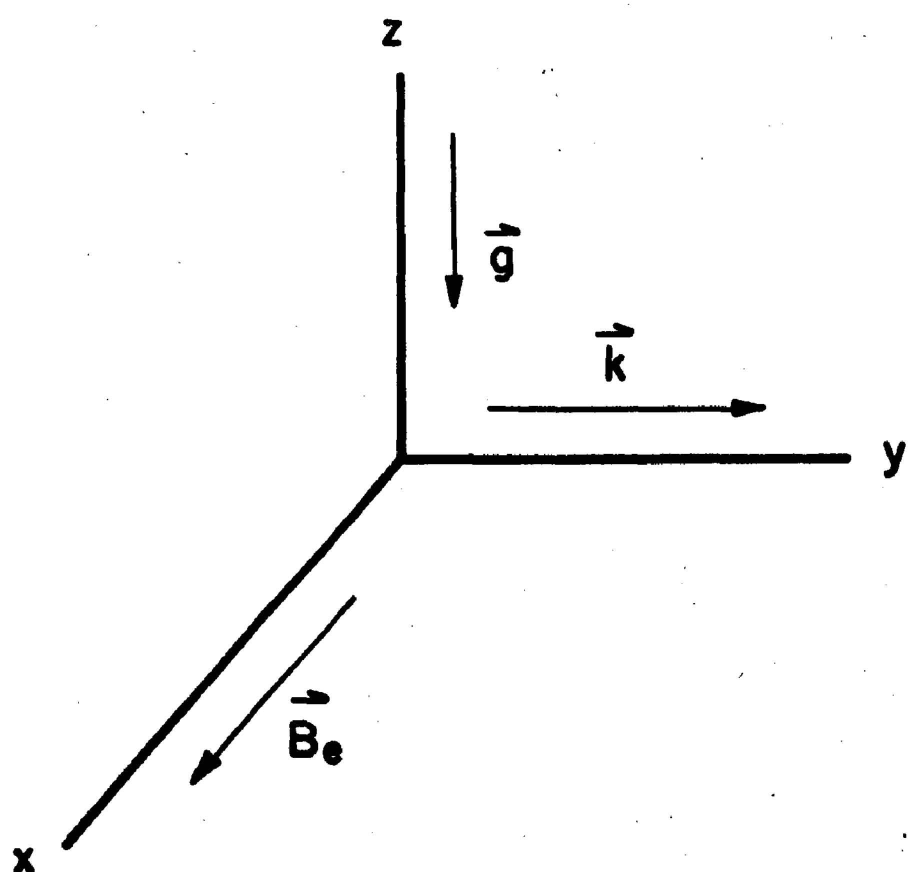


Figure 8