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NON-LINEAR FEEDBACK CONFIGURATIONS IN THE
THEORY OF INVARIANCE

by

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A thesis submitted in partial fulfilment
of the requirements for the degree of
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UNIVERSITY OF TORONTO

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MOSTAFA AHMED HASSAN

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School of Graduate Studies

NON-LINEAR FEEDBACK CONFIGURATIONS IN THE THEORY OF INVARIANCE

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THESIS

NON-LINEAR FEEDBACK CONFIGURATIONS IN THE THEORY OF INVARIANCE

(Summary)

The problem of parameter sensitivity is placed within the frame-work of the more general problem of adaptivity. The use of non-linear elements to reduce the depadence of the system response on plant parameters is studied.

A non-linear feedback approach to performance invariance is analyzed in detail, and new configurations to produce approximate invariance are presented. The schemes depend on the extraction of information about the current plant situation and the use of this data to modulate the system input so that the plant drive will compensate for the parameter variations.

The components of the proposed schemes are system models, phase coordinate transducers and arithmetic comparators including dividers and multipliers. As compared with the conventional approach of closing high-gain feedback loops around the plant, the proposed schemes present no stability problems with high order plants, and allow the use of appropriate filters to attenuate transducer noise in the system output.

The schemes are grouped into two classes - the one dealing with gain variations, the second with the variations in any other parameters. Members of each group are similar in construction. The simple one-parameter scheme is relatively insensitive to slight deviations in the non-compensated parameters. Compounding can be used to extend the schemes to handle situations in which two parameters vary substantially. Extension to certain types of linear systems with many inputs and outputs is also included.

The Laplace transform is used for synthesis of these schemes with the assumptions that the parameter variations are restricted to be piece-wise constant and the initial conditions on all other variables are zero. A time domain analysis is also included. The systems were described by differential and integro-differential equations, and in some special cases through the use of simplifying assumptions they were reduced to known forms.

A simulation study was conducted to verify the analytic conclusions and to examine situations that are difficult to analyze. Both analogue and digital techniques were employed and the results substantiated the conjectures and the analysis concerning the adequacy of the configurations in coping with parameter variations.

BIOGRAPHY

1933 --	Born, Cairo, Egypt
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PUBLICATIONS

On the description of multivariable systems (Correspondence)
Transactions of IRE on AC, Vol. AC-8, pp. 62, January, 1963.

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TABLE OF CONTENTS

NOMENCLATURE AND SYMBOLS

These are the terms and notations most oftenly used in the text. Other terms are defined as they appear.

y = output of plant

x = output of model

ξ = difference between actual and desired outputs

k = plant gain

t = real time

s = Laplace transform variable

m = plant drive

r = system input

θ = drive modulating signal

D = differentiation operator

γ = noise

P = transfer operator of a plant

Subscript d pertains to model

$+\!\!\!\!/\!\!\!+$ on the figures indicates the signal is a divider's denominator

$'$ indicates differentiation with respect to time

$\rightarrow$ = vector

$\underline{}$ = matrix

$|\ |$ = determinant of a matrix.

CHAPTER 1

INTRODUCTION

1.1 Motivation

The present trend in automatic control theory and practice is towards more sophisticated schemes. More computing and logical devices are now in common use, whereas in the past the designer had only high gain amplifiers and shaping filters to choose from. By allowing more freedom of element choice, it is hoped to improve the two basic aspects of control technology: information extraction and control signal shaping. Two basic questions must be answered before we can assess how well the system performs: 1) how general a situation can the system handle, and 2) how close to the desired response will it function. The answers to these questions may seem mutually exclusive since the search for the optimum solution needs accurate specification of the situation. But somehow the system that will carry the name adaptive has to handle such a diverse condition successfully. Despite the abundance of literature on adaptive control systems, the extreme difficulty of the problem still warrants further investigations. This thesis sets as a preliminary aim, the study of the problem of adaptivity in control systems, and the clarification of the possible approaches to its solution. This has led to the recognition of parameter insensitivity, an aspect of great engineering importance, as a special case of adaptivity. To this subclass the core of the thesis is devoted, and a search for systems especially suited

1-2

for parameter sensitivity reduction became the objective of the present research.

1.2 Summary

The thesis is divided into seven chapters. After the introductory Chapter, Chapter 2 is a general exposition of the adaptive problem and the different approaches to its solution. It is intended to give the essence of the great amount of literature on adaptive control systems. The relative position of the parameter sensitivity reduction approach is pointed out.

Chapter 3 is a bridge to the rest of the thesis. The model ratio systems which constructionally seem to be particularly suited for multiplicative disturbances are analyzed. The form of the dependence of the over-all transformation on the controlled plant parameters is expounded. As a result of the failure to ascribe specific insensitivity properties to the above mentioned class of feedback systems a more radical approach is considered for the rest of the work. In Chapter 4 we pose the question of how to build a deterministic system where the over-all transformation is independent of the plant parameters. This action is sought through proper modulation of the plant drive. The principle of non-linear invariance provides an approximate solution. The schemes required are relatively easy to implement. In this chapter the modern state space approach is used to develop such systems for the two general situations of gain and inertia changes (if the plant is given by $y' = ay + km$ then changes in k are called gain changes

and in a inertia changes). A basic assumption underlying the analysis in this chapter is piece-wise constancy of the parameter changes.

Chapter 5 is devoted to the investigation of the deviations caused by the non-adherence to the above assumption and to deriving the exact equations representing the systems. A comparison between the proposed schemes and high gain feedback schemes is included. Other particularly important problems such as the effect of deviations in non-compensated parameters are also studied.

Since an exact solution of the describing equations is practically impossible, the analogue simulation study covered in Chapter 6 is aimed at the solution of the over-all response to different practical situations. A case of particular interest is the third order system which cannot be handled by a feedback scheme.

Lastly, Chapter 7 is devoted to a discussion of the different uses of the proposed schemes and to situations where it can offer the only possible control variant.

1.3 Appraisal of the Results

The thesis provides a concise review of the available adaptive literature. It gives a clearer understanding of the parameter sensitivity problem. However, the main contribution is the introduction of the concept of non-linear invariance and schemes for its implementation. The proposed schemes are relatively simple. They use the phase coordinates of the plant

together with a combination of plant models and non-linear comparators to generate a shaping signal. This signal is to affect the input in such a way as to generate a plant drive which tends to compensate for changes in plant conditions. Functionally, this is different from the older high gain feedback concept where one tries to drive the plant as hard as possible to make the output reach the steady-state quickly under all plant conditions. This particular drive-generation mechanism makes our schemes more suited to handling previously troublesome plants such as non-minimum phase and pure delay ones.

CHAPTER 2

ADAPTIVITY AS THE MOST GENERAL CONTROL PROBLEM

2.1 External Manifestation of Adaptivity

In this section we shall discuss briefly how an adaptive system should behave externally. The description given will be of a general nature and aims at encountering the broadest possible class. Generally speaking, an adaptive system should behave in an acceptable way under diversified situations. Such a feature is possessed to some degree or other by all control systems. More specific requirements together with their methods of implementation will be discussed in the following sections. However, we can point out two salient features that are likely to be associated with the given notion of adaptivity: 1) learning to improve the behaviour with the unfolding of time for stochastic systems and, 2) being relatively insensitive to the current plant conditions for deterministic systems. We note here that under the given broad definition optimality is not a necessary condition for adaptivity. This view of adaptivity agrees with the formal definition given in [1].

2.2 A Formal Approach Involving Learning

The approach that will be briefly outlined in this section is that of Richard Bellman [2]. However, it is important to keep in mind that, in its present shape, it is more of a way of thinking than a specified technique that can be used directly to get working schemes for adaptive processes.

To give an idea about the elements of such a system we assume that we have a process, which is to be controlled optimally, the behaviour or environment of which are unknown or vaguely known. The controller which is generally a large scale digital computer, has to continuously collect all the information available about the process and its environment, and construct their current models. From this knowledge it generates the corresponding optimum control signal. A block diagram for such a control system is shown in Fig. 2.1. To formulate the learning and control policy, Bellman suggests the use of the following tools:

(a) <u>Information Pattern</u> which is a collection of data that gives some knowledge about the system from the consideration of its past history. This means that the pattern, which could be a set of changing numbers, enables the learning element to give a more precise model to the system. These numbers may be considered as <u>information coordinates</u>, thus increasing the dimensionality of the state space of the system.

(b) <u>The Functional Equation</u>: This is an analytical representation of the behaviour of the system, in the extended state space, under the action of the control signal. The decision element uses this equation to generate the optimum control signal. For the very simple cases, some working schemes are available in [2]. There is also some other simple worked examples along the same line of thinking in [3] and [4].
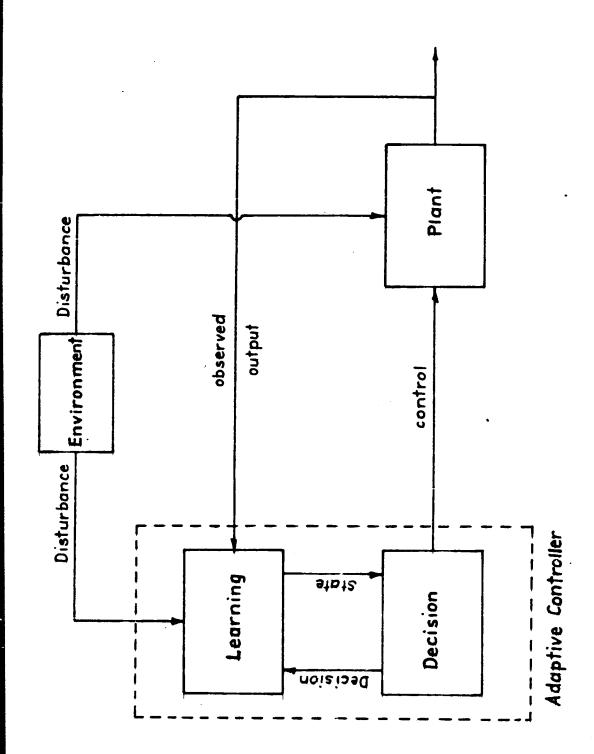
Fig. 2.1. Block Diagram of an Optimum Adaptive System

The basic difficulties with this approach are, the need to represent the information pattern by a small number of coordinates, and to devise sophisticated methods to handle the problem of searching for the optimum control over the extended state space.

2.3 An Empirical Approach Involving Searching

This approach is typically represented by an old class of systems known as extremal-seeking. A controller is said to be extremal-seeking if it will progressively move towards an extremal-operating point of the controlled object by a combination of experimentation, calculation and logic. The specific field where this subclass of adaptive systems finds application, is that of complex processes where a complete mathematical description of the process is very cumbersome and sometimes not possible. The extremal-seeking approach can be considered to involve learning in the sense of exploring the vicinity of the operating point to determine the next control step. The control system will also keep on driving the plant to its extremal-operating condition irrespective of its current state. The plant under control has a process function whose value represents the cost or profit to be extremalized. This function may have one or more extremal points. The process function may depend explicitly on part of the process variables. Fig. 2.2 represents a block diagram for a general extremalizing system. The noise and statistical errors in measuring the process function, and the ignorance of the form of the functional
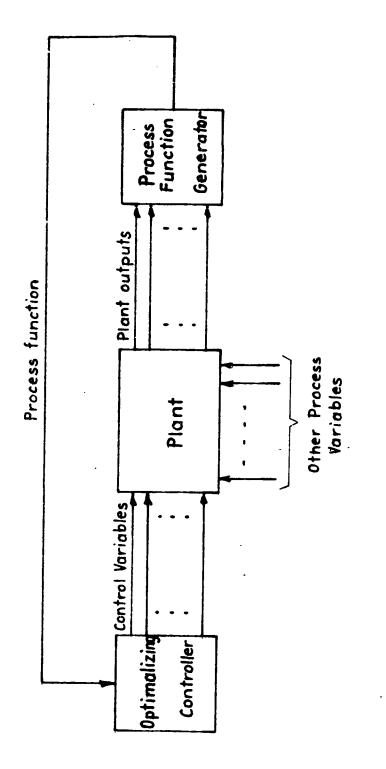
Fig. 2.2 Block Diagram of an Optimalizing System

dependence of the process variables, are two factors that complicate the operation. A factor usually neglected in the analysis available in the literature is the process dynamics, i.e. the fact that the effect of a control variable change does not reflect itself instantaneously on the process function. A simple example of the above general concept is the case where the process function is the integral of the square of the output deviation from the desired shape, and the control variable is a compensator setting[16].

The operation logic of the optimalizer depends on the kind of disturbance present in the system. If we have a deterministic system with only a slow drift in the process function, the situation involved will be very similar to on-line solution of a mathematical programming problem. Most of the available techniques for extremal-seeking are adaptations of mathematical programming techniques. A very good review of such techniques is found in [5].

The random fluctuations in the process function may make it impossible to determine the actual effects of each process variable on it, and resorting to statistical methods is necessary. Reference [6] contains illustrations for the use of these statistical methods. A recent approach used in the field of optimalizing processes is the stochastic approximation technique which is explained in [7]. An interesting formulation of the optimalizing methods is the dual control, i.e. experimentation and decision, which uses a combination of statistical decision and dynamic programming. See reference [8] for complete information about this formulation.

2.4 A Parameter Sensitivity Approach

A basic feature of this subclass is that optimality is not the objective. Simplicity is certainly a very desirable aspect. What we want is an acceptable over-all system performance which is more or less independent of the current plant parameters. Such performance has been long considered desirable and classical high gain feedback systems were always designed with the objective of sensitivity reduction. Referring to Fig. 2.3 which shows the two degrees of freedom linear feedback configuration, we may choose G and H so as to satisfy any two non-contradicting goals (e.g. a specified over-all transfer function and a specified sensitivity to variations in P). Let us assume that:

P = the current plant transfer function

P_n = the nominal plant transfer function

T = the current over-all transfer function

T_d = the desired over-all transfer function

In order to obtain the desired over-all transfer function T_d, and a specified sensitivity ratio $\frac{T_d}{T}$, we must choose G and H according to the following relations:

$$T_d = \frac{G\,P_n}{1 + P_n\,H} \tag{2.1}$$

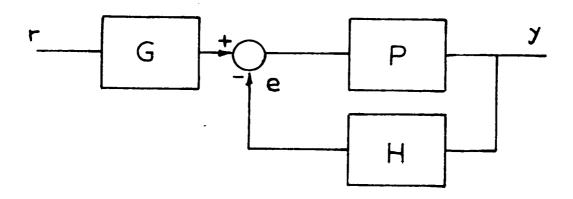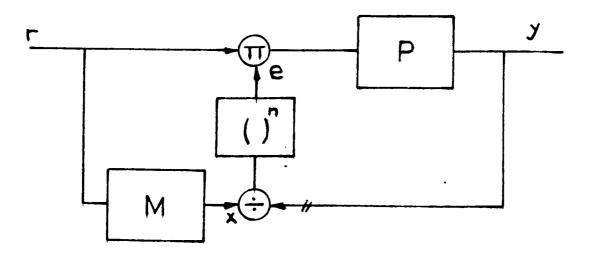$$\frac{T_d}{T} = \frac{\frac{P_n}{P} + H\,P_n}{1 + H\,P_n} \tag{2.2}$$

Fig. 2.3 Two Degrees of Freedom Linear Feedback Configuration

M = Transfer function of the plant model

$(\ )^n$ = Power raising amplifier.
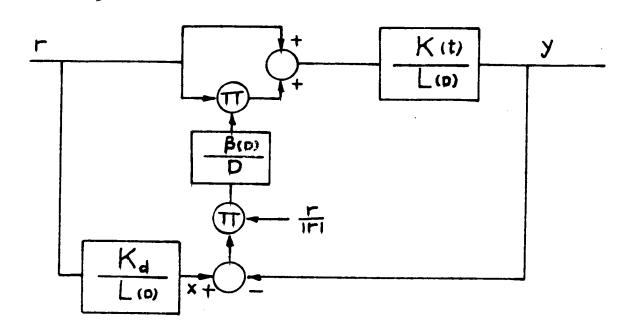
Fig. 2.4 Model Ratio System

Conceptually the reduction in sensitivity obtainable is at the expense of high gain feedback. This is always associated with high drives and unless saturation is considered in the analysis the results obtained can be deceiving. Other limitations on the high gain concept are that broad band, closed loop systems are more susceptible to feedback transducer noise and the increasingly difficult problem of stability of higher order systems. A compromise approach [9] would provide high gain only for low and intermediate frequencies and decrease the gain, as rapidly as allowable by stability considerations, at high frequencies. The procedure is of the cut and try type with no guarantee of attaining the objectives in general.

A straight-forward approach to sensitivity reduction is often encountered in the adaptive literature, namely that of inserting noise to identify the plant conditions and using the results to adjust a variable compensator [15]. This is sometimes referred to as active parameter insensitivity.

A more implicit direction to the reduction of sensitivity is through the introduction of non-linearities in the system. This type of compensation may be called passive sensitivity reduction. One of the earliest such systems is the ratio system [10]. This system, shown in Fig. 2.4 was suggested heuristically to be less sensitive to multiplicative disturbances (e.g. change of plant gain) than its linear counterpart, but no study of the transient dependence on the plant parameters was ever conducted.

More recently a similar system, with the division comparator replaced by subtraction and an integration type operator used in the difference branch, was introduced [11]. This scheme is shown in Fig. 2.5. Analysis of the transient dependence on the plant parameters are provided in the above-mentioned reference, but an interesting fact to note, is that if the system is stable then it will converge to the desired model steady-state output only because of the difference line integrator.

It is to this particular subclass (parameter insensitive schemes) that the rest of the present study is devoted.

$K(t)$ = Variable plant gain

K_d = Model gain

$L(D)$ = Plant & model dynamics (a polynomial in the
differential operator D)

$\beta(D)$ = Dynamics compensation operator (a polynomial in D
with a free term)

Fig. 2.5 A Passive Adaptive System

CHAPTER 3

THE FORM OF PARAMETER DEPENDENCE IN A
MODIFIED MODEL RATIO SYSTEM

3.1   Introduction

The author had the idea since the start of the work, that the introduction of simple non-linearities in feedback systems would reduce the dependence of the over-all response on the plant parameters more than a purely linear design can do. To carry on investigation aimed at testing this idea, the ratio system was used as a starting point since it seemed to be intuitively designed to handle multiplicative disturbance.

Consideration of the results of this chapter together with those of a similar work in [11] has served as a stimulus for subsequently introducing the concept of non-linear invariance.

3.2   The Exact Describing Equation of the Modified System

As was pointed out before, an integrating type operator is needed in the difference line if we want the steady-state error in our model schemes to be zero. This fact suggested the scheme shown in Fig. 3.1 as a modification for the model ratio systems. The power raising amplifier previously used was replaced by a proportional plus integral operator. The proportional element is needed here to reduce the system's tendency to overshoot, which is inherent in the use of integrators.
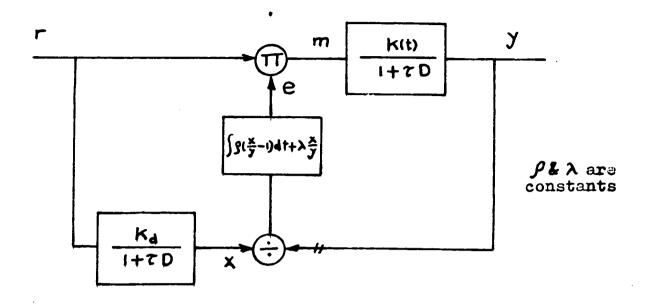
Fig. 3.1   A Modified Model Ratio System

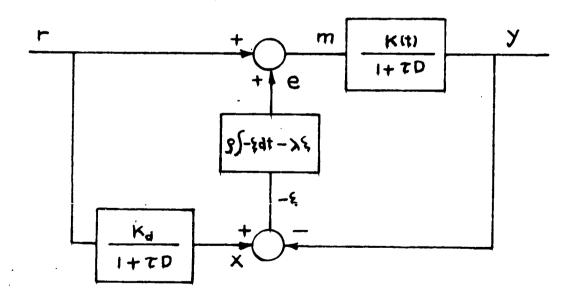ρ & λ are constants

Fig. 3.2   A Linear Model System

A thorough consideration of the situation reveals the need for an external negative unity source to be included with the integrand of the difference operator. This is needed to attain stability, since the convergence of the plant output to a value equal to the model output will give unity divider output. A similar situation does not arise in the linear version of Fig. 3.2, since the subtractor output will be zero by then.

With the understanding that the gain variations are the kind of parameter variations that would best represent multiplicative disturbances, we will proceed to derive the exact over-all equation for the case of a first-order plant with variable gain controlled by a modified model-ratio system.

Consider a plant with the following transfer operator $\frac{k(t)}{1 + \tau D}$, D being the differentiation operator, τ a fixed time constant and k(t) a variable plant gain. If we denote $\alpha(t) \triangleq \frac{k(t)}{\tau}$ and $\beta = \frac{1}{\tau}$ then we can write referring to Fig. 3.1, the following equation for the plant output:

$$y' = \alpha(t) \, m - \beta y \qquad (3.1)$$

where m is the plant drive.

The output of the model can be written as a known function of t ;

$$x = x(t)$$

But

$$L(u) = \int_0^t \rho (u-1) dt + \lambda u \qquad (3.2)$$

where L is the difference line operator and u $\triangleq \frac{x}{y}$.

ρ, λ are constants, and m = r.e, r is the system input.

Thus

$$e = \int_0^t \rho\left(\frac{x}{y} - 1\right)dt + \lambda \frac{x}{y} \qquad (3.3)$$

where e is the input modulating signal.

Dividing (3.1) throughout by $\alpha(t)r$ we have

$$\frac{y'}{\alpha(t)r} = e - \frac{\beta}{\alpha(t)r} y \qquad (3.4)$$

substituting $\mu \triangleq \frac{1}{\alpha(t)r}$ we can write

$$\mu y' = e - \beta \mu y \qquad (3.5)$$

Differentiating the above equation w.r.t. time we have

$$\mu y'' + \mu' y' = e' - \beta (\mu y' + \mu' y) \qquad (3.6)$$

We have from (3.3) by differentiation,

$$e' = \rho\left(\frac{x}{y} - 1\right) + \lambda \frac{d}{dt}\left(\frac{x}{y}\right) \qquad (3.7)$$

By eliminating e' between (3.6) and (3.7) we have

$$(\mu y'' + \mu' y') + \beta (\mu y' + \mu' y) = \rho\left(\frac{x}{y} - 1\right) + \lambda \frac{yx' - xy'}{y^2} \qquad (3.8)$$

grouping together similar terms we get

$$\mu y^2 y'' + \left[(\mu' + \beta\mu)y^2 + \lambda x\right]y' + \mu'\beta y^3 + \rho y^2 - (\rho x + \lambda x')y = 0 \qquad (3.9)$$

The above equation can be written in the following form:

$$A(y,t) \frac{d^2 y}{dt^2} + B(y,t) \frac{dy}{dt} + C(y,t) = 0 \qquad (3.10)$$

where

$$A(y,t) = A_2(t)y^2 \; ; \qquad A_2(t) = \mu$$

$$B(y,t) = B_0(t) + B_2(t)y^2; \quad B_0(t) = \lambda x; \quad B_2(t) = \mu' + \beta\mu$$

$$C(y,t) = C_1(t)y + C_2(t)y^2 + C_3(t)y^3$$

$$C_1(t) = -(\rho x + \lambda x'); \qquad C_2(t) = \rho \; ; \quad C_3(t) = \mu'\beta$$

This type of equation is known in mathematics as a second order non-linear polynomial equation and has been of interest to mathematicians; some properties are ascribed to it in [12]. But no general theory is available and explicit solutions in terms of elementary functions are unlikely. Hence if we want to reduce the complexity of the situation in a way to allow comparison with the linear counterpart, some simplifying assumptions must be introduced.

3.3 A Comparison Between the Linear and Non-linear Variants

In this section we will carry out analytic comparison between the systems of Fig. 3.1 and Fig. 3.2. However for the non-linear case the following approximations will be used:

$$\frac{x}{y} \simeq (1 - \frac{f}{x}) \qquad\qquad (3.11)$$

where $f = y - x$, $\widetilde{f}$ represents the difference between the actual and the desired responses.

The approximation can be justified as follows:

$$\frac{x}{y} = \frac{x}{f + x} \simeq \left[1 - \frac{f}{x} + \frac{f^2}{x^2} \cdots \right] \tag{3.12}$$

and thus if f is assumed small compared to x, this being natural if the system is to be of any use, then we can take only the first two terms of the series.

a) The equations of the non-linear scheme are:

$$e = \int_{o}^{t} \rho \left(\frac{x}{y} - 1 \right) dt + \lambda \frac{x}{y} \tag{3.13}$$

$$e' = \rho \left(\frac{x}{y} - 1 \right) + \lambda \frac{d}{dt} \left(\frac{x}{y} \right) \tag{3.14}$$

Using the relation (3.11):

$$e' = \left(1 - \frac{f}{x} - 1 \right) \rho + \lambda \frac{d}{dt} \left(1 - \frac{f}{x} \right) \tag{3.15}$$

or

$$e' = -\frac{f}{x} \rho - \lambda \left(\frac{x f' - x' f}{x^2} \right) = -\frac{f}{x} \rho - \lambda \frac{f'}{x} + \lambda \frac{x' f}{x^2} \tag{3.16}$$

Since $x' = \alpha r - \beta x$, $y' = \alpha(t) m - \beta y$, where $\alpha = \frac{k_d}{\tau}$, we can write the relationship for e' as follows:

$$e' = -\frac{f}{x} \rho - \frac{\lambda}{x} \left[\alpha(t) \, er - \alpha r - \beta f \right] + \frac{\lambda f}{x^2} \left[\alpha r - \beta x \right] \tag{3.17}$$

$$e' = \left[-\frac{\rho}{x} + \frac{\lambda \alpha r}{x^2} \right] f - \frac{\lambda \alpha(t) r}{x} e + \frac{\alpha r \lambda}{x} \tag{3.18}$$

Also

$$f' = -\beta f + \alpha(t) \, re - \alpha r \tag{3.19}$$

Equations (3.18) and (3.19) are sufficient to describe the system.

b) The equations of the linear scheme are:

$$e = -\left[\rho \int_0^t f \, dt + \lambda f\right] \tag{3.20}$$

$$e' = -(\rho f + \lambda f') \tag{3.21}$$

and

$$y' = \alpha(t) \, m - \beta y \tag{3.22}$$

$$x' = \alpha r - \beta x \tag{3.23}$$

$$\therefore \quad (y' - x') = \alpha(t) \, m - \alpha r - \beta f \tag{3.24}$$

or

$$f' = -\beta f + \alpha(t) \, e + r \, (\alpha(t) - \alpha) \tag{3.25}$$

and

$$e' = -\rho f - \lambda \left[-\beta f + \alpha(t) \, e + r \, (\alpha(t) - \alpha)\right] \tag{3.26}$$

Equations (3.26) and (3.25) are the counter part of (3.18) and (3.19).

The above-derived equations can be used to verify that both linear and non-linear cases have the trivial solution $f = 0$. But there is no reason to suggest a general superiority of the transient provided by either. Actually a computed example given in the next section is a case where the linear variant has a generally better transient.

To simplify the derived equation so as to be able to have some sort of qualitative comparison we can assume that

r = 1 and that the model output is to converge to x = 1, that is $\alpha = \beta$; then the equation for the non-linear case will be

$$f' = -\beta f + \alpha(t)\ e - \alpha \tag{3.27}$$

$$e' = (-\rho + \lambda\beta\ )f - \alpha(t)\lambda\ e + \lambda\alpha \tag{3.28}$$

and those for the linear case will be

$$f' = -\beta f + \alpha(t)\ e + (\alpha(t) - \alpha) \tag{3.29}$$

$$e' = (-\rho + \lambda\beta)f - \alpha(t)\lambda\ e - \lambda(\alpha(t) - \alpha) \tag{3.30}$$

In the above equations we notice that the homogeneous part (i.e. the part which is linearly dependent on e, f) is the same in both, but the non-homogeneous part of the linear system is dependent on the changes in the gain while that of the non-linear is not. However nothing can be attributed to this fact since the kernel will be dependent on the changing gain, and the non-homogeneous part in the linear case has opposite signs for each of the error and modulation equation.

3.4 Computationally Evaluated Performance Curves and Conclusions

As an illustration of comparative responses of the linear and non-linear cases, a worked example is exposed in this section. The control object is a first order variable gain plant.

First the describing equations are written in their canonical form and then solved numerically using a modified Runge-Kutta sub-routine on the IBM 7090. The initial conditions are set at corresponding values for the two cases.

We choose a plant with $\tau = 0.2$ and a gain that will vary from 2 to 3.5 to 0.5. The model has $\tau = 0.2$ and fixed gain of 2. The system input is unity and it was supposed to be at rest at the start of the test. We set $\lambda = \tau$ and $\rho = 1$.

If we denote $e \triangleq y_1$; $x \triangleq y_2$; $y \triangleq y_3$, the canonical equations for the linear case are (from Fig. 3.2):

$$y_1' = k_d r - k(y_1 + r) \tag{3.31}$$

$$y_2' = k_d \frac{r}{\tau} - \frac{y_2}{\tau} \tag{3.32}$$

$$y_3' = \frac{k(y_1 + r)}{\tau} - \frac{y_3}{\tau} \tag{3.33}$$

at $t = 0_-$; $y_1 = 0$, $y_2 = y_3 = 2$, and those of the non-linear case are (from Fig. 3.1):

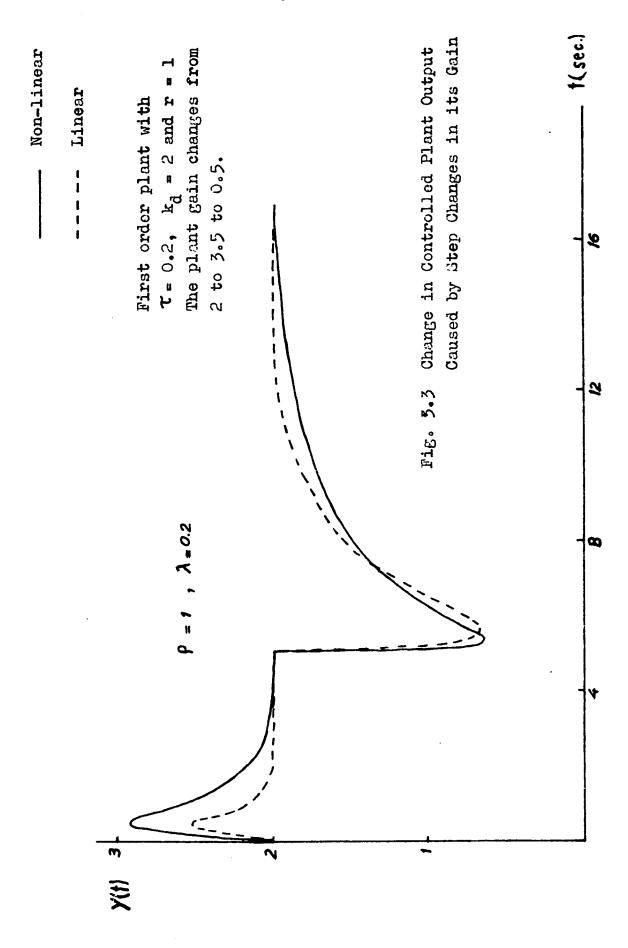$$y_1' = \frac{y_2}{y_3} + \frac{k_d r}{y_3} - \frac{k y_1 y_2 r}{y_3^2} - 1 \tag{3.34}$$

$$y_2' = \frac{k_d r}{\tau} - \frac{y_2}{\tau} \tag{3.35}$$

$$y_3' = \frac{k y_1 r}{\tau} - \frac{y_3}{\tau} \tag{3.36}$$

at $t = 0_-$; $y_1 = 1$, $y_2 = y_3 = 2$.

The results of the computer run are shown in Fig. 3.3. And it serves to emphasize the fact that the non-linear version is not in general better suited for variable gains.

Since the only reason for the choice of the model-ratio system is based on its steady-state response, and since the above study of the modified version (modification being necessary for zero steady-state error) reveals no general transient superiority over the linear variant, it might be necessary to follow a more formal line in choosing non-linear passive adaptive schemes. This led the author to the idea of trying to achieve absolute invariance through the use of stand-ard system components, which is the guiding philosophy for the rest of the work.

Non-linear

Linear

First order plant with
$\tau = 0.2$, $k_d = 2$ and $r = 1$
The plant gain changes from
2 to 3.5 to 0.5.

$\rho = 1$, $\lambda = 0.2$

$Y(t)$

$t(sec.)$

Fig. 3.3   Change in Controlled Plant Output
Caused by Step Changes in its Gain

CHAPTER 4

NON-LINEAR INVARIANCE

4.1 What is the Problem?

In the previous chapter the discussed system had an input-output equation dependent on the current value of the plant gain. This is certainly an undesirable fact. It raises the important question: Is it possible, without using a priori information about the variable parameter, to devise a system working around the plant so that the overall input-output relationship will be independent of the plant parameters? One can think of such a system as a sort of generalized feedback (Fig. 4.1) where N is a general non-linear operator. But what is N exactly? The synthesis of N may be accomplished rigorously or through intuitive reasoning.

This type of situation has arisen previously in the control field when the possibility of external disturbance rejection in linear feedback systems was considered. It was first rigorously analyzed in mathematical language by Academician Kulebakin [13]. However, the practical schemes proposed by him seemed to have been known at quite an earlier date [18]. The actual value of the theory of linear invariance discussed in [13] lies in its exploration of an aspect hitherto neglected by applied mathematicians. The idea behind the formulation of the linear invariance theory is the use of matrix differential operators to describe the overall system, and then the formulation of conditions on the elements of these
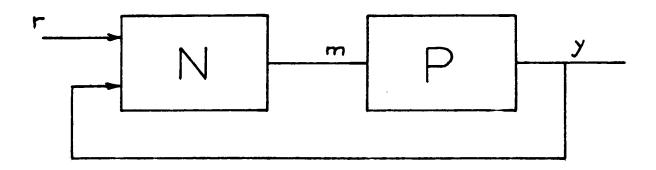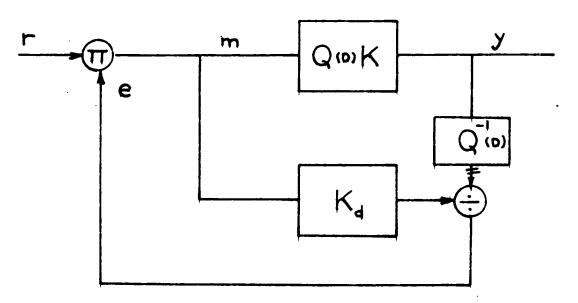
Fig. 4.1   Generalized Feedback

Fig. 4.2   Absolute Invariance System

$$Q(D) = \frac{1}{1 + \tau D}$$

$$Q^{-1}(D) = 1 + \tau D$$

matrices which will render the expression for certain plant outputs independent of some particular disturbance. If, after solving the matrix relationship for the particular output y_1, we have the following scalar equation

$$F(D) \, y_1 = A(D) \, \gamma(t) + B(D) \, r(t) \qquad (4.1)$$

where $F(D)$, $A(D)$ and $B(D)$ are polynomials in the differential operator D, $r(t)$ is the system input and $\gamma(t)$ is a disturbance acting on the system. The condition for invariance of y_1 with respect to $\gamma(t)$ is that $A(D) \, \gamma(t) \equiv 0$ which can be realized by considering the relationships among the transfer functions of the different system blocks. What actually limits the practical value of the schemes is that the conditions necessary for invariance usually involve the need for impractical differentiation operations.

If we reconsider our parameter problem, two conditions are necessary to form an invariance scheme with respect to a parameter: 1) It should be possible, using signals available for measurement, to obtain instantaneous information about the current parameter values; and 2) It should be possible, using this information, to affect the plant drive so as to cancel the effect of the parameter change.

Clearly a rigorous formulation of such a problem is not an easy matter. Even if it is done, its solution might lead (as it already did in the linear invariance case) to the need for impractical elements unless constraints on the feasible operators are included. A very simple case that will

illustrate this argument is that of the variable gain plant of Fig. 4.2 where, due to the need for the inverse operator $Q^{-1}(D)$, the practical value of the system is limited. However, the r and y relationship is completely independent of k, which in the case of a first order plant having a transfer function $\frac{k}{1 + \tau D}$ is

$$y = \frac{k_d}{1 + \tau D} \, r \qquad\qquad (4.2)$$

where $Q^{-1}(D) = (1 + \tau D)$ which involves the differentiation operator.

The line of thought that will be entertained in this thesis is to relax the conditions of absolute invariance in favour of a condition of simple practical system components. In our case the components will be chosen as plant models and arithmetic comparators.

4.2   A First Order Plant

Let us for simplicity consider a first order plant represented by the scalar equation

$$y' = -a(t)y + k(t)m \qquad\qquad (4.3)$$

where $a(t)$ and $k(t)$ are unknown functions of time. As a special case of this plant, one with only $k(t)$ as a function of time will be referred to in the text as a gain plant and one with only $a(t)$ as a function of time will be referred to as an inertia plant. The control schemes proposed for these two types

of systems will be structurally different, but the principles involved are the same (namely the ones explained in the previous section). To arrive at these schemes requires a thorough digestion of the chosen compensation mechanism and some trial-and-error procedures. But in a later section the schemes will be generalized to eliminate this need. A basic assumption behind the simplified analysis of the present chapter is that we can use Laplace transforms and neglect initial conditions. This is approximately true for slow parameter changes.

i) A gain plant; Using the plant describing equation:

$$y' = -a\,y + k(t)\,m, \qquad (4.4)$$

consider the scheme of Fig. 4.3, and assume that over the period $t_n < t < t_{n+1}$ where n is a point on the time axis, we can take k(t) to be fixed at $k \neq k_d$. Then we can write the following using Laplace transformed variables

$$y(s) = \frac{k}{a + s}\,m(s) \qquad (4.5)$$

$$x(s) = \frac{k_d}{a + s}\,m(s) \qquad (4.6)$$

From equations (4.5) and (4.6) we get:

$$x(s) = \frac{k_d}{k}\,y(s)$$

$$\therefore \quad e = \frac{x}{y} = \frac{k_d}{k} = \text{Const.} \qquad (4.7)$$

and

$$m(s) = er(s) = \frac{k_d}{k} r(s) \qquad (4.8)$$

Thus

$$y(s) = \frac{k_d}{a + s} r(s) \qquad (4.9)$$

which indicates an overall performance approximately independent of $k(t)$ because of the assumption of piece-wise constancy.

ii) An inertia plant; Using the plant describing equation:

$$y' = -a(t) y + km, \qquad (4.10)$$

consider the scheme of Fig. 4.4 and assume that, over the interval $t_n < t < t_{n+1}$, we can take $a(t)$ to be fixed at $a \ne a_d$ (a_d is the model value of a). Then we can write the following using Laplace transformed variables (from now on the dependence of the variables on s is implicitly understood).

$$z = y - x = k\left[\frac{1}{s+a} - \frac{1}{s+a_d}\right] m = \frac{k[a_d-a]\, m}{(s+a)(s+a_d)} \qquad (4.11)$$

$$q = \frac{ky}{(s+a_d)} = \frac{k^2 m}{(s+a_d)(s+a)} \qquad (4.12)$$

From equations (4.11) and (4.12) we get

$$l = \frac{z}{q} = \frac{1}{k}(a_d - a) = \text{Const.} \qquad (4.13)$$

$$\therefore \quad e = \frac{1}{k}(a_d - a)\, y \qquad (4.14)$$
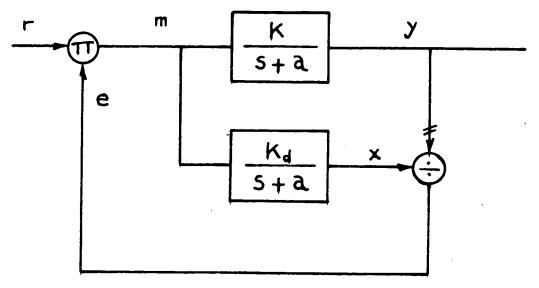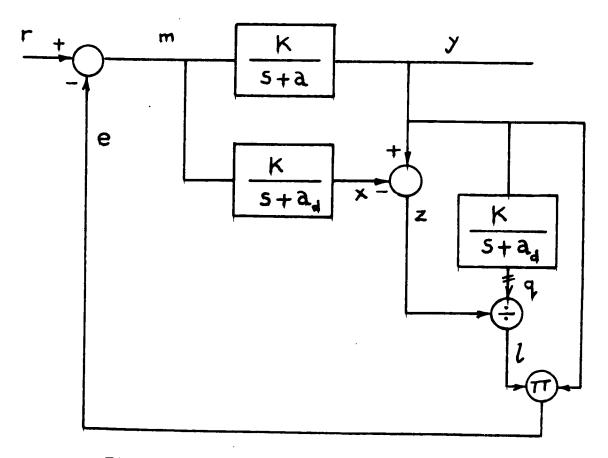
Fig. 4.3   The Variable Gain Scheme

Fig. 4.4   The Variable Inertia Scheme

and
$$m = r + \frac{1}{k}(a - a_d)\, y \tag{4.15}$$

$$\therefore \quad y = \frac{k}{s+a}\left[r + \frac{1}{k}(a - a_d)\, y\right] \tag{4.16}$$

or
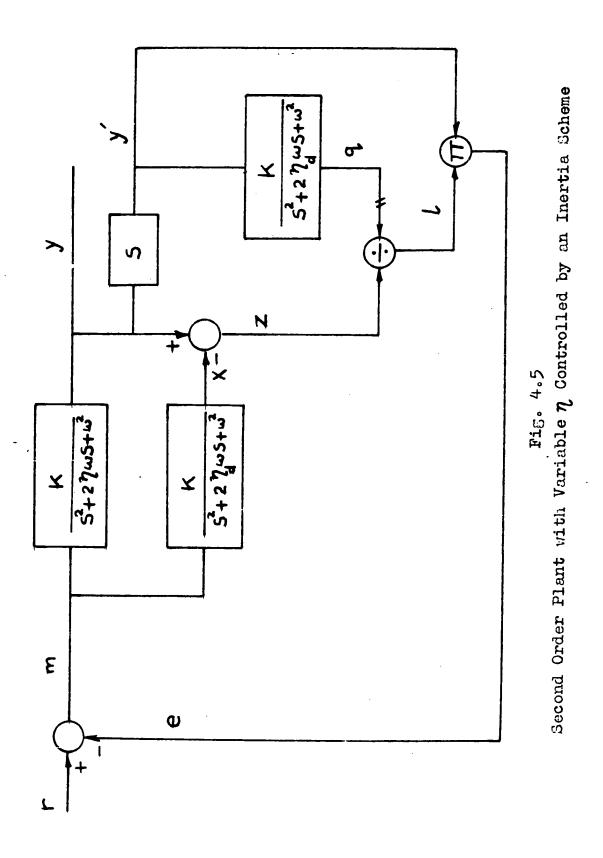$$y = \frac{k}{s+a_d}\, r \tag{4.17}$$

again approximately independent of $a(t)$.

4.3 Plants of Higher Order

Under the previous simplified analysis it is clear that, for a gain plant, adding poles or zeros to the plant transfer function does not change the situation and hence the configuration will remain the same. As for an inertia scheme, it can handle any plant as long as only one pole changes value. However, we will consider in this section a case of a second order plant with a variable damping $\eta(t)$. The plant equation will be the following:

$$y'' + 2\eta(t)\omega y' + \omega^2 y = km \tag{4.18}$$

Then the scheme of Fig. 4.5 shows how the plant model and the arithmetic comparators can be used to operate on the output phase coordinates so as to produce the desired parameter invariance system. With the introduction of $\eta \neq \eta_d$ (η_d is the model value of η) we can write:

Fig. 4.5

Second Order Plant with Variable η Controlled by an Inertia Scheme

$$z = y - x = k\left[\frac{1}{s^2 + 2\eta\omega s + \omega^2} - \frac{1}{s^2 + 2\eta_d\omega s + \omega^2}\right]m$$

$$= \frac{2\omega k(\eta_d - \eta)\ sm}{(s^2 + 2\eta\omega s + \omega^2)(s^2 + 2\eta_d\omega s + \omega^2)} \qquad (4.19)$$

$$q = k\ \frac{sy}{s^2 + 2\eta_d\omega s + \omega^2}$$

$$= k^2\ \frac{sm}{(s^2 + 2\eta_d\omega s + \omega^2)(s^2 + 2\eta\omega s + \omega^2)} \qquad (4.20)$$

From equations (4.19) and (4.20) we get

$$l = \frac{z}{q} = \frac{2\omega(\eta_d - \eta)}{k} = \text{Const.} \qquad (4.21)$$

and

$$e = \frac{2\omega(\eta_d - \eta)}{k}\ sy \qquad (4.22)$$

$$\therefore \quad (s^2 + 2\eta\omega s + \omega^2)y = k\left[r - \frac{2\omega(\eta_d - \eta)}{k}\ sy\right] \qquad (4.23)$$

$$\text{or} \quad y = \frac{k}{s^2 + 2\eta_d\omega s + \omega^2}\ r \qquad (4.24)$$

which is independent of $\eta(t)$. Like the case of gain plants the inclusion of additional non-varying poles or zeros will not change the prescribed scheme, nor the above simplified analysis. However, it must be always remembered that this analysis fails to account for a transient caused by an abrupt change in the plant conditions.

4.4   The Generalization to Multidimensional Cases

We now generalize the previous ideas to the case of a multidimensional plant. Suppose a plant is given by the following vector-matrix equation:

$$\frac{d}{dt} \vec{y} = \underline{A}(t) \vec{y} + \underline{K}(t) \vec{m} \tag{4.25}$$

Let us consider first the case where $\underline{A}$ is not dependent on time (the gain case). We also assume $\underline{K}$ to be a diagonal matrix. Then referring to Fig. 4.6, and again using the approximation of piece-wise constancy, we have in the Laplace domain:

$$\vec{y} = \underline{P}(s) \underline{K} \vec{m} \tag{4.26}$$

where

$$\underline{P}(s) \triangleq (Is - \underline{A})^{-1}$$

and I is the unit matrix

$$\vec{v} = \underline{P}_{adj}(s) \vec{y} = \underline{P}_{adj} \underline{P} \underline{K} \vec{m} \tag{4.27}$$

where $\underline{P}_{adj}(s)$ is the adjoint matrix of $\underline{P}(s)$.
Since $\underline{P}^{-1} = \frac{P_{adj}}{|P|}$, then we can write

$$\vec{v} = |P| \underline{K} \vec{m} \tag{4.28}$$

$$\vec{w} = |P| \underline{K}_d \vec{m} \tag{4.29}$$

From equations (4.28) and (4.29) we get

$$\vec{e} = \underline{K}_d \underline{K}^{-1} \vec{I} = \text{Const.} \tag{4.30}$$
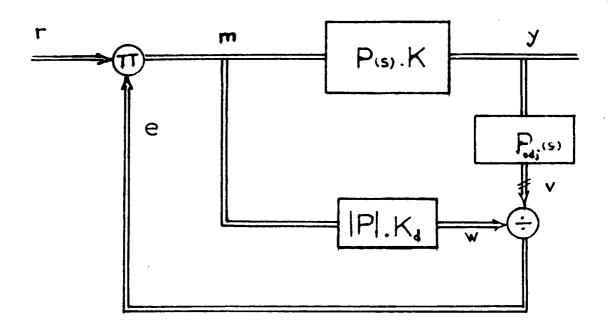
Fig. 4.6
Multidimentional Gain Scheme
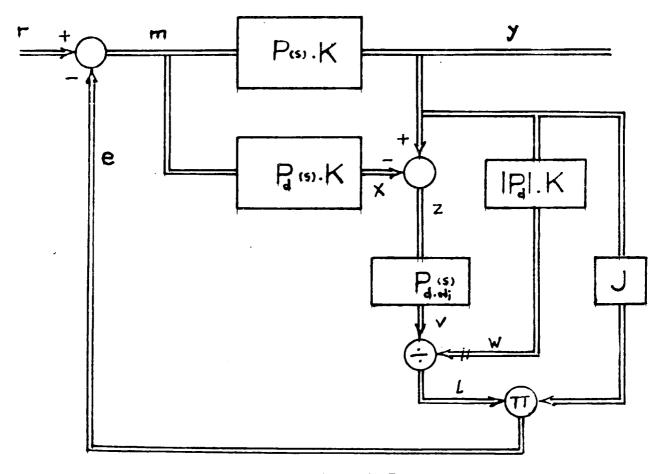
Fig. 4.7
Multidimentional Inertia Scheme

where $\vec{i}$ is the unit vector with all of its elements equal to unity.

$$\therefore \quad \vec{y} = \underline{P}(s) \; \underline{K}_d \; \vec{r} \tag{4.31}$$

Now we turn to inertia schemes (Fig. 4.7). With the assumption that $\underline{A}(t)$ has only one variable parameter per row and $\underline{K}$ is diagonal we can write the following equations:

$$[sI - \underline{A}] \; \vec{y} = \underline{K} \; \vec{m} \tag{4.32}$$

$$[sI - \underline{A}_d] \; \vec{x} = \underline{K} \; \vec{m} \tag{4.33}$$

$$\therefore \quad [sI - \underline{A}] \; \vec{y} = [sI - \underline{A}_d] \; \vec{x} \tag{4.34}$$

or

$$sI [\vec{y} - \vec{x}] = \underline{A} \; \vec{y} - \underline{A}_d \; \vec{x} \tag{4.35}$$

Since

$$\vec{z} = \vec{y} - \vec{x} \tag{4.36}$$

$$\therefore \quad sI\vec{z} = \underline{A}_d \; \vec{z} + [\underline{A} - \underline{A}_d] \; \vec{y} = \underline{A}_d \; \vec{z} + \underline{\Delta A} \; \vec{y} \tag{4.37}$$

where $\underline{\Delta A}$ is equal to the difference between the actual value of the variable coefficient matrix and its desired value. From equation (4.37), and by defining $\underline{P}_d$ to be equal to $(sI - \underline{A}_d)^{-1}$, we get

$$\vec{z} = \underline{P}_d \; \underline{\Delta A} \; \vec{y} \tag{4.38}$$

Referring to Fig. 4.7 we can also write

$$\vec{v} = \underline{P}_d \text{ adj } \vec{z}$$

$$= |P_d| \underline{\Delta A} \vec{y} \qquad (4.39)$$

and

$$\vec{w} = |P_d| \underline{K} \vec{y} \qquad (4.40)$$

By dividing the elements of the vector $\vec{v}$ by the corresponding ones of $\vec{w}$ we obtain a vector signal $\vec{l}$, each of whose components is the desired difference in one of the plant coefficients divided by the corresponding gain. These coefficients are multiplied by the adequate components of $\vec{y}$ obtained through the use of a matrix $\underline{J}$, $\underline{J}$ having unit entries for the proper elements and zero entries elsewhere. The result is the vector compensating signal $\vec{e}$ which is subtracted from the input to eliminate the effect of parameter changes.

A particular aspect of the multidimensional schemes is the use of adjoint and determinant operators rather than a system model. This is needed for efficient information separation, a problem raised by the interaction of multivariables. However, the single variable case can be derived as a special case of the multivariable one without much difficulty.

The method of analysis used previously is very helpful in developing the invariance schemes topologically. Its accuracy might be in some cases low. This is due to its inability to anticipate the transient deviations in the plant output. Exact analysis of the performance of the developed

schemes is very difficult. But a few analytical predictions are included in the following chapter. Also a detailed simulation study of the systems is included in Chapter 6.

CHAPTER 5

ANALYSIS OF THE PERFORMANCE OF INVARIANCE SCHEMES

5.1 Equations of the First Order Cases and Conditions for the Previously Used Approximation

In this section the invariance schemes will be studied in the time domain. This will give us a better qualitative understanding of the transient response under the effect of a parameter disturbance. The exact equations describing the input-output relationships for the gain scheme of Fig. 4.3 and the inertia scheme of Fig. 4.4 will be derived. As will be seen the equations are highly non-linear and only through the use of adequate approximations and assumptions can we justify the previously used relationship.

1) The gain scheme:

Referring to Fig. 4.3 we have:

Object equation:

$$ay + y' = k(t)\, m \tag{5.1}$$

Model equation:

$$ax + x' = k_d\, m \tag{5.2}$$

Modulation signal equation:

$$e = \frac{x}{y} = \frac{k_d \int_0^t m(\tau)e^{-a(t-\tau)}d\tau}{\int_0^t k(\tau)m(\tau)e^{-a(t-\tau)}d\tau} = \frac{m}{r} \tag{5.3}$$

The above equation is correct only under the assumption that $x = y = 0$ at $t = 0$. In a later section, using a different approach, this assumption will not be needed. Equation (5.3) is an integral equation, and when solved, gives the time dependence of m. We would point out here that this equation retains the same form for higher order plants except that $e^{-a(t-\tau)}$ should be replaced by $h(t-\tau)$, where $h(t)$ is the plant impulse response.

Now instead of attempting to solve equation (5.3) we shall show that it will approximately give the relationships of the last chapter provided $k'(t)$ (the rate of change of gain w.r.t. time) can be assumed negligibly small.

Consider the denominator of equation (5.3). Applying integration by parts, we have:

$$y = \int_0^t k(\tau)\, m(\tau) e^{-a(t-\tau)}\, d\tau = \left[k(\tau) \int_0^\tau m(u) e^{-a(t-u)}\, du \right]_0^t$$

(5.4)

$$- \int_0^t k'(\tau) \int_0^\tau m(u) e^{-a(t-u)}\, du\, d\tau \, .$$

With the above-mentioned condition of $k'(\tau) \simeq 0$ we can neglect the second term in the expression for y. But

$$\left[\int_0^\tau m(u) e^{-a(t-u)}\, du \right]_{\tau = 0} = 0$$

since $m(0)$ is finite. So, we can write:

$$y = k(t) \int_0^t m(\tau) e^{-a(t-\tau)}\, d\tau \qquad (5.5)$$

Thus we have for e:

$$e = \frac{k_d \int_o^t m(\tau)e^{-a(t-\tau)}d\tau}{k(t)\int_o^t m(\tau)e^{-a(t-\tau)}d\tau} = \frac{k_d}{k(t)} \qquad (5.6)$$

And substituting for e in the object equation, we have the following approximate relationship for the over-all behaviour of the system:

$$ay + y' = k(t)\frac{k_d}{k(t)} r = k_d r \qquad (5.7)$$

which is the result reached in the last chapter using Laplace methods.

The fact that the behaviour of the system is governed by the above relationship whenever we can assume $k'(t) \approx o$ and $x(o) = y(o) = o$ indicates the inherent stability of our schemes. That is to say, after an initial surge caused by a step change in a parameter the system output will eventually be described by the above equation. The above conclusion coincides with the results of the simulation given in the following chapter.

ii) The inertia scheme:

Referring to Fig. 4.4, we have:

Difference equation:
$$z' = -a_d z + (a_d - a(t))y \qquad (5.8)$$

Subsidiary equation:
$$q' = -a_d q + ky \qquad (5.9)$$

Plant equation:

$$y' = -a(t) \, y + km \qquad (5.10)$$

Modulation signal equations:

$$e = \frac{z \cdot y}{q} \qquad (5.11)$$

and
$$m = -e + r. \qquad (5.12)$$

In the following we will derive a single integro-differential equation for y as compared to the integral equation for m derived for the gain scheme in the above section.

Integrating the difference and subsidiary equation, we can write:

$$z = \int_0^t e^{-a_d(t-\tau)} (a_d - a(\tau)) \, y(\tau) \, d\tau \qquad (5.13)$$

and

$$q = k \int_0^t e^{-a_d(t-\tau)} y(\tau) \, d\tau, \qquad (5.14)$$

the above being true only under the assumption that

$$z = q = 0 \quad \text{at} \quad t = 0.$$

$$m = -\frac{y}{k} \cdot \frac{\int_0^t e^{-a_d(t-\tau)} y(\tau) \, (a_d - a(\tau)) \, d\tau}{\int_0^t e^{-a_d(t-\tau)} y(\tau) \, d\tau} + r \qquad (5.15)$$

and

$$y' = -a(t)y - \frac{\int_0^t e^{-a_d(t-\tau)} y(\tau) \, (a_d - a(\tau)) \, d\tau}{\int_0^t e^{-a_d(t-\tau)} y(\tau) \, d\tau} y + kr \qquad (5.16)$$

Considering the numerator of the fraction in the above equation and integrating it by parts, we have:

$$\int_0^t e^{-a_d(t-\tau)} y(\tau)(a_d-a(\tau)) \, d\tau = \left[(a_d-a(\tau)) \int_0^\tau y(u)e^{-a_d(t-u)} \, du \right]_0^t$$

$$(5.17)$$

$$+ \int_0^t a'(\tau) \int_0^\tau y(u)e^{-a_d(t-u)} \, du \, d\tau.$$

With the assumption that $a'(\tau) \simeq 0$, the last term of the above relationship can be dropped off. And

$$\left[\int_0^\tau y(u)e^{-a_d(t-u)} \, du \right]_{\tau=0} = 0$$

if $y(o)$ is finite.

We can write:

$$\int_0^t e^{-a_d(t-\tau)} y(\tau)(a_d-a(\tau)) \, d\tau \simeq (a_d-a(t)) \int_0^t y(\tau)e^{-a_d(t-\tau)} \, d\tau$$

$$(5.18)$$

and the integro-differential equation (5.16) can be approximated by the following differential equation

$$y' = -a(t) \, y - (a_d-a(t)) \, y + kr$$

or

$$y' = -a_d y + kr \qquad (5.19)$$

which is the equation derived for the inertia scheme in the last chapter.

As in the gain case, any change in the dynamics of the plant would require replacing $e^{-a_d t}$ by the adequate model impulse response without affecting the above explained mechanism. Again inherent stability can be ascribed to the configuration since, after the initial transient, the system output will be described by the (stable) model differential equation.

In the above we have studied the behaviour of the systems under multiplicative parameter changes. Now we will investigate the behaviour of the proposed invariance schemes under additive disturbances (eg. load disturbance).

Assume a load disturbance $\ell(t)$ to act at the output of the object in the gain scheme (Fig. 5.1). We will proceed to deduce the describing equation under the assumption that the plant has a fixed gain equal to that of the model. This will be a simplification of the situation in which both disturbances act together. But it serves to show the mechanism of the invariance scheme reaction to an additive type of disturbance.
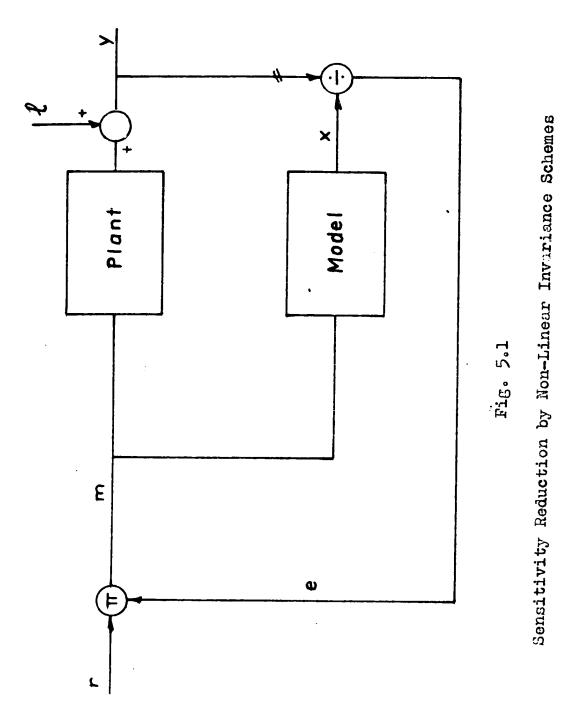
$$y = \frac{km}{a + D} + \ell \tag{5.20}$$

$$x = \frac{km}{a + D} \tag{5.21}$$

$$y = x + \ell \tag{5.22}$$

and

$$e = \frac{x}{y} = \frac{y - \ell}{y} = 1 - \frac{\ell}{y} \tag{5.23}$$

$$m = re = r - \frac{\ell r}{y} \tag{5.24}$$

Fig. 5.1

Sensitivity Reduction by Non-Linear Invariance Schemes

$$y = \frac{k}{a + D}\left[r - \frac{\ell r}{y}\right] + \ell \tag{5.25}$$

$$y' + ay = kr - \frac{k\ell r}{y} + (a + D)\ell \tag{5.26}$$

or finally

$$y' = -ay - k\ell r \cdot \frac{1}{y} + kr + a\ell + \ell' \tag{5.27}$$

The above equation is of Abel's type [12] which can be solved in closed form, given the analytic expression for $\ell(t)$. It has a trivial solution at $y = \frac{k}{a} r$ when $\ell' = o$ which implies zero steady-state error if the solution converges. It is worth noting from the above equation that the shape of the transient depends on the sign of ℓ. This is to be compared with the independence of the transient shape on the direction of change in k which will be shown in the next section.

The ability of parameter invariance schemes to handle simultaneously gain and load disturbances will not be discussed here. This is due to the possibility of using other schemes that can handle load disturbances in conjunction with invariance schemes for the parameter disturbances.

5.2 A Study of the Exact Equations for the First Order Gain Plant

We will consider now in greater detail the exact describing equations of the gain system. To begin, we take the most general situation where $k(t)$, $a(t)$ are both unknown functions of time.

$$y' = -a(t)y + k(t)r \frac{x}{y} \tag{5.28}$$

$$x' = -a_d x + k_d r \frac{x}{y} \tag{5.29}$$

or
$$\frac{dy}{dx} = \frac{-a(t)y^2 + k(t)r\ x}{-a_d\ xy + k_d r\ x} \tag{5.30}$$

This is a first order differential equation connecting the plant and model outputs which cannot be solved in closed form. Notice here that no simplifying assumptions were made yet in the above equation. However, if the time parameter is not present some properties can be attributed to the solution [17]. We now take an alternative route by solving the model equation for x in terms of y and then substituting the result in the plant equation.

$$x' = \left[\frac{k_d r}{y} - a_d \right] x \tag{5.31}$$

where its solution could be written as follows:

$$x = x_o\ e^{\displaystyle\int_0^t (\frac{k_d r}{y} - a_d)\ dt} \tag{5.32}$$

By substituting this value of x in equation (5.28) we get

$$y' + a(t)y = k(t)\ \frac{r}{y}\ x_o\ e^{\displaystyle\int_0^t (\frac{k_d r}{y} - a_d)\ dt} \tag{5.33}$$

or

$$\frac{y'y + a(t)y^2}{k(t)r\ x_0} = e^{\int_0^t (\frac{k_d r}{y} - a_d)\ dt}$$

(5.34)

which could be written as

$$Ln\left[\frac{y'y + a(t)y^2}{k(t)r\ x_0}\right] = \int_0^t (\frac{k_d r}{y} - a_d)\ dt$$

(5.35)

The above equation is now ready to be transformed to a non-linear differential equation in y. To simplify the analysis which otherwise would be very cumbersome, we introduce the following assumption:

$$a(t) = a = a_d \quad \text{and} \quad k(t) = k \neq k_d$$

and

$$r = \text{const.}$$

Differentiating both sides of equation (5.35) w.r.t. time, we have

$$\frac{k\ x_0\ r}{y'y + a\ y^2} \cdot \frac{y''y + y'^2 + 2ayy'}{k\ x_0\ r} = \frac{k_d r - a_d y}{y}$$

(5.36)

$$y''y + y'^2 + 2ayy' = (y' + ay)(k_d r - ay)$$

(5.37)

$$yy'' + \left[3ay - k_d r\right]y' + y'^2 + \left[a^2 y^2 - k_d\ ray\right] = 0$$

(5.38)

The usefulness of the above equation lies in its ability to predict the initial form of the transient. There is no need to assume that $x = y = 0$ at $t = 0$ as was necessary before.

writing

$$y' = q$$

The above equation can be written in the following form:

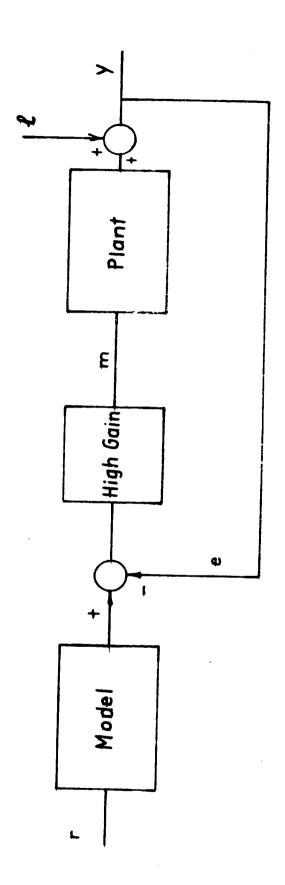$$q \frac{dq}{dy} + \frac{1}{y} q^2 + (3a - \frac{k_d r}{y}) q + a^2 y - k_d ar = 0 \qquad (5.39)$$

which is Abel's equation of the type already encountered in the case of additive disturbance.

$y(t)$ is a function of the initial values $y(o)$ and $y'(o)$. $y'(o)$ depends on k, thus explaining the dependence of the transient of $y(t)$ on k. The form of the solution is independent of changes in the sign and amplitude of k.

5.3 A Comparison of Invariance and High Gain Schemes

Consider the systems shown in Fig. 5.2 where a first order plant is controlled by a high gain feedback scheme. We shall explain qualitatively how this system differs in its approach to parameter invariance, as an objective, from the gain invariance scheme discussed in the previous section and represented in Fig. 5.1. Suppose that the plant and its model have identical dynamics. In the high gain feedback scheme the input signal will be shaped by means of a model which is then applied to the high gain feedback scheme so that this latter part of the system exhibits a unity transfer function.

The gain invariance scheme on the other hand, will not use this type of mechanism, but rather will apply the proper drive directly to the plant, the drive being generated by

Fig. 5.2

Sensitivity Reduction by High Gain Feedback Schemes

multiplication of the input signal with a correcting factor generated by the invariance scheme (to compensate for the gain difference).

Referring to Fig. 5.2 and assuming the model transfer function to be $\frac{k_d}{1 + \tau s}$, while that of the plant is $\frac{k}{1 + \tau s}$, and the high gain is k_o, we can write:

$$m = (\frac{k_d}{1 + \tau s} r - y) k_o \qquad \text{(with } \ell = 0) \qquad (5.40)$$

$$y = \frac{k}{1 + \tau s} m \qquad (5.41)$$

From equations (5.40) and (5.41) we can write

$$y = \frac{k}{1 + \tau s} (\frac{k_d}{1 + \tau s} r - y) k_o \qquad (5.42)$$

$$\therefore \left[1 + \frac{kk_o}{1 + \tau s}\right] y = \frac{kk_o \, k_d}{(1 + \tau s)^2} r \qquad (5.43)$$

$$\frac{(1 + kk_o) + \tau s}{1 + \tau s} y = \frac{kk_o \, k_d}{(1 + \tau s)^2} r \qquad (5.44)$$

$$\therefore y = \frac{kk_o \, k_d}{(1 + \tau s) \left[(1 + kk_o) + \tau s\right]} r \qquad (5.45)$$

and if k_o is sufficiently large, we have:

$$y \approx \frac{k_d}{1 + \tau s} r \qquad (5.46)$$

From equation (5.45) we can write

$$m = \frac{k_o k_d}{[(1 + kk_o) + \tau s]} r \tag{5.47}$$

If we assume $r = \frac{r_o}{s}$ (i.e. a step function of magnitude r_o), we have at $t = o_+$ (using the initial value theorem):

$$m = o$$

and at $t = \infty$ (using the final value theorem):

$$m \approx \frac{k_d}{k} r_o \tag{5.48}$$

The above equations show how the drive signal will be generated in the high gain, two-degrees of freedom configuration.

However, we shall now discuss some limitations and compromises that are always connected with the high gain schemes. With the use of high values of gain, the feedback transducer noise cannot be suppressed. Actually it will face a transfer function very close to unity. The situation with load disturbances is deceiving, since a linear type of analysis would indicate that high gain feedback can eliminate load effects instantaneously. However, this is not the case since a change in the output will cause an error magnified k_o times which will saturate the plant and make linear predictions impossible, such saturation will not happen in invariance schemes. Finally, high gain feedback is not always feasible. Examples are high-order plants, non-minimum phase plants, and pure delay plants. Even where its use is possible, the design of the above type of compensator is very cumbersome and involves

tedious trial and error methods [9].

Let us sum up the differences between high gain and invariance schemes; differences exist in the drive signal generation mechanism, in the effect of output additive disturbances (no generation of saturating signals in the invariance schemes), and in the method of synthesis. Therefore we can say that invariance systems are more suited for parameter sensitivity problems than linear feedback systems, by virtue of the afore-mentioned reasons.

5.4 Effect of Discrepancies in Fixed Parameters and Problems Associated with the Divider

The divider seems to be the most troublesome component of the invariance scheme and its presence in the schemes gives rise to some peculiar phenomena to be discussed in this section. The output of the divider both in the gain and inertia schemes yields information about the unknown parameter. The divider arrives at this information by evaluating the ratio of its two inputs. And since the unknown parameter is never infinite it seems that the denominator should not pass through a zero unless the numerator does so simultaneously. However, discrepancies in fixed parameters greatly distort this picture. With these discrepancies, the zero crossings may not be simultaneous, and the region around the zero crossing is usually one of high error. This might suggest for example, an averaging device or a finite memory, delay unit to follow the divider. The choice is left until the preliminary study on an analogue

computer is completed.

The dividers in the case of gain schemes have no zero crossings unless the system input does so. For the inertia schemes, zero crossing would occur even without the system input changing sign. Referring to Fig. 5.3, impulses and distorted information are liable to appear at the time of operation in the high error region. The use of delayed storage to supplement the division process (Fig. 5.4) offers an attractive solution. The delayed storage mechanism would work as follows: The output of the divider is fed simultaneously to the delayed storage unit, and the "upperbound-lowerbound cut off" unit, the latter one will generate an inhibit signal to the storage unit if the value of $\frac{x}{y}$ lies within a predetermined range (depending on a knowledge of the range of parameter variation). If the output of the divider exceeds the limits of the given range a release signal is generated which interrupts the main information path and connects the delayed storage unit. Thus the output l will be the most recent divider value (within the range defined before). The proper sign would be given to l if known. This may be done in both the main and delayed paths, since a wrong sign represents distortion in the information extraction. To show quantitatively how the discrepancy effects a rise let us assume that there is a difference between the weighting functions of the plant and its model.

$$\triangle h(t) \triangleq h_d(t) - h(t) \tag{5.49}$$

Then referring to Fig. 4.3, we have:
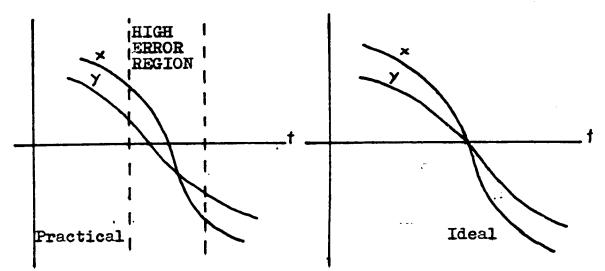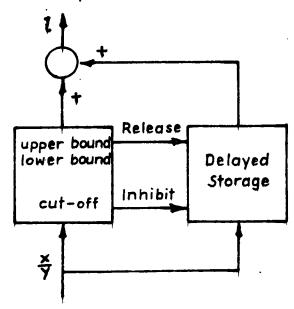
Fig. 5.3 Effect of Discrepancies in Modeling
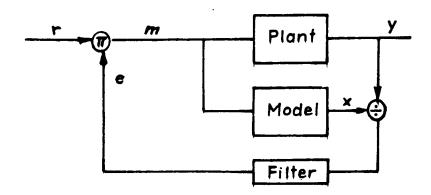

Fig. 5.4 Block Diagram of Delayed Storage Mechanism

Fig. 5.5 Divider Output Filtering

$$e = \frac{k_d \int_o^t h_d(t-\tau)\, m(\tau)\, d\tau}{k \int_o^t h(t-\tau)\, m(\tau)\, d\tau}$$

$$= \frac{k_d \left[\int_o^t h(t-\tau)\, m(\tau)\, d\tau + \int_o^t \triangle h(t-\tau)\, m(\tau)\, d\tau \right]}{k \int_o^t h(t-\tau)\, m(\tau)\, d\tau}$$

$$= \frac{k_d}{k} + \left[\frac{k_d \int_o^t \triangle h(t-\tau)\, m(\tau)\, d\tau}{k \int_o^t h(t-\tau)\, m(\tau)\, d\tau}\right] \tag{5.50}$$

the last term being the error term caused by the discrepancy. It can be easily shown to approach zero as $t \rightarrow \infty$.

The use of an averaging filter after the divider is an alternative to a delayed storage unit, and in the following analysis its effect on the output is explained.

Referring to Fig. 5.5 and assuming the plant gain to be k and that of the model k_d, we take the approximate value of the divider output as:

$$q = \frac{k_d}{k} \tag{5.51}$$

Then,

$$e = \frac{k_d}{k}\left(1 - e^{-\frac{t}{\tau}}\right) \tag{5.52}$$

where τ is the time constant of the filter. The overall closed loop equation can be written as:

$$y' + ay = k_d r(1 - e^{-\frac{t}{\tau}}) \tag{5.53}$$

This would mean the equivalent overall gain will not be that of the model except after a length of time dependent on τ. Similar arguments applied to the inertia system would lead to the following equation for the scheme with a filter (after the divider):

$$y' + \left[a_d - (a_d - a(t))e^{-\frac{t}{\tau}} \right]y = kr \tag{5.54}$$

5.5 The Composite Gain Inertia Scheme:

The desire to broaden the sphere of application of invariance schemes would raise the question: Is there a method of instantaneous identification that would handle more than one parameter simultaneously? The answer to this question seems to be negative at the moment. It seems that delays are needed to evaluate more than one parameter. A more direct two-parameter invariance approach is to investigate the possibility of compound schemes with both the gain and inertia scheme involved. If we consider the steady-state conditions, it is easy to see that the gain schemes are not affected by changes in the coefficients of the denominator polynomial of the plant transfer function (except the free term). On the other hand, for inertia schemes, a change in plant gain would affect the transient behaviour as well as making the system unstable. This is due to the steady difference between the outputs of the plant and its model, (see Fig. 4.5),

which yields a constant numerator in the divider, while the denominator of the divider will tend to zero with the increase of time. Thus if we want to use both schemes around a plant with gain and inertia variations, we have to use the gain scheme from the inside. The mechanism of action will be as follows: the gain scheme around the plant would provide a system with variable inertia (this is only ideally) and the inertia scheme will handle such a system to produce the over-all invariance.

Such a scheme is shown in Fig. 5.6. The performance of this compound scheme is very difficult to investigate analytically, but the analogue studies that will be exposed in the next chapter have shown the over-all system to be stable and its transient to involve improvement over the case with only the gain scheme employed.

5.6 Some Numerically Evaluated Performance Curves

The transient response curves given in this section are accurately evaluated through the numerical solution of the system canonical equations. This was done using the IBM 7090 and a Modified Runge-Kutta equation integrator sub-routine. Most of the examples that will be given here are connected with a first order system. Digital accuracy was sought to give correct evaluation of the response under ideal circumstances. A second order system was also worked out for comparison purpose with the simulation results of the next chapter using an analogue computer.

Fig. 5.6 The Composite Gain-Inertia Scheme

i Connected with
  Inertia
g Connected with
  Gain
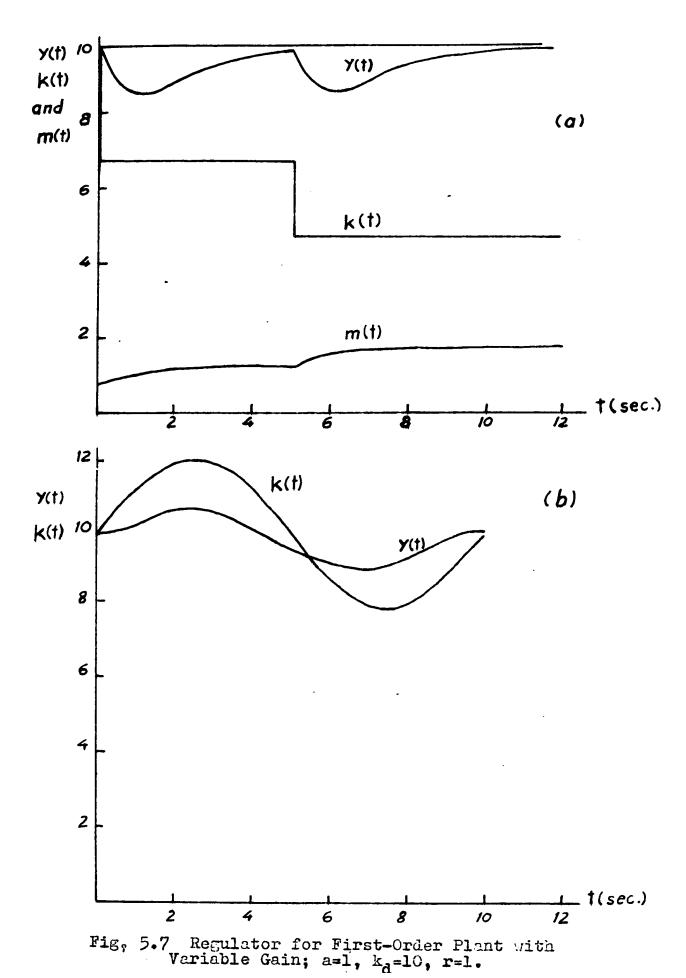
To begin with, the response of a first order gain scheme with model parameters $a=1$ and $k_d=10$ is investigated under step-type and sinusoidal-type variations in $k(t)$ (Fig. 5.7a and b). The step change is $k(t) = 10 \rightarrow 7 \rightarrow 5$ applied at $t = -\infty, 0, 5$ seconds respectively and the sinusoidal change has the equation $k(t) = 10 + 2 \sin 0.63t$. The input to the system in all the above tests is unity.
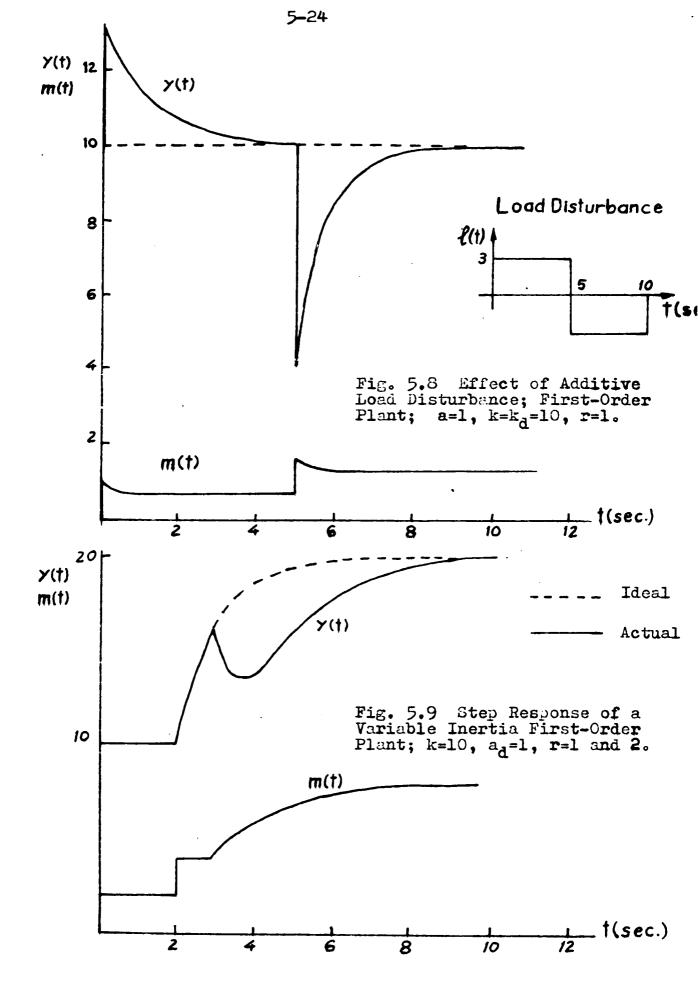
The effect of an additive disturbance (step-type) acting on the plant output as in Fig. 5.1 is then evaluated (Fig. 5.8). The plant under consideration is the same as the one used before except that its parameters are in complete correspondence with those of the model. The disturbance has the shape of a rectangular wave of 3 units in amplitude and 10 seconds period (see Fig. 5.8). The input to the system is unity.

All the above tests are for the system acting in the regulator mode. The changes in the gain are seen to cause a transient after which the system output converges to the nominal output with a time constant equal to that of the plant.

The second type of test is for the response to inertia changes. The same plant is subjected to a step increase in its inertia, while its output was still rising under the effect of a step input. The particulars of the test are; $k = k_d = 10$, $a_d = 1$ and the step inertia change is $a(t) = 1 \rightarrow 2$ applied at $t = -\infty, 3$. The system response is plotted in Fig. 5.9. The test was done two times for input signal levels of $r=1$ and 2 and the plot in the above mentioned figure is for

Fig. 5.7 Regulator for First-Order Plant with
Variable Gain; a=1, k_d=10, r=1.

Fig. 5.8 Effect of Additive Load Disturbance; First-Order Plant; a=1, k=k$_d$=10, r=1.

Fig. 5.9 Step Response of a Variable Inertia First-Order Plant; k=10, a$_d$=1, r=1 and 2.
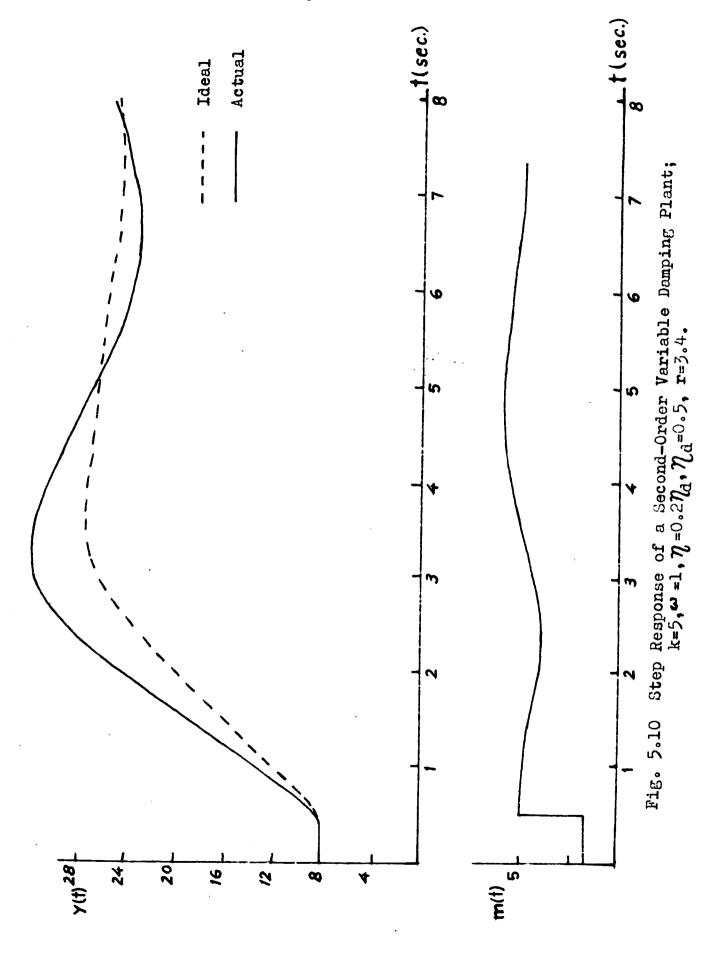
the two cases with the latter at half the scale of the former and as is seen they completely coincide, thus showing that the input level does not play any part in the conducted test. The step increase in inertia seems to cause a transient after which the overall output will rise again to the required steady-state value with approximately the nominal time constant.

Now, we consider the second order plant having the following parameters, $k = 5$, $\omega = 1$, $\eta = 0.1$, and a model $\eta_d = 0.5$. The input signal = 5 units and the initial output = 8 units. The response of the invariance scheme is given in Fig. 5.10.

A second order plant with $k = k_d = 5$, $\eta = 0.1\eta_d$, $\eta_d = 0.5$ and an additional deviation of $T = \frac{1}{\omega} = 1.42T_d$, $T_d = 1$, was then controlled by an inertia scheme supplemented by a delayed storage unit. The initial values are all set to zero. The input signal = 2 units and the starting value for the signal in the delayed storage is 0.12 units. The upper and lower bounds imposed on the signal l are ±0.24 units. The value of l if the zero crossings of the input signals to the divider coincide (the case of a deviation in η only) is 0.18 units. The resulting responses for the cases of corrected and uncorrected sign of l (see page 5.16) are shown in Fig. 5.11.

The general conclusions from the examples tested are: i) a step change of parameters will cause a relatively low amplitude deviation from the desired shape which will disappear with time; ii) sinusoidal gain variations would cause attenuated, periodic changes in the output; iii) the signal

Fig. 5.10 Step Response of a Second-Order Variable Damping Plant; $k=5$, $\omega_n=1$, $\eta=0.2$, η_d, $\eta_d=0.5$, $r=3.4$.

Ideal

Actual

$Y(t)$

$m(t)$

t(sec.)

t(sec.)
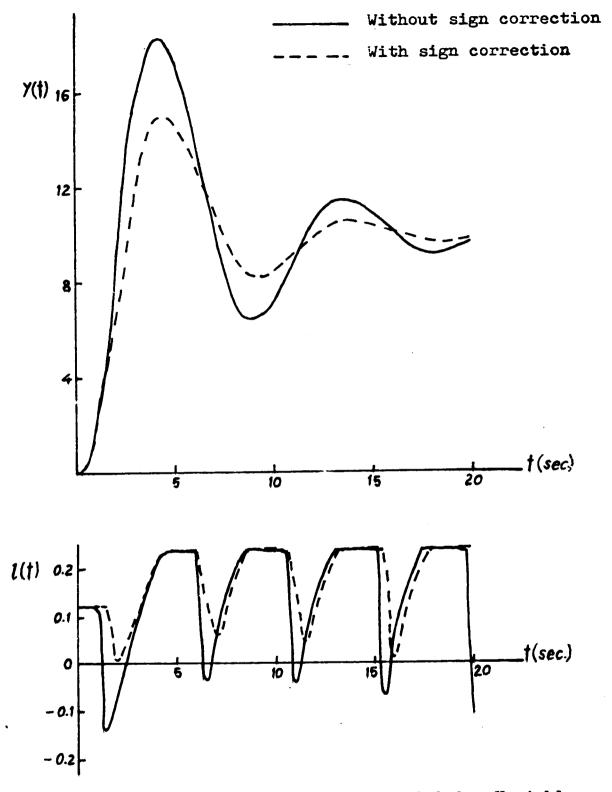
Fig. 5.11 Step Response of Second-Order Variable Damping Plant with Discrepancies in the Uncompensated Parameters, Scheme Encountering a Delayed Storage Unit; $k=5$, $T=1.42T_d$, $T_d=1$, $\eta=0.1\eta_d$, $\eta_d=0.5$, $r=2$ units.

level does not cause any changes in the shape of the transients; and iv) the modulation of the drive signal caused by the gain schemes is in the form of smooth changes towards the constant level needed to cope with the steady-state situation.

CHAPTER 6

A SIMULATION STUDY OF THE SCHEMES

6.1 Description of the Equipment Used

The object of the study conducted in this Chapter is to investigate, as closely to the practical situation as possible, the behaviour of the proposed non-linear invariance schemes. Such situations that we have in mind are: generated noise in various blocks and signal transmission paths, inaccuracies in measuring the fixed parameters of the plant, and time delays that can occur in the division and multiplication comparators or that may be inherent in the control object. All these factors will be encountered automatically to a small extent in the analogue simulation because of the nature of the simulation process. In case particular emphasis is desired on one of the above aspects specifically, additional circuitry and signals can always be used to stress the required effect. This study will also establish the stability and the freedom from limit cycles for our schemes, since no rigorous mathematical analysis is possible at this time.

The computer used is the Berkeley Ease, Model 1032, an electronic analogue type. For the representation of the plant and its models, high gain feedback amplifiers are used throughout according to circuits recommended by [14]. For the linear comparator, summators are used and for a multiplication comparator, an all-quadrant electronic multiplier is used. The

only element whose description is worth detailed discussion is
the division comparator. As already shown, it is the most
sensitive and troublesome element of the proposed schemes.

The divider used in the simulation is built up from
a servo-multiplier and a high gain feedback amplifier according
to Fig. 6.1 The equation governing their action is the follow-
ing:

$$e = -k(x + ey)$$

$$= \frac{-kx}{1 + ky}$$

and if k is kept high enough we have

$$e \approx -\frac{x}{y}$$

The servo-multiplier uses an automatic potentiometer
driven by a servo loop with the following specifications: for
a 10 volt step input the rise time is 20 milli-seconds; from
10 per cent to 90 per cent of the final value, the settling
time is 60 milli-seconds before output remains within 5 per cent
of final value and the overshoot is 1.5 volts.

The circuit recommended by the computer manufacturers,
for building the divider, uses the multiplier only as a two
quadrant type. A four quadrant divider is not needed for gain
schemes, but for inertia schemes it is frequently needed, an
example is the case of the second order variable damping
scheme simulated on the digital computer in the last chapter.
The fact that in our analogue study the divider used is a two
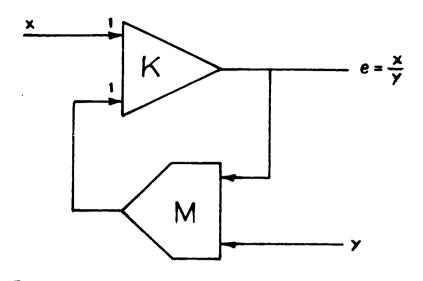
Fig. 6.1   Block Diagram of the Divider
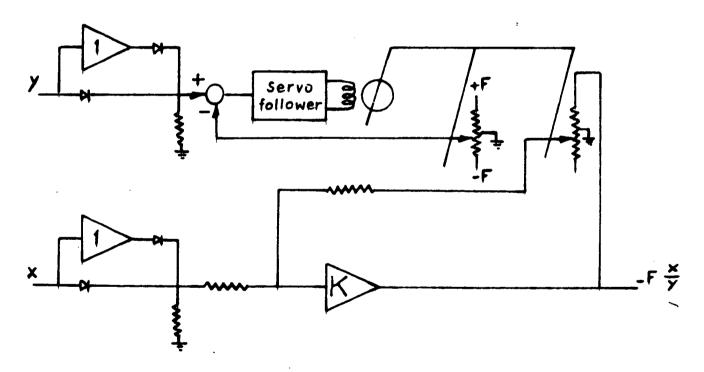
Fig. 6.2   Scheme of the Divider with Pre-divider Rectifiers

quadrant one, leads to the need (with inertia schemes) for full wave rectifiers at its two inputs (Fig. 6.2). This would be accompanied by the proper choice of input signal polarity and plant parameter change direction (relative to the model) so that one is able to compensate for the possibly incorrect sign in the output of the division process. In the scheme of Fig. 6.2 the rectifiers are arranged for the case when $\eta < \eta_d$. However, when $\eta > \eta_d$ the rectifiers should be reversed. Hence, this type of simulation necessitates a previous knowledge of the direction of change of η. A four quadrant divider would eliminate the need for this knowledge provided that there is no deviations in the uncompensated parameters.

Due to the deviations in the uncompensated parameters, the zero crossings of the two divider inputs will not coincide giving rise to high signals. However the fact that the amplifier would saturate at 100 volts limits those signals to values slightly higher than 100 volts, an effect quite similar to the action of delayed storage schemes.

6.2 The Study of First-Order Systems

As the behaviour of the first-order system was previously studied using the digital computer, only two tests will be conducted here to provide a sort of comparison. The plant chosen here has $P(s) = \frac{k}{1 + \tau s}$ where $\tau = 10$. The model has a gain $k_d = 4$, a time constant $\tau = 10$, and the input step is 10 volts in magnitude. The records of Fig. 6.3a show the step
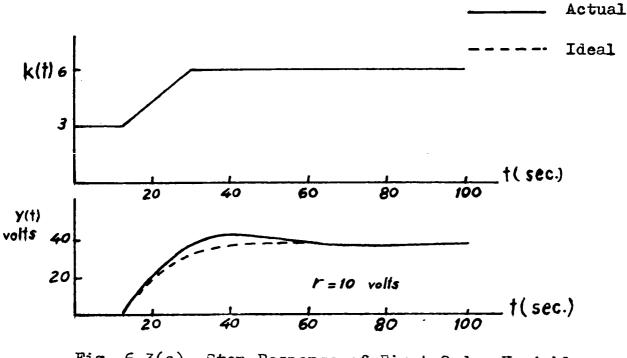
Fig. 6.3(a)   Step Response of First-Order Variable
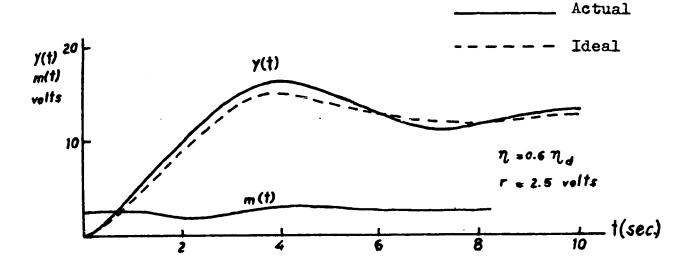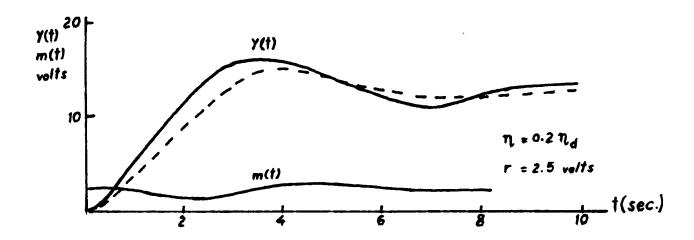Gain Scheme; $\tau=10$, $k_d=4$.
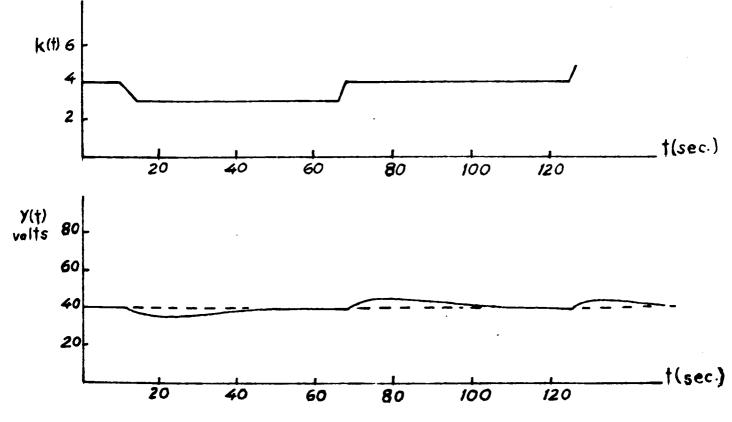
Fig. 6.3(b)   Regulator Action of Variable Gain Scheme;
$\tau=10$, $k_d=4$.

response, when k of the plant varies linearly from 3 to 6, during the transient period, and Fig. 6.3b shows the change in the output as a result of gain variations after the system was brought to the steady-state. The last curves have the same form as the ones obtained previously using digital methods.

6.3 The Study of Second-Order Systems

The plant with the following transfer function $P(s) = \dfrac{k}{T^2 s^2 + 2 \eta T s + 1}$ is to be studied. The model parameters are $k_d = 5$, $\eta_d = 0.5$, $T_d = \dfrac{1}{\omega_d} = 1$. First we investigate a control scheme of the inertia type. A similar test was conducted in the last chapter (Fig. 5.10) using digital techniques. The parameter that will vary widely is η, the damping of the plant. Fig. 6.4 represents the step response for η equal to 0.6, 0.2 and 2 of η_d respectively, but the rest of the plant parameters are in complete correspondence with those of the model. The results show close correspondence between the overall system and the desired nominal response. Without the invariance scheme the plant is heavily oscillating when η reaches a value such as $0.2\,\eta_d$ (Fig. 6.5). The first two tests with an additional deviation, $T = 1.2\,T_d$, included to simulate errors in the evaluation of one of the fixed parameters, is shown in Fig. 6.7. The input in the above two groups of tests were r = 2.5 volts and 3 volts respectively. The records of Fig. 6.7 as compared to those of Fig. 6.4 show more deviation of the output from the nominal response and the phenomena of drive impulses already predicted in the analysis is seen in
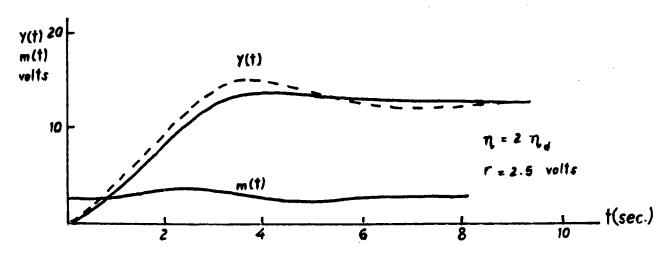
Fig. 6.4 Step Response of Second-Order Inertia Scheme
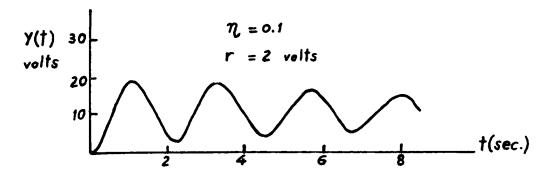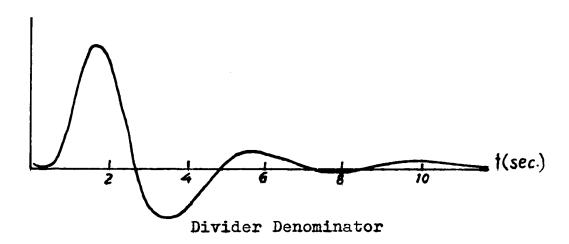with Discrepancy in η only;
$k=k_d=5$, $\eta_d=0.5$, $T=T_d=1$.

Fig. 6.5 Step Response of Second-Order
Uncompensated Plant; k=5, T=1.
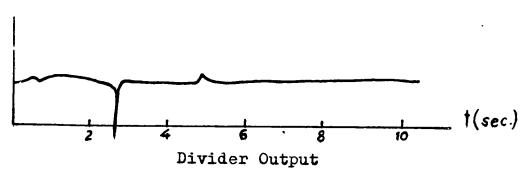

Divider Denominator

Divider Output

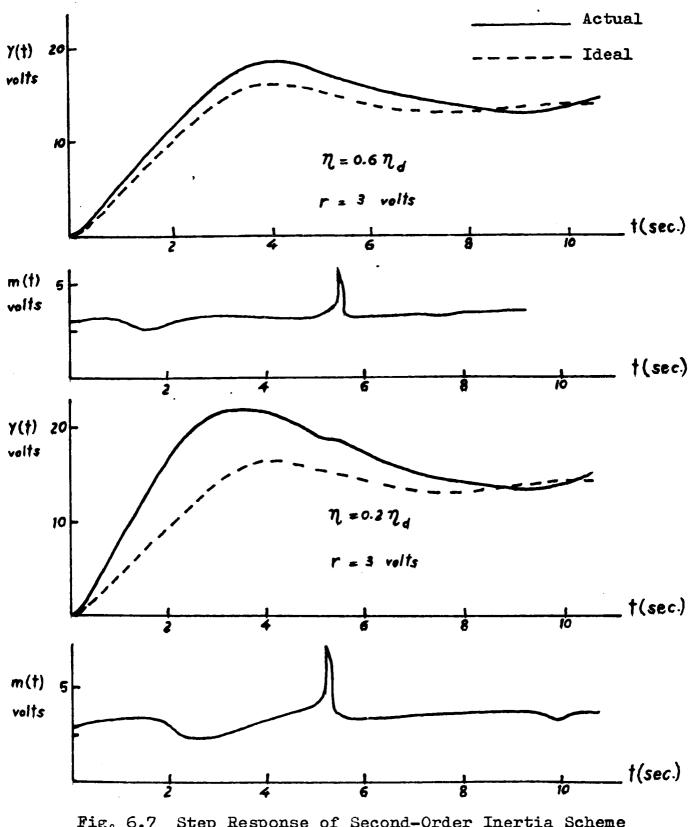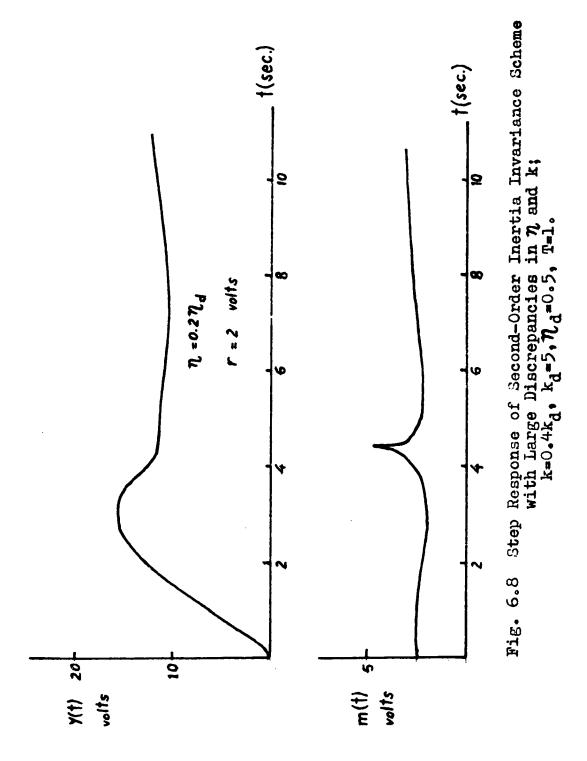Fig. 6.6 Effect of Non-Coincidence in Zero Crossing

Fig. 6.7 Step Response of Second-Order Inertia Scheme with Discrepancies in Both η and T; $T=1.2T_d$, $T_d=1$, $k=k_d=5$, $\eta_d=0.5$.

these records. Fig. 6.6 shows the coincidence of the impulse appearance with the divider's denominator zero crossing.

Fig. 6.8 shows that the inertia invariance scheme is inadequate for handling gain discrepancies (the output is steadily increasing, being limited only by the saturation of the amplifiers). This fact is also predicted in the analysis. Finally, in the test of the inertia scheme, two cases of extreme deviation are shown: i) in Fig. 6.9 where $\eta = 0.1\eta_d$ and $T = 0.71 T_d$; ii) in Fig. 6.10 where $\eta = 0.1\eta_d$ and $T = 1.42 T_d$. The gain is fixed at $k = 5$, and the input signal in both tests is $r = 2$ volts. The response in both cases is already far from the nominal, but more or less exponential in shape. Much of this shape may be attributed to the pre-divider rectifiers combined with saturation of the divider output at non-corresponding zero crossings. Case (ii) is studied in detail at the end of section 6.6.

We now proceed to a study of gain schemes in conjunction with the second-order plant that was used above. The step response of such a scheme for a case of $k = 4 k_d$ is shown in Fig. 6.11, where $k_d = 10$, and all the other parameters are in complete correspondence with those of the model. The input was $r = 5$ volts. Finally, curve (a) in Fig. 6.12 shows the response of the previous inertia controlled plant with $\eta = 0.125\eta_d$, and $r = 3.5$ volts when a filter with the transfer function $P(s) = \frac{1}{1 + s}$ is included in the path of the modulating signal e and compares with the filter-free curve (b).
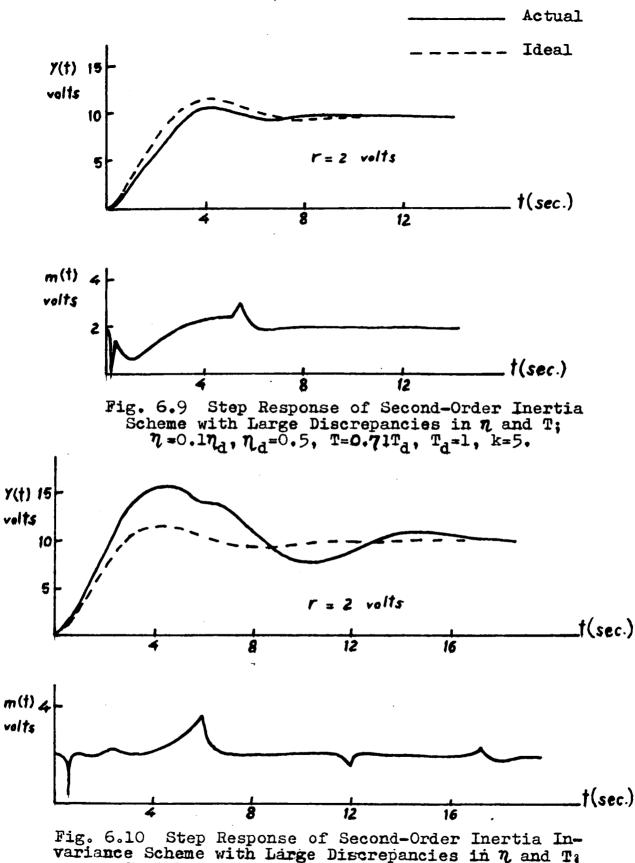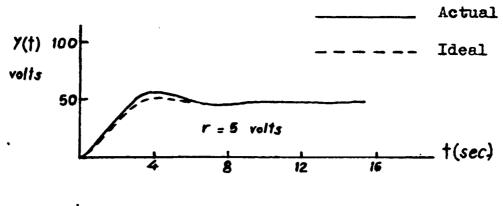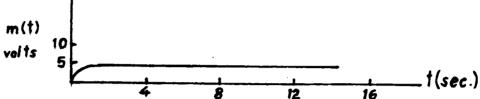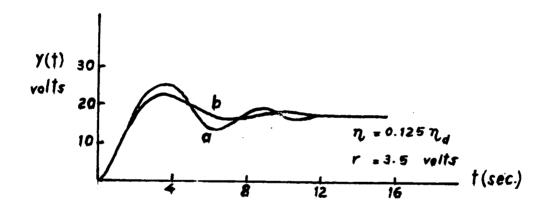
Fig. 6.8 Step Response of Second-Order Inertia Invariance Scheme with Large Discrepancies in η and k; $k=0.4k_d$, $k_d=5$, $\eta_d=0.5$, $T=1$.

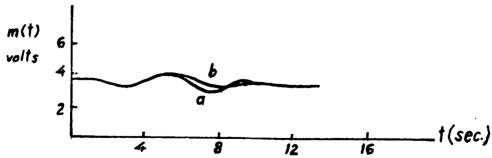Fig. 6.9 Step Response of Second-Order Inertia Scheme with Large Discrepancies in η and T; $\eta = 0.1\eta_d$, $\eta_d = 0.5$, $T = 0.71T_d$, $T_d = 1$, k=5.

Fig. 6.10 Step Response of Second-Order Inertia Invariance Scheme with Large Discrepancies in η and T; $\eta = 0.1\eta_d$, $\eta_d = 0.5$, $T = 1.42T_d$, $T_d = 1$, k=5.

Fig. 6.11   Step Response of Second-Order Gain
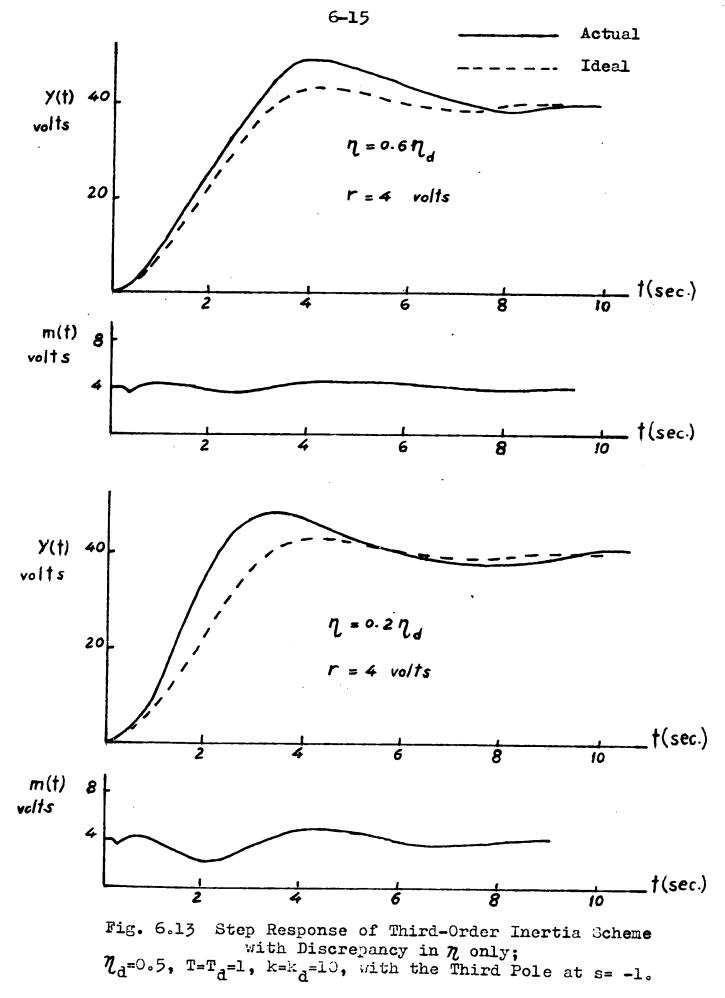Scheme with Discrepancy in k only;
$k=4k_d$, $k_d=10$, $\eta=0.5$, $T=1$.

Fig. 6.12   Effect of Filters in the Modulation Path
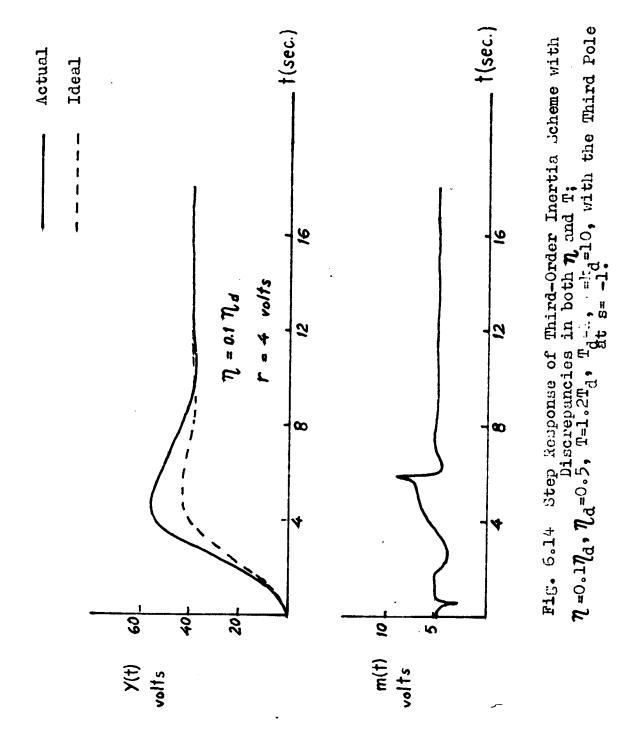on Step Response of Second-Order Inertia Scheme;
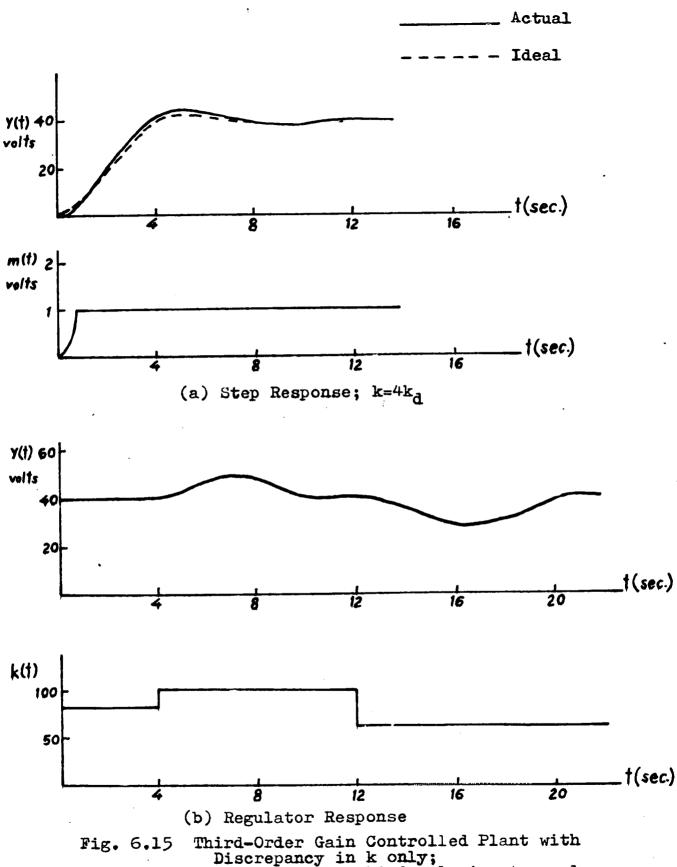$k=5$, $\eta_d=0.5$, $T=1$
(a) With filter
(b) Without filter

6.4 The Study of Two Feedback Unstable Systems

i) A third-order system: The third-order system that will be used for this investigation is made up by adding a pole to the second-order plant of the last section. If in addition, the model gain was increased from 5 to 10 while $T_d = 1$ and $\eta_d = 0.5$, the situation would become feedback unstable. Thus we have for our objective the following nominal transfer function $\dfrac{k}{(s+1)(T^2 s^2 + 2\eta Ts + 1)}$ which cannot be achieved through feedback around the variable parameter plant. Unity feedback around such a nominal plant would give rise to the following characteristic equation $s^3 + 2s^2 + 2s + 11 = 0$ which does have "right half poles". The above building-up of the plant would also serve to show the effect of adding poles on the proper functioning of the identification scheme.

The tests carried on in this case are of the same nature and scope as those done before in conjunction with the second-order plant. First we review tests on the plant when controlled by the inertia scheme. Fig. 6.13 shows the step response for the cases of damping values of $\eta = 0.6$ and 0.2 of η_d respectively, the other parameters being in complete correspondence. The input signal is 4 volts. Fig. 6.14 is for the case of $\eta = 0.1\eta_d$ and $T = 1.2\ T_d$, and no deviation in the plant gain since this would lead to instability (as explained before). Secondly, we review the tests on the plant when controlled by a gain scheme. Fig. 6.15a shows the step response and modulated drive for a given deviation of $k = 4\ k_d$, and Fig. 6.15b shows the regulator response of the invariance

Fig. 6.13 Step Response of Third-Order Inertia Scheme
with Discrepancy in η only;
$\eta_d=0.5$, $T=T_d=1$, $k=k_d=10$, with the Third Pole at s= -1.

Fig. 6.14 Step Response of Third-Order Inertia Scheme with
Discrepancies in both η and T;
$\eta = 0.1 \eta_d$, $\eta_d = 0.5$, $T = 1.2 T_d$, $T_d = ?$, $?d = 10$, with the Third Pole
at $s = -1$.

_____ Actual

_ _ _ _ _ Ideal

(a) Step Response; $k=4k_d$

(b) Regulator Response

Fig. 6.15 Third-Order Gain Controlled Plant with
Discrepancy in k only;
$k_d=10, \eta=0.5$, T=1, and the third pole is at s= -1,
r=4 volts.

scheme. Fig. 6.16a shows the step response for a given deviation of $k = 8k_d$ when other deviations $\eta = 0.8\eta_d$, $T = 1.2T_d$ are also present, and Fig. 6.16b shows the regulator response under the same conditions. Notice the tendency for a prolongation of the transient period caused by the inertia deviations.

The following general observations can be made from the above results: the invariance schemes basically functioned independently of the additional pole in the plant; they were not particularly sensitive to tolerances in the fixed parameters, but the presence of such tolerances had a detrimental effect on the transient process.
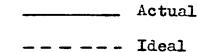
ii) A non-mini-phase plant: The plant chosen for this investigation is one of particularly queer dynamics. It is not chosen for its practical value but as an extreme case to be used in the demonstration of the relative independence of invariance schemes from plant dynamics. Such a plant would not lend itself to the use of high gain feedback.
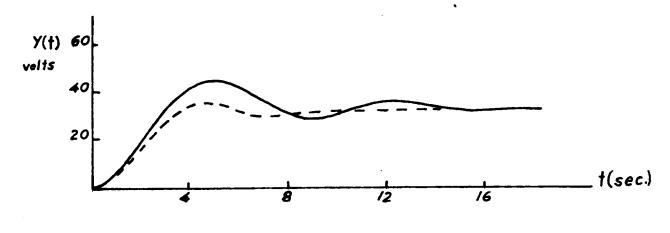
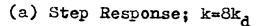Consider the plant with the following transfer function:

$$P(s) = \frac{k(T_2 s - 1)^2}{(T_1 s + 1)(T_3 s + 1)}$$

where $T_{1_d} = 3$, $T_{2_d} = 2$, $T_{3_d} = 1$.

The circuitry used for its simulation is that recommended by [14] and shown in Fig. 6.18. The plant will be controlled
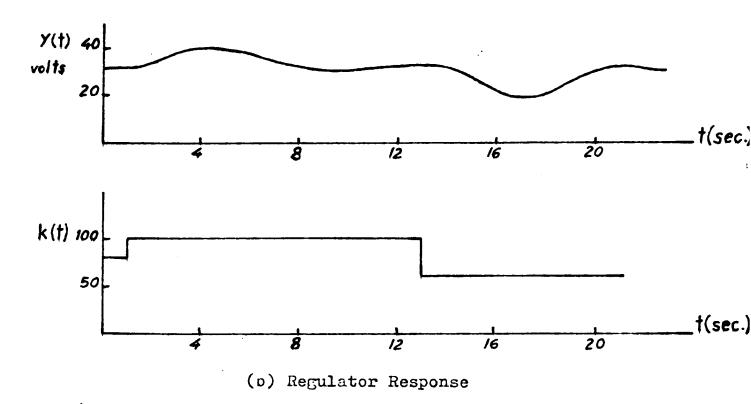
(a) Step Response; $k=8k_d$

(b) Regulator Response

Fig. 6.16 Third-Order Gain Plant with Discrepancies in k, η and T; $k_d=10$, $\eta=0.8\eta_d$, $\eta_d=0.5$, $T=1.2T_d$, $T_d=1$, with the Third Pole at $s=-1$, $r=4$ volts.
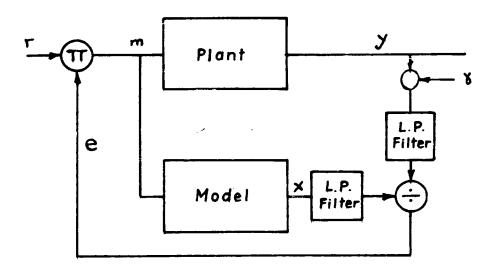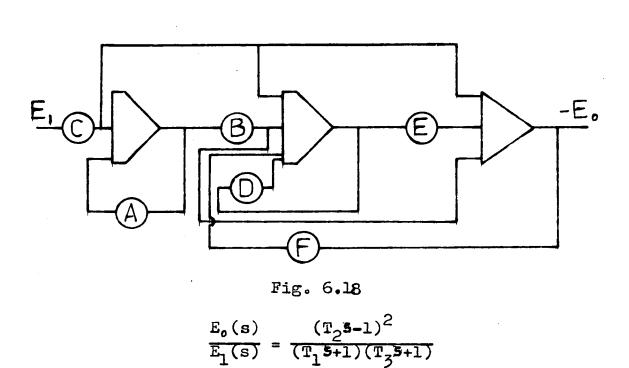
Fig. 6.17

Scheme for a Non-Min-Phase Plant

Fig. 6.18

$$\frac{E_o(s)}{E_1(s)} = \frac{(T_2 s - 1)^2}{(T_1 s + 1)(T_3 s + 1)}$$
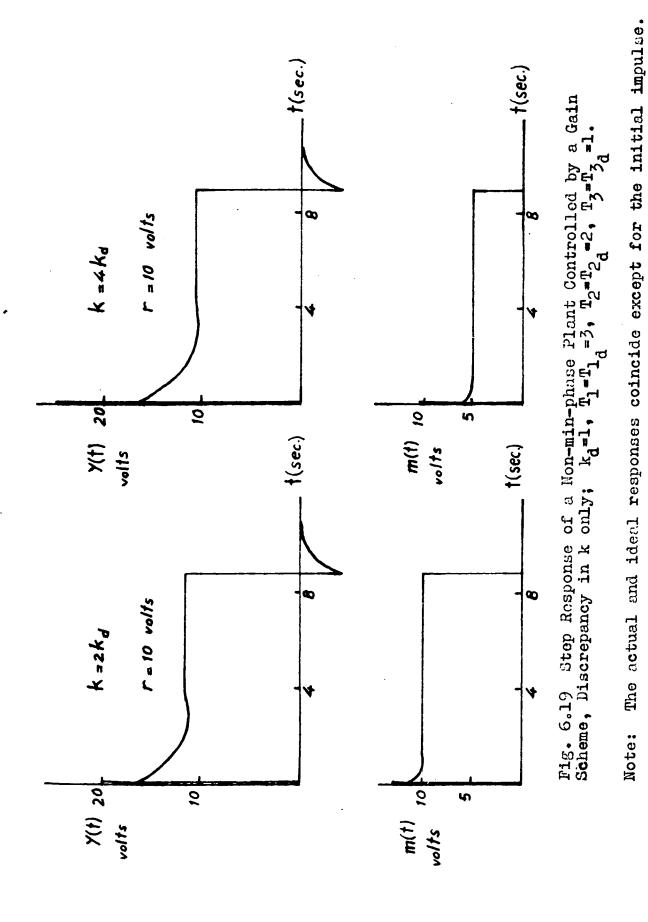
$$T_1 = \frac{1}{D - EF} , \qquad T_2 = \frac{1}{D - E} , \qquad T_3 = \frac{1}{A}$$

through a gain scheme. Because of the initial simulation study, practical reasons (instability of the simulated system) dictated the use of a L.P.F. with the transfer function $\frac{1}{1+s}$ before the inputs to the divider as shown in Fig. 6.17. From the figure it is easy to see that the situation as far as identification is concerned is identical with the case of a plant with an additional pole at s = -1.

The response of the system to step inputs is shown under two cases of parameter deviations in Fig. 6.19. It shows the responses for cases where k = 2 and 4 k_d (and k_d = 1) but all the other parameters are in complete correspondence. The input is taken as r = 10 volts. Fig. 6.20 shows the transients caused by gain changes when the above system is already in the steady-state. Fig. 6.21 shows the step response and the gain change transients in the case of a pole movement of T_1 = 0.5 T_{1_d} combined with a gain deviation of k = 2 k_d.

The shown results tend to emphasize the relative freedom of the invariance mechanism from the plant dynamic effects. This is what actually makes these systems more suitable for the parameter problems than the high-gain schemes, where the plant dynamics play a decisive role.

It is interesting to note that the signal at the output of the filter is identical with the output of a more practical plant with the number of poles exceeding the number of zeros by one.

Fig. 6.19 Step Response of a Non-min-phase Plant Controlled by a Gain Scheme, Discrepancy in k only; $k_d=1$, $T_1=T_{1_d}=3$, $T_2=T_{2_d}=2$, $T_3=T_{3_d}=1$.

Note: The actual and ideal responses coincide except for the initial impulse.
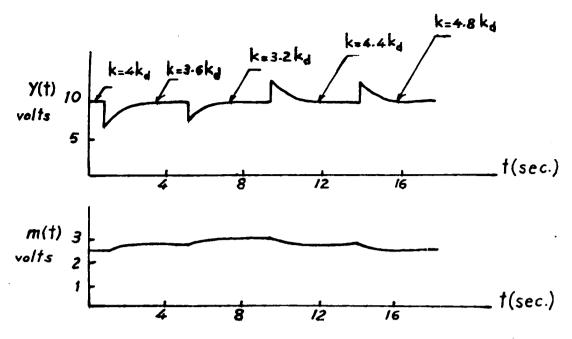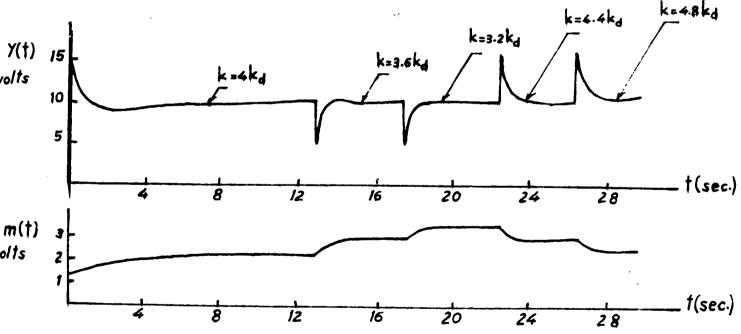
Fig. 6.20   Gain Scheme Working as a Regulator Around the Non-min-phase Plant with Changes in k only; $k_d=1$, $T_1=T_{1_d}=3$, $T_2=T_{2_d}=2$, $T_3=T_{3_d}=1$, r=10 volts.

Fig. 6.21   Response of the Regulated Plant to Changes in k and an error in T; $k=2k_d$, $T_1=0.5T_{1_d}$, $T_{1_d}=3$, $T_2=T_{2_d}=2$, $T_3=T_{3_d}=1$, r=10 volts.

6.5 A Study of the Composite Scheme

The schemes to be tested in the following are especially suited for plants with essential variations in two parameters (one gain and one inertia parameter). The block diagram is shown in Fig. 5.6. In the following, results of the different experiments conducted on the above configuration, in conjunction with a second and a third-order plant, are given. The second-order plant has the transfer function

$P(s) = \dfrac{k}{T^2 s^2 + 2\eta Ts + 1}$ where $T = \dfrac{1}{\omega}$ and the third-order plant

is made from it by adding the pole $(s + 1)$ to the second-order plant. For the second-order plant the variations will be $k = 8k_d$ and $\eta = 1, 0.5, 0.1\eta_d$ respectively where $k_d = 10$, $\eta_d = 0.5$ and $T = T_d = 1$. Fig. 6.22a, b and c shows a comparison between the transients of each of the three chosen cases using a composite scheme and the case when the plant is controlled by a gain scheme only. Fig. 6.23 shows the transient of the third-order plant in the case of complete correspondence with the model as compared to the case of $k = 5k_d$ and $\eta = 0.5\eta_d$.

The composite use of two schemes is seen to introduce improvement over the use of only one of them. However, for small tolerances in the fixed parameters with gain being the predominant variable, a gain scheme could handle the situation successfully.

6.6 Miscellaneous Tests

In this part some tests of general character and different practical significance will be conducted. The aim
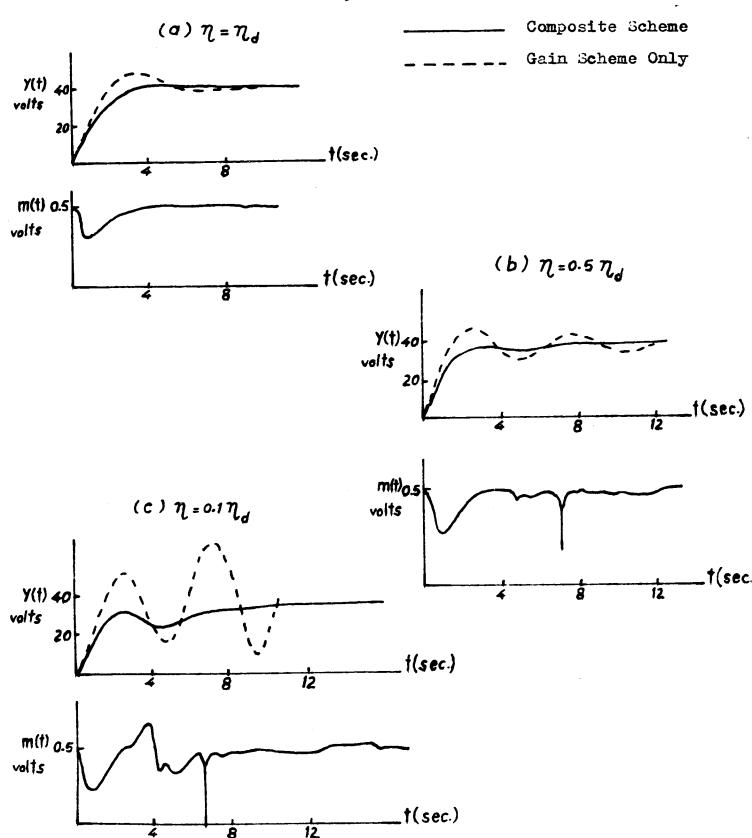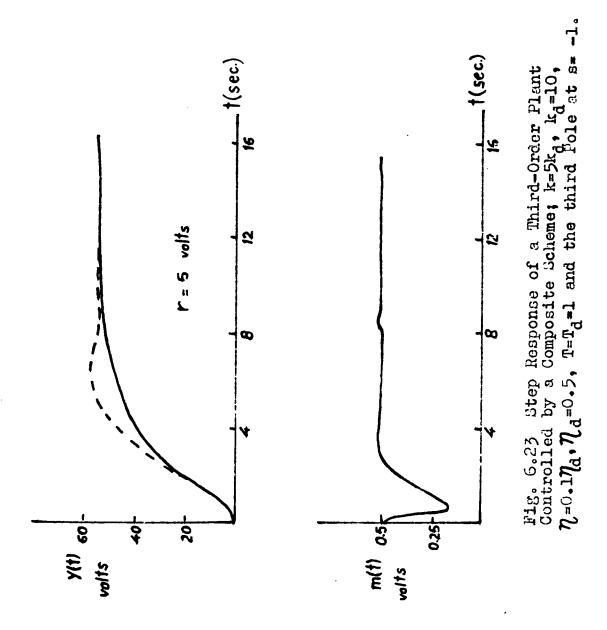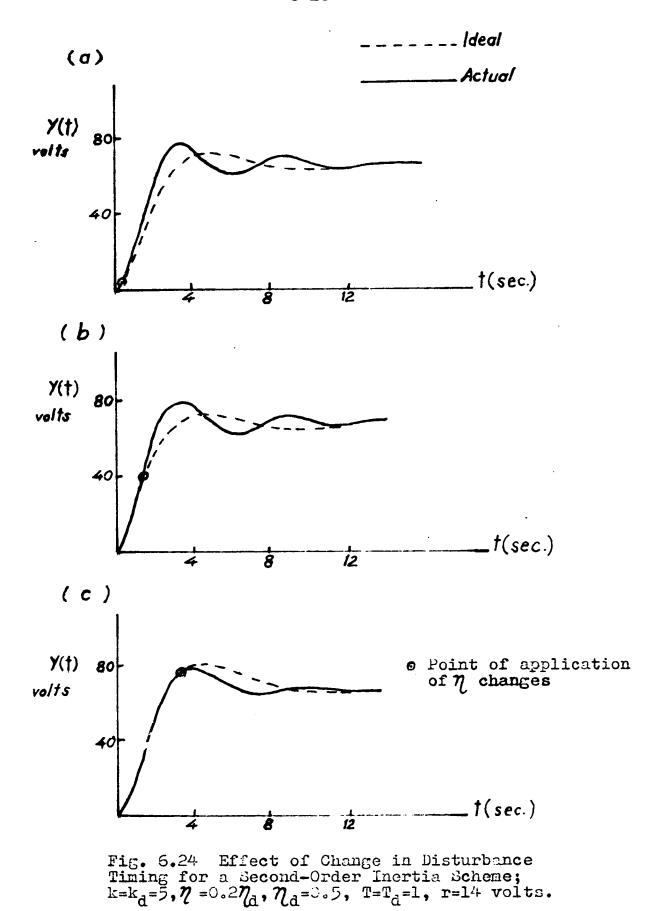
Fig. 6.22 Step Response of a Second-Order Plant Controlled by a Composite Scheme; $k=8k_d$, $k_d=10$, $\eta_d=0.5$, $T=T_d=1$, $r=4$ volts.

Fig. 6.23 Step Response of a Third-Order Plant
Controlled by a Composite Scheme; $k=5k_d$, $k_d=10$,
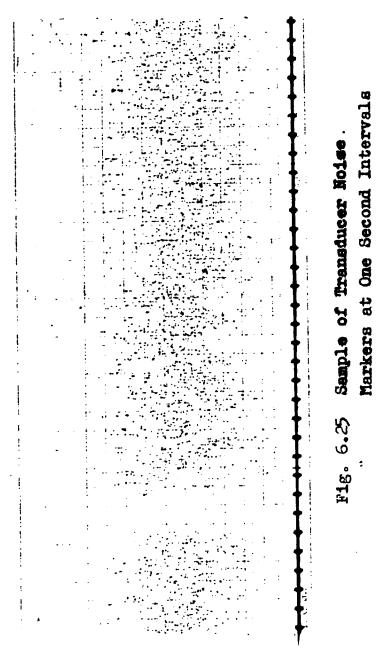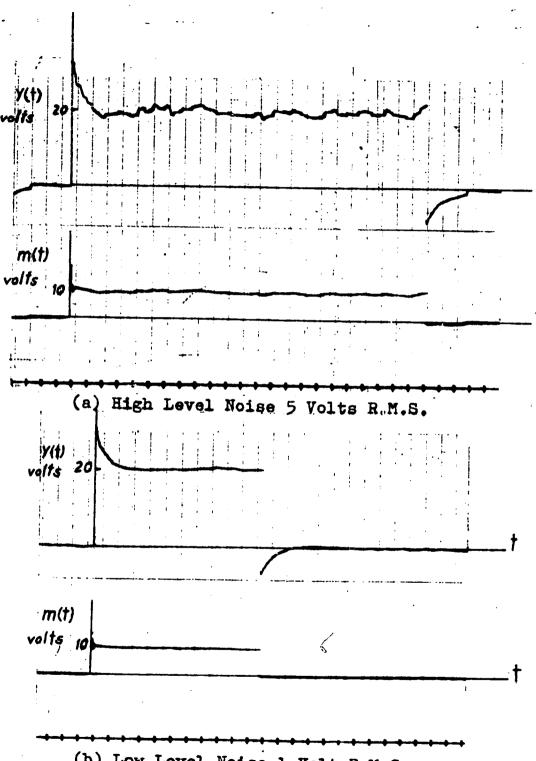$\eta=0.1\eta_d$, $\eta_d=0.5$, $T=T_d=1$ and the third pole at $s=-1$.

is to ascertain further the reliability of the proposed schemes under a practical environment.

i) First, the effect of the disturbance timing relative to the timing of the input is to be studied. Fig. 6.24 shows the transient response of a second-order system considered previously and controlled by an inertia scheme for cases where the parameter change (step change in the damping) occurs with the input application near the middle of the transient, and in the last curve at the end of the transient. The results indicate that the first case is the most critical. This coincides with the intuitive estimation.

ii) Secondly, the effect of feedback transducer noise Y, Fig. 6.17 (the noise used has a bandwidth from 0 to 10 cps., a sample of which is given in Fig. 6.25) in conjunction with the non-mini-phase plant considered previously and the gain invariance scheme is shown in Fig. 6.26(a) and (b) for low (1 volt R.M.S.) and high (5 volt R.M.S.) noise levels respectively. (The steady plant output is adjusted to 20 volts and the plant parameters adjusted to complete correspondence with those of the model $k_d=2$, $T_1=3$, $T_2=2$, $T_3=1$.) The chosen plant represents an extreme case for a response with relatively flat frequency characteristics. The results show that the noise is apparent in the system output, but reasonably attenuated because of the filters at the input to the divider.

The monotonic increase in the noise level leads to monotonic increase in the plant output distortion; this is seen from Fig. 6.27 where the noise level is decreased from

Fig. 6.24 Effect of Change in Disturbance
Timing for a Second-Order Inertia Scheme;
$k=k_d=5$, $\eta = 0.2\eta_d$, $\eta_d = 0.5$, $T=T_d=1$, $r=14$ volts.

Fig. 6.25 Sample of Transducer Noise.
Markers at One Second Intervals

(a) High Level Noise 5 Volts R.M.S.

(b) Low Level Noise 1 Volt R.M.S.

Fig. 6.26 Step Response and Drive Shape for a Non-min-phase System with Feedback Transducer Noise. Input Step 10 volts, k_d=2. Markers at One Second Intervals.
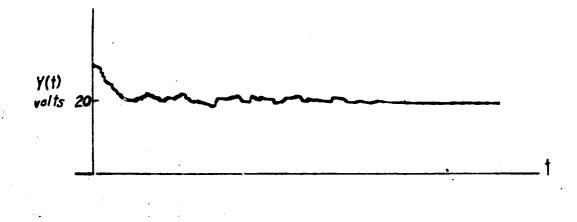
Fig. 6.27 System Output Change as a Result of Monotonic Decrease in Transducer Noise from Starting Value of 5 Volts R.M.S. to Zero. Markers at One Second Intervals.
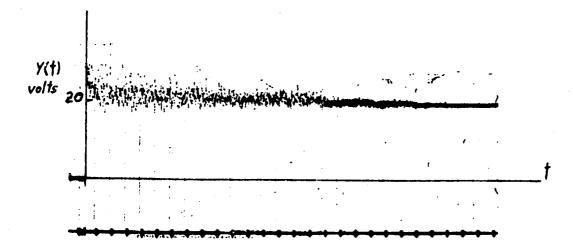

Fig. 6.28 System Output Change as a Result of Monotonic Decrease in Noise Entering After the Output Filter from Starting Value of 3 Volts R.M.S. to Zero.
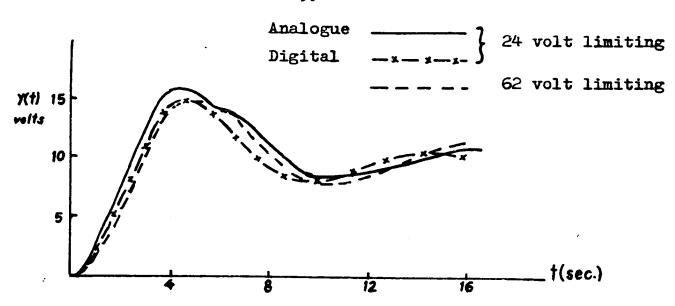
5 to zero volts R.M.S.
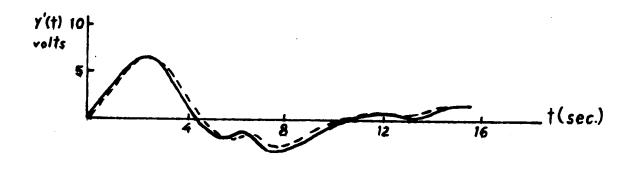
For comparison purposes the same test was carried out with the noise added after the filter (instead of before it as previously done). The resulting output is shown in Fig. 6.28. In this case the noise level was decreased from 3 to zero volts R.M.S. It is seen from the provided record that the system did not provide any noise reduction or smoothing as in the last case.

To conclude we can state that being able to insert filters in both the system and model output signal paths, before going to the divider (without affecting the quality of the information extraction mechanism), offers a great help in reducing the feedback transducer noise influence on the system output.

iii) Finally, the effect of predivider rectifiers and high signal limiting, in the case of a variable inertia scheme with deviations in the uncompensated parameters, is studied.

The plant whose response was given in Fig. 6.10 is reexamined with the divider output limited to low and intermediate values of 24 and 62 volts respectively. The ideal divider output with the current deviation in damping, only, should be 18 volts. These three divider outputs in volts correspond to the 0.24, 0.62 and 0.18 units used in the digital simulation discussed on page 5.25. The plant step response, the plant output rate of change and the divider output are shown in Fig.6.29 for both limiting levels. The response obtained digitally
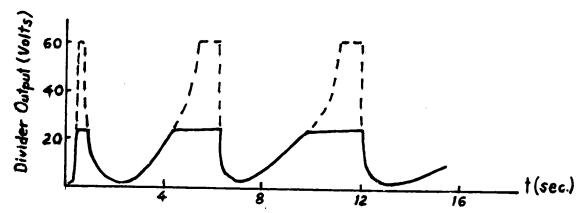
Fig. 6.29 Effect of Divider Output Limiting.
Second-Order Inertia Invariance Scheme with
Large Discrepancies in η and T; $\eta = 0.1\eta_d$,
$\eta_d = 0.5$, T=1.42T_d, $T_d = 1$, k=5.

before is also shown for comparison. The input signal is 2 volts. From these curves it is seen that in this case the limiting level has only a small effect on the response shape, since the high values occur at the low values of y'.

As for the predivider rectifiers, it is clear from the digitally simulated case of Fig. 5.11 that they provide improvement in the system response.

6.7 Some Quantitative Estimates Based on the Collected Data

In the simulation study reported earlier in this chapter, the variable parameter changed over a wide range, in some cases as far as five times its nominal value. However, the overall performance remained within close limits of the desired model. This can be judged by comparing the dotted and continuous curves given before. It can be concluded safely that the proposed invariance schemes offer a reduction in sensitivity.

In the present section we will give quantitative measures of the sensitivity and explore its dependence on the parameter deviation. These values are extracted from the simulation data. It would be very difficult to compare these sensitivity measures with corresponding ones evaluated for high gain schemes, since the figures for the latter schemes are dependent on the specific design employed.

Two measures of invariance will be used here; the percentage ratio of the peak difference between the actual and

the nominal outputs to the steady-state value of the output, and the integral of the magnitude of this difference.

The above measures will be evaluated for the second order inertia plant previously tested. All its parameters are in perfect matching with those of the model except for its damping factor. The input step is 2.5 volts. In Fig. 6.30 the percentage ratio of the peak deviation to the steady output is plotted against the damping ratio η/η_d, and Fig. 6.31 shows the normalized integral of the deviation in volt.sec. per volt input plotted also against the damping ratio. A typical peak deviation is 13% for a change of the damping to one-half its nominal value.

The values of the above invariance measures depend on the type of schemes used and the kind of parameter disturbance involved, but a common property to all invariance schemes is that no steady-state error exists.
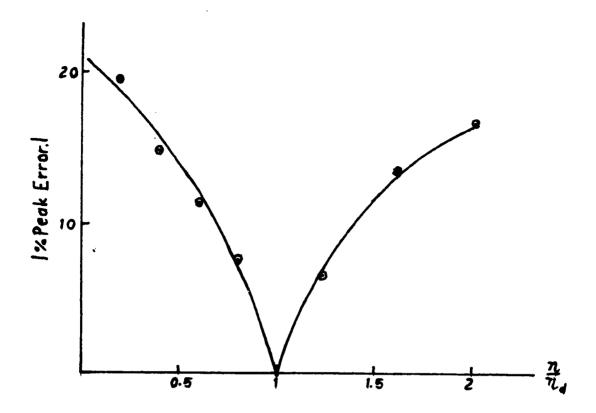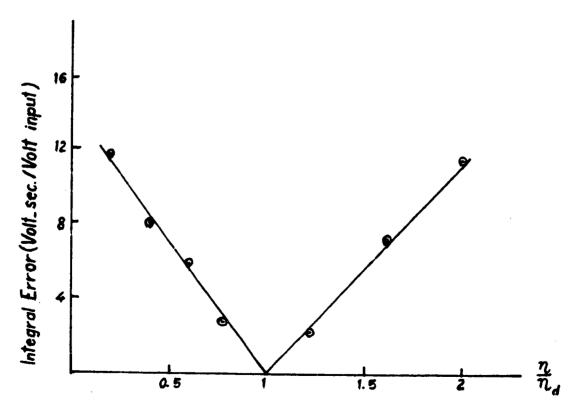
Fig. 6.30 % Peak Error Against Damping Ratio

Fig. 6.31 Normalized Integral of Error Magnitude
Against Damping Ratio.

CHAPTER 7

CONCLUSIONS

7.1 The Value of the Schemes for Previously Difficult Situations

These schemes, as previously explained, differ conceptually from high gain feedback types. There is no high gain involved and thus they are not susceptible to saturation or wide-band problems. They are especially suited for parameter invariance control irrespective of the dynamics of the plant. Thus, adding a pole to the plant will not alter the mechanism or cause stability problems provided it is added simultaneously to the model. Gain schemes can handle multiplicative as well as additive disturbance successfully without the appearance of excessive drives which are common with high gain feedback. Inertia schemes can be used alone or compoundly with gain schemes to make the overall scheme invariant of plant dynamics.

However, apart from this conceptual difference, the proposed schemes offer the only available means for sensitivity reduction in case of feedback unstable plants; e.g. higher order plants, non-min-phase plants and plants with pure delay. The fact that unity feedback, however limited its value may be, would lead to instability with some plants, had made these plants a standing problem [9].

Feedback transducer noise (broadband) does not represent a problem in our case as it usually does with high gain

feedback, since the necessary filters can be added without affecting the invariance mechanism.

7.2 The Use of Invariance Schemes in Conjunction with Other Schemes

Since the proposed systems are designed for parameter sensitivity, it might be useful to use them in conjunction with other schemes designed for dynamic improvement or disturbance rejection. Two examples of such combinations are shown in Fig. 7.1 and Fig. 7.2.

The first of these schemes uses a gain configuration on the plant and an optimum switching circuit to drive the constant parameter system in a minimum-time fashion. This application seems to be very desirable since optimum controls are usually parameter-sensitive.

The second scheme is for a reversed set-up. In this case a linear feedback compensator is first applied around the plant to improve its dynamics. Around the resulting system an invariance scheme can take care of the parameter insensitivity requirements.

7.3 Final Remarks

The proposed schemes are of fundamental interest, since they represent a different approach to the sensitivity problem and give simple solutions for long standing difficulties.
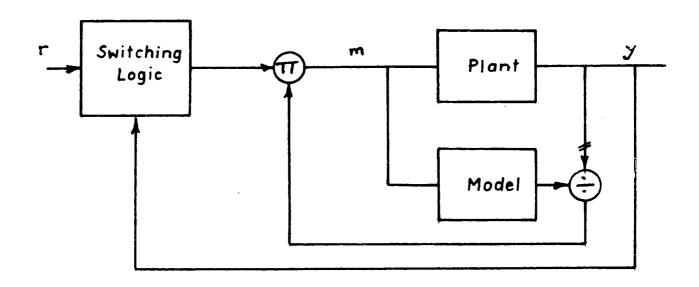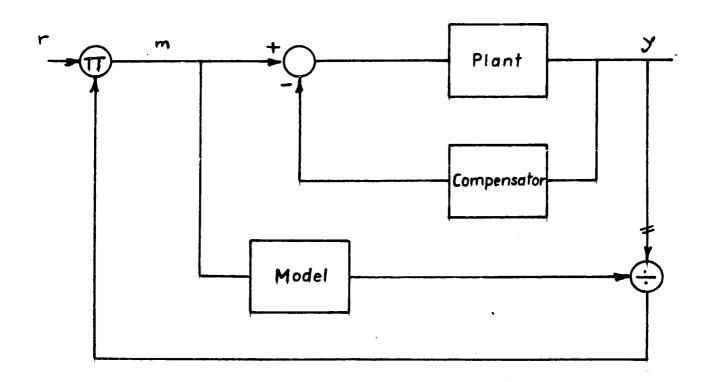
Fig. 7.1   Internal Use of Invariance Schemes

Fig. 7.2   External Use of Invariance Schemes

Identification is done implicitly in the schemes, as compared to the explicit identification approach usually encountered in the literature [15].

The simplicity of the proposed schemes allows an appropriate qualitative evaluation of the overall response and an intuitive understanding of the mechanism of action.

Finally the proposed schemes offer practical interest structurally. The fact that no signal shaping is necessary and the only controller components are models and comparators is a practical advantage in itself.

However, the divider remains the heart of any inconvenience that may arise. It exhibits high errors around the zero crossings and is at the present time an expensive component. The use of logical circuits in conjunction with the division comparator is certain to alleviate some of its problems. Due to the non-linearity of the proposed schemes analogue simulation is a necessity for their final design.

The work presented in the thesis raises some questions of theoretical interest. The most relevant is that of a formal mathematical approach to the generalized invariance problem. Introduction of constraints on the allowable operators is necessary if the theoretical achievements are to be of practical value. In the field of differential equations, the study gives practical significance to some non-linear differential and integro-differential equations. At the present time there is no general solution for such equations. But there is a revived

interest in such a problem [17]. Due to the nature of the problem, the interest should not only be confined to obtaining analytical expressions for specified equations, but rather to a better understanding of the asymptotic behaviour. The use of the divider in the schemes sometimes results in the co-efficients of the describing equation having singular points. Such equations are very little studied because of the apparent complications caused by this singularity.

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