WARD, James Robert, Jr., 1944-
CONTRIBUTIONS TO THE EXISTENCE AND STABILITY OF SOLUTIONS TO NONLINEAR VOLTERRA INTEGRAL EQUATIONS.

University of South Florida, Ph.D., 1974
Mathematics

CONTRIBUTIONS TO THE EXISTENCE AND STABILITY OF SOLUTIONS TO NONLINEAR VOLTERRA INTEGRAL EQUATIONS

by

James R. Ward

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in The University of South Florida

March, 1975

Thesis supervisor:
Associate Professor Athanassios G. Kartsatos

Certificate of Approval - Ph.D. Thesis

Graduate Council
University of South Florida
Tampa, Florida

CERTIFICATE OF APPROVAL

Ph.D. Thesis

This is to certify that the Ph.D. thesis of
James R. Ward

with a major in Mathematics
has been approved by the Examining Committee as
satisfactory for the thesis requirement for the
Ph.D. degree at the convocation of
March, 1975

Thesis committee:

Thesis Supervisor:
Athanassios G. Kartsatos

Member:
H.K. Eichhorn-von-Wurmb

M. N. Manougian
Member:
Manoug N. Manougian

Arunava Mukherjea
Member:
Arunava Mukherjea

Member:
Chris P. Tsokos

ACKNOWLEDGEMENT

The author here takes the opportunity to thank Professor Athanassios G. Kartsatos, the author's thesis advisor, for first suggesting the topic of this thesis and for his many conversations with the author.

TABLE OF CONTENTS

CONTRIBUTIONS TO THE EXISTENCE AND STABILITY OF SOLUTIONS TO NONLINEAR VOLTERRA INTEGRAL EQUATIONS

by

James R. Ward

An Abstract

Of a thesis submitted in partial fulfillment of
the requirements for the degree of Doctor of Philosophy
in the Department of Mathematics in
The University of South Florida

March, 1975

Thesis supervisor:
Associate Professor Athanassios G. Kartsatos

The existence and stability properties of solutions to the n-dimensional nonlinear Volterra integral equation

$$\text{(E)} \qquad x(t) = f(t) + \int_0^t K(t,s,x(s))ds, \quad t\varepsilon[0,\infty)$$

are studied. The Hildebrandt-Graves implicit function theorem for equations in a Banach space is used extensively. Let B be a Banach space contained in $C([0,\infty),R^n)$ with topology stronger than that of uniform convergence on compact sets in $[0,\infty)$. Sufficient conditions are given for the existence of a solution $x\varepsilon B$ for $f\varepsilon B$ in a ball containing the origin. The solution x is also Fréchet differentiable with respect to f, and the variational equation for (E) is studied. Several specific instances for such spaces B are considered in more detail. The spaces considered are of interest in applications.

The results obtained are used to establish existence and stability of solutions to a perturbed form of (E). The Schauder-Tychonov theorem is used in this context.

The results are also applied to the nonlinear n-dimensional boundary value problem

$$\text{(B1)} \qquad x' + A(t)x = F(t,x), \quad t\varepsilon[0,\infty)$$

$$\text{(B2)} \qquad Jx = r,$$

where J is a bounded linear operator mapping the bounded continuous functions on $[0,\infty)$ into R^n. Sufficient conditions for the existence of a unique solution to the problem ((B1),(B2)) are given. The solution x is shown to

depend continuously on changes in $r \in R^n$.

Finally, iterative procedures for approximating the solution to (E) are discussed.

Abstract approved:

Thesis supervisor:
Athanassios G. Kartsatos

Associate Professor

Department of Mathematics

March, 1975

CHAPTER ONE

1. Introduction

This work is concerned with nonlinear Volterra integral equations of the form

$$(1.1) \qquad x(t) = f(t) + \int_0^t K(t,s,x(s))ds$$

where $f(t)$ is a real n-vector continuous for $t \geq 0$ and $K(t,s,x)$ is an n-vector continuous for all (t,s,x) such that $0 \leq s \leq t < +\infty$ and $x \in R^n$ with $\|x\| < r$ $(0 < r \leq +\infty)$. We study solutions $x(t)$ which are defined and continuous for all $t \geq 0$.

Krasnosel'skii [20]* has written in the introduction to his book Topological Methods in the Theory of Nonlinear Integral Equations, " Since nonlinear equations occur in many problems of contemporary physics and technology, the importance of studying such equations needs no explanation." Indeed, nonlinear integral equations arise in a wide variety of problems in mathematics, science, and engineering. The applications to both ordinary and partial differential equations are abundant; see, e.g., Pogorzelski [26]. More

---

* Numbers and page references in brackets refer to entries in the bibliography.

recently Volterra integral equations have been used in systems theory, network theory, and in the study of nuclear reactor dynamics. Two recent books of particular interest on the subject are those of Corduneanu [8] and Miller [23].

In this work we are essentially interested in stability type questions, in the sense that generally we will assume the existence of a solution x_0 to (1.1) for a particular function f_0 and provide conditions for the existence of solutions in a neighborhood of x_0 for all free terms f in a neighborhood of f_0. The notion of neighborhood will be made precise later. Essential to the approach made here is the study of the Fréchet differentiability of the operator K defined by

$$K(\mu)(t) = \int_0^t K(t,s,\mu(s))ds$$

where K acts as an operator between Banach spaces of continuous functions. We use the Fréchet differentiability of K to provide conditions for the Fréchet differentiability of the solutions of (1.1) with respect to variations in f. This in turn provides us both with a method to study stability for (1.1) and a method of linearization of the equation to approximate the solutions.

Equation (1.1) may be written in abstract form

$$x = f + K(x) \tag{1.2}$$

Let B and D be Banach spaces of functions continuous (not necessarily bounded) on $R^+ = [0, +\infty)$. Suppose that for a given $f_0 \in B$ there does exist a unique solution $x_0 \in D$ to equation (1.2). Then part of this work is to provide sufficient conditions for stability of the solution x_0 in the following sense: to show that if $f_1 \in B$ is sufficiently close to f_0 in norm, then there is a corresponding unique solution $x_1 \in D$ which is close to x_0 in the norm on D. This type of stability has been investigated by a number of authors, including Corduneanu [7], Grossman [15], and Miller, Nohel, and Wong [24].

Let $G(f,x) = f + K(x) - x$; then equation (1.2) is equivalent to

$$G(f,x) = 0 \tag{1.3}$$

Suppose $(f_0,x_0) \in B \times D$ is a solution to (1.3) and G maps a neighborhood Ω of (f_0,x_0) in $B \times D$ into a Banach space E such that G is continuously Fréchet differentiable on Ω. If the partial Frechet derivative of G with respect to its second variable at (f_0,x_0) is a linear homeomorphism from D onto E then by an implicit function theorem of Hildebrandt and Graves [10] there is a continuously Fréchet differentiable mapping T from a neighborhood N_B of f_0 into a neighborhood N_D of x_0 such that

$$G(f,T(f)) = 0 \tag{1.5}$$

for all $f \in N_B$. This theorem is applied extensively in

Chapter 2 of this work, where we provide sufficient conditions for the existence and Fréchet differentiability of the solutions to (1.1), for various pairs of Banach spaces B and D, examine the Fréchet differential of the solution x and the resulting linearization of equation (1.1).

The approach outlined above was used by Graves [14] to study bounded solutions on finite intervals for Volterra integral equations satisfying Caratheodory type conditions. More recently it was used by Bennett [3] to study stability and boundedness on $[0,\infty)$ for continuous solutions to equations of the form

$$x(t) = f(t) + \int_0^t K(t,s)g(s,x(s))ds. \tag{1.6}$$

This approach apparently has seen little other use in the study of Volterra integral equations, especially as applied to showing the stability and Fréchet differentiability of the solutions in the general setting here.

In Chapter 3 we will apply the results of Chapter 2 to the perturbed equation

$$G(f,x) = P(x) \tag{1.7}$$

for which the operator P will not be assumed to be differentiable. We again provide sufficient conditions for existence and stability of solutions in various Banach spaces. Use is made of the Banach and Schauder-Tychonov fixed point theorems.

In Chapter 4 we examine some examples and apply the earlier results to a non-linear boundary value problem on $[0,\infty)$. Chapter 5 discusses how the solutions to equation (1.1) might be obtained by iterative methods, including Newton's method.

2. Preliminaries

2.1 Notation and Conventions

Let R denote the set of real numbers, $R^+ = \{t \varepsilon R \mid t \geq 0\}$ and R^n the set of n dimensional real collumn vectors. For each $x \varepsilon R^n$, $x = (x_1, x_2, \ldots, x_n)^T$, let $\|x\| = \sum_{i=1}^{n} |x_i|$ and for every nxn real matrix $A = [a_{ij}]$ take $\|A\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$. It is well known that these norms are compatible; that is, if A is an nxn real matrix and $x \varepsilon R^n$ then $\|Ax\| \leq \|A\| \cdot \|x\|$.

We let $\Delta[a,b] = \{(t,s) \mid a \leq s \leq t \leq b\}$ for $a < b$ and $\Delta[a,\infty) = \{(t,s) \mid a \leq s \leq t < \infty\}$. If B is any Banach space with norm $\|\cdot\|_B$ and $x_0 \varepsilon B$, $r > 0$, then by $N(B, x_0, r)$ we mean $\{x \varepsilon B \mid \|x_0 - x\|_B < r\}$, the open ball in B of radius r centered at x_0. We will also write $N(B,r)$ for $N(B,0,r)$, where 0 denotes the null vector of B.

In this work we will be concerned with various Banach spaces of functions defined and continuous on R^+ with values in R^n. If A and B are topological spaces by $C(A,B)$ we mean the set of continuous functions mapping A into B. By $C_c(R^+,R^n)$ we mean the space of all functions defined and continuous on R^+ with values in R^n with the topology of uniform convergence on compact subsets of R^+. $C_c(R^+,R^n)$ is a locally convex Frechét space. $BC(R^+,R^n)$ will be used to denote the Banach space of bounded continuous R^n valued functions with norm

$$\|x\|_0 = \sup_{t\geq 0} \|x(t)\|.$$

By $C^\ell(R^+,R^n)$ we mean the subspace of $BC(R^+,R^n)$ consisting of those functions which possess a finite limit as $t \to +\infty$. That is, $x \in C^\ell(R^+,R^n)$ if $x \in BC(R^+,R^n)$ and there exists $\ell_x \in R^n$ with $\lim_{t\to\infty} x(t) = \ell_x$. The norm for C^ℓ is the same as that of $BC(R^+,R^n)$. The subspace of $C^\ell(R^+,R^n)$ consisting of those $x \in C^\ell(R^+,R^n)$ such that $\lim_{t\to\infty} x(t) = 0$ will be denoted by $C^\circ(R^+,R^n)$. It too is a Banach space under the sup norm.

Another space of interest is $BCL(R^+,R^n) = \{x\varepsilon BC(R^+,R^n) \mid \int_0^\infty \|x(t)\|dt < \infty\}$ with norm $\|x\|_+ = \|x\|_0 + \int_0^\infty \|x(t)\|dt$. $BCL(R^+,R^n)$ also is a Banach space.

Let $D(t)$, $t \geq 0$, be an nxn matrix of continuous functions such that $D^{-1}(t)$ exists and is continuous for all $t \geq 0$. Then C_D consists of those $x \in C_c(R^+,R^n)$ for which there exists a constant $m_x \geq 0$ such that $\|D^{-1}(t)x(t)\| \leq m_x$ for all $t \geq 0$. C_D forms a Banach space with $\|x\|_D = \sup_{t\geq 0} \|D^{-1}(t)x(t)\|$. For any such D, C_D is easily seen to be isomorphic to $BC(R^+,R^n)$ under the mapping $BC(R^+,R^n) \to C_D$ given by $x \to Dx$. It is however of interest since C_D may contain unbounded functions, unlike the other Banach spaces considered here. C_D spaces and integral equations have been investigated by several authors, notably Corduneau [6], Kartsatos [18] and Gollwitzer [12].

We wish to point out also that all of the Banach spaces in the above listing have topologies stronger than that of $C_c(R^+,R^n)$. That is, if B is one of these spaces and $\{x_n\}$ is a sequence in B converging in norm to $x \in B$ then $x_n \to x$ uniformly on compact subsets of R^+.

2.2 The Fréchet Derivative

In this section we recall the concept of Fréchet differential, introduce some relevant notation, and state an implicit function theorem of Hildebrandt and Graves [10] which will be used extensively in the sequel. If N_1 and N_2 are normed real linear spaces we will denote by $L(N_1,N_2)$ the linear space of bounded linear operators mapping N_1 into N_2.

Let N_1 and N_2 be normed linear spaces with norms $|\cdot|_1$ and $|\cdot|_2$, respectively. Suppose U is an open subset of N_1 and F is an operator mapping U into N_2. F is said to be Fréchet differentiable at $x_0 \varepsilon U$ if there exists an operator $dF(x_0) \varepsilon L(N_1,N_2)$ such that for every $h \varepsilon N_1$ with $x_0+h \varepsilon U$,

$$(1.8) \qquad F(x_0+h) - F(x_0) = dF(x_0)h + \omega(x_0;h)$$

where

$$(1.9) \qquad \lim_{h\to 0} \frac{\|\omega(x_0;h)\|_2}{\|h\|_1} = 0.$$

$dF(x_0)$ is called the (first) Fréchet derivative of F at x_0. If F is differentiable at every point of U, then F is said to be <u>differentiable on U</u>. We will also use the notation $F'(x_0)h$ for $dF(x_0)h$. If the mapping $F': x \to F'(x)$ is a continuous function from U into $L(N_1,N_2)$ then F is continuously Fréchet differentiable on U. We recall that when $dF(x_0)$ exists it is unique.

Let N_3 be a third normed linear space with norm $\|\cdot\|_3$, and let $N_1 \times N_2$ have norm $\|(x_1,x_2)\|_B = \|x_1\|_1 + \|x_2\|_2$. Let U be an open set in $N_1 \times N_2$. Then an operator $G:U \to N_3$ has a first Fréchet partial derivative with respect to x_1 at x^0, $x^0 = (x_1^0,x_2^0) \in U$ if the operator G_1 given by $G_1(x_1) = G\ (x_1,x_2)$, defined in a neighborhood of x_1^0, has a Frechet derivative at x_1^0. It will be denoted by $d_1G(x_1^0,x_2^0)$. $d_2G(x_1^0,x_2^0)$ is defined analogously. The following theorem will be needed; for a proof see Dieudonné [9], p. 167.

<u>Theorem 1.1</u> Let G be a continuous mapping of an open subset U of $N_1 \times N_2$ into N_3. F is continuously differentiable on U if and only if F has both partial derivatives at each point of U and the mappings $(x_1,x_2) \to d_iG(x_1,x_2)$ of U into

$L(N_i,N_3)(i = 1,2)$ are continuous. Then, at each point $(x_1,x_2) \in U, dG(x_1,x_2)(h_1,h_2) = d_1G(x_1,x_2)h_1 + d_2G(x_1,x_2)h_2$.

We now state an implicit function theorem of Hildebrandt and Graves [10] ; see also Kantorovich and Akilov [16].

<u>Theorem 1.2</u> Let B_1, B_2, B_3 be three Banach spaces, U an open subset of $B_1 \times B_2$, and G a mapping of U into B_3. Let $(x_1^0, x_2^0) \in U$ such that $G(x_1^0, x_2^0) = x_3^0$. If

(1) G is continuously differentiable on U;

(2) $d_2G(x_1^0, x_2^0)$ is a linear homeomorphism of B_2 onto B_3;

then there exist positive numbers α and β and a function $T: N(B_1, x_1^0, \alpha) \to N(B_2, x_2^0, \beta)$ with the following properties:

(C_1) The region $N(B_1, x_1^0, \alpha) \times N(B_2, x_2^0, \beta)$ is contained in U.

(C_2) The point $(x_1, T(x_1))$ is a solution of the equation

$$G(x_1, x_2) = x_3^0$$

for every $x_1 \in N(B_1, x_1^0, \alpha)$, and there is no other solution with the same x_1, having x_2 in $N(B_2, x_2^0, \beta)$.

(C_3) The differential $d_2G(x_1, T(x_1))$ is a linear homeomorphism onto B_3 for every $x_1 \in N(B_1, x_1^0, \alpha)$.

(C_4) The function T is continuously Fréchet differentiable on $N(B_1, x_1^0, \alpha)$ and its derivative is given by

$$(1.10)\quad dT(x_1) = -[d_2G(x_1, T(x_1))]^{-1} \circ [d_1G(x_1, T(x_1))].$$

2.3 Preliminary Results

The primary interest in this work is the equation

$$\text{(E)} \qquad x(t) = f(t) + \int_0^t K(t,s,x(s))ds$$

in which $f:R^+ \to R^n$ and $K:\Delta[0,\infty) \times \Omega \to R^n$ are continuous functions, Ω being a non-empty open subset of R^n. A local solution of equation (E) is an R^n-valued function x continuous on a non-degenerate interval $[0,\omega)$ such that $x(t) \in \Omega$ for $t \in [0,\omega)$ and $x(t)$ satisfies (E) for all $t \in [0,\omega)$. By a solution to (E) we mean a function $x(t)$ which is a local solution of (E) on $[0,\omega)$ for all $\omega > 0$. Two important lemmas which follow immediately from theorems in Miller [23] (pp. 87-98) will now be stated.

<u>Lemma 1.3</u> If $K:\Delta[0,\infty) \times \Omega \to R^n$ is continuous, Ω an open set in R^n and $f \in C_c(R^+,R^n)$ is such that there exist numbers $a > 0$ and $b > 0$ for which

$$\{x \in R^n \mid \|x - f(t)\| \le b \text{ for some } t \in [0,a]\}$$

is a subset of Ω. Then there exists a number $t_0 > 0$ and a continuous function $x(t)$ such that $x(t)$ satisfies (E) when $0 \le t \le t_0$.

<u>Lemma 1.4</u> If $K \in C(\Delta[0,\infty) \times R^n, R^n)$ and $f \in C_c(R^+,R^n)$ then each continuous local solution $x(t)$ of (E) can be extended to an interval $[0,\omega)$ where either $\omega = +\infty$ or $\limsup \|x(t)\| = +\infty$ as $t \to \omega^-$.

Thus, if there exists an "a priori" bound on $x(t)$, $x(t)$ may be extended to the right to a bounded function on $[0,\infty)$ which satisfies (E). Also, the following result is an immediate consequence of Lemma 1.3.

<u>Lemma 1.5</u> Let $N(R^n,\alpha) = \{x\varepsilon R^n \mid \|x\| < \alpha\}$. If $K:\Delta[0,\infty) \times N(R^n,\alpha) \to R^n$ is continuous and $K(t,s,0) = 0$ for all $(t,s) \varepsilon \Delta[0,\infty)$ then there exists a number $\delta > 0$ such that $f \varepsilon BC(R^+,R^n)$ and $\|f\|_0 < \delta$ implies the existence of a local solution $x(t)$ to (E) on some interval $[0,t_0)$, $0 < t_0 \leq \infty$.

Proof: Let $f \varepsilon BC(R^+,R^n)$ with $\|f\|_0 < \alpha/2$. Then if $x \varepsilon R^n$ and $\|x - f(t)\| < \alpha/2$ for some $t \varepsilon [0,1]$ we have $\|x\| - \|f(t)\| < \alpha/2$ and $\|x\| < \alpha/2 + \|f\|_0 < \alpha$. Thus $\{x\varepsilon R^n \mid \|x - f(t)\| < \alpha/2$ for some $t \varepsilon [0,1]\} \subsetneq N(R^n,\alpha)$ and by Lemma 1.3 this lemma follows.

Suppose now that $K \varepsilon C(\Delta[0,\infty) \times \Omega,R^n)$ where Ω is a non-empty subset of R^n, and let B and D be normed spaces contained in $C(R^+,R^n)$ with norms $\|\cdot\|_B$ and $\|\cdot\|_D$ respectively. Suppose also that $f_0 \varepsilon B$, and $x_0 \varepsilon D$ is the unique continuous solution to the equation (E) with $f = f_0$, for all $t \geq 0$. We make the following definition:

Definition 1.6 Let K, f_0, and x_0 be as described above. The solution $x_0 \varepsilon D$ will be said to be <u>stable</u> (B,D) if for each number $\varepsilon > 0$ there corresponds a number $\delta > 0$ such that whenever $f \varepsilon B$ and $\|f - f_0\|_B < \delta$ there is a unique continuous solution x of (E) and $x \in D$ with $||x - x_0||_D < \varepsilon$.

This definition is a generalization of the definition of stability for Volterra integral equations given by Bounds and Cushing [4], who consider the cases $D = BC(R^+,R^n)$ and $D = C^0(R^+,R^n)$, referring to the case of stability $(B,C^0(R^+,R^n))$ as asymptotic stability. The definition is in fact simply the most general notion of stability, and is considered in the case of linear systems by Corduneanu [6a].

We now show there is no loss of generality in assuming $K(t,s,0) = 0$ for $(t,s) \in \Delta[0,\infty)$. Let B and D be normed spaces contained in $C(R^+,R^n)$ with topologies stronger than that of $C_c(R^+,R^n)$. Suppose $K \in C(\Delta[0,\infty) \times \Omega,R^n)$ where $\Omega \subsetneq R^n$ is open, and that for $f_0 \in B$ there exists a unique $x_0 \in D$, a solution of (E).
$L(t,s,y) = K(t,s,y + x_0(s)) - K(t,s,x_0(s))$. Then $L(t,s,0) = 0$ for all $(t,s) \in \Delta[0,\infty)$ and if we assume that the boundary of Ω does not intersect the limit set of $x_0(t)$ (that is, if x^* is on the boundary of Ω, then there is no sequence $\{t_n\} \in R^+$, $t_n \to \infty$, such that $\lim_{n\to\infty} x_0(t_n) = x^*$) then $L(t,s,y)$ is defined and continuous on $\Delta[0,\infty) \times N(R^n,\alpha)$ for some $\alpha > 0$. We now consider the relation between solutions of (E) for f in a neighborhood of $f_0 \in B$ and solutions of

$$(E^0) \qquad y(t) = h(t) + \int_0^t L(t,s,y(s))ds$$

for $h \in B$ in a neighborhood of $f = 0$. Suppose that for each $\varepsilon > 0$ there corresponds a $\delta_\varepsilon > 0$ such that

$h \in B$ and $||h||_B < \delta_\varepsilon$ implies the existence and uniqueness of a solution $y \in D$ of (E^0) such that $||y||_D < \varepsilon$.

Now suppose $\bar{f} \in B$ with $||\bar{f} - f_0||_B < \delta_\varepsilon$. We claim there is a unique $\bar{x} \in D$ such that $\bar{x}$ satisfies (E) with $f = \bar{f}$, and $||x_0 - \bar{x}||_B < \varepsilon$. Let $h = \bar{f} - f_0$. Then $h \in B$ and $||h||_D < \delta_\varepsilon$. Thus there exists a unique $\bar{y} \in D$ with $||\bar{y}||_D < \varepsilon$ such that $\bar{y}$ satisfies (E^0). Let $\bar{x} = \bar{y} + x_0$; then $\bar{x}$ satisfies

$$\bar{y}(t) = h(t) + \int_0^t L(t,s,\bar{y}(s))ds$$

$$\bar{y}(t) + x_0(t) = h(t) + x_0(t) + \int_0^t L(t,s,\bar{y}(s))ds$$

$$\bar{y}(t) + x_0(t) = h(t) + x_0(t) + \int_0^t K(t,s,\bar{y}(s) + x_0(s))ds$$

$$- \int_0^t K(t,s,x_0(s))ds$$

$$\bar{y}(t) + x_0(t) = h(t) + f_0(t) + \int_0^t K(t,s,\bar{y}(s))+ x_0(s))ds$$

or

$$\bar{x}(t) = \bar{f}(t) + \int_0^t K(t,s,\bar{x}(s))ds$$

since

$$\bar{x}(t) = \bar{y}(t) + x_0(t).$$

Thus $\bar{x}$ satisfies (E) for $f = \bar{f}$, and

$||x_0 - \bar{x}||_D = ||x_0 - (\bar{y}+x_0)||_D = ||\bar{y}||_D < \varepsilon$. Moreover, there can be no other solution of (E) with $f = \bar{f}$, for suppose there were two distinct solutions, $\bar{x}$ and x^*. Then $\bar{y} = \bar{x}-x_0$ and $y^* = x^*-x_0$ would be two distinct solutions of (E^0) for $h = \bar{f} - f_0$, $||h||_B < \delta_\varepsilon$, which cannot be. Thus $\bar{x}$ is unique. Conversely, if for each $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that $\bar{f} \in B$ with $||\bar{f} - f_0||_B < \delta_\varepsilon$ implies the existence of a unique $\bar{x} \in D$ with $||\bar{x} - x_0||_D < \varepsilon$ satisfying (E), then for each $h \in D$ with $||h||_D < \delta_\varepsilon$ there corresponds a unique $\bar{y} \in D$ satisfying (E^0) and $||\bar{y}||_D < \varepsilon$. Because of this relation between (E) and (E^0) we will assume in the sequel that $K \in C(\Delta[0,\infty)\times\Omega,R^n)$ where $\Omega \subseteq R^n$ is open with $0 \in \Omega$ and $K(t,s,0) = 0$ for all $(t,s) \in \Delta[0,\infty)$.

We now wish to discuss the linear Volterra integral equation

$$(L) \qquad x(t) = f(t) + \int_0^t K(t,s)x(s)ds$$

where $f \in C(R^+,R^n)$ and K is an $n\times n$ matrix whose components $k_{ij}(t,s)$ are continuous functions from $\Delta[0,\infty)$ into R, and $K_{ij}(t,s) = 0$ for $s > t$. It can be shown that the mapping $K:\Delta[0,\infty) \to L(R^n,R^n)$ given by $(t,s) \to K(t,s)$ is continuous in the operator norm on $L(R^n,R^n)$. $K(t,s)$ is called the kernel of the equation (L).

It is well known (see Miller, [23]) that for each $f \in C_c(R^+,R^n)$ there is a unique $x \in C_c(R^+,R^n)$ satisfying (L) for all $t \geq 0$.

The solution of (L) may be expressed in terms of the <u>resolvent kernel</u> of the equation, which is the unique continuous solution $R(t,s)$ of the equation

$$(1.11) \quad R(t,s) = -K(t,s) + \int_s^t R(t,\mu)K(\mu,s)d\mu$$

The resolvent $R(t,s)$ also satisfies

$$(1.12) \quad R(t,s) = -K(t,s) + \int_s^t K(t,\mu)R(\mu,s)d\mu$$

The solution of (L) is given by

$$(1.13) \quad x(t) = f(t) - \int_0^t R(t,s)f(s)ds$$

which is a convenient form in which to study the properties of $x(t)$.

$R(t,s)$ is also given by Neumann's series which is defined as follows:

$$K_1(t,s) = K(t,s)$$

$$K_n(t,s) = \int_s^t K_{n-1}(t,\mu)K(\mu,s)d\mu, \quad (n=1,2,\ldots,)$$

Then

$$(1.14) \quad R(t,s) = -\sum_{n=1}^{\infty} K_n(t,s)$$

It follows from equations (1.11) and (1.12) that being a resolvent is a reciprocal relation. That is, if $R(t,s)$ is the resolvent kernel corresponding to $K(t,s)$, then also $K(t,s)$ is the resolvent kernel corresponding to $R(t,s)$.

We thus have the related equations

$$(1.5) \qquad x(t) = f(t) + \int_0^t K(t,s)x(s)ds$$

$$(1.16) \qquad z(t) = f(t) + \int_0^t R(t,s)z(s)ds$$

whose solutions are given respectively by

$$(1.17) \qquad x(t) = f(t) - \int_0^t R(t,s)f(s)ds$$

$$(1.18) \qquad z(t) = f(t) - \int_0^t K(t,s)f(s)ds.$$

The following well known result (see Corduneanu [8]) will be needed.

<u>Lemma 1.7</u> Let B and D be two Banach spaces of functions each contained in $C_c(R^+,R^n)$ with topologies stronger than that of $C_c(R^+,R^n)$. Assume $K:C_c(R^+,R^n) \to C_c(R^+,R^n)$ is a continuous linear mapping. If $KB \subseteq D$, then K is continuous from B into D.

We also will need

<u>Lemma 1.8</u> Let B and D be Banach spaces contained in $C_c(R^+,R^n)$ with topologies stronger than that of $C_c(R^+,R^n)$. If

1. $K(t,s)$ is an $n\times n$ matrix of functions continuous on $[0,\infty)$ with resolvent $R(t,s)$;
2. $T_k(B) \subseteq D$ where T_k is given by

$$(1.19) \qquad T_K(\mu)(t) = \mu(t) - \int_0^t K(t,s)\mu(s)ds$$

3. $T_R(D) \subseteq B$ where T_R is given by

$$(1.20) \qquad T_R(v)(t) = v(t) - \int_0^t R(t,s)v(s)ds$$

Then

1. T_K is a linear homeomorphism from B onto D and $T_K^{-1} = T_R$.

2. The mapping $F:f \to x$ from D into B where x is the unique solution of

$$(1.21) \qquad x(t) = f(t) + \int_0^t K(t,s)x(s)ds$$

is a linear homeomorphism and $F = T_R$.

Proof: If $x_n \to x$ uniformly on compact sets of R^+ then it is easy to see that $T_K x_n \to T_K x$ and $T_R x_n \to T_R x$ uniformly on compact sets. Since $T_K(B) \subseteq D$ and $T_R(D) \subseteq B$, by Lemma 1.7 T_K is a bounded linear operator from B into D and T_R is a bounded linear operator from D into B. The equation

$$y(t) = \mu(t) + \int_0^t R(t,s)\ y(s)ds$$

has a unique solution given by (1.18)

$$y(t) = \mu(t) - \int_0^t K(t,s)\mu(s)ds$$

It follows that T_K is one to one. Now let $z_0 \in D$ and $T_R(z_0) = \mu_0 \in B$. Then $T_K(\mu_0) = T_K(T_R(z_0))$ is given by

$$T_K(\mu_0)(t) = \mu_0(t) - \int_0^t K(t,s)\mu_0(s)ds$$

$$T_K(\mu_0)(t) = z_0(t) - \int_0^t R(t,s)z_0(s)ds$$

$$- \int_0^t K(t,s)\left[z_0(s) - \int_0^s R(s,\mu)z_0(\mu)d\mu\right]ds$$

$$= z_0(t) - \int_0^t R(t,s)z_0(s)ds - \int_0^t K(t,s)z_0(s)ds$$

$$+ \int_0^t \int_0^s K(t,s)R(s,\mu)z_0(\mu)d\mu ds$$

$$= z_0(t) - \int_0^t R(t,s)z_0(s)ds$$

$$- \int_0^t K(t,s)\ z_0(s)\ ds$$

$$+ \int_0^t \int_\mu^t K(t,s)R(s,\mu)ds\ z_0(\mu)d\mu$$

$$= z_0(t) - \int_0^t (R(t,s) + K(t,s))z_0(s)ds$$

$$+ \int_0^t (R(t,\mu) + K(t,\mu))z_0(\mu)d\mu$$

$$= z_0(t)$$

where in the second to the last step we made use of (1.12).

Thus T_K maps B onto D, and T_K^{-1} exists.

We have already shown that $T_K \circ T_R = I_D$, the identity operator on D. Similarly one can show $T_R \circ T_K = I_B$. Thus

$T_K^{-1} = T_R$ and T_K is a linear homeomorphism from B onto D. Since the solution of (1.21) is
$T_R f(t) = f(t) - \int_0^t R(t,s)f(s)ds$, (2) follows.

Let B and D be Banach spaces contained in $C_c(R^+,R^n)$ with topologies stronger than the topology of $C_c(R^+,R^n)$. If for each $f \in B$ the solution x of (L) is in D the pair (B,D) is said to be admissible for the equation (L), a concept which has been investigated by a number of authors, including Corduneanu [6] and Miller [22]. In the case of the linear system (2) the zero solution is stable (B,D) if and only if the pair (B,D) is admissible for (L). The proof is similar to that of Lemma 2 in Miller [22] and will be omitted.

CHAPTER TWO

1. A Condition for Existence and Fréchet Differentiability of Solutions.

In this chapter, we establish conditions for the existence and Fréchet differentiability of solutions to the equation

$$(E) \qquad x(t) = f(t) + \int_0^t K(t,s,x(s))ds.$$

The first result will give conditions for the existence and Fréchet differentiability of solutions to (E) for every bounded continuous function f. For this we will not use the implicit function theorem (Theorem 1.2 of Chapter 1.) We will then prove a number of similar results by using the implicit function theorem. These results will generally hold only for f in an open ball containing the origin.

If $K \varepsilon C(\Delta[0,\infty) x R^n, R^n)$ has a partial derivative with respect to the variable $x \varepsilon R^n$ at the point $(t,s,\mu) \varepsilon \Delta[0,\infty) x R^n$ then it will be denoted by $K_x(t,s,\mu)$; it is represented by the Jacobian matrix

$$\left[\frac{\partial k_i}{\partial x_j}\right] = \begin{pmatrix} \frac{\partial k_1}{\partial x_1} & \frac{\partial k_1}{\partial x_2} & \cdots\cdots & \frac{\partial k_1}{\partial x_n} \\ \frac{\partial k_2}{\partial x_1} & \frac{\partial k_2}{\partial x_2} & \cdots\cdots & \frac{\partial k_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial k_n}{\partial x_1} & \frac{\partial k_n}{\partial x_2} & \cdots\cdots & \frac{\partial k_n}{\partial x_n} \end{pmatrix}$$

where $K(t,s,x) = (K_1(t,s,x),K_2(t,s,x),\ldots,K_n(t,s,x))^T \in R^n$ and $x = (x_1,x_2,\ldots,x_n)^T \in R^n$.

The following hypotheses concerning $K(t,s,x)$ will be used.

(H1) $K \in C(\Delta[0,\infty)\times R^n,R^n)$

(H2) $K_x(t,s,\mu)$ exists and is continuous on a set of the form $\Delta[0,\infty)\times N(R^n,\omega)$, where $N(R^n,\omega) = \{x \in R^n \,|\, ||x|| < \omega\}$, with $0 < \omega \le +\infty$.

(H3) $K(t,s,0) = 0$ for all $(t,s) \in \Delta[0,\infty)$.

We need the following lemma.

<u>Lemma 2.1</u> If $\alpha \in C(\Delta[0,\infty),R^+)$, $f \in C(R^+,R^+)$ and $\mu \in C(R^+,R\ )$ and $r(t,s)$ is the resolvent kernel for $\alpha(t,s)$ then

$$\mu(t) \le f(t) + \int_0^t \alpha(t,s)\mu(s)ds, \ t \in R^+$$

implies

$$\mu(t) \le f(t) - \int_0^t r(t,s)f(s)ds.$$

Proof: From Neuman's series

$$r(t,s) = -\sum_{n=1}^{\infty} \alpha_n(t,s), \ 0 \le s \le t < \infty$$

where

$$\alpha_n(t,s) = \int_s^t \alpha_{n-1}(t,\mu)\alpha(\mu,s)d\mu$$

for $n = 2,3,\cdots$, and $\alpha_1(t,s) = \alpha(t,s)$. Thus $\alpha_n(t,s) \ge 0$ for all $(t,s) \in \Delta[0,\infty)$, and $r(t,s) \le 0$ for all such (t,s). For each $s \in R^+$ we have

$$\mu(s) \le f(s) + \int_0^s \alpha(s,y)\mu(y)dy$$

so for $t \ge s$

$$-r(t,s)\mu(s) \le -r(t,s)f(s) - \int_0^s r(t,s)\alpha(s,y)\mu(y)dy$$

$$-\int_0^t r(t,s)\mu(s)ds \le \int_0^t -r(t,s)f(s)ds$$

$$-\int_0^t \int_0^s r(t,s)\alpha(s,y)\mu(y)dyds.$$

Thus

$$-\int_0^t r(t,s)\mu(s)ds \le -\int_0^t r(t,s)f(s)ds$$

$$-\int_0^t \int_y^t r(t,s)\alpha(s,y)ds\mu(y)dy.$$

The right side of the last inequality equals, by (1.11)

$$-\int_0^t r(t,s)f(s)ds - \int_0^t [r(t,y) + \alpha(t,y)]\mu(y)dy$$

Therefore

$$0 \leq -\int_0^t r(t,s)f(s)ds - \int_0^t \alpha(t,s)\mu(s)ds$$

and

$$f(t) + \int_0^t \alpha(t,s)\mu(s)ds \leq f(t) - \int_0^t r(t,s)f(s)ds$$

which proves the lemma.

<u>Theorem</u> 2.2. Assume that (H1), (H2), and (H3) hold, and that $\omega = +\infty$ in (H2). We also make the following hypotheses:

(1) There is a function $\alpha \in C(\Delta[0,\infty),R^+)$ such that

$$||K_x(t,s,\mu)|| \leq \alpha(t,s).$$

for all $(t,s,\mu) \in \Delta[0,\infty)\times R^n$, and there is a number $\alpha^* > 0$ such that

$$\sup_{t\geq 0} \int_0^t \alpha(t,s)ds < \alpha^*$$

(2) There is a function $\beta \in C(\Delta[0,\infty),R^+)$ and for each $\varepsilon > 0$ there is a number $\delta > 0$ such that $\mu_1,\mu_2 \in R^n$ and $||\mu_1 - \mu_2|| < \delta$ imply

$$||K_x(t,s,\mu_1) - K_x(t,s,\mu_2)|| < \varepsilon\, \beta(t,s).$$

Also there is a number $\beta^* > 0$ for which

$$\sup_{t\geq 0} \int_0^t \beta(t,s)ds < \beta^*$$

(3) Let $r(t,s)$ denote the resolvent kernel for $\alpha(t,s)$. There is a number $r^* > 0$ such that

$$\sup_{t\geq 0} \int_0^t |r(t,s)|ds < r^*.$$

The following conclusions hold:

(C1) For each $f \in BC(R^+,R^n)$ there is a unique solution $x_f \in BC(R^+,R^n)$ to equation (E).

(C2) Let T denote the mapping $f \to x_f$ from $BC(R^+,R^n)$ into itself. T is continuously Fréchet differentiable at each $f \in BC(R^+,R^n)$ and for $h \in BC(R^+,R^n)$, $dT(f)h$ is the solution y of the linear equation

$$\text{(VE)} \qquad y(t) = h(t) + \int_0^t K_x(t,s,x_f(s))y(s)ds$$

Proof: Since $K(t,s,\mu)$ is continuous on $\Delta[0,\infty)\times R^n$ by Lemma 1.3 for each $f \in BC(R^+,R^n)$ there is a local solution $x(t)$ of (E) defined on an interval $[0,t_0)$, $t_0 > 0$. For $t \in [0,t_0)$ we have

$$||x(t)|| \leq ||f(t)|| + \int_0^t ||K(t,s,x(s))||ds$$

$$= ||f(t)|| + \int_0^t ||\int_0^1 K_x(t,s,\lambda x(s))x(s)d\lambda||ds$$

by Taylor's theorem, since $K(t,s,0) = 0$. This quantity is bounded above by

$$||f(t)|| + \int_0^t \alpha(t,s)||x(s)||ds$$

$$\leq ||f(t)|| - \int_0^t r(t,s)||f(s)||ds$$

$$\leq ||f||_0 + r^*||f||_0 = (1+r^*)||f||_0$$

by Lemma 2.1. $x(t)$ is therefore bounded on $[0,t_0)$ and by Lemma 1.4 the solution may be continued to the right of t_0. Since t_0 is arbitrary, $x(t)$ may be extended as a solution for all $t \geq 0$, and $x \in BC(R^+,R^n)$, the bound $(1+r^*)||f||_0$ being independent of t_0. Moreover, x is the only solution of (E) for this f.

Suppose y is a solution of (E) for $t \in [0,t^*)$, $t^* > 0$, such that $y(t) \neq x(t)$ for some $t \in [0,t^*)$, and let $t_1 = \inf\{0 \leq t < t^* | x(t) \neq y(t)\}$. Since $y(o) = x(o) = f(o)$ and x and y are continuous on $[0,t^*)$, $x(t_1) = y(t_1)$, but there exists t_2, $t_1 < t_2 \leq t^*$ such that $x(t) \neq y(t)$ for all $t \in (t_1,t_2)$. However, for $t \in (t_1,t_2)$ we have

$$||x(t) - y(t)|| \leq \int_0^t ||K(t,s,x(s)) - K(t,s,y(s))||ds$$

$$= \int_0^t ||\int_0^1 K_x(t,s,y(s)+\lambda(x(s)-y(s))$$

$$\cdot(x(s)-y(s))d\lambda||ds$$

by Taylor's theorem. Thus

$$||x(t) - y(t)|| \leq \int_0^t \alpha(t,s)||x(s) - y(s)||ds$$

and

$$||x(t) - y(t)|| \le - \int_0^t r(t,s)\cdot 0\cdot ds = 0$$

by Lemma 2.1. Thus we must have $x(t) = y(t)$ for $t \in [0,t^*)$, and $x(t)$ is the unique solution of (E) on any interval $[0,\alpha)$, $\alpha > 0$.

Let T denote the mapping $f \to x_f$. We first show that T satisfies a Lipschitz condition. Let $f,g \in BC(R^+,R^n)$, then

$$\begin{aligned}
||x_f(t) - x_g(t)|| &= ||f(t) - g(t) \\
&\quad + \int_0^t [K(t,s,x_f(s)) - K(t,s,x_g(s))]ds|| \\
&\le ||f - g||_0 + \int_0^t ||\int_0^1 (K_x(t,s,x_g(s) \\
&\quad + \lambda(x_f(s) - x_g(s))d\lambda(x_f(s) - x_g(s))||ds \\
&\le ||f - g||_0 + \int_0^t \alpha(t,s)||x_f(s)-x_g(s)||ds \\
&\le ||f - g||_0 - \int_0^t r(t,s)||f - g||_0 ds \\
&\le ||f - g||_0 + r^*||f - g||_0 \\
&= (1 + r^*)||f - g||_0.
\end{aligned}$$

where in the third line from the end we used Lemma 2.1

again. We have

$$(2.1) \qquad ||x_f - x_g||_0 \le (1+r^*)||f - g||_0.$$

If y is the solution of (VE) for $h \in BC(R^+,R^n)$ then $y \in BC(R^+,R^n)$:

$$||y(t)|| \le ||h(t)|| + \int_0^t ||K_x(t,s,x_f(s))y(s)||ds$$

$$\le ||h(t)|| + \int_0^t \alpha(t,s)||y(s)||ds$$

$$||y(t)|| \le ||h(t)|| - \int_0^t r(t,s)||h(s)||ds$$

$$\le ||h||_0 + r^*||h||_0$$

$$= (1+r^*)||h||_0.$$

This also shows that the mapping $h \to y$ is a continuous linear map by Lemma 1.7. We now examine $x_{f+h} - x_f - y$ and show that it is $o(||h||_0)$ as $h \to 0$.

$$||x_{f+h}(t) - x_f(t) - y(t)||$$

$$= ||f(t)+h(t) + \int_0^t K(t,s,x_{f+h}(s))ds$$

$$- f(t) - \int_0^t K(t,s,x_f(s))ds - h(t)$$

$$- \int_0^t K_x(t,s,x_f(s))y(s)ds||$$

$$\leq \int_0^t ||K(t,s,x_{f+h}(s)) - K(t,s,x_f(s)) - K_x(t,s,x_f(s))y(s)||ds$$

$$\leq \int_0^t \int_0^1 ||K_x(t,s,x_f(s) + \lambda(x_{f+h}(s) - x_f(s)))$$

$$\cdot(x_{f+h}(s)-x_f(s)) - K_x(t,s,x_f(s))y(s)||d\lambda ds$$

$$= \int_0^t \int_0^1 ||K_x(t,s,x_f(s) + \lambda(x_{f+h}(s)-x_f(s)))(s_{f+h}(s)-x_f(s))$$

$$- K_x(t,s,x_f(s))(x_{f+h}(s)-x_f(s))$$

$$+ K_x(t,s,x_f(s))(x_{f+h}(s)-x_f(s))$$

$$- K_x(t,s,x_f(s))y(s)||d\lambda ds$$

$$\leq \int_0^t \int_0^1 ||K_x(t,s,x_f(s) + \lambda(x_{f+h}(s)-x_f(s)))$$

$$- K_x(t,s,x_f(s))||\cdot||x_{f+h}(s) - x_f(s)||d\lambda ds$$

$$+ \int_0^t ||K_x(t,s,x_f(s))||\cdot||x_{f+h}(s) - x_f(s) - y(s)||ds.$$

Choose $\varepsilon > 0$ and let $\varepsilon^0 = \varepsilon(\beta^* + 2\beta^* r^* + r^* \cdot r^* \cdot \beta^*)^{-1}$ and pick $\delta^0 > 0$ so that $\mu_1, \mu_2 \in R^n$ and $||\mu_1 - \mu_2|| < \delta^0$ imply

(2.3) $||K_x(t,s,\mu_1) - K_x(t,s,\mu_2)|| < \varepsilon^0 \beta(t,s)$,

by (2). Choose $\delta(\varepsilon) = (1+r^*)^{-1}\delta^0$. Then for $h \in BC(R^+,R^n)$

with $||h||_0 < \delta(\varepsilon)$ we have from (2.1)

$$||x_{f+h} - x_f||_0 < (1+r^*)||h||_0$$

$$\leq (1+r^*)^{-1}\delta^0 = \delta^0$$

and from (2.2) and (2.3)

$$||x_{f+h}(t) - x_f(t) - y(t)|| \leq \int_0^t \int_0^1 \varepsilon^0 \beta(t,s) d\lambda ds$$

$$\cdot ||x_{f+h} - x_f||_0 + \int_0^t \alpha(t,s)||x_{f+h}(s) - x_f(s) - y(s)||ds$$

$$||x_{f+h}(t) - x_f(t) - y(t)|| \leq \varepsilon^0 \beta^* ||x_{f+h} - x_f||_0$$

$$+ \int_0^t \alpha(t,s)||x_{f+h}(s) - x_f(s) - y(s)||ds$$

and by Lemma 2.1 we obtain

$$||x_{f+h}(t) - x_f(t) - y(t)||$$

$$\leq \varepsilon^0 \beta^* ||x_{f+h} - x_f||_0 - \int_0^t r(t,s)\varepsilon^0 \beta^* ||x_{f\ h} - x_f||_0 ds$$

$$\leq \varepsilon^0 \beta^* ||x_{f+h} - x_f||_0 + r^* \varepsilon^0 \beta^* ||x_{f+h} - x_f||_0$$

$$\leq \varepsilon^0 \{\beta^*(1+r^*)||h||_0 + r^*\beta^*(1+r^*)||h||_0$$

$$= \varepsilon\{\beta^* + 2\beta^* r^* + r^* r^* \beta^*\}^{-1}$$

$$\cdot \{\beta^* + r^*\beta^* + r^*\beta^* + \beta^* r^* r^*\}||h||_0$$

$$= \varepsilon||h||_0$$

so that

$$(2.4) \qquad ||x_{f+h} - x_f - y||_0 < \varepsilon||h||_0$$

whenever $||h||_0 < \delta(\varepsilon)$, and $x_{f+h} - x_f - y = o(||h||_0)$ as $h \to 0$, and y is the Fréchet derivative of T at f, evaluated at h.

We complete the proof of the theorem by showing that the map $f \to dT(f)$ from $BC(R^+,R^n) = BC$ into $L(BC,BC)$, the space of bounded linear operators on $BC(R^+,R^n)$, is a continuous mapping.

Let $y_1 = dT(f)h$ and $y_2 = dT(g)h$. Then

$$\begin{aligned}
||y_1(t)-y_2(t)|| &= ||h(t) + \int_0^t K_x(t,s,x_f(s))y_1(s)ds \\
&\qquad - h(t) - \int_0^t K_x(t,s,x_g(s))y_2(s)ds|| \\
&\le \int_0^t ||K_x(t,s,x_f(s))y_1(s) - K_x(t,s,x_g(s)) \\
&\qquad \cdot y_2(s)||ds \\
&\le \int_0^t ||K_x(t,s,x_f(s)) - K_x(t,s,x_g(s))|| \\
&\qquad \cdot ||y_1(s)||ds
\end{aligned}$$

(inequality cont. next page)

$$+ \int_0^t ||K_x(t,s,x_g(s))||\cdot||y_1(s) - y_2(s)||ds.$$

As in the argument above showing differentiability of T, given $\varepsilon > 0$ we may choose $\delta > 0$ so small that $||f - g||_0 < \delta$ implies the last quantity above has as an upper bound

$$\int_0^t \varepsilon\beta(t,s)||y_1(s)||ds + \int_0^t \alpha(t,s)||y_1(s) - y_2(s)||ds$$

$$\leq \varepsilon\beta^*||y_1||_0 + \int_0^t \alpha(t,s)||y_1(s) - y_2(s)||ds.$$

Therefore

$$||y_1(t) - y_2(t)|| \leq \varepsilon\beta^*||y_1||_0 - \int_0^t r(t,s)\varepsilon\beta^*||y_1||_0$$

$$\leq \varepsilon\beta^*||y_1|| + r^*\varepsilon\beta^*||y_1||_0$$

$$= \varepsilon(\beta^*+r^*\beta^*)||y_1||_0$$

$$= \varepsilon(\beta^*+r^*\beta^*)||dT(f)h||_0$$

$$\leq \varepsilon(\beta^*+r^*\beta^*)||dT(f)||\cdot||h||_0$$

$dT(f)$ being a bounded linear operator. Let $M = (\beta^*+r^*\beta^*)\cdot ||dT(f)||$. We have

$$||y_1 - y_2||_0 \leq \varepsilon M||h||_0 \tag{2.5}$$

for $||f-g||_0<\delta$. M is a constant independent of h, so we may choose $\delta^* > 0$ so small that $||f-g||_0 < \delta^*$ implies

$$(2.6) \qquad ||y_1 - y_2||_0 \le \varepsilon||h||_0$$

and since $y_1 = dT(f)h$ and $y_2 = dT(g)h$

$$(2.7) \qquad ||dT(f)h - dT(g)h||_0 \le \varepsilon||h||_0$$

which shows that

$$(2.8) \qquad ||dT(f) - dT(g)|| < \varepsilon$$

whenever $||f - g||_0$ is sufficiently small, and thus T is continuously Fréchet differentiable at all $f \in BC(R^+,R^n)$. This completes the proof of the theorem.

As an example to illustrate the theorem, let $x = (x_1,x_2)$ and

$$K(t,s,x) = \begin{pmatrix} \dfrac{\alpha e^{-t+s} x_1}{1+x_1^2} \\ \\ \beta e^{-t+s} x_2 \end{pmatrix}$$

$K(t,s,x)$ satisfies the hypothesis of Theorem 2.2 if $|\alpha| + |\beta| < 1$. This condition is needed to insure that the resolvent of $\alpha(t,s) = (|\alpha| + |\beta|)e^{-(t-s)}$ satisfies the integrability condition of the theorem. The function $\alpha(t,s)$ dominates $||K_x(t,s,\mu)||$ for all $\mu \in R^2$, as can be easily verified.

Remark: The condition $K(t,s,0) = 0$ is not completely necessary for this type of theorem; the condition

$$\sup_{t\geq 0} \int_0^t ||K(t,s,0)||ds < +\infty$$

could also be used in showing the existence of solutions in $BC(R^+,R^n)$.

Corollary 2.3 Let $A(t,s)$ be an $n\times n$ matrix continuous for $(t,s) \varepsilon \Delta[0,\infty)$ and $g \varepsilon C(R^+,R^n)$ such that $g(0) = 0$ and

(1) $g(\mu)$ is differentiable for all $\mu \varepsilon R^n$ and its derivative $g_x(\mu)$ is uniformly continuous on R^n and there is a number $N > 0$ such that $||g_x(\mu)|| \leq N < +\infty$ for all $\mu \varepsilon R^n$.

(2) $$\sup_{t\geq 0} \int_0^t ||A(t,s)||ds = M < +\infty.$$

(3) Let $R_0(t,s)$ denote the resolvent of the function $||A(t,s)||\cdot N$. There is a number $R^* > 0$ such that

$$\sup_{t\geq 0} \int_0^t |R_0(t,s)| \, ds < R^*.$$

Then (C1),(C2) of Th. 2.2 hold with $K(t,s,x)=A(t,s)g(x)$ in (E).

Proof: Let $K(t,s,x) = A(t,s)g(x)$; $K(t,s,x)$ satisfies the hypotheses of Theorem 2.2: it is continuously differentiable in x on all of $\Delta[0,\infty)\times R^n$ and

$$\sup_{\mu\varepsilon R^n} ||K_x(t,s,\mu)|| \leq \sup_{\mu\varepsilon r^n} ||A(t,s)||\cdot||g_x(\mu)||$$

$$\leq ||A(t,s)||\cdot N,$$

so that (1) of Theorem 2.2 are satisfied. Also

$$||K_x(t,s,\mu_1) - K_x(t,s,\mu_2)||$$

$$\leq ||A(t,s)||\cdot||g_x(\mu_1) - g_x(\mu_2)||$$

By hypothesis $g_x(\mu)$ is uniformly continuous on R^n. Therefore given $\varepsilon > 0$ there does exist a number $\delta > 0$ such that $\mu_1,\mu_2 \in R^n$ and $||\mu_1 - \mu_2|| < \delta$ imply $||g_x(\mu_1) - g_x(\mu_2)|| < \varepsilon$. We have for such μ_1,μ_2

$$||K_x(t,s,\mu_1) - K_x(t,s,\mu_2)|| < \varepsilon||A(t,s)||$$

so that (2) of Theorem 2.2 is satisfied and (3) of Theorem 2.2 is immediate from (3) of this corollary; the result follows.

2. Solutions and Stability by the Implicit Function Theorem

In this section the Hildebrandt-Graves implicit function theorem (Theorem 1.2) will be applied to obtain results similar to those of Theorem 2.2 in a variety of situations.

Unless otherwise restricted, we will assume that $K(t,s,x)$ satisfies (H1) - (H3). We introduce some notation. K will denote the nonlinear operator mapping $C_c(R^+,R^n)$ into itself given by

$$K(\mu)(t) = \int_0^t K(t,s,\mu(s))ds \tag{2.9}$$

for $\mu \in C_c(R^+,R^n)$ and $t \in R^+$.

For fixed $\mu \in C_c(R^+,R^n)$, $K_x(\mu)$ will denote the linear operator mapping $C_c(R^+,R^n)$ into itself given by

$$(2.10) \qquad K_x(\mu)h(t) = \int_0^t K_x(t,s,\mu(s))h(s)ds$$

for $h \in C_c(R^+,R^n)$ and $t \in R^+$.

For $f,\mu \in C_c(R^+,R^n)$ we define the operator G by

$$(2.11) \qquad G(f,\mu) = f + K(\mu) - \mu$$

so that

$$(2.12) \qquad G(0,0) = 0$$

and any solution $x \in C_c$ to the equation

$$(2.13) \qquad G(f,x) = 0$$

for fixed $f \in C_c$ also solves

$$(E) \qquad x(t) = f(t) + \int_0^t K(t,s,x(s))ds$$

and conversely.

Lemma 2.4 Assume (H1) - (H3) and let $B,C,D,$ be Banach spaces with topologies stronger than that of $C_c(R^+,R^n)$ and $B \subseteq D$ with stronger topology than that of D. If there is a number $p > 0$ such that

$$G(f,\mu) \in D \quad \text{for all } (f,\mu) \in N(B\times C,p)$$

then G has a continuous Fréchet partial derivative with respect to its first variable on $N(B\times C,p)$, and for $h \varepsilon B$, $d_1G(f,\mu)h = h$.

Proof: Let $(f,\mu) \varepsilon N(B\times C,p)$ and $h \varepsilon B$ with $(f+h,\mu) \varepsilon N(B\times C,p)$.

$$G(f+h,\mu) - G(f,\mu) = f+h + K(\mu) - \mu - f - k(\mu) + \mu = h.$$

Thus

$$||G(f+h,\mu) - G(f,\mu) - h||_D = 0$$

and we need to show that the mapping $h \to h$ considered as a mapping from B into D is a bounded operator. Since the topology of B is stronger that that of D, $h_n \to h$ in B implies that $h_n \to h$ in D, so the injection $h \to h$ from B into D is continuous, and therefore bounded.

Let (f_1,μ_1), $(f_2,\mu_2) \varepsilon N(B\times C,p)$:

$$||d_1G(f_1,\mu_1)h - d_1G(f_2,\mu_2)h||_D = ||h - h||_D = 0$$

for any $h \varepsilon B$. Thus the map $(f,\mu) \to d_1G(f,\mu)$ is continuous. This proves the lemma.

Lemma 2.5 Assume (H1) - (H3) and let B,C,D be Banach spaces contained in $C_c(R^+,R^n)$ with topologies stronger than that of $C_c(R^+,R^n)$. If there is a number $p > 0$ such that

(1) $G(f,\mu) \varepsilon D$ for all $(f,\mu) \varepsilon N(B\times C,p)$

(2) $\mu \varepsilon N(C,p)$ and $h \varepsilon C$ imply $K_x(\mu)h - h \varepsilon D$.

(3) For every $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that $\mu_1, \mu_2 \in N(C,p)$ and $||\mu_1 - \mu_2||_C < \delta_\varepsilon$ implies

$$||K_x(\mu_1)h - K_x(\mu_2)h||_D < \varepsilon ||h||_C$$

Then G possesses a continuous Fréchet partial derivative with respect to its second variable on $N(B\times C,p)$ and

$$d_2G(f,\mu)h = K_x(\mu)h - h$$

for $(f,\mu) \in N(B\times C,p)$ and $h \in C$.

Proof: Let $(f,\mu) \in N(B\times C,p)$ and $h \in C$. Then

$$||(f,\mu)||_{B\times C} = ||f||_B + ||\mu||_C < p$$

so that $\mu \in N(C,p)$ and by (2) $K_x(\mu)h - h \in D$. Since

$$(K_x(\mu)h - h)(t) = \int_0^t K_x(t,s,\mu(s))h(s)ds - h(t)$$

and $K_x(t,s,\mu)$ is continuous, the mapping $h \to K_x(\mu)h - h$ is easily seen to be continuous as a mapping from $C_c(R^+,R^n)$ into itself. By Lemma 1.7 the mapping is bounded linear operator from C into D.

Now let $\varepsilon > 0$ be given, and let δ_ε be chosen as in (3) so that if $(f,\mu) \in N(B\times C,p)$ and $h \in C$ such that $(f,\mu+h) \in N(B\times C,p)$ and $||h||_C < \delta_\varepsilon$ then

$$G(f,\mu+h) - G(f,\mu) - (K_x(\mu)h-h)$$

$$= (f+K(\mu+h)-\mu-h) - (f+K(\mu)-\mu) - K_x(\mu)h + h$$

$$= K(\mu+h) - K(\mu) - K_x(\mu)h.$$

For any $t \geq 0$

$$K(\mu+h)(t) - K(\mu)(t) - K_x(\mu)h(t)$$

$$= \int_0^t \{K(t,s,\mu(s)+h(s)) - K(t,s,\mu(s)) - K_x(t,s,\mu(s))h(s)\}ds$$

$$= \int_0^t \int_0^1 [K_x(t,s,\mu(s)+ h(s)) - K_x(t,s,\mu(s))]h(s)d\ ds$$

by Taylor's theorem. Since the functions considered are all continuous we may interchange the order of integration to obtain

$$\int_0^1 \{\int_0^t [K_x(t,s,\mu(s)+ h(s)) - K_x(t,s,\mu(s))]h(s)ds\}d\lambda$$

$$= \int_0^1 [K_x(\mu+\lambda h) - K_x(\mu)]h(t)d\,\lambda$$

$$= \int_0^1 [K_x(\mu+\lambda h) - K_x(\mu)h]d\lambda(t)$$

where the latter integral is an abstract Riemann integral (see Appendix). Therefore

$$||K(\mu+h)-K(\mu)-K_x(\mu)h||_D = ||\int_0^1 [K_x(\mu+\lambda h)-K_x(\mu)]hd\lambda||_D$$

$$\leq \int_0^1 ||K_x(\mu+\lambda h)h-K_x(\mu)h||_D d\lambda < \int_0^1 \varepsilon||h||_c d\lambda = \varepsilon||h||_c$$

by (3), since $||\lambda h||_c = \lambda||h||_c < \delta_\varepsilon$ for $0 \leq \lambda \leq 1$. This shows $d_2G(f,\mu)h = K_x(\mu)h-h$ on $N(B\times c,p)$.

We now show the mapping $(f,\mu) \to d_2G(f,\mu)$ is continuous for $(f,\mu) \in N(B\times C,p)$. Let $\varepsilon > 0$ be given and choose $\delta > 0$ by (3) so that $\mu_1,\mu_2 \in N(C,p)$ and $||\mu_1 - \mu_2||_C < \delta$ implies $||K_x(\mu_1)h - K_x(\mu_2)h||_D < \varepsilon||h||_C$. Let $(f_0,\mu_0) \in N(B\times C,p)$ and $(f,\mu) \in N(B\times C,p)$ with $||(f_0-f,\mu_0-\mu)||_{B\times C} = ||f_0-f||_B + ||\mu_0-\mu||_C < \delta$. Then $||\mu_0 - \mu||_C < \delta$ and thus for $h \in C$

$$
\begin{aligned}
||d_2G(f,\mu)h - d_2G(f_0,\mu_0)h||_D &= ||K_x(\mu)h-h+K_x(\mu_0)h+h||_D \\
&= ||K_x(\mu)h - K_x(\mu_0)h||_D \\
&< \varepsilon||h||_C.
\end{aligned}
$$

This completes the proof of the lemma.

<u>Lemma 2.6</u> Assume (H1) - (H3) and let B,C,D be Banach spaces contained in $C_c(R^+,R^n)$ with topologies stronger than that of $C_c(R^+,R^n)$ and B contained in D with topology stronger than that of D. If there is a number $p > 0$ such that

(1) $G(f,\mu) \in D$ for all $(f,\mu) \in N(B\times C,p)$

(2) $\mu \in N(C,p)$ and $h \in C$ imply $K_x(\mu)h - h \in D$.

(3) For every $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that $\mu_1,\mu_2 \in N(C,p)$ and $||\mu_1 - \mu_2||_C < \delta_\varepsilon$ implies $||K_x(\mu_1)h - K_x(\mu_2)h||_D < \varepsilon||h||_C$ for all $h \in C$. Then G is continuously Fréchet differentiable for all $(f,\mu) \in N(B\times C,p)$. For $(h_1,h_2) \in B\times C$

$$dG(f,\mu)(h_1,h_2) = d_1G(f,\mu)h_1 + d_2G(f,\mu)h_2.$$

Proof: By Lemmas 2.4 and 2.5 G possesses continuous Fréchet partial derivatives in each variable for all $(f,\mu) \in N(B\times C,p)$. The result follows by an application of Theorem 1.1.

Theorem 2.7 Assume the hypotheses of Lemma 2.6 hold and let $R_0(t,s)$ be the resolvent kernel for $K_x(t,s,0)$, and let R_0 denote the operator $R_0\mu(t) = \int_0^t R_0(t,s)\mu(s)ds$. If for all $\mu \in D$, $\mu - R_0\mu \in C$ then there exist positive constants α and β and a function $T:N(B,\alpha) \to N(C,\beta)$ with the following properties:

(C1) The pair $(f,T(f))$ is a solution to the equation

$$G(f,x) = 0$$

for every $f \in N(b,\alpha)$, and there is no other solution with the same f having $x \in N(C,\beta)$.

(C2) The differential $d_2G(f,T(f))$ is a linear homeomorphism mapping C into D for every $f \in N(B,\alpha)$.

(C3) The function T is continuously Fréchet differentiable on $N(B,\alpha)$, and for each $f \in N(B,\alpha)$ and $h \in B$ $dT(f)h = y$ is the solution to the equation

$$\text{(VE)} \qquad y = h + K_x(T(f))y$$

Proof: By Lemma 2.6 G is continuously Fréchet differentiable on $N(B\times C,p)$. Let T_k and T_R be defined by

$$T_K\mu = K_x(0)\mu - \mu$$

and

$$T_R\mu = R_0\mu - \mu$$

T_K and T_R are continuous mappings from $C_c(R^+,R^n)$ into itself since $K_x(t,s,0)$ and $R_0(t,s)$ are continuous. By hypothesis $T_K(C) \subseteq D$ and $T_R(D) \subseteq C$. It is an immediate consequence of Lemma 1.8 that T_K is a linear homeomorphism of C onto D. By Lemma 2.5 $T_K = d_2G(0,0)$, and by hypothesis $G(0,0) = 0$. By the Hildebrandt-Graves implicit function theorem (Theorem 1.2) (C1) and (C2) follow immediately, as does the first part of (C3). To prove the second part, by Theorem 1.2

$$dT(f) = -[d_2G(f,T(f))]^{-1} \circ d_1G(f,T(f))$$

so that

$$d_2G(f,T(f)) \circ dT(f) = -d_1G(f,T(f))$$

and for $h \in C$

$$d_2G(f,T(f))dT(f)h = -d_1G(f,T(f))h$$

using Lemma 2.4 and letting $y = dT(f)h$ we have

$$d_2G(f,T(f))y = -h$$

and

$$K_x(T(f))y - y = -h$$

by Lemma 2.5.

Therefore

$$y = h + K_x(T(f))y$$

as claimed. This completes the proof of the theorem.

According to Theorem 2.7 for each $f \varepsilon N(B,\alpha)$ there is a unique $x \varepsilon S(C,\beta)$ such that $x = f + K(x)$. However, this does not guarantee that there is not another solution y lying outisde the ball $N(C,\beta)$. That this cannot happen is essentially because K satisfies locally a Lipschitz condition in x.

<u>Theorem 2.8</u> Assume the hypotheses of Theorem 2.7 and let $\alpha > 0$ and $\beta > 0$ and T be the numbers and the function whose existence is guaranteed by that theorem. Then for each $f \varepsilon N(B,\alpha)$ the solution $T(f) \varepsilon C$ is the only continuous solution to equation (E).

Proof: Let $x = T(f)$ and suppose that $y(t)$ is a solution defined on an interval $[0,t_1)$ with $x(t) \neq y(t)$ for some $t \varepsilon [0,t_1)$. Let $t_0 = \inf \{t\varepsilon[0,t_1) | x(t) \neq y(t)\}$. Then since $y(0) = x(0)$ and x and y are continuous it follows that $y(t) = x(t)$ for $0 \leq t \leq t_0$ and $t_0 < t_1$. Thus there is a $\delta > 0$ such that $y(t)$ is continuous on $[0,t_0+\delta]$, with $t_0 + \delta < t_1$. By hypothesis $K(t,s,\mu)$ is continuously differentiable in μ for all $\mu \in R^n$ and $(t,s) \in \Delta[0,\infty)$.

Let

$$q = \sup_{0\le t\le t_0+\delta} ||x(t)|| + \sup_{0\le t\le t_0+\delta} ||y(t)||$$

then the set

$$F = \{(t,s,\mu) |\ 0\le s\le t\le t_0+\delta \text{ and } ||\mu|| \le q\}$$

is compact.

Let

$$M = \sup_{(t,s,\mu)\varepsilon F} ||K_x(t,s,\mu)||$$

We have, for $t \varepsilon [0,t_0+\delta]$:

$$||x(t)-y(t)|| = ||\int_0^t [K(t,s,x(s))-K(t,s,y(s))ds||$$

$$= ||\int_0^t \int_0^1 K_x(t,s,\lambda x(s)+(1-\lambda)y(s))$$

$$\cdot(x(s)-y(s))d\lambda ds||$$

$$\le \int_0^t \int_0^1 ||K_x(t,s,\lambda x(s)+(1-\lambda)y(s))||$$

$$\cdot||x(s)-y(s)||d\lambda ds$$

$$\le \int_0^t M||x(s) - y(s))||ds, \quad t \varepsilon [0,t_0)\delta]$$

By Gronwall's inequality we obtain $||x(t) - y(t)|| = 0$ for for $t \varepsilon [0,t_0+\delta]$. This contradicts the assumption that

$x(t) \neq y(t)$ for some $t \in [0,t_1)$. Therefore there is only one continuous solution to (E) and it is given by $T(f)$ and lies in the Banach space C.

It follows from Theorem 2.4 that equation (E) is stable as the following theorem indicates.

Theorem 2.9 Assume the hypotheses of Theorem 2.7. For each $\varepsilon > 0$ there corresponds a $\delta_\varepsilon > 0$ such that if $f \in B$ with $||f||_B < \delta_\varepsilon$ then a unique solution x_f to (E) exists and $x_f \in C$ with $||x_f||_C < \varepsilon$.

Proof: Let α, β be the numbers whose existence is guaranteed by Theorem 2.7 and let T be the function whose existence also is guaranteed by that theorem. Then $T:N(B,\alpha) \to N(C,\beta)$ is continuous. Since $T(0) = 0$ (by the uniqueness part of Theorem 2.4) there is a δ_ε, $0 < \delta_\varepsilon \leq \alpha$ such that if $f \in B$ and $||f||_B < \delta_\varepsilon$, $x_f = T(f) \in C$ and $||x_f||_C < \varepsilon$. By Theorem 2.8 x_f is the unique solution of (E). This proves the theorem.

A differential or integral equation which is satisfied to within terms of the first order by the difference between two neighboring solutions of a given differential or integral equation is called the variational equation of the original equation. (Gel'fand and Fomin [11], p. 113). For this reason equation (VE) is called the variational equation of equation (E), as the following theorem justifies.

<u>Theorem 2.10</u> Assume the hypotheses of Theorem 2.7, and let $\alpha > 0$ and $\beta > 0$ be the numbers whose existence is guaranteed by that theorem. If $f \varepsilon N(B,\alpha)$, $h \varepsilon B$ and y is the solution of

$$\text{(VE)} \qquad y = h + K_x(x_f)y$$

then for each $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that whenever $||h||_B < \delta_\varepsilon$ then $y \varepsilon C$ and

$$||x_{f+h} - x_f - y||_C \leq \varepsilon||h||_B.$$

Proof: Let $||h||_B < \alpha - ||f||_B$ so that $f + h \varepsilon N(B,\alpha)$. Then $x_{f+h} = T(f+h)$, $x_f = T(f)$, and $y = dT(f)h$ by Theorem 2.7. Thus, if $||h||_B$ is sufficiently small

$$||x_{f+h} - x_f - y||_C = ||T(f+h) - T(f) - dT(f)h||_C$$

$$\leq ||h||_B,$$

since $dT(f)$ is the Fréchet derivative of $T(f)$.

Thus the variational equation (VE) serves as a linearization of the original equation (E). Another result along this line is

<u>Theorem 2.11</u> Assume the hypotheses of Theorem 2.7. Let ε be any positive number. There exist numbers $\delta > 0$ and $\eta > 0$ such that $f \varepsilon N(B,\delta)$ and $g \varepsilon B$ with $||f - g||_B < \eta$ imply a unique solution $x_f \varepsilon C$ of

(E) $$x = f + K(x)$$

and a unique solution $y \in C$ of

(VE) $$y = g + K_x(o)y$$

exist and satisfy

$$||x_f - y||_C < \varepsilon.$$

Proof: Assume $f \in N(B,\alpha)$ then x_f and y exist and are unique by Theorems 2.7 and 2.8, and x_f, $y \in C$. Also if $||g||_B < \alpha$:

$$\begin{aligned} ||x_f - y||_C &= ||T(f) - dT(o)g||_C \\ &= ||T(f) - dT(o)f + dT(0)f - dT(0)g||_C \\ &\leq ||T(f) - dT(0)f||_C + ||dT(0)(f-g)||_C \end{aligned}$$

We may choose δ, $0 < \delta \leq 1$, so that $||f||_B < \delta$ implies

$$||T(f) - dT(0)f||_C \leq \frac{\varepsilon}{2}||f||_B$$

by Theorem 2.4. Choose $\eta = \frac{1}{2}\varepsilon||dT(0)||^{-1}$. Then $||f||_B < \delta$ and $||f - g||_B < \eta$ imply

$$||T(f) - dT(0)f||_C + ||dT(0)(f-g)||_C$$

$$\leq \frac{1}{2}\varepsilon||f||_B + ||dT(0)||\ ||f - g||_B$$

$$\leq \tfrac{1}{2}\varepsilon + ||dT(0)|| \cdot \tfrac{1}{2}\varepsilon||dT(0)||^{-1} = \varepsilon.$$

This proves the theorem.

We will now particularize the results of Theorem 2.7 for several concrete pairs of Banach spaces B, C of general interest. First we will examine the situation in which B and C are contained in the space $BC(R^+,R^n)$ with topologies stronger than that of $BC(R^+,R^n)$. Examples of such spaces are $C^{\ell}(R^+;R^n)$, $C^0(R^+,R^n)$, $BCL(R^+,R^n)$, all discussed in Chapter One, and the space consisting of those functions in $BC(R^+,R^n)$ which possess first derivatives which lie in the same space. We denote this space by $BD(R^+,R^n)$ with norm $||\cdot||_{BD}$ given by $||x||_{BD} = ||x||_0 + ||x^1||_0$, where $x' = dx/dt$.

We will need the following additional hypothesis concerning the function $K(t,s,x)$ in (E).

(H4) There is a number $\bar{\omega} > 0$ such that for each $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ for which $x_1, x_2 \in R^n$ with $||x_1||$, $||x_2|| \leq \bar{\omega}$ and $||x_1 - x_2|| < \delta_\varepsilon$ imply

$$||K_x(t,s,x_1) - K_x(t,s,x_2)|| < \varepsilon\beta(t,s),$$

where $\beta(t,s)$ is a continuous function for $(t,s) \in \Delta[0,\infty)$, independent of x_1, x_2, and ε, such that

$$\sup_{t\geq 0} \int_0^t \beta(t,s)ds = \beta^* < +\infty.$$

Theorem 2.12 Let B and D be Banach spaces contained in $BC(R^+,R^n)$ with topologies stronger than that of $BC(R^+,R^n)$; also B is contained in D with stronger topology. Assume (H1)-(H4) (with $\omega = +\infty$ in (H2)).

(1) $||x||_D = \sup\limits_{t\geq 0} ||x(t)||$

(2) There is a number $r > 0$ such that for all $\mu \in N(B,r)$ and for all $h \in B$, $K(\mu) \in D$ and $K_x(\mu)h \in D$.

(3) Let R_0 denote the resolvent for $K_x(0)$. Then $h - R_0h \in B$ for all $h \in D$.

Then there exist numbers $\alpha > 0$ and $\beta > 0$ and a function $T:N(B,\alpha) \to N(B,\beta)$ with the following properties:

(1) $T(f) = x_f$ is a solution to (E), and there is no other solution for this f.

(2) The function T is continuously Fréchet differentiable on $N(B,\alpha)$ and for each $f \in N(B,\alpha)$ and $h \in B$, $dT(f)h = y$ is the solution of the equation

$$y = h + K_x(T(f))y.$$

Proof: Since B has a topology stronger than that of $BC(R^+,R^n)$, there is a constant $m > 0$ such that $\mu \in B$ implies $\sup\limits_{t\geq 0} ||\mu(t)|| = ||\mu||_0 \leq m||\mu||_B$.

Let $\varepsilon > 0$ be given and let $\varepsilon^* = \varepsilon/m\beta^*$, β^* being the number in (H4), and choose δ for this ε^* as in (H4), and define $\delta^* = \delta/m$.

Let $P^* = \bar{\omega}/m$ and let $\mu_1, \mu_2 \in N(B, P^*)$ with $||\mu_1 - \mu_2||_B < \delta^*$. Then for all $t \geq 0$ and $i = 1,2$

$$||\mu_i(t)|| \leq m||\mu_i||_B < mP^* = \omega$$

and

$$||\mu_1(t) - \mu_2(t)|| \leq m||\mu_1 - \mu_2||_B < m\delta^* = \delta.$$

Therefore by (H4) we have

$$||K_x(t,s,\mu_1(s)) - k_x(t,s,\mu_2(s))|| < \varepsilon^*\beta(t,s).$$

Take $r_0 = \min(r, P^*)$. Then for $\mu_1, \mu_2 \in N(B, r_0)$ we have $k_x(\mu_1)h$, $k_x(\mu_2)h \in D$ for all $h \in B$ and

$$||K_x(\mu_1)h - K_x(\mu_2)h||_D$$

$$= \sup_{t\geq 0} ||K_x(\mu_1)h(t) - k_x(\mu_2)h(t)||$$

$$= \sup_{t\geq 0} ||\int_0^t (K_x(t,s,\mu_1(s)) - K_x(t,s,\mu_2(s)))h(s)ds||$$

$$\leq \sup_{t\geq 0} ||\int_0^t ||K_x(t,s,\mu_1(s)) - K_x(t,s,\mu_2(s))||ds \cdot \sup_{t\geq 0}||h(t)||$$

$$\leq \sup_{t\geq 0} \int_0^t \varepsilon^*\beta(t,s)ds \cdot ||h||_0$$

$$\leq \varepsilon^*\beta^* m||h||_B = \varepsilon||h||_B.$$

This shows that (3) of Lemma 2.6 is satisfied. We now establish (1) and (2) of that lemma.

Let $(f,\mu) \in B \times B$ with $||(f,\mu)|| = ||f||_B + ||\mu||_B < r$. Then $G(f,\mu) = f + k(\mu) - \mu \in D$ since B is contained in D and $K(B) \subsetneq D$ by (2). This proves (1) of Lemma 2.6. Hypothesis (2) holds if $K_x(\mu)h - h \in D$ for $\mu \in N(B,r_0)$ and $h \in B$. But this is immediate also from our hypotheses for this theorem.

Thus the hypotheses of Lemma 2.6 hold and this together with (3) of this theorem are all of the conditions for Theorem 2.7. Uniqueness follows by Theorem 2.8. This proves the theorem.

Remark: It is not essential to take $\omega = +\infty$ in (H2) for Theorem 2.12. This was done for ease in application of Theorem 2.7. Since B is a space of bounded functions, it is easy to show that $k_x(\mu)$ would be defined for $\mu \in B$ even if $k_x(t,s,x)$ exists only for $||x|| < \omega < +\infty$. This is the reason ω was taken to be $+\infty$ in Theorem 2.7, that we make no assumption concerning the boundedness of the functions in the spaces B or C.

We now present some particular cases of the last result which may be of general interest.

<u>Corollary 2.13</u> Let (H1)-(H4) hold, with ω in (H2) taken to equal $\tilde{\omega}$ in (H4).

(1) There is a number $r > 0$ and a continuous function $m(t,s)$ such that for $\mu \in R^n$ and $||\mu|| < r$

$$||K_x(t,s,\mu)|| \leq m(t,s)$$

and

$$\sup_{t \geq 0} \int_0^t m(t,s)ds = m^* < +\infty.$$

(2) $R_0(t,s)$ denotes the integral resolvent of $K_x(t,s,0)$.

$$\sup_{t \geq 0} \int_0^t ||R_0(t,s)||ds = R^* < +\infty.$$

Then the conclusions of Theorem 2.12 hold, taking the spaces B and D to be the space of bounded continuous functions, $BC(R^+,R^n)$.

Proof: We establish the hypotheses of Theorem 2.12. We have explicitly assumed (H1)-(H4), and (1) of that theorem holds trivially. As for (2): let $\mu \in BC(R^+,R^n)$. Then if $||\mu||_0 < \min(\bar{\omega},r)$ where $\bar{\omega}$ is the constant in (H4) we have

$$||K(\mu)(t)|| = ||\int_0^t K(t,s,\mu(s))ds||$$

$$\leq \int_0^t || \int_0^1 K_x(t,s,\lambda\mu(s))\mu(s)d\lambda||ds$$

$$\leq \int_0^t \int_0^1 ||K_x(t,s,\lambda\mu(s))||d\lambda ds||\mu||_0$$

$$\leq \int_0^t m(t,s)ds||\mu||_0 \leq m^*||\mu||_0.$$

This together with the continuity of $K(t,s,\mu)$ implies that $K(\mu) \in BC(R^+,R^n)$.

Now let $h \in BC(R^+,R^n)$, and μ as before. Then $K_x(\mu)h \in C_c(R^+,R^n)$ by the continuity of $K_x(t,s,\mu(s))$ and $h(s)$, and

$$||K_x(\mu)h(t)|| = ||\int_0^t K_x(t,s,\mu(s))h(s)ds||$$

$$\le m^*||h||_0$$

so that $K_x(\mu)h$ is also bounded.

For (3) of Theorem 2.12, let $\mu \in BC(R^+,R^n)$. Then since $K_x(t,s,0)$ is continuous for $(t,s) \in \Delta[0,+\infty)$, so is its resolvent $R_0(t,s)$ and thus $\mu - R_0\mu$ is continuous. Also

$$||\mu(t) - R_0\mu(t)|| = ||\mu(t) - \int_0^t R_0(t,s)\mu(s)ds||$$

$$\le ||\mu||_0 + R^*||\mu||_0$$

and (3) is established. The result follows immediately.

The next result provides conditions for the existence and Fréchet differentiability of bounded solutions of (E) which tend to 0 as $t \to +\infty$.

<u>Corollary 2.14</u>. Let (H1) - (H4) hold with ω in (H2) equal to $\bar{\omega}$ in (H4) (possibly finite).

(1) There is a continuous function $m(t,s)$ such that for $(t,s) \in \Delta[0,\infty)$ and $\mu \in R^n$ with $||\mu|| < \omega$

$$||K_x(t,s,\mu)|| < m(t,s)$$

where

$$\sup_{t\geq 0} \int_0^t m(t,s)ds = m^* < \infty$$

and for all finite $T > 0$,

$$\lim_{t\to\infty} \int_0^T m(t,s)ds = 0.$$

(2) Let $R_0(t,s)$ denote the integral resolvent of $K_\chi(t,s,0)$. Then

$$\sup_{t\geq 0} \int_0^t ||R_0(t,s)||ds = R^* < \infty$$

and for all finite $T > 0$

$$\lim_{t\to\infty} \int_0^T ||R_0(t,s)||ds = 0.$$

Then the conclusions of Theorem 2.12 hold with $C^0(R^+,R^n)$ in place of the spaces B and D.

<u>Proof</u>: It is sufficient to show K maps $N(C^0,\omega)$ into C^0, $K_\chi(\mu)$ maps C^0 into C^0 if $\mu \in N(C^0,\omega)$, and the operator R given by

$$R(\mu)(t) = \int_0^t R_0(t,s)\mu(s)ds$$

maps C^0 into itself. It then follows as before that $G(f,\mu)$ maps $N(C^0 \times C^0,\omega)$ into C^0 and is continuously Fréchet differentiable, $d_2G(0,0)$ being a linear homeomorphism of C^0 onto itself.

Let $\mu \in C^0(R^+,R^n)$, $||\mu||_0 < \omega$, $\mu \neq 0$. Then $||\mu(s)|| < \omega$ for all $s \geq 0$ and

$$
\begin{aligned}
||K(\mu)(t)|| &= ||\int_0^t K(t,s,\mu(s))ds|| \\
&\le \int_0^t ||K(t,s,\mu(s))\ ds \\
&= \int_0^t ||\int_0^1 K_\chi(t,s,\lambda\mu(s))\mu(s)d\lambda||ds \\
&\le \int_0^t m(t,s)||\mu(s)||ds < m^*||\mu||_0
\end{aligned}
$$

so that $K(\mu)$ is bounded. $K(\mu)(t)$ is continuous as in the previous results. We must show $\lim_{t\to\infty}||K(\mu)(t)|| = 0$. Let $\varepsilon > 0$ be given. Then there exists a number $T > 0$ such that $t > T$ implies $||\mu(t)|| < \varepsilon/2m^*$. There also exists a number $T_1 > 0$ such that $t > T_1$ implies

$$\int_0^T m(t,s)ds < \frac{\varepsilon}{2||\mu||_0}.$$

Let $t > \max(T,T_1)$; then

$$
\begin{aligned}
||K(\mu)(t)|| &\le \int_0^t m(t,s)||\mu(s)||ds \\
&= \int_0^T m(t,s)||\mu(s)||ds + \int_T^t m(t,s)||\mu(s)||ds \\
&\le \int_0^T m(t,s)ds||\mu||_0 + \int_T^t m(t,s)\varepsilon/(2m^*)ds \\
&< \varepsilon/(2\cdot||\mu||_0) \cdot ||\mu||_0 + m^*\varepsilon/(2m^*) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus $K(\mu) \varepsilon C^0$. Since for all $\mu \varepsilon C^0$ with $||\mu|| < \omega$ and for all $h \varepsilon C^0$

$$||K_X(\mu)h(t)|| = ||\int_0^t K_X(t,s,\mu(s))h(s)ds||$$

$$\leq \int_0^t m(t,s)||h(s)||ds$$

A similar argument shows $K_X(\mu)h \in C^0$. Similarly the mapping $v(t) \to v(t) - \int_0^t R_0(t,s)v(s)ds$ takes C^0 into itself, by (2). The result follows.

We now consider the case in which we seek solutions to (E) in $C^{\ell}(R^+,R^n) = C^{\ell}$. If $x \in C^{\ell}$ we denote $\lim_{t\to\infty} x(t)$ by ℓ_x.

<u>Corollary 2.15</u>. We assume (H1) - (H4) and take $\omega = \overline{\omega}$, where ω and $\overline{\omega}$ are the numbers in (H2) and (H4), respectively, and (1), (2) of Corollary 2.14 and

(a) For $x \in R^n$, $||x|| < \omega$, there exists a vector $\overline{K}(x) \in R^n$ and $n\times n$ real matrix $M(x)$ such that

$$\lim_{t\to\infty} \int_0^t K(t,s,x)ds = \overline{K}(x)$$

$$\lim_{t\to\infty} \int_0^t K_X(t,s,x)ds = M(x).$$

(b) Let $R_0(t,s)$ denote the integral resolvent of $K_X(t,s,0)$. There exists an $n\times n$ real matrix $\overline{R}$ such that

$$\lim_{t\to\infty} \int_0^t R_0(t,s)ds = \overline{R}.$$

Then the conclusions of Theorem 2.12 hold for $B = D = C^{\ell}$.

<u>Proof</u>: Since C^{ℓ} is a subspace of $BC(R^+,R^n)$ with stronger topology, as in Corollary 2.14, it is sufficient to show that the nonlinear integral operation K maps $N(C^{\ell},\omega)$ into

C^{ℓ}, $K_X(\mu)$ maps C^{ℓ} into itself if $\mu \in N(C^{\ell},\omega)$, and the resolvent maps C^{ℓ} into C^{ℓ}.

As in Corollary 2.14, $K(\mu) \in BC(R^+,R^n)$ for all $\mu \in N(C^{\ell},\omega)$. Let $m(t,s)$ be the function in (1) of Corollary 2.14, and let $\mu \in N(C\ ,\omega)$. Choose $T_1 > 0$ so that $t > T_1$ implies $||\mu(t) - \ell_\mu|| < \varepsilon/(3m^*)$. Let T_2 be such that $t > T_2$ implies $||\int_0^t K(t,s,\ell_\mu)ds - \overline{K}(\ell_\mu)|| < \varepsilon/3$. Finally choose $T_3 = \max(T_1,T_2)$ and choose $T_4 > T_3$ so that

$t > T_4$ implies $\int_0^{T_3} m(t,s)ds||\mu - \ell_\mu|| < \varepsilon/3$.

Then for $t > T_4$ we have

$$||\int_0^t K(t,s,\mu(s))ds - \overline{K}(\ell_\mu)||$$

$$= ||\int_0^t K(t,s,\mu(s))ds - \int_0^t K(t,s,\ell_\mu)ds$$

$$+ \int_0^t K(t,s,\ell_\mu)ds - \overline{K}(\ell_\mu)||$$

$$\le ||\int_0^{T_3} (K(t,s\ \mu(s)) - K(t,s,\ell_\mu))ds$$

$$+ ||\int_{T_3}^t (K(t,s,\mu(s)) = K(t,s,\ell_\mu))ds$$

$$+ ||\int_0^t K(t,s,\ell_\mu)ds - \overline{K}(\ell_\mu)||$$

$$\le \int_0^{T_3} m(t,s)ds||\mu-\ell_\mu||$$

$$+ \int_{T_3}^t m(t,s)||\mu(s) - \ell_\mu||ds + \varepsilon/3$$

$\leq \varepsilon/3 + m^*\varepsilon/(3m^*) + \varepsilon/3 = \varepsilon.$

Thus $\lim_{t\to\infty} K(\mu)(t) = \overline{K}(\ell_\mu)$ and $K(\mu) \in C^\ell$.

Now let $\mu \in C^\ell$, $||\xi||_0 < \omega$, and $h \in C^\ell$. We show that

$$\lim_{t\to\infty} K_\chi(\mu)h(t) = M(\ell_\mu)\ell_h.$$

Let $\varepsilon > 0$ be given. Then

$$||K_\chi(\mu)h(t) - M(\ell_\mu)\ell_h||$$

$$= ||K_\chi(\mu)h(t) - K_\chi(\mu)(t)\cdot\ell_h + K_\chi(\mu)(t)\cdot\ell_h$$

$$- K_\chi(\ell_\mu)(t)\ell_h + K_\chi(\ell_\mu)(t)\ell_h - M(\ell_\mu)\ell_h||.$$

Choose $T_1 > 0$ so that $t > T_1$ implies

$$||\int_0^t K_\chi(t,s,\ell_\mu)ds - M(\ell_\mu)||\cdot||\ell_h|| < \varepsilon/5$$

Choose T_2 so that $t > T_2$ implies $||\mu(t) - \ell_\mu|| < \delta^*$ where $\delta^* > 0$ is such that $||\mu_1 - \mu_2|| < \delta^*$, $||u_1||, ||\mu_2|| < \omega$ implies $||\ell_h||\cdot||K_\chi(t,s,\mu_1) - K_\chi(t,s,\mu_2)|| < \beta(t,s)\cdot\varepsilon/(5\cdot\beta^*)$, as guaranteed by (H4). Choose $T_3 > \max(T_1,T_2)$ so that $t > T_3$ implies

$$2\int_0^{T_2} m(t,s)ds||\ell_h|| < \varepsilon/5.$$

Let T_4 be such that $t > T_4$ implies $||h(t) - \ell_h|| < \varepsilon/(5m^*)$. Finally, choose $T_5 > T_4$ so that $t > T_5$ implies

$$\int_0^{T_4} m(t,s)ds||h - \ell_\mu|| < \varepsilon/5.$$

Now let $T^* = \max(T_1, T_2, T_3, T_4, T_5)$. For $t > T^*$ we have:

$$||K_\chi(\mu)h(t) - M(\ell_\mu)\ell_h||$$

$$\le ||\int_0^{T_4} K_\chi(t,s,\mu(s))(h(s)-\ell_h)ds$$

$$+ ||\int_{T_4}^{t} K_\chi(t,s,\mu(s))(h(s)-\ell_h)ds$$

$$+ ||\int_0^{T_2}(K_\chi(t,s,\mu(s)) - K_\chi(t,s,\ell_\mu))\ell_h ds||$$

$$+ ||\int_{T_2}^{t}(K_\chi(t,s,\mu(s)) - K_\chi(t,s,\ell_\mu))\ell_h ds||$$

$$+ ||\int_0^{t} K_\chi(t,s,\ell_\mu)\ell_h ds - M(\ell_\mu)\ell_h||$$

$$< \int_0^{T_4} m(t,s)ds||h-\ell_h||_0 + \int_{T_4}^{t} m(t,s)||h(s) - \ell_h||ds$$

$$+ \int_0^{T_2}||K_\chi(t,s,\mu(s)) - K_\chi(t,s,\ell_\mu)||ds\cdot||\ell_h||$$

$$+ \int_{T_2}^{t}||K_\chi(t,s,\mu(s)) - K_\chi(t,s,\ell_\mu)||ds||\ell_h||$$

$$+ ||\int_0^{t} K_\chi(t,s,\ell_\mu)ds - M(\ell_\mu)||\cdot||\ell_h||$$

$$< \varepsilon/5 + m^* \varepsilon/(5m^*)$$

$$+ \int_0^{T_2} 2m(t,s)ds||\ell_h||$$

$$+ \int_{T_2}^{t} \beta(t,s)ds\ \varepsilon/(5m^*) + \varepsilon/5$$

$$< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon.$$

Thus $K_x(\mu)$ maps $C^{\ell}(R^+,R^n)$ into itself, for $\mu \in C^{\ell}$ with $||\mu|| < \omega$. By a similar argument, R maps C^{ℓ} into itself, where

$$R\mu(t) = \int_0^t R_0(t,s)\mu(s)ds.$$

The result follows by Theorem 2.12.

If $A \in L_1(R^+,M^n)$, M^n being the class of all nxn real matrices, then for all finite T 0

$$\lim_{t\to\infty} \int_0^T ||A(t-s)||ds = 0. \tag{2.15}$$

For $A \in L_1(R^+,M^n)$ let $\hat{A}$ denote the Laplace transforma- of A:

$$\hat{A}(s) = \int_0^\infty e^{-st}A(t)dt$$

Then a result due to Paley and Weiner [25] (see Miller [23], p. 407-409) states that if $A \in L_1(R^+,M^n)$ and $\mathrm{Det}[I-\hat{A}(s)] \neq 0$ for $\mathrm{Re}(s) \geq 0$, where I is the n×n identity matrix, the resolvent is also in $L_1(R^+,R^n)$. We have the following corollary to Corollary 2:

<u>Corollary 2.16</u> Assume $A \in BCL(R^+,M^n)$, $g \in C(R^n,R^n)$ such that $g(0) = 0$ and $g_x(\mu)$ exists and is continuous for $\mu \in N(R^n,\omega)$, $0 < \omega \leq \infty$, and

$$\mathrm{Det}[I-\hat{A}(s)g_x(0)] \neq 0 \tag{2.16}$$

for $\mathrm{Re}(s) \geq 0$.

Then there are numbers $\alpha,\beta > 0$ such that for each $f \in C^{\ell}(R^+,R^n)$ with $||f||_0 < \alpha$ there is a unique solution x_f to

$$x(t) = f(t) + \int_0^t A(t-s)g(x(s))ds$$

and $x_f \in C^{\ell}(R^+,R^n)$ with $||x||_0 < \beta$. Moreover, the mapping $T:f \to x_f$ is Fréchet differentiable and $dT(f) = y \in C^{\ell}$ is the solution of

$$y(t) = f(t) + \int_0^t A(t-s)g_x(x_f(s))y(s)ds$$

Proof: By (2.15) and (2.16) the special conditions of Corollary 2.15 are met. (H1)-(H4) are also readily verified. We omit any details.

We will now consider some cases in which the norm of the space D is not the sup norm. In these cases, Theorem 2.12 does not apply.

The first space we consider is $BCL(R^+,R^n)$, the continuous bounded integrable functions, with norm $||x||_+ = ||x||_0 + \int_0^\infty ||x(t)||dt$. We will not use (H4) but will need

(H5) There is a number $\omega > 0$ such that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $x_1,x_2 \in R^n$ with $||x_1||,||x_2|| < \omega$ and $||x_1 - x_2|| < \delta$ imply

$$||K_x(t,s,x_1) - K_x(t,s,x_2)|| \le \varepsilon F(t-s),$$

where $F \varepsilon BCL(R^+,R^+)$, and is independent of x_1,x_2, and ε.

Also we will use

(H6) Let ω be as in (H5). Then for $\mu \varepsilon R^n$, $||\mu|| < \omega$,

$$||K_x(t,s,\mu)|| \leq H(t-s)$$

where $H \varepsilon BCL(R^+,R^+)$.

We begin by showing that the hypotheses of Lemma 2.6 hold.

Lemma 2.17 Assume (H1)-(H3) and (H5)-(H6), with ω in (H2) the same as that in (H5)-(H6) ($\leq \int_0^t \int_0^1 ||K_x(t,s,\lambda\mu(s))\mu(s)||d\lambda ds$ $\mu \varepsilon BCL(R^+,R^n) = BCL$ with $||\mu||_+ < \omega$ and for all $h \varepsilon BCL$, $K(\mu) \varepsilon BCL$ and $K_x(\mu)h \varepsilon BCL$.

Proof: $K(\mu)$ is a continuous function by (H1). Since $||\mu||_+ = ||\mu||_0 + \int_0^\infty ||\mu(t)||dt$ we have $||\mu(t)|| < \omega$ for $t \varepsilon R^+$ if $||\mu||_+ < \omega$. Thus

$$\begin{aligned} ||K(\mu)(t)|| &\leq \int_0^t ||K(t,s,\mu(s))||ds \\ &\leq \int_0^t \int_0^1 ||K_x(t,s,\lambda\mu(s))\mu(s)||d\lambda ds \\ &\leq \int^t G(t-s)ds||\mu||_0 \\ &\leq \int_0^\infty G(s)ds||\mu||_+ < +\infty. \end{aligned}$$

and $K(\mu) \varepsilon BC$.

It is well known that the Laplace convolution of two L_1 functions is again an L_1 function. Thus

$$\int_0^\infty ||K(\mu)(t)||dt \le \int_0^\infty \int_0^t ||K(t,s,\mu(s))||dsdt$$

$$\le \int_0^\infty \int_0^t G(t-s)||\mu(s)||ds$$

$$= \int_0^\infty G*||\mu||(s)ds < +\infty$$

where * denotes the convolution. We have, then $K(\mu) \in BCL(R^+,R^n)$. By a very similar argument $K_x(\mu)h \in BCL$.

<u>Lemma 2.18</u> Assume (H1)-(H3) and (H5)-(H6), taking the number ω in each of (H2), (H5), and (H6) to be the same. Then for each $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that $\mu_1,\mu_2 \in BCL$, $||\mu_1||_+,||\mu_2||_+ < \omega$ and $||\mu_1 - \mu_2||_+ < \delta_\varepsilon$ imply

$$||K_x(\mu_1)h - K_x(\mu_2)h||_+ < \varepsilon||h||_+$$

for all $h \in BCL$.

Proof: Let $\varepsilon > 0$ be given, and let $\varepsilon^* = \varepsilon/\left(2\int_0^\infty F(s)ds\right)$. Choose $\delta = \delta(\varepsilon^*)$ corresponding to the δ in (H5) for ε^*. If $\mu_1,\mu_2 \in BCL$ with $||\mu_1||_+,|||\mu_2||_+ < \omega$ and $||\mu_1 - \mu_2||_+ < \delta(\varepsilon^*)$ we have $||\mu_1(t)||$, $||\mu_2(t)|| < \omega$ and $||\mu_1(t) - \mu_2(t)|| < \delta(\varepsilon^*)$ for all $t \ge 0$. Thus for $h \in BCL$

$$\left\|\int_0^t K_x(t,s,\mu_1(s))h(s))ds - \int_0^t K_x(t,s,\mu_2(s))h(s)ds\right\|$$

$$\leq \int_0^t \|K_x(t,s,\mu_1(s)) - K_x(t,s,\mu_2(s))\| \ \|h(s)\|ds$$

$$\leq \int_0^t \varepsilon^* F(t-s)\|h(s)\|ds$$

$$\leq \varepsilon^* \int_0^\infty F(s)ds\|h\|_+ = \frac{\varepsilon}{2} \|h\|_+.$$

Thus

$$\sup_{t\geq 0} \|K_x(\mu_1)h(t) - K_x(\mu_2)h(t)\| \leq \frac{\varepsilon}{2} \|h\|_+.$$

Furthermore,

$$\int_0^\infty \left\|\int_0^t [K_x(t,s,\mu_1(s)) - K_x(t,s,\mu_2(s))]h(s)ds\right\|dt$$

$$\leq \int_0^\infty \int_0^t \varepsilon^* F(t-s)\|h(s)\|dsdt$$

$$= \varepsilon^* \int_0^\infty \int_s^\infty F(t-s)dt\|h(s)\|ds$$

$$= \varepsilon^* \int_0^\infty \int_0^\infty F(y)dy\|h(s)\|ds$$

$$= \varepsilon^* \int_0^\infty F(y)dy \int_0^\infty \|h(s)\|ds$$

$$= \frac{\varepsilon}{2} \int_0^\infty \|h(s)\|ds \leq \frac{\varepsilon}{2}\|h\|_+.$$

Combining these two results we have

$$||K_x(\mu_1)h - K_x(\mu_2)h||_+ \le \frac{\varepsilon}{2}||h||_+ + \frac{\varepsilon}{2}||h||_+ = \varepsilon||h||_+.$$

This proves the lemma.

<u>Theorem 2.19</u> Assume (H1)-(H3) and (H5)-(H6) taking ω to be the same in (H2) and (H5)-(H6). Let $R_0(t,s)$ be the resolvent kernel for $K_x(t,s,0)$. If there are finite numbers r_1 and r_2 such that

$$\sup_{t\ge 0} \int_0^t ||R_0(t,s)||ds = r_1 \tag{2.17}$$

and

$$\sup_{s\ge 0} \int_s^\infty ||R_0(t,s)||dt = r_2$$

then the conclusions of Theorem 2.7 hold with $BCL(R^+,R^n)$ in place of each of the spaces B,C, and D.

Proof: Although in Theorem 2.7 it is assumed that $K_x(t,s,\mu)$ exists and is continuous for all $\mu \in R^n$, that assumption is only used to ensure that $K_x(t,s,\mu(s))$ is defined and is continuous for all functions μ in a ball about the origin in the space C. Since the functions in the space considered here (BCL) are actually bounded, that assumption is not needed here.

By Lemma 2.17 $G(f,\mu) = f+K(\mu)-\mu \in$ BCL for all $(f,\mu) \in N(BCL,BCL,\omega)$ and $K_x(\mu)h - h \in$ BCL for all $\mu,h \in$ BCL with $||\mu||_+ < \omega$. Thus (1) and (2) of Lemma 2.6

are satisfied. Condition (3) of Lemma 2.6 is immediate from Lemma 2.18. By (2.16) for $\mu \in BCL(R^+,R^n)$ we have

$$\sup_{t\geq 0}\left|\left|\int_0^t R_0(t,s)\mu(s)ds\right|\right| \leq r_1||\mu||_+$$

and

$$\int_0^\infty\left|\left|\int_0^t R_0(t,s)\mu(s))ds\right|\right|dt \leq \int_0^\infty \int_0^t||R_0(t,s)\mu(s)||dsdt$$

$$= \int_0^\infty \int_s^\infty||R_0(t,s)||dt||\mu(s)||ds$$

$$\leq \int_0^\infty r_2||\mu(s)||ds \leq r_2||\mu||_+$$

so that for $\mu \in BCL$ the function given by

$$\mu(t) - \int_0^t R_0(t,s)\mu(s)ds$$

is also in BCL.

This establishes all the hypotheses of Theorem 2.7. The result follows immediately by that Theorem.

Let $D(t)$ be an invertible $n\times n$ matrix for each $t \geq 0$ such that the mapping $t \to D(t)$ is continuous. Since the mapping $A \to A^{-1}$ for invertible matrices is continuous, it follows that $t \to D^{-1}(t)$ is also continuous.

The space C_D with norm

$$||x||_D = \sup_{t\geq 0}||D^{-1}(t)x(t)||$$

was discussed in Chapter 1. Here we will provide conditions

For the existence and Fréchet differentiability of solutions to (E) in C_D for $f \in C_D$. We will place certain restrictions on the matrix $D(t)$ and the vector $K(t,s,\mu)$ as follows.

(H7) Let $K(t,s,x)$ be as in (H1)-(H3), ($\omega = +\infty$ in (H2)) with $K(t,s,x) = (k_1(t,s,x),\ldots,k_n(t,s,x))^T$. Then K satisfies

$$\frac{\partial ki}{\partial xj} = 0 \text{ for } i \neq j. \tag{2.18}$$

(H8) $D(t)$ is an $n\times n$ diagonal matrix: $D(t) = [d_{ij}(t)]$ with $d_{ij}(t) = 0$ for $i \neq j$ and $d_{ii}(t) \neq 0$ and continuous for $t \geq 0$, for $i = 1,2,\ldots,n$. and

$$\sup_{t\geq 0} \int_0^t ||D^{-1}(t)D(s)||ds = D^* < \infty.$$

(H9) There exists a function $M \in C([0,p),R^+)$, $p > 0$, such that

$$\sup_{t\geq 0} \int_0^t ||D^{-1}(t)K_x(t,s,\mu(s))D(s)||ds \leq M(||\mu||_D)$$

for all $\mu \in C_D$ with $||\mu||_D < p$.

(H10) For $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that for $\mu,h \in C_D$ with $||h||_D < \delta(\varepsilon)$ and $||\mu||_D < p$ the inequality

$$||k_x(t,s,\mu(s)+h(s)) - k_x(t,s,\mu(s))|| < \varepsilon$$

holds for $(t,s) \in \Delta[0,\infty)$. ($p$ here is taken to be the same as in (H9)).

Remark: If A and B are diagonal matrices $AB = BA$. Thus the matrices $D(t),D^{-1}(s)$, and $k_x(t,s,\mu)$ all commute with each other as a result of (H7) and (H8).

Lemma 2.20 Assume (H1)-(H3) and (H8)-(H9). Then for all $\mu \in N(C_D, p)$ and $h \in C_D$, $K(\mu) \in C_D$ and $K_x(\mu)h \in C_D$.

Proof: $K(\mu)$ is certainly a continuous function. Also

$$||D^{-1}(t)k(\mu)(t)|| = ||\int_0^t D^{-1}(t)K(t,s,\mu(s))ds||$$

$$= ||\int_0^t D^{-1}(t)\int_0^1 K_x(t,s,\lambda\mu(s))D(s)D^{-1}(s)\mu(s)d\lambda ds||$$

$$\leq \int_0^1 \int_0^t ||D^{-1}(t)K_x(t,s,\lambda\mu(s))D(s)||\cdot||D^{-1}(s)\mu(s)||dsd\lambda$$

$$\leq \int_0^1 M(\lambda||\mu||_D)d\lambda||\mu||_D < +\infty$$

since $M(r)$ is continuous for all $0 \leq r < p$, and $||\mu||_D < p$.

It follows that

$$\sup_{t\geq 0}||D^{-1}(t)K(\mu)(t)|| \leq \int_0^1 M(\lambda||\mu||_D)d\lambda \; ||\mu||_D$$

and $K(\mu) \in C_D$.

Also

$$||D^{-1}(t)K_x(\mu)h(t)||$$

$$= ||\int_0^t D^{-1}(t)K_x(t,s,\mu(s))h(s)ds||$$

$$\leq \int_0^t ||D^{-1}(t)K_x(t,s,\mu(s))D(s)||\cdot||D^{-1}(s)\mu(s)||$$

$$\leq M(||\mu||_D)||h||_D \text{ by (H9).}$$

This proves the lemma.

Lemma 2.21 Assume (H1)-(H3) and (H8)-(H10). For each $\varepsilon > 0$ there exists $\delta > 0$ such that $h \in C_D$ and $\mu_1, \mu_2 \in N(C_D, p)$ with $||\mu_1 - \mu_2||_D < \delta$ implies

$$||K_x(\mu_1)h - K_x(\mu_2)h||_D < \varepsilon ||h||_D$$

Proof: Let $\varepsilon > 0$ be given and let $\varepsilon^* = \varepsilon/D^*$, where D^* is the constant in (H8). Let $\delta^* = \delta(\varepsilon^*)$ be chosen as in (H10). By Lemma 2.20. $K(\mu+h) - K(\mu) - K_x(\mu)h \in C_D$ for all $\mu \in N(C_D, p)$ and $h \in C_D$. If $||h||_D < \delta^*$. We have

$$||K(\mu+h) - K(\mu) = K_x(\mu)h||_D$$

$$= \sup_{t \geq 0} ||\int_0^t D^{-1}(t)[K(t,s,\mu(s)+h(s)) - k(t,s,\mu(s))]$$

$$- K_x(t,s,\mu(s))h(s)]ds||$$

$$\leq \sup_{t \geq 0} \int_0^t ||D^{-1}(t)\int_0^1 [K_x(t,s,\mu(s)+\lambda h(s))$$

$$- K_x(t,s,\mu(s))]d\lambda h(s)||ds$$

$$= \sup_{t \geq 0} \int_0^t ||D^{-1}(t)D(s)\int_0^1 [K_x(t,s,\mu(s)+\lambda h(s))$$

$$- K_x(t,s,\mu(s))]d\lambda D^{-1}(s)h(s)||ds$$

since diagonal matrices commute. Since $||h||_D < \delta^*$ by (H8) and (H10) we have the last term above bounded by

$$D^* \ \varepsilon^*||h||_D = \varepsilon||h||_D.$$

This proves the lemma.

Theorem 2.22. Assume (H1)-(H3) and (H8)-(H10) hold, with $\omega = +\infty$ in (H2). Let $R_0(t,s)$ denote the resolvent kernel for $K_x(t,s,0)$

$$\text{(2.18)} \quad \sup_{t\geq 0} \int_0^t ||D^{-1}(t)R_0(t,s)D(s)||ds = R^* < \infty.$$

Then the conclusions of Theorem 2.7 hold, taking each of the spaces in Theorem 2.7 to be C_D.

Proof: The hypotheses of Lemma 2.6 follow immediately from Lemmas 2.20 and 2.21. The remaining hypothesis of Theorem 2.7 we must verify is that the mapping $\mu \to \mu - R_0\mu$ maps C_D into itself, where R_0 is again given by

$$R_0\mu(t) = \int_0^t R_0(t,s)\mu(s)ds.$$

This follows from (2.18) since if $\mu \in C_D$ then

$$||D^{-1}(t)\int_0^t R_0(t,s)\mu(s)ds||$$

$$= \int_0^t ||D^{-1}(t)R_0(t,s)D(s)D^{-1}(s)\mu(s)||ds$$

$$\leq \int_0^t ||D^{-1}(t)R_0(t,s)D(s)||ds \cdot \sup_{t\geq 0}||D^{-1}(t)\mu(t)||$$

$$\leq R^*||\mu||_D.$$

Thus $R_0\mu \in C_D$ and $\mu - R_0\mu \in C_D$ for all $\mu \in C_D$. The conclusions follow.

Perhaps the most interesting aspect of Theorem 2.22 is that functions in C_D may not be bounded. An example of a C_D space and a kernel $K(t,s,\mu)$ which satisfy the requirements of Theorem 2.22 will be given in Chapter 4, along with other examples of results in this chapter.

It should be noted that Theorems 2.9, 2.10, and 2.11 all apply to the special cases of Theorem 2.7 concerning the spaces $BC(R^+,R^n)$, $C^0(R^+,R^n)$, $C^\ell(R^+,R^n)$, $BCL(R^+,R^n)$ and C_D. It is clear that this approach can be used to provide sufficient conditions for the existence, (B,D) stability, and Fréchet differentiability of the solutions of (E) for many pairs of spaces (B,D), although the approach taken here has been generally to have B = D. In this manner one can obtain about solutions the sort of information which directly relates to the properties which characterize Banach spaces. Such properties as boundedness, tendency to zero, integrability, etc. are thus amenable to this approach, as has been demonstrated here. Some other desirable properties which the solutions might have, such as equiasymptotic stability, or equiboundedness (see Yoshizawa [27] for definitions) apparently cannot be studied by this approach.

CHAPTER THREE

In this chapter we will study the stability of equation (E) under perturbations. Let B be a Banach space of functions in $C_c(R^+,R^n)$ with stronger topology. For $q > 0$ suppose P is a function mapping $N(B,q)$ into B, with $P(0) = 0$, continuous but not necessarily Fréchet differentiable on $N(B,q)$. Then the equation of interest in this chapter is

$$\text{(P)} \qquad x(t) = f(t) + \int_0^t K(t,s,x(s))ds + P(x)(t)$$

where $f \in B$ and K satisfies (H1)-(H3) of Chapter 2. Writing as before $G(f,\mu) = f+K(\mu) -\mu$ equation (P) may be written in abstract form

$$(3.1) \qquad\qquad x = f + k(x) + P(x)$$

or equivalently

$$(3.2) \qquad\qquad G(f,x) = P(x).$$

1. Solutions and Stability of the Perturbed Equation by Contraction Mappings

Theorem 3.1. Assume (H1)-(H3) and let B be a Banach space contained in $C_c(R^+,R^n)$ with stronger topology. Assume further that there is a number $p > 0$ such that

(1) $K(\mu) \in B$ for all $\mu \in N(B,p)$.

(2) $\mu \in N(B,p)$ and $h \in B$ imply $K_x(\mu)h \in B$.

(3) For every $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that $\mu_1,\mu_2 \in N(B,p)$ and $||\mu_1 - \mu_2||_B < \varepsilon$ implies $||K_x(\mu_1)h - K_x(\mu_2)h||_B < \varepsilon||h||_B$ for all $h \in B$.

(4) Let $R_0(t,s)$ be the resolvent kernel for $K_x(t,s,0)$, and let R_0 denote the operator $R_0\mu(t) = \int_0^t R_0(t,s)\mu(s)ds$. For all $\mu \in B$, $R_0\mu \in B$.

(5) P maps $N(B,p)$ into B with $P(0) = 0$ and there is a number $\rho > 0$ such that

$$||P(\mu_1) - P(\mu_2)||_B \le \rho||\mu_1 - \mu_2||_B$$

for all $\mu_1,\mu_2 \in N(B,p)$.

(6) Let I denote the identity operator on B. Assume $\rho||I - R_0|| < 1$, where ρ is the number in (5).

Note: By (4), $R_0(B) \subseteq B$. Since $R_0(t,s)$ is continuous on $\Delta[0,\infty)$ it follows from Lemma 1.7 that R_0 is a bounded linear operator on B, and $||I - R_0||$ in (6) is defined, where the norm is the operator norm for B.

Conclusions: There exist numbers $m^* > 0$ and $\omega^* > 0$ such that for each $f \in N(B,m^*)$ there is a unique $x_f \in N(B,\omega^*)$ which solves (P). Moreover, x_f is a continuous function of changes in f, for $f \in B$.

Proof: Assumptions (1)-(4) immediately imply those of Theorem 2.7, taking the spaces B,C,D of that theorem all to be our space B. Thus there are numbers $\alpha > 0$ and $\beta > 0$ and a continuously Fréchet differentiable operator T mapping $N(B,\alpha)$ into $N(B,\beta)$ such that $G(f,T(f)) = 0$ for $f \in N(B,\alpha)$. Moreover $dT(f)h$ is the solution of $y = h + k_x(T(f))h$ so that $dT(0)h$ is the solution of $y = h + k_x(0)h$. But this is given by $y = h - R_0 h$ or

$$(3.3) \qquad dT(0)h = y = (I - R_0)h$$

so that $||dT(0)|| = ||I - R_0||$.

Let $f \in N(B,\alpha)$ and $\mu \in B$. Then $f + P(\mu) \in N(B,\alpha)$ if $||f + P(\mu)||_B \leq ||f||_B + \rho||\mu||_B < \alpha$ or

$$(3.4) \qquad ||\mu||_B < (\alpha - ||f||_B)/\rho.$$

Assume, then, that $\mu \in B$ satisfies (3.4). Then the equation

$$(3.5) \qquad x = f + P(\mu) + K(x)$$

has a solution $x_\mu = T(f+P(\mu))$ with $||x_\mu|| < \beta$. Define an operator $Q(\mu) = T(f+P(\mu))$ for μ satisfying (3.4); any $x \in B$ with $Q_f(x) = x$ and $||x||_B < \beta$ must therefore be a solution to (P), and the only solution with $||x||_B < \beta$, by the uniqueness part of Theorem 2.7.

Since $\rho||I - R_0|| = \rho||dT(0)|| < 1$ we may choose a number $\gamma_0 > 0$ such that $\rho||dT(0)|| + \rho\gamma_0 \equiv \lambda_0 < 1$. Since T is continuously Fréchet differentiable on $N(B,\alpha)$ there is a number δ_0, $0 < \delta_0 \le \alpha$ such that $f \in N(B,\delta_0)$ implies

$$(3.6) \qquad ||dT(f)|| < ||dT(0)|| + \gamma_0$$

Let f, $\mu \in B$ with $||f||_B < \frac{1}{2}\delta_0$ and $||\mu||_B < \frac{1}{2}\delta_0\rho^{-1}$. Then

$$||f + P(\mu)||_B < ||f||_B + \rho||\mu||_B$$

$$\le \frac{1}{2}\delta_0 + \rho\frac{1}{2}\delta_0\rho^{-1} = \delta_0.$$

Let

$$(3.7) \qquad \omega^* = \min(\alpha,\beta, \tfrac{1}{2}\delta_0\rho^{-1})$$

and let

$$(3.8) \quad m^* = \min\left(\tfrac{1}{2}\delta_0, (1-\lambda_0)\cdot\omega^*\cdot(||dT(0)||+\gamma_0)^{-1}\right)$$

Then for $f \in N(B,m^*)$ fixed, Q_f maps $N(B,\omega^*)$ into itself: for $\mu \in N(B,\omega^*)$, $||f + P(\mu)||_B \le \alpha$ and

$$||T(f+P(\mu))||_B = ||\int_0^1 dT(\lambda(f+P(\mu))\ (f+P(\mu))d\lambda||_B$$

where the integral is an abstract Riemann integral, which exists and satisfies the equality by Taylor's theorem for Banach space valued functions (Graves [13]). The last term above is bounded by

$$\int_0^1 ||dT(\lambda(f+p(\mu))||d\lambda||f + P(\mu)||_B$$

$$\le \int_0^1 (||dT(0)||+\gamma_0)d\lambda||f + P(\mu)||_B$$

since $||\lambda(f+\rho(\mu))||_B < \delta_0$. Thus

$$||Q_f(\mu)||_B \leq (||dT(0)|| + \gamma_0)(||f||_B + \rho||\mu||_B)$$

$$\leq (||dT(0)|| + \gamma_0)\Big[(1-\lambda_0)\omega^*(||dT(0)|| + \gamma_0)^{-1} + \rho||\mu||_B\Big]$$

$$= (1-\lambda_0)\omega^* + (||dT(0)|| + \gamma_0)\rho||\mu||_B$$

$$= (1-\lambda_0)\omega^* + \lambda_0||\mu||_B$$

$$\leq (1-\lambda_0)\omega^* + \lambda_0\omega^* = \omega^*.$$

Thus for $f \in N(B,m^*)$, $Q_f[N(B,\omega^*)]$ is contained in $N(B,\omega^*)$. Finally, Q_f is a contraction on $N(B,\omega^*)$:

$$||Q_f(\mu_1) - Q_f(\mu_2)||_B = ||T(f+P(\mu_1)) - T(f+P(\mu_2))||_B$$

$$= ||\int_0^1 dT[f+P(\mu_2)+\lambda(P(\mu_1) - P(\mu_2)]d\lambda \cdot [P(\mu_1)-P(\mu_2)]||$$

again by Taylor's theorem. Now

$$||f + P(\mu_2) + \lambda(P(\mu_1) - P(\mu_2))||_B \leq \delta_0$$

Thus we have

$$||Q_f(\mu_1) - Q_f(\mu_2)||_B \le (||dT(0)||+\gamma_0)||P(\mu_1) - P(\mu_2)||_B$$

$$\le (||dT(0)||+ \gamma_0)\rho||\mu_1 - \mu_2||_B$$

$$= \lambda_0||\mu_1 - \mu_2||_B.$$

Since $0 < \lambda_0 < 1$, Q is a contraction mapping on $N(B,\omega^*)$ for $f \in N(B,m^*)$, and Q_f has a unique fixed point $x \in N(B,\omega^*)$.

For the remainder of this proof we will denote the unique fixed point of Q_f in $N(B,\beta)$ by x_f. We now show x_f is a continuous function of $f \in N(B,m^*)$.

Let f, $g \in N(B,m^*)$.

$$||x_f - x_g||_B = ||Q_f(x_f) - Q_f(x_g) + Q_f(x_g) - Q_g(x_g)||_B$$

$$\le ||Q_f(x_f) - Q_f(x_g)||_B + ||Q_f(x_g) - Q_g(x_g)||_B$$

$$= ||T(f+P(x_f)) - T(f+P(x_g))||_B$$

$$+ ||T(f+P(x_g)) - T(g+P(x_g))||_B$$

$$= ||\int_0^1 dT(f+P(x_g) + \lambda(P(x_f) - P(x_g))\cdot(P(x_f)$$

$$- P(x_g))d\lambda||_B$$

$$+ ||\int_0^1 dT(g+P(x_g)+ (f-g))(f-g)d\lambda||_B$$

by Taylor's theorem. By (3.7) and (3.8)
$||f + P(x_g) + \lambda(P(x_f) - P(x_g)||_B < \delta^0$ and
$||g + P(x_g) + \lambda(f-g)||_B < \delta_0$. By (3.6) and the last term above we obtain

$$||x_f - x_g||_B \le (||dT(0)||+\gamma_0)\rho||x_f - x_g||_B$$
$$+ (||dT(0)||+\gamma_0)||f - g||_B$$
$$= \lambda_0||x_f - x_g||_B + (||dT(0)||+\gamma_0)||f - g||_B$$

And because $0 < \lambda_0 < 1$

$$||x_f - x_g||_B \le (1-\lambda_0)^{-1}(||dT(0)||+\gamma_0)||f - g||_B$$

Thus $||f - g||_B \to 0$ implies $||x_f - x_g||_B \to 0$. This proves the theorem.

Corollary 3.2 Assume (H1)-(H3) and (1)-(4) of Theorem 3.1. If P maps N(B,p) into B with P(0) = 0 and for each $\varepsilon > 0$ there is a number δ, $0 < \delta \le p$ such that $\mu_1,\mu_2 \in B$ with $||\mu_1||_B, ||\mu_2||_B < \delta$ implies $||P(\mu_1) - P(\mu_2)||_B < \varepsilon||\mu_1 - \mu_2||_B$ then the conclusions of Theorem 3.1 follow.

Proof: Fix $\varepsilon^* > 0$ so that $\varepsilon^*||I - R_0|| < 1$. Pick δ^*, $0 < \delta^* \le p$ so that $\mu_1,\mu_2 \in N(B,\delta^*)$ implies $||P(\mu_1) - P(\mu_2)||_B < \varepsilon^*||\mu_1 - \mu_2||_B$. Then the hypotheses of Theorem 3.1 are satisfied on $N(B,\delta^*)$ and the conclusions follow.

We will apply Theorem 3.1 to only one case where we take B to be one of the Banach spaces which were used in Chapter 2. It will be apparent that Theorem 3.1 could be applied readily to many other cases with appropriate conditions.

<u>Corollary 3.3</u> Assume (H1)-(H4) and

(1) There is annumber $r > 0$ and a function m, $m \in C(\Delta[0,\infty),R^+)$ such that for $\mu \in R^n$ with $||\mu|| < r$ $||K_x(t,s,\mu)|| \leq m(t,s)$ and

$$\sup_{t\geq 0} \int_0^t m(t,s)ds = m^* < +\infty$$

(2) $R_0(t,s)$ is the integral resolvent kernel for $K_x(t,s,0)$ and

$$\sup_{t\geq 0} \int_0^t ||R_0(t,s)||ds = R^* < +\infty.$$

(3) P is an operator defined by

$$P(\mu)(t) = \int_0^t p(t,s,\mu(s))ds$$

for $\mu \in C_c(R^+,R^n)$, where $p \in C(\Delta[0,\infty)\times R^n,R^n)$ and $p(t,s,0) \equiv 0$.

(4) There exists $\alpha \in C(\Delta[0,\infty),R^+)$ such that $x_1,x_2 \in R^n$ implies

$$||p(t,s,x_1) - p(t,s,x_2)|| \leq \alpha(t,s)||x_1 - x_2||$$

for $(t,s) \in \Delta[0,\infty)$ and

$$\sup_{t \geq 0} \int_0^t \alpha(t,s)ds = \alpha^* < +\infty.$$

(5) $\alpha^*||I - R_0|| < 1$, where R_0 is the bounded linear integral operator mapping $BC(R^+,R^n)$ into itself with kernel $R_0(t,s)$.

Then there exist numbers $m^* > 0$ and $\omega^* > 0$ such that for each $f \in N(BC,m^*)$ there is a unique $x_f \in N(BC,\omega^*)$ which solves (P). Moreover, x_f is a continuous function of changes in f.

Proof: As in Corollary 2.14, (H1)-(H4), (1), and (2) imply the hypotheses of Theorem 2.7, which are also (1)-(4) of Theorem 3.1. Also if $\mu_1,\mu_2 \in BC(R^+,R^n)$ then

$$||P(\mu_1)(t)|| \leq \int_0^t ||p(t,s,\mu_1(s))||ds$$

$$\leq \int_0^t \alpha(t,s)||\mu_1(s)||ds \leq \alpha^*||\mu_1||_0$$

and similarly

$$||P(\mu_1)(t) - P(\mu_2)(t)|| \leq \alpha^*||\mu_1 - \mu_2||_0$$

so that P(BC) is contained in BC and P satisfies a Lipschitz condition with constant α^*. Since $\alpha^*||I - R_0|| < 1$, the hypotheses of Theorem 3.1 are met and the conclusions follow.

2. Solutions and Stability of Perturbations by the Schauder-Tychonov Theorem

We recall that a subset of a topological space X is called precompact if its closure in X is compact. Let X be a Banach space and U an open subset of X. A map $T:U \to X$ is said to be completely continuous if T is continuous and maps bounded sets into precompact sets.

We will use the following form of the Schauder-Tychonov fixed point theorem, from Collatz [5].

Theorem 3.4 Let Q be a (nonlinear) operator defined in a bounded, closed, convex subset S of a Banach space B, and suppose (1) $Q(\mu) \in S$ for every $\mu \in S$, and (2) Q is completely continuous. Then there exists at least one $\mu \in S$ such that $Q(\mu) = \mu$.

Theorem 3.5 Assume (H1)-(H3), (1)-(4) of Theorem 3.1, and

(i) P is a completely continuous operator mapping N(B,r) into B.

(ii) There is a positive number P* such that $||P(\mu)||_B \leq P^*||\mu||_B$ for all $\mu \in N(B,r)$.

(iii) $||I - R_0||\cdot P^* < 1$.

Then for each $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that for $f \in N(B,\delta_\varepsilon)$ there exists a solution $x_f \in N(B,\varepsilon)$ to equation (P).

Proof: As in Theorem 3.1, the hypotheses of Theorem 2.7 are met. Let $\alpha > 0$, $\beta > 0$ and T be the numbers and the operator whose existences are guaranteed by Theorem 2.7. Choose $m > 0$ so that $(||I - R_0|| + m)P^* < 1$. Since $dT(0) = I - R_0$ we have

$$(||dT(0)||+m)P^* < 1. \tag{3.9}$$

T is continuously Fréchet differentiable on $N(B,\alpha)$ so that there is a number n, $0 < n \leq \alpha$, such that $g \in N(B,n)$ implies

$$||dT(g)|| < ||dT(0)|| + m. \tag{3.10}$$

For convenience we define a function γ by

$$\gamma(s) = \frac{(||dT(0)||+m)s}{1 - (||dT(0)||+m)P^*} \tag{3.11}$$

Fix $b > 0$ so that for $s \in [0,b)$

$$\gamma(s) < \min(r,(n-s)/P^*,n) \tag{3.12}$$

Then for $f \in N(B,b)$ and $\mu \in N(B,\gamma(||f||_B))$ we have $||\mu||_B \in r$ and

$$||f + P(\mu)||_B \leq ||f||_B + P^*||\mu||_B$$

$$\leq ||f||_B + P^*\gamma(||f||_B)$$

which is, by (3.12), bounded by

$$||f||_B + P^*(n - ||f||_B)/P^* = n.$$

Hence by (3.10),

(3.13) $$||dT(f+P(\mu))|| < ||dT(0)|| + m.$$

For $f \in N(B,b)$ define an operator Q_f on $N(B,\gamma(||f||_B))$ by

(3.14) $$Q_f(\mu) = T(f + P(\mu)).$$

T is continuous and by hypothesis P is completely continuous. Thus $\mu \to f + P(\mu)$ is completely continuous, and it follows that Q_f is completely continuous. ($f + P(\cdot)$ takes bounded sets to precompact sets; T takes precompact sets to precompact sets).

Q_f maps $N(B,\gamma(||f||_B))$ into itself: by Taylor's theorem

$$||Q_f(\mu)||_B = ||T(f+P(\mu))||_B$$

$$\leq \int_0^1 ||dT(\lambda(f+P(\mu)))||d\lambda||f + P(\mu)||_B$$

which by (3.13) is bounded by

$$(||dT(0)|| + m)(||f||_B + P^*||\mu||_B).$$

Let $\alpha_0 = ||dT(0)|| + m$; we have

$$||Q_f(\mu)|| \leq \alpha_0||f||_B + \alpha_0 P^*||\mu||_B$$

$$\leq \alpha_0||f||_B + \alpha_0 P^*\gamma(||f||_B)$$

$$\leq \alpha_0||f||_B + \alpha_0 P^*\cdot\alpha_0||f||_B(1 - \alpha_0 P^*)^{-1}$$

$$= \frac{\alpha_0 ||f||_B - \alpha_0^2 ||f||_B P^* + \alpha_0^2 ||f||_B P^*}{(1 - \alpha_0 P^*)}$$

$$= \frac{\alpha_0 ||f||_B}{(1 - \alpha_0 P^*)} = \gamma(||f||_B).$$

Thus Q_f maps $N(B,\gamma(||f||_B))$ into itself. $N(B,\gamma(||f||_B))$ is bounded, closed, and convex. By Theorem 3.4 Q_f has a fixed point in $N(B,\gamma(||f||_B))$. This fixed point is a solution of equation (P).

The function γ has the form $\gamma(s) = ks$. Let $\varepsilon > 0$ be given and let $\delta = \min(b,\varepsilon/k)$. Then for $f \varepsilon N(B,\delta)$ there is a solution $x \varepsilon N(B,\gamma(||f||_B))$, i.e., $||x||_B \leq \gamma(||f||_B) \leq k||f||_B \leq \varepsilon$. This completes the proof.

There are a number of corollaries to the latest theorem. We will present one of them, providing for the existence of solutions in $C^{\ell}(R^+,R^n)$ to equation (P).

A family V of functions in C^{ℓ} is precompact in C^{ℓ} if V is uniformly bounded, equicontinuous, and equi-convergent (Avaramescu [1]). By equicontinuous it is meant that for all $\varepsilon > 0$ there is a number $\delta > 0$ such that for all $t_1,t_2 \varepsilon R^+$, with $|t_1 - t_2| < \delta$, one has $||\mu(t_1) - \mu(t_2)|| < \varepsilon$ for all $\mu \varepsilon V$. By equi-convergence it is meant that for all $\varepsilon > 0$ there exists a positive number $T = T(\varepsilon)$ such that for all $\mu \varepsilon V$ and all $t \geq T(\varepsilon)$, one has $||\mu(t) - \ell_\mu|| < \varepsilon$.

<u>Theorem 3.6</u> Assume (H1)-(H4) and hypothesis of Corollary 2.15. If

(1) $A(t,s)$ is a continuous $n\times n$ matrix for $(t,s) \in \Delta[0,\infty)$ satisfying

$$\lim_{t\to\infty} \int_0^t ||A(t,s)||ds = 0.$$

(2) $g \in C(R^n,R^n)$ for which there are numbers $M > 0$, and $r > 0$ such that $||g(x)|| \leq M||x||$ for all $x \in R^n$ with $||x|| \leq r$.

(3) The operator P on $C_c(R^+,R^n)$ is defined by

$$P(x)(t) = \int_0^t A(t,s)g(x(s))ds$$

for $x \in C_c(R^+,R^n)$.

(4) Let $N = \sup_{t\geq 0} \int_0^t ||A(t,s)||ds$; then $||I - R_0||\cdot MN < 1$.

Then for every $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that for each $f \in N(C^{\ell},\delta(\varepsilon))$ there exists an $x \in N(C^{\ell},\varepsilon)$ which is a solution to equation (P).

Proof: From the proof of Theorem 2.15 concerning the solutions of (E) in C^{ℓ}, it follows that hypotheses (1)-(4) of Theorem 3.1 are met with $B = C^{\ell}$. According to Theorem 3.5, we have only to verify that P is a completely continuous operator mapping $N(C^{\ell},r)$ into C^{ℓ} and verify (ii),(iii) of that same theorem.

Let $\mu \in N(C^{\ell},r)$. Then

$$||P(\mu)(t)|| \leq \int_0^t ||A(t,s)|| \; ||g(\mu(s))||ds$$

$$\leq \int_0^t ||A(t,s)||ds \cdot M \cdot \sup_{t\geq 0} ||\mu(t)|| \leq MNr$$

and

$$\lim_{t\to\infty} ||P(\mu)(t)|| \leq \lim_{t\to\infty} \int_0^t ||A(t,s)||dsMr = 0.$$

Thus $P(\mu) \varepsilon C^{\ell}$; indeed, $P(\mu) \varepsilon C^0$, and $||P(\mu)||_0 \leq MNr$ for all $\mu \varepsilon N(C^{\ell},r)$.

The operator P is continuous on C^{ℓ}. Since $g \varepsilon C(R^n,R^n)$, given $\varepsilon > 0$ there is a number $\delta_1 > 0$ such that if $x_1,x_2 \varepsilon R^n$ and $||x_1 - x_2|| < \delta_1$ then $||g(x_1) - g(x_2)|| < \varepsilon/N$. If $\mu_1,\mu_2 \varepsilon C^{\ell}$ and $||\mu_1 - \mu_2||_0 < \delta_1$ then $||\mu_1(s)-\mu_2(s)|| < \delta_1$ for all $s \varepsilon R^+$ so that

$$||P(\mu_1)(t) - P(\mu_2(t)|| \leq \int_0^t ||A(t,s)||$$

$$\cdot ||g(\mu_1(s)) - g(\mu_2(s))||ds < N \cdot \varepsilon/N = \varepsilon.$$

This shows P is continuous on C^{ℓ}.

Let $V = P(N(C^{\ell},r))$. V is an equi-convergent family: choose $\varepsilon > 0$; there is a number $T(\varepsilon) > 0$ such that $t \geq T(\varepsilon)$ implies

$$\int_0^t ||A(t,s)||dsMr < \varepsilon$$

by (1) of this theorem. Thus for $\mu \varepsilon N(C \;,r)$, and $t \geq T(\varepsilon)$,

$$||P(\mu)(t)|| \le \int_0^t ||A(t,s)||ds\ Mr < \varepsilon.$$

V is also an equicontinuous family. Fix $\varepsilon > 0$, and choose $T^* > 0$ so that $t \ge T^*$ implies

$$\int_0^t ||A(t,s)||ds \cdot Mr < \varepsilon/2. \tag{3.15}$$

$A(t,s)$ is uniformly continuous on the compact set $F = \Delta[0,T^*+1]$. Let

$$Q = \sup_{(t,s)\varepsilon F} ||A(t,s)||. \tag{3.16}$$

There is a number $\eta > 0$ for which $|t_1 - t_2| < \eta$ and $t_1,t_2 \ \varepsilon\ [0,T^*+1]$ imply

$$||A(t_1,s) - A(t_2,s)|| < \varepsilon/(2Mr(T^*+1)) \tag{3.17}$$

Choose

$$\delta = \min(\tfrac{1}{2},\ \eta, \varepsilon/(2QMr)) \tag{3.18}$$

Let $t_1,t_2 \ \varepsilon\ R^+$ with $|t_1 - t_2| < \delta$. Without loss of generality we assume $t_1 < t_2$. There are three possibilities: either $0 \le t_1 < t_2 \le T^*$, $0 \le t_1 \le T^* \le t_2 < T^* + 1$, or $T^* \le t_1 < t_2$. For any $\mu\ \varepsilon\ N(C^{\ell},r)$ we have, in the first case and the second case,

$$||P(\mu)(t_1)-P(\mu)(t_2)|| = ||\int_0^{t_2} A(t_2,s)g(\mu(s))ds$$
$$-\int_0^{t_1} A(t_2,s)g(\mu(s))ds$$

$$+ \int_0^{t_1} A(t_2,s)g(\mu(s)ds$$

$$- \int_0^{t_1} A(t_1,s)g(\mu(s))ds||$$

$$\leq \int_{t_1}^{t_2} ||A(t_2,s)g(\mu(s))||ds$$

$$+ \int_0^{t_1} ||A(t_2,s)-A(t_1,s)||$$

$$\cdot\ ||g(\mu(s))||ds$$

$$\leq \int_{t_1}^{t_2} QMrds + \int_0^{t_1} \varepsilon/(2Mr(T^*+1))dsMr$$

by (3.16) and (3.17). Thus

$$||P(\mu)(t_1)-P(\mu)(t_2)|| \leq (t_2-t_1)QMr$$

$$+ (T^*+1)Mr\cdot\varepsilon/(2Mr(T^*+1))$$

$$< \varepsilon/(2QMr)\cdot QMr + \varepsilon/2 = \varepsilon,$$

using (3.18).

In the third case, $t_1,t_2 \geq T^*+1$ and by (3.15)

$$||\int_0^{t_2} A(t_2,s)g(\mu(s))ds - \int_0^{t_1} A(t_1,s)g(\mu(s))ds$$

$$\leq \int_0^{t_2} ||A(t_2,s)||dsMr + \int_0^{t_1} ||A(t_1,s)||dsMr$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus V is an equicontinuous family of functions.

Hypothesis (4) of this theorem immediately implies (iii) of Theorem 3.5. The result follows by Theorem 3.5.

Since the Schauder-Tychonov fixed point theorem does not imply uniqueness of the fixed point, we cannot conclude uniqueness for the solution of (P) where we have made use of this theorem without other conditions. If we do have uniqueness, however, (B,B) stability of the zero solution of (P) follows from Theorem 3.5.

CHAPTER FOUR

1 Application to a Boundary Value Problem

In this section we apply some of our previous results to the boundary value problem

(4.1) $$x' + A(t)x = F(t,x)$$

(4.2) $$Jx = r$$

We make the following assumptions

(B1) $t \to A(t)$ is a continuous function from R^+ into the space of $n\times n$ matrices.

(B2) $F \in C(R^+\times R^n, R^n)$, $F(t,0) = 0$ for $t \in R^+$, and $\frac{\partial}{\partial x} F(t,x) \equiv F_x(t,x)$ is a continuous $n\times n$ matrix for $(t,x) \in R^+\times R^n$ whose norm is bounded by a number N for all (t,x), and is uniformly continuous in x for all $t \geq 0$. That is, for each $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that $x_1, x_2 \in R^n$ with $||x_1 - x_2|| < \delta_\varepsilon$ implies

$$||F_x(t,x_1) - F_x(t,x_2)|| < \varepsilon$$

for all $t \geq 0$.

(B3) J is a bounded linear operator from $BC(R^+,R^n)$ into R^n.

Let $X(t)$ denote the fundamental matrix for the homogenous linear system

$$x' + A(t)x = 0$$

with the property $X(0) = I$, the $n \times n$ identity matrix, and let J_0 denote the constant matrix whose collumns are the values of J on the corresponding collumns of $X(t)$. If J_0 is nonsingular then any solution of the boundary value problem ((4.1),(4.2)) is a solution to the equation

$$(4.3) \qquad x(t) = X(t)J_0^{-1}(r - Jp(\cdot,x)) + p(t,x)$$

where

$$(4.4) \qquad p(t,f) = \int_0^t X(t)X^{-1}(s)F(s,f(s))ds$$

as long as $p(\cdot,x) \in BC(R^+,R^n)$. The converse is also true (see Kartsatos, [19]).

Theorem 4.1 For the problem ((4.1),(4.2)) asume (B1)-(B3) hold and the following hypotheses:

(1) There is a number $B \geq 0$ such that

$$||X(t)|| \leq B \text{ for } t \geq 0.$$

(2) There is a number $M \geq 0$ such that

$$\int_0^t ||X(t)X^{-1}(s)||ds \leq M, \text{ for } t \geq 0.$$

(3) The constant matrix J_0 is invertible.

(4) Let $R_0(t,s)$ denote the resolvent kernel for $X(t)X^{-1}(s)F_x(s,0)$. Then

$$\sup_{t\geq 0} \int_0^t ||R_0(t,s)||ds < +\infty$$

Let R_0 be the operator from $BC(R^+,R^n)$ into itself defined by

$$(R_0f)(t) = \int_0^t R_0(t,s)f(s)ds$$

and let $\rho > 0$ be a number such that

$$||I - R_0|| < \rho$$

where I is the identity operator on $BC(R^+,R^n)$.

If the hypotheses above hold and

$$\rho B||J_0^{-1}|| + ||J||MN \leq 1 \tag{5}$$

then there is a number $\omega > 0$ such that for all $r \in R^n$ with $||r|| < \omega$ there exists a function $x \in BC(R^+,R^n)$ which solves problem ((4.1),(4.2)) and this solution is unique. Moreover, the solution x is a continuous function of changes in r.

Proof: For $g \in BC(R^+,R^n)$ consider the integral equation

$$x(t) = g(t) + \int_0^t X(t)X^{-1}(s)F(s,x(s))ds \tag{4.5}$$

we show that (4.5) satisfies the hypotheses of Corollary 2 2.18. (H1) through (H3) are obviously satisfied, with $K(t,s,x) = X(t)X^{-1}(s)F(s,x)$. For (H4): for $x_1,x_2 \in R^{hn}$ and $(t,s) \in \Delta[0,\infty)$ we have

$$||X(t)X^{-1}(s)F_x(s,x_1) - X(t)X^{-1}(s)F_x(s,x_2)||$$

$$\leq ||X(t)X^{-1}(s)||\cdot||F_x(s,x_1) - F_x(s,x_2)||$$

and by (B2) and (ii) it follows that (H4) holds.

Furthermore, since

$$||X(t)X^{-1}(s)F_x(s,x)|| \leq ||X(t)X^{-1}(s)||\cdot N$$

and

$$\sup_{t\geq 0} \int_0^t ||X(t)X^{-1}(s)||ds \leq M < +\infty$$

(1) of Corollary 2.18 (and $p(\cdot,f) \in BC$ for $f \in BC$) is verified. (2) of Corollary 2.18 is an immediate consequence of (4). The hypotheses of Corollary 2. are thus satisfied.

Let α, $\beta > 0$, and T be the numbers and the operator guaranteed to exist by Corollary 2.18. Then the solution of equation (4.5) is $x_g = T(g)$ for all $g \in N(BC,\alpha)$.

For $f \in BC(R^+,R^n)$ and $r \in R^n$ define $D(r,f)$ by

$$D(r,f)(t) = X(t)J_0^{-1}(r-Jp(\cdot,f)).$$

$D(r,f) \in BC(R^+,R^n)$ for all $r \in R^n$, $f \in BC(R^+,R^n)$ since $p(\cdot,f) \in BC(R^+,R^n)$ and $X(t)$ is continuous and bounded by B. $D(r,f)$ also is continuous in each variable. Let $f \in BC(R^+,R^n)$ be fixed, and $r_1,r_2 \in R^n$.

$$\begin{aligned}
&||D(r_1,f) - D(r_2,f)||_0 \\
&= \sup_{t\geq 0}||X(t)J_0^{-1}(r_1-Jp(\cdot,f)) \\
&\quad - X(t)J_0^{-1}(r_2-Jp(\cdot,f))|| \\
&\leq B||J_0^{-1}||\ ||r_1 - r_2||
\end{aligned}$$

which shows $D(r,f)$ to be continuous in its first variable. Now fix $r \in R^n$ and let $f_1,f_2 \in BC(R^+,R^n)$.

$$\begin{aligned}
&||D(r,f_1) - D(r,f_2)||_0 \\
&= \sup_{t\geq 0}||X(t)J_0^{-1}(r-Jp(\ ;f_1)) \\
&\quad - X(t)J_0^{-1}(r-Jp(\cdot,f_2))|| \\
&\leq B||J_0^{-1}||\ ||J||\ ||p(\cdot,f_1) - p(\cdot,f_2)||.
\end{aligned}$$

Since $p(\cdot,f)$ is a continuous function of $f \in BC(R^+,R^n)$ by virtue of its satisfying the hypotheses of Corollary 2.18 it follows that $D(r,f)$ is also a continuous function of changes in F.

$D(0,0) = 0$, so that there exists a number $a > 0$ such that $||r|| < a$ and $||f||_0 < a$ implies $||D(r,f)||_0 < \alpha$, where α is the number whose existence is guaranteed by Corollary 2.18. Let $x(r,f) = T(D(r,f))$ for $||r|| < a$ and $||f||_0 < a$.

x(r,f) is unique and is a continuous function of (r,f), since T and D are each continuous functions of their variables.

Now $||I - R_0|| < \rho$ by hypothesis. But $I - R_0 = dT(0)$, and because $dT(\cdot)$ is continuous there is a number $\delta(\rho) > 0$ such that $||g||_0 < \delta(\rho)$ implies $||dT(g)|| < \rho$. Pick $b > 0$, smaller than the constant a chosen above, so that $||r|| \leq b$ and $||f||_0 \leq b$ imply $||D(r,f)|| < \delta(\rho)$. Then we have for $\lambda \in [0,1]$:

$$||dT(\lambda D(r,f))|| < \rho$$

and thus

$$\begin{aligned}
||T(D(r,f))|| &= ||\int_0^1 dT(\lambda D(r,f))D(r,f)d\lambda|| \\
&\leq \int_0^1 ||dT(\lambda D(r,f))||d\lambda||D(r,f)|| \\
&\leq \rho||D(r,f)||_0 \\
&= \rho \sup_{t\geq 0} ||X(t)J_0^{-1}(r-Jp(\cdot,f))|| \\
&\leq \rho B||J_0^{-1}||(||r|| + ||J|| \; ||p(\cdot,f)||) \\
&\leq \rho B||J_{00}^{-1}||(||r|| + ||J||MN||f||_0) \\
&\leq (\rho B||J_0^{-1}|| + ||J||MN)b
\end{aligned}$$

$\leq b$ for $||r|| < b$ and $||f||_0 < b$.

Thus for $r \in R^{hn}$ fixed with $||r|| < b$, $T(D(r,f))$ maps $N(BC,b)$ into itself.

We now will show that $T(D(r,f))$ is equicontinuous for $f \in N(BC,b)$ at each point $t_0 \in R^+$. Without loss of generality assume $t > t_0$. Then

$$||x(r,f)(t) - x(r,f)(t_0)||$$

$$= ||X(t)J_0^{-1}(r-J(\cdot,f)) + \int_0^t X(t)X^{-1}(s)$$

$$\cdot F(s,x(r,f)(s))ds - X(t_0)J_0^{-1}(r-J(\cdot,f))$$

$$- \int_0^{t_0} X(t_0)X^{-1}(s)F(s,x(r,f)(s))ds||$$

$$\leq ||X(t)-X(t_0)||(||J_0^{-1}|| \; ||r||+||J||\cdot MN\cdot ||f||_0)$$

$$+ ||\int_0^t X(t)X^{-1}(s)F(s,x(r,f)(s))ds$$

$$- \int_0^{t_0} X(t)X^{-1}(s)F(s,x(r,f)(s))ds||$$

$$+ ||\int_0^{t_0} X(t)X^{-1}(s)F(s,x(r,f)(s)ds$$

$$- \int_0^{t_0} X(t_0)X^{-1}(s)F(s,x(r,f)(s))ds||$$

$$\leq ||X(t) - X(t_0)|| \; Qb$$

$$+ \int_{t_0}^t ||X(t)X^{-1}(s)||\cdot||F(s,x(r,f)(s))||ds$$

$$+ \int_0^{t_0} ||X(t) - X(t_0)|| \; ||X^{-1}(s)||$$

$$\cdot \; ||F(s,x(r,f)(s))||ds$$

where $Q = ||J_0^{-1}|| + ||J||MN$.

Since $X^{-1}(s)$ is continuous on $[0,t_0]$ and

$$\begin{aligned} &||F(s,x(r,f))(s)|| \\ &\le ||\int_0^1 F_x(s,\lambda x(r,f)(s))d\lambda|| \; ||x(r,f)(s)|| \le Nb \end{aligned}$$

we have that the quantity on the last line above is bounded above by

$$\begin{aligned} &||X(t) - X(t_0)|| \; Qb \\ &+ \int_{t_0}^{t} ||X(t)X^{-1}(s)|| \; ds \; Nb + ||X(t) \\ &- X(t_0)|| \cdot Nb \cdot \sup_{0\le s\le t} \cdot ||X^{-1}(s)||. \end{aligned}$$

Because $X(t)$ and $||X(t)X^{-1}(s)||$ are continuous functions of t and (t,s) and because the last term above is independent of $f \in N(BC,b)$ it follows that $T(D(r,f)) = x(r,f)$ is an equicontinuous family at each $t_0 \in R^+$ for all $f \in N(BC,b)$. By the Schauder-Tychonov Theorem (Coppel [6]) $T(D(r,\cdot))$ has a fixed point X_r in $N(BC,b)$,

$$X_r = T(D(r,x_r))$$

and x_r solves

$$x(t) = X(t)J_0^{-1}(r-Jp(\cdot,x)) + p(t,x)$$

therefore x_r is a solution to the boundary value problem ((4.1),(4.2)).

Uniqueness is an immediate consequence of the facts that J is bounded and linear and $P(\cdot x)$ satisfies a Lipschitz con-condition in x. For continuity with respect to changes in r, we have $r_1, r_2 \in R^n$, $||r_1||$, $||r_2|| < b$:

$$
\begin{aligned}
||x_{r_1} - x_{r_2}||_0 &= ||T(D(r_1, x_{r_1})) - T(D(r_2, x_{r_2}))||_0 \\
&= \left|\left|\int_0^1 dT(D(r_2, x_{r_2}) + \lambda(D(r_1, x_{r_1}) \right.\right. \\
&\qquad \left.\left. - D(r_2, x_{r_2}))\, d\lambda[D(r_1, x_{r_1}) - D(r_2, x_{r_2})]\right|\right|_0 \\
&\le \rho||D(r_1, x_{r_1}) - D(r_2, x_{r_2})||_0 \\
&\le \rho \sup_{t \ge 0} ||X(t)J_0^{-1}((r_1 - Jp(\cdot, x_{r_1})) \\
&\qquad - (r_2 - Jp(\cdot, x_{r_2}))|| \\
&\le \rho \quad B||J_0^{-1}||\ ||r_1 - r_2||\ ||J||\ ||p(\cdot, x_{r_1}) \\
&\qquad - p(\cdot, x_{r_2})||_0 \\
&\le ||r_1 - r_2||\ \rho B||J_0^{-1}||\ ||J||MN||x_{r_1} - x_{r_2}|| \\
&\le ||r_1 - r_2||\ \rho B||J_0^{-1}||\,||J||MN2b
\end{aligned}
$$

which shows x_r to be a continuous function of r. This completes the proof of the theorem.

An example illustrating the theorem above is the boundary value problem.

$$(4.6)\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' + \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.05 \begin{bmatrix} \sin(x_1-x_2) \\ \sin(x_1+x_2) \end{bmatrix}$$

$$(4.6)\quad Jx = 5\int_0^\infty e^{-s}x(s)ds = r\varepsilon R^2 \quad \text{for } x\varepsilon BC(R^+,R^2).$$

(B1) , (B2), and (B3) are easily satisfied.

Let $x = (x_1,x_2)^T$ and A the matrix in (4.6). Then the fundamental matrix X(t) for the homogeneous system $x'+Ax = 0$ with the property that $X(0) = I$ is

$$(4.8)\quad X(t) = \exp(-At) = \begin{bmatrix} e^{-\frac{3}{2}t} & 0 \\ 0 & e^{-2t} \end{bmatrix}.$$

To study this problem we will use the R^2 norm $||(x_1,x_2)^T|| = \max(|x_1|,|x_2|)$ and 2×2 matrix norm $||B|| = \max(|b_{11}|+|b_{12}|,|b_{21}|+|b_{22}|)$ for $B = [b_{ij}]_{2\times 2}$, where i denotes the row and j the column of b_{ij}. These norms are compatible, i.e., $||Bx|| \le ||B||\cdot||x||$ for $x\varepsilon R^2$.

We have $F_x(\mu) = 0.05\begin{bmatrix} \cos(x_1-x_2) & -\cos(x_1-x_2) \\ \cos(x_1+x_2) & \cos(x_1+x_2) \end{bmatrix}$ so that $||F_x(\mu)|| \le 0.1$ for all $\mu\varepsilon R^2$, so we take $N = 0.1$ in the theorem. Also $||X(t)|| \le 1$ for $t\varepsilon R^+$, so we may take $B = 1$.

$$\int_0^t ||X(t)X^{-1}(s)||ds \le \frac{2}{3} \text{ and } M = \frac{2}{3}.$$

Since $X(t)X^{-1}(s)F_x(0) = X(t-s)F_x(0)$, Laplace transform techniques may be used to obtain the resolvent $R_0(t-s)$; it is given by $R_0(t) = [R_{ij}(t)]$ where

$$R_{11}(t) = \frac{\beta}{r_1 - r_2} \quad (r_1-\alpha_2\beta)e^{r_1 t}+(-r_2+\alpha_2\beta)e^{r_2 t}$$

$$R_{12}(t) = \frac{\beta}{r_1 - r_2} \quad (r_1-\alpha_2\beta)e^{r_1 t}+(-r_2+\alpha_2\beta)e^{r_2 t}$$

$$R_{21}(t) = \frac{\beta}{r_1 - r_2} \quad (r_1+\alpha_1\beta)e^{r_1 t}+(-r_2-\alpha_1\beta)e^{r_2 t}$$

$$R_{22}(t) = \frac{\beta}{r_1 - r_2} \quad (r_1+\alpha_1-\alpha\beta)e^{r_1 t}+(-r_2-\alpha_1+\alpha\beta)e^{r_2 t}$$

With $\beta = 0.05$, $\alpha_1 = \frac{3}{2}$, $\alpha_2 = 2$, $r_1 \doteq -1.945$, and $r_2 \doteq -1.455$. It was found that

$$||I-R_0|| \le 1.02 < 1.1 \;\; = \rho.$$

It is easy to see that

$$||J_0^{-1}|| = 0.6 \text{ and } ||J|| = 5.$$

We have

$$\rho B||J_0^{-1}||+||J||M\cdot N = 1.1 \times 1.\times 0.6$$

$$+ 5.\times 0.667\times 0.1 = 0.9935 < 1.$$

Thus, for $||r||$ sufficiently small, ((4.6),(4.7)) has a unique solution $x\varepsilon BC(R^+,R^n)$.

2 Examples Illustrating the Results in Chapters Two and Three.

The first example in this section is an illustration (in particular) of Corol. 2.15 and Corollary 2.16, concerning solutions to (E) in $C^{\ell}(R^+,R^n)$

Consider the system

$$(4.9) \qquad x(t) = f(t) + \int_0^t A(t-s)g(x(s))ds \text{ with}$$

$$A(t) = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

and $g\in C(R^2,R^2)$ given by, for $x = (x_1,x_2)^T$,

$$g(x) = \begin{bmatrix} 1 - e^{-x_1} \\ x_1 - x_2 \end{bmatrix} .$$

The system (4.9) satisfies the hypotheses of Theorem 2.15 and Corollary 2.16. Since g_x is continuous on R^2, and A is integrable (H1) - (H4) are certainly satisfied. The Laplace transform of $A(t)g_x(0)$ is $\hat{A}(s)g_x(0)$ given by

$$\hat{A}(s)g_x(0) = \begin{bmatrix} (s+3)(s+2)^{-2} & -(s+2)^{-2} \\ (s+2)^{-1} & -(s+2)^{-1} \end{bmatrix}$$

and $\mathrm{Det}[I-\hat{A}(s)g_x(0)] \neq 0$ for $\mathrm{Re}(s) \geq 0$. Thus the hypotheses of Corollary 2.16 are met, and for $(f_1,f_2)^T \in C^{\ell}(R^+,R^2)$ with sufficiently small norm, the solution to (4.9) exists, lies in C^{ℓ}, is stable (C^{ℓ},C^{ℓ}), and the solutions are Fréchet differentiable with respect to $(f_1,f_2)^T$.

The next example (4.10) is a system which satisfies the hypotheses of Theorem 2.19, and thus there exists a Fréchet differentiable solution $x \in BCL(R^+, R^2)$ for each $f \in BCL(R^+, R^2)$ with small norm.

(4.10) $x(t) = f(t) + \int_0^t K(t,s,x(s))ds$ where

$$f \in BCL(R^+,R^2) \text{ and } K(t,s,x) = \begin{bmatrix} e^{-2(t-s)} \cdot x_1^2 + e^{-3(t-s)} \cdot x_2^2 \\ (t-s)e^{-2(t-s)}(x_1 - x_2)^2 \end{bmatrix}$$

for $(t,s) \in \Delta[0,\infty)$ and $x = (x_1, x_2)^T \in R^2$.

It is not difficult to show that the hypotheses of Theorem 2.19 are met by (4.10). In particular, $K_x(t,s,0) \equiv 0$, so that $R_0(t,s) \equiv 0$.

The following example illustrates Theorem 2.22 concerning C_D spaces.

Let $D(t) = e^t I$, where I is the 2×2 identity matrix. Take $K(t,s,x)$ for $x = (x_1, x_2)^T \in R^2$ to be

$$K(t,s,x) = \begin{bmatrix} \cos(t-s)(x_1 + \sin^2(e^{-s} x_1)) \\ e^{-2t+s} x_2^2 \end{bmatrix}$$

It is easily seen that $D(t)$ satisfies (H8) and $K(t,s,x)$ satisfies (H1) - (H3) and (H7). Slightly more difficult to check, perhaps, are (H9) and (H10), but the proof of this we also omit. The resolvent of

$$K_x(t,s,0) = \begin{bmatrix} \cos(t-s) & 0 \\ 0 & 0 \end{bmatrix}$$

is $R_0(t) = [r_{ij}(t)]_{2\times 2}$ where

$$r_{11}(t) = -e^{\frac{1}{2}t}\left[\cos(t\sqrt{3}/2) + \frac{\sqrt{3}}{3}\sin(t\sqrt{3}/2)\right]$$

and $r_{ij}(t) = 0$ otherwise Because for $t\varepsilon R^+$

$$\int_0^t |e^{-t} r_{11}(t-s)e^s|ds \leq 2,$$

$R_0(t-s)$ satisfies (2.18), the final hypothesis of Theorem 2.22. Thus for $f\varepsilon C_D$ with sufficiently small norm, i.e., if $\sup_{s\geq 0} ||e^{-s}f(s)||$ is small enough, there exist a unique solution to (E) in C_D, and the solutions are Fréchet differentiable with respect to changes in f.

We next consider the scalar integral equation

$$(4.11)\quad x(t) = f(t) + \int_0^t e^{-(t^2-s^2)}\log(1+x(s))ds+P(x)(t)$$

where, for $\mu\varepsilon C_c(R^+,R)$

$$(4.12)\quad P(\mu)(t) = 0.1\int_0^t e^{-t^2+s^2}\int_0^{\mu(s)} [y]dyds$$

where $[y]$ means the greatest integer not exceeding y.

Thus $g(\mu) = 0.1\int_0^{\mu} [y]dy$, for $\mu\varepsilon R$, is not differentiable at $\mu = 0, \pm 1, \pm 2, \ldots$ However, $|g(\mu)| \leq 0.1\,|\mu|$ for $|\mu| \leq 1$.

For each number $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that $f\varepsilon C^{\ell}(R^+,R)$, $||f||_0 < \delta_\varepsilon$ implies the existence of a solution $x\varepsilon C^{\ell}(R^+,R)$ to (4.11) with $||x||_0 < \varepsilon$. We apply Theorem 3.6. First of all, the unperturbed equation

(4.13) $x(t) = f(t) = \int_0^t e^{-t^2+s^2}\log(1+x(s))ds$

satisfies the hypotheses of Corollary 2.15; it follows by l'Hospital's rule that

$$\lim_{t\to\infty}\int_0^t e^{-t^2+s^2}ds = 0.$$

Moreover, the resolvent $R_0(t,s)$ of $\frac{\partial}{\partial x}\left[e^{-t^2+s^2}\log(1+x)\right]_{x=0} = e^{-t^2+s^2}$ also has this property: $R_0(t,s)$ must satisfy for all $(t,s)\epsilon\Delta[0,\infty)$

$$R_0(t,s) = -e^{-t^2+s^2} + \int_s^t e^{-t^2+\mu^2}R_0(\mu,s)d\mu$$

so that

$$e^{t^2}\left|R_0(t,s)\right| \le e^{s^2} + \int_s^t e^{\mu^2}\left|R_0(\mu,s)\right|d\mu$$

by Gronwall's inequality (Bellman [2] p.35)

$$e^{t^2}|R_0(t,s)| \le e^{s^2}\cdot e^{t-s}$$

$$|R_0(t,s)| \le e^{-t^2+s^2+t-s}$$

and by another application of l'Hospital's rule,

(4.14) $\lim_{t\to\infty}\int_0^t |R_0(t,s)|ds = 0.$

Since for $0 < s \le t < \infty$

$$t^2-s^2 = (t+s)(t-s) \ge (t-s)^2$$

and

$$\int_0^t e^{-(t^2-s^2)}ds \le \int_0^t e^{-(t-s)^2}ds$$

(4.15)

$$\le \int_0^\infty e^{-\mu^2}du = \frac{\sqrt{\pi}}{2}.$$

By a similar argument

$$(4.16) \quad \int_0^t |R_0(t,s)|ds \le \tfrac{1}{2}e^{\frac{1}{4}}(1+\sqrt{\pi}) \text{ for all } t\varepsilon R^+.$$

It is now clear that, if $||\mu||_0 < p$, where p is any number such that $0 < p < 1$ the operator K given by

$$K(\mu)(t) = \int_0^t e^{-t^2+s^2}\log(1+\mu(s))ds$$

maps $N(C^\ell,p)$ into C^ℓ, and $K_x(\mu)$ given by

$$K_x(\mu)h(t) = \int_0^t e^{-t^2+s^2}(1+\mu(s))^{-1}h(s)ds$$

maps $C^\ell(R^+,R)$ into itself. In fact, all the hypotheses of Corollary 2.15 now readily follow.

In applying Theorem 3.6 to (4.11) let $M = 0.1$ from (4.12) and $N = \frac{1}{2}\sqrt{\pi}$, by (4.15). Thus

$$||I-R_0||M\cdot N \le (1+2^{-1}\,e^{\frac{1}{4}}\,(1+\pi^{\frac{1}{2}}))\cdot(0.1)\cdot(2^{-1}\pi^{\frac{1}{2}})$$

$$< (1+0.5\times1.4)\times(1+1.8))\times(0.1)\times(0.5\times1.8)$$

$$< 0.3$$

and (4) of Theorem 3.6 is satisfied. Theorem 3.6 therefore applies to this equation, and the conclusions follow.

CHAPTER FIVE

1. Methods of Obtaining the Solutions.

In this chapter we discuss the problem of obtaining, by iteration, the solution to the equation

$$\text{(E)} \qquad x(t) = f(t) + \int_0^t K(t,s,x(s))ds$$

when the existence of the solution has been guaranteed by the use of the implicit function theorem (Theorem 1.2) as was the case in Chapter Two.

We begin by examining the proof of the implicit function theorem, following the proof given in Kantorovich and Akilov [16].

Suppose the function G satisfies the hypotheses of the implicit function theorem as follows: B_1, B_2 and B_3 are Banach spaces and there is a number $\omega > 0$ for which G maps $N(B_1 \times B_2, \omega)$ into B_3, $G(0,0) = 0$, G is continuously differentiable on $N(B_1 \times B_2, \omega)$ and $d_2G(0,0)$ is a linear homeomorphism from B_2 onto B_3. Then one takes $x \varepsilon N(B_1, \omega)$ and defines $\Omega^{(x)} = \{y \varepsilon B_2 | (x,y) \varepsilon N(B_1 \times B_2, \omega)\}$. For any such x, $\Omega^{(x)} \neq \emptyset$, and one defines the operation $Q^{(x)}$ on $\Omega^{(x)}$ by

$$\text{(5.1)} \qquad Q^{(x)}(y) = y - [d_2G(0,0)]^{-1}G(x,y).$$

For every small $\varepsilon > 0$, a $\phi > 0$ can be found such that the function $Q^{(x)}$ maps $N(B_2,\varepsilon)$ into itself for $x\varepsilon N(B_1,\delta)$ and $Q^{(x)}$ is a contraction on $N(B_1,\varepsilon)$. The proof of this may be found in the book by Kantorovich and Akilov [16], p. 687. Therefore $Q^{(x)}$ has a unique fixed point y^* in $N(B_2,\varepsilon)$

$$y^* = y^* - [d_2G(0,0)]^{-1}G(x,y^*)$$

and

$$G(x,y^*) = 0.$$

For $x\varepsilon N(B_1,\delta)$ we therefore define the operator T of Theorem 1.2 by $T(x) = y^*$.

If we can determine ε and δ then, since $Q^{(x)}$ is a contraction on $N(B_2,\varepsilon)$ we may use the iterations.

(5.2) $\qquad y_{n+1} = Q^{(x)}(y_n) \quad (n = 0,1,\ldots)$

to obtain y^*, for $||y_0||_2 < \varepsilon$.

For the function $G(f,\mu) = f+K(\mu)-\mu$ used in Theorem 2.7, $Q^{(f)}$ is given by (5.4) where $K_x(0)-I$ and R_0-I simply denote the mappings $h \to K_x(0)h-h$ and $g \to R_0g-g$, respectively, the first being a bounded map from C onto D, the second its inverse, a bounded map from D onto C.

(5.4) $$Q^{(f)}(\mu) = \mu-[K_x(0)-I]^{-1}(f+K(\mu)-\mu)$$
$$= \mu-[R_0-I](f+K(\mu)-\mu)$$

(5.5) $$Q^{(f)}(\mu) = f+K(\mu) - R_0(f+K(\mu)-\mu)$$

where R_0 is defined in Theorem 2.7.

Therefore, if the conditions of Theorem 2.7 hold and one can determine explicitly numbers ε and δ which satisfy the conditions such that the operator $Q^{(f)}$ in (5.5) is a contraction on $N(C,\varepsilon)$ for each $f\varepsilon N(B,\delta)$, then the iterations

(5.6) $\qquad \mu_{n+1} = f+K(\mu_n) - R_0(f+K(\mu_n)-\mu_n)$

will converge to the unique function $x\varepsilon N(C,\varepsilon)$ which solves (E), whenever μ_0 $N(C,\varepsilon)$.

The following theorem from Holtzman [15a], provides a criteria for obtaining explicit ε and δ.

<u>Theorem 5.1</u>. X, Y, and Z are Banach spaces with norms $||\cdot||_1$, $||\cdot||_2$, and $||\cdot||_3$, respectively. G is a continuously Fréchet differentiable operation mapping an open subset Ω of $X\times Y$ into Z, and $(x_0,y_0)\varepsilon\Omega$. Assume also

(1) $G(x_0,y_0) = 0$

(2) The operator $d_2G(x_0,y_0)$ is a linear homeomorphism mapping Y onto Z, with $\Gamma = [d_2G(x_0,y_0)]^{-1}$ and $||\Gamma|| \le C_1$

(3) $N(X,x_0,\delta) \times N(Y,y_0,\varepsilon) = S$ is contained in the set Ω.

(4) There is a real valued function $g_1(\mu,v)$ defined for $\mu\varepsilon[0,\delta]$, and $v\varepsilon[0,\varepsilon]$ and nondecreasing in each argument with the other fixed such that $(x,y)\varepsilon S$ implies

$$||d_2G(x,y) - d_2G(x_0,y_0)|| < g_1(||x-x_0||_1, ||y-y_0||_2).$$

(5) There is a nondecreasing function g_2 defined on $[0,\delta]$ such that $(x,y)\varepsilon S$ implies

$$||G(x,y_0)||_3 \le g_2(||x-x_0||_1)$$

(6) $C_1 g_1(\delta,\varepsilon) \le \gamma < 1$ and $C_1 g_2(\delta) \le \varepsilon(1-\gamma)$.

Then there is an operator T mapping $N(X,x_0,\delta)$ into $\{y\varepsilon Y \mid ||y-y_0||_2 \le \varepsilon\}$ such that the point $(x,T(x))$ is a solution to the equation $G(x,y) = 0$ for every $x\varepsilon N(X,x_0,\delta)$, and there is no other solution for this x, having $y\varepsilon Y$, $||y-y_0||_2 \le \varepsilon$. Moreover, T is continuously Fréchet differentiable for $||x-x_0||_1 < \delta$, and $d_2G(x,T(x))$ is a linear homeomorphism onto Z.

Thus the new conditions (4), (5) and (6) allow the stronger conclusion. In the case of interest here, with $G(f,\mu) = f+K(\mu)-\mu$, $G(0,0) = 0$, and $\Gamma = [d_2G(0,0)]^{-1} = R_0-I$, if we add the following hypotheses to Theorem 2.7, we may form the stronger conclusions of Theorem 5.1:

(a) There is a number ε, $0 < \varepsilon \le p$ (p is given in Theorem 2.7) and a real-valued function $g_1(v)$ defined for $v \,\varepsilon\, [0,\varepsilon]$ and nondecreasing such that if $\mu\varepsilon N(C,\varepsilon)$, then $||K_x(\mu) - K_x(0)|| \le g_1(||\mu||_C)$ and

(b) $||R_0 - I||g_1(\varepsilon) \le \gamma < 1$.

To see that (4), (5), (6) of Theorem 5.1 follow from (a) and (b) we note that $||G(f,0)||_D = ||f||_D < m_1||f||_B$ for some $m_1 > 0$ since B is contained in D with stronger topology. We may take the function g_2 of Theorem 5.1 to be $g_2(\mu) = m_1\mu$, and we may take δ so that

$$||I-R_0||m_1\delta = \varepsilon(1-\gamma) \text{ so that}$$

(5.7) $$\delta = \varepsilon(1-\gamma)m_1^{-1} \; ||I-R_0||^{-1}$$

(we note that $||I-R_0|| \neq 0$, since R_0 must satisfy the operator equation $R_0 = -K_x(0) + K_x(0)R_0$; see (1.12)). Statements (a) and (b) above are, therefore, the only additional hypotheses needed for the conclusions of Theorem 5.1 to hold.

If the mapping $\mu \to K_x(\mu)$ satisfies a Lipschitz condition in a neighborhood of the origin, then (a) and (b) both hold.

<u>Theorem 5.2</u>. Assume the hypotheses of Theorem 2.7, and there are numbers η, $0 < \eta \leq p$, and L such that μ_1, $\mu_2 \in N(C,\eta)$ implies

$$||K_x(\mu_1) - K_x(\mu_2)|| < L||\mu_1-\mu_2||_C.$$

For any $\gamma \in (0,1)$ define $\varepsilon = \min(\eta, \gamma L^{-1}||R_0-I||^{-1})$
$\delta = \varepsilon(1-\gamma)m_1^{-1}||R_0-I||^{-1}$ (where $m_1 > 0$ satisfies $||\mu||_B \leq m_1||\mu||_D$ for all $\mu \varepsilon B$).

Then the operator T of Theorem 2.7 is defined and continuously Fréchet differentiable on $N(B,\delta)$, and for each $f \in N(B,\delta)$ the solution of (E), T(f), satisfies $||T(f)||_C \leq \varepsilon$.

<u>Proof</u>: We first show hypotheses (a) and (b) above are satisfied. Define the function g_1 by $g_1(\mu) = L\mu$ for $\mu \varepsilon [0,\varepsilon]$. Then g_1 is nondecreasing on $[0,\varepsilon]$ and $||K_x(\mu)-K_x(0)|| \leq L||\mu||_C = g_1(||\mu||_C)$ for $\mu \varepsilon N(C,\varepsilon)$, since $0 < \varepsilon \leq \eta$. Thus (a) is satisfied. For (b): $||R_0-I||g_1(\varepsilon) \leq ||R_0-I||L\gamma L^{-1}||R_0-||^{-1} = \gamma < 1$, and (b) is satisfied. By (a), (b), (5.7), and Theorem

5.1 the operator T will be defined and continuously differentiable on $N(B,\delta)$, and $T(f) \in N(C,\varepsilon)$ for each $f \in N(B,\delta)$.

The mapping $\mu \to K_x(\mu)$ may satisfy a Lipschitz condition if the function $K_x(t,s,\mu)$ satisfies a similar condition. As an illustration, we have

Theorem 5.3. Suppose $A(t,s)$ is a continuous $n\times n$ matrix, defined for $(t,s)\in\Delta[0,\infty)$, $g \in C(R^n,R^n)$, g is continuously differentiable on R^n and there are numbers A^*, R^*, q, and M, all positive, such that

(1) for all $t\in R^+$, $\int_0^t ||A(t,s)||ds \leq A^*$

and $\int_0^t ||R_0(t,s)||ds \leq R^*$

where $R_0(t,s)$ is the resolvent kernel for $A(t,s)g_x(0)$.

(2) $||g_x(\mu_1) - g_x(\mu_2)|| \leq M||\mu_1-\mu_2||$ for μ_1, $\mu_2 \in R^n$ and $||\mu_1||$, $||\mu_2|| \leq q$.

For each $\gamma \in (0,1)$ let $\varepsilon = \min(q, \gamma(A^*M)^{-1}||R_0-I||^{-1})$ $\delta = \varepsilon(1-\gamma)m^{-1}_1||R_0-I||^{-1}$.

Then for each $f\in BC(R^+,R^n)$ with $||f||_0 < \delta$ there is a unique $x_f \in BC(R^+,R^n)$ with $||x_f||_0 \leq \varepsilon$ such that $x_f(t)$ solves

$$x(t) = f(t) + \int_0^t A(t,s)g(x(s))ds$$

and the mapping $f \to x_f$ is continuously Fréchet differentiable. Moreover, for each such f the solution y of

$$y(t) = h(t) + \int_0^t A(t,s)g_x(x_f(s))y(s)ds$$

is in $BC(R^+,R^n)$, for any $h \in BC(R^+,R^n)$, and $y = dT(f)h$.

We omit the proof, which follows from Theorem 5.2 (cf. Corollary 2.13).

When the conditions of Theorem 5.2 are satisfied, then one may use the iteration scheme (5.6):

(5.8) $$\mu_{n+1} = f+K(\mu_n) - R_0(f+K(\mu_n)-\mu_n)$$

to approximate the solution, using μ_0 any function in the ball of radius ε.

The scheme (5.2) from which (5.8) derives is also known as the modified Newton's method (Katorovich and Akilov [16], p. 696).

If P is an operator differentiable on a set Ω, then the process of forming the sequence

(5.9) $\mu_{n+1} = \mu_n - [dP(\mu_n)]^{-1}P(\mu_n)$, $(n = 0,1,2,\ldots)$ is known as Newton's method. The dissertation of McCandless [21] contains an introduction to the use of Newton's method, as does the book of Kantorovich and Akilov [16].

There are two fundamental problems which are likely to arise in using Newton's method, before convergence can even be discussed. One, we do not know that $\mu_{n+1} \in \Omega$ just because $\mu_n \in \Omega$; secondly, even if $\mu_{n+1} \in \Omega$, $dP(\mu_{n+1})$ may not be invertible

The latter difficulty can be circumvented by using the scheme

(5.10) $\mu_{n+1} = \mu_n - [dP(\mu_0)]^{-1}P(\mu_n)$ $(n = 0,1,2,\ldots)$

when $dP(\mu_0)$ is known to have an inverse. This is the general form for the so-called modified Newton's method, mentioned above. However, μ_{n+1} still may be out of Ω for some n. This problem is generally solved either through assumptions such as (4), (5) and (6) in Theorem 5.1, which may be viewed as sort of "slope" restrictions on the derivative, or by assuming the existence of a second derivative together with some restrictions on the second derivative. An example is the following from Kantorovich and Akilov [16].

Theorem 5.4. Let the operation P be defined on a set Ω and have a continuous second derivative in $N(B,x_0,r)$. Moreover, let

(1) the linear operation $\Gamma_0 = [P'(x_0)]^{-1}$ exists;

(2) $||\Gamma_0(P(x_0))|| \leq \eta$;

(3) $||\Gamma_0 P''(x)|| \leq K$ $(x \in N(B,x_0,r))$.

Now if,

$$h = K_\eta \leq \frac{1}{2}$$

and

$$r \geq r_0 = \frac{1-\sqrt{1-2h}}{h}\,\eta,$$

the equation

$$P(x) = 0$$

will have a solution x^* to which the Newton method (original or modified) is convergent.. Here,

$$||x^*-x_0|| \leq r_0.$$

Furthermore, if for $h < \frac{1}{2}$

$$r < r_1 = \frac{1+\sqrt{1+2h}}{h}\eta$$

or for $h = \frac{1}{2}$

$$r \leq r_1$$

the solution x^* will be unique in the sphere $N(B,x_0,r)$.

The speed of convergence of the original method is characterized by the inequality

$$||x^*-x_n||_B \leq \frac{1}{2^n}(2h)^{2^n}\frac{\eta}{h}\ (n=0,1,\ldots),$$

and that of the modified method, for $h < \frac{1}{2}$, by

$$||x^*-x_n||_B \leq \frac{\eta}{h}(1-\sqrt{1-2h})^{n+1}$$

$$(n=0,1,\ldots).$$

This theorem, and results similar to it, should have many interesting uses in the study of Volterra integral equations. Because the hypothesis of a continuous second derivative would begin to bring us to far from the main part of this work, we will not investigate its usefulness here, but postpone that for a later work.

Another topic which would be of interest is to use the techniques of this work to study the existence, stability, and Fréchet differentiability of solutions to equation (E) when the function $K(t,s,x)$ possesses a singularity; e.g., functions of the form

$$K(t,s,x) = |t-s|^{\alpha} g(x)$$

where $-1 < \alpha < 0$.

APPENDIX

The purpose of this appendix is to justify the step on page 38 in the proof of Lemma 2.5 where it is asserted that

$$\text{(A.1)} \qquad \int_0^1 [K_x(\mu+\lambda h)-K_x(\mu)]h(t)d\lambda = \int_0^1 [K_x(\mu+\lambda h) -K_x(\mu)]hd\lambda(t);$$

the latter integral is taken to be an abstract Riemann integral. We first establish the existence of this integral and then the equality (A.1).

By hypothesis (2) of Lemma 2.5, $\mu\varepsilon N(C,p)$ and $h\varepsilon C$ imply $K_x(\mu)h-h\varepsilon D$. Therefore if μ, $\mu+h\varepsilon N(C,p)$ (as is the case is Lemma 2.5), then for each $\lambda\varepsilon[0,1]$

$$F(\lambda) = K_x(\mu+\lambda h)h-K_x(\mu)h\varepsilon D.$$

Moreover, the mapping $\lambda\to F(\lambda)$ from [0,1] into D is continuous since, for λ_1,λ_2 [0,1], $||F(\lambda_1)-F(\lambda_2)||_D = ||K_x(\mu+\lambda_1 h)h-K_x(\mu+\lambda_2 h)h||_D$, and (3) of Lemma 2.5 states that for every $\varepsilon>0$ there is a number $\delta_\varepsilon>0$ such that $\mu_1,\mu_2\varepsilon N(C,p)$ and $||\mu_1-\mu_2||_C < \delta_\varepsilon$ implies

$$||K_x(\mu_1)h-K_x(\mu_2)h||_D < \varepsilon||h||_C.$$

Thus if 0 is given and

$$||\lambda_1 h-\lambda_2 h||_C = |\lambda_1-\lambda_2|\ ||h||_C < \delta_\varepsilon$$

then

$$||F(\lambda_1)-F(\lambda_2)||_D < \varepsilon||h||_C$$

which shows $F(\lambda)$ to be a continuous function of $[0,1]$. It follows (see Graves [13]) that $\int_0^1 F(\lambda)d$ exists as an abstract Riemann integral, and

$$\text{(A.2)} \qquad \int_0^1 F(\lambda)d\lambda \in D.$$

We must now establish that (A.1) holds for each $t \in R^+$. Let P_n be a partition of $[0,1]$ into n subintervals $[\lambda_i,\lambda_{i+1}]$, $i=1,2,\ldots,n$. Let $|P_n| = \max_{1\leq i\leq n} |\lambda_{i+1}-\lambda_i|$, and let $\lambda_i^* \in [\lambda_i,\lambda_{i+1}]$; then

$$\lim_{|P_n|\to 0} \left|\left| \sum_{i=1}^{n} F(\lambda_i^*)(\lambda_{i+1}-\lambda_i) - \int_0^1 F(\lambda)d\lambda \right|\right|_D = 0$$

since the integral in (A.2) exists. Thus, since D is contained in $C_c(R^+,R^n)$ with stronger topology for each t R^+

$$\lim_{|P_n|\to 0} \sum_{i=1}^{n} F(\lambda_i^*) \ (\lambda_{i+1}-\lambda_i)(t) = \int_0^1 F(\lambda)d\lambda(t)$$

and, in fact, the convergence is uniform on compact sets. For each n

$$\sum_{i=1}^{n} F(\lambda_i^*)(\lambda_{i+1}-\lambda_i)(t) = \sum_{i=1}^{n} F(\lambda_i^*)(t)(\lambda_{i+1}-\lambda_i).$$

Thus

$$\text{(A.3)} \qquad \lim_{|P_n|\to 0} \sum_{i=1}^{n} F(\lambda_i^*)(t)(\lambda_{i+1}-\lambda_i) = \int_0^1 F(\lambda)d\lambda(t).$$

But also

$$\text{(A.4)} \qquad \lim_{|P_n|\to 0} \sum_{i=1}^{n} F(\lambda_i^*)(t)(\lambda_{i+1}-\lambda_i) = \int_0^1 F(\lambda)(t)\lambda$$

so by (A.3) and (A.4)

$$\int_0^1 F(\lambda)(t)d\lambda = \int_0^1 F(\lambda)d\lambda(t)$$

which is just (A.1).

BIBLIOGRAPHY

1. C. Avramescu, Sur l'existence des solutions convergentes pour des équations intégrales, Anal. Univ. Craiova, Seria Matematica, Vol. 1, No. 1 (1972).

2. R. Bellman, "Stability Theory of Differential Equations," McGraw-Hill Book Company, New York (1953).

3. John B. Bennett, "Volterra Integral Equations and Fréchet Differentials," Doctoral dissertation, University of Oklahoma (1974).

4. J. M. Bownds and J. M. Cushing, Stability of Systems of Volerra Integral Equations, J. Applicable Analysis (to appear).

5. L. Collatz, Some applications of functional analysis to analysis, particularly to nonlinear integral equations, in "Nonlinear Functional Analysis and Applications," ed. by Louis B. Rall, Academic Press, New York, 1971.

6. W. A. Coppel, "Stability and Asymptotic Behavior of Differential Equations," Heath, Boston, Mass., 1965.

6a. C. Corduneanu, Problèmes globaux dans le théorie des equations intégrales de Volterra, Ann. Mat. Pura Appl. (4) V. 67 (1965), 349-363.

7. C. Corduneanu, Some Perturbation Problems in the Theory of Integral Equations, Math. Systems Theory, V. 1, No. 2 (1967), 143-155.

8. C. Corduneanu, "Integral Equations and Stability of Feedback Systems," Academic Press, New York, 1973.

9. J. Dieudonné, "Foundations of Modern Analysis," Academic Press, New York, 1960.

10. T. H. Hildebrandt and L. M. Graves, Implicit Functions and their differentials in general analysis, Trans. Am. Math. Soc. 29 (1927), 127-153.

11. I. M. Gelfand and S. V. Fomin, "Calculus of Variations," Tr. by Richard A. Silverman, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1963.

12. H. E. Gollwitzer, Admissiblity and integral operators, Math. Systems Theory, 7 (1973), 219-231.

13. L. M. Graves, Riemann integration and Taylor's theorem in general analysis, Trans. Am. Math. Soc. 29 (1927), 163-177.

14. L. M. Graves, Implicit functions and differential equations in general analysis, Trans. Am. Math. Soc. 29 (1927), 514-552

15. Stanley I. Grossman, Existence and stability of a class of nonlinear Volterra integral equations, Trans. Am. Math. Soc., 150 (1970), 541-556.

15a. Jack M. Holtzman, "Nonlinear System Theory: A Functional Analysis Approach," Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1970.

16. L. V. Kantorovich and G. P. Akilov, "Functional Analysis in Normed Spaces", Tr. by D. E. Brown and A. P. Robertson, Pergamon Press, New York, 1964.

17. A.G. Kartsatos and G. J. Michaelides, Existence of convergent solutions to quasilinear systems and asymptotic equivalence, J. of Diff. Eq., 13 (1973), 481-489.

18. A.G. Kartsatos, Existence of bounded solutions and asymptotic relationships for nonlinear Volterra integral equations, (to appear).

19. A.G. Kartsatos, The Leray-Schauder Theorem and the existence of solutions of boundary value problems on infinite intervals, Indiana Univ. Math. J., 23 (1974), 1021-1029.

20. M. A. Krasnosel'skii, "Topological Methods in the Theory of Nonlinear Integral Equations," Tr. by A. H. Armstrong and J. Burlak, Macmillan, New York, 1964.

21. William L. McCandless, "Nonlinear Boundary Value Problems for Ordinary Differential Equations," Doctoral Dissertation, University of Waterloo, Waterloo, Ontario, 1972.

22. R. K. Miller, Admissiblity and nonlinear Volterra integral equations, Proc. Am. Math. Soc., 25 (1970), 65-71.

23. R. K. Miller, "Nonlinear Volterra Integral Equations," W. A. Benjamin, Inc., Menlo Park, California, 1971.

24. R. K. Miller, J. A. Nohel and J. S. W. Wong, Perturbations of Volterra integral equations, J. Math Anal. Appl. 25 (1969), 676-691.

25. R. E. A. C. Paley and N. Wiener, "Fourier Transforms in the Complex Domain," Amer. Math. Soc. Colloquium Publications, 134.

26. W. Pogorzelski, "Integral Equations and their Applications," Vol. 1, Pergammon Press, New York, 1966.

27. T. Yoshizawa, "The Stability Theory by Liapunov's Second Method," Math. Soc. Japan, Tokyo, 1967.

SUPPLEMENTARY BIBLIOGRAPHY

The following works, while not used in this thesis, contain closely related material.

1. Athanassios G. Kartsatos, A boundary value problem on an infinite interval, (to appear).

2. Athanassios G. Kartsatos, The Hildebrandt-Graves theorem and the existence of solutions of boundary value problems on infinite intervals, (to appear).

3. Richard K. Miller and George R. Sell, Existence, uniqueness and continuity of solutions of integral equations, Ann. Mat. Pura Appl. 80 (1968), 135-152.

4. Richard K. Miller, On linearization of Volterra integral equations, J. Math. Anal. Appl. 23 (1968), 198-208.

5. John A. Nohel, Remarks on nonlinear Volterra equations, in "Proceedings United States-Japan Seminar on Differential and Functional Equations," W. A. Benjamin, New York, 1967.

6. John A. Nohel, Asymptotic equivalence of Volterra equations, Ann. Mat. Pura Appl. 96 (1973), 339-347.

7. Aaron Strauss, On a perturbed Volterra integral equation, J. Math. Anal. Appl. 30 (1970), 564-575.